Math 249 Notes

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Notation

Let $\langle x^n \rangle f(x)$ denote the coefficient of x^n in $f(x)$.

Lectures 1-6

Definition - Multinomial Coefficient: Define the *multinomial coefficient* $\binom{n}{r_1, \dots, r_k}$ to be the number of permutations of $1^{r_1} \dots k^{r_k}$.

Then consider the $S_n \curvearrowright \{\text{permutations of } 1^{r_1} \dots k^{r_k}\}$, then $\mathrm{Stab}(1^{r_1} \dots k^{r_k}) = S_{r_1} \times \dots \times S_{r_k}$, and so $\mathrm{Stab}(1^{r_1} \dots k^{r_k}) = r_1! \dots r_k!$. Hence, by the Orbit-Stabilizer theorem, the number of orbits (permutations) of $1^{r_1} \dots k^{r_k}$ is $\binom{n}{r_1, \dots, r_k} = \dfrac{n!}{r_1! \dots r_k!}$.

Theorem - Multinomial Theorem: For intermediates $x_1, \dots, x_k$,

$$(x_1 + x_2 + \dots + x_k)^n = \sum_{r_1, \dots, r_k} \binom{n}{r_1, \dots, r_k} x_1^{r_1} \dots x_1^{r_1} \dots x_k^{r_k}.$$

Definition - Multiset (Coefficient): A *multiset* is a set with repetition. Define the *multiset coefficient* $\left\langle {n \atop k} \right\rangle = \binom{n+k-1}{n}$ to be the number of k-element multi-subsets of $[n]$.

Alternatively, the multiset coefficient $\left\langle {n \atop k} \right\rangle$ can be understood as the number of weak compositions of n into k parts, i.e. the number of sequences $(x_1, \dots, x_k), x_i \geq 0$ such that $x_1 + \dots + x_k = n$. This is equivalent to placing unlabelled balls into labelled boxes.

Definition - Stirling Numbers of the 2nd Kind: Define *Stirling numbers of the 2nd kind* $S(n, k)$ to be the number of partitions of $[n]$ into k non-empty subsets. An important property of Stirling numbers is

$$S(n, k) = kS(n-1, k) + S(n-1, k-1).$$

The exponential generating functions for Sterling numbers of the second kind is

$$\sum_n S(n, k) \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}.$$

Using this, we can derive the closed form

$$S(n, k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n.$$

Definition - (Signless) Stirling Numbers of the 1st Kind: Define

$$\sum_k s(n,k)x^k = (x)_n = x(x-1)\dots(x-n+1),$$

where $s(n,k)$ are Stirling numbers of the 1st kind. Similarly,

$$\sum_k (-1)^{n-k} s(n,k)x^k = x(x+1)\dots(x+n-1).$$

Define the *signless Stirling numbers of the first kind* to be $c(n,k) = (-1)^{n-k}s(n,k) = |s(n,k)|$. Then

$$c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1).$$

Additionally, signless Stirling numbers of the 1st kind, $c(n,k)$, count the number of permutations $\sigma \in S_n$ with k cycles.

Generating Function for Integer Partitions: The ordinary generating function for integer partitions is

$$\sum_n p(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i},$$

where $p(n)$ denotes the number of integer partitions of n.

Partition Identities:

1. The number of partitions with odd parts $p_o(n)$ is equal to the number of partitions with distinct parts $p_d(n)$.

   An explicit bijection for showing this identity is

   (a) **Distinct → odd**: Turn each distinct part $r = 2^k l$ (where l is odd) into 2^k copies of l. Then the resulting partition is necessarily composed only of odd parts.

   (b) **Odd → Distinct**: Group like parts together, then if a partition has m parts of size l (l odd), write m in terms of its binary expansion $m = 2^{k_1} + \dots + 2^{k_j}$, then $ml = 2^{k_1}l + \dots + 2^{k_j}l$, which are all distinct.

2. Let $p(n,k)$ be the number of partitions of n into k non-zero parts, then

$$\sum_{n,k} p(n,k)q^k x^n = \prod_{i=1}^{\infty} \frac{1}{1-qx^i} = \sum_k \frac{q^k x^k}{(1-x)\dots(1-x^k)}.$$

Rogers-Ramanujan Identities:

$$\prod_{i\equiv 1,4 \pmod 5} \frac{1}{1-x^i} = \sum_{k=0}^{\infty} \frac{x^{k^2}}{(1-x)\dots(1-x^k)} \tag{1}$$

$$\prod_{i\equiv 2,3 \pmod 5} \frac{1}{1-x^i} = \sum_{k=0}^{\infty} \frac{x^{k(k+1)}}{(1-x)\dots(1-x^k)} \tag{2}$$

