Species and Tree-like Structures Notes

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A. Theory of Combinatorial Species

A.1 Introduction to Species of Structures

Definition - Structure, Underlying Set: Loosely, a *structure* s is a construction γ on a set U, where

$$s = (\gamma, U)$$

denotes the structure of γ on U. We often call U the *underlying set* of the structure s.

We may also abuse notation by writing $s = \gamma$ to mean $s = (\gamma, U)$, if there is no ambiguity of the underlying set U.

Definition - Isomorphism (Type), Unlabelled Structures: Given a structure $s = (\gamma, U)$, if we want to relabel $U = \{a, b, c\}$ to $V = \{1, 2, 3\}$, where $\sigma : U \to V$ takes

$$\begin{aligned} a &\mapsto 1 \\ b &\mapsto 2 \\ c &\mapsto 3, \end{aligned}$$

then we write that $t = \sigma \cdot s$. In this case, we say that s and t are *isomorphic*. Alternatively, we say that s, t have the same *isomorphism type*, in which the elements are indistinguishable points and the structure is said to be *unlabelled*.

Definition - Automorphism: If $\sigma : U \to V = U$ is a permutation of U, then σ is said to be an *automorphism* of the structure on which it was defined. In this case, we write $s = \sigma \cdot s$ and the transported (post-image) structure is identical to the original structure.

Informal Exploration of Species of Structures: Let $\mathcal{G}$ denote the *species* of simple graphs (undirected graphs without loops or multiple edges). Then for each finite set U, $\mathcal{G}[U]$ denotes the set of all structures of simple graph on U, or

$$\mathcal{G}[U] = \{g | g = (\gamma, U), \gamma \subseteq \wp^{[2]}[U]\},$$

where $\wp^{[2]}[U]$ stands for the collection of (unordered) pairs of elements of U. In the simple graph $g = (\gamma, U)$, the elements of U are the vertices and y is the set of edges. Clearly $G[U]$ is a finite set. The following three expressions are considered to be equivalent:

1. g is a simple graph on U;
2. $g \in \mathcal{G}[U]$;
3. g is a $\mathcal{G}$-structure on U.

Then $\sigma : U \to V$ defined as above induces a function

$$\mathcal{G}[\sigma] : \mathcal{G}[U] \to \mathcal{G}[V], g \mapsto \sigma \cdot g$$

describing the transport of graphs along σ. Formally, for $g = (\gamma, U)$, $\mathcal{G}[\sigma](g) = \sigma \cdot g = (\sigma \cdot \gamma, V)$, where $\sigma \cdot \gamma$ is the set of all $\{\sigma(x), \sigma(y)\}$, obtained from pairs $\{x, y\} \in \gamma$.

Then it follows that for bijections $\sigma : U \to V, \tau : V \to W$,

$$\mathcal{G}[\tau \circ \sigma] = \mathcal{G}[\tau] \circ \mathcal{G}[\sigma],$$

and $\mathcal{G}[\mathrm{Id}_U] = \mathrm{Id}_{\mathcal{G}[U]}$. These two equations above make $\mathcal{G}$ a functor.

Definition 1.1.3 - Species of Structures, (Transport of) F-Structure: A species of structures is a rule F which

i) produces, for each finite set U, a finite set $F[U]$,

ii) produces, for each bijection $\sigma : U \to V$, a function

$$F[\sigma] : F[U] \to F[V].$$

The functions $F[\sigma]$ should further satisfy the following functorial properties:

a) for all bijections $\sigma : U \to V$ and $\tau : V \to W$,

$$F[\tau \circ \sigma] = F[\tau] \circ F[\sigma].$$

b) for the identity map $\mathrm{Id}_U : U \to U$,

$$F[\mathrm{Id}_U] = \mathrm{Id}_{F[U]}\,.$$

An element $s \in F[U]$ is called an *F-structure on U* (or a *structure of species F on U*). The function $F[\sigma]$ is called a *transport of F-structures along U*. The following three expressions are considered to be equivalent:

1. s is a structure of species F on U;
2. $s \in F[U]$;
3. s is an F-structure on U.

It immediately follows from the functorial properties that each transport function $F[\sigma]$ is necessarily a bijection, and we use $\sigma \cdot s$, or $\sigma \cdot_F s$, to designate $F[\sigma](s)$.

Definition 1.1.4 - Isomorphism (Type), Automorphism: Consider two F-structures $s_1 \in F[U]$ and $s_2 \in F[V]$. A bijection $\sigma : U \to V$ is called an *isomorphism* of s_1 to s_1 if $s_2 = \sigma \cdot s_1 = F[\sigma](s_1)$. One says that these structures have the same *isomorphism type*. Moreover, an isomorphism from s to s is said to be an *automorphism* of s.

