Photogrammetry & Robotics Lab

Relative Orientation, Fundamental and Essential Matrix

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The slides have been created by Cyrill Stachniss.

Winter term 2024 – Cyrill Stachniss

5 Minute Preparation for Today

https://www.youtube.com/watch?v=auhpPoAqprk

Camera Pair

- So far, we computed the camera orientation for a **single camera**
- We are now considering situations in which we have **two images**

Two Common "Camera Pairs"

- A stereo camera
- One camera that moves

Camera pair = two configurations from which an image each has been taken

Orientation Parameters for the Camera Pair and Relative Orientation

Orientation Parameters

- The orientation of the camera pair can be described using two independent orientations

How many parameters are needed?

- Calibrated cameras: **?** parameters (angle preserving mapping)
- Uncalibrated cameras: **?** parameters (straight-line preserving mapping)

Orientation Parameters

- The orientation of the camera pair can be described using two independent orientations

How many parameters are needed?

- Calibrated cameras: **12** parameters (angle preserving mapping)
- Uncalibrated cameras: **22** parameters (straight-line preserving mapping)

Orientation with Control Points

- The orientation of the camera pair can be described using independent orientations for each camera

- Calibrated camera pair: 12 parameters
- Can be computed via two separate spatial resection/P3P steps
- Requires 3(4) known control points

Orientation with Control Points

- The orientation of the camera pair can be described using independent orientations for each camera

- Uncalibrated pair: 22 parameters
- Can be computed via two separate DLT steps
- Requires 6 known control points

Can We Estimate the Camera Orientations WITHOUT Knowing the Scene?

Which Parameters Can We Obtain and Which Not?

Cameras Measure Directions

- We cannot obtain the (global) **translation** and **rotation** of the pair as well as the **scale** of the scene

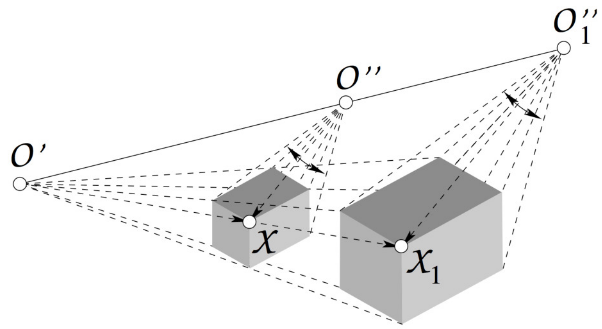


Image courtesy: Förstner & Wrobel

What We Can Compute

- The **rotation** R of the second camera w.r.t. the first one (3 parameters)
- The **direction** B of the line connecting the to centers of projection (2 params)
- We do **not know** their **distance** (the length of B)

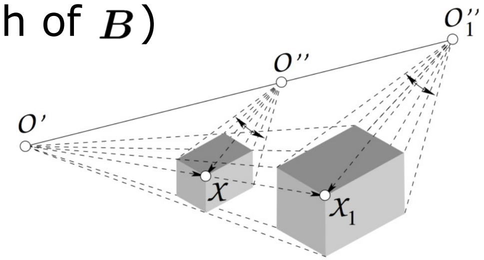


Image courtesy: Förstner & Wrobel

For **Calibrated** Cameras

- We need 2x6=12 parameters for two calibrated cameras for the orientation
- With a calibrated camera, we obtain an angle-preserving model of the object
- Without additional information, we can **only obtain** 12-7 = **5 parameters** (not 7=translation, rotation, scale)

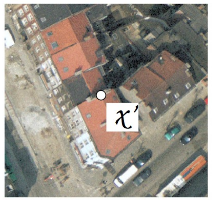
first camera

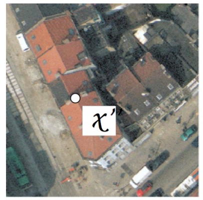
distance between cameras

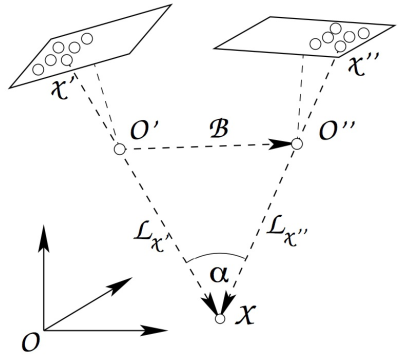
Photogrammetric Model

- Given two images, we can reconstruct the object only **up to a similarity transform**
- Called a **photogrammetric model**
- The **orientation of the photogrammetric model** is called the **absolute orientation**
- For obtaining the absolute orientation, we need at least **3 points** in 3D (to estimate the 7 parameters)

What Is Needed for Computing an **3D Model** of a Scene?







