Math 104 Notes

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Random

$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$.

Introduction

In this book, $\mathbb{N}$ denotes the set of all positive integers.

The **successor** of a positive integer n is $n + 1$.

Peano Axioms/Postulates of $\mathbb{N}$:

N1. 1 belongs to $\mathbb{N}$.

N2. If n belongs to $\mathbb{N}$, then its successor $n + 1$ belongs to $\mathbb{N}$.

N3. 1 is not the successor of any element in $\mathbb{N}$.

N4. If n and m in $\mathbb{N}$ have the same successor, then $n = m$.

N5. A subset of $\mathbb{N}$ which contains 1, and which contains $n + 1$ whenever it contains n, must equal $\mathbb{N}$.

A number is called an **algebraic number** if it satisfies a polynomial equation

$$c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0,$$

where the coefficients $c_0, c_1, \ldots, c_n$ are integers, $c_n \neq 0$ and $n \geq 1$.

Algebraic numbers are all numbers containing any combination of any radical forms ($\sqrt{\ }$, $\sqrt[3]{\ }$, etc).

SOLVING* ** How to show that a number x is not rational: find the polynomial equation x is a solution to and use the **Rational Zeros Theorem**(roots) to find all the possible rational roots, and test that each of these do not work.

Algebraic properties in $\mathbb{Q}$:

A1. $a + (b + c) = (a + b) + c$ for all a, b, c.
A2. $a + b = b + a$ for all a, b.
A3. $a + 0 = a$ for all a.
A4. For each a, there is an element $-a$ such that $a + (-a) = 0$.
M1. $a(bc) = (ab)c$ for all a, b, c.
M2. $ab = ba$ for all a, b.
M3. $a \cdot 1 = a$ for all a.
M4. For each $a = 0$, there is an element a^{-1} such that $aa^{-1} = 1$.
DL. $a(b + c) = ab + ac$ for all a, b, c.

A system that has more than one element and satisfies these nine properties is called a **field**.

Order structure of $\mathbb{Q}$:

O1. Given a and b, either $a \leq b$ or $b \leq a$.

O2. If $a \leq b$ and $b \leq a$, then $a = b$.

O3. If $a \leq b$ and $b \leq c$, then $a \leq c$.

O4. If $a \leq b$, then $a + c \leq b + c$.

O5. If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

Property O3 is called the **transitive law**. This is the characteristic property of an ordering. A field with an ordering satisfying properties O1 through O5 is called an **ordered field**.

Theorem 3.1: The following are consequences of the field properties

1. $a + c = b + c$ implies $a = b$;
2. $a \cdot 0 = 0$ for all a;
3. $(-a)b = -ab$ for all a, b;
4. $(-a)(-b) = ab$ for all a, b;
5. $ac = bc$ and $c \neq 0$ imply $a = b$;
6. $ab = 0$ implies either $a = 0$ or $b = 0$;
   for $a, b, c \in \mathbb{R}$.

Theorem 3.2: The following are consequences of the properties of an ordered field

1. If $a \leq b$, then $-b \leq -a$;
2. If $a \leq b$ and $c \leq 0$, then $bc \leq ac$;
3. If $0 \leq a$ and $0 \leq b$, then $0 \leq ab$;
4. $0 \leq a^2$ for all a;
5. $0 < 1$;
6. If $0 < a$, then $0 < a - 1$;
7. If $0 < a < b$, then $0 < b - 1 < a - 1$;
   for $a, b, c \in \mathbb{R}$.

Note $a < b$ means $a \leq b$ and $a \neq b$.

Theorem 3.5: Basic properties of the absolute value

1. $|a| \geq 0$;
2. $|ab| = |a| \cdot |b|$;
3. **Theorem 3.7 -** (*triangle inequality*): $|a + b| \leq |a| + |b|$;
   for all $a, b \in \mathbb{R}$.

Corollary 3.6: $\text{dist}(a,c) \le \text{dist}(a,b) + \text{dist}(b,c)$, for all $a, b, c \in \mathbb{R}$.

Definition 4.1: Let S be a nonempty subset of $\mathbb{R}$.

1. If S contains a largest element s_0 [that is, s_0 belongs to S and $s \le s_0$ for all $s \in S$], then we call s_0 the **maximum** of S and write $s_0 = \max S$.
2. If S contains a smallest element, then we call the smallest element the **minimum** of S and write it as $\min S$.

Definition 4.2: Let S be a nonempty subset of $\mathbb{R}$.

1. If a real number M satisfies $s \le M$ for all $s \in S$, then M is called an **upper bound** of S and the set S is said to be **bounded above**.
2. If a real number m satisfies $m \le s$ for all $s \in S$, then m is called a **lower bound** of S and the set S is said to be **bounded below**.
3. The set S is said to be **bounded** if it is bounded above and bounded below. Thus, S is bounded if there exist real numbers m and M such that $S \subseteq [m, M]$.

Definition 4.3: Let S be a nonempty subset of $\mathbb{R}$.

1. If S is bounded above and S has a least upper bound, then we will call it the **supremum**, or *least upper bound*, of S and denote it by $\sup S$.
2. If S is bounded below and S has a greatest lower bound, then we will call it the **infimum**, or *greatest lower bound*, of S and denote it by $\inf S$.

4.4 - (*Completeness Axiom*): Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. In other words, $\sup S$ exists and is a real number.

Corollary 4.5: Every nonempty subset S of $\mathbb{R}$ that is bounded below has a greatest lower bound $\inf S$.

