MATH 307: Individual Homework 15

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Problem 1

Textbook page 80, problem 1.

W.T.S. $\|Ax\| \leq \|A\| \, \|x\|$

$$\begin{aligned} \text{Since } \|A\| &= \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} \\ \text{There must be } \frac{\|Ax\|}{\|x\|} &\leq \|A\| \\ &\implies \|Ax\| \leq \|A\| \, \|x\| \end{aligned}$$

W.T.S. $\|AB\| \leq \|A\| \, \|B\|$

$$\begin{aligned} \|AB\| &= \sup_{x \neq 0} \frac{\|(AB)x\|}{\|x\|} = \sup_{\|x\|=1} \|A(Bx)\| \\ &\leq \sup_{\|x\|=1} \|A\| \, \|B(x)\| \\ &\leq \sup_{\|x\|=1} \|A\| \, \|B\| \\ \|x\| &= \|A\| \, \|B\| \\ \|AB\| &\leq \|A\| \, \|B\| \end{aligned}$$

Problem 2

Textbook page 80, problem 2.

∞ norm As $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$, we will take the maximum row sum of the matrix, which is $|-20| + |20| + |2| + |-2| + |0| = 44$.

1-norm Similar to the ∞ norm above, as $\|A\|_1 = \max_j \sum_{i=1}^{m} |a_{ij}|$, we will take the maximum column sum of the matrix, which is $|4| + |-1| + |5| + |20| = 30$.

Frobenius norm As $\|A\|_F = \sqrt{\sum_i^m \sum_j^n |a_{ij}|^2}$, we will take the square of each entries, add them together, and square root it. For simplicity I will omit the copying, the result should be $\sqrt{1279}$.

Problem 3

Textbook page 80, problem 3.

W.T.S. $\|PA\|_2 = \|A\|_2$
Known that for an orthogonal matrix A, there must be $A^T A = I$; and also known $\|A\| = \sqrt{A^T A}$. Note

$$\begin{aligned}
\|QA\| &= \sqrt{(QA)^T QA} = \sqrt{A^T Q^T QA} \\
&= \sqrt{A^T IA} = \sqrt{A^T A} \\
&= \|A\|
\end{aligned}$$

Since this relationship is not restricted to any p-norm, we may have $\|QA\|_2 = \|A\|_2$.

W.T.S. $\|AQ\|_2 = \|A\|_2$
Known that $\|Qx\|_2 = \sqrt{< Qx, Qx >} = \sqrt{< x, Q^T Qx} = \sqrt{< x, x >} = \|x\|_2$, we have:

$$\begin{aligned}
\|AQ\|_2 &= \sup_{\|x\|_2=1} \|AQx\|_2 = \sup_{\|Qx\|_2=1} \|A(Qx)\|_2 \\
&= \sup_{\|y\|_2=1} \|Ay\|_2 = \|A\|_2
\end{aligned}$$

Problem 4

Textbook page 80, problem 4.

$\max(A) = \max\{a_{ij}\}$ is not a norm as it does not support *non-negetivity* $\max(A) > 0$ when $A \neq 0$. Take a matrix with all negative entries, say $A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$. We have $\max(A) = -1 < 0$, non-negetivity is violated and therefore not a norm.

$\max(|AB|) = \max\{|a_{ij}|\}$ is also not a norm as it does not support *submultiplicative* $\max(AB) \leq \max(A)\max(B)$. Take $A = B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, we have:

$$\begin{aligned}
\max(|AB|) &= \max(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}) = \max(\begin{bmatrix} |2| & |2| \\ |2| & |2| \end{bmatrix}) = 2 \\
\max(|A|)\max(|B|) &= \max(\begin{bmatrix} |1| & |1| \\ |1| & |1| \end{bmatrix}) \max(\begin{bmatrix} |1| & |1| \\ |1| & |1| \end{bmatrix}) = 1 \\
&\implies \max(AB) > \max(A)\max(B)
\end{aligned}$$

As submultiplicative is violated, it is also not a norm.