Voronoi Diagrams and Delaunay Triangulations

Anne Driemel and Herman Haverkort updated: November 5, 2024

In this lecture we will define Voronoi diagrams and Delaunay triangulations, two concepts that are independent from one another, while having surprisingly strong relationships.

1 Voronoi diagrams

We first give the general definition of Voronoi diagrams in $\mathbb{R}^d$. For most of this lecture we will only be concerned with Voronoi diagrams in the plane. We will get back to higher-dimensional Voronoi diagrams later in the course. Consider a finite set $P \subset \mathbb{R}^d$. For each $p \in P$ we define a region $\mathrm{reg}(p)$ which consists of the points $x \in \mathbb{R}^d$ for which p is the closest among points of P. Formally,

$$\mathrm{reg}(p) = \left\{ x \in \mathbb{R}^d \;\middle|\; \|x-p\| \leq \|x-q\| \text{ for all } q \in P \right\}$$

The Voronoi diagram of P is the set of all regions $\mathrm{reg}(p)$ for $p \in P$.

For any two distinct $p, q \in \mathbb{R}^d$, we call the set

$$b(p,q) = \left\{ x \in \mathbb{R}^d \;\middle|\; \|x-p\| = \|x-q\| \right\}$$

the bisector of p and q. This is the set of points that have the same distance to both p and q. The bisectors are hyperplanes and they can be used to characterize the structure of the Voronoi diagram.

Observation 8.1. *For $p \in P$, the region $\mathrm{reg}(p)$ is the intersection of $|P|-1$ halfspaces in $\mathbb{R}^d$.*

Proof. The bisector $b(p,q)$ is a hyperplane, since the condition for a point x to be in the set $b(p,q)$ can be rewritten as follows

$$\begin{array}{rcl}
& \|x-p\| & = \|x-q\| \\
\Leftrightarrow & \langle x-p, x-p\rangle & = \langle x-q, x-q\rangle \\
\Leftrightarrow & \langle x,x\rangle + \langle p,p\rangle - 2\langle p,x\rangle & = \langle x,x\rangle + \langle q,q\rangle - 2\langle q,x\rangle \\
\Leftrightarrow & 2\langle q,x\rangle - 2\langle p,x\rangle & = \langle q,q\rangle - \langle p,p\rangle \\
\Leftrightarrow & \langle 2(q-p), x\rangle & = \langle q,q\rangle - \langle p,p\rangle
\end{array}$$

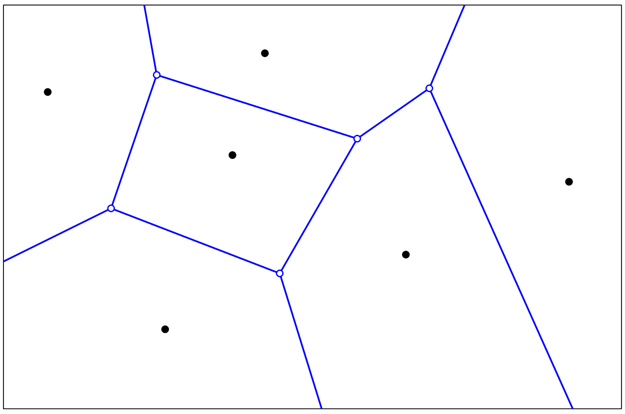
Let $a = 2(q-p)$ and $b = \langle q,q\rangle - \langle p,p\rangle$. It follows that the condition for a point x to be in the bisector set $b(p,q)$ is equivalent to $\langle a,x\rangle = b$ for some $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$.

Now, consider the Voronoi region of a point $p \in P$. By definition,

$$\begin{aligned}
\mathrm{reg}(p) &= \bigcap_{q \in P\setminus\{p\}} \left\{ x \in \mathbb{R}^d \;\middle|\; \|x-p\| \leq \|x-q\| \right\} \\
&= \bigcap_{q \in P\setminus\{p\}} \left\{ x \in \mathbb{R}^d \;\middle|\; \langle 2(q-p), x\rangle \leq \langle q,q\rangle - \langle p,p\rangle \right\}
\end{aligned}$$

This is an intersection of $|P|-1$ halfspaces, each halfspace bounded by a bisector between p and a point $q \in P$. $\square$