For Uncalibrated Cameras

- **Straight-line preserving** but **not angle preserving**
- Objects can only be reconstructed up to a straight-line preserving mapping
- Projective transform (15 parameters)
- Thus, for uncalibrated cameras, we can only **obtain 22-15=7** parameters given two images
- We need at **least 5 points** in 3D (15 coordinates) for the absolute o.

Relative Orientation Summary

Cameras	#params /img	#params /img pair	#params for RO	#params for AO	min #P
calibrated	**6**	**12**	**5**	**7**	**3**
not calibrated	**11**	**22**	**7**	**15**	**5**

RO = relative orientation

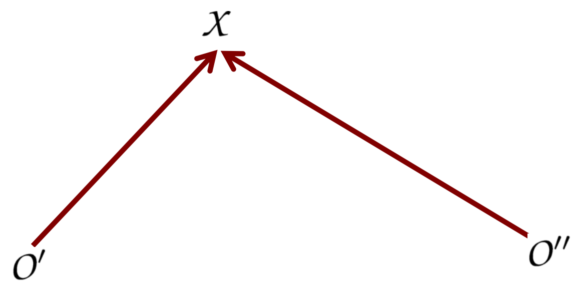
AO = absolute orientation

min #P = min. number of control points

Coplanarity Constraint for Straight-Line Preserving (Uncalibrated) Cameras to Obtain the Fundamental Matrix

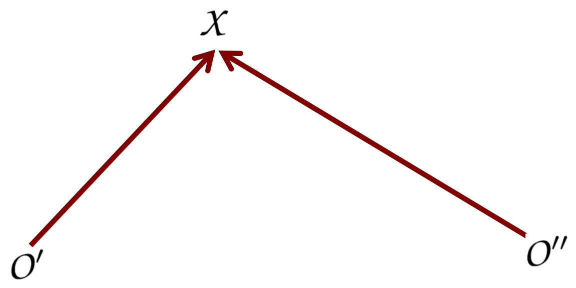
Which Parameters Can We Compute **Without Additional** Information About the Scene?

Start with a perfect orientation and the intersection of two corresponding rays



Coplanarity Constraint

- Consider perfect orientation and the intersection of two corresponding rays
- Rays lie within one plane in 3D

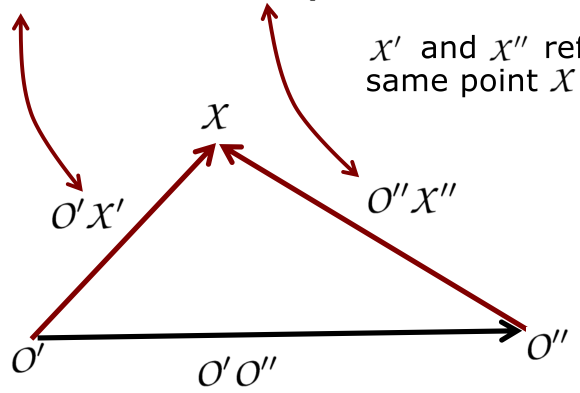


Coplanarity Constraint for Uncalibrated Cameras

Coplanarity can be expressed by

$$[O'X' \quad O'O'' \quad O''X''] = 0$$

x' and x'' refer to the same point X in space



Scalar Triple Product

- The operator $[\cdot,\cdot,\cdot]$ is the triple product
- Dot product of one of the vectors with the cross product of the other two
  $$[A, B, C] = (A \times B) \cdot C$$
- It is the volume of the parallelepiped of three vectors
- $[A, B, C] = 0$ means that the vectors lie in one plane