Basic Species:

- the species $\mathcal{A}$, of *rooted trees*;
- the species $\mathcal{G}$, of *simple graphs*;
- the species $\mathcal{G}^c$, of *connected simple graphs*;
- the species $\mathfrak{a}$, of *trees* (connected simple graphs without cycles);
- the species $\mathcal{D}$, of *directed graphs*;

- the species Par, of *set partitions*;
- the species $\wp$, of *subsets*, i.e.,
$$p[U] = \{S | S \subseteq U\};$$
- the species End, of *endofunctions*, i.e.,
$$\text{End}[U] = \{\psi | \psi : U \to U\};$$
- the species Inv, of *involutions*, i.e., those endofunctions ψ such that $\psi \circ \psi = \text{Id}$;
- the species S, of *permutations* (i.e., bijective endofunctions);
- the species C, of *cyclic permutations* (or oriented cycles);
- the species L, of *linear (or total) orders*;
- The species E, of *sets*, defined by $E[U] = \{U\}$. For each finite set U, there is a unique E-structure, namely the set U itself;
- The species ϵ, of *elements*, defined by $\epsilon[U] = U$, where the structures on U are the elements of U;
- The species X, *characteristic of singletons*, defined by
$$X[U] = \begin{cases} \{U\}, & \text{if } |U| = 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$
Here $\emptyset$ denotes the empty set. As a consequence, there are no X-structures on a set U when $|U| \neq 1$.
- The species 1, *characteristic of the empty set*, defined by
$$1[U] = \begin{cases} \{U\}, & \text{if } U = \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}$$
- The species 0, the *empty* species, defined by $0[U] = \emptyset$ for all U.
- The species E_2, *characteristic of sets of cardinality 2*, defined by
$$E_2[U] = \begin{cases} \{U\}, & \text{if } |U| = 2, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Remark 1.1.6: The reader who is familiar with category theory will have observed that a species is simply a functor
$$F : \mathbb{B} \to \mathbb{E}$$
from the category $\mathbb{B}$ of finite sets and bijections to the category $\mathbb{E}$ of finite sets and functions.

Definition - (Un)labeled Structure: An F-structure $s \in F[U]$ on a set U is often referred to as a *labeled structure*, whereas an *unlabeled structure* is an isomorphism class of F-structures.

Definition 1.2.1 - Exponential/Ordinary Generating Series: The *exponential generating series* of a species of structures F is the formal power series
$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!},$$
where $f_n = |F[n]| =$ the number of F-structures on a set of n elements (labeled structures), where $F[n] = F[\{1, 2, \ldots, n\}] = F[[n]]$.

The *ordinary generating series* of a species of structures F is defined similarly as the formal power series

$$G(x) = \sum_{n=0}^{\infty} g_n x^n.$$

Notation - $[x^n]$:

$$[x^n]G(x) = g_n, \text{ and}$$
$$n![x^n]F(x) = f_n = \left.\frac{d^n F(x)}{dx^n}\right|_{x=0}.$$

Example 1.2.2 - (Exponential) Generating Series of Species:

a) $L(x) = \dfrac{1}{1-x}$,

b) $S(x) = \dfrac{1}{1-x}$,

c) $C(x) = -\log(1-x)$,

d) $E(x) = e^x$,

e) $\epsilon(x) = xe^x$,

f) $\wp(x) = e^{2x}$,

g) $X(x) = x$,

h) $1(x) = 1$,

i) $0(x) = 0$,

j) $\mathcal{G}(x) = \displaystyle\sum_{n\geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!}$

k) $\mathcal{D}(x) = \displaystyle\sum_{n\geq 0} 2^{n^2} \frac{x^n}{n!}$

l) $\text{End}(x) = \displaystyle\sum_{n\geq 0} n^n \frac{x^n}{n!}$

Definition - Equivalence Relation: We can define an *equivalence relation* $s \sim t$ if and only if s and t have the same isomorphism type. Alternatively, $s \sim t$ if and only if there exists a permutation $\pi : [n] \to [n]$ such that $F[\pi](s) = t$.