Consider a finite set $P \subset \mathbb{R}^2$ in general position. In particular, we assume that no three points lie on the same line and no four points lie on the same circle. The Voronoi diagram of P is a subdivision of the plane into n convex polygons (some of them can be unbounded). We can think of it as a drawing of a planar graph with straight edges. In particular, an edge (which we call a Voronoi edge) of the graph is formed by the intersection of two neighboring Voronoi regions, and a vertex (which we call Voronoi vertex) of the graph is formed by the common intersection of three pairwise neighboring Voronoi regions. In this context we call the points of P Voronoi sites. If no more than three points lie on a common circle, then the maximum degree of a vertex of the Voronoi diagram is 3.



2 Delaunay triangulations

Let $P \subset \mathbb{R}^2$ be a finite set of points in general position. We use the same assumption as in the previous section (no three points lie on the same line and no four points lie on the same circle).

Definition 8.2 (Triangulation)**.** *A triangulation of P is a graph G that has P as its vertex set, and the set E is maximal in the following sense. When each edge is drawn as a straight line segment, then no two edges of E cross each other and any edge $e \in (P \times P) \setminus E$ would cross at least one edge of E. We can observe that such a graph induces a subdivision of the plane where all the faces are triangles (except for the infinite face). This is why we call the graph a triangulation.*

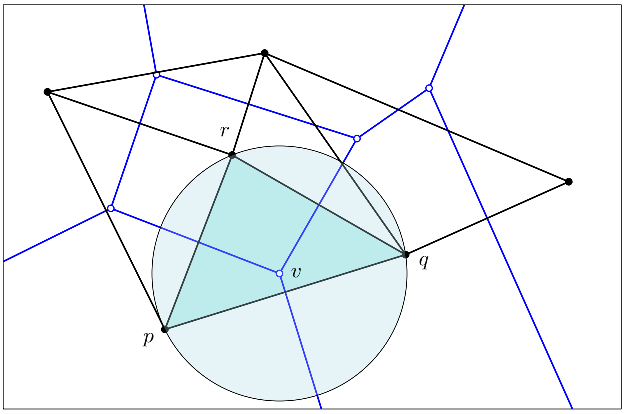
Definition 8.3 (Delaunay edge)**.** *Let $p, q \in P$. If there exists a circle with p and q on its boundary that contains no point of P in its interior, then the edge (p, q) has the Delauney property with respect to P. A triangulation where all edges are Delaunay edges, is a Delaunay triangulation.*

Definition 8.4 (Delaunay triangle)**.** *Let $p, q, r \in P$. If the circle through p, q, and r does not contain any point of P in its interior, then the triangle (p, q, r) has the Delaunay property with respect to P.*

Lemma 8.5. *Delaunay edges do not cross.*

Proof. Assume for the sake of contradiction that there are two edges (a, b) and (p, q) that have the Delaunay property. By the Delaunay property, there exists a circle C_1 through a and b that does not contain either p or q in its interior, and there exists a circle C_2 through p and q that does not contain either a or b in its interior. However, since the edges cross, there is a point contained in both edges that is contained in the interior of both circles. Since the interior of a circle is a connected set, this would imply that the two circles intersect in at least four points, which is not possible. □

Any finite set of points $P \subseteq \mathbb{R}^2$ has a Delaunay triangulation. To see this, we can construct the graph G of such a triangulation directly from the Voronoi diagram of P. For every Voronoi edge that lies on the boundary of two Voronoi regions $\text{reg}(p)$ and $\text{reg}(q)$, we create an edge between p and q in G. This way, we obtain a graph that has P as its vertex set, and an edge between two points $p, q \in P$ is present if and only if the Voronoi regions $\text{reg}(p)$ and $\text{reg}(q)$ are neighboring regions in the Voronoi diagram (they share a part of their boundary). Every vertex of G is associated with a unique Voronoi region and every face of this graph (except for the infinite face) corresponds to a unique Voronoi vertex. For every Voronoi vertex that lies at the intersection of three Voronoi regions $\text{reg}(p), \text{reg}(q)$ and $\text{reg}(r)$, the graph G contains a triangle (a cycle of three edges) in G with vertices p,q and r. By construction, the circle through p,q, and r does not contain any point of P in its interior, since the Voronoi vertex v has p, q and r as its closest points among all points of P. Therefore, every face of the graph (except for the infinite face) corresponds to a Delaunay triangle. This also implies, that every edge of G has the Delaunay property with respect to P and by Lemma 8.5 no two such edges cross.



