Algorithmic Game Theory, Summer 2025 Lecture 1 (4 pages)

Introduction to Congestion Games

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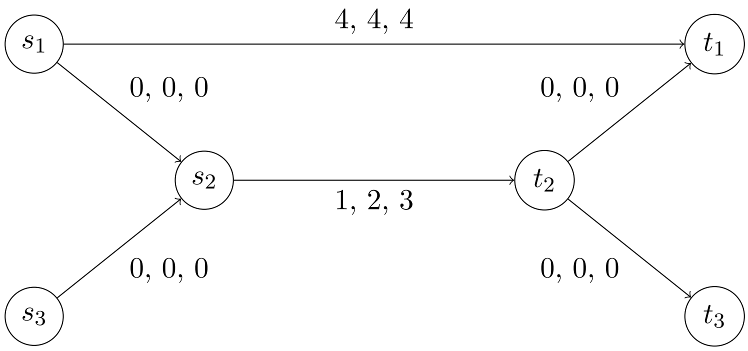
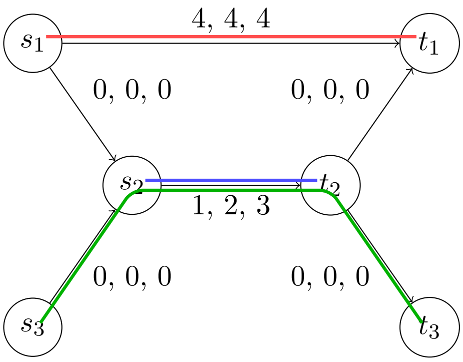
In this lecture, we get to know *congestion games*, which will be our running example for many concepts in game theory. Before coming to the formal definition, let us consider the following example.

We are given the following directed graph; there are three players, who each want to reach their respective destination node from their start node. Edge labels indicate the cost *each* player incurs if this edge is used by one, two, or all three players. So, if the edge label is a, b, c and the edge is used by two players, then each player has cost b for this edge.



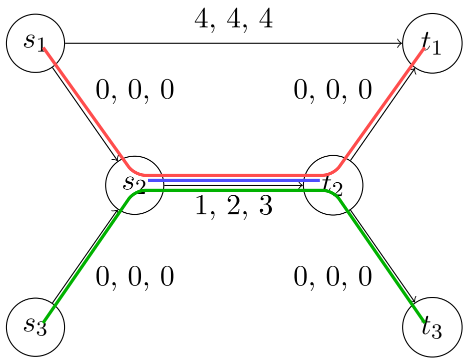
Players 2 and 3 do not have any choice, but player 1 has. He can either use the direct edge or go via s_2 and t_2. That is, we have the following two states.

State A:



social cost: $4 + 2 + 2 = 8$

State B:



social cost: $3 + 3 + 3 = 9$

We observe that State A has a smaller *social cost* than State B. However, player 1 prefers State B because his *individual cost* is smaller there. In contrast to State A, State B is stable because every player is happy with his choice; it is an equilibrium.

We will introduce a general model that allows us to capture these effects. We will ask questions such as: Are there equilibria? How can these equilibria be found? How much performance is lost due to selfishness?

1 Formal Definition

The above example is an example of a *congestion game*, which is generally defined as follows.

Definition 1.1 (Congestion Game (Rosenthal 1973))**.** *A* congestion game *is a tuple* $\Gamma = (N, R, (\Sigma_i)_{i \in N}, (d_r)_{r \in R})$. *The set* $N = \{1, \dots, n\}$ *is a set of players; the set* R, $|R| = m$, *is a set of resources. For each player* $i \in N$, $\Sigma_i \subseteq 2^R$ *denotes the strategy space of player* i. *Every resource* $r \in R$ *has delay function* $d_r \colon \{1, \dots, n\} \to \mathbb{Z}$.

In general, the strategy spaces may be arbitrary subsets of the resources. In the above example, there is much more structure, namely the subsets correspond to paths between sources and sinks in a graph. Such a congestion game is called *network congestion game*.

Example 1.2 (Network Congestion Game)**.** *In a* network *congestion game, there is a graph $G = (V, E)$. The resource set R corresponds to the set of edges E. For each player $i \in N$, there is a dedicated source-sink pair (s_i, t_i) such that Σ_i is the set of paths from s_i to t_i.*

In particular, in the above example

$$N = \{1, 2, 3\} \qquad \textit{and} \qquad R = \{(s_1, t_1), (s_1, s_2), (s_2, t_2), (s_3, s_2), (t_2, t_1), (t_2, t_3)\} \ .$$

Player 1's strategy set is given by $\Sigma_1 = \{\{(s_1, t_1)\}, \{(s_1, s_2), (s_2, t_2), (t_2, t_1)\}\}$. These are two strategies: The first one uses only a single resource/edge, the second one uses three. Players 2 and 3 only have one strategy each.

