Algorithmic Game Theory, Summer 2024	Lecture 14 (4 pages)

General Combinatorial Auctions in Polynomial Time

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Last time, we got to know the famous VCG mechanism. It almost gives us everything we ask for: It is reasonably easy to understand, dominant-strategy incentive compatible (truthful), and in the dominant-strategy equilibrium the social welfare is maximized. Unfortunately, there is one drawback, which is that it requires us to maximize the declared welfare exactly. Approximate solutions are not enough.

There are many ways to still design truthful mechanisms outside the VCG framework—we have seen examples for single-parameter settings, in which we only had to come up with monotone algorithms. Beyond single-parameter settings, things become more difficult. However, there are still many ideas to build truthful mechanisms. We will get to know one such approach for combinatorial auctions today.

1 Setting

Recall that in a *combinatorial auction*, we have a set N of n bidders and a set M of m items. Each bidder i has a private valuation function $v_i\colon 2^M \to \mathbb{R}_{\geq 0}$, defining a non-negative value of each subset of items. We will not make any assumptions on v_i today except for normalization ($v_i(\emptyset) = 0$) and monotonicity: If $S \subseteq S'$, then $v_i(S) \leq v_i(S')$. The set of feasible allocations is given as $X = \{(S_1, \ldots, S_n) \mid S_i \cap S_{i'} = \emptyset \text{ for } i \neq i'\}$. We would like to maximize social welfare $\sum_{i \in N} v_i(S_i)$. Recall that this problem is NP-hard even for single-minded bidders.

We would like to come up with a truthful mechanism $M = (f, p)$, consisting of an allocation function $f\colon V \to X$ and a payment rule $p\colon V \to \mathbb{R}^n$, where $V = V_1 \times \ldots \times V_n$ and V_i is the set of all monotone normalized functions $2^M \to \mathbb{R}_{\geq 0}$.

Our mechanism will be randomized. That is, both $f(b)$ and $p(b)$ are random variables. The approximation guarantee will be with respect to the *expected* social welfare $\mathbf{E}\left[\sum_{i \in N} v_i(f_i(v))\right]$. Our point of comparison will be the allocation $(\mathrm{OPT}_1, \ldots, \mathrm{OPT}_n)$ of maximum social welfare, that is, $\sum_{i \in N} v_i(\mathrm{OPT}_i)$ is maximized.

The mechanism will run in polynomial time in n and m given “appropriate” access to the valuation functions. This latter point is actually important: A valuation function itself can take 2^m different values but it will usually be represented more succinctly. We will not define the kind of access we need formally; it is implicit from the formulation of our mechanism.

2 A Fixed-Price Mechanism

At the center of our mechanism, there will be a fixed-price mechanism: We can think of it as offering items at a price p per item to the bidders one after the other. Each bidder chooses whatever set maximizes his/her utility. Indeed, we will not make such an offer to all bidders but only to a subset $\mathit{FIXED} \subseteq N$ of them.

So the fixed-price mechanism parameterized by $\mathit{FIXED} \subseteq N$ and $p \geq 0$ reads.

- Initialize $\mathit{Sold} = \emptyset$.
- Approach bidders $i \in \mathit{FIXED}$ in order $1, \ldots, n$.
  - Assign bidder i the set $S_i \subseteq M \setminus \mathit{Sold}$ that maximizes $b_i(S_i) - |S_i| \cdot p$.
  - Set payment $p_i(b) = |S_i| \cdot p$. Add all items from S_i to Sold.

Lemma 14.1. *If $b = v$, for $p = \frac{\max_{i \in N} v_i(M)}{\sqrt{m}}$, the fixed-price mechanism obtains welfare at least*

$$\sum_{i \in N} v_i(S_i) \geq \frac{1}{\sqrt{m}} \sum_{i \in FIXED} v_i(\mathrm{OPT}_i) - \max_{i \in N} v_i(M) \enspace .$$

Proof. Note that we can rewrite the social welfare also as the sum of utilities plus the sum of payments as (already using that $b = v$)

$$\sum_{i \in N} v_i(S_i) = \sum_{i \in N} u_i(v, v_i) + \sum_{i \in N} p_i(v) \enspace .$$

We will now bound these two parts of the social welfare separately.

