MATH 307: Group Homework 6

Group 8
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Problem 1

See HW instruction.

Assume the original vector $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ with a length of r has a degree of θ, we first reflect it about y-axis by swapping the x value with $-x$ and make $A \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -x_0 \\ y_0 \end{bmatrix}$, this means $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

Say the $\begin{bmatrix} -x_0 \\ y_0 \end{bmatrix}$, we call it $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, got a degree of θ, we then rotate it ϕ (in this case $\phi = \frac{\pi}{4}$) degrees more couterclock wisely to have $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$:

$$\begin{aligned}
x_2 &= r\cos(\theta + \phi) = r(\cos\theta\cos\phi - \sin\theta\sin\phi) \\
&= r\cos\theta\cos\phi - r\sin\theta\sin\phi \\
&= x_1\cos\phi - y_1\sin\phi \\
y_2 &= r\sin(\theta + \phi) = r(\sin\theta\cos\phi + \cos\theta\sin\phi) \\
&= r\sin\theta\cos\phi + r\cos\theta\sin\phi \\
&= y_1\cos\phi + x_1\sin\phi
\end{aligned}$$

As we need $B \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, we must have:

$$
\begin{aligned}
x_2 &= x_1 \cos\phi - y_1 \sin\phi \\
y_2 &= y_1 \cos\phi + x_1 \sin\phi \\
\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \\
\text{Since } \phi &= \frac{\pi}{4} \\
\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \cdot (\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}) \\
&= (\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}
\end{aligned}
$$

Thus, $C = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, for $C \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ will conduct the proposed operations on $\begin{bmatrix} x \\ y \end{bmatrix}$.

Problem 2

See HW instruction.

LHS: The ij-th entry of $(A + B)$ is $a_{ij} + b_{ij}$
The ij-th entry of $(A + B)^2 = (A + B)(A + B)$ is $\sum_{k=1}^{n}(a_{ik} + b_{ik})(a_{kj} + b_{kj}) = \sum_{k=1}^{n} a_{ik}a_{kj} + a_{ik}b_{kj} + b_{ik}a_{kj} + b_{ik}b_{kj}$

RHS: The ij-th entry of A^2 is $\sum_{k=1}^{n} a_{ik}a_{kj}$
It follows that the ij-th entry of B^2 is $\sum_{k=1}^{n} b_{ik}b_{kj}$
The ij-th entry of AB is $\sum_{k=1}^{n} a_{ik}b_{kj}$
Therefore the ij-th entry of $A^2 + 2AB + B^2$ is $\sum_{k=1}^{n} a_{ik}a_{kj} + 2a_{ik}b_{kj} + b_{ik}b_{kj}$

Conclusion: The respective ij-th entry of the LHS and the RHS of the equation are not equivalent unless $AB = BA$, which is not guaranteed; therefore $(A + B)^2 = A^2 + 2AB + B^2$ is not true for two square matrices, A and B, of the same size.

Problem 3

See HW instruction.

Known that $M^*_{ij} = \overline{M_{ji}}$, we have:

$$LHS = (AB)^*_{ij} = \overline{(AB)_{ji}} = \sum_k^n \overline{A_{jk}B_{ki}}$$

$$RHS = (B^*A^*)_{ij} = \sum_k^n B^*_{ik}A^*_{kj} = \sum_k^n \overline{B_{ki}A_{jk}}$$

$$\implies (AB)^*_{ij} = (B^*A^*)_{ij}$$

Problem 4

See HW instruction.

Known that $(AB)_{ij} = \sum_{k=1}^n A_{ik}B_{kj}$. In the case that $i > j$, we have either $A_{ik} = 0$ (when $i > k$) or $B_{jk} = 0$ (when $k < j$). So the sum will always be zero for $(AB)_{ij}$ where $i > j$, and thus an upper-triangular matrix.

Problem 5

See HW instruction.

$A\vec{x} = 0$ implies $\sum_i^n \sum_{k=1}^n A_{ik}x_k = 0$, which can be rearranged to $\sum_{k=1}^n x_k v_k = 0$ for v_i being the i-th column of A. If all the v_i of A are linearly independent, we must have $A\vec{x} = 0$ only when $\vec{x} = 0$

W.T.S. A^{-1} exists $\implies A\vec{x} = 0$ only when $\vec{x} = 0$.

$$\begin{aligned} A\vec{x} &= 0 \\ A^{-1}A\vec{x} &= A^{-1}0 \\ I\vec{x} &= 0 \\ \vec{x} &= 0 \end{aligned}$$

Thus, the first direction is therefore proven.

W.T.S. $A\vec{x} = 0$ only when $\vec{x} = 0 \implies A^{-1}$ exists.

Since all columns x_i of A are linearly independent, by definition any $\vec{b}$ within the defined vector space ($\mathbb{R}^n$ or $\mathbb{C}^n$) can be achived by a linear combination of columns of A, which implies there must be a solution for $A\vec{x} = \vec{b}$ for all legal $\vec{b}$.

Now we want to show that $AM = I$, since $I \in \mathbb{R}^n$ (or $I \in \mathbb{C}^n$, W.L.O.G.), there must be a solution M for this equality. Thus, A is invertable given all of its columns are linearly independent.

As both directions are proven, the iff-relationship-in-question is therefore proven.