Definition - q-analog: We define the q-analog

$$\binom{n}{r_1,\dots,r_k}_q = \sum_{w\sim 1^{r_1}\dots k^{r_k}} q^{\mathrm{inv}(w)},$$

where $\text{inv}(w) = \#\{(i,j) | i < j \text{ and } w(i) > w(j)\}$. Then

$$\binom{n}{k}_q = \sum_{w \sim 0^k 1^{n-k}} q^{\text{inv}(w)}.$$

Alternatively, this can be understood as boundary paths of a partition contained inside a box (from the top-left to bottom-right corner), where a line across is a 0 and a line down is a 1. Then

$$\binom{n}{k}_q = \sum_{\substack{l(\lambda) \leq k \\ \lambda_1 \leq n-k}} q^{\text{inv}(w)}.$$

More explicitly,

$$\binom{n}{r_1, \ldots, r_k}_q = \frac{[n]_q!}{[r_1]_q! \ldots [r_k]_q!},$$

where $[n]_q! = [n]_q [n-1]_q \ldots [1]_q$ and $[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$.

Lemma: For permutations w of $1, 2, \ldots, n$ (or any word with n distinct totally ordered letters),

$$\sum_w q^{\text{inv}(w)} = [n]_q!.$$

Definition - Grassmannian: Define the *Grassmannian* $G_k^n(\mathbb{F}_q)$ as the set of k-subspaces (k-dimensional) of $\mathbb{F}^n$. This is equivalent (bijective) to reduced $k \times n$ echelon matrices (leading, aka left-most, entry in each row is a 1, and 0's in the column above each 1). Then the number of free entries of each reduced echelon matrix is equal to $\text{inv}(w), w \in S_n$, where we consider a sequence of 1-indexed pivot columns $\{2, 4, 7, 9\}$, where $k = 4, n = 9$, as $w = 010100101$. Then

$$\binom{n}{k}_q = |G_k^n(\mathbb{F}_q)|,$$

for q the order of the finite field $\mathbb{F}$.

Consider $k+1$ flags $0 = V_0 \subset V_1 \subset \cdots \subset V_k = \mathbb{F}_q^n$, where $\dim V_i / V_{i-1} = r_i$, then

$$\#\text{ of flags } = \binom{n}{r_1}_q \binom{n-r_1}{r_2}_q \cdots = \binom{n}{r_1, r_2, \ldots, r_k}_q.$$

q-Binomial Theorem(s):

1. $$\sum_{k=0}^{n} q^{\binom{k}{2}} \binom{n}{k}_q x^k = (1+x)(1+qx)\ldots(1+q^{n-1}x)$$

2. $$\sum_{k=0}^{\infty} \binom{n+k-1}{n}_q x^k = \frac{1}{(1-x)(1-qx)\ldots(1-q^{n-1}x)}$$

Lectures 7-15 (Species and Plethystic Evaluation)

Note that the notes about species are incomplete, as the notes are taken on the document titled "Species and Tree-like Structures Notes."

Definition - Species: A species is essentially a functor $\underline{\underline{E}} : I \to I$, where $I =$ (finite sets, bijections). In words, a species is a "structure" (generally combinatorial in nature) that can be assigned to any finite set.

Examples of species: Some common examples of species that we will use include:

1. $\underline{\underline{\pi}}(S) = \{\text{partitions of } S\}$;
2. $\underline{\underline{L}}(S) = \{\text{linear orderings of } S\}$;
3. $\underline{\underline{P}}(S) = \{\text{permutations of } S\}$, i.e. bijective maps $S \to S$;
4. $\underline{\underline{T}}(S) = \{\text{labeled trees with vertex set } S\}$;
5. $\underline{\underline{B}}(S) = \{\text{subsets of } S\}$;
6. $\underline{\underline{M}}_A(S) = \{\text{maps } S \to A\}$;
7. $\underline{\underline{O}}(S) = \{\text{ordered, rooted trees with labels } [n]\}$, for $|S| = n$;
8. $\underline{\underline{x}}_J(S) = \begin{cases} \{\cdot\}, \text{ if } |S| \in J, \\ \emptyset, \text{ otherwise} \end{cases}$, for some $J \subseteq \mathbb{N}$. This is the indicator species.
9. $1 = \underline{\underline{x}}_{\{0\}}(S) = \begin{cases} \{\cdot\}, \text{ if } S = \emptyset, \\ \emptyset, \text{ otherwise} \end{cases}$
10. $x = \underline{\underline{x}}_{\{1\}}(S) = \begin{cases} \{\cdot\}, \text{ if } |S| = 1, \\ \emptyset, \text{ otherwise} \end{cases}$
11. $\underline{\underline{C}}(S) = \{\text{cyclic orderings (single cycle permutations) of } S\}$;
12. $\underline{\underline{E}}(S) = \{\cdot\}$, the trivial species on S, which has exactly 1 structure on every set (including the null set);

