MATH 307: Group Homework 10

Group 8

Shaochen (Henry) ZHONG, Zhitao (Robert) CHEN, John MAYS, Huaijin XIN

{sxz517, zxc325, jkm100, hxx200}@case.edu

Due and submitted on 04/26/2021

Spring 2021, Dr. Guo

Problem 1

See HW instruction.

(a) LU decomposition

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} x = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \Rightarrow \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & \frac{5}{2} \end{bmatrix} x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\text{Let } \begin{bmatrix} 2 & 3 \\ 0 & \frac{5}{2} \end{bmatrix} x = y$$

$$\text{We have } \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \Rightarrow y = \begin{bmatrix} 1 \\ \frac{5}{2} \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 2 & 3 \\ 0 & \frac{5}{2} \end{bmatrix} x = y = \begin{bmatrix} 1 \\ \frac{5}{2} \end{bmatrix} \qquad \Rightarrow x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(b) Gaussian elimination

$$\underbrace{\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & 4 & 3 \end{array}\right]}_{R_2 - \frac{1}{2}R_1 \ \to R_2} = \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & \frac{5}{2} & \frac{5}{2} \end{array}\right] \Rightarrow \begin{cases} \frac{5}{2}x_2 = \frac{5}{2} & x_2 = 1 \\ 2x_1 + 3(1) = 1 & x_1 = -1 \end{cases}$$

$$\Longrightarrow x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(c) Gauss-Jordan elimination

$$\underbrace{\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]}_{\frac{1}{2}R_1 \to R_1} = \underbrace{\left[\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]}_{R_2 - R_1 \to R_2} = \underbrace{\left[\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & \frac{5}{2} & -\frac{1}{2} & 1 \end{array}\right]}_{\frac{5}{2}R_2 \to R_2} = \underbrace{\left[\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{10} & \frac{2}{5} \end{array}\right]}_{R_1 - \frac{3}{2}R_2 \to R_1} = \left[\begin{array}{cc|cc} 1 & 0 & \frac{8}{10} & -\frac{3}{5} \\ 0 & 1 & -\frac{2}{10} & \frac{2}{5} \end{array}\right]$$

$$\Rightarrow A^{-1} = \left[\begin{array}{cc|cc} 1 & 0 & \frac{8}{10} & -\frac{3}{5} \\ 0 & 1 & -\frac{2}{10} & \frac{2}{5} \end{array}\right]$$

Known that $x = A^{-1}b$

$$x = \left[\begin{array}{cc|cc} 1 & 0 & \frac{8}{10} & -\frac{3}{5} \\ 0 & 1 & -\frac{2}{10} & \frac{2}{5} \end{array}\right] \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{8}{10} - \frac{18}{10} \\ -\frac{2}{10} + \frac{12}{10} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

To verify, we have $Ax = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2+3 \\ -1+4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = b.$

Problem 2

See HW instruction.

First we try to find $rref(A)$:

$$\begin{aligned}
A &= \begin{pmatrix} 1 & 2 & 1 & 3 \\ -3 & 2 & 1 & 0 \\ 3 & 2 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 8 & 4 & 9 \\ 0 & -4 & -2 & -8 \end{pmatrix}, (R_2 = R_2 + 3R_1, R_3 = R_3 - 3R_1) \\
&= \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & \frac{1}{2} & \frac{9}{8} \\ 0 & -4 & -2 & -8 \end{pmatrix}, (R_2 = \frac{R_2}{8}) \\
&= \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & \frac{1}{2} & \frac{9}{8} \\ 0 & 0 & 0 & -\frac{7}{2} \end{pmatrix}, (R_3 = -\frac{2}{7}R_3) \\
&= \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, (R_1 = R_1 - 3R_3, R_2 = R_2 - \frac{9}{8}R_3) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, (R_1 = R_1 - 2R_2) \\
\Longrightarrow rref(A) &= \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}
\end{aligned}$$

So the three pivots are the **bold 1** showed above. The rank of A is the number of pivots, thus

$rank(A) = 3$. And per the rank-nullity theorem, we have the nullity $dim(A) - rank(A) = 4 - 3 = 1$.

Now for column space, since we have pivots on the fisrt, second, and fourth columns, we have $range(A) = span(\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix})$. And for nullspace, let $Ax = 0$, we have :

$$\begin{cases} x_1 = 0 \\ x_2 + \frac{1}{2}x_3 = 0 \\ x_4 = 0 \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ -2x_2 \\ 0 \end{bmatrix} = r \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}$$

$$null(A) = span(\begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix})$$