# MATH 307: Individual Homework 23

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## Problem 1

*See HW instruction.*

First we need to find $\det(A) = (-2) * (2) - 1(-1) = -3$, then we apply the Cramer's rule:

$$x_1 = \frac{\det(\begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix})}{\det(A)} = \frac{6-3}{-3} = -1$$
$$x_2 = \frac{\det(\begin{bmatrix} -2 & 3 \\ -1 & 3 \end{bmatrix})}{\det(A)} = \frac{-6-(-3)}{-3} = 1$$

So we have $x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$; and we may verify that $Ax = \begin{bmatrix} 2+1 \\ 1+2 \end{bmatrix} = b$.

## Problem 2

*See HW instruction.*

**(a) W.T.S.** $Ax = 0$ has nontrivial solutions $\Longrightarrow 0$ is an eigenvalue of $A$.
Assume $Ax = 0 = \lambda x$ with $x \neq 0$, we must have $\lambda = 0$.

**(b) W.T.S.** $0$ is an eigenvalue of $A \Longrightarrow \det(A) = 0$.
In **Individual Homework 22, Problem 2** we have already established that determinant of $A$ is equals to the product of its eigenvalues. Given that 0 is an eigenvalue of $A$, we must have $\det(A) = 0 \cdot \lambda_1 \ldots \lambda_n = 0$.

**(c) W.T.S.** $\det(A) = 0 \Longrightarrow Ax = 0$ has nontrivial solutions.
Known that $\det(A)$ is the product of diagonal entries of $rref(A)$, for $\det(A) = 0$ we know that $A$ does not have full rank. So there must be $null(A) \neq 0$ according to the rank-nullity theorem, which implies $Ax = 0$ has a nontrivial solution.

Since we have showed $(a) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a)$, we may say that (a), (b), and (c) are all equivalent.

## Problem 3

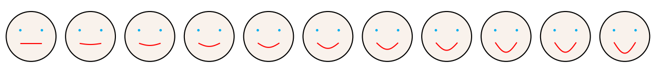
*See HW instruction.*

In **Individual Homework 17, Problem 2** we have established that for $A^*A$ is invertible for $A$ with positive singular values. As I didn't solve that problem perfectly, I'd like to go through it again here:

$$
\begin{aligned}
A &= (U\Sigma V^*) \\
A^*A &= (U\Sigma V^*)^*(U\Sigma V^*) \\
&= V\Sigma^* U^* U\Sigma V^* \\
&= V\Sigma^*\Sigma V^*
\end{aligned}
$$

Known that $V$ and $V^*$ are both invertible for being orthogonal, we only interested in if $\Sigma^*\Sigma$ is invertible. Also known that $A$ is invertible, so the diagonal entries of $\Sigma$ are nonzero, which implies $\Sigma^*\Sigma$ is simply a square matrix with diagonal entries being the squares of diagonal entries of $\Sigma$ and therefore also invertible. Thus, $A^*A$ is invertible.

Then we may simply have $A^*Ax = A^*b \Longrightarrow x = (A^*A)^{-1}A^*b$. As $x$ only depends on $A$ and $b$, the solution will be unique.

ALL DONE  THANK YOU !!!!!!!!!!!!!!!!!!!