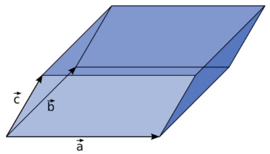
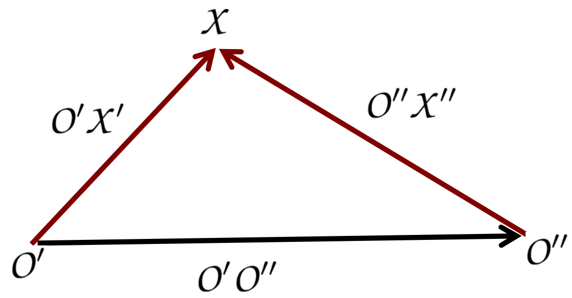


Image courtesy: Wikipedia (Niabot)

Coplanarity Constraint for Uncalibrated Cameras

Coplanarity can be expressed by

$$[O'X' \quad O'O'' \quad O''X''] = 0$$



Coplanarity Constraint for Uncalibrated Cameras

- The directions of the vectors $O'X'$ and $O''X''$ can be derived from the image coordinates $\mathbf{x}', \mathbf{x}''$

$$\mathbf{x}' = \mathsf{P}'\mathbf{X} \qquad \mathbf{x}'' = \mathsf{P}''\mathbf{X}$$

- with the projection matrices

$$\mathsf{P}' = \mathsf{K}'R'[I_3 | -\boldsymbol{X}_{O'}] \qquad \mathsf{P}'' = \mathsf{K}''R''[I_3 | -\boldsymbol{X}_{O''}]$$

Reminder: $[I_3 | -\boldsymbol{X}_{O''}] = \begin{bmatrix} 1 & 0 & 0 & -X_{O''} \\ 0 & 1 & 0 & -Y_{O''} \\ 0 & 0 & 1 & -Z_{O''} \end{bmatrix}$

Directions to a Point

- The normalized directions of the vectors $O''X''$ and $O'X'$ are
  $$\boxed{{}^n\mathbf{x}' = (R')^{-1}(\mathsf{K}')^{-1}\mathbf{x}'} \longleftarrow \text{image coord.}$$
- as the normalized projection
  $${}^n\mathbf{x}' = [I_3 | -\boldsymbol{X}_{O'}]\mathbf{X} \longleftarrow \text{world coord.}$$
- provides the direction to from the center of projection to the point in 3D
- Analogous:
  $${}^n\mathbf{x}'' = (R'')^{-1}(\mathsf{K}'')^{-1}\mathbf{x}''$$

Base Vector

- The base vector $O'O''$ directly results from the coordinates of the projection centers

$$\boxed{\boldsymbol{b} = \boldsymbol{X}_{O''} - \boldsymbol{X}_{O'}}$$

Coplanarity Constraint

- Using the previous relations, the coplanarity constraint

$$[O'X' \quad O'O'' \quad O''X''] = 0$$

- can be rewritten as

$$\begin{aligned} [\,{}^n\mathbf{x}' \quad \boldsymbol{b} \quad {}^n\mathbf{x}''] &= 0 \\ {}^n\mathbf{x}' \cdot (\boldsymbol{b} \times {}^n\mathbf{x}'') &= 0 \\ {}^n\mathbf{x}'^{\mathsf{T}} S_b \, {}^n\mathbf{x}'' &= 0 \end{aligned}$$

skew-symmetric matrix

Derivation

- Why is this correct?

$$\begin{aligned} {}^n\mathbf{x}' \cdot (\boldsymbol{b} \times {}^n\mathbf{x}'') &= 0 \\ {}^n\mathbf{x}'^{\mathsf{T}} S_b \, {}^n\mathbf{x}'' &= 0 \end{aligned}$$

Derivation

- Why is this correct?

$$\begin{aligned} {}^n\mathbf{x}' \cdot (\boldsymbol{b} \times {}^n\mathbf{x}'') &= 0 \\ {}^n\mathbf{x}'^{\mathsf{T}} S_b \, {}^n\mathbf{x}'' &= 0 \end{aligned}$$

