Posted Prices with Incomplete Information

Thomas Kesselheim Last Update: June 17, 2025

Our question is still "How well can prices coordinate markets?" Last time, we got a glimpse at the classic economic theory: If there is a Walrasian equilibrium, it defines prices, which make everybody happy and yield maximum social welfare.

Today, we will turn to much more recent results, in which the perspective of computer science comes into play. We turn to a setting of incomplete information. Our goal is to post prices for items without knowing which buyers will be present eventually. Buyers then show up one after the other and buy their preferred item(s).

1 Setting and Result

Recall the definition of a combinatorial auction. There are n buyers $N = \{1, \ldots, n\}$ and m items M. Each buyer has a private valuation function $v_i \colon 2^M \to \mathbb{R}_{\geq 0}$. Each item can be assigned to at most one buyer.

We assume that the valuation functions v_i are *unit-demand*, that is, they are of the form $v_i(S) = \max_{j \in S} v_{i,j}$.

Buyer i's valuation v_i is drawn independently from a publicly known distribution $\mathcal{D}_i$. The outcome v_i, however, is private. We use the knowledge of the distributions $(\mathcal{D}_i)_{i \in N}$ to compute item prices $(p_j)_{j \in M}$. The mechanism then looks as follows:

- Approach the buyers in order $i = 1, \ldots, n$
- Buyer i buys whatever set S_i of unsold items maximizes $v_i(S_i) - \sum_{j \in S_i} p_j$, pays $\sum_{j \in S_i} p_j$

Note that this mechanism still consists of an allocation function f and a payment function p. Of course, buyers could decide to lie about their valuation v_i and buy another set. But this can only reduce the utility because the choice of the set S_i is exactly so that it maximizes utility.

Observation 18.1. *The posted-prices mechanism is truthful for any choice of prices.*

We are interested to what extent such a mechanism can optimize social welfare. That is, how does $\sum_{i \in N} v_i(S_i)$ compare to $\mathrm{OPT}(v)$, which is the allocation that maximizes $\sum_{i \in N} v_i(S_i)$. We will show the following theorem.

Theorem 18.2 (Feldman/Gravin/Lucier, 2015)**.** *For all distributions over unit-demand valuations, there is a choice of prices such that the expected social welfare of the posted-prices mechanism is a $\frac{1}{2}$ fraction of the expected optimal social welfare.*

The major difference to Walrasian equilibria is that we now are in an incomplete information setting. Walrasian prices are always adjusted to the respective valuation profile. Now, we cannot do this. We are not guaranteed to get optimal social welfare but only a constant fraction.

There is one more detail, which is not extremely important for us: Tie breaking may be arbitrary. In a Walrasian equilibrium this is different: Buyers might (and usually will) be indifferent between multiple choices.

2 Single-Item Case

Let's first talk about the case in which there is only a single item. Let's define an arbitrary prices p for the item and bound the social welfare we obtain.

To this end, let q be the probability that the item is sold.

If is straightforward to bound the expected revenue of the mechanism. That is, how much do buyers pay in expectation, namely

$$\mathbf{E}\left[\text{revenue}(v)\right] = pq$$

Next, we will consider the utility of buyer i. We will use the following notation

$$x^+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \mathbf{1}_{\mathcal{E}} = \begin{cases} 1 & \text{if } \mathcal{E} \text{ takes place} \\ 0 & \text{otherwise} \end{cases}$$

Note that buyer i will buy the item if $v_i \geq p$ and if nobody before i buys it. Then buyer i's utility is $v_i - p$. By this, we get

$$u_i(v, v_i) = (v_i - p)^+ \cdot \mathbf{1}_{\text{nobody before } i \text{ buys}} \enspace .$$

Taking the expectation and using the fact that v_i and $v_1, \ldots, v_{i-1}$ are independent, we get

$$\begin{aligned} \mathbf{E}\left[u_i(v, v_i)\right] &= \mathbf{E}\left[(v_i - p)^+ \cdot \mathbf{1}_{\text{nobody before } i \text{ buys}}\right] \\ &= \mathbf{E}\left[(v_i - p)^+\right] \cdot \mathbf{Pr}\left[\text{nobody before } i \text{ buys}\right] \\ &\geq \mathbf{E}\left[(v_i - p)^+\right] \cdot (1 - q) \end{aligned}$$

Recall, that we can write the social welfare as revenue $+ \sum_{i \in N} u_i(v, v_i)$. So, we get

$$\begin{aligned} \mathbf{E}\left[\text{welfare}\right] &= \mathbf{E}\left[\text{revenue}\right] + \sum_i \mathbf{E}\left[u_i(v, v_i)\right] \\ &\geq pq + \sum_i \mathbf{E}\left[(v_i - p)^+\right] \cdot (1 - q) \\ &\geq pq + \mathbf{E}\left[\max_i v_i - p\right] \cdot (1 - q) \enspace , \end{aligned}$$

where the second inequalities uses that the maximum of non-negative terms is no more than their sum.

