MATH 307: Group Homework 7

Group 8
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Problem 1

See HW instruction.

W.T.S For all matrices A in $F^{m \times n}$ with all of its row sums being 1, the vector $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in F^{n \times 1}$ can be the eigenvector.

$$Av = \lambda v$$
$$\begin{bmatrix} a_1^* \\ a_2^* \\ \vdots \\ a_n^* \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$
$$\text{Suppose } \lambda = 1 \quad \text{we have}$$
$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Since the $Av = \lambda v$ equality is fulfilled with $\lambda = 1$, this is the corresponding eigenvalue for v.

Problem 2

See HW instruction.

(a)

$$
\begin{aligned}
Au &= \lambda u \\
(\lambda + c)u &= \lambda u + cu \\
(A + cI)u &= Au + cIu = Au + cU = \lambda u + cu \\
(A + cI)u &= (\lambda + c)u
\end{aligned}
$$

(b)

In **Problem 1** we have proved that for a matrix with row sum 1, we may have $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ to be the eigenvector. This conclusion can in fact be expand to matrices with row sum s, as:

$$
\begin{aligned}
Av &= \lambda v \\
A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} &= \begin{bmatrix} s \\ s \\ \vdots \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \\
\lambda &= s \\
v &= \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}
\end{aligned}
$$

Thus, for matrix with row sum $s = 0$ we have the eigenvalue being $\lambda = s = 0$.

Known that $Av = \lambda v = 0v = 0$. For A to be invertable we must have $AB = BA = I$, which implies there should be $BAv = Iv = 0 \Longrightarrow v = 0$. This contradicts $v = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ and therefore A is not invertible.

Problem 3

See HW instruction.

Known that Q is an unitary matrix, we must have $Q^*Q = QQ^* = I$. Now to investigate its eigenvalue λ:

$$
\begin{aligned}
Qv &= \lambda v \\
(Qv)^* &= (\lambda v)* \\
v^* Q^* &= \lambda^* v^* \\
(v^* Q^*) Qv &= (\lambda^* v^*) \lambda v \\
v^* I v &= (\lambda^* \lambda) v^* v \\
\|v\|^2 &= |\lambda|^2 \|v\|^2 \\
1 &= |\lambda|^2 \\
|\lambda| &= 1
\end{aligned}
$$

The equality in question is therefore proven.