Then, by definition, an isomorphism type of F-structures of order n is an equivalence class (modulo $\sim$) of F-structures on $[n]$. Such an equivalence class is also called an unlabeled F-structure of order n. Denote by $T(F_n)$ the quotient set $F[n]/\sim$ of types of F-structures of order n and let

$$T(F) = \sum_{n\geq 0} T(F_n).$$

Definition 1.2.3 - (Isomorphism) Type Generating Series: The *(isomorphism) type generating series* of a species of structures F is the formal power series

$$\tilde{F}(x) = \sum_{n\geq 0} \tilde{f}_n x^n,$$

where $\tilde{f_n} = |T(F_n)|$ is the number unlabeled F-structures of order n.

Example 1.2.4 - Type Generating Series Examples:

a) $\tilde{L}(x) = \dfrac{1}{1-x}$,

b) $\tilde{S}(x) = \displaystyle\prod_{k=1}^{\infty} \frac{1}{1-x^k}$,

c) $\tilde{C}(x) = \dfrac{x}{1-x}$,

d) $\tilde{E}(x) = \dfrac{1}{1-x}$,

e) $\tilde{\epsilon}(x) = \dfrac{x}{1-x}$,

f) $\tilde{\wp}(x) = \dfrac{1}{(1-x)^2}$,

g) $\tilde{X}(x) = x$,

h) $\tilde{1}(x) = 1$,

i) $\tilde{0}(x) = 0$,

Definition 1.2.5 - Cycle Type, Fixed Points: Let U be a finite set and σ, a permutation of U. The *cycle type* of the permutation σ is the sequence $(\sigma_1, \sigma_2, \dots)$ where for $k \geq 1, \sigma_k$ is the number of cycles of length k in the decomposition of a into disjoint cycles.

Then by the definition above, σ_1 is the number of *fixed points* of σ. Additionally, define

$$\begin{aligned} \operatorname{Fix}\sigma &= \{u \in U | \sigma(u) = u\}, \\ \operatorname{fix}\sigma &= |\operatorname{Fix}\sigma| = \sigma_1, \end{aligned}$$

where Fix denotes the set of fixed points of σ, and fix denotes the number of fixed points of σ.

Definition 1.2.6 - Cycle Index Series: The *cycle index series* of a species of structures F is the formal power series (in an infinite number of variables $x_1, x_2, x_3, \dots$)

$$Z_F(x_1, x_2, x_3, \dots) = \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in S_n} \operatorname{fix} F[\sigma] x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} \dots \right),$$

where S_n denotes the group of permutations of $[n]$ (i.e., $S_n = S[n]$) and $\operatorname{fix} F[\sigma] = (F[\sigma])_1 =$ is the number of F-structures on $[n]$ fixed by $F[\sigma]$, i.e., the number of F-structures on $[n]$ for which σ is an automorphism.

Example 1.2.7 - Cycle Index Series Examples: We give some examples of cycle index series for species that we have seen so far:

1. $Z_0(x_1, x_2, x_3, \dots) = 0$,
2. $Z_1(x_1, x_2, x_3, \dots) = 1$,
3. $Z_X(x_1, x_2, x_3, \dots) = x_1$,
4. $Z_L(x_1, x_2, x_3, \dots) = \dfrac{1}{1-x_1}$,

5. $Z_S(x_1, x_2, x_3, \dots) = \dfrac{1}{(1-x_1)(1-x_2)(1-x_3)\dots}$,
6. $Z_E(x_1, x_2, x_3, \dots) = \exp\Big(x_1 + \dfrac{x_2}{2} + \dfrac{x_3}{3} + \dots\Big)$,
7. $Z_\epsilon(x_1, x_2, x_3, \dots) = x_1 \exp\Big(x_1 + \dfrac{x_2}{2} + \dfrac{x_3}{3} + \dots\Big)$.

Theorem 1.2.8: For any species of structures F, we have

1. $F(x) = Z_F(x, 0, 0, \dots)$,
2. $\tilde{F}(x) = Z_F(x, x^2, x^3, \dots)$.

We have that

$$Z_F(x_1, x_2, x_3, \dots) = \sum_{n_1+2n_2+3n_3+\dots<\infty} \operatorname{fix} F[n_1, n_2, n_3, \dots] \frac{x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots}{1^{n_1} n_1! 2^{n_2} n_2! 3^{n_3} n_3! \dots},$$

where $\operatorname{fix} F[n_1, n_2, n_3, \dots]$ denotes the number of F-structures on a set of $n = \sum_{i\geq 1} i n_i$ elements which are fixed under the action of any (given) permutation of type $n := (n_1, n_2, n_3, \dots)$. Additionally

$$\operatorname{Aut}(n) = 1^{n_1} n_1! 2^{n_2} n_2! 3^{n_3} n_3! \dots,$$

and

$$\operatorname{fix} F[n] = \operatorname{coeff}_n Z_F = \operatorname{Aut}(n)[x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots] Z_F(x_1, x_2, x_3, \dots).$$

