Advanced Topics in Computer Graphics II
Geometry Processing

Laplace Beltrami Operator

November 14, 2024

Let Ω be an open subset of $\mathbb{R}^2$ and $g\colon \partial\Omega \to \mathbb{R}$ a mapping.

Question: How to find a function $f\colon \Omega \to \mathbb{R}$ that equals g on $\partial\Omega$ and is "as flat as possible"?

Idea 1: For each surface f that agrees with g on $\partial\Omega$, we assign a numerical measure of flatness:

$$E[f] = \frac{1}{2} \int_{\Omega} \|\nabla f(\boldsymbol{u})\|_2^2 \, d\boldsymbol{u}$$

This integral is often referred to as the **Dirichlet Energy Integral**, in analogy with the kinetic energy $\frac{1}{2}\, m\, v^2$.

Idea 2:

$$\begin{aligned} &\text{minimize}_f \;\; E[f] \\ &\text{subject to } f(\boldsymbol{u}) = g(\boldsymbol{u}) \quad \forall \boldsymbol{u} \in \partial\Omega \end{aligned}$$

Note, the unknown in this minimization task is a function f rather than a point in $\mathbb{R}^n$.

If f minimizes E subject to the boundary conditions, then the **directional derivatives** $\frac{d}{d\epsilon}E[f+\epsilon\, h]|_{\epsilon=0} = 0$ for all functions h with $h(\boldsymbol{u}) = 0$, $\forall\, \boldsymbol{u} \in \partial\Omega$.

$$
\begin{aligned}
E[f+\epsilon h] &= \frac{1}{2}\int_{\Omega} \|\nabla f(\boldsymbol{u}) + \epsilon\, \nabla h(\boldsymbol{u})\|_2^2 \, d\boldsymbol{u} \\
&= \frac{1}{2}\int_{\Omega} \|\nabla f(\boldsymbol{u})\|_2^2 + 2\,\epsilon \,\langle\, \nabla f(\boldsymbol{u}) \,|\, \nabla h(\boldsymbol{u})\,\rangle + \epsilon^2\, \|\nabla h(\boldsymbol{u})\|_2^2 \, d\boldsymbol{u}
\end{aligned}
$$

Differentiating with respect to ϵ and using Green's first idendity leads to

$$
\begin{aligned}
\frac{d}{d\epsilon}E[f+\epsilon\, h] &= \frac{1}{2}\int_{\Omega} 2\,\langle\, \nabla f(\boldsymbol{u}) \,|\, \nabla h(\boldsymbol{u})\,\rangle + 2\,\epsilon\, \|\nabla h(\boldsymbol{u})\|_2^2 \, d\boldsymbol{u} \\
\Rightarrow \frac{d}{d\epsilon}E[f+\epsilon\, h]|_{\epsilon=0} &= \int_{\Omega} \langle\, \nabla f(\boldsymbol{u}) \,|\, \nabla h(\boldsymbol{u})\,\rangle \, d\boldsymbol{u} \\
&= \int_{\partial\Omega} h(\boldsymbol{u})\,\langle\, \nabla f(\boldsymbol{u}) \,|\, \hat{\nu}\,\rangle \, d\boldsymbol{u} - \int_{\Omega} h(\boldsymbol{u})\,\Delta f(\boldsymbol{u})\, d\boldsymbol{u} \\
&= -\int_{\Omega} h(\boldsymbol{u})\,\Delta f(\boldsymbol{u})\, d\boldsymbol{u}
\end{aligned}
$$

where $\hat{\nu}$ is the outward surface normal to $\partial\Omega$.

To express $\nabla f(\boldsymbol{u}) \cdot \nabla h(\boldsymbol{u})$ in a form suitable for integration by parts:

$$\begin{aligned} \nabla \cdot (f(\boldsymbol{u}) \nabla h(\boldsymbol{u})) &= \underbrace{\nabla \nabla f(\boldsymbol{u})}_{\Delta f(\boldsymbol{u})} h(\boldsymbol{u}) + \nabla f(\boldsymbol{u}) \nabla h(\boldsymbol{u}) \\ \Leftrightarrow \qquad \nabla f(\boldsymbol{u}) \nabla h(\boldsymbol{u}) &= \nabla \cdot (f(\boldsymbol{u}) \nabla h(\boldsymbol{u})) - h(\boldsymbol{u}) \Delta f(\boldsymbol{u}) \end{aligned}$$

Integrating both side over the domain Ω leads to

$$\int_{\Omega} \nabla f(\boldsymbol{u}) \nabla h(\boldsymbol{u}) d\boldsymbol{u} = \int_{\Omega} \nabla \cdot (f(\boldsymbol{u}) \nabla h(\boldsymbol{u})) - h(\boldsymbol{u}) \Delta f(\boldsymbol{u}) d\boldsymbol{u}$$

Applying the divergence theorem to the first term on the right hand side we get

$$\int_{\Omega} \nabla f(\boldsymbol{u}) \nabla h(\boldsymbol{u}) d\boldsymbol{u} = \int_{\partial\Omega} h(\boldsymbol{u}) \left\langle \nabla f(\boldsymbol{u}) \,\middle|\, \hat{\nu} \right\rangle d\boldsymbol{u} - \int_{\Omega} h(\boldsymbol{u}) \Delta f(\boldsymbol{u}) d\boldsymbol{u}$$

Requiring that all directional derivatives of the ernergy functional vanish, i.e. that

$$-\int_{\Omega} h(\boldsymbol{u})\, \Delta f(\boldsymbol{u})\, d\boldsymbol{u} = 0,\ \forall \text{ functions } h \text{ with } h(\boldsymbol{u}) = 0,\ \forall\, \boldsymbol{u} \in \partial\Omega$$

we get $\Delta f(\boldsymbol{u}) = 0\, \forall \boldsymbol{u} \in \Omega \setminus \partial\Omega$.

