In the previous lectures we adopted a somewhat passive perspective: We considered a given game, and analyzed strategic outcomes of this game. So in some sense we assumed the rules of the game to be fixed. What if we could change the rules of the game in order to achieve some objective in strategic equilibrium?

This is the grand question of a field called mechanism design, which we will explore next. As a warm-up we will consider single-item auctions. We will identify an auction mechanism with great properties; these properties will henceforth serve as a gold standard against which we will evaluate our solutions.

1 First-Price Auction

We first consider the (sealed-bid) first-price auction, which is an easy and natural way to sell an item among n players.

(Sealed-Bid) First-Price Auction

1. Each bidders writes their bid b_i on one side of a piece of paper, folds it, writes their name on it, and hands it to the auctioneer.

2. Once the auctioneer has received all bids, the winner is determined as the the bidder with the highest bid and has to pay *what he/she has bid*.

One also has to establish a tie-breaking rule, for example, breaking ties lexicographically: That is, if they bid the same, Anna would be favored over Paul and so on.

In order to reason about what to do in an auction, we need a model of bidder behavior. We will assume that there is a set of n players N and a single item for sale. Each player $i \in N$ has a willingness-to-pay (or *value*) $v_i \in \mathbb{R}_{\geq 0}$. We assume players seek to maximize their utility. If for a given bid profile b, player $i \in N$ wins he/she has a *utility* of $u_i(b, v_i) = v_i - b_i$ and he/she has a utility of $u_i(b, v_i) = 0$ if he/she loses.

Bidding in a first-price auction is not easy. Of course, a player could just bid what the item is worth to them But then, no matter what the other players were to bid, a player would never make a bargain or positive utility in the terminology that we just defined.

How would you bid if your goal was to maximize your utility? Wouldn't you shade your bid in order to achieve a lower price? But by how much should you shade your bid? The problem is that this depends on what you know about the bids of the others!

In the simplest model, the *complete information model*, one assumes that the players know each other's values.

Observation 10.1. *In the first-price auction, the state in which all players bid their true value is generally not a Nash equilibrium.*

Observation 10.2. *Consider a first-price auction in which ties are broken in favor of a player of maximum value. Then there is the following pure Nash equilibrium b. Let i be this player of highest value and set $b_i = \max_{j \neq i} v_j$ and $b_j = v_j$ for $j \neq i$.*

2 Second-Price Auction

The (sealed-bid) second-price auction is very similar, except for what the winner has to pay:

(Sealed-Bid) Second-Price Auction

1. Each bidders writes their bid b_i on one side of a piece of paper, folds it, writes their name on it, and hands it to the auctioneer.

2. Once the auctioneer has received all bids, the winner is determined as the the bidder with the highest bid and has to pay *the next highest bid*.

To model bidding behavior here, the only difference is the utility of the auction winner. It will now be $u_i(b, v_i) = v_i - \max_{j \neq i} b_j$.

It turns out that in the second-price auction bidding is much easier. A bit of thinking reveals that bidding your true value is not only a Nash equilibrium. It is, in fact, the best you can do, independent of what other players bid.

Definition 10.3. *A bid profile b is a* dominant strategy equilibrium *for value profile v if for each player $i \in N$ bid b_i is a (weakly) dominant strategy. A bid b_i is a* (weakly) dominant strategy *for player i with value v_i if for all possible bids b_i' by that player and all possible bids b_{-i} of the other players, $u_i((b_i, b_{-i}), v_i) \geq u_i((b_i', b_{-i}), v_i)$.*

Theorem 10.4 (Vickrey, 1961)**.** *In a second-price auction, for each player $i \in N$ it is a dominant strategy to bid truthfully.*

Proof. Fix a player i, his/her value v_i, and the bids b_{-i} of the other players. We need to show that player i's utility is maximized by setting $b_i = v_i$.

Let $b_{max} = \max_{j \neq i} b_j$ denote the highest bid by a player other than i. Note that even though there is an infinite number of bids that i could make, only two distinct outcomes can result. For this we can without loss of generality assume that player i loses if he/she bids $b_i = b_{max}$. In this case if $b_i \leq b_{max}$, then i loses and receives utility 0. If $b_i > b_{max}$, then i wins at price b_{max} and receives utility $v_i - b_{max}$.

We now consider two cases. First, if $v_i \leq b_{max}$, the highest utility that bidder i can get is $\max\{0, v_i - b_{max}\} = 0$, and he/she achieves this by bidding truthfully (and losing). Second, if $v_i > b_{max}$, the highest utility that bidder i can get is $\max\{0, v_i - b_{max}\} = v_i - b_{max}$, and he/she achieves this by bidding truthfully (and winning). $\square$

Observation 10.5. *In a second-price auction, if each player bids his/her true value, then he/she never has negative utility.*

3 Mechanism Design with Money

The single-item auction is a mechanism-design problem with money. More generally, there is a set N of n players (or agents) that interacts with a mechanism to select a feasible outcome. The set of outcomes is denoted by X.

Each agent i has a private *valuation* function $v_i \colon X \to \mathbb{R}$ and has a set B_i of possible *bids*. We let $B = B_1 \times \ldots \times B_n$ denote the set of all bid profiles. A mechanism $\mathcal{M} = (f, p)$ consists of an outcome rule $f \colon B \to X$ and a payment rule $p \colon B \to \mathbb{R}^n$. It asks the players to report their bids, which we will denote by b, and computes from it an outcome $f(b)$ and a vector of payments $(p_1(b), \ldots, p_n(b))$ that the agents have to make to the mechanism.