4.6 - (*Archimedean Property*): If $a > 0$ and $b > 0$, then for some positive integer n, we have $na > b$.
Setting $a = 1$ and $b = 1$ separately gives very useful conclusions.

4.7 - (*Denseness of* $\mathbb{Q}$): If $a, b \in \mathbb{R}$ and $a < b$, then there is a rational $r \in \mathbb{Q}$ such that $a < r < b$.

$[a, \infty)$ and $(-\infty, b]$ are called **closed intervals** or **unbounded closed intervals**, while (a, ∞) and $(-\infty, b)$ are called **open intervals** or **unbounded open intervals**.

We define $\sup S = +\infty$ if S is not bounded above and $\sup S = -\infty$ if S is not bounded below.

*****NOTATION** We agree that $-\infty \le a \le +\infty$ for all $a \in R \cup \{-\infty, \infty\}$. (page 28)

Sequences

Let $(s_n)_{n=m}^{\infty}$ denote $(s_m, s_{m+1}, \dots)$.

Definition 7.1: A sequence (s_n) of real numbers is said to **converge** to the real number s provided that for each $\epsilon > 0$, there exists a number N such that $n > N$ implies $|s_n - s| < \epsilon$.

If (s_n) converges to s, we write $\lim_{n\to\infty} s_n = s$, or $s_n \to s$.

We say a limit **exists** if it converges or diverges to $+\infty$ or $-\infty$.

Theorem 9.1: Convergent sequences are bounded.

Theorem 9.2: If the sequence (s_n) converges to s and k is in $\mathbb{R}$, then the sequence (ks_n) converges to ks. That is, $\lim(ks_n) = k \lim s_n$.

Theorem 9.3: If (s_n) converges to s and (t_n) converges to t, then $(s_n + t_n)$ converges to $s + t$. That is, $\lim(s_n + t_n) = \lim s_n + \lim t_n$.

Theorem 9.4: If (s_n) converges to s and (t_n) converges to t, then $(s_n t_n)$ converges to st.

Lemma 9.5: If (s_n) converges to s, if $s_n \neq 0$ for all n, and if $s \neq 0$, then $(1/s_n)$ converges to $1/s$.

Theorem 9.6: Suppose (s_n) converges to s and (t_n) converges to t. If $s \neq 0$ and $s_n \neq 0$ for all n, then (t_n/s_n) converges to t/s.

Theorem 9.7:

1. $\lim_{n\to\infty}(\frac{1}{n^p}) = 0$ for $p > 0$.
2. $\lim_{n\to\infty} a^n = 0$ if $|a| < 1$.
3. $\lim_{n\to\infty}(n^{1/n}) = 1$.
4. $\lim_{n\to\infty}(a^{1/n}) = 1$ for $a > 0$.

*** **Definition 9.8**: For a sequence (s_n), we write $\lim s_n = +\infty$ provided for each $M > 0$ there is a number N such that $n > N$ implies $s_n > M$. In this case we say the sequence diverges to $+\infty$.
Similarly, we write $\lim s_n = -\infty$ provided for each $M < 0$ there is a number N such that $n > N$ implies $s_n < M$.

Theorem 9.9: Let (s_n) and (t_n) be sequences such that $\lim s_n = +\infty$ and $\lim t_n > 0$ [$\lim t_n$ can be finite or $+\infty$]. Then $\lim s_n t_n = +\infty$.

Theorem 9.10: For a sequence (s_n) of positive real numbers, we have $\lim s_n = +\infty$ if and only if $\lim(1/s_n) = 0$.

Definition 10.1: A sequence (s_n) of real numbers is called an **increasing sequence** if $s_n \leq s_{n+1}$ for all n, and (s_n) is called a **decreasing sequence** if $s_n \geq s_{n+1}$ for all n. Note that if (s_n) is increasing, then $s_n \leq s_m$ whenever $n < m$. A sequence that is increasing or decreasing will be called a **monotone/monotonic sequence**.

Theorem 10.2: All bounded monotone sequences converge.

Theorem 10.4:

1. If (s_n) is an unbounded increasing sequence, then $\lim s_n = +\infty$.
2. If (s_n) is an unbounded decreasing sequence, then $\lim s_n = \infty$.

Theorem 10.5: If (s_n) is a monotone sequence, then the sequence either converges, diverges to $+\infty$, or diverges to $-\infty$. Thus $\lim s_n$ is always meaningful for monotone sequences.

Definition 10.6: Let (s_n) be a sequence in $\mathbb{R}$. We define

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n : n > N\}$$

and

$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n : n > N\}.$$

If (s_n) is not bounded above, we say $\sup\{s_n : n > N\} = \infty$. If (s_n) is not bounded below, we say $\inf\{s_n : n > N\} = -\infty$.

In non-math terms, $\limsup s_n$ is the largest value that infinitely many s_n's can get close to.

Theorem 10.7: Let (s_n) be a sequence in $\mathbb{R}$.

1. If $\lim s_n$ is defined [as a real number, $+\infty$ or $-\infty$], then $\liminf s_n = \lim s_n = \limsup s_n$.
2. If $\liminf s_n = \limsup s_n$, then $\lim s_n$ is defined and $\lim s_n = \liminf s_n = \limsup s_n$.