- Results from the cross product as

$$\underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}} \times \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} -b_3 x_2 + b_2 x_3 \\ b_3 x_1 - b_1 x_3 \\ -b_2 x_1 + b_1 x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}}_{S_b} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}}$$

- with S_b being a skew-symmetric matrix

Coplanarity Constraint

- By combining ${}^n\mathbf{x}' = (R')^{-1}(\mathsf{K}')^{-1}\mathbf{x}'$ and ${}^n\mathbf{x}'^{\mathsf{T}} S_b \, {}^n\mathbf{x}'' \quad = \quad 0$
- we obtain

$$\mathbf{x}'^{\mathsf{T}}(\mathsf{K}')^{-\mathsf{T}}(R')^{-\mathsf{T}}S_b(R'')^{-1}(\mathsf{K}'')^{-1}\mathbf{x}'' = 0$$

Coplanarity Constraint

- By combining ${}^n\mathbf{x}' = (R')^{-1}(\mathsf{K}')^{-1}\mathbf{x}'$ and ${}^n\mathbf{x}'^{\mathsf{T}} S_b \, {}^n\mathbf{x}'' \quad = \quad 0$
- we obtain

$$\mathbf{x}'^{\mathsf{T}}\underbrace{(\mathsf{K}')^{-\mathsf{T}}(R')^{-\mathsf{T}}S_b(R'')^{-1}(\mathsf{K}'')^{-1}}_{\mathsf{F}}\mathbf{x}'' = 0$$

$$\begin{aligned}\mathsf{F} &= (\mathsf{K}')^{-\mathsf{T}}(R')^{-\mathsf{T}}S_b(R'')^{-1}(\mathsf{K}'')^{-1} \\ &= (\mathsf{K}')^{-\mathsf{T}}R'S_bR''^{\mathsf{T}}(\mathsf{K}'')^{-1}\end{aligned}$$

Fundamental Matrix

- The matrix F is the **fundamental matrix** (for uncalibrated cameras):

$$\boxed{\mathsf{F} \quad = \quad (\mathsf{K}')^{-\mathsf{T}}R'S_bR''^{\mathsf{T}}(\mathsf{K}'')^{-1}}$$

- It allow for expressing the **coplanarity constraint** by

$$\boxed{\mathbf{x}'^{\mathsf{T}}\mathsf{F}\mathbf{x}'' \quad = \quad 0}$$

Fundamental Matrix

- The **fundamental matrix** is the matrix that fulfills the equation

$$\mathbf{x}'^{\mathsf{T}}\mathsf{F}\mathbf{x}'' \quad = \quad 0$$

for corresponding points
- Fundamental matrix contains all the **information** available **about the relative orientation** of **two images** from uncalibrated cameras

Fundamental Matrix From the Camera Projection Matrices

- If the projection matrices are given, we can derive the fundamental matrix?

$$\mathsf{P}', \mathsf{P}'' \rightarrow \mathsf{F}$$

- Let the projection matrices be partitioned into a left 3×3 matrix and a 3-vector as $\mathsf{P}' = [\mathsf{A}'|\mathbf{a}']$.

Fundamental Matrix From the Camera Projection Matrices

- We have $\mathsf{P}' = [\mathsf{A}'|\mathbf{a}'] = [\underbrace{\mathsf{K}'\mathsf{R}'}_{\mathsf{A}'}|\underbrace{-\mathsf{K}'\mathsf{R}'\boldsymbol{X}_{O'}}_{\mathbf{a}'}]$
- and can recover the projection center

$$\mathsf{A}'^{-1}\mathbf{a}' = (\mathsf{K}'\mathsf{R}')^{-1}\mathbf{a}' = -\mathsf{R}'^{\top}\mathsf{K}'^{-1}\mathsf{K}'\mathsf{R}'\boldsymbol{X}_{O'} = -\boldsymbol{X}_{O'}$$

$$\boldsymbol{X}_{O'} = -\mathsf{A}'^{-1}\mathbf{a}'$$

- so that the base line is given by

$$\boldsymbol{b}'_{12} = -\mathsf{A}''^{-1}\mathbf{a}'' + \mathsf{A}'^{-1}\mathbf{a}'$$