Definition - $c_n(p_1, p_2, \dots)$: We define

$$c_n(p_1, p_2, \dots) = \sum_{\sigma \in S_n} \prod_k p_k^{\#k-\text{cycles}},$$

where $\underline{\underline{c}}_p$ is the species of cycles, with k-cycles weighted by p_k. Similarly, define

$$C(x; p) = \sum_k p_k \frac{x^k}{k}.$$

Definition - z_λ: For some partition λ, define

$$z_\lambda = \prod_k r_k! k^{r_k},$$

if $\lambda = (1^{r_1}, 2^{r_2}, \dots)$. Another way to understand this is that z_λ is the size of the centralizer of a permutation of cycle type λ.

Definition - $\Omega(p; x)$: Define

$$\Omega(p; x) = \sum_n c_n(p) \frac{x^n}{n!} = \exp\left(\sum_{k=1}^{\infty} p_k \frac{x_k}{k}\right).$$

As a shorthand, define

$$\Omega(p) = \Omega(p; 1) = \exp\left(\sum_{k=1}^{\infty} \frac{p_k}{k}\right) = \sum_\lambda \frac{p_\lambda}{z_\lambda}.$$

Then

$$\Omega[X + Y] = \Omega[X]\,\Omega[Y] \text{ and } \Omega[-X] = \Omega[X]^{-1}.$$

Additionally, for $X = x_1 + x_2 + \dots,$

$$\Omega[X] = \prod_i \frac{1}{1 - x_i}.$$

Additionally, we define XY by element-wise multiplication, for $Y = y_1 + y_2 + \dots$.

Definition - Isomorphism of Species: We say species $\underline{\underline{F}}, \underline{\underline{G}}$ are *isomorphic*, i.e. $\underline{\underline{F}} \cong \underline{\underline{G}}$ to mean $\underline{\underline{F}}(S) \cong \underline{\underline{G}}(S)$ for all sets S, or that there exists a natural isomorphism of $\underline{\underline{F}}, \underline{\underline{G}}$ as functors.

Definition - Exponential generating function for $\underline{\underline{F}}$: We define the exponential generating function for $\underline{\underline{F}}$ as

$$F(x) = \sum_n |\underline{\underline{F}}([S])| \frac{x^n}{n!},$$

where $|S| = n$, for each n in the summation.

Theorem - Cayley's Theorem: Let $c_T(i)$ denote the number of children of i in tree T. Then *Cayley's theorem* states that

$$\sum_{T \in \underline{\underline{T}}([n])} \prod x_i^{c_T(i)} = (x_1 + \dots + x_n)^{n-1}.$$

Corollary:

1. (Number of trees $T \in \underline{\underline{T}}([n])$ with given $c_T(i) = d_i$, for all i) $= \binom{n-1}{d_1, \dots, d_n}$,

2.

$$\begin{aligned}
\sum_n \sum_{T \in \underline{\underline{T}}([n])} \prod_i \mu_{c_T(i)} \frac{x^n}{n!} &= \sum_n \sum_{d_1 + \dots + d_n = n-1} \binom{n-1}{d_1, \dots, d_n} \mu_{d_1} \dots \mu_{d_n} \frac{x^n}{n!} \\
&= \sum_n \frac{1}{n} \langle z^{n-1} \rangle H(z)^n x^n \\
&= \sum_n \left\langle \frac{z^{n-1}}{(n-1)!} \right\rangle H(z)^n \frac{x^n}{n!},
\end{aligned}$$

where $H(z) = \sum_{k=0}^{\infty} h_k \frac{z^k}{k!}$.

Theorem - Lagrange Inversion: The *Lagrange inversion formula* is (where $f(x)^{\langle -1 \rangle}$ denotes the inverse of f)

$$\begin{aligned}
\left(\frac{x}{H(x)}\right)^{\langle -1 \rangle} &= \sum_n \left(\frac{1}{n} \langle z^{n-1} \rangle H(z)^n\right) x^n \\
&= \sum_n \left(\left\langle \frac{z^{n-1}}{(n-1)!} \right\rangle H(z)^n\right) \frac{x^n}{n!},
\end{aligned}$$

for any formal power series $H(x)$ with invertible constant term $H(0)$.