Thus solving the minimization problem

$$\begin{aligned} &\text{minimize }_f E[f] \\ &\text{subject to } f(\boldsymbol{u}) = g(\boldsymbol{u})\, \forall \boldsymbol{u} \in \partial\Omega \end{aligned}$$

amounts to solving the **Laplace's equation**:

$$\begin{aligned} \Delta f(\boldsymbol{u}) &= 0 && \forall\, \boldsymbol{u} \in \Omega \setminus \partial\Omega \\ f(\boldsymbol{u}) &= g(\boldsymbol{u}) && \forall\, \boldsymbol{u} \in \partial\Omega \end{aligned}$$

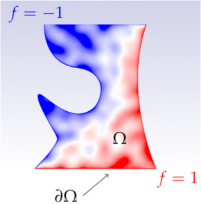
From the derivation it follows that $-\Delta f$ is the **gradient of the Dirichlet energy**. The gradient is also called **functional derivative** and denoted by $\frac{\delta E}{\delta f}$. It is implicitly defined by

$$\int_{\Omega} \frac{\delta E}{\delta f}\, \phi(\boldsymbol{u})\, d\boldsymbol{u} = \frac{d}{d\epsilon} E(f + \epsilon\, \phi)|_{\epsilon=0}$$

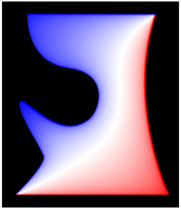
Laplace's equation

Physical Interpretation: Applying heating/cooling g to the boundary of a metal plate. Interior temperatur will reach *some* **steady state**, where it is no longer dependent on time. In the steady state it is assumed that $\frac{\partial f}{\partial t} = 0$. The **steady state** of the heat equation without a heat source within Ω results in the **Laplace equation**

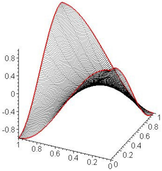
$$\frac{df}{dt} = \Delta f(\boldsymbol{x}) = 0$$



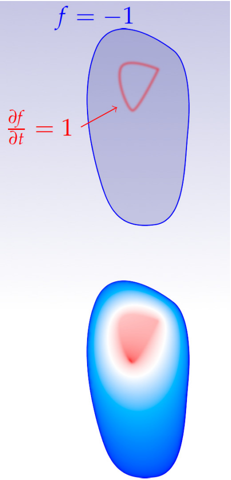
(a) Non-smooth $f(\boldsymbol{x})$



(b) Solution $\Delta f(\boldsymbol{x}) = 0$



(c) Boundary interpolation with $\Delta f(\boldsymbol{x}) = 0$ on a rectangle shown as a graph



If in addition prescibe the gradient of f by a vector field $\boldsymbol{v}$ te get the following variational problem:

$$\text{minimize}_f \int_{\Omega} \|\nabla f - \boldsymbol{v}\|_2^2 \, du$$
$$\text{subject to } f(u) = f_0(u) \quad \forall u \in \partial\Omega.$$

Minimizing leads to the **Poisson equation**:

$$\begin{aligned} \Delta f(\boldsymbol{u}) &= g(\boldsymbol{u}) & \boldsymbol{u} &\in \Omega \\ f(\boldsymbol{u}) &= f_0(\boldsymbol{u}) & \boldsymbol{u} &\in \partial\Omega, \end{aligned}$$

where $\operatorname{div} \boldsymbol{v}(\boldsymbol{u}) = g(\boldsymbol{u}), \boldsymbol{u} \in \Omega$.
It is the steady state of the following equation

$$\frac{df}{dt}(\boldsymbol{u}) = \Delta f(\boldsymbol{u}) - g(\boldsymbol{u})$$

which can be interpreted as adding a heat source g causing the fixed vector field $\boldsymbol{v}$ inside Ω.

Derivation of the Poisson equation:

$$\begin{aligned} E[f+\epsilon h] &= \frac{1}{2}\int_{\Omega} \|\nabla f(\boldsymbol{u}) + \epsilon\,\nabla h(\boldsymbol{u}) - v\|_2^2\, d\boldsymbol{u} \\ &= \frac{1}{2}\int_{\Omega} \|\nabla f(\boldsymbol{u}) + v(\boldsymbol{u})\|_2^2 + 2\,\epsilon\,\langle\, \nabla f(\boldsymbol{u}) - v(\boldsymbol{u}) \,|\, \nabla h(\boldsymbol{u})\,\rangle - 2\langle\, v(\boldsymbol{u}) \,|\, \nabla f(\boldsymbol{u})\,\rangle \\ &\quad + \epsilon^2\, \|\nabla h(\boldsymbol{u})\|_2^2\, d\boldsymbol{u} \end{aligned}$$

Differentiating with respect to ϵ, evaluating at $\epsilon = 0$ and using Green's first idendity leads to

$$\begin{aligned} \frac{d}{d\epsilon} E[f+\epsilon\, h]|_{\epsilon=0} &= \int_{\Omega} \langle\, \nabla f(\boldsymbol{u}) - v \,|\, \nabla h(\boldsymbol{u})\,\rangle\, d\boldsymbol{u} \\ &= \int_{\partial\Omega} h(\boldsymbol{u})\,\langle\, \nabla f(\boldsymbol{u}) - v \,|\, \hat{\nu}\,\rangle\, d\boldsymbol{u} - \int_{\Omega} h(\boldsymbol{u})\ \mathrm{div}\,(\nabla f(\boldsymbol{u}) - v)\, d\boldsymbol{u} \\ &= -\int_{\Omega} h(\boldsymbol{u})\ \mathrm{div}\,(\nabla f(\boldsymbol{u}) - v)\, d\boldsymbol{u} \end{aligned}$$

where $\hat{\nu}$ is the outward surface normal to $\partial\Omega$. Following the arguments above we get

$$\mathrm{div}\,(\nabla f(\boldsymbol{u}) - v) = 0 \Longleftrightarrow \boldsymbol{\Delta f}(\boldsymbol{u}) = \mathrm{div}\,\boldsymbol{v}(\boldsymbol{u}) = g(\boldsymbol{u})$$

Weak solutions: A weak solution f for the Poisson equation $\Delta f(\boldsymbol{x}) = g(\boldsymbol{x}),\ \forall\, \boldsymbol{x} \in \Omega$ is a function that satisfies

$$\int_\Omega h(\boldsymbol{u})\, \Delta f(\boldsymbol{u})\, d\boldsymbol{u} = \int_\Omega h(\boldsymbol{u})\, g(\boldsymbol{u})\, d\boldsymbol{u}$$

forall reasonable $h\colon \Omega \to \mathbb{R}$ with $h|_{\partial\Omega} = 0$. The functions h are called **test functions**.

