MATH 307: Group Homework 9

Group 8
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Due and submitted on 04/16/2021
Spring 2021, Dr. Guo

Problem 1

See HW instruction.

$\forall x = \begin{bmatrix} Re(x_1) + Im(x_1) \\ Re(x_2) + Im(x_2) \end{bmatrix} \in \mathbb{C}^2$, we have:

$$\begin{aligned}
\begin{bmatrix} Re(x_1) + Im(x_1) \\ Re(x_2) + Im(x_2) \end{bmatrix} &= \underbrace{Re(x_1)e_1 + Im(x_1)e_1}_{\in V} + \underbrace{Re(x_2)e_2 + Im(x_2)e_2}_{\in W} \\
&= \underbrace{-Re(x_1)(e_2 - e_1) - Im(x_1)(e_2 - e_1)}_{\in V} + \underbrace{(Re(x_1) + Re(x_2) + Im(x_1) + Im(x_2))e_2}_{\in W} \\
&= -Re(x_1)e_2 + Re(x_1)e_1 - Im(x_1)e_2 + Im(x_1)e_1 \\
&\quad + Re(x_1)e_2 + Re(x_2)e_2 + Im(x_1)e_2 + Im(x_2)e_2 \\
&= Re(x_1)e_1 + Re(x_2)e_2 + Im(x_1)e_1 + Im(x_2)e_2 \\
\implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= V + W
\end{aligned}$$

Although we may have $\mathbb{C}^2 = V + W$, the solution is not unique and therefore not a direct sum.

Problem 2

See HW instruction.

Let $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. We have $\forall x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ to be:

$$\begin{aligned}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \underbrace{x_1e_1 + x_2e_2}_{\in V} + \underbrace{x_3e_3}_{\in W} = V + W \\
&= \underbrace{x_1e_1}_{\in V} + \underbrace{x_2e_2 + x_3e_3}_{\in W} = V + W
\end{aligned}$$

As shown above, although we may have $\mathbb{R}^3 = V + W$, the solution is not unique and therefore not a direct sum.

Problem 3

See HW instruction.

For $A \in C^{m\times n}$ where $A = U\Sigma V^*$ and $A^* = V\Sigma^* U^*$. Assume there are r number of nonzero singular values in Σ, and known that U is an unitary matrix with orthonormal columns, we have:

$$\begin{aligned} range(A) &= span(u_1, u_2, \ldots, u_r) \\ null(A^*) &= span(u_{r+1}, \ldots, u_m) \\ \implies range(A) + null(A^*) &= span(u_1, u_2, \ldots, u_m) \end{aligned}$$

This implies $\forall x \in \mathbb{C}^m, x = \underbrace{\sum_{i=1}^{r} c_i u_i}_{\in range(A)} + \underbrace{\sum_{j=r+1}^{m} c_j u_j}_{\in null(A^*)}$, thus $\mathbb{C}^m = range(A) + null(A^*)$.

Now to show uniqueness, we know that $range(A) \perp null(A^*)$ from **Midterm 2**'s last question. This implies $V \cap W = \{0\}$ and therefore $\mathbb{C}^m = range(A) \oplus null(A^*)$