Let $T = \{i \in FIXED \mid \mathrm{OPT}_i \cap Sold \neq \emptyset\}$ be the set of bidders with the property that at least one of the items from OPT_i is also in *Sold*. Note that if $i \in FIXED \setminus T$, then one option for set S_i would be OPT_i. That is, bidder i would be able to buy OPT_i and pay $|\mathrm{OPT}_i| \cdot p$. The bidders get whatever set maximizes their utility. So, we get

$$u_i(v, v_i) \geq v_i(\mathrm{OPT}_i) - |\mathrm{OPT}_i| \cdot p \quad \text{ if } i \in FIXED \setminus T$$

Similarly, for every $i \in N$, it would be an option to assign nothing. So, another lower bound for the utility is given by

$$u_i(v, v_i) \geq 0 \enspace .$$

Furthermore, we have $|T| \leq |Sold|$ because each item that is allocated can move at most one bidder to T. Consequently, we have

$$\begin{aligned}
\sum_{i \in FIXED} u_i(v, v_i) &\geq \sum_{i \in FIXED \setminus T} (v_i(\mathrm{OPT}_i) - |\mathrm{OPT}_i| \cdot p) \\
&= \sum_{i \in FIXED \setminus T} v_i(\mathrm{OPT}_i) - \sum_{i \in FIXED \setminus T} |\mathrm{OPT}_i| \cdot p \\
&\geq \sum_{i \in FIXED} v_i(\mathrm{OPT}_i) - m \cdot p - |T| \cdot \max_i v_i(M) \enspace ,
\end{aligned}$$

where we use that $\sum_{i \in FIXED \setminus T} |\mathrm{OPT}_i| \leq m$ because each item is in at most one set OPT_i and $v_i(\mathrm{OPT}_i) \leq v_i(M)$ by monotonicity.

Furthermore, the revenue, that is the sum of the payments is exactly $\sum_{i \in N} p_i(v) = |Sold| \cdot p$. Using that $u_i(v, v_i) \geq 0$ for all i, the social welfare is at least

$$\sum_{i \in N} v_i(S_i) \geq \frac{1}{\sqrt{m}} \sum_{i \in FIXED} u_i(v, v_i) + \sum_{i \in N} p_i(v) \geq \frac{1}{\sqrt{m}} \sum_{i \in FIXED} v_i(\mathrm{OPT}_i) - \sqrt{m} \cdot p \enspace . \qquad \square$$

3 The Mechanism

If we didn't care about truthfulness, Lemma 14.1 would already tell us what to do: We could use the fixed-price mechanism on all bidders, i.e., $FIXED = N$, and get welfare $\frac{1}{\sqrt{m}}\mathrm{OPT} - \max_i v_i(M)$. Alternatively, we assign all items to only one bidder. Then we would get welfare $\max_i v_i(M)$. If we computed both solutions and took the better of the two, we would always get welfare at least $\frac{1}{2\sqrt{m}}\mathrm{OPT}$ (by a simple case distinction).

Unfortunately, we usually won't get a truthful mechanism this way. One of the reasons is that we do not know how to set the price p without asking the bidders. Therefore our mechanism is a little more complicated: We divide the bidders into two sets. Only half of the bidders are in the set *FIXED* and we use the *other* bidders to determine p. Also we do not choose the better of two outcomes but a random one.

- For each $i \in N$, flip an independent coin and add it to *FIXED* with probability $\frac{1}{2}$.
- Let $i^* \in N \setminus \mathit{FIXED}$ be the bidder of maximum $b_i(M)$ that is not in *FIXED*.
- With probability $\frac{2}{3}$: Assign all items to bidder i^*, that is $S_{i^*} = \emptyset$, and let him/her pay $p_{i^*}(b) = \max_{i \in N \setminus \mathit{FIXED}, i \neq i^*} b_i(M)$ (second-price auction).
  Otherwise: Run fixed-price mechanism on *FIXED* with price $p = \frac{b_{i^*}(M)}{\sqrt{m}}$.
- All remaining bidders are assigned no items and no payment.

4 Truthfulness

Let's first observe that this mechanism is indeed truthful.

Theorem 14.2. *Regardless of the outcome of the random coin flips, for every bidder $i \in N$, for all v_i and all b, we have*

$$u_i((v_i, b_{-i}), v_i) \geq u_i(b, v_i) \enspace .$$

Proof. There are the two cases that $i \in \mathit{FIXED}$ but the second-price auction is run or $i \notin \mathit{FIXED}$ and the fixed-price mechanism is run. In both cases, bidder i will not be allocated anything and will not have to pay anything, regardless of the bid. So $u_i((v_i, b_{-i}), v_i) = 0 = u_i(b, v_i)$.

Now consider $i \notin \mathit{FIXED}$ and the second-price auction is run. Then, indeed, $u_i((v_i, b_{-i}), v_i) \geq u_i(b, v_i)$ follows exactly from the argument that all that matters is bidder i's bid for all items. From bidder i's perspective, the mechanism looks exactly like a second-price auction against $N \setminus \mathit{FIXED}$ with the property that all items are always sold as one bundle.

Finally, consider $i \in \mathit{FIXED}$ and the fixed-price mechanism is run. First of all, observe that bidder i will not influence p. Bidder i's bid will neither influence the allocation made to bidders $1, \ldots, i-1$, so the set at the time of allocation *Sold* is also fixed. By design, the fixed-price mechanism allocates S_i that maximizes $b_i(S_i) - |S_i| \cdot p$. Let S_i be the allocation under bid b_i, S_i' the one under v_i. Now we have

$$u_i((v_i, b_{-i}), v_i) = v_i(S_i') - |S_i'| \cdot p \geq v_i(S_i) - |S_i| \cdot p = u_i(b, v_i) \enspace ,$$

where the inequality is coming from the fact that S_i' was chosen exactly so that this term was maximized. So, this also shows truthfulness in this case. □

5 Approximation Guarantee

Now, we come to the approximation guarantee. Given that the mechanism is truthful, we will assume that all bidders bid their true values and derive a lower bound on the social welfare for this case.