Definition - Catalan Number: Let the nth *Catalan number*,

$$\begin{aligned}
C_n &= \text{the number of unlabelled binary trees on } n \text{ nodes} \\
&= \text{the number of unlabelled ordered rooted trees on } n+1 \text{ nodes} \\
&= \text{the number of unlabelled ordered forests on } n \text{ nodes.}
\end{aligned}$$

A closed form for Catalan numbers is

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Definition - Z_F: Define

$$Z_F(p_1, p_2, \dots) = \sum_n \frac{1}{n!} \sum_{\sigma \in S_n} |F([n])^\sigma| p_{\tau(\sigma)},$$

where the partition $\tau(\sigma)$ is the cycle shape of the permutation σ. Some properties of this function include that

1. $Z_F(x, 0, \dots) = F(x)$,
2. $Z_F(x, x^2, \dots)$ is the ordinary generation function for unlabelled F-structures, i.e. this is the type generating series of F,
3. $Z_{F+G} = Z_F + Z_G$.

Definition - Plethystic Evaluation: Define the *plethystic evaluation* $z[A] = z|_{p_k \mapsto p_k[A]}$, where $p_k[A] = A|_{a \mapsto a^k}$ for all variables $a \in A$.

Definition - Decorated F-Structures: Let $\mathcal{A}$ be a set with weight monomials $x_a (a \in \mathcal{A})$ in variables x, then

$$A(x) = \sum_{a \in \mathcal{A}} x.$$

Definition - $\mathcal{A}$-Decorated Species: Given a species F, an $\mathcal{A}$-*decorated species* $F_{\mathcal{A}}(S) = F(S) \times \{\text{maps } \alpha : S \to \mathcal{A}\}$. In words, an $\mathcal{A}$-decorated species $F_{\mathcal{A}}(S)$ is an F-structure on S together with a decoration $\alpha : S \to A$.

Proposition: $Z_F[A]$ is the ordinary generating function for unlabelled $\mathcal{A}$-decorated F-structures, weighted by $\prod_{s \in S} x_{\alpha(s)}$ for decoration $\alpha : S \to \mathcal{A}$, where $A = A(x) = \sum_{a \in \mathcal{A}} x_a$ is the ordinary generating function for $\mathcal{A}$.

Definition - Plethysm: We define a plethysm to be

$$Z(p_1, p_2, \dots) * W(p_1, p_2, \dots) = Z|_{p_k \mapsto W(p_k, p_{2k}, \dots)}.$$

Some properties of plethysm are that

1. $Z * W$ is linear and multiplicative in Z,
2. $p_k * W$ is linear and multiplicative in W,
3. $p_k * p_l = p_{kl}$.

Lemma: $(Z * W)[A] = Z[W[A]]$, in other words, $*$ is associative.

$\Omega[A + B] = \Omega[A]\Omega[B]$

For species F, E, T, if $F = E \circ T$, then $Z_F = Z_E * Z_T$. Additionally, for E the trivial species, $Z_E = \Omega$. For species product, $T = x \cdot F$, we have that $Z_T = Z_x \cdot Z_F$.

Species generating function examples:

1. $Z_C[x, 0, \dots] = C(x) = \log \dfrac{1}{1-x}$,
2. $Z_C[X] = \sum_{n=1}^{\infty} x^n$,

Definition - Composition with Trivial Species: Composition with the trivial species E, for $F = E \circ G$ means that F are disjoint unions of connected components, where G is the species of connected structures. In other words, an F structure is some collection of disjoint unordered G structures on some set.

Definition - Plethystic Logarithm: Define the *Plethystic logarithm* $\Lambda(p_1, p_2, \dots)$ by

$$\Omega * \Lambda = 1 + p_1,$$

where $\Omega = Z_E$ is the plethystic exponentiation.

Definition - Möbius/Inversion: Define the *Möbius function*

$$\mu(n) = \begin{cases} (-1)^r, \text{ if } n = p_1 \dots p_r, \text{ for distinct primes } p_i, 1 \leq i \leq r, \\ 0, \text{ otherwise.} \end{cases}$$

Möbius inversion states that

$$f(n) = \sum_{d|n} g(d) \iff g(n) = \sum_{d|n} \mu(n/d) f(d).$$

In particular, taking $g(n) = \delta_{1,n}$

$$\delta_{i,j} = \begin{cases} 1, \text{ if } i = j, \\ 0, \text{ if } i \neq j. \end{cases},$$

we get $f(n) = 1$.

