MATH 307: Individual Homework 12

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Problem 1

See HW instruction.

For $A = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$, we have:

$$B = A^{-1} = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}^{-1} = \frac{1}{-2i} \begin{bmatrix} -i & -i \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2}i & \frac{1}{2}i \end{bmatrix}$$

Now to confirm that $AB = BA = I$:

$$\begin{aligned} AB &= \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2}i & \frac{1}{2}i \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{2} - \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} & \frac{1}{2} + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \\ BA &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2}i & \frac{1}{2}i \end{bmatrix} \cdot \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{2}i - \frac{1}{2}i \\ -\frac{1}{2}i + \frac{1}{2}i & \frac{1}{2} + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \end{aligned}$$

Problem 2

See HW instruction.

(a)

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we have:

$$
\begin{aligned}
Ax &= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b = \begin{bmatrix} 0 \\ 2\sqrt{2} \end{bmatrix} \\
&= \begin{cases} \frac{\sqrt{2}}{2}x_1 - \frac{\sqrt{2}}{2}x_2 = 0 \\ \frac{\sqrt{2}}{2}x_1 + \frac{\sqrt{2}}{2}x_2 = 2\sqrt{2} \end{cases} \quad = \begin{cases} x_1 = 2 \\ x_2 = 2 \end{cases} \\
\implies x &= \begin{bmatrix} 2 \\ 2 \end{bmatrix}
\end{aligned}
$$

(b)

If A is to rotate x for 45° to b, reversely, we may counter rotate b −45° to get x. As we have shown in *Group HW5* and *Group HW6*, to rotate a vector for ϕ degree, we need to left multiply $\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ to the target vector.

In this case, we have ϕ to be $-\frac{\pi}{4}$, thus the rotation matrix should be $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$. We call this matrix C and now we try to left multiply it with b to get x:

$$
\begin{aligned}
Cb &= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2\sqrt{2} \end{bmatrix} = x \\
x &= \begin{bmatrix} 2 \\ 2 \end{bmatrix}
\end{aligned}
$$

(c)

We know A is an orthogonal matrix as:

$$
AA^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{2} - \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} & \frac{1}{2} + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I
$$

Known this, we also know that $A^{-1} = A^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$. Now we may left multiply A^{-1} to the both sides of $Ax = b$:

$$
\begin{aligned}
A^{-1}Ax &= A^{-1}b \\
Ix &= A^{-1}b \\
x &= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}
\end{aligned}
$$