Fundamental Matrix From the Camera Projection Matrices

- We have $\mathsf{P}' = [\mathsf{A}'|\mathbf{a}'] = [\underbrace{\mathsf{K}'\mathsf{R}'}_{\mathsf{A}'}|\underbrace{-\mathsf{K}'\mathsf{R}'\boldsymbol{X}_{O'}}_{\mathbf{a}'}]$
- and $\boldsymbol{b}'_{12} = \mathsf{A}'^{-1}\mathbf{a}' - \mathsf{A}''^{-1}\mathbf{a}''$
- and thus can compute F by

$$\mathsf{F} \quad = \quad (\mathsf{K}')^{-\mathsf{T}}R'\mathsf{S}_b R''^{\mathsf{T}}(\mathsf{K}'')^{-1} \quad = \quad \mathsf{A}'^{-\top}\mathsf{S}_{b'_{12}}\mathsf{A}''^{-1}$$

- with $S_b = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}$

Alternative Definition

- In the context of many images, we will call F_{ij} that fundamental matrix which yields the constraint $\mathbf{x}'^{\mathsf{T}}_i \mathsf{F}_{ij}\mathbf{x}''_j = 0$
- Thus in our case, we have $\mathsf{F} = \mathsf{F}_{12}$
- Our definition of F is not the same as in the book by Hartley and Zisserman
- The definition in Hartley and Zisserman is based on $\mathbf{x}''^{\top}_i \mathsf{F}_{ij}\mathbf{x}'_j = 0$, i.e. $\mathsf{F} = \mathsf{F}_{21} = \mathsf{F}^{\mathsf{T}}_{12}$
- The transposition needs to be taken into account when comparing expressions

The Fundamental Matrix Song

Video courtesy: Daniel Wedge
http://danielwedge.com/fmatrix/

Essential Matrix
("F for Calibrated Cameras")

Using Calibrated Cameras

- Most photogrammetric systems rely on calibrated cameras
- Calibrated cameras simplify the orientation problem
- Often, we assume that both cameras have the same calibration matrix
- Assumption here: no distortions or other imaging errors

Coplanarity Constraint

- For calibrated cameras the coplanarity constraint can be simplified
- Based on the calibration matrices, we obtain the **directions** as

$$\underbrace{{}^k\mathbf{x}'}_{\text{direction in camera frame}} = \mathsf{K}'^{-1} \underbrace{\mathbf{x}'}_{\text{coordinates in the image}} \qquad {}^k\mathbf{x}'' = \mathsf{K}''^{-1}\mathbf{x}''$$

- This relation results from

$$\mathbf{x}' = \mathsf{P}'\mathbf{X}' = \mathsf{K}'R'[I_3| - \boldsymbol{X}'_O]\mathbf{X}' = \mathsf{K}'\,{}^k\mathbf{x}'$$

Coplanarity Constraint

- Exploiting the fundamental matrix

$$\begin{aligned} \mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' &= 0 \\ \mathbf{x}'^{\mathsf{T}} \underbrace{(\mathsf{K}')^{-\mathsf{T}} (R')^{-\mathsf{T}} \mathsf{S}_b (R'')^{-1} (\mathsf{K}'')^{-1}}_{\mathsf{F}} \mathbf{x}'' &= 0 \end{aligned}$$

Coplanarity Constraint

- Exploiting the fundamental matrix

$$\begin{aligned} \mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' &= 0 \\ \mathbf{x}'^{\mathsf{T}} \underbrace{(\mathsf{K}')^{-\mathsf{T}} (R')^{-\mathsf{T}} \mathsf{S}_b (R'')^{-1} (\mathsf{K}'')^{-1}}_{\mathsf{F}} \mathbf{x}'' &= 0 \\ \underbrace{\mathbf{x}'^{\mathsf{T}} (\mathsf{K}')^{-\mathsf{T}}}_{{}^k\mathbf{x}'^{\mathsf{T}}} R' \mathsf{S}_b R''^{\mathsf{T}} \underbrace{(\mathsf{K}'')^{-1} \mathbf{x}''}_{{}^k\mathbf{x}''} &= 0 \end{aligned}$$