The delay function of the resource/edge (s_2, t_2) is $d_{(s_2,t_2)}(x) = x$ for all x.

Next, we have to add semantics by formalizing the notion of an individual player's cost.

Definition 1.3. *For any state $S = (S_1, \ldots, S_n) \in \Sigma_1 \times \cdots \Sigma_n$, let $n_r(S) = |\{i \in N \mid r \in S_i\}|$ denote the number of players with $r \in S_i$, that is, who use resource r in S. The delay of resource r in state S is given by $d_r(n_r(S))$. Player i's cost, $i \in N$, is defined to be $c_i(S) = \sum_{r \in S_i} d_r(n_r(S))$. That is, it is the sum of delays of the resources the player uses.*

Example 1.4. *In the above example, there are two different states. We have $n_{(s_2,t_2)}(A) = 2$ and $n_{(s_2,t_2)}(B) = 3$.*

Player 1's cost can be computed as $c_1(A) = d_{(s_1,t_1)}(n_{(s_1,t_1)}(A)) = 4$ in state A and $c_1(B) = d_{(s_1,s_2)}(n_{(s_1,s_2)}(B)) + d_{(s_2,t_2)}(n_{(s_2,t_2)}(B)) + d_{(t_2,t_1)}(n_{(t_2,t_1)}(B)) = 0 + 3 + 0 = 3$.

Now, we are ready for the main definition. Consider a player $i \in N$ and any *fixed* choice of strategies of the other players. The strategies that player i can choose from usually yield different costs. One or multiple minimize the cost. These are called best responses. A pure Nash equilibrium is a state in which each player is choosing such a best response.

Definition 1.5. *A strategy S_i is called a* best response *for player $i \in N$ against a profile of strategies $S_{-i} := (S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n)$ if $c_i(S_i, S_{-i}) \leq c_i(S'_i, S_{-i})$ for all $S'_i \in \Sigma_i$. A state $S \in \Sigma_1 \times \cdots \times \Sigma_n$ is called a* pure Nash equilibrium *if S_i is a best response against the other strategies S_{-i} for every player $i \in N$.*

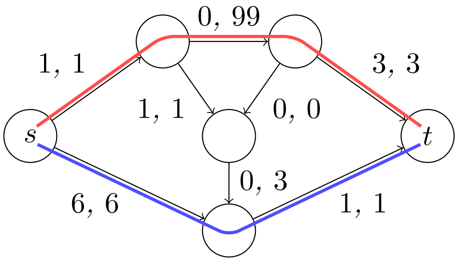
So, in other words, a pure Nash equilibrium is a state in which no player can unilaterally decrease his cost by deviating to a different strategy. It is possible, however, that other strategies have the same cost. Also, equilibria need not be unique.

2 Existence of Pure Nash Equilibria

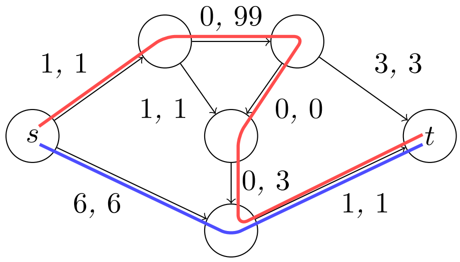
As our first result, we will show that every congestion game has a pure Nash equilibrium. We will talk about *improvement steps*. The pair of states (S, S') is an improvement step if there is some player $i \in N$ such that $c_i(S') < c_i(S)$ and $S'_{-i} = S_{-i}$.

Example 1.6. *A sequence of (best response) improvement steps:*

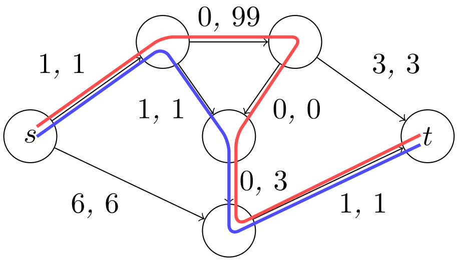
start:



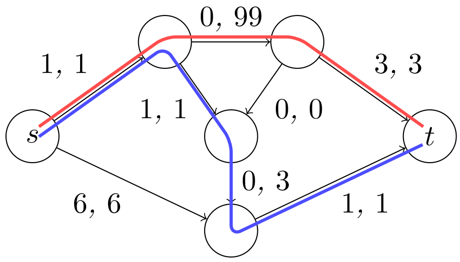
after first improvement (red player):



after second improvement (blue player):



after third improvement (red player):



We will show the following theorem.