*****NOTATION** Let $u_N = \inf\{s_n : n > N\}$ and $v_N = \sup\{s_n : n > N\}$.

Definition 10.8: A sequence (s_n) of real numbers is called a **Cauchy sequence** if for each $\epsilon > 0$ there exists a number N such that $m, n > N$ implies $|s_n - s_m| < \epsilon$.

Lemma 10.9: Convergent sequences are Cauchy sequences.

Lemma 10.10: Cauchy sequences are bounded.

Theorem 10.11: A sequence is a convergent sequence if and only if it is a Cauchy sequence.

Definition 11.1: Suppose $(s_n)_{n\in\mathbb{N}}$ is a sequence. A **subsequence** of this sequence is a sequence of the form $(t_k)_{k\in\mathbb{N}}$ where for each k there is a positive integer n_k such that

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \ldots$$

and $t_k = s_{n_k}$. Thus (t_k) is just a selection of some [possibly all] of the s_n's taken in order.

An alternate definition of a subsequence is

$$t_k = t(k) = s \circ \sigma(k) = s(\sigma(k)) = s(n_k) = s_{n_k}, k \in \mathbb{N}.$$

Theorem 11.2: Let (s_n) be a sequence.

1. If $t \in \mathbb{R}$, then there is a subsequence of (s_n) converging to t iff the set $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$ is infinite for all $\epsilon > 0$.
2. If the sequence (s_n) is unbounded above, it has a subsequence with limit $+\infty$.
3. Similarly, if (s_n) is unbounded below, a subsequence has limit $-\infty$.

In each case, the subsequence can be taken to be monotonic.

Theorem 11.3: If the sequence (s_n) converges, then every subsequence converges to the same limit.

Theorem 11.4: Every sequence (s_n) has a monotonic subsequence.

Theorem 11.5 - (Bolzano-Weierstrass Theorem): Every bounded sequence has a convergent subsequence.

Let (s_n) be a sequence in $\mathbb{R}$. A **subsequential limit** is any real number or symbol $+\infty$ or $-\infty$ that is the limit of some subsequence of (s_n).

Theorem 11.7: Let (s_n) be any sequence. There exists a monotonic subsequence whose limit is $\limsup s_n$, and there exists a monotonic subsequence whose limit is $\liminf s_n$.

Theorem 11.8: Let (s_n) be any sequence in $\mathbb{R}$, and let S denote the set of subsequential limits of (s_n).

1. S is nonempty.
2. $\sup S = \limsup s_n$ and $\inf S = \liminf s_n$.
3. $\lim s_n$ exists if and only if S has exactly one element, namely $\lim s_n$.

Theorem 11.9: Let S denote the set of subsequential limits of a sequence (s_n). Suppose (t_n) is a sequence in $S \cap \mathbb{R}$ and that $t = \lim t_n$. Then t belongs to S.

Exercise 11.8: $\liminf s_n = -\limsup(-s_n)$ for every sequence (s_n).

Theorem 12.1: If (s_n) converges to a positive real number s and (t_n) is any sequence, then

$$\limsup s_n t_n = s \cdot \limsup t_n.$$

Here we allow the conventions $s \cdot (+\infty) = +\infty$ and $s \cdot (-\infty) = -\infty$ for $s > 0$.

Theorem 12.2: Let (s_n) be any sequence of nonzero real numbers. Then we have

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{\frac{1}{n}} \leq \limsup |s_n|^{\frac{1}{n}} \leq \limsup \left|\frac{s_{n+1}}{s_n}\right|.$$

Corollary 12.3: If $\lim \left|\frac{s_{n+1}}{s_n}\right|$ exists [and equals L], then $\lim |s_n|^{1/n}$ exists [and equals L].

To assign meaning to $\sum_{n=m}^{\infty} a_n$, we consider the sequences $(s_n)_{n=m}^{\infty}$ of **partial sums**:

$$s_n = a_m + \cdots + a_n = \sum_{k=m}^{n} a_k.$$

An infinite series is said to **converge** if it's partial sums converge to some real number S. In this case, we say any of the following

$$\sum_{k=m}^{\infty} a_k = \lim s_n = \lim_{n\to\infty} (\sum_{k=m}^{n} a_k) = S$$

Consider an empty sum (no starting or ending bounds) to be an infinite series.

The series $\sum a_n$ is said to **converge absolutely** or to be **absolutely convergent** if $\sum |a_n|$ converges.

Definition 14.3: We say a series $\sum a_n$ satisfies the **Cauchy criterion** if its sequence (s_n) of partial sums is a Cauchy sequence. That is, for each $\epsilon > 0$ there exists a number N such that

$$m, n > N \to |s_n - s_m| < \epsilon.$$

Theorem 14.4: A series converges if and only if it satisfies the Cauchy criterion.

Corollary 14.5: If a series $\sum a_n$ converges, then $\lim a_n = 0$.

14.6 - (Comparison Test): Let $\sum a_n$ be a series where $a_n \geq 0$ for all n.

1. If $\sum a_n$ converges and $|b_n| \leq a_n$ for all n, then $\sum b_n$ converges.
2. If $\sum a_n = +\infty$ and $b_n \geq a_n$ for all n, then $\sum b_n = +\infty$.

Corollary 14.7: Absolutely convergent series are convergent.

14.8 - (Ratio Test): A series $\sum a_n$ of nonzero terms

1. converges absolutely if $\limsup |\frac{a_{n+1}}{a_n}| < 1$,
2. diverges if $\liminf |\frac{a_{n+1}}{a_n}| > 1$.
3. Otherwise $\liminf |\frac{a_{n+1}}{a_n}| \leq 1 \leq \limsup |\frac{a_{n+1}}{a_n}|$ and the test gives no information.