Coplanarity Constraint

- Exploiting the fundamental matrix

$$\begin{aligned} \mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' &= 0 \\ \mathbf{x}'^{\mathsf{T}} \underbrace{(\mathsf{K}')^{-\mathsf{T}} (R')^{-\mathsf{T}} \mathsf{S}_b (R'')^{-1} (\mathsf{K}'')^{-1}}_{\mathsf{F}} \mathbf{x}'' &= 0 \\ \underbrace{\mathbf{x}'^{\mathsf{T}} (\mathsf{K}')^{-\mathsf{T}}}_{{}^k\mathbf{x}'^{\mathsf{T}}} R' \mathsf{S}_b R''^{\mathsf{T}} \underbrace{(\mathsf{K}'')^{-1} \mathbf{x}''}_{{}^k\mathbf{x}''} &= 0 \\ {}^k\mathbf{x}'^{\mathsf{T}} R' \mathsf{S}_b R''^{\mathsf{T}} \, {}^k\mathbf{x}'' &= 0 \end{aligned}$$

same form as the fundamental matrix but for calibrated cameras

Essential Matrix

- From F to the essential matrix E

$$\begin{aligned} \mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' &= 0 \\ \mathbf{x}'^{\mathsf{T}} \underbrace{(\mathsf{K}')^{-\mathsf{T}} (R')^{-\mathsf{T}} \mathsf{S}_b (R'')^{-1} (\mathsf{K}'')^{-1}}_{\mathsf{F}} \mathbf{x}'' &= 0 \\ \underbrace{\mathbf{x}'^{\mathsf{T}} (\mathsf{K}')^{-\mathsf{T}}}_{{}^k\mathbf{x}'^{\mathsf{T}}} R' \mathsf{S}_b R''^{\mathsf{T}} \underbrace{(\mathsf{K}'')^{-1} \mathbf{x}''}_{{}^k\mathbf{x}''} &= 0 \\ {}^k\mathbf{x}'^{\mathsf{T}} \underbrace{R' \mathsf{S}_b R''^{\mathsf{T}}}_{\mathsf{E}} \, {}^k\mathbf{x}'' &= 0 \\ {}^k\mathbf{x}'^{\mathsf{T}} \mathsf{E} \, {}^k\mathbf{x}'' &= 0 \end{aligned}$$

essential matrix
(DE: Essentielle Matrix)

Essential Matrix

- We derived a **specialization of the fundamental matrix**
- For the calibrated cameras, it is called the **essential matrix**

$$\boxed{\mathsf{E} = R' \mathsf{S}_b R''^{\mathsf{T}}}$$

- We can write the coplanarity constraint for calibrated cameras as

$$\boxed{{}^k\mathbf{x}'^{\mathsf{T}} \mathsf{E}\, {}^k\mathbf{x}'' = 0}$$

Essential Matrix

- The essential matrix as **five** degrees of freedom
- There are **five** parameters that determine the relative orientation of the image pair for calibrated cameras
- This means: there are 4=9-5 constraints to its 9 elements (3x3)
- The essential matrix is homogenous and singular

$$ {}^k\mathbf{x}'^{\mathsf{T}} \mathsf{E}\, {}^k\mathbf{x}'' = 0$$

Popular Parameterizations for the Relative Orientation

Five Parameters – How?

- Five parameters that determine the relative orientation of the image pair

How to parameterize the essential matrix?

The Popular Parameterizations

- Five parameters that determine the relative orientation of the image pair
- Three popular parameterizations

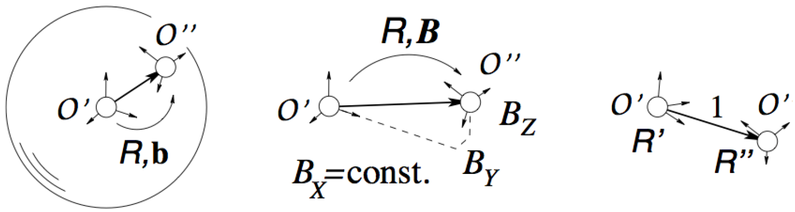


Image courtesy: Förstner

The Popular Parameterizations

1. General parameterization of dependent images
2. Photogrammetric parameterization of dependent images
3. Parameterization of independent images