Definition 1.2.11 - Equipotent: Let F and G be two species of structures. An *equipotence* α of F to G is a family of bijections α_U, where for each finite set U,

$$\alpha_U : F[U] \xrightarrow{\sim} G[U].$$

The two species F and G are then called *equipotent*, and one writes $F \equiv G$. Clearly, $F \equiv G \iff F(x) = G(x)$, however, $F \equiv G \not\Longrightarrow F(x) = G(x)$

Definition 1.2.12 - Naturality Condition, Isomorphism, Combinatorial Equality: Let F and G be two species of structures. An *isomorphism* of F to G is a family of bijections $\alpha_U : F[U] \to G[U]$ which satisfies the following *naturality condition*: For any bijection $\sigma : U \to V$ between two finite sets, the following diagram commutes:

```tikzcd
F[U] \arrow[r, "\alpha_U"] \arrow[d, "F[\sigma]"'] & G[U] \arrow[d, "G[\sigma]"] \\
F[V] \arrow[r, "\alpha_V"] & G[V]
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In other words, for any F-structures $s \in F[U]$, one must have $\sigma \cdot \alpha_U(s) = \alpha_V(\sigma \cdot s)$. The two species F and G are then said to be *isomorphic*, and one writes $F \simeq G$.

$$F \simeq G \implies \begin{cases} F(x) = G(x), \\ \tilde{F}(x) = \tilde{G}(x), \\ Z_F(x_1, x_2, x_3, \dots) = Z_G(x_1, x_2, x_3, \dots). \end{cases}$$

As a result, we may use $F = G$ interchangeable with $F \simeq G$ to represent *combinatorial equality* between species F and G.

Definition - Contact of Order n: Let $a(x) = \sum_{n\geq 0} a_n x^n$ and $b(x) = \sum_{n\geq 0} b_n x^n$, one says that $a(x)$ and $b(x)$ have *contact of order* n, and one writes $a(x) =_n b(x)$, if for all $k \leq n$, $[x^k]a(x) = [x^k]b(x)$. In other words, $a_{\leq n}(x) = b_{\leq n}(x)$.

Contact of order n for index series of the form

$$h(x_1, x_2, x_3, \dots) = \sum_{n_1+2n_2+3n_3+\dots} h_{n_1 n_2 n_3 \dots} x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots$$

is defined to be

$$h_{\leq n}(x_1, x_2, x_3, \dots) = \sum_{n_1+2n_2+3n_3+\dots \leq n} h_{n_1 n_2 n_3 \dots} x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots$$

Definition 1.2.14 - Contact of Order n: Let F and G be two species of structures and n, an integer ≥ 0. One says that F and G have contact of order n, and one writes

$$F =_n G$$

if the combinatorial equality $F_{\leq n} = G_{\leq n}$, where $F_{\leq n}$ denotes the restriction of F to sets of cardinality $\leq n$. Then

$$F_{\leq n}[U] = \emptyset, \text{ if } |U| > n,$$
$$F_{\leq n}[U] = F[U] \text{ and } F_{\leq n}[\sigma] = F[\sigma], |U| \leq n.$$

Definition 1.2.15 - Limit of A Sequence of Species of Structures: A sequence $(F_n)_{n\geq 0}$ of species of structures is said to *converge* to a species F, written as

$$\lim_{n\to\infty} F_n = F,$$

if for any integer $N \geq 0$, there exists $K \geq 0$ such that for all $n \geq K$, $F_n =_N F$.

The following table (Table 1) describes the coefficients h_n of $h(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!}$, for f, g, similarly, exponential power series.

Table 1

Operation	Coefficient h_n
$h = f + g$	$h_n = f_n + g_n$
$h = f \cdot g$	$h_n = \sum_{i+j=n} \frac{n!}{i!\,j!} f_i g_j$
$h = f \circ g$ $(g(0) = 0)$	$h_n = \sum_{\substack{0\leq k\leq n \\ n_1+\cdots+n_k=n}} \frac{n!}{k!n_1!\dots n_k!} f_k g_{n_1} \cdots g_{n_k}$
$h = f'$	$h_n = f_{n+1}$

The algebra of species should, intuitively, follow certain rules

1. The number of $(F+G)$-structures on n elements is

$$|(F+G)[n]| = |F[n]| + |G[n]|.$$

2. The number of $(F \cdot G)$-structures on n elements is

$$|(F \cdot G)[n]| = \sum_{i+j=n} \frac{n!}{i!j!} |F[i]||G[j]|.$$