So far, these calculations hold for all prices p. If we choose $p = \frac{1}{2}\mathbf{E}\left[\max_i v_i\right]$, we get

$$\mathbf{E}\left[\text{welfare}\right] \geq \frac{1}{2}\mathbf{E}\left[\max_i v_i\right] q + \frac{1}{2}\mathbf{E}\left[\max_i v_i\right] (1 - q) = \frac{1}{2}\mathbf{E}\left[\max_i v_i\right] \enspace .$$

In particular, observe that by the choice of the price p, the probability of selling the item q does actually not play a role in the lower bound anymore.

In addition, it is easy to see that this bound is even optimal. Consider the following example. Buyer 1 has value 1, buyer 2 has value $\frac{1}{\epsilon}$ with probability ϵ, 0 otherwise. The optimal social welfare is achieved by giving the item to buyer 2 if he has high value, to buyer 1 otherwise. The expected optimal value is hence

$$\epsilon \cdot \frac{1}{\epsilon} + (1 - \epsilon) \cdot 1 = 2 - \epsilon \enspace .$$

With any price, the best expected social welfare of the mechanism however is no more than 1.

3 Multiple Items

Now, we will turn to the case of multiple items. We will proceed in a similar way as in the single-item case and first fix item prices $p_1, \dots, p_m$ arbitrarily and derive a bound on social welfare.

For these prices, define $T_i(v)$ as the set of items that are sold to buyers $1, \dots, i$ on v and let q_j be the probability that item j is sold to any of the buyers. That is, $q_j = \mathbf{Pr}\left[j \in T_n(v)\right]$.

We again get a similar bound on the expected revenue.

Lemma 18.3. *The expected revenue is $\sum_{j \in M} p_j q_j$.*

Proof. Whenever item j gets sold, the respective buyer pays p_j. So, we get

$$\text{revenue}(v) = \sum_{j \in M} p_j \mathbf{1}_{j \in T_n(v)} \enspace .$$

Therefore, by linearity of expectation

$$\mathbf{E}\left[\text{revenue}(v)\right] = \mathbf{E}\left[\sum_{j \in M} p_j \mathbf{1}_{j \in T_n(v)}\right] = \sum_{j \in M} p_j \mathbf{E}\left[\mathbf{1}_{j \in T_n(v)}\right] = \sum_{j \in M} p_j q_j \enspace .$$

□

To derive a lower bound on the expected utilities, we introduce a little more notation referring to the optimal allocation. Recall that $\mathrm{OPT}(v)$ denotes the optimal allocation. Without loss of generality, each buyer gets only one item in $\mathrm{OPT}(v)$. We write $(i, j) \in \mathrm{OPT}(v)$ if buyer i gets item j. Furthermore, let $v_{\mathrm{OPT},j} = v_{i,j}$ if $(i, j) \in \mathrm{OPT}(v)$; $v_{\mathrm{OPT},j} = 0$ if item j is not allocated. That is, $v_{\mathrm{OPT},j}$ denotes the contribution of item j to the optimal social welfare on valuation profile v. Succinctly, we can also write

$$v_{\mathrm{OPT},j} = \sum_{i \in N} v_{i,j} \mathbf{1}_{(i,j) \in \mathrm{OPT}(v)} \enspace . \tag{1}$$

Lemma 18.4. *The expected sum of utilities is lower-bounded by*

$$\mathbf{E}\left[\sum_{i \in N} u_i(v, v_i)\right] \geq \sum_{j \in M} (1 - q_j) \left(\mathbf{E}\left[v_{\mathrm{OPT},j}\right] - p_j\right)$$

Proof. To lower-bound the utility of buyer i, we observe that this buyer could choose any item that is not in $T_{i-1}(v)$ or no item at all (which results in zero utility). This gives us

$$u_i(v, v_i) = \max_{j \notin T_{i-1}(v)} (v_{i,j} - p_j)^+ \enspace .$$

To bound the maximum, we now perform a thought experiment: One way to choose an item would be to draw valuations $v'_{i'}$ for buyers $i' \neq i$ from the probability distributions $\mathcal{D}_{i'}$. Now, see what item j buyer i gets in $\mathrm{OPT}(v_i, v'_{-i})$. The maximum is at least the term for this j if $j \notin T_{i-1}(v)$. Also, the maximum is never negative. Therefore

$$\max_{j \notin T_{i-1}(v)} (v_{i,j} - p_j)^+ \geq \sum_{j \in M} (v_{i,j} - p_j)^+ \mathbf{1}_{(i,j) \in \mathrm{OPT}(v_i, v'_{-i})} \mathbf{1}_{j \notin T_{i-1}(v)} \enspace .$$