Note, that weak solutions to PDEs may exist even when a strong solution does not.

When $h|_{\partial\Omega} = 0$, Green's first identity implies that

$$\int_\Omega h(\boldsymbol{u})\, \Delta f(\boldsymbol{u})\, d\boldsymbol{u} = -\int_\Omega \langle\, \nabla h(\boldsymbol{u}) \,|\, \nabla f(\boldsymbol{u}) \,\rangle\, d\boldsymbol{u}$$

For functions h, f we can define a bilinear operator

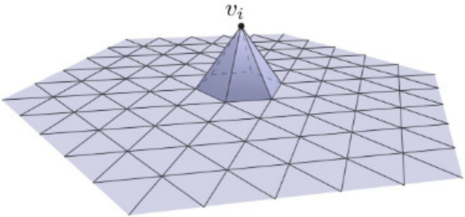
$$\langle\, h \,|\, f \,\rangle_\Delta := \int_\Omega \langle\, \nabla h(\boldsymbol{u}) \,|\, \nabla f(\boldsymbol{u}) \,\rangle\, d\boldsymbol{u}$$

The **finite element method (FEM)** makes the problem tractable by restricting the solution to a subspace of functions that can be represented by a finite basis $\Phi_1(\boldsymbol{u}), \ldots, \Phi_n(\boldsymbol{u})$ and writing

$$f(\boldsymbol{u}) = \sum_{i \in I} a_i \, \Phi_i(\boldsymbol{u})$$

$$g(\boldsymbol{u}) = \sum_{i \in I} b_i \, \Phi_i(\boldsymbol{u})$$

where I is the set of all mesh vertices and $b_1, \ldots, b_n$ are real coefficients representing g and $a_1, \ldots, a_n$ are the unknown real coefficients that determine the function f.



The best-known Finite element approximation is the **Galerkin method**.
In this method, it is required that the test functions h can also be written in the Φ_i basis:

$$h(\boldsymbol{u}) = \sum_{i \in I} b_i \, \Phi_i(\boldsymbol{u}) \quad , \quad b_i|_{\partial\Omega} = 0$$

For a weak solution of the Poisson equation the linearity of the integral implies

$$\begin{aligned} \int_\Omega \Phi_i(\boldsymbol{u}) \, \Delta f(\boldsymbol{u}) \, d\boldsymbol{u} &= \int_\Omega \Phi_i \, g(\boldsymbol{u}) \, d\boldsymbol{u} \qquad && i \in I_{\Omega \setminus \partial\Omega} \\ \Leftrightarrow -\int_\Omega \langle \, \nabla\Phi_i(\boldsymbol{u}) \, | \, \nabla f(\boldsymbol{u}) \, \rangle \, d\boldsymbol{u} &= \int_\Omega \Phi_i \, g(\boldsymbol{u}) \, d\boldsymbol{u} \qquad && i \in I_{\Omega \setminus \partial\Omega} \end{aligned}$$

Writing f and g in the Φ basis and expanding it we get

$$-\int_\Omega \left\langle \nabla \Phi_i(\boldsymbol{u}) \,\middle|\, \nabla \left(\sum_{j \in I} a_j \, \Phi_j(\boldsymbol{u}) \right) \right\rangle d\boldsymbol{u} = \int_\Omega \Phi_i \left(\sum_{j \in I} b_j \, \Phi_j(\boldsymbol{u}) \right) d\boldsymbol{u} \quad , \quad i \in I_{\Omega \setminus \partial\Omega}$$

Moving integrals, gradients and products inside the summation gives

$$\sum_{j \in I} a_j \int_\Omega -\langle \nabla \Phi_i(\boldsymbol{u}) \,|\, \nabla \Phi_j(\boldsymbol{u}) \rangle \, d\boldsymbol{u} = \sum_{j \in I} b_j \int_\Omega \Phi_i(\boldsymbol{u}) \, \Phi_j(\boldsymbol{u}) \, d\boldsymbol{u} \quad , \quad i \in I_{\Omega \setminus \partial\Omega}$$

Because the values of f are known on the boundary, we can move these to the right hand side:

$$\begin{aligned} \sum_{j \in I_{\Omega \setminus \partial\Omega}} a_j \int_\Omega -\langle \nabla \Phi_i(\boldsymbol{u}) \,|\, \nabla \Phi_j(\boldsymbol{u}) \rangle \, d\boldsymbol{u} = \sum_{j \in I} b_j \int_\Omega \Phi_i(\boldsymbol{u}) \, \Phi_j(\boldsymbol{u}) \, d\boldsymbol{u} \\ + \sum_{j \in I_{\partial\Omega}} a_j \int_\Omega \langle \nabla \Phi_i(\boldsymbol{u}) \,|\, \nabla \Phi_j(\boldsymbol{u}) \rangle \, d\boldsymbol{u} \end{aligned}$$

which holds for all $i \in I_{\Omega \setminus \partial\Omega}$.

By declaring that

$$S_{ij} := -\int_{\Omega} \langle\, \nabla \Phi_i(\boldsymbol{u}) \,|\, \nabla \Phi_j(\boldsymbol{u}) \,\rangle \, d\boldsymbol{u}$$

$$b_j := \sum_{i \in I} b_j \int_{\Omega} \Phi_i(\boldsymbol{u})\, \Phi_j(\boldsymbol{u}) \, d\boldsymbol{u} + \sum_{i \in I_{\partial \Omega}} a_j \int_{\Omega} \langle\, \nabla \Phi_i(\boldsymbol{u}) \,|\, \nabla \Phi_j(\boldsymbol{u}) \,\rangle \, d\boldsymbol{u}$$

we arrive at an $\left| I_{\Omega \setminus \partial \Omega} \right| \times \left| I_{\Omega \setminus \partial \Omega} \right|$ linear system in matrix form:

$$\boldsymbol{S}\, \boldsymbol{a} = \boldsymbol{b}$$

The system matrix $\boldsymbol{S}$ is called the **stiffness matrix** of this elliptic PDE.