Lemma 8.6. *Let G be the graph of a Delaunay triangulation of P.*

(i) If and only if an edge (p, q) for $p, q \in P$ has the Delaunay property with respect to P, then it is present as an edge in G.

(ii) If and only if a triangle (p, q, r) for $p, q, r \in P$ has the Delaunay property with respect to P, then it is present as a face in G.

Proof. Since G is a Delaunay triangulation, all edges of G have the Delaunay property. We argue that any edge which has the Delaunay property must be present in G. Assume for the sake of contradiction that (p, q) is a Delaunay edge of P, but it is not present in G. If this edge is not in G, then there must be some other edge (s, t) in G that crosses (p, q) (otherwise G cannot be maximal, since we could add (p, q) without introducing a crossing). By Lemma 8.5 the other edge (s, t) cannot be a Delaunay edge, since Delaunay edges do not cross. Therefore, G is not a Delaunay triangulation. A contradiction. This proves (i).

It remains to prove (ii). Let (p, q, r) be a triangle that has the Delaunay property with respect to P. The above implies that all its edges are present in G. By its Delaunay property, the triangle does not contain any point of P in its interior. Therefore, the triangle is also present as a face of the triangulation. On the other hand, if the triangle does not have the Delaunay property with respect to P, then one can show that either it contains a point of P in its interior, or one of its edges does not have the Delaunay property, or both. In either case, it cannot be present as a face in a Delaunay triangulation of P, by (i). □

3 Algorithm

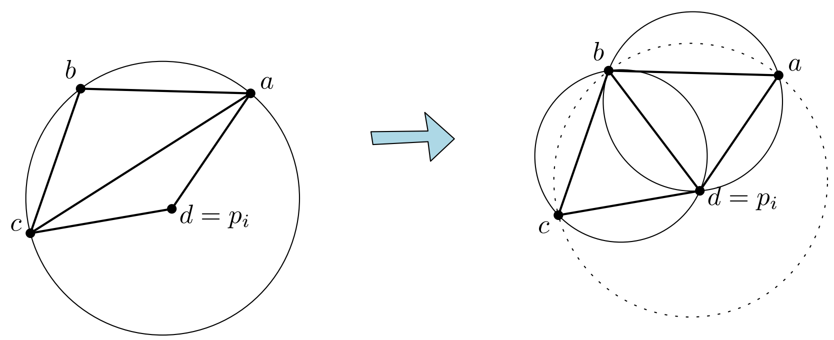
Consider the following incremental algorithm for computing the Delaunay triangulation. The algorithm starts by computing a sufficiently large triangle that contains the input points. The bounding triangle should be such that none of its vertices invalidates the Delaunay property of any Delaunay triangles. Then, in every step, the algorithm inserts a new point by creating three new triangles in the triangle of the current triangulation that contains the point. The Delaunay property might now be violated. Therefore, the algorithm recursively "flips" individual edges of the triangulation until the Delaunay property is "repaired".

Algorithm 8.1

procedure DELAUNAY-TRIANGULATION(Set of points P in $\mathbb{R}^2$)
    Let $p_1, \ldots, p_n$ be the points of P
    Compute a suitable bounding triangle t containing P
    Initialize a triangulation T as a DCEL with t
    for $i \leftarrow 1$ **to** n **do**
        Find the triangle (a, b, c) in T that contains p_i
        Insert edges (p_i, a), (p_i, b), and (p_i, c) into T
        LEGALIZEEDGE((a, b), T)
        LEGALIZEEDGE((b, c), T)
        LEGALIZEEDGE((c, a), T)
    end for
    Remove the vertices of t and all incident edges from T
    return T
end procedure

Algorithm 8.2

procedure LEGALIZE-EDGE(Edge e, triangulation T) // *p_i lies to the left of e*
    $a \leftarrow Start(e)$
    $b \leftarrow End(Next(Twin(e)))$ // *vertex of the triangle to the right of e*
    $c \leftarrow End(e)$
    $d \leftarrow End(Next(e))$ // *this should be p_i*
    if e is not a Delaunay edge with respect to $\{a, b, c, d\}$ **then**
        Replace edge (a, c) with the edge (d, b) in T
        LEGALIZEEDGE((a, b), T)
        LEGALIZEEDGE((b, c), T)
    end if
end procedure



