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Due date: June 25, 2025

Algorithmic Game Theory

Summer Term 2025

Exercise Set 7

If you would like to present one of the solutions in class, please also send an email to rlehming@uni-bonn.de *containing the* ***task*** *which you would like to present and in* ***which of the tutorials*** *you would like to do so. Deadline for the email is Tuesday, 10:00 pm. Please note that the tasks will be alloecated on a first-come-first-served basis, so sending this email earlier than Tuesday evening is highly recommended.*

Exercise 1: (4 Points)

Recall the valuation functions of single-minded bidders from Definition 12.2. Let the maximum bundle size be defined by $d = \max_{i \in \mathcal{N}} |S_i^*|$. Show that in the case of single-minded bidders with maximum bundle size d, item bidding with first price payments is $(\frac{1}{2}, 2d)$-smooth.
Hint: In order to define deviation bids $b_{i,j}^*$, consider a welfare-maximization allocation on v. If bidder i does not get his bundle in the optimal allocation, then define $b_{i,j}^* = 0$ for all items $j \in M$. Otherwise, define $b_{i,j}^* = \frac{v_i}{2d}$ for all $j \in S_i^*$ and $b_{i,j}^* = 0$ if $j \notin S_i^*$. That is, each winner in the optimal allocation equally divides the value for his bundle among all items of the bundle and bids half of it.

Exercise 2: (4 Points)

An *all-pay auction* is a single-item auction defined in almost the same manner as a first-price auction: Each bidder reports a bid $b_i \geq 0$. The bidder with the highest bid wins the item. However, every bidder pays their own bid regardless of whether they win the item or not.
Derive the symmetric Bayes-Nash equilibrium of an all-pay auction with 2 bidders whose valuations are distributed uniformly on the unit interval, i.e. $v_i \sim Unif([0, 1])$ for $i = 1, 2$. You may assume that β is invertible and differentiable.

Exercise 3: (4 Points)

Consider m items and n bidders. We define a generalization of Walrasian equilibria: Let $S = (S_1, \ldots, S_n)$ be an allocation of items to bidders and $q \in \mathbb{R}^m_{\geq 0}$ a price vector. We call the pair (q, S) an ϵ*-approximate Walrasian equilibrium* if unallocated items have price 0, every bidder i has non-negative utility $v_i(S_i) - \sum_{j \in S_i} q_j \geq 0$, and every bidder receives items within ϵ of their favorite bundle, i.e., $v_i(S_i) - \sum_{j \in S_i} q_j \geq v_i(S_i') - \sum_{j \in S_i'} q_j - \epsilon$ for every bundle S_i'.
Prove an approximate version of the First Welfare Theorem: If (q, S) is an ϵ-approximate Walrasian equilibrium, then the social welfare of an optimal allocation S^* cannot surpass the one of S by more than $\min\{m, n\} \cdot \epsilon$.

Exercise 4:

Consider the setting from lecture 18, where we want to sell a single item under limited information (for $v_i \sim \mathcal{D}_i$, we know the distributions $\mathcal{D}_i$ but not the realizations v_i).
We offer the item for a price p. The customers arrive sequentially and buy the item if $v_i \geq p$. Show that if we set p so that $\mathbf{Pr}\left[\text{do not sell item}\right] = \frac{1}{2}$, we attain at least half of the optimal welfare.