14.9 - (Root Test): Let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$. The series $\sum a_n$

1. converges absolutely if $\alpha < 1$,
2. diverges if $\alpha > 1$.
3. Otherwise $\alpha = 1$ and the test gives no information.

Theorem 15.1: $\sum 1/n^p$ converges if and only if $p > 1$.

15.2 - (Integral Tests): Use integral tests when:

1. The tests in §14 do not seem to apply.
2. The terms an of the series $\sum a_n$ are nonnegative.
3. There is a nice decreasing function f on $[1, \infty)$ such that $f(n) = a_n$ for all n [f is decreasing if $x < y$ implies $f(x) \geq f(y)$].
4. The integral of f is easy to calculate or estimate.

The series (a_n) converges if the integral is $< \infty$, diverges otherwise.

15.3 - (Alternating Series Theorem): If $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0$ and $\lim a_n = 0$, then the alternating series $\sum(-1)^{n+1}a_n$ converges. Moreover, the partial sums $s_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$ satisfy $|s - s_n| \leq a_n$ for all n.

Continuous Functions

Natural domain is the largest subset of $\mathbb{R}$ on which the function is a well-defined real-valued function.

Defintion 17.1: Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. The function f is **continuous** at x_0 in $\text{dom}(f)$ if, for every sequence (x_n) in $\text{dom}(f)$ converging to x_0, we have $\lim_n f(x_n) = f(x_0)$.

If f is continuous at each point of a set $S \subseteq \text{dom}(f)$, then f is said to be **continuous on** S. The function

f is said to be continuous if it is continuous on $\mathrm{dom}(f)$.

Theorem 17.2: Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Then f is continuous at x_0 in $\mathrm{dom}(f)$ if and only if for each $\epsilon > 0$, there exists $\delta > 0$ such that $x \in \mathrm{dom}(f)$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \epsilon$.

Theorem 17.3: Let f be a real-valued function with $\mathrm{dom}(f) \subseteq \mathbb{R}$. If f is continuous at x_0 in $\mathrm{dom}(f)$, then $|f|$ and $kf, k \in \mathbb{R}$, are continuous at x_0.

Theorem 17.4: Let f and g be real-valued functions that are continuous at x_0 in $\mathbb{R}$. Then

1. $f + g$ is continuous x_0;
2. fg is continuous at x_0;
3. f/g is continuous at x_0 is $g(x_0) \neq 0$.

Theorem 17.5: If f is continuous at x_0 and g is continuous at $f(x_0)$, then the composite function $g \circ f$ is continuous at x_0.

A real-valued function f is said to be **bounded** if $\{f(x) : x \in \mathrm{dom}(f)\}$ is a bounded set, i.e., if there exists a real number M such that $|f(x)| \leq M$ for all $x \in \mathrm{dom}(f)$.

Theorem 18.1: Let f be a continuous real-valued function on a closed interval $[a, b]$. Then f is a bounded function. Moreover, f assumes its maximum and minimum values on $[a, b]$; that is, there exist x_0, y_0 in $[a, b]$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [a, b]$.

18.2 - (Intermediate Value Theorem): If f is a continuous real-valued function on an interval I, then f has the **intermediate value property** on I: Whenever $a, b \in I, a < b$ and y lies between $f(a)$ and $f(b)$ [i.e., $f(a) < y < f(b)$ or $f(b) < y < f(a)$], there exists at least one x in (a, b) such that $f(x) = y$.

Corollary 18.3: If f is a continuous real-valued function on an interval I, then the set $f(I) = \{f(x) : x \in I\}$ is also an interval or a single point.

Theorem 18.4: Let f be a continuous strictly increasing function on some interval I. Then $f(I)$ is an interval J by Corollary 18.3 and f^{-1} represents a function with domain J. The function f^{-1} is a continuous strictly increasing function on J.

Theorem 18.5: Let g be a strictly increasing function on an interval J such that $g(J)$ is an interval I. Then g is continuous on J.

Theorem 18.6: Let f be a one-to-one continuous function on an interval I. Then f is strictly increasing [$x_1 < x_2$ implies $f(x_1) < f(x_2)$] or strictly decreasing [$x_1 < x_2$ implies $f(x_1) > f(x_2)$].

Definition 19.1: Let f be a real-valued function defined on a set $S \subseteq \mathbb{R}$. Then f is **uniformly continuous** on S if for every $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in S$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \epsilon$.

Theorem 19.2: If f is continuous on a closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$.

We call $S = [a, b]$ a **closed set** if any convergent sequence in $[a, b]$ converges to an element in S.

*** Generalization of theorem 19.2; If f is continuous on a closed and bounded set S, then f is uniformly continuous on S.

Theorem 19.4: If f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then $(f(s_n))$

is a Cauchy sequence.

We say a function $\widetilde{f}$ is an **extension** of a function f if $\operatorname{dom} f \subseteq \operatorname{dom} \widetilde{f}$ and $f(x) = \widetilde{f}(x)$ for all $x \in \operatorname{dom} f$.

Theorem 19.5: A real-valued function f on (a, b) is uniformly continuous on (a, b) if and only if it can be extended to a continuous function $\widetilde{f}$ on $[a, b]$.

Theorem 19.6: Let f be a continuous function on an interval I [I may be bounded or unbounded]. Let I° be the interval obtained by removing from I any endpoints that happen to be in I. If f is differentiable on I° and if f' is bounded on I°, then f is uniformly continuous on I.

