MATH 307: Group Homework 8

Group 8
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Problem 1

See HW instruction.

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 0 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} \\
&= -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = -(\lambda - 2)^2(\lambda - 3) = 0 \\
&\implies \begin{cases} \lambda_1 = 2 \\ \lambda_2 = 3 \end{cases}
\end{aligned}$$

$$\begin{aligned}
0 &= (A - \lambda_1 I)v_{\lambda_1} \\
&= \begin{bmatrix} 3-\lambda_1 & 0 & 1 \\ 0 & 2-\lambda_1 & 1 \\ 0 & 0 & 2-\lambda_1 \end{bmatrix} v \\
&= \begin{bmatrix} 3-2 & 0 & 1 \\ 0 & 2-2 & 1 \\ 0 & 0 & 2-2 \end{bmatrix} v \Rightarrow \begin{cases} v_1 + v_3 = 0 \\ 0v_2 + v_3 = 0 \\ 0v_3 = 0 \end{cases} \\
\implies v_{\lambda_1} &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}$$

Known that $\lambda_1 = 2$ has an algebraic multiplicity of 2 (appered twice as the root); but the eigenspace of λ_1 is spanned by $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, which means it only has a geometric multiplicity of 1. Thus, the matrix is defective.

Problem 2

See HW instruction.

Known that $A = A^*$, let $Av = \lambda v$, $Aw = \mu w$ and we must have:

$$\begin{aligned}
Aw &= \mu w \\
v^* A w &= v^* \mu w = \mu(v^* w) \\
Aw &= A^* w \\
v^* A^* w &= (Av)^* w = (\lambda v)^* w = \lambda(v^* w) \\
\implies \mu(v^* w) &= \lambda(v^* w)
\end{aligned}$$

Since $\lambda, \mu \neq 0$ and $\lambda \neq \mu$ for being distinct eigenvalues, there must be $v^* w = 0 =< w, v >$. Thus, the eigenvectors-in-question are orthogonal.

Problem 3

See HW instruction.

$A \in F^{n \times n} = U \Sigma V^*$ where U, V are unitary and Σ is diagonal, we must have:

$$\begin{aligned}
A^* A &= (U \Sigma V^*)^* (U \Sigma V^*) \\
&= (V \Sigma^* U^*)(U \Sigma V^*) \\
&= V \Sigma^* U^* U \Sigma V^* \\
&= V \Sigma^* I \Sigma V^* \\
&= V \Sigma^* \Sigma V^* \\
&= V (\Sigma^* \Sigma) V^*
\end{aligned}$$

Since V is a unitary matrix, it is also invertible, and $V^* = V^{-1}$. Therefore:

$$V(\Sigma^* \Sigma) V^* = V(\Sigma^* \Sigma) V^{-1}$$

Furthermore, the resultant matrix of $\Sigma^* \Sigma$ will also be a diagonal matrix $= \text{diag}(|\sigma_1|^2, |\sigma_2|^2, \ldots, |\sigma_n|^2)$.

Since V is invertible, $V^* = V^{-1}$, and $\Sigma^* \Sigma$ is a diagonal matrix, $V(\Sigma^* \Sigma) V^*$ is an eigendecomposition of $A^* A$.

Problem 4

See HW instruction.

$$
\begin{aligned}
Ax &= b \\
U\Sigma V^T x &= b \\
(V\Sigma^{-1}U^T)U\Sigma V^T x &= (V\Sigma^{-1}U^T)b \\
V\Sigma^{-1}(U^T U)\Sigma V^T x &= V\Sigma^{-1}U^T b \\
V(\Sigma^{-1}\Sigma)V^T x &= V\Sigma^{-1}U^T b \\
VV^T x &= V\Sigma^{-1}U^T b \\
x &= V\Sigma^{-1}U^T b
\end{aligned}
$$

Subsitute the known values into above equation, we have:

$$
\begin{aligned}
x &= V\Sigma^{-1}U^T b \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^T \begin{bmatrix} -\frac{3}{2}\sqrt{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} -\frac{3}{2}\sqrt{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}
$$

Thus, $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.