Theorem: The solution of $\Omega * \Lambda = 1 + p_1$ is

$$\Lambda = \sum_{\ell} \frac{\mu(\ell)}{\ell} \log(1 + p_\ell).$$

Definition - Euler's Totient Function: Define *Euler's totient function*

$$\varphi(n) = |\{j \in [n] | \gcd(j, n) = 1\}|.$$

In other words, $\varphi(n)$ is the number of positive integers less than n that are coprime to n. By Möbius inversion, $\frac{\varphi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}$.

Lectures 16-24 (Symmetric Functions)

Definition - $\Lambda_R(x_1, \dots, x_n)$: Denote by $\Lambda_R(x_1, \dots, x_n) = R[x_1, \dots, x_n]^{S_n}$ the ring of symmetric polynomials in n variables, for $R = \mathbb{R}$ or $\mathbb{Q}$.

$\Lambda_R(x) = \oplus_{d \geq 0} \Lambda_R(x)_d$ is graded, where $\Lambda_R(x)_d = \{f \in \Lambda_R \text{ homogenous of degree } d\}$.

Definition - m_λ: In $\Lambda_R(x_1, x_2, x_3)$,

1. $m_\emptyset = 1$,
2. $m_{(1)} = x_1 + x_2 + x_3$,
3. $m_{(21)} = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$.

Proposition: $\Lambda_R(x_1, \dots, x_n)_d$ is a free R-module with basis $\{m_\lambda | \ell(\lambda) \le n, |\lambda| = d\}$.

To formalize some notations of symmetric functions in finitely many variables, we note some things. Firstly, in infinitely many variables, any non-constant symmetric polynomial is a formal infinite series, but we can think of it as a polynomial if f has bounded degree. Additionally, S_∞ can be understood as permutations $\mathbb{N} \to \mathbb{N}$, or the subgroup generated by transpositions, i.e. the union of $S_1 \subset S_2 \subset \dots$ belonging in S_∞. Additionally, $\Lambda_R(x_1, x_2, \dots) \twoheadrightarrow \Lambda_R(x_1, \dots, x_n)$, define by $f(x) \mapsto f(x_1, \dots, x_n, 0, 0, \dots)$ is a surjective R-algebra homomorphism, and bijective on $(\Lambda_R)_d$, for $d \le n$ (though currently not sure how to prove the bijective part).

Proposition:

1. In infinitely many variables, $(\Lambda_R)_d$ is a free R-module with basis $\{m_\lambda \,|\, |\lambda| = d\}$.
2. $\Lambda_R(x_1, x_2, \dots) \to \Lambda_R(x_1, \dots, x_n)$ is $m_\lambda \to \begin{cases} m_\lambda, \ell(\lambda) \le n, \\ 0, \text{ otherwise} \end{cases}$ Note that what this is saying is the mapping of functions. In particular, for some $\ell(\lambda) \le n$, $m_\lambda(x_1, x_2, \dots) \mapsto m_\lambda(x_1, \dots, x_n)$.

Definition - e_k, h_k, p_k:

1. Define
$$e_k = m_{(1^k)} = \sum_{i_1 < \dots < i_k} x_{i_1} \dots x_{i_k}$$
to be the kth *elementary symmetric function.*
2. Define
$$h_k = \sum_{|\lambda|=k} m_\lambda = \sum_{i_1 \le \dots \le i_k} x_{i_1} \dots x_{i_k}$$
to be the kth *complete homogenous symmetric function.*
3. Define
$$p_k = m_{(k)} = x_1^k + x_2^k + \dots$$
to be the kth *power sum symmetric function.*

Finally, we define $e_\lambda = e_{\lambda_1} \dots e_{\lambda_\ell}$, for $\ell = \ell(\lambda)$; h_λ and p_λ are defined similarly.

Generating functions for the above basis: For e_k,
$$E(t) = \sum e_n t^n = \prod_i (1 + t x_i).$$

For h_k,
$$H(t) = \sum h_n t^n = \prod_i \frac{1}{1 - t x_i} = E(-t)^{-1}.$$

Finally, for p_k,
$$P(t) = \frac{d}{dt} \log \prod_{i \ge 1} (1 - x_i t)^{-1} = \frac{d}{dt} \log H(t) = H'(t)/H(t).$$

Additionally,
$$\Omega[X] = \prod_i \frac{1}{1 - x_i}, \text{ and } H(t) = \Omega[tX].$$

Proposition: Clearly, $H(t)E(-t) = 1$, which implies that $h_n - h_{n-1} e_1 + \dots + (-1)^n e_n = 0$.