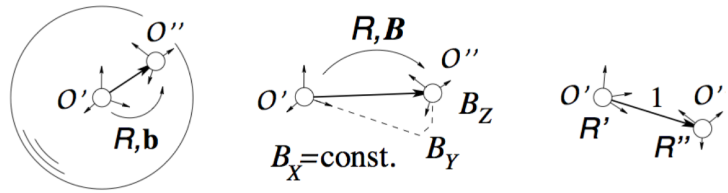


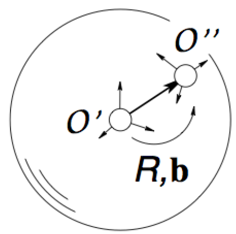
Image courtesy: Förstner

General Parameterization of Dependent Images

The general parameterization of dependent images uses a

- **normalized direction vector** b
- **rotation matrix** R

DE: Allgemeine Parametrisierung des Folgebildanschluss

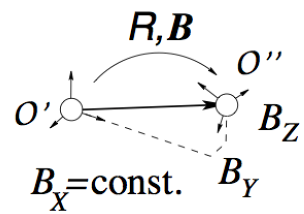


Photogrammetric Parametrizat. of Dependent Images

Photogrammetric parameterization of dependent images uses

- **two components B_Y and B_Z of the base direction (B_X=const)**
- a **rotation matrix** R

DE: Klassich-photogrammetrische Parametrisierung des Folgebildanschluss

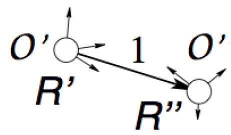


Parameterization of Independent Images

The parameterization with independent images uses

- a **rotation matrix** $R'(\omega', \phi', \kappa')$
- a **rotation matrix** $R''(\omega'', \phi'', \kappa'')$
- a fixed basis of constant length

DE: Parametrisierung mit Bilddrehungen

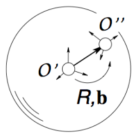


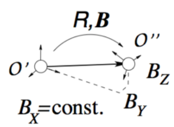
Parameterization of Dependent Images

(DE: Parametrisierung des Folgebildanschluss)

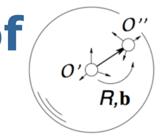
For the Parameterizations of Dependent Images

- The reference frame is the frame of the first camera
- Describe the second camera relative to the first one
- Rotation mat. of the first cam is $R' = I_3$
- The rotation of the R.O. is then $R = R''$





General Parameterization of Dependent Images

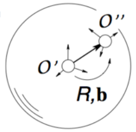


- The orientation of the second camera is $R = R''$ and we obtain from the coplanarity constraint

$$^{k}\mathbf{x}'^{\mathsf{T}} S_b R^{\mathsf{T}} \, ^{k}\mathbf{x}'' = 0 \qquad \text{with} \qquad |\boldsymbol{b}| = 1$$

- 6 parameters + 1 constraint $|\boldsymbol{b}| = 1$

General Parameterization of Dependent Images



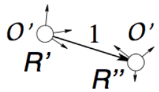
- The resulting 5 degree of freedom are

$$\underbrace{(B_X, B_Y, B_Z}_{\mathbf{b}}, \underbrace{\omega, \phi, \kappa)}_{R} \quad \text{with} \quad B_X^2 + B_Y^2 + B_Z^2 = 1$$

Parameterization of Independent Images

(DE: Parametrisierung mit Bilddrehungen)

Parameterization of Independent Images

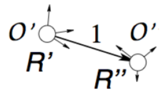


- The center of the reference frame is the projection center $\mathcal{O}'$ of the 1st cam
- The x-axis $\boldsymbol{e}_1^{[3]}$ of the object c.s. is the basis

$$\boldsymbol{B}_r = \begin{bmatrix} B_{X_r} \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{X}_{O''_r} - \boldsymbol{X}_{O'_r}$$

- with $\mathbf{b} = \boldsymbol{B}_r = (B_{X_r}, 0, 0)^{\mathsf{T}},\ B_{X_r} = const.$