Theorem 1.7 (Rosenthal 1973)**.** *For every congestion game, every sequence of improvement steps is finite.*

This property is sometimes also called *finite improvement property*. It immediately implies the following corollary.

Corollary 1.8. *Every congestion game has at least one pure Nash equilibrium.*

The reason is as follows: Start from an arbitrary state $S^{(0)}$ and generate an improvement sequence $S^{(0)}, S^{(1)}, \ldots$. If there is no improvement step $(S^{(t)}, S')$, then $S^{(t)}$ is a pure Nash equilibrium. Otherwise, there is improvement step $(S^{(t)}, S')$ and we can set $S^{(t+1)} = S'$. After only finitely many steps, we have to have reached a pure Nash equilibrium, otherwise we would be generating in infinite sequence of improvement steps.

Proof of Theorem 1.7. Rosenthal's analysis is based on a potential function argument. For every state S, let

$$\Phi(S) = \sum_{r \in R} \sum_{k=1}^{n_r(S)} d_r(k) \enspace .$$

This function is called *Rosenthal's potential function.*

Lemma 1.9. *Let S be any state and let S'_i be an alternative strategy for player i. Then $\Phi(S'_i, S_{-i}) - \Phi(S) = c_i(S'_i, S_{-i}) - c_i(S)$.*

Proof. In order to prove the lemma, we need a better understanding of the potential. To this end, let's write $n_r(S_{-i})$ for the number of players other than i that use resource r. That is, $n_r(S) = n_r(S_{-i}) + 1$ if $r \in S_i$ and $n_r(S) = n_r(S_{-i})$ if $r \notin S_i$. By this definition and using that $\sum_{r \in S_i} d_r(n_r(S)) = c_i(S)$, we have

$$\Phi(S) = \sum_{r \in R} \sum_{k=1}^{n_r(S_{-i})} d_r(k) + \sum_{r \in S_i} d_r(n_r(S)) = \sum_{r \in R} \sum_{k=1}^{n_r(S_{-i})} d_r(k) + c_i(S) \enspace .$$

We can write $\Phi(S'_i, S_{-i})$ analogously as

$$\Phi(S'_i, S_{-i}) = \sum_{r \in R} \sum_{k=1}^{n_r(S_{-i})} d_r(k) + c_i(S'_i, S_{-i}) \ .$$

Note that the first part of the expression for $\Phi(S)$ and $\Phi(S'_i, S_{-i})$ is exactly the same because it is independent of player i's choice. It also has an intuitive meaning: It is the Rosenthal potential of state S_{-i} in a hypothetical game in which player i does not participate. It is completely independent of player i's choice. When taking the difference, the first part cancels and we get that $\Phi(S'_i, S_{-i}) - \Phi(S) = c_i(S'_i, S_{-i}) - c_i(S)$. □

A maybe more intuitive explanation of the proof is as follows: The potential $\Phi(S)$ can be calculated by inserting the players one after the other in any order, and summing the costs of the players at the point of time at their insertion. As this is true for any order, we can consider what happens when player i is inserted last. Then the potential accounted for player i in $\Phi(S)$ corresponds to $c_i(S)$, that is, the cost of player i in state S. When calculating $\Phi(S'_i, S_{-i})$, everything is the same before inserting player i. Now, the potential accounted for player i is $c_i(S'_i, S_{-i})$. So, the difference is exactly $c_i(S'_i, S_{-i}) - c_i(S)$

The lemma shows that Φ is a so-called *exact potential*, i.e., if a single player changes its strategy, making the cost change by a value of Δ, then Φ decreases by exactly the same amount.

Further observe that

(i) the delay values are integers so that, for every improvement step, $c_i(S'_i, S_{-i}) - c_i(S) \leq -1$,

(ii) for every state S, $\Phi(S) \leq \sum_{r \in R} \sum_{i=1}^{n} |d_r(i)|$,

(iii) for every state S, $\Phi(S) \geq -\sum_{r \in R} \sum_{i=1}^{n} |d_r(i)|$.

Consequently, the number of improvements is upper-bounded by $2 \cdot \sum_{r \in R} \sum_{i=1}^{n} |d_r(i)|$ and hence finite. □

Acknowledgments

Parts of these lecture notes are based on earlier versions by Berthold Vöcking, Martin Hoefer and Paul Dütting.

References and Further Reading

- R. W. Rosenthal. A class of games possessing pure-strategy Nash equilibria, Intl. J. Game Theory, 2:65–67, 1973. (Definition of Congestion Games, Existence of Equilibria)