Definition 20.1: Let S be a subset of $\mathbb{R}$, let a be a real number or symbol $+\infty$ or $-\infty$ that is the limit of some sequence in S, and let L be a real number or symbol $+\infty$ or $-\infty$. We write $\lim_{x \to a^S} f(x) = L$ if f is a function defined on S and for every sequence (x_n) in S with limit a, we have $\lim_{n \to \infty} f(x_n) = L$.

The expression $\lim_{x \to a^S} f(x) = L$ is read "limit, as x tends to a along S, of $f(x)$."

Remarks 20.2:

1. From defintion 17.1, we see that a function f is continuous at a in $\operatorname{dom}(f) = S$ if and only if $\lim_{x \to a^S} f(x) = L$.
2. Observe that limits, when they exist, are unique. This follows from the fact that limits of sequences are unique.

Definition 20.3:

1. For $a \in \mathbb{R}$ and a function f we write $\lim_{x \to a} f(x) = L$ provided $\lim_{x \to a^S} f(x) = L$ for some set $S = J \backslash \{a\}$, where J is an open interval containing a. $\lim_{x \to a} f(x) = L$ is called the **two-sided limit of f at** a. Note f not need to be defined at a and, even if f is defined at a, the value $f(a)$ need not equal $\lim_{x \to a} f(x) = L$. In fact, $f(a) = \lim_{x \to a} f(x) = L$ if and only if f is defined on an open interval containing a and f is continuous at a.
2. For $a \in \mathbb{R}$ and a function f we write $\lim_{x \to a^+} f(x) = L$ provided $\lim_{x \to a^S} f(x) = L$ for some open interval $S = (a, b)$. $\lim_{x \to a^+} f(x)$ is the **right-handed limit of f at** a. Again f need not be defined at a.
3. For $a \in \mathbb{R}$ and a function f we write $\lim_{x \to a^-} f(x) = L$ provided $\lim_{x \to a^S} f(x) = L$ for some open interval $S = (c, a)$. $\lim_{x \to a^-} f(x)$ is the **left-handed limit of f at** a. Again f need not be defined at a.
4. For a function f we write $\lim_{x \to \infty} f(x) = L$ provided $\lim_{x \to \infty^S} f(x) = L$ for some interval $S = (c, \infty)$. Likewise, we write $\lim_{x \to -\infty} f(x) = L$ provided $\lim_{x \to -\infty^S} f(x) = L$ for some interval $S = (-\infty, b)$.

Theorem 20.4: Let f_1 and f_2 be functions for which the limits $L_1 = \lim_{x \to a^S} f_1(x)$ and $L_2 = \lim_{x \to a^S} f_2(x)$ exist and are finite. Then

1. $\lim_{x \to a^S} (f_1 + f_2)(x)$ exists and equals $L_1 + L_2$.
2. $\lim_{x \to a^S} (f_1 f_2)(x)$ exists and equals $L_1 L_2$.
3. $\lim_{x \to a^S} (f_1 / f_2)(x)$ exists and equals L_1 / L_2 provided $L_2 \neq 0$ and $f_2(x) \neq 0, \forall x \in S$.

Theorem 20.5: Let f be a function for which the limit $L = \lim_{x \to a^S} f(x)$ exists and is finite. If g is a function defined on $\{f(x) : x \in S\} \cup \{L\}$ that is continuous at L, then $\lim_{x \to a^S} g \circ f(x)$ exists and equals $g(L)$.

Theorem 20.6: Let f be a function defined on a subset S of $\mathbb{R}$, let a be a real number that is the limit of some sequence in S, and let L be a real number. Then $\lim_{x\to a^S} f(x) = L$ if and only if for each $\epsilon > 0$, there exists $\delta > 0$ such that $x \in S$ and $|x - a| < \delta$ imply $|f(x) - L| < \epsilon$.

Corollary 20.7: Let f be a function defined on $J\backslash\{a\}$ for some open interval J containing a, and let L be a real number. Then $\lim_{x\to a} f(x) = L$ if and only if for each $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$.

Corollary 20.8: Let f be a function defined on some interval (a, b), and let L be a real number. Then $\lim_{x\to a^+} f(x) = L$ if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that $0 < x < a+\delta$ implies $|f(x)-L| < \epsilon$.

Theorem 20.10: Let f be a function defined on $J\backslash\{a\}$ for some open interval J containing a. Then $\lim_{x\to a} f(x)$ exists if and only if the limits $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$ both exist and are equal, in which case all three limits are equal.

Differentiation

Definition 28.1: Let f be a real-valued function defined on an open interval containing a point a. We say f **is differentiable at** a, or f **has a derivative at** a, if the limit

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite. If it exists and is finite,

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

Since f' exists whenever f is differentiable, $\operatorname{dom} f' \subseteq \operatorname{dom} f$.

Theorem 28.2: If f is differentiable at a point a, then f is continuous at a.

Theorem 28.3: Let f and g be functions that are differentiable at the point a. Each of the functions cf [c is a constant], $f + g, fg$ and f/g is also differentiable at a, except f/g if $g(a) = 0$ since f/g is not defined at a in this case. The formulas are

1. $(cf)'(a) = c \cdot f'(a)$;
2. $(f + g)'(a) = f'(a) + g'(a)$;
3. (product rule) $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$;
4. (quotient rule) $(f/g)'(a) = [f'(a)g(a) - f(a)g'(a)]/g^2(a)$

Theorem 28.4 - (Chain Rule): If f is differentiable at a and g is differentiable at $f(a)$, then the composite function $g \circ f$ is differentiable at a and we have $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.

