MATH 307: Individual Homework 22

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Problem 1

See HW instruction.

W.T.S. $\sigma_i \in \Sigma$ are nonzero $\Longrightarrow A$ is invertible.

For A to be not invertible we must have $\det(A) = 0$, which implies:

$$\det(A) = \det(U\Sigma V^T) = \det(u)\det(\Sigma)\det(V^T) = 0$$

Since U, V are orthognal matrices, we must have ± 1 to be their determinants[1]. So we must have $\det(\Sigma) = 0$ for A to be not invertible.

Also since Σ is a diagonal matrix, we have $\det(\Sigma)$ to be the product of its diagonal valus. So for $\det(\Sigma) = 0$ we must have at least on of its singular value to be zero. So by contrapositive, we have showed that for A to be invertible, there must be $\sigma_i \in \Sigma$ are nonzero.

W.T.S. A is invertible $\Longrightarrow \sigma_i \in \Sigma$ are nonzero.

The reverse is rather simple. Again by contrapositive, if Σ has zero singular values, we have $\det(\Sigma) = 0$ and therefore $\det(A) = 0$, thus A is not invertible by the determinant definition.

As both directions are showed, the proposed iff statmenet is therefore proven.

Problem 2

See HW instruction.

$$\begin{aligned} \det(A - \lambda I) = p(\lambda) &= (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)\dots(\lambda_4 - \lambda) \\ \text{Set } \lambda &= 0 \\ \det(A) &= \lambda_1\lambda_2\lambda_3\dots\lambda_n \end{aligned}$$

Problem 3

See HW instruction.

[1] because $\det(QQ^T) = \det(I) \Longrightarrow \det(Q)\det(Q^{-1}) = 1 \Longrightarrow \det(Q)^2 = 1 \Longrightarrow \det(Q) = \pm 1$, assume Q is a orthognal matrix.

W.T.S. A is invertible $\implies \det(A) \neq 0$.

$$\begin{aligned} Av &= \lambda v \\ A^{-1}Av &= \lambda A^{-1}v \\ v &= \lambda A^{-1}v \\ \implies A^{-1}v &= \frac{1}{\lambda}v \end{aligned}$$

As A is invertible, it must has nonzero eigenvalues as otherwise we can't have $\frac{1}{\lambda}$. And per **Problem 2**, we know that $\det(A)$ is the product of A's eigenvalues, so $\det(A) \neq 0$.

W.T.S. $\det(A) \neq 0 \implies A$ is invertible.

From **Problem 1**, assume $A = U\Sigma V^T$ to be not invertible, we know that A must have zero value in its Σ. Since Σ is a diagonal matrix, we have $\det(\Sigma)$ to be the product of its diagonal valus, so $\Sigma = 0$

Also known that U, V^T are orthognal matrices and their determinants must therefore be ± 1. So $\det(A) = \det(U\Sigma V^T) = \det(u)\det(\Sigma)\det(V^T) = (\pm 1)0(\pm 1) = 0$. We have proven the statement-in-question by contrapositive.

As both directions are showed, the proposed iff statmenet is therefore proven.