MATH 307: Individual Homework 18

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Problem 1

See HW instruction.

The question worded as *"find a basis for both $range(A)$ and $range(A^*)$"*, I am a bit unsure if I should find one basis for $range(A)$ and another basis $range(A^*)$, or should I find a single basis for both of them – it seems like the question is asking the latter one, but I don't think that is possible given that $A \in F^{m \times n}$; and if $m \neq n$, the dimension of row and column space is intrinctly different.

Let $\sigma_1, \sigma_2, \ldots, \sigma_r$ be the nonzero singular values in Σ of $A = U\Sigma V^*$, we know that $\{u_1, u_2, ..., u_r\}$ is a basis for $range(A)$ because of the following.

W.T.S. $range(A) \subset span(u_1, u_2, ..., u_r)$

$\forall y \in range(A)$, $y = Ax$ for some $x \in F^n$. Since we know that V is unitary, columns of V form a basis for F^n. Thus, we have $x = \sum \alpha_i v_i$; also known that $AV = U\Sigma$, we have:

$$\begin{aligned} y = Ax &= A(\sum_{i=1}^{r} \alpha_i v_i) \\ &= \sum_{i=1}^{r} \alpha_i A v_i \\ &= \sum_{i=1}^{r} \alpha_i \sigma_i u_i \in span(u_1, u_2, ..., u_r) \end{aligned}$$

This implies $range(A) \subset span(u_1, u_2, \ldots, u_r)$

W.T.S. $span(u_1, u_2, \ldots, u_r) \subset range(A)$

$$\begin{aligned} &AV = U\Sigma \\ &Av_i = \sigma_i u_i \text{ for } i = 1, \ldots, r \\ &u_i = A(\frac{v_i}{\sigma_i}) \\ &\Longrightarrow u_i \in range(A) \end{aligned}$$

This implies $span(u_1, u_2, \ldots, u_r) \subset range(A)$. Combine both findings, we have $range(A) = span(u_1, u_2, \ldots, u_r)$. As $u_1, u_2, \ldots, u_r$ are linearly independent, it is a basis for $range(A)$ the column space of A.

Similarily, we have $A^* = V\Sigma U^*$ for V being unitary (implies orthogonal and linearly independent columns). So, we have $\{v_1, v_2, \ldots, v_r\}$ to be basis for $range(A^*)$ the row space of A.

Since both $\{u_1, u_2, \ldots, u_r\}$ and $\{v_1, v_2, \ldots, v_r\}$ have a dimension of r. The column rank is same as the row rank of A.

Problem 2

See HW instruction.

(a)

As we have previously established in the above **Problem 1**. The row rank of A is the number of nonzero singular values in Σ of $A = U\Sigma V^*$. In this case, we have 5 nonzero singular values, so the row rank of A is 5. The orthonormal basis of $range(A^*)$ will therefore be $\{v_1, v_2, \ldots, v_5\}$.

For proving. We know that $\{v_1, v_2, \ldots, v_5\}$ is a orthonormal set as it came out from an unitary matrix V. We also know there must be $span(v_1, v_2, \ldots, v_5) = range(A^*)$ as we have previously showned in **Problem 1**. Thus, $\{v_1, v_2, \ldots, v_5\}$ is a basis of $range(A^*)$.

(b)

The rank-nullity theorem tells us that $n = r + l$ for r stands for rank and l stands for nullity. In this case $n = 6$ as $A^* \in F^{8\times 6}$, we have the nullity being $l = n - r = 6 - 5 = 1$.

As $A^* = V\Sigma U^*$, we have $\{u_{r+1}, \ldots, u_n\} = \{u_6\}$ to be the basis of $null(A^*)$. As we have $null(A^*) \subset span(u_6)$ because they are both $\in F^m$. We also have $span(u_6) \subset null(A^*)$ as:

$$\begin{aligned}
&A^*U = V\Sigma \\
&A^*u_i = \sigma_i v_i \text{ for } i = r+1, \ldots, n \\
&A^*u_i = 0 \\
&\implies span(u_{r+1}, \ldots, u_n) \subset null(A^*)
\end{aligned}$$

Which is equivalent to $span(u_6) \subset null(A^*)$. Combine the two findings we have $span(u_6) = null(A^*)$. As $\{u_6\}$ is obviously linearly independent, $\{u_6\}$ is the basis of $null(A^*)$.