3. The number of $(F \circ G)$-structures on n elements is

$$|(F \cdot G)[n]| = \sum_{j=0}^{n} \sum_{\substack{n_1+n_2+\cdots+n_j=n \\ n_i>0}} \frac{1}{j!} \binom{n}{n_1, n_2, \ldots, n_j} |F[j]| \prod_{i=1}^{j} |G[n_i]|.$$

4. The number of F'-structures on n elements is

$$|F'[n]| = |F[n+1]|.$$

Definition 1.3.1 - Species Sum: Let F and G be two species of structures. The species $F+G$, called the *sum of F and G*, is defined as follows: an $(F+G)$-structure on U is an F-structure on U or (exclusive) a G-structure on U. In other words, for any finite set U, one has

$$(F+G)[U] = F[U] + G[U] \qquad \text{(disjoint union)}.$$

The transport along a bijection $\sigma : U \to V$ is carried out by setting, for any $(F+G)$-structure s on U,

$$(F+G)[\sigma](s) = \begin{cases} F[\sigma](s), & \text{if } s \in F[U], \\ G[\sigma](s), & \text{if } s \in G[U]. \end{cases}$$

Additionally, note that which species, F or G, is selected matters, which is why we often define

$$(F+G)[\sigma](s) = \begin{cases} F[\sigma](s) \times \{1\}, & \text{if } s \in F[U], \\ G[\sigma](s) \times \{2\}, & \text{if } s \in G[U]. \end{cases}$$

Proposition 1.3.3: Given two species of structures F and G, the associated series of the species $F+G$ satisfy the equalities

a) $(F+G)(x) = F(x) + G(x)$,

b) $(\widetilde{F+G})(x) = \tilde{F}(x) + \tilde{G}(x)$,

c) $Z_{F+G} = Z_F + Z_G$.

Definition 1.3.5 - Summable: A family $(F_i)_{i\in I}$ of species of structures is said to be *summable* if for any finite set U, $F_i[U] = \emptyset$, except for a finite number of indices $i \in I$. The sum of a summable family $(F_i)_{i\in I}$ is the species $\sum_{i\in I} F_i$ defined by the equalities

a) $\left(\sum_{i\in I} F_i\right)[U] = \sum_{i\in I} F_i[U] = \bigcup_{i\in I} F_i[U] \times \{i\}$,

b) $\left(\sum_{i\in I} F_i\right)[\sigma](s,i) = (F_i[\sigma](s), i)$,

where $\sigma : U \to V$ is a bijection and $(s,i) \in (\sum_{i\in I} F_i)[U]$. Additionally,

1. $\left(\sum_{i\in I} F_i\right)(x) = \sum_{i\in I} F_i(x)$,

2. $\widetilde{\left(\sum_{i\in I} F_i\right)}(x) = \sum_{i\in I} \tilde{F}_i(x)$,

3. $Z_{(\sum_{i\in I} F_i)} = \sum_{i\in I} Z_{F_i}$.

Definition - Species Restricted to Cardinality n: Each species F gives rise to a *canonically decomposition* of an enumerable family $(F_n)_{n\geq 0}$ of species defined by setting, for each $n \in \mathbb{N}$,

$$F_n[U] = \begin{cases} F[U], & \text{if } |U| = n, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then clearly $F = F_0 + F_1 + F_2 + \dots$. We then say that F_n *is the species* F *restricted to cardinality* n. In the case that $F = F_k$ (i.e., $F_n = 0$, for $n \neq k$), then we say that F *is concentrated on the cardinality* k.

Definition - Finite Sum: The *finite sum* $F + F + \cdots + F$ of n copies of the same F is often denoted by $nF = n \cdot F$ (discussed later), and we have

a) $(nF)(x) = nF(x)$,

b) $(\widetilde{nF})(x) = n\tilde{F}(x)$,

c) $Z_{nF} = nZ_F$.

Definition 1.3.7 - Species Product: Let F and G be two species of structures. The species $F \cdot G$ (also denoted FG), called the *product* of F and G, is defined as follows: an $(F \cdot G)$-structure on U is an ordered pair $s = (f, g)$, where

1. f is an F-structure on a subset $U_1 \subseteq U$;
2. g is a G-structure on a subset $U_2 \subseteq U$;
3. (U_1, U_2) is a decomposition of U, i.e., $U = U_1 \sqcup U_2$.

In other words, for any finite set U,

$$(F \cdot G)[U] = \sum_{U_1 \sqcup U_2 = U} F[U_1] \times G[U_2],$$

the disjoint sum being taken over all decompositions (U_1, U_2) of U. The transport along a bijection $\sigma : U \to V$ is done by setting, for each $(F \cdot G)$-structure $s = (f, g)$ on U,