Definition - Dominance Partial Ordering on Partitions of n: We say that $\lambda \leq \mu$ if

$$|\lambda| = |\mu| \text{ and } \lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k, \forall k \leq \max(\ell(\lambda), \ell(\mu)),$$

where we pad the shorter partition with 0's. Note that this is a partial ordering, as neither of the following two partitions of 6 are greater than another: $(2,2,2)$ and $(3,1,1,1)$, as $2 \leq 3$ but $2+2+2 = 6 \geq 5 = 3+1+1$.

Proposition: $\leq$ is the transitive closure of the raising operator relation $\lambda \to \mu$ if $\mu - \lambda = \epsilon_i - \epsilon_j$, for $i < j$, where $\epsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the ith standard basis vector.

Corollary: $\lambda \leq \mu \iff \lambda^* \geq \mu^*$, where $|\lambda| = |\mu|$.

Proposition: $\ell_\lambda = \sum a_{\lambda\mu} m_\mu$, where $a = \#$ 0-1 matrices with row sum λ and column sum μ.

Proposition: $a_{\lambda\mu} \neq 0 \iff \mu \leq \lambda^*(\mu^* \geq \lambda)$, and $a_{\lambda\lambda^*} = 1$.

Corollary:

(a). $e_\lambda = m_{\lambda^*} + \sum_{\mu < \lambda^*} a_{\lambda\mu} m_\mu$.

(b). $\{e_\lambda\}$ is a graded R-basis of Λ_R. Similarly, $\{e_\lambda | \lambda_1 \leq n\}$ is a graded R-basis of $\Lambda_R(x_1, \ldots, x_n)$. Additionally, $e_k(x_1, \ldots, x_n) = 0$ for $k > n$.

(c). $\Lambda_R \cong R[e_1, e_2, \ldots]$ as a graded R-algebra, with $\deg e_k = k$.

(d). There is a unique R-algebra homomorphism

$$\begin{aligned} w : \Lambda_R &\to \Lambda_R \\ e_k &\mapsto h_k, \end{aligned}$$

where $w^2 = id$ (hence w is an isomorphism).

(e). $\Lambda_R \cong R[h_1, h_2, \ldots]$; $\{h_\lambda\}$ is a graded R-basis.

Proposition: $h_\lambda = \sum b_{\lambda\mu} m_\mu$ where $b_{\lambda\mu} = \#$ of $\mathbb{N}$-matrices with row sums λ and column-sums μ.

Definition/Corollary - Hall Inner Product:

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}$$

is symmetric, graded, and is a perfect pairing on each $(\Lambda_R)_d$.

Using the fact that $\Omega = \sum_\lambda \frac{p_\lambda}{z_\lambda}$, we know that $h_n = \sum_{|\lambda|=n} \frac{p_\lambda}{z_\lambda}$.

Corollary: $\{p_\lambda : |\lambda| = d\}$ is a basis of $(\Lambda_R)_d$ if $\mathbb{Q} \subseteq R$.

Corollary: Any polynomial or power series $Z(p_1, p_2, \ldots)$ in variables p_k is determined by the symmetric polynomial of series $Z[X]$, where $x = x_1 + x_2 + \ldots$.

Definition - ϵ: Define

$$\epsilon f[-x] = wf[X],$$

where $\epsilon f[X] = f(-x_1, -x_2, \ldots) = (-1)^d f(x)$ for $f \in \Lambda_d$.

Corollary:

$$wp_k = \epsilon p_k[-X] = -\epsilon p_k[X] = (-1)^{n-1} p_k$$
$$wp_\lambda = (-1)^{n-\ell(\lambda)} p_\lambda, \quad |\lambda| = n,$$

where $(-1)^{n-\ell(\lambda)}$ is the sign of $\sigma \in S_n$ if $\tau(\sigma) = \lambda$.

Proposition - Cauchy Formula, Dual Basis: Let $\{u_\lambda\}, \{v_\lambda\}$ be graded basis of Λ. Then

$$\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu} \iff \Omega[XY] = \sum_\lambda u_\lambda(x) v_\lambda(y),$$

where $\delta_{\lambda\mu}$ is the Kronecker Delta ($= 1$ if $\lambda = \mu$, and 0 otherwise). $\{u_\lambda\}$ is the dual basis of $\{v_\lambda\}$. This also implies that $\langle f[X], \Omega[XY]\rangle_x = f[Y]$.

