MATH 307: Individual Homework 21

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Due and submitted on 04/28/2021
Spring 2021, Dr. Guo

Problem 1

See HW instruction.

Know that $L_1 = I + \ell_1 e_1^T$ and W.T.S. that $L_1^{-1} = I - \ell_1 e_1^T$, we may simply check if $L_1(I - \ell_1 e_1^T) = I$.

$$
\begin{aligned}
L(I - \ell_1 e_1^T) &= (I + \ell_1 e_1^T)(I - \ell_1 e_1^T) = I - \ell_1 e_1^T + \ell_1 e_1^T - \ell_1 e_1^T \ell_1 e_1^T \\
&= I - \ell_1 (e_1^T \ell_1) e_1^T \\
&= I - \ell_1 (\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{a_{21}}{a_{11}} \\ \vdots \\ -\frac{a_{n1}}{a_{11}} \end{bmatrix}) \\
&= I - ell_1(0) e_1^T \\
&= I
\end{aligned}
$$

As $L(I - \ell_1 e_1^T) = I$, there must be $L_1^{-1} = I - \ell_1 e_1^T$ as $L_1 L_1^{-1} = I$.

Problem 2

See HW instruction.

Start with Gaussian elimination:

$$
\underbrace{\left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 2 \\ -3 & 2 & 1 & 0 & -5 \\ 3 & 2 & 1 & 1 & 2 \end{array}\right]}_{R_2 + 3R_1 \to R_2,\ R_3 - 3R_1 \to R_3} = \underbrace{\left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 2 \\ 0 & 8 & 4 & 9 & 1 \\ 0 & -4 & -2 & -8 & -4 \end{array}\right]}_{R_3 + \frac{R_2}{2} \to R_3} = \underbrace{\left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 2 \\ 0 & 8 & 4 & 9 & 1 \\ 0 & 0 & 0 & -\frac{7}{2} & -\frac{7}{2} \end{array}\right]}_{-\frac{2}{7} R_3 \to R_3} = \left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 2 \\ 0 & 8 & 4 & 9 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]
$$

So we have a system of:

$$
\begin{cases} x_1 + 2x_2 + x_3 + 3x_4 = 2 \\ 8x_2 + 4x_3 + 9x_4 = 1 \\ x_4 = 1 \end{cases}
$$

We have the first, second, and fourth columns being the pivot columns. Let the free column $x_3 = 0$, we have:

$$\begin{cases} x_1 + 2x_2 + 0 + 3 = 2 \\ 8x_2 + 0 + 9 = 1 \end{cases}$$
$$\implies \begin{cases} x_1 = 1 \\ x_2 = -1 \end{cases}$$

Thus, $x_p = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. Now to find $null(A)$, which we know is same as $null(\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 8 & 4 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix})$. We let the $x_3 = 1$ and have:

$$\begin{cases} x_1 + 2x_2 + 1 = 0 \\ 8x_2 + 4 = 0 \\ x_4 = 0 \end{cases}$$
$$\implies \begin{cases} x_1 = 0 \\ x_2 = -\frac{1}{2} \end{cases}$$

Thus, we have $null(A) = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$; which we may confirm by checking $null(A) + rank(A) = dim(A) \iff 1 + 3 = 4$. And we may there have the general solution as $x_g = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$.

Problem 2

See HW instruction.

(a)

$$\underbrace{\left[\begin{array}{ccccc|c} -1 & 2 & 1 & 0 & 2 & -1 \\ 2 & 0 & 0 & 3 & -1 & 0 \\ -1 & 6 & 3 & 3 & 5 & c \end{array}\right]}_{2R_1+R_2 \to R_2,\ -R_1+R_3 \to R_3} = \left[\begin{array}{ccccc|c} -1 & 2 & 1 & 0 & 2 & -1 \\ 0 & 4 & 2 & 3 & 3 & -2 \\ 0 & 4 & 2 & 3 & 3 & c+1 = -2 \Rightarrow c = -3 \end{array}\right]$$

So we must have $c = -3$ as otherwise there must be a conflict between R_2 and R_3.

(b)

To find x_p we continue the row deduction:

$$\underbrace{\left[\begin{array}{ccccc|c} -1 & 2 & 1 & 0 & 2 & -1 \\ 0 & 4 & 2 & 3 & 3 & -2 \\ 0 & 4 & 2 & 3 & 3 & -2 \end{array}\right]}_{-R_2+R_3\to R_3} = \left[\begin{array}{ccccc|c} -1 & 2 & 1 & 0 & 2 & -1 \\ 0 & 4 & 2 & 3 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

By inspecting the reduced matrix we know that columns 3, 4, 5 are all free variables, thus we may have:

$$\begin{cases} -x_1 + 2x_2 + x_3 + 2x_5 = -1 \\ 4x_2 + 2x_3 + 3x_4 + 3x_5 = -2 \end{cases}$$

$$\text{Let } x_3 = x_4 = x_5 = 0$$

$$\implies \begin{cases} -x_1 + 2x_2 = -1 \iff -x_1 - 1 = -1 & x_1 = 0 \\ 4x_2 = -2 & x_2 = -\frac{1}{2} \end{cases}$$

Thus, we may have $x_p = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$.