$$(F \cdot G)[\sigma](s) = (F[\sigma_1](f), G[\sigma_2](g)),$$

where $\sigma_i = \sigma|_{U_i}$, for $i = 1, 2$. Extending the definition of species product of finite families of species follows similarly. We can also define $F \cdot F \cdot \cdots \cdot F$ (k times) as F^k. An F^k-structure on a set U is therefore a k-tuple $(s_1, s_2, \dots, s_k)$ of disjoint F-structures whose disjoint union of underlying sets is U.

The neutral element of species multiplication is the 1 species, i.e. for any species F,

$$1 \cdot F = F \cdot 1 = F.$$

Additionally,

$$0 \cdot F = F \cdot 0 = 0.$$

Proposition 1.3.8: Let F and G be two species of structures. Then the series associated with the species $F \cdot G$ satisfy the equalities

1. $(F \cdot G)(x) = F(x)G(x)$,

2. $\widetilde{(F \cdot G)}(x) = \tilde{F}(x)\tilde{G}(x)$,

3. $Z_{F \cdot G}(x_1, x_2, \dots) = Z_F(x_1, x_2, \dots)Z_G(x_1, x_2, \dots)$.

Definition - (Non)Recurrent Points: Consider an endofunction $\varphi \in \mathrm{End}[U]$ of a set U. Then we define

1. the *recurrent points*, i.e., those $x \in U$ for which there exists a $k > 0$ such that $\varphi^k(x) = x$. Equivalently, these are the elements located on cycles.

2. the *non-recurrent points*, i.e., those x for which $\varphi^k(x) \neq x$ for all $k > 0$.

Definition 1.4.1 - (Partitional) Composite/Substitution, F-Assembly of G-Structures: Let F and G be two species of structures such that $G[\emptyset] = \emptyset$ (i.e., there is no G-structure on the empty set). The species $F \circ G$, also denoted $F(G)$, called the *(partitional) composite, or substitution, of G in F*, is defined as follows: An $(F \circ G)$-structure on U is a triplet $s = (\pi, \varphi, \gamma)$, where

1. π is a partition of U,

2. φ is an F-structure on the set of classes of π,

3. $\gamma = (\gamma_p)_{p \in \pi}$, where for each class p of π, γ_p is a G-structure on p.

In mathematical notation, for any finite set U, we have

$$(F \circ G)[U] = \sum_{\pi \text{ partition of } U} F[\pi] \times \prod_{p \in \pi} G(p),$$

where the (disjoint) sum is taken over the set of partitions π of U (i.e., $\pi \in \mathrm{Par}[U]$).

The transport along a bijection $\sigma : U \to V$ is carried out by setting, for any $(F \circ G)$-structure $s = (\pi, \varphi, (\gamma_p)_{p \in \pi})$ on U,

$$(F \circ G)[\sigma](s) = (\overline{\pi}, \overline{\varphi}, (\overline{\gamma}_{\overline{p}})_{\overline{p} \in \overline{\pi}}),$$

where

1. $\overline{\pi}$ is the partition of V obtained by transport of π along σ,

2. for each $\overline{p} = \sigma(p) \in \overline{\pi}$, the structure $\overline{\gamma}_{\overline{p}}$ is obtained from the structure γ_p by G-transport along $\sigma|_p$,

3. the structure $\overline{\varphi}$ is obtained from the structure φ by F-transport along the bijection $\overline{\sigma}$ induced on π by σ.

We may also call an $(F \circ G)$-structure an *F-assembly of (disjoint) G-structures.* When $F = E$, the species of sets, an $(F \circ G)$-structure is simply called an *assembly of G-structures.*

Theorem 1.4.2 - Plethystic Substitution: Let F and G be two species of structures and suppose that $G[\emptyset] = \emptyset$. Then the series associated to the species $F \circ G$ satisfy the equalities

1. $(F \circ G)(x) = F(G(x))$,

2. $\widetilde{(F \circ G)}(x) = Z_F(\tilde{G}(x), \tilde{G}(x^2), \tilde{G}(x^3), \dots)$,

3. $Z_{F \circ G}(x_1, x_2, x_3, \dots) = Z_F(Z_G(x_1, x_2, \dots), Z_G(x_2, x_4, \dots), \dots)$