Read full proof of theorem 28.4 (Chain Rule) in the book.

Theorem 29.1: If f is defined on an open interval containing x_0, if f assumes its maximum or minimum at x_0, and if f is differentiable at x_0, then $f'(x_0) = 0$.

Theorem 29.2 - Rolle's Theorem: Let f be a continuous function on $[a, b]$ that is differentiable on (a, b) and satisfies $f(a) = f(b)$. There exists [at least one] x in (a, b) such that $f'(x) = 0$.

Theorem 29.3 - Mean Value Theorem: Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). Then there exists [at least one] x in (a, b) such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Corollary 29.4: Let f be a differentiable function on (a, b) such that $f'(x) = 0$ for all $x \in (a, b)$. Then f is a constant function on (a, b).

Corollary 29.5: Let f and g be differentiable functions on (a, b) such that $f' = g'$ on (a, b). Then there exists a constant c such that $f(x) = g(x) + c$ for all $x \in (a, b)$.

We make distinctions between **strictly increasing** and **increasing**, and **strictly decreasing** and **decreasing** when talking about a function over an interval I.

Corollary 29.7: Let f be a differentiable function on an interval (a, b). Then

1. f is **strictly increasing** if $f'(x) > 0, \forall x \in (a, b)$;
2. f is **strictly decreasing** if $f'(x) < 0, \forall x \in (a, b)$;
3. f is **increasing** if $f'(x) \geq 0, \forall x \in (a, b)$;
4. f is **decreasing** if $f'(x) \geq 0, \forall x \in (a, b)$.

Theorem 29.8 - Intermediate Value Theorem for Derivatives: Let f be a differentiable function on (a, b). If $a < x_1 < x_2 < b$, and if c lies between $f'(x_1)$ and $f'(x_2)$, there exists [at least one] x in (x_1, x_2) such that $f'(x) = c$.

Theorem 29.9: Let f be a one-to-one continuous function on an open interval I, and let $J = f(I)$. If f is differentiable at $x_0 \in I$ and if $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Theorem 30.1 - Generalized Mean Value Theorem: Let f and g be continuous functions on $[a, b]$ that are differentiable on (a, b). Then there exists [at least one] x in (a, b) such that

$$f'(x)[g(b) - g(a)] = g'(x)[f(b) - f(a)].$$

Theorem 30.2 - L'Hospital's Rule: Let s signify a, a^+, a^-, ∞ or $-\infty$ where $a \in \mathbb{R}$, and suppose f and g are differentiable functions for which $\lim_{x\to s} \frac{f'(x)}{g'(x)} = L$. If

$$\lim_{x\to s} f(x) = \lim_{x\to s} g(x) = 0$$

or if

$$\lim_{x\to s} |g(x)| = +\infty,$$

then $\lim_{x\to s} \frac{f(x)}{g(x)} = L$.

Topological Concepts in Metric Spaces

Definition 13.1: Let S be a set, and suppose d is a function defined for all pairs (x, y) of elements from S satisfying

D1. $d(x, x) = 0$ for all $x \in S$ and $d(x, y) > 0$ for distinct $x, y \in S$.

D2. $d(x, y) = d(y, x)$ for all $x, y \in S$.

D3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in S$.

Such a function d is called a **distance function** or a **metric** on S. A metric space S is a set S together with a metric on it. Properly speaking, the **metric space** is the pair (S, d) since a set S may well have more than one metric on it.

Definition 13.2: A sequence (s_n) in a metric space (S, d) **converges** to s in S if $\lim_{n\to\infty} d(s_n, s) = 0$. A sequence (s_n) in S is a **Cauchy sequence** if for each $\epsilon > 0$ there exists an N such that $m, n > N$ implies $d(s_m, s_n) < \epsilon$.
The metric space (S, d) is said to be **complete** if every Cauchy sequence in S converges to some element in S.

*** **Notation** We will write $(x^{(n)})$ for a sequence of (x_j).

Lemma 13.3: A sequence $(x^{(n)})$ in $\mathbb{R}^k$ converges if and only if for each $j = 1, 2, \ldots, k$, the sequence $(x_j^{(n)})$ converges in $\mathbb{R}$. A sequence $(x^{(n)})$ in $\mathbb{R}^k$ is a Cauchy sequence if and only if each sequence $(x_j^{(n)})$ is a Cauchy sequence in $\mathbb{R}$.

Theorem 13.4: Euclidean $k-$space $\mathbb{R}^k$ is complete.

Theorem 13.5 - Bolzano-Weierstrass Theorem: Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

Definition 13.6: Let (S, d) be a metric space. Let E be a subset of S. An element $s_0 \in E$ is **interior** to E if for some $r > 0$ we have
$$\{s \in S : d(s, s_0) < r\} \subseteq E.$$
We write $E\circ$ for the set of points in E that are interior to E. The set E is **open in** S if every point in E is interior to E, i.e., if $E = E\circ$.

A point is interior to E if it is just some point that is not along the edge (an endpoint) of E.

Discussion 13.7: For some metric space (S, d),

(i) S is open in S.

(ii) The empty set $\emptyset$ is open in S.

(iii) The union of any collection of open sets is open.

(iv) The intersection of finitely many open sets is again an open set.