Taking the expectation, we get

$$\begin{aligned} \mathbf{E}\left[u_i(v, v_i)\right] &\geq \mathbf{E}\left[\sum_{j \in M} (v_{i,j} - p_j)^+ \mathbf{1}_{(i,j) \in \mathrm{OPT}(v_i, v'_{-i})} \mathbf{1}_{j \notin T_{i-1}(v)}\right] \\ &= \sum_{j \in M} \mathbf{E}\left[(v_{i,j} - p_j)^+ \mathbf{1}_{(i,j) \in \mathrm{OPT}(v_i, v'_{-i})} \mathbf{1}_{j \notin T_{i-1}(v)}\right] \enspace . \end{aligned}$$

Observe that the first part of the expectation only depends on v_i and v'_{-i} whereas $T_{i-1}(v)$ only depends on $v_1, \ldots, v_{i-1}$. Therefore, we can write

$$\begin{aligned}&\mathbf{E}\left[(v_{i,j} - p_j)^+ \mathbf{1}_{(i,j)\in \mathrm{OPT}(v_i, v'_{-i})} \mathbf{1}_{j \notin T_{i-1}(v)}\right] \\ &= \mathbf{E}\left[(v_{i,j} - p_j)^+ \mathbf{1}_{(i,j)\in \mathrm{OPT}(v_i, v'_{-i})}\right] \mathbf{E}\left[\mathbf{1}_{j \notin T_{i-1}(v)}\right] \quad .\end{aligned}$$

Now, we use the fact that v'_{-i} and v_{-i} are drawn from the same probability distribution and are independent of v_i. Therefore we can replace v'_{-i} in the first term by v_{-i}, which did not occur before, giving us

$$\mathbf{E}\left[(v_{i,j} - p_j)^+ \mathbf{1}_{(i,j)\in \mathrm{OPT}(v_i, v'_{-i})}\right] = \mathbf{E}\left[(v_{i,j} - p_j)^+ \mathbf{1}_{(i,j)\in \mathrm{OPT}(v_i, v_{-i})}\right] \quad .$$

Furthermore

$$\mathbf{E}\left[\mathbf{1}_{j \notin T_{i-1}(v)}\right] = \mathbf{Pr}\left[j \notin T_{i-1}(v)\right] \geq \mathbf{Pr}\left[j \notin T_n(v)\right] = 1 - q_j$$

So overall

$$\mathbf{E}\left[u_i(v, v_i)\right] \geq \sum_{j \in M} \mathbf{E}\left[(v_{i,j} - p_j) \mathbf{1}_{(i,j)\in \mathrm{OPT}(v)}\right] (1 - q_j) \quad .$$

Now, we take the sum over all $i \in N$

$$\begin{aligned}\mathbf{E}\left[\sum_{i\in N} u_i(v, v_i)\right] &= \sum_{i \in N} \mathbf{E}\left[u_i(v, v_i)\right] \\ &\geq \sum_{i\in N}\sum_{j \in M} \mathbf{E}\left[(v_{i,j} - p_j) \mathbf{1}_{(i,j)\in \mathrm{OPT}(v)}\right] (1 - q_j) \\ &= \sum_{j \in M} (1 - q_j) \sum_{i\in N} \mathbf{E}\left[(v_{i,j} - p_j) \mathbf{1}_{(i,j)\in \mathrm{OPT}(v)}\right] \\ &= \sum_{j \in M} (1 - q_j) \left(\mathbf{E}\left[\sum_{i\in N} v_{i,j} \mathbf{1}_{(i,j)\in \mathrm{OPT}(v)}\right] - p_j \mathbf{E}\left[\sum_{i\in N} \mathbf{1}_{(i,j)\in \mathrm{OPT}(v)}\right]\right) \\ &\geq \sum_{j \in M} (1 - q_j) \left(\mathbf{E}\left[v_{\mathrm{OPT},j}\right] - p_j\right) \quad ,\end{aligned}$$

where the last step uses Equation (1) and $\sum_{i\in N} \mathbf{1}_{(i,j)\in \mathrm{OPT}(v)} \leq 1$ because each item is allocated at most once. □

Proof of Theorem 18.2. Combining Lemmas 18.3 and 18.4, the social welfare is at least

$$\sum_{j\in M} p_j q_j + \sum_{j \in M} (1 - q_j) \left(\mathbf{E}\left[v_{\mathrm{OPT},j}\right] - p_j\right) \quad .$$

We notice that setting $p_j = \frac{1}{2}\mathbf{E}\left[v_{\mathrm{OPT},j}\right]$, this is exactly $\frac{1}{2}\mathbf{E}\left[\sum_{j\in M} v_{\mathrm{OPT},j}\right]$. □

References

- M. Feldman, N. Gravin, B. Lucier, Combinatorial Auctions via Posted Prices, SODA 2015. (Original proof in a more general form.)
- P. Dütting, M. Feldman, T. Kesselheim, B. Lucier, Prophet Inequalities Made Easy: Stochastic Optimization by Pricing Non-Stochastic Inputs, FOCS 2017. (Improved and generalized proof.)