Insertion Sort

1 Algorithm

The approach is to sort the first `i-1` elements, then insert element `i` into its proper location in that subarray. Repeat until the array is sorted.

There are two versions of the algorithm: first one assumes a sentinel value `A[0] = -inf`, and the second one doesn't use this sentinel.

```
A[0] = -inf
for i in [2, n]:
    t, j = A[i], i-1
    while A[j] > t:
        A[j+1] = A[j]
        j--
    A[j+1] = T
```

```
for i in [2, n]:
    t, j = A[i], i-1
    while j > 0 and A[j] > t:
        A[j+1] = A[j]
        j--
    A[j+1] = t
```

2 Analysis

2.1 Comparisons

The following calculations are for the sentinel version. They can be redone for the non-sentinel version pretty simply by counting how many fewer operations are done every iteration.

Best case In the best case, $n-1$ comparisons are done because the while loop checks once every time the for loop runs.

Worst case Because of the presence of the sentinel value, the worst case is a reverse-sorted array in which case there are i comparisons done in each iteration of the for loop.

$$\sum_{i=2}^{n} i = \left(\sum_{i=1}^{n} i\right) - 1 = \frac{(n-1)(n+2)}{2} \tag{1}$$

Average case Consider a snapshot of the algorithm at iteration i, j. `t` can potentially be inserted into any of i possible spots, and at the j^{th} iteration of the `while`-loop there will have been $i-j+1$ comparisons conducted.

$$\begin{aligned}
\sum_{i=2}^{n}\sum_{j=1}^{i}\frac{1}{i}(i-j+1) &= \sum_{i=2}^{n}\frac{1}{i}\cdot\sum_{j=1}^{i}(i-j+1) && (2)\\
&= \sum_{i=2}^{n}\frac{1}{i}\cdot\sum_{j=1}^{i}j && (3)\\
&= \sum_{i=2}^{n}\frac{1}{i}\cdot\frac{i(i+1)}{2} && (4)\\
&= \frac{1}{2}\sum_{i=2}^{n}(i+1) && (5)\\
&= \frac{(n-1)(n+4)}{4} && (6)
\end{aligned}$$

2.2 Exchanges

Best case $2n-1$, with the extra 1 being from moving the sentinel into position 0 at the beginning of the algorithm.

Worst case The worst case is a reverse-sorted array, which means there will be $i+1$ exchanges done every iteration.

$$\begin{aligned}
1+\sum_{i=2}^{n}(i+1) &= 1+\sum_{i=1}^{n-1}(i+2) && (7)\\
&= \frac{(n-1)(n+4)}{2}+1 && (8)
\end{aligned}$$

Average case It's not worth it to do this calculation.