Definition 13.8: Let (S, d) be a metric space. A subset E of S is **closed** if its complement $S\backslash E$ is an open set. In other words, E is closed if $E = S\backslash U$, where U is an open set.

The **closure** E^- of a set E is the intersection of all closed sets containing E. The **boundary** of E is the set $E^-\backslash E\circ$; points in this set are called **boundary points** of E.

Proposition 13.9: Let E be a subset of a metric space (S, d).

(a) The set E is closed if and only if $E = E^-$.

(b) The set E is closed if and only if it contains the limit of every convergent sequence of points in E.

(c) An element is in E^- if and only if it is the limit of some sequence of points in E.

(d) A point is in the boundary of E if and only if it belongs to the closure of both E and its complement.

Theorem 13.10: Let (F_n) be a decreasing sequence [i.e., $F1 \supseteq F2 \supseteq \dots$] of closed bounded nonempty sets in $\mathbb{R}^k$. Then $F = \cap_{n=1}^{\infty} F_n$ is also closed, bounded and nonempty.

Definition 13.11: Let (S, d) be a metric space. A family $\mathcal{U}$ of open sets is said to be an **open cover for a set** E if each point of E belongs to at least one set in $\mathcal{U}$, i.e.,

$$E \subseteq \cup\{U : U \in \mathcal{U}\}.$$

A **subcover** of U is any subfamily of U that also covers E. A cover or subcover is **finite** if it contains only finitely many sets; the sets themselves may be infinite.

A set E is **compact** if every open cover of E has a finite subcover of E.

Theorem 13.12 - Heine-Borel Theorem: A subset E of $\mathbb{R}^k$ is compact if and only if it is closed and bounded.

A set F is a $k-$**cell** if there exist closed intervals $[a_1, b_1], [a_2, b_2], \dots, [a_k, b_k]$ so that

$$F = \{x \in \mathbb{R}^k : x_j \in [a_j, b_j], j = 1, 2, \dots, k\}.$$

F is sometimes written as $F = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k]$.

The **diameter** of F is

$$\delta = \left[\sum_{j=1}^{k} (b_j - a_j)^2\right]^{1/2};$$

that is, $\delta = \sup\{d(x, y) : x, y \in F\}$.

Proposition 13.13: Every $k-$cell F in $\mathbb{R}^k$ is compact.

Sequences and Series of Functions

Given a sequence $(a_n)_{n=0}^{\infty}$ of real numbers, the series $\sum_{n=0}^{\infty} a_n x^n$ is called a **power series**.

We use the convention that $0^0 = 1$.

It turns out that, given any sequence (a_n), one of the following holds for its power series:

(a) The power series converges for all $x \in \mathbb{R}$;

(b) The power series converges only for $x = 0$;

(c) The power series converges for all x in some bounded interval centered at 0; the interval may be open, half-open or closed.

Theorem 23.1: For the power series $\sum_{n=0}^{\infty} a_n x^n$, let $\beta = \limsup |a_n|^{1/n}$ and $R = \frac{1}{\beta}$. [If $\beta = 0$ we set $R = +\infty$, and if $\beta = +\infty$ we set $R = 0$.] Then

1. The power series converges for $|x| < R$;
2. The power series diverges for $|x| > R$.

R is called the **radius of convergence** for the power series.

$\lim_{n\to\infty} \sum_{k=0}^{n} a_k x^k$ **converges uniformly** to $\sum_{k=0}^{\infty} a_k x^k$ on sets $[-R_1, R_1]$ such that $0 \leq R_1 < R$.

Definition 24.1: Let (f_n) be a sequence of real-valued functions defined on a set $S \subseteq \mathbb{R}$. The sequence (f_n) **converges pointwise** [i.e., at each point] to a function f defined on S if

$$\lim_{n\to\infty} f_n(x) = f(x), \forall x \in S.$$

We often write $\lim f_n = f$ pointwise [on S] or $f_n \to f$ pointwise [on S].

Definition 24.2: Let (f_n) be a sequence of real-valued functions defined on a set $S \subseteq \mathbb{R}$. The sequence (f_n) **converges uniformly** on S to a function f defined on S if for each $\epsilon > 0$ there exists a number N such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in S$ and all $n > N$.

We write $\lim f_n = f$ **uniformly on** S or $f_n \to f$ **uniformly on** S.

Theorem 24.3: The uniform limit of continuous functions is continuous. More precisely, let (f_n) be a sequence of functions on a set $S \subseteq \mathbb{R}$, suppose $f_n \to f$ uniformly on S, and suppose $S = \mathrm{dom}(f)$. If each f_n is continuous at x_0 in S, then f is continuous at x_0. [So if each f_n is continuous on S, then f is continuous on S.]

A generalization of Theorem 24.3 is that limits can be interchanged (order of limits can be changed).

Famous "$\frac{\epsilon}{3}$ argument":

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|.$$

Remark 24.4: Uniform convergence can be reformulated as follows. A sequence (f_n) of functions on a set $S \subseteq \mathbb{R}$ converges uniformly to a function f on S if and only if $\lim_{n\to\infty} \sup\{|f(x) - f_n(x)| : x \in S\} = 0$.

Theorem 25.2: Let (f_n) be a sequence of continuous functions on $[a, b]$, and suppose $f_n \to f$ uniformly on $[a, b]$. Then

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Definition 25.3: A sequence (f_n) of functions defined on a set $S \subseteq \mathbb{R}$ is **uniformly Cauchy on** S if for each $\epsilon > 0$, there exists a number N such that $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in S$ and all $m, n > N$.