The elements

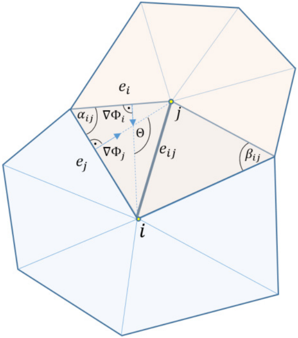
$$M_{ij} := \int_{\Omega} \Phi_i(\boldsymbol{u})\, \Phi_j(\boldsymbol{u}) \, d\boldsymbol{u}$$

needed on the right hand side of the equation are collected in the **mass matrix**.

For linear basis functions the elements of the stiffness matrix

$$S_{ij} := -\int_{\Omega} \langle \nabla\Phi_i(\boldsymbol{u}) \,|\, \nabla\Phi_j(\boldsymbol{u}) \rangle \, d\boldsymbol{u}$$

can be derived as follows[1]:



- We first notice that $\nabla\Phi_i$ is constant on each triangle and only nonzero on triangles incident to vertex i.
- For a triangle incident to vertex i, $\nabla\Phi_i$ points perpendicularly from the opposite edge $\boldsymbol{e}_i$ with inverse magnitude equal to the height h of the triangle treating $\boldsymbol{e}_i$ as base:

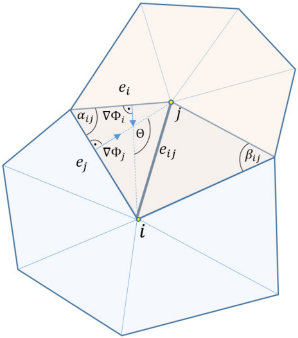
$$\|\nabla\Phi_i\| = \frac{1}{h} = \frac{\|\boldsymbol{e}_i\|}{2\,A},$$

where A is the area of the triangle.

[1] A. Jacobson. "Algorithms and interfaces for real-time deformation of 2d and 3d shapes". PhD thesis. ETH, 2013.

First we consider the case where $i \neq j$ and let θ be the angle between $\nabla \Phi_i$ and $\nabla \Phi_j$. Then we get

$$\langle \nabla \Phi_i \,|\, \nabla \Phi_j \rangle = \|\nabla \Phi_i\| \, \|\nabla \Phi_j\| \cos(\theta) = \frac{\|\boldsymbol{e}_i\|}{2\,A} \, \frac{\|\boldsymbol{e}_j\|}{2\,A} \cos(\theta)$$

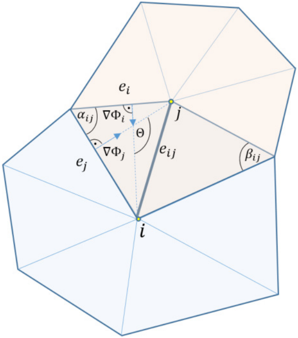


- Let α_{ij} and β_{ij} be the angle between the edges opposite to $\boldsymbol{e}_{ij}$, respectively. Then $\cos(\theta) = -\cos(\pi - \theta) = -\cos(\alpha_{ij})$ and we get

$$\langle \nabla \Phi_i \,|\, \nabla \Phi_j \rangle = -\frac{\|\boldsymbol{e}_i\|}{2\,A} \, \frac{\|\boldsymbol{e}_j\|}{2\,A} \cos(\alpha_{ij})$$

- Using $\sin(\alpha_{ij}) = \frac{h_j}{\|\boldsymbol{e}_i\|} = \frac{h_i}{\|\boldsymbol{e}_j\|}$ where h_i is the height of the triangle over base $\boldsymbol{e}_i$ and h_j the height of the triangle over base $\boldsymbol{e}_j$, we rewrite one of the $\|\boldsymbol{e}_i\|$ and get

$$\langle \nabla \Phi_i \,|\, \nabla \Phi_j \rangle = -\frac{\frac{h_j}{\sin(\alpha_{ij})}}{2\,A} \, \frac{\|\boldsymbol{e}_j\|}{2\,A} \cos(\alpha_{ij})$$



- Combining cosine and sine terms leads to

$$\langle \nabla\Phi_i \,|\, \nabla\Phi_j \rangle = -\frac{h_j}{2\,A}\,\frac{\|\boldsymbol{e}_j\|}{2\,A}\,\cot\alpha_{ij}$$

- Since $\|\boldsymbol{e}_j\|\,h_j = 2\,A$, we get

$$\langle \nabla\Phi_i \,|\, \nabla\Phi_j \rangle = -\frac{\cot(\alpha_{ij})}{2\,A}$$

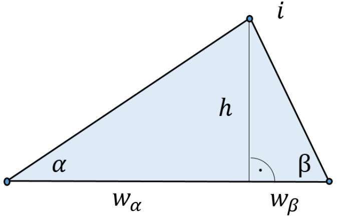
and similarly, on the opposite triangle with area B

$$\langle \nabla\Phi_i \,|\, \nabla\Phi_j \rangle = -\frac{\cot(\beta_{ij})}{2\,B}$$

Using these results and observing that $\Phi_i(u)\,\Phi_j(u)$ is nonzero inside these two triangles we get the **cotangent weights**

$$\int_\Omega \langle \nabla\Phi_i(\boldsymbol{u}) \,|\, \nabla\Phi_j(\boldsymbol{u}) \rangle \, d\boldsymbol{u} = A\,\langle \nabla\Phi_i \,|\, \nabla\Phi_j \rangle|_{T_A} + B\,\langle \nabla\Phi_i \,|\, \nabla\Phi_j \rangle|_{T_B}$$
$$= -\frac{1}{2}\left(\cot\alpha_{ij} + \cot\beta_{ij}\right)$$

What remains is the case $i = j$, i.e. the computation of $\int_\Omega \langle \nabla \Phi_i(\boldsymbol{u}) \,|\, \nabla \Phi_i(\boldsymbol{u}) \rangle \, d\boldsymbol{u}$.



