MATH 307: Individual Homework 14

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Problem 1

See HW instruction.

$\forall x \in range(A)$, we have $x = Ak$; $\forall y \in N(A^*)$, we must have $A^*y = 0$. Which means:

$$\begin{aligned} < x, y > &= y^*x = y^*Ak \\ &= ((y^*A)^*)^*k \\ &= (A^*y)^*k \\ &= 0^*k = 0 \end{aligned}$$

As the inner product yields zero, the two spaces in question are orthogonal to each other.

Problem 2

See HW instruction.

$$
\begin{aligned}
u_1 &= a_1 = (1,3,1) \\
e_1 &= \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{11}}(1,3,1) = (\frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}) \\
u_2 &= a_2 - < a_2 \cdot e_1 > e_1 = (2,-1,1) - (\frac{2}{\sqrt{11}} - \frac{3}{\sqrt{11}} + \frac{1}{\sqrt{11}})e_1 \\
&= (2,-1,1) - 0e_1 = (2,-1,1) \\
e_2 &= \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{6}}(2,-1,1) = (\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}) \\
u_3 &= a_3 - < a_3 \cdot e_1 > e_1 - < a_3 \cdot e_2 > e_2 \\
&= (1,1,2) - (\frac{1}{\sqrt{11}} + \frac{3}{\sqrt{11}} + \frac{2}{\sqrt{11}})e_1 - < a_3 \cdot e_2 > e_2 \\
&= (1,1,2) - \frac{6}{\sqrt{11}}(\frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}) - < a_3 \cdot e_2 > e_2 \\
&= (1,1,2) - (\frac{6}{11}, \frac{18}{11}, \frac{6}{11}) - < a_3 \cdot e_2 > e_2 \\
&= (1,1,2) - (\frac{6}{11}, \frac{18}{11}, \frac{6}{11}) - (\frac{2}{\sqrt{6}} + \frac{-1}{\sqrt{6}} + \frac{2}{\sqrt{6}})e_2 \\
&= (1,1,2) - (\frac{6}{11}, \frac{18}{11}, \frac{6}{11}) - \frac{3}{\sqrt{6}}(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}) \\
&= (1,1,2) - (\frac{6}{11}, \frac{18}{11}, \frac{6}{11}) - (1, \frac{-1}{2}, \frac{1}{2}) \\
&= (\frac{-6}{11}, \frac{-3}{22}, \frac{21}{22}) \\
e_3 &= \frac{u_3}{\|u_3\|} = (-2\sqrt{\frac{2}{33}}, \frac{-1}{\sqrt{66}}, \frac{7}{\sqrt{66}})
\end{aligned}
$$

Now we may represent A as:

$$
\begin{aligned}
A = QR = [e_1 \mid e_2 \mid e_3] &\begin{bmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 & a_3 \cdot e_1 \\ 0 & a_2 \cdot e_2 & a_3 \cdot e_2 \\ 0 & 0 & a_3 \cdot e_3 \end{bmatrix} \\
= \begin{bmatrix} \frac{1}{\sqrt{11}} & \frac{2}{\sqrt{6}} & -2\sqrt{\frac{2}{33}} \\ \frac{3}{\sqrt{11}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{66}} \\ \frac{1}{\sqrt{11}} & \frac{1}{\sqrt{6}} & \frac{7}{\sqrt{66}} \end{bmatrix} &\begin{bmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 & a_3 \cdot e_1 \\ 0 & a_2 \cdot e_2 & a_3 \cdot e_2 \\ 0 & 0 & a_3 \cdot e_3 \end{bmatrix} \\
= \begin{bmatrix} \frac{1}{\sqrt{11}} & \frac{2}{\sqrt{6}} & -2\sqrt{\frac{2}{33}} \\ \frac{3}{\sqrt{11}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{66}} \\ \frac{1}{\sqrt{11}} & \frac{1}{\sqrt{6}} & \frac{7}{\sqrt{66}} \end{bmatrix} &\begin{bmatrix} \sqrt{11} & 0 & \frac{6}{\sqrt{11}} \\ 0 & \sqrt{6} & 2\sqrt{\frac{2}{3}} - \frac{1}{\sqrt{6}} \\ 0 & 0 & 3\sqrt{\frac{3}{22}} \end{bmatrix}
\end{aligned}
$$

Problem 3

See HW instruction.

We must have $A = QR = AI$, where $Q = A$ and $R = I$. Since every column in A are orthogonal, we have $Q = A$ being the orthogonal matrix; and since I is an upper-triangular matrix, we may have $R = I$.

It is my understanding that this is good to go as the Q in QR decomposition only requires being an orthogonal matrix, regardless if its entries are normalized. But in case the preferred answer wants a normalized Q (as normalization is mentioned in the question), we may have A' being a column-normalized version of A, where each entry of A' is $\frac{a_{ij}}{\|a_i\|}$ in contrast to $a_i j$ in A (where a_{ij} refers to the j-th row entry in i-th column in A); and I' being a diagonal matrix similar to I, but instead of the 1 entries in I, it will have $\|a_1\|, \|a_2\|, ..., \|a_n\|$ respectively along the diagonal line.