Theorem 25.4: Let (f_n) be a sequence of uniformly Cauchy functions defined on a set $S \subseteq \mathbb{R}$. Then there exists a function f on S such that $f_n \to f$ uniformly on S.

If the sequence of partial sums converges uniformly on a set S to $\sum_{k=0}^{\infty} g_k$, then we say the **series is uniformly convergent on** S.

Theorem 25.5: Consider a series $\sum_{k=0}^{\infty} g_k$ of functions on a set $S \subseteq \mathbb{R}$. Suppose each g_k is continuous on S and the series converges uniformly on S. Then the series $\sum_{k=0}^{\infty} g_k$ represents a continuous function on S.

Theorem 25.6: If a series $\sum_{k=0}^{\infty} g_k$ of functions satisfies the Cauchy criterion uniformly on a set S, then the series converges uniformly on S.

25.7 - Weierstrass M-Test: Let (M_k) be a sequence of nonnegative real numbers where $\sum M_k < \infty$. If $|g_k(x)| \leq M_k$ for all x in a set S, then $\sum g_k$ converges uniformly on S.

Integration

Definition 32.1: Let f be a bounded function on a closed interval $[a,b]$. For $S \subseteq [a,b]$, we adopt the notation

$$M(f,S) = \sup\{f(x) : x \in S\} \text{ and } m(f,S) = \inf\{f(x) : x \in S\}.$$

A **partition** of $[a,b]$ is any finite ordered subset P having the form

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}.$$

The **upper Darboux sum** $U(f,P)$ of f with respect to P is the sum

$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1}),$$

and the **lower Darboux sum** $L(f,P)$ is

$$L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1}).$$

The **upper Darboux integral** $U(f)$ of f over $[a,b]$ is defined by

$$U(f) = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\},$$

and the **lower Darboux integral** is defined by

$$L(f) = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}.$$

$$\int_a^b f = \int_a^b f(x)dx = L(f) = U(f)$$

is called the **Darboux integral**.

*** If the upper and lower integrals agree, then the function is integrable.

Lemma 32.2: Let f be a bounded function on $[a,b]$. If P and Q are partitions of $[a,b]$ and $P \subseteq Q$, then

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P).$$

Lemma 32.3: If f is a bounded function on $[a,b]$, and if P and Q are partitions of $[a,b]$, then $L(f,P) \leq U(f,Q)$.

$L(f)$ is the supremum (least upper bound) of $\{L(f,P) : P \text{ is a partition of } [a,b]\}$ and $U(f)$ is the infimum of $\{U(f,P) : P \text{ is a partition of } [a,b]\}$.

Theorem 32.4: If f is a bounded function on $[a,b]$, then $L(f) \leq U(f)$.

Theorem 32.5: A bounded function f on $[a,b]$ is integrable if and only if for each $\epsilon > 0$ there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$.

Definition 32.6: The **mesh** of a partition P is the maximum length of the subintervals comprising P. Thus if $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$, then

$$\text{mesh}(P) = \max\{t_k - t_{k-1} : k = 1, 2, \ldots, n\}.$$

Theorem 32.7: A bounded function f on $[a, b]$ is integrable if and only if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\text{mesh}(P) < \delta \to U(f, P) - L(f, P) < \epsilon,$$

for all partitions P of $[a, b]$.

Definition 32.8: Let f be a bounded function on $[a, b]$, and let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a, b]$. A **Riemann sum** of f associated with the partition P is a sum of the form

$$\sum_{k=1}^{n} f(x_k)(t_k - t_{k-1}),$$

where $x_k \in [t_{k-1}, t_k]$ for $k = 1, 2, \ldots, n$.

In the above definition, x_k is arbitrary, so there are infinitely many valid vices of x_k.

The function f is **Riemann integrable** on $[a, b]$ if there exists a number r with the following property. For each $\epsilon > 0$ there exists $\delta > 0$ such that

$$|S - r| < \epsilon$$

for every Riemann sum S of f associated with a partition P having $\text{mesh}(P) < \delta$. The number r is the **Riemann integral** of f on $[a, b]$ and will be provisionally written as $\mathcal{R} \int_a^b f$.

Theorem 32.9: A bounded function f on $[a, b]$ is Riemann integrable if and only if it is [Darboux] integrable, in which case the values of the integrals agree.

Corollary 32.9: Let f be a bounded Riemann integrable function on $[a, b]$. Suppose (S_n) is a sequence of Riemann sums, with corresponding partitions P_n, satisfying $\lim_n \text{mesh}(P_n) = 0$. Then the sequence (S_n) converges to $\int_a^b f$.

Rudin's Principles of Mathematical Analysis

Uniform Convergence and Integration

Theorem 7.16: Let (a_n) be monotonically increasing on $[a, b]$. Suppose f_n on $[a, b]$ for $n = 1, 2, 3, \ldots,$ and suppose $f_n \to f$ uniformly on $[a, b]$. Then f exists on $[a, b]$, and

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx.$$

Uniform Convergence and Differentiation

Theorem 17.7: Suppose (f_n) is a sequence of differentiable functions on $[a, b]$ such that $(f_n(x_0))$ converges for some point x_0, on $[a, b]$. If (f_n') converges uniformly on $[a, b]$, then (f_n) converges uniformly on $[a, b]$, to a function f, and

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$