- The aspect ratio of a triangle can be expressed as the sum of the cotangents of the interior angles at its base:

$$\frac{w}{h} = \frac{w_\alpha + w_\beta}{h} = \cot\alpha + \cot\beta$$

- For each triangle T incident on vertex i we get

$$\int_T \langle \nabla \Phi_i(\boldsymbol{u}) \,|\, \nabla \Phi_i(\boldsymbol{u}) \rangle \, d\boldsymbol{u} = \frac{1}{h^2} A_T$$
$$= \frac{1}{h^2} \frac{w\,h}{2} = \frac{1}{2} \frac{w}{h}$$
$$= \frac{1}{2}(\cot\alpha + \cot\beta)$$

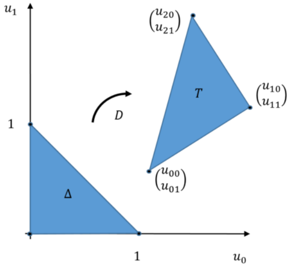
Summing around all triangles in the one-ring of vertex i we get

$$\int_\Omega \langle \nabla \Phi_i(\boldsymbol{u}) \,|\, \nabla \Phi_i(\boldsymbol{u}) \rangle \, d\boldsymbol{u} = \frac{1}{2} \sum_{j \in \mathcal{N}(i)} \cot\alpha_{ij} + \cot\beta_{ij}$$

To compute the entries of the mass matrix we divide the integral to contributions of each triangle simultaneously incident on node i and node j:

$$M_{ij} = \sum_{T \in \mathcal{N}(i) \cap \mathcal{N}(j)} \int_{\Omega} \Phi_i(\boldsymbol{u})\, \Phi_j(\boldsymbol{u})\, d\boldsymbol{u}$$

To integrate $\Phi_i(\boldsymbol{u})\, \Phi_j(\boldsymbol{u})$ over an arbitrary triangle T, we map the unit reference triangle Δ using the affine transformation D onto T:



$$D(x, y) = \begin{pmatrix} u_{10} - u_{00} & u_{20} - u_{00} \\ u_{11} - u_{01} & u_{21} - u_{01} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} u_{00} \\ u_{01} \end{pmatrix}$$

On the reference triangle we compute

$$\int_{\Delta} \Phi_{\Delta i}(\boldsymbol{u})\,\Phi_{\Delta j}(\boldsymbol{u})\,d\boldsymbol{u} = \int_0^{1-u_1}\int_0^1 \Phi_{\Delta i}(u_1,u_2)\,\Phi_{\Delta j}(u_1,u_2)\,du_1\,du_2$$
$$= \begin{cases} \int_0^{1-u_1}\int_0^1 u_1^2\,du_1\,du_2 = \frac{1}{12} & \text{if } i=j \\ \int_0^{1-u_1}\int_0^1 u_1\,u_2\,du_1\,du_2 = \frac{1}{24} & \text{otherwise} \end{cases}$$

Using the transformation theorem for integrals we get:

$$\int_T \Phi_i(\boldsymbol{u})\,\Phi_j(\boldsymbol{u})\,d\boldsymbol{u} = \det\big(D'\big)\int_{\Delta} \Phi_i(D(\boldsymbol{u}))\,\Phi_j(D(\boldsymbol{u}))\,d\boldsymbol{u}$$
$$= 2\,A_T \int_{\Delta} \Phi_{\Delta i}(\boldsymbol{u})\,\Phi_{\Delta j}(\boldsymbol{u})\,d\boldsymbol{u}$$
$$= \begin{cases} 2\,A_T \int_0^{1-u_1}\int_0^1 u_1^2\,du_1\,du_2 = \frac{A_T}{6} & \text{if } i=j \\ 2\,A_T \int_0^{1-u_1}\int_0^1 u_1\,u_2\,du_1\,du_2 = \frac{A_T}{12} & \text{otherwise} \end{cases}$$

where $A_T = \frac{1}{2}\det(D')$ is the area of the triangle T.

The mass matrix can be used for calculating integrals over the mesh M. Let $f\colon M \to \mathbb{R}$ be expanded in the Φ-basis: $f(\boldsymbol{u}) = \sum_{i \in I} a_i \, \Phi_i(\boldsymbol{u})$. Using $\sum_{i \in I} \Phi_i(\boldsymbol{u}) = 1$ we get

$$\begin{aligned}
\int_M f(\boldsymbol{u}) \, d\boldsymbol{u} &= \int_M \sum_{i \in I} a_i \, \Phi_i(\boldsymbol{u}) \\
&= \int_M \sum_{i \in I} a_i \, \Phi_i(\boldsymbol{u}) \sum_{j \in I} \Phi_j(\boldsymbol{u}) \\
&= \sum_{i \in I} \sum_{j \in I} \left(\int_M \Phi_i(\boldsymbol{u}) \, \Phi_j(\boldsymbol{u}) \, d\boldsymbol{u} \right) a_i \\
&= \sum_{i \in I} \sum_{j \in I} M_{ij} \, a_i \\
&= (1, \ldots, 1) \, M \, \boldsymbol{a}
\end{aligned}$$

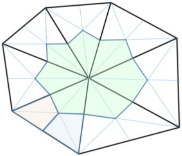
Setting $\boldsymbol{a} = (1, \ldots, 1)^T$ gives the surface area.

A lumped mass matrix concentrates the masses on diagonal elements making numerical operations more efficient. Setting

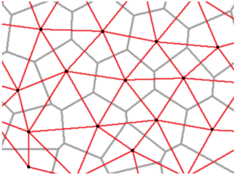
$$M_{ij} := \begin{cases} \text{Area}(\text{cell } i) & i = j \\ 0 & \text{else} \end{cases}$$

where cell i is the area corresponding to vertex i (dual cell).

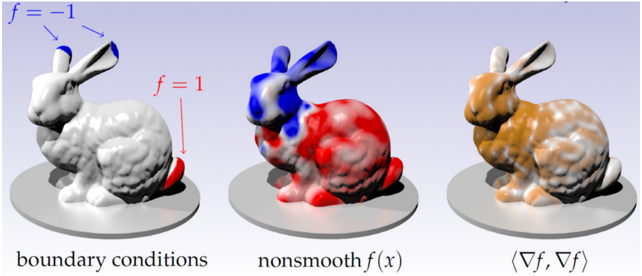
- The lumped mass matrix still allows the computation of the total surface area and as the mesh gets more and more refined, the original mass matrix will converge towards the diagonal one.
- There are many ways to choose the dual cell. Examples are the Voronoi lumped mass matrix and the barycentric lumped mass matrix.