The index series given in the part 3 is called the *plethystic substitution* of Z_G in Z_F, and is denoted $Z_F \circ Z_G$, or $Z_F(Z_G)$.

Definition 1.4.3 - Plethystic Substitution: Let $f = f(x_1, x_2, x_3, \dots)$ and $g = g(x_1, x_2, x_3, \dots)$ be two formal power series. Then the *plethystic substitution* $f \circ g$ is defined by

$$(f \circ g)(x_1, x_2, x_3, \dots) = f(g_1, g_2, g_3, \dots),$$

where $g_k = g(x_k, x_{2k}, x_{3k}, \dots)$, for $k = 1, 2, \dots$. Then $g_k = x_k \circ g = g \circ x_k$.

Equation 1.4.(17): From $\mathcal{S} = E \circ \mathcal{C}$ (the species of permutations is isomorphic to a set of cyclically ordered disjoint subsets of the original set of elements), we achieve the following identities:

1. $\displaystyle\prod_{k\geq 1} \frac{1}{1-x^k} = \tilde{S}(x) = Z_E(\tilde{C}(x), \tilde{C}(x^2), \dots) = \exp\left(\sum_{n\geq 1} \frac{1}{k} \frac{x^k}{1-x^k}\right)$;

2. $\displaystyle\frac{1}{1-x_1} \frac{1}{1-x_2} \frac{1}{1-x_3} \dots = Z_S(x_1, x_2, x_3, \dots) = \exp\left(\sum_{k\geq 1} \frac{1}{k} Z_C(x_k, x_{2k}, x_{3k}, \dots)\right)$.

Identity 2 above gives way to an explicit calculation of the index series Z_C,

$$Z_C(x_1, x_2, x_3, \dots) = \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \log \frac{1}{1-x_k},$$

where ϕ is the arithmetic Euler's totient function.

Definition - Connected F-Structures: If two species F, F^c are related by a combinatorial equation of the form

$$F = E \circ F^c,$$

we say that F^c is the species of connected F-structures. Then we have that

1. $F(x) = e^{F^c(x)}$,
2. $\displaystyle\tilde{F}(x) = \exp \sum_{k\geq 1} \frac{1}{k} \tilde{F}^c(x^k)$,
3. $\displaystyle Z_F(x_1, x_2, x_3, \dots) = \exp \sum_{k\geq 1} \frac{1}{k} Z_{F^c}(x_k, x_{2k}, \dots)$.

We use $F^{\langle n \rangle}$ to denote n compositions of F with itself. For example, $F^{\langle 3 \rangle} = F(F(F))$.

Properties of Substitution:

1. The species X of singletons is the neutral element (composition identity) for the substitution of species
$$F = F(X) = X(F).$$
2. Substitution is associative up to isomorphism of species.
3. If $G[\emptyset] = \emptyset, G(0) = 0$.

Definition 1.4.5 - Derivative of Species: Let F be a species of structures. The species F' (also denoted by $\frac{d}{dX}F(X)$), called the *derivative* of F, is defined as follows: An F'-structure on U is an F-structure on $U^+ = U \cup \{*\}$, where $* = *_U$ is a element chosen outside of U. In other words, for any finite set U, one sets

$$F'[U] = F[U^+], \text{ where } U^+ = U + \{*\}.$$

The transport along a bijection $\sigma : U \to V$ is carried out by setting, for any F'-structure s on U,

$$F'[\sigma](s) = F[\sigma^+](s),$$

where $\sigma^+ : U^+ = U + \{*\} \to V + \{*\}$ is the canonical extension of σ obtained by setting

$$\sigma^+(u) = \sigma(u), u \in U, \text{ and } \sigma^+(*) = *.$$

Additionally, note that general properties of derivatives also hold for derivatives of species, namely the chain rule, product rule, and distribution of derivatives over addition.

Proposition 1.4.8: Let F be a species of structures. One has the equalities

1. $F'(x) = \dfrac{d}{dx} F(x)$,
2. $\tilde{F}'(x) = \left(\dfrac{\partial}{\partial x_1} Z_F \right)(x, x^2, x^3, \dots)$, (I think this is a typo it should be derivative, not partial, with respective to x),
3. $Z_{F'}(x_1, x_2, x_3, \dots) = \left(\dfrac{\partial}{\partial x_1} Z_F \right)(x_1, x_2, x_3, \dots)$.

A.2 Complements on Species of Structures

Definition 2.1.1 - F Dot: Let F be a species of structures. The species $F^\bullet$, called F *dot*, is defined as follows: An $F^\bullet$-structure on U is a pair $s = (f, w)$, where