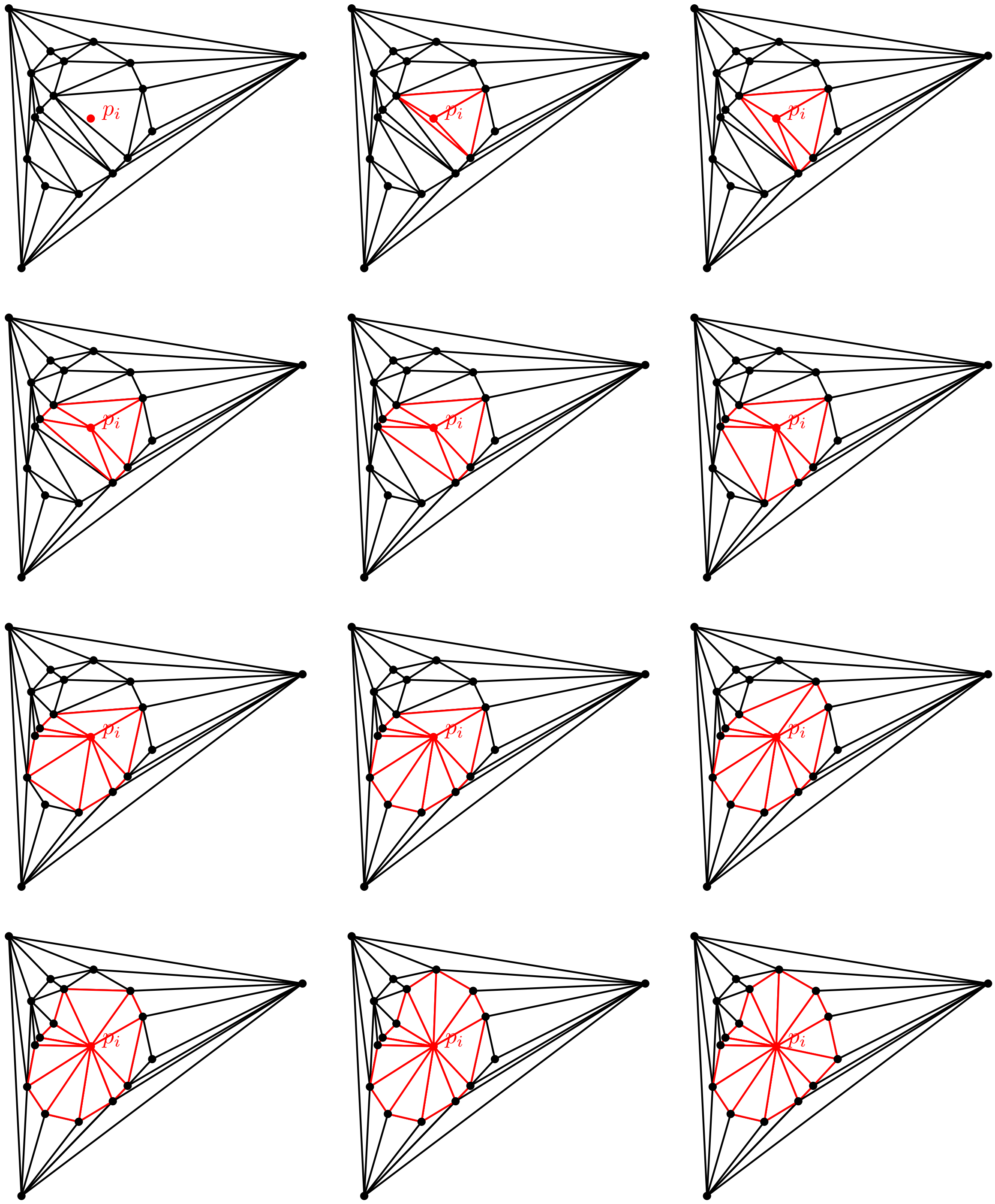


Figure 1: Example of sequence of edge flips done during the recursive calls to LegalizeEdge when inserting one point.

References

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