We make the standard assumption of quasi-linear utilities. That is, player i's utility if her valuation is v_i and the bids are b is $u_i^{\mathcal{M}}(b, v_i) = v_i(f(b)) - p_i(b)$. When it is clear from the context, which mechanism we are referring to we will drop the superscript $\mathcal{M}$. Sometimes we also drop the valuation function v_i.

Example 10.6 (Single-Item Auction)**.** *In a single-item auction n bidders compete for the assignment of an item. Each player can get the item or not, so feasible assignments are vectors $x \in X \subseteq \prod_i X_i = \{0,1\}^n$ with $\sum_i x_i = 1$, where we interpret $x_i = 1$ as bidder i gets the item. We overload notation and write v_i for both the function $X \to \mathbb{R}$ as well as the value. This way, $v_i(x) = v_i \cdot x_i$. Our goal is to allocate the item to the bidder with the highest value.*

Note that in our definition so far, the bids can be arbitrary. In many cases, it makes sense that the bids exactly correspond to possible valuations. So, letting V_i denote the set of all possible valuation functions for player i, we would set $B_i = V_i$. Such a mechanism is called a *direct mechanism*.

Example 10.7. *The first price auction is a direct mechanism, in which $V_i = B_i = \mathbb{R}$. Given a bid vector b, the allocation function f assigns the item to a bidder with the highest bid. That is, it returns an allocation x that maximizes $\sum_i b_i(x)$. The payment function is simply $p_i(b) = b_i$ if $(f(b))_i = 1$ and $p_i(b) = 0$ otherwise, or equivalently $p_i(b) = b_i(f(b))$, again interpreting b_i as a function $X \to \mathbb{R}$.*

The basic dilemma of mechanism design is that the mechanism designer (think of a government or company) wants to optimize some global objective such as the social welfare $\sum_{i \in \mathcal{N}} v_i(x)$ by computing an allocation x based on the bids b, while the players choose their bids b_i so as to maximize their utilities $u_i(b, v_i)$.

Example 10.8 (Combinatorial Auction)**.** *There are again n bidders but now there is a set M of m item. Each bidder i has a private valuation function $v_i \colon 2^M \to \mathbb{R}_{\geq 0}$, defining a non-negative value of each subset of items. The set of feasible allocations is given as $X = \{(S_1, \ldots, S_n) \mid S_i \subseteq M, S_i \cap S_{i'} = \emptyset \text{ for } i \neq i'\}$.*

We will consider direct mechanisms, in which each bidder reports her v_i, as well as indirect mechanisms, in which, for example, all items are sold separately.

Example 10.9 (Sponsored Search Auction)**.** *In a sponsored search auction we have n bidders and k positions. Each position has an associated click-through rate α_j, where we assume that positions are sorted such that $\alpha_1 > \alpha_2 > \cdots > \alpha_k > 0$. Feasible allocations are $x \in X$ for which $x_i \in \{0, \alpha_k, \ldots, \alpha_1\}$ for all i and for $i \neq j$ we can only have $x_i = x_j$ if $x_i = x_j = 0$. The valuation of player i is given as $v_i(x) = v_i \cdot x_i$ for some $v_i \geq 0$.*

Example 10.10 (Scheduling on Related Machines)**.** *There are n machines, each of which is a player, and each player has a private speed s_i. The inverse of the speed $t_i = 1/s_i$ is the time that machine i takes to process a job of unit length. There are m jobs with loads $\ell_1, \ldots, \ell_m$, which need to be allocated to the machines. An allocation induces a work load $W_1, \ldots, W_n$ for each machine. Player i's cost for processing work load W_i is $W_i \cdot t_i$, so the valuation for load vector W is $v_i(W) = -W_i \cdot t_i$. Each machine is interested in maximizing $u_i(b) = v_i(W(b)) - p_i(b)$, while the mechanism designer wants to minimize the makespan $\max_i W_i \cdot t_i$. Note that now payments will usually be negative because the mechanism compensates the players for the cost they incur.*

4 Desirable Properties of Auctions

We have seen a remarkable property of the second-price auction, or Vickrey auction: To maximize the utility, it is enough to bid your true value. We can generally state this property for direct mechanisms as follows.

Definition 10.11. *A direct mechanism $\mathcal{M} = (f, p)$ is called* dominant-strategy incentive compatible *(DSIC) (or just* truthful*), if for each player i bidding $b_i = v_i$ is a weakly dominant strategy. That is, for all $i \in \mathcal{N}$, $v_i \in V_i$, and all $b \in V$ it holds that*

$$u_i^{\mathcal{M}}((v_i, b_{-i}), v_i) \geq u_i^{\mathcal{M}}((b_i, b_{-i}), v_i) \ .$$

However, truthfulness is not the only requirement that we have in a mechanism. For example, not allocating the item at all has the same property. If all bidders bid truthfully, then it also allocates the item to one of the bidders who has the maximum value. Furthermore, it is very easy to compute the allocation and the payments.

So, overall, we have:

1. Strong incentive guarantees. The Vickrey auction is *dominant strategy incentive compatible* (DSIC), i.e., truthtelling is a dominant strategy equilibrium.

2. Strong performance guarantees. At equilibrium, the Vickrey auction maximizes *social welfare* $\sum_{i \in N} v_i(x)$.

3. Computational efficiency. The Vickrey auction can be computed in *polynomial time*.

We will see that when trying to generalize this result to more complex settings, we often find that obtaining all three properties at once is impossible and we will have to relax one or more of these goals.

Acknowledgments

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References and Further Reading

- Chapter 9 in the AGT book. (Introduction to the topic)

- Tim Roughgarden's lecture notes `http://timroughgarden.org/f13/l/l2.pdf`

- William Vickrey. Counterspeculation, auctions, and competitive sealed tenders. Journal of Finance, 16(1):8–37, 1961.