Theorem:
$$\langle u_\lambda(x)\rangle \Omega[XY] = \langle u_\lambda(x), \Omega[XY]\rangle_X = v_\lambda(y), \text{ i.e. } \Omega[XY] = \sum_\lambda u_\lambda(x) v_\lambda(y).$$

We also have that $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$.

Corollary: $\langle wf, wg \rangle = \langle f, g \rangle$.

$\Omega[AX]^\perp g[X] = g[A + X]$.

Vandermonde's Identity: *Vandermonde's Identity* states that

$$\Delta(x_1, \ldots, x_n) = \prod_{i<j} (x_i - x_j) = \det \begin{pmatrix} x_1^{n-1} & \ldots & x_1 & 1 \\ x_2^{n-1} & \ldots & x_2 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_n^{n-1} & \ldots & x_n & 1 \end{pmatrix}$$

Definition - a_μ: Define

$$a_\mu = \det \begin{pmatrix} x_1^{\mu_1} & \ldots & x_1^{\mu_n} \\ x_2^{\mu_1} & \ldots & x_2^{\mu_n} \\ \vdots & \ddots & \vdots \\ x_n^{\mu_1} & \ldots & x_n^{\mu_n} \end{pmatrix} = \sum_{w \in S_n} \epsilon(w) w(x^\mu),$$

where $\mu_1 > \cdots > \mu_n$, then $\Delta(x_1, \ldots, x_n) = a_p$, for $p = (n-1, \ldots, 1, 0)$. Note that $a_\mu = 0$ if μ is not a strictly decreasing partition.

Schur Functions: The *Schur function*

$$s_\lambda = s_\lambda(x_1, \ldots, x_n) = \frac{a_{\lambda+p}(x)}{a_p(x)} \in \mathbb{Z}[x_1, \ldots, x_n]^{S_n} = \Lambda_{\mathbb{Z}}(x_1, \ldots, x_n).$$

$\{a_{\lambda+p}\}$ forms a $\mathbb{Z}$-basis of $\mathbb{Z}[X]^\epsilon = \Delta(x)\, \mathbb{Z}[X]^{S_n}$, the anti-symmetric polynomials, and thus $\{s_\lambda\}$ forms a basis of $\Lambda(x_1, \ldots, x_n)$.

Properties of Schur Functions:

1. $s_\lambda(x_1, \ldots, x_n, 0 \ldots, 0) = \begin{cases} s_\lambda(x_1, \ldots, x_n), \text{ if } \ell(\lambda) \leq n, \\ 0, \text{ if } \ell(\lambda) > n. \end{cases}$
2. $s_{(1^k)}(x_1, \ldots, x_n) = e_k(x_1, \ldots, x_n)$
3. $s_{(k)}(x_1, \ldots, x_n) = h_k(x_1, \ldots, x_n)$

Corollary/Definition: There exists a unique Schur function $s_\lambda(x_1, x_2, \dots) \in \Lambda$ such that for all n:

$$s_\lambda(x_1, \dots, x_n, 0, \dots) = \begin{cases} s_\lambda(x_1, \dots, x_n), \text{ if } \ell(\lambda) \leq n, \\ 0, \text{ if } \ell(\lambda) > n. \end{cases}$$

and $\{s_\lambda\}$ is a graded basis of Λ. Explicitly, $\langle m_\mu \rangle s_\lambda = \langle x^\mu \rangle s_\lambda(x_1, \dots, x_n)$ for any $n \geq \ell(\mu), \ell(\lambda)$, independent of n.

Weyl Character Formula:

$$\begin{aligned} s_\lambda(x_1, \dots, x_n) &= \sum_{w \in S_n} \frac{\epsilon(w) w(x^{\lambda+p})}{\Delta(x)} \\ &= \sum_{w \in S_n} w \left(\frac{x^{\lambda+p}}{\prod_{i<j}(x_i - x_j)} \right) \\ &= \sum_{w \in S_n} w \left(\frac{x^{\lambda}}{\prod_{i<j}(1 - x_j/x_i)} \right) \\ &= x_\lambda(x_1, \dots, x_n) = \mathrm{tr}_{v_\lambda} \begin{pmatrix} x_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & x_n \end{pmatrix} = \sum \dim(v_\lambda)_\mu x^\mu, \end{aligned}$$

where v_λ is the irreducible representation of $GL_n(\mathbb{C})$ with highest weight λ.

Definition - Pieri Rule, Skew Diagram, Horizontal/Vertical Strips: A *skew diagram* λ/μ is the difference of partition diagrams. A *horizontal strip* is a skew diagram with no 2 blocks with one above the other. Similarly, a *vertical strip* is a skew diagram with no 2 blocks with one to the right of the other. Then λ/μ is a vertical strip $\iff$ $\lambda - \mu$ is a $0, 1$ vector ϵ_I, which is the sum of all the e_i standard basis vectors, for $i \in I$. But $\lambda = \mu + \epsilon_I$ is only a partition of $\mu_{i-1} > \mu_i$, or $i = 1$, for all $i \in I$.

