MATH 307: Group Homework 12

Group 8
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Due and submitted on 05/07/2021
Spring 2021, Dr. Guo

Problem 1

Textbook page 132, problem 1.

So we will try to identify the $\|b - Ax\|_2^2$ among the three candidates.

$$\left\| b - A \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} \right\|_2^2 = \left\| b - \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} \right\|_2^2 = \left\| \begin{bmatrix} 8 \\ 8 \\ 4 \\ 16 \end{bmatrix} - \begin{bmatrix} 10 \\ 5 \\ 8 \\ 11 \end{bmatrix} \right\|_2^2 = \left\| \begin{bmatrix} -2 \\ 3 \\ -4 \\ 5 \end{bmatrix} \right\|_2^2 = 54$$

$$\left\| b - A \begin{bmatrix} 8 \\ 1 \\ -2 \end{bmatrix} \right\|_2^2 = \left\| \begin{bmatrix} 8 \\ 8 \\ 4 \\ 16 \end{bmatrix} - \begin{bmatrix} 5 \\ 8 \\ 7 \\ 6 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 3 \\ 0 \\ -3 \\ 10 \end{bmatrix} \right\| = 118$$

$$\left\| b - A \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\|_2^2 = \left\| \begin{bmatrix} 8 \\ 8 \\ 4 \\ 16 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} \right\|_2^2 = \left\| \begin{bmatrix} 7 \\ 8 \\ 5 \\ 18 \end{bmatrix} \right\|_2^2 = 462$$

Thus, $\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$ is the least squares solution.

Problem 2

Textbook page 132, problem 2.

Problem 2

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \; b = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

A^T would obviously be:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Calculating P :

$$P = A(A^TA)^{-1}A^T = A(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix})^{-1}A^T$$
$$= A(\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix})^{-1}A^T = A(\frac{1}{2}\begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix})A^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1.5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0.5 \\ 1 & -1 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = P = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

Calculating Pb:

$$Pb = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

Problem 3

Textbook page 132, problem 3.

$$Ax = A(A^*A^{-1})A^*b = Pb$$
$$P = A(A^*A^{-1})A^* = A(A^*A)^{-1}A^*$$
$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} (\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix})^{-1} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \frac{1}{14} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{14} & \frac{1}{7} & \frac{3}{14} \\ \frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ \frac{3}{14} & \frac{3}{7} & \frac{9}{14} \end{bmatrix}$$
$$\Rightarrow Pb = \begin{bmatrix} \frac{1}{14} & \frac{1}{7} & \frac{3}{14} \\ \frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ \frac{3}{14} & \frac{3}{7} & \frac{9}{14} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} \frac{29}{14} \\ \frac{29}{7} \\ \frac{87}{14} \end{bmatrix} = Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} x$$
$$\Longrightarrow x = \frac{29}{14}$$

Thus, the least squares solution for this system is $\frac{29}{14}$.