Summation Approximations

1 Fundamental Theorem of Calculus

Not an approximation, but still a useful method of solving certain summations. For a constant variable x:

$$\begin{aligned}
\sum_{i=1}^{n-1} i x^{i-1} &= \frac{d}{dx} \int \sum_{i=1}^{n-1} i x^{i-1} dx && (1)\\
&= \frac{d}{dx} \left(\sum_{i=1}^{n-1} \int i x^{i-1} dx \right) && (2)\\
&= \frac{d}{dx} \sum_{i=1}^{n-1} x^i && (3)\\
&= \frac{d}{dx} \left(\frac{x^n - 1}{x - 1} - 1 \right) && (4)\\
&= \frac{(n-1)x^n - n x^{n-1} + 1}{(x-1)^2} && (5)
\end{aligned}$$

2 Harmonic Sum

$$\sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \dots \tag{6}$$

To approximate this sum, take the first element as-is. Now because the function is strictly decreasing, the sum of any block of consecutive terms in the sequence can be upper-bounded by the length of the block times the first element.

Using this approach, upper-bound $1/3$ to $1/2$, $(1/5, 1/6, 1/7)$ each to $1/4$, and so on and so forth. For every element $(1/2)^j$, upper-bound the next $2^j - 1$ elements to the first. This divides the entire summation into a number of blocks, where each block adds up to 1. The number of blocks is $\lg(n+1)$, so therefore

$$\sum_{i=1}^{n} \leq \lg(n+1) \tag{7}$$

3 Geometric Approximation

For a strictly increasing or decreasing function, its summation can be approximated by a geometric series.

$$\sum_{i=1}^{\infty} \frac{i^2}{2^i} = \frac{1}{2} + \frac{4}{4} + \frac{9}{8} + \frac{16}{16} + \frac{25}{32} + \frac{36}{64} + \dots \tag{8}$$

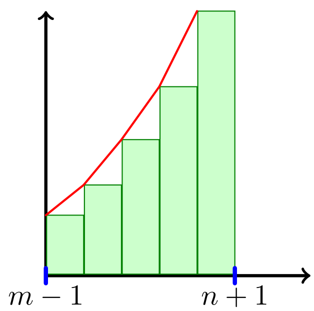
Notice that for this summation, the terms begin decreasing at $i = 3$. An approximation can be taken by bounding the summation at this point with a geometric series. Taking the ratio of consecutive terms,

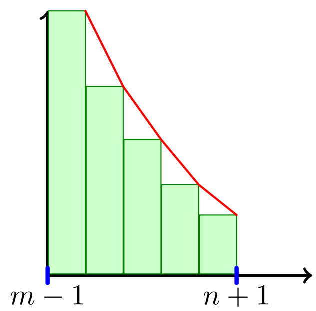
$$r = \frac{\frac{(i+1)^2}{2^{i+1}}}{\frac{i^2}{2^i}} = \frac{1}{2}\left(1 + \frac{1}{i}\right)^2 \tag{9}$$

Better approximations can be achieved by increasing i and adding in the sum of excluded terms.

4 Integral Approximation

For strictly increasing or decreasing functions, summations can be bounded both above and below by integrals.





For an increasing f,

$$\int_{m-1}^{n} f(x)dx \leq \sum_{i=m}^{n} f(i) \leq \sum_{m}^{n+1} f(x)dx \tag{10}$$

and for a decreasing f,

$$\int_{m}^{n+1} f(x)dx \leq \sum_{i=m}^{n} f(i) \leq \int_{m-1}^{n} f(x)dx \tag{11}$$