Definition - Semi-standard Young Tableaux, $K_{\lambda\mu}$: A semi-standard Young tableaux is a map $T : \lambda/\mu \to \mathbb{Z}_+$. Additionally, let $K_{\lambda\mu} = |SSYT(\lambda, \mu)|$ denote the number of semi-standard Young tableau's of shape λ of digits $1^{\mu_1}, 2^{\mu_2}, \dots$.

Proposition:

$$e_k \cdot s_\lambda = \sum_{\substack{|\mu/\lambda| = k \\ \text{vertical strip}}} s_\mu$$

Corollary:

$$e_\mu = \sum_\lambda K_{\lambda^* \mu} s_\lambda,$$

where μ does not have to be in decreasing order; $K_{\lambda\mu}$ is constant with respect to permuting μ.

Bernstein Operators:

$$B_m f(x_1, \dots, x_{n-1}) = \sum_{w \in S_n / S_1 \times S_{n-1}} w \left(\frac{x_1^m f(x_2, \dots, x_n)}{\prod_{j \neq 1}(1 - x_j/x_i)} \right),$$

for some symmetric polynomial f.

By the Weyl character formula,

$$s_\lambda(x_1, \dots, x_n) = B_{\lambda_1} \dots B_{\lambda_n}(1).$$

Stable Formula:

$$B_m f[X] = \sum_i \frac{x_i^m f[X - x_i]}{\prod_{j \neq i}(1 - x_j/x_i)},$$

for $X = x_1 + \cdots + x_n$.

$\langle z^0 \rangle \Omega[X/z] z^m f[X - z] = B_m f[X]$, for $m \geq 0$, then

$$B_m = \langle z^{-m} \rangle \Omega[X/z]^\bullet \Omega[-zX]^\perp,$$

using the fact that $f[X + A] = \Omega[AX]^\perp$.

Lemma - Dual Pieri Rule: $\Omega[AX]^\perp \Omega[BX]^\bullet = \Omega[AB]\Omega[BX]^\bullet \Omega[AX]^\perp$.

Proposition:

1. $e_k^\perp B_m = B_m e_k^\perp + B_{m-1} e_{k-1}^\perp$
2. $B_r B_{s+1} = -B_s B_{r+1}$ ($= 0$ if $r = s$)

Proposition:

$$e_k^\perp s_\lambda = \sum_{\substack{|\lambda/\mu| = k \\ \text{vertical strip}}} s_\mu$$

Proposition: Schur functions are orthonormal, i.e. $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$.

Lemma: $K_{\lambda\mu} = 0$ if $\mu \not\leq \lambda$; $K_{\lambda\lambda} = 1$.

Proposition: $w s_\lambda = s_{\lambda^*}$.

Corollary:

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu = \sum_{T \in SSYT(\lambda)} x^T,$$

where $x^T = \sum_{c \in \lambda} x_{T(c)}$.

Corollary: Pieri rules

$$h_k s_\lambda = \sum_{\substack{|\mu/\lambda| = k \\ \text{horizontal strip}}} s_\mu$$

$$h_k^\perp s_\lambda = \sum_{\substack{|\lambda/\mu| = k \\ \text{horizontal strip}}} s_\mu$$

Corollary: $h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$.

Definition - Schur Functor: For $|\lambda| = d$, V a vector space, the Schur functor

$$S_\lambda(V) = V_\lambda = \operatorname{Im} \psi,$$

for $\psi : \Lambda^{\lambda_1^*}(V) \otimes \cdots \otimes \Lambda^{\lambda_k^*}(V) \mapsto S^{\lambda_1}(V) \otimes \cdots \otimes S^{\lambda_\ell}(V)$.

Example: For $V = \mathbb{C}^n$, $S(V) = \mathbb{C}[x_1, \ldots, x_n]$, and $S^d(V)$ be the ring of homogenous polynomials of degree d,

$$S(V) \otimes \cdots \otimes S(V) = \mathbb{C}[x] \otimes \cdots \otimes \mathbb{C}[x] = \mathbb{C}[x_1^{(1)}, \ldots, x_n^{(1)}, \ldots, x_1^{(\ell)}, \ldots, x_n^{(\ell)}],$$

where $x^{(j)}$ denotes the jth tensor factor.