MATH 307: Individual Homework 16

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Due and submitted on 04/05/2021
Spring 2021, Dr. Guo

Problem 1

See HW instruction.

First to find eigenvalues:

$$\det(A - \lambda I) = \det \begin{bmatrix} -2-\lambda & -2 \\ -1 & -3-\lambda \end{bmatrix} = 0$$
$$0 = (-2-\lambda)(-3-\lambda) - (-2 \cdot -1)$$
$$= \lambda^2 + 5\lambda + 4 = (\lambda+1)(\lambda+4)$$
$$\Longrightarrow \begin{cases} \lambda_1 = -1 \\ \lambda_2 = -4 \end{cases}$$

Then to find the corresponding eigenvectors:

$$Av_1 = \lambda_1 v_1$$
$$\begin{bmatrix} -2x-2y \\ -1x-3y \end{bmatrix} = \begin{bmatrix} \lambda_1 x \\ \lambda_1 y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$
$$x = -2y$$
$$\Longrightarrow v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$Av_2 = \lambda_2 v_2$$
$$\begin{bmatrix} -2x-2y \\ -1x-3y \end{bmatrix} = \begin{bmatrix} \lambda_2 x \\ \lambda_2 y \end{bmatrix} = \begin{bmatrix} -4x \\ -4y \end{bmatrix}$$
$$2x = 2y$$
$$\Longrightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus, we have eigenvalues being $\lambda_1 = -1, \lambda_2 = -4$ and their corresponding eigenvectors being $\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 2

See HW instruction.

First we want to find out what will be the eigenvalue and its corresponding eigenvector for A^-1

$$\begin{aligned}
Av &= \lambda v \\
A^{-1}Av &= A^{-1}\lambda v \\
v &= A^{-1}\lambda v \\
\lambda^{-1}v &= \lambda^{-1}A^{-1}\lambda v \\
\lambda^{-1}v &= Av
\end{aligned}$$

Known that the eigenvalue for A^{-1} is λ^{-1} and its corresponding eigenvector is still v. Assume that $Bv = \lambda v$ we want to show B^k has eigenvalue λ^k and eigenvector v.

$$\begin{aligned}
B^{k-1}Bv &= B^{k-1}\lambda v \\
&= B^{k-2}\lambda(Bv) = B^{k-2}\lambda(\lambda v) \\
B^k v &= B^{k-2}\lambda^2 v \\
&= B^{k-3}\lambda^2(Bv) = B^{k-3}\lambda^3 v \\
&= B^{k-4}\lambda^3(Bv) = B^{k-4}\lambda^4 v \\
&\dots \\
B^k v &= B^{k-k}\lambda^{k-1}(Bv) = \lambda^k v
\end{aligned}$$

So in this case we have $B = A^-1$ and $k = 3$, so we have $(A^{-1})^3 v = (\lambda^{-1})^3 v$. As $(\lambda^{-1})^3 = \lambda^{-3}$ is the eigenvalue and v is the corresponding eigenvector for $(A^{-1})^3$.

Problem 3

See HW instruction.

In the previous **Problem 2** we have established that $Av = \lambda v \iff A^k v = \lambda^k v$, so we must have $Pv = \lambda v \iff P^2 v = \lambda^2 v$.

Since $P = P^2$, we have $Pv = P^2 v = \lambda^2 v = \lambda v$, which implies $\lambda^2 = \lambda$. Thus, there must be $\lambda = \{0, 1\}$.

Problem 4

See HW instruction.

Assume $A^*Av = \lambda v$ with $v \neq 0$, we have:

$$\begin{aligned} A^*Av &= \lambda v \\ v^*A^*Av &= v^*\lambda v \\ < Av, Av > &= \lambda v^* v \\ \|Av\|_2^2 &= \lambda \|v\|_2^2 \\ \implies \lambda &= \frac{\|Av\|_2^2}{\|v\|_2^2} \end{aligned}$$

Since we know that norm are non-negative and real, we know that $\lambda = \frac{\|Av\|_2^2}{\|v\|_2^2}$ must also be non-negative and real.