(a) Barycentric cells.



(b) Voronoi cells.

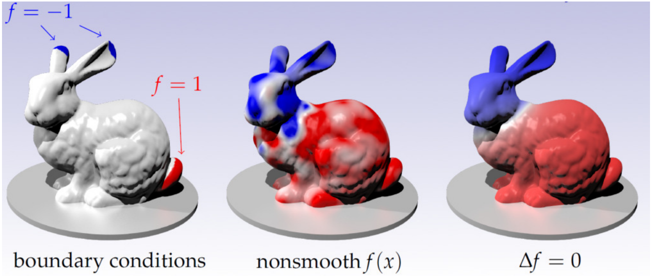


Dirichlet energy[2] on a surface $\mathcal{M}$:

$$E[f] = \frac{1}{2} \int_{\mathcal{M}} \|\nabla_M f(\boldsymbol{u})\|_2^2 \, d\boldsymbol{u} \quad , \quad \nabla_M f(\boldsymbol{u}) = \nabla f(\boldsymbol{u}) - \langle \nabla f(\boldsymbol{u}) \,|\, \boldsymbol{n}_T \rangle \, \boldsymbol{n}_T$$

where $\mathcal{M}$ is the mesh (embedded in $\mathbb{R}^3$), ∇_M is the **tangential gradient** on $\mathcal{M}$, ∇ is the three-dimensional gradient and $\boldsymbol{n}_T$ is the normal vector of $\mathcal{M}$ on the triangle T.

[2] G. Dziuk. "Finite elements for the Beltrami operator on arbitrary surfaces". In: *Partial differential equations and calculus of variations*. Springer, 1988, pp. 142–155.



Laplace equation: The Dirichlet Energy is minimized by solving the Laplace equation

$$\begin{aligned} \Delta_M f(\boldsymbol{u}) &= 0 && \forall \boldsymbol{u} \in \mathcal{M} \setminus \partial\mathcal{M} \\ f(\boldsymbol{u}) &= g(\boldsymbol{u}) && \forall \boldsymbol{u} \in \partial\mathcal{M} \end{aligned}$$

resulting in exactly the same coefficients of the stiffness matrix and mass matrix as in the 2D case (see above).

Note, that for closed meshes $\mathcal{M}$, $\partial\mathcal{M} = \emptyset$ and therefore the equation

$$\sum_{j \in I_{\mathcal{M}\setminus\partial\mathcal{M}}} S_{ij}\, a_j = \sum_{j \in I} M_{ij}\, b_j + \sum_{j \in I_{\partial\mathcal{M}}} S_{ij}\, a_j \quad , \quad \forall i \in I_{\mathcal{M}\setminus\partial\mathcal{M}}$$

simplifies to

$$\sum_{j \in I_{\Omega}} S_{ij}\, a_j = \sum_{j \in I} M_{ij}\, b_j \quad , \quad \forall i \in I_{\mathcal{M}}$$

This can be written in matrix form

$$\boldsymbol{S}\, \boldsymbol{a} = \boldsymbol{M}\, \boldsymbol{b}$$

where $\boldsymbol{a}$ represents f and $\boldsymbol{b}$ represents g. This motivates the definition of the Laplacian matrix L:

$$\Delta f = g \Leftrightarrow \underbrace{\boldsymbol{M}^{-1}\boldsymbol{S}}_{:=\boldsymbol{L}}\, \boldsymbol{a} = \boldsymbol{b}$$

For meshes with boundary we sort the indices of the vertices into the set of interior vertices $\tilde{I} := I \setminus \partial I$ and the set of boundary vertices ∂I. Using this notation the equation

$$\sum_{j \in I_{\mathcal{M} \setminus \partial \mathcal{M}}} S_{ij}\, a_j = \sum_{j \in I} M_{ij}\, b_j + \sum_{j \in I_{\partial \mathcal{M}}} S_{ij}\, a_j \quad , \quad \forall i \in I_{\mathcal{M} \setminus \partial \mathcal{M}}$$

can be written in matrix form

$$\begin{pmatrix} S_{\tilde{I}\,\tilde{I}} & S_{\tilde{I}\,\partial I} \\ S_{\partial I\,\tilde{I}} & S_{\partial I\,\partial I} \end{pmatrix} \begin{pmatrix} a_{\tilde{I}} \\ a_{\partial I} \end{pmatrix} = \begin{pmatrix} M_{\tilde{I}\,\tilde{I}} & 0 \\ 0 & M_{\partial I\,\tilde{I}} \end{pmatrix} \begin{pmatrix} b_{\tilde{I}} \\ b_{\partial I} \end{pmatrix}$$

Since $a_{\partial I}$ is known (boundary values), just solve

$$S_{\tilde{I}\,\tilde{I}}\, a_{\tilde{I}} = M_{\tilde{I}\,\tilde{I}}\, b_{\tilde{I}} - S_{\tilde{I}\,\partial I}\, a_{\partial I}$$

for $a_{\tilde{I}}$.

There are different approaches to define the area around a vertex $\boldsymbol{v}_i$.

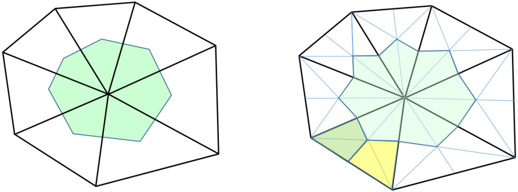


Figure: Left: Voronoi area, Right: Barycentric area.