1. f is an F-structure on U,
2. $u \in U$ (a *distinguished* element).

The pair (f, u) is called *pointed* F-structure (pointed at the distinguished element u). In other words, for any finite set U,

$$F^\bullet[U] = F[U] \times U \qquad \text{(set-theoretic Cartesian product)}.$$

The transport along a bijection $\sigma : U \to V$ is carried out by setting

$$F^\bullet[\sigma](s) = (F[\sigma](f), \sigma(u)),$$

for any $F^\bullet$-structure $s = (f, u)$ on U.

Then clearly, an $F^\bullet$-structure satisfies

$$|F^\bullet[n]| = n|F[n]|, n \geq 0.$$

Additionally, it satisfied the following combinatorial equation

$$F^\bullet = X \cdot F'.$$

Proposition 2.1.2: Let F be a species of structures. One has the equalities

1. $F^\bullet(x) = x \frac{d}{dx} F(x)$,
2. $\tilde{F}^\bullet(x) = x \left(\frac{\partial}{\partial x_1} Z_F \right)(x, x^2, \dots)$,
3. $Z_{F^\bullet}(x_1, x_2, \dots) = x_1 \left(\frac{\partial}{\partial x_1} Z_F \right)(x_1, x_2, \dots)$.

Example 2.1.3 - Vertebrates: Let

$$\mathcal{V} = a^{\bullet\bullet} = A^{\bullet}$$

denote the species of *vertebrates* (doubly rooted trees, or *bipointed* trees), where $v \in \mathcal{V}$ is called *degenerate* if both pointers/roots are the same element. Note that the roots are numbered, i.e. distinct.

Definition 2.1.4 - Cartesian Product: Let F and G be two species of structures. The species $F \times G$, called the *Cartesian product* of F and G, is defined as follows: An $(F \times G)$-structure on a finite set U is a pair $s = (f, g)$, where

1. f is an F-structure on U,
2. g is a G-structure on U.

In other words, for all finite sets U, one has

$$(F \times G)[U] = F[U] \times G[U] \qquad \text{(Cartesian product)}.$$

The transport along a bijection $\sigma : U \to V$ is carried out by setting

$$(F \times G)[\sigma](s) = (F[\sigma](f), G[\sigma](g)),$$

for any $(F \times G)$-structure $s = (f, g)$ on U.

Then clearly,

$$|(F \times G)[n]| = |F[n]| \cdot |G[n]|, n \geq 0.$$

Definition - Hadamard Product: We define the *Hadamard product* $f \times g$ of two index series

$$f(x) = \sum f_n \frac{x^n}{\mathrm{Aut}(n)}, \quad g(x) = \sum g_n \frac{x^n}{\mathrm{Aut}(n)},$$

where $x = (x_1, x_2, \dots)$, $n = (n_1, n_2, \dots)$, coefficient-wise as

$$(f \times g)(x) = \sum f_n g_n \frac{x^n}{\mathrm{Aut}(n)}.$$

Proposition 2.1.7: Let F and G be two species of structures. Then the series associated to the species $F \times G$ satisfy the equalities

1. $(F \times G)(x) = F(x) \times G(x)$,
2. $(\widetilde{F \times G})(x) = (Z_F \times Z_G)(x, x^2, \dots)$,
3. $Z_{F \times G}(x_1, x_2, \dots) = Z_F(x_1, x_2, \dots) \times Z_G(x_1, x_2, \dots)$.

Properties of Cartesian Product:

- The species E of sets is the neutral element for the Cartesian product, i.e. for any species F

$$E \times F = F \times E = F.$$

- If we simply restrict E to cardinality n, then we have that

$$E_n \times F = F \times E_n = F_n.$$

- The Cartesian product distributes over addition, i.e.

$$\begin{aligned} F &= F \times E \\ &= F \times (E_0 + E_1 + E_2 + \dots) \\ &= F_0 + F_1 + F_2 + \dots \\ &= F \end{aligned}$$

- $(F \times G)^\bullet = F^\bullet \times G = F \times G^\bullet$.
- $(F \circ G)^\bullet = (F' \circ G) \cdot G^\bullet$.

Definition 2.2.1 - Functorial Composite: Let F and G be two species of structures. The species FG (also denoted by $F[G]$), called the *functorial composite* of F and G, is defined as follows: An (FG)-structure on U is an F-structure placed on the set $G[U]$ of all the G-structures on U. In other words, for any finite set U,