Theorem 14.3. *If all bidders bid truthfully, that is, $b = v$, the mechanism's allocation fulfills*

$$\mathbf{E}\left[\sum_{i \in N} v_i(S_i)\right] \geq \frac{1}{12\sqrt{m}} \sum_{i \in N} v_i(\mathrm{OPT}_i) \enspace .$$

That is, we compute an $O(\sqrt{m})$-approximation. Recall that under usual complexity-theoretic assumptions, this is (almost) the best we can hope for.

Proof. Let i^* be a bidder such that $v_{i^*}(M)$ is maximized. We will bound the social welfare for two (disjoint) events, both assuming that $i^* \in \mathit{FIXED}$.

Let $\mathcal{E}_1$ be the event that $i^* \notin \mathit{FIXED}$ and the second-price auction is run. By definition $\mathbf{Pr}\left[\mathcal{E}_1\right] = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$.

Let $\mathcal{E}_2$ be the event that $i^* \notin \mathit{FIXED}$ and the fixed-price auction is run. By definition $\mathbf{Pr}\left[\mathcal{E}_2\right] = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$.

Observe that in both these cases, bidder i^* as defined in the mechanism coincides with the one defined above, namely the one that maximizes $v_{i^*}(M)$.

We will entirely ignore the social welfare obtained when $i^* \in \mathit{FIXED}$.

If event $\mathcal{E}_1$ takes place, bidder i^* will get all items. So

$$\mathbf{E}\left[\sum_{i \in N} v_i(S_i) \;\middle|\; \mathcal{E}_1\right] = v_{i^*}(M) = \max_i v_i(M) \enspace .$$

Now, let's talk about even $\mathcal{E}_2$. We can make use of Lemma 14.1 to get

$$\begin{aligned}
\mathbf{E}\left[\sum_{i \in N} v_i(S_i) \;\middle|\; \mathcal{E}_2\right] &\geq \mathbf{E}\left[\frac{1}{\sqrt{m}} \sum_{i \in \mathit{FIXED}} v_i(\mathrm{OPT}_i) - \max_i v_i(M) \;\middle|\; \mathcal{E}_2\right] \\
&= \frac{1}{\sqrt{m}} \sum_{i \in N} v_i(\mathrm{OPT}_i) \mathbf{Pr}\left[i \in \mathit{FIXED} \mid \mathcal{E}_2\right] - \max_i v_i(M) \\
&= \frac{1}{2\sqrt{m}} \sum_{i \neq i^*} v_i(\mathrm{OPT}_i) - \max_i v_i(M) \\
&\geq \frac{1}{2\sqrt{m}} \sum_{i \in N} v_i(\mathrm{OPT}_i) - 2 \max_i v_i(M) \enspace .
\end{aligned}$$

Here, we use that $\mathbf{Pr}\left[i \in \mathit{FIXED} \mid \mathcal{E}_2\right] = \frac{1}{2}$ for $i \neq i^*$.

In combination (using that $\sum_{i \in N} v_i(S_i) \geq 0$ when neither $\mathcal{E}_1$ nor $\mathcal{E}_2$ takes place), we get

$$\begin{aligned}
\mathbf{E}\left[\sum_{i \in N} v_i(S_i)\right] &\geq \mathbf{Pr}\left[\mathcal{E}_1\right] \max_i v_i(M) + \mathbf{Pr}\left[\mathcal{E}_2\right] \left(\frac{1}{2\sqrt{m}} \sum_{i \in N} v_i(\mathrm{OPT}_i) - 2 \max_i v_i(M)\right) \\
&= \frac{1}{12\sqrt{m}} \sum_{i \in N} v_i(\mathrm{OPT}_i) \enspace .
\end{aligned}$$

□

6 Outlook

This mechanism is not nearly as elegant and clean as VCG. However, it is the best that people have come up with when it comes to designing truthful mechanisms that run in polynomial time. For example, it is still an open question whether randomization is actually necessary. An alternative to such complicated truthful mechanisms is to analyze simpler, non-truthful ones. This will be our next topic in this course.

References

- Shahar Dobzinski, Noam Nisan, Michael Schapira: Truthful randomized mechanisms for combinatorial auctions. STOC 2006: 644-652 (Original paper with a more complicated version of this mechanism)

- Sepehr Assadi, Thomas Kesselheim, Sahil Singla: Improved Truthful Mechanisms for Subadditive Combinatorial Auctions: Breaking the Logarithmic Barrier. SODA 2021: 653-661 (Recent paper that includes some of the techniques used here but for a different setting)