Lumping the mass matrix $\boldsymbol{M}$ using leads to the famous formula also derived by[3]

$$\boldsymbol{L}(\boldsymbol{v}_i) = \frac{1}{2\,A_i} \sum_{j \in \mathcal{N}(i)} (\cot \alpha_{ij} + \cot \beta_{ij})\,(\boldsymbol{v}_j - \boldsymbol{v}_i)$$

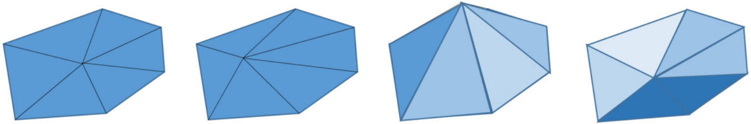
Note that A_i depends on which approach is used, e.g. barycentric cells.

[3] M. Meyer et al. "Discrete differential-geometry operators for triangulated 2-manifolds". In: *Visualization and mathematics* 3.2 (2002), pp. 52–58.

This formula can be also derived using the definition of the Laplace Beltrami Operator used in differential geometry:

$$\Delta_M \boldsymbol{v}_i := \lim_{diam(A)\to 0} -\frac{\nabla A(\boldsymbol{v}_i)}{A(\boldsymbol{v}_i)} = -2\,\kappa_M\,\boldsymbol{n}_i$$

where A is the area of a surface around vertex $\boldsymbol{v}_i$ and is based on the observation that the area around a vertex $\boldsymbol{v}_i$ lying in the same plane as its 1-ring neighbors does not change if the vertex moves within the plane, and can only increase otherwise.



Following the idea of Desbrun et al. (1999), we compute $\nabla A(\boldsymbol{v}_i)$ around the vertex $\boldsymbol{v}_i$.

The area of the one-ring of $\boldsymbol{v}_i$ is given by

$$A(\boldsymbol{v}_i) = \sum_{j \in \mathcal{N}(i)} A_j(\boldsymbol{v}_i)$$

where A_j is the area of tringle T_j given by the vertices $\boldsymbol{v}_i, \boldsymbol{v}_j, \boldsymbol{v}_{j+1}$. Instead of computing ∇A_j directly, we first compute $\frac{\partial A_j^2(\boldsymbol{v}_i)}{\partial \boldsymbol{v}_i}$ and then use $\frac{\partial A_j^2(\boldsymbol{v}_i)}{\partial \boldsymbol{v}_i} = 2\, A_j\, \frac{\partial A_j(\boldsymbol{v}_i)}{\partial \boldsymbol{v}_i}$ to get ∇A_j.

We start with the computation of $A_j^2(\boldsymbol{v}_i)$:

$$A_j^2(\boldsymbol{v}_i) = \frac{1}{4} \left\langle\, (\boldsymbol{v}_j - \boldsymbol{v}_i) \times (\boldsymbol{v}_{j+1} - \boldsymbol{v}_i) \,\middle|\, (\boldsymbol{v}_j - \boldsymbol{v}_i) \times (\boldsymbol{v}_{j+1} - \boldsymbol{v}_i) \,\right\rangle$$

Using the identity $\langle\, \boldsymbol{a} \times \boldsymbol{b} \,|\, \boldsymbol{c} \times \boldsymbol{d} \,\rangle = \langle\, \boldsymbol{a} \,|\, \boldsymbol{c} \,\rangle \langle\, \boldsymbol{b} \,|\, \boldsymbol{d} \,\rangle - \langle\, \boldsymbol{a} \,|\, \boldsymbol{d} \,\rangle \langle\, \boldsymbol{b} \,|\, \boldsymbol{c} \,\rangle$ we get

$$A_j^2(\boldsymbol{v}_i) = \frac{1}{4} \left(\langle\, \boldsymbol{v}_j - \boldsymbol{v}_i \,|\, \boldsymbol{v}_j - \boldsymbol{v}_i \,\rangle \langle\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_i \,|\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_i \,\rangle - \langle\, \boldsymbol{v}_j - \boldsymbol{v}_i \,|\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_i \,\rangle^2 \right)$$

$$A_j^2(\boldsymbol{v}_i) = \frac{1}{4}\Big(\langle\, \boldsymbol{v}_j - \boldsymbol{v}_i \,|\, \boldsymbol{v}_j - \boldsymbol{v}_i \,\rangle \langle\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_i \,|\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_i \,\rangle - \langle\, \boldsymbol{v}_j - \boldsymbol{v}_i \,|\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_i \,\rangle^2\Big)$$

Taking the derivative with respect to $\boldsymbol{v}_i$ we get

$$\begin{aligned}
&\frac{\partial\Big(A_j^2(\boldsymbol{v}_i)\Big)}{\partial \boldsymbol{v}_i} \\
&= \frac{1}{4}\left(-2\left(\boldsymbol{v}_j - \boldsymbol{v}_i\right)\langle\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_i \,|\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_i \,\rangle - 2\left(\boldsymbol{v}_{j+1} - \boldsymbol{v}_i\right)\langle\, \boldsymbol{v}_j - \boldsymbol{v}_i \,|\, \boldsymbol{v}_j - \boldsymbol{v}_i \,\rangle\right. \\
&\qquad \left. +2\,\langle\, \boldsymbol{v}_j - \boldsymbol{v}_i \,|\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_i \,\rangle((\boldsymbol{v}_{j+1} - \boldsymbol{v}_i) + (\boldsymbol{v}_j - \boldsymbol{v}_i))\right) \\
&= \frac{1}{2}\left(\boldsymbol{v}_i\left(\langle\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_i \,|\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_i \,\rangle + \langle\, \boldsymbol{v}_j - \boldsymbol{v}_i \,|\, \boldsymbol{v}_j - \boldsymbol{v}_i \,\rangle - 2\,\langle\, \boldsymbol{v}_j - \boldsymbol{v}_i \,|\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_i \,\rangle\right)\right. \\
&\qquad -\boldsymbol{v}_j\left(\langle\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_i \,|\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_i \,\rangle - \langle\, \boldsymbol{v}_j - \boldsymbol{v}_i \,|\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_i \,\rangle\right) \\
&\qquad \left. -\boldsymbol{v}_{j+1}(\langle\, \boldsymbol{v}_j - \boldsymbol{v}_i \,|\, \boldsymbol{v}_j - \boldsymbol{v}_i \,\rangle - \langle\, \boldsymbol{v}_j - \boldsymbol{v}_i \,|\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_i \,\rangle)\right) \\
&= \frac{1}{2}\left(\boldsymbol{v}_i\,\langle\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_j \,|\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_j \,\rangle\right. \\
&\qquad \left. -\boldsymbol{v}_j\,\langle\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_j \,|\, \boldsymbol{v}_{j+1} - \boldsymbol{v}_i \,\rangle - \boldsymbol{v}_{j+1}\,\langle\, \boldsymbol{v}_j - \boldsymbol{v}_{j+1} \,|\, \boldsymbol{v}_j - \boldsymbol{v}_i \,\rangle\right)
\end{aligned}$$

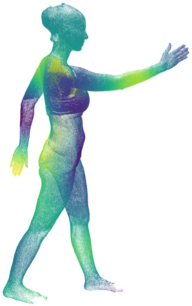
Dividing by $2\,A_j$ we get

$$\begin{aligned}
\frac{1}{2\,A_j}\,\frac{\partial\Big(A_j^2(\boldsymbol{v}_i)\Big)}{\partial \boldsymbol{v}_i} &= \frac{1}{4}\left(\boldsymbol{v}_i\,\frac{\langle\,\boldsymbol{v}_{j+1}-\boldsymbol{v}_j\,|\,\boldsymbol{v}_{j+1}-\boldsymbol{v}_j\,\rangle}{A_j}\right.\\
&\qquad\left.-\boldsymbol{v}_j\,\frac{\langle\,\boldsymbol{v}_{j+1}-\boldsymbol{v}_j\,|\,\boldsymbol{v}_{j+1}-\boldsymbol{v}_i\,\rangle}{A_j}-\boldsymbol{v}_{j+1}\,\frac{\langle\,\boldsymbol{v}_j-\boldsymbol{v}_{j+1}\,|\,\boldsymbol{v}_j-\boldsymbol{v}_i\,\rangle}{A_j}\right)\\
&= \frac{1}{4}\left(\boldsymbol{v}_i\,\frac{w_j^2}{\frac{1}{2}\,w_j\,h_j}-\boldsymbol{v}_j\,\frac{\cos(\beta_{ij})\,w_j\,h_j}{\frac{1}{2}\,w_j\,h_j\sin(\beta_{ij})}-\boldsymbol{v}_{j+1}\,\frac{\cos(\alpha_{ij})\,w_j\,h_j}{\frac{1}{2}\,w_j\,h_j\sin(\alpha_{ij})}\right)\\
&= \frac{1}{2}\left(\boldsymbol{v}_i\left(\cot\alpha_{ij}+\cot\beta_{ij}\right)-\boldsymbol{v}_j\,\cot\beta_{ij}-\boldsymbol{v}_{j+1}\,\cot\alpha_{ij}\right)
\end{aligned}$$

Summing around the one-ring of $\boldsymbol{v}_i$ and dividing by the area A_i corresponding to the vertex $\boldsymbol{v}_i$ delivers the famous formula

$$\Delta_M\boldsymbol{v}_i \approx \frac{1}{A(\boldsymbol{v}_i)}\sum_{j\in\mathcal{N}(i)}-\nabla A_j(\boldsymbol{v}_i) = \frac{1}{2\,A_i}\sum_{j\in\mathcal{N}(i)}\left(\cot\alpha_{ij}+\cot\beta_{ij}\right)\left(\boldsymbol{v}_j-\boldsymbol{v}_i\right)$$

The Laplace operator can also be defined on point clouds[4,5].



- In real data point clouds often have no connectivity (but contain noise and holes)
- **Idea:** Use heat flow to discretize Δ_M

$$\frac{1}{dt} f = \Delta_M f \Rightarrow \Delta_M f \approx + \frac{f(T) - f(0)}{T}$$

- How to get $f(T)$? Convolve the point cloud with a (euclidean) heat kernel $\frac{1}{4\,\pi\,T^2}\, e^{\frac{r^2}{4\,T}}$:

$$\Delta_M f(v) = \lim_{T \to 0} \frac{1}{4\,\pi\,T^2} \left(\int_{\mathcal{M}} e^{\frac{\|\boldsymbol{v}-\boldsymbol{w}\|_2^2}{4\,T}} f(v)\, d\mu_w - \int_{\mathcal{M}} e^{\frac{\|\boldsymbol{v}-\boldsymbol{w}\|_2^2}{4\,T}} f(w)\, d\mu_w \right)$$

[4] M. Belkin et al. "Constructing Laplace operator from point clouds in R d". In: *Proceedings of the twentieth Annual ACM-SIAM Symposium on Discrete Algorithms*. Society for Industrial and Applied Mathematics. 2009, pp. 1031–1040.

[5] Y. Liu et al. "Point-based manifold harmonics". In: *IEEE Transactions on Visualization and Computer Graphics* 18.10 (2012), pp. 1693–1703.

References I

[1] M. Belkin, J. Sun, and Y. Wang. "Constructing Laplace operator from point clouds in R d". In: *Proceedings of the twentieth Annual ACM-SIAM Symposium on Discrete Algorithms*. Society for Industrial and Applied Mathematics. 2009, pp. 1031–1040.

[2] M. Desbrun, M. Meyer, P. Schröder, and A. H. Barr. "Implicit fairing of irregular meshes using diffusion and curvature flow". In: *Proceedings of the 26th annual conference on Computer graphics and interactive techniques*. ACM Press/Addison-Wesley Publishing Co. 1999, pp. 317–324.

[3] G. Dziuk. "Finite elements for the Beltrami operator on arbitrary surfaces". In: *Partial differential equations and calculus of variations*. Springer, 1988, pp. 142–155.

[4] A. Jacobson. "Algorithms and interfaces for real-time deformation of 2d and 3d shapes". PhD thesis. ETH, 2013.

[5] Y. Liu, B. Prabhakaran, and X. Guo. "Point-based manifold harmonics". In: *IEEE Transactions on Visualization and Computer Graphics* 18.10 (2012), pp. 1693–1703.

[6] M. Meyer, M. Desbrun, P. Schröder, A. H. Barr, et al. "Discrete differential-geometry operators for triangulated 2-manifolds". In: *Visualization and mathematics* 3.2 (2002), pp. 52–58.