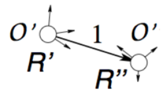
Parameterization of Independent Images



- We have 6 rotation parameters but one rotation around the basis cannot be obtained
- It would result in a change in the exterior orientation of the camera pair
- Thus, one omits the rotation ω' or uses the difference $\Delta\omega = \omega' - \omega''$

$${}^k\mathbf{x}'^{\mathsf{T}} R' S R''^{\mathsf{T}}\, {}^k\mathbf{x}'' = 0 \quad \text{with} \quad \omega', S = const.$$

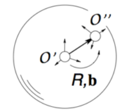
Parameterization of Independent Images

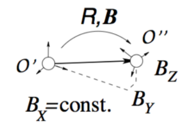


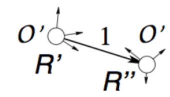
- The resulting 5 parameters are

$$(\Delta\omega, \phi', \kappa', \phi'', \kappa'')$$

Parameterizations Summary

1. General parameterization of dependent images
   

   $(B_X, B_Y, B_Z, \omega, \phi, \kappa)$ with $B_X^2 + B_Y^2 + B_Z^2 = 1$
2. Photogrammetric parameterization of dependent images
   

   $(B_Y, B_Z, \omega, \phi, \kappa)$
3. Parameterization of independent images
   

   $(\Delta\omega, \phi', \kappa', \phi'', \kappa'')$

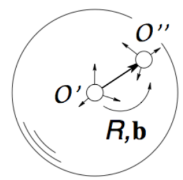
Remark

- Two parameterizations are general and can represent **all geometric configurations**
- The classical photogrammetric parameterization **has a singularity**
- Singularity: If the base vector is directed orthogonal to the X axis, the base components B_Y and B_Z will be infinitely large in general
- This parameterization therefore leads to instabilities

Commonly Used: General Parameterization of Dependent Images

- This **general parameterization** is the **most frequently used one**
- The resulting parameters are

$$(\underbrace{B_X, B_Y, B_Z}_{\mathbf{b}}, \underbrace{\omega, \phi, \kappa}_{R}) \quad \text{with} \quad \underbrace{B_X^2 + B_Y^2 + B_Z^2 = 1}_{(|\mathbf{b}| = 1)}$$



Summary

- Parameters of image pairs
- Relative orientation
- Fundamental matrix F
- Coplanarity constraint $\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' = 0$
- Essential matrix E (F for the calibrated camera pair)
- Coplanarity constraint ${}^k\mathbf{x}'^{\mathsf{T}} \mathsf{E}\, {}^k\mathbf{x}'' = 0$
- Parameterization of the relative orientation

Testing Point Correspondences (optional material)

Correspondence Test for Two Points in the Image Plane

- We can exploit the coplanarity constraint to test for the correspondence of two points
- For correspondence, the residual

$$\begin{aligned} w &= \mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' = \mathrm{vec}(\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'') \\ &= (\mathbf{x}'' \otimes \mathbf{x}')^{\mathsf{T}} \mathrm{vec}\mathsf{F} = (\mathbf{x}'' \otimes \mathbf{x}')^{\mathsf{T}} \mathbf{f} \end{aligned}$$

should be zero (the operator $\otimes$ is the Kronecker product, see next slide)

Kronecker Product

- The Kronecker product is a special product of matrices and defined as

$$A \otimes B = \begin{bmatrix} A_{11}B & \dots & A_{1n}B \\ \dots & \dots & \dots \\ A_{m1}B & \dots & A_{mn}B \end{bmatrix}$$

- Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \otimes \begin{pmatrix} 7 & 8 \\ 9 & 0 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 7 & 8 \\ 9 & 0 \end{pmatrix} & 2\begin{pmatrix} 7 & 8 \\ 9 & 0 \end{pmatrix} \\ 3\begin{pmatrix} 7 & 8 \\ 9 & 0 \end{pmatrix} & 4\begin{pmatrix} 7 & 8 \\ 9 & 0 \end{pmatrix} \\ 5\begin{pmatrix} 7 & 8 \\ 9 & 0 \end{pmatrix} & 6\begin{pmatrix} 7 & 8 \\ 9 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 7 & 8 & 14 & 16 \\ 9 & 0 & 18 & 0 \\ 21 & 24 & 28 & 32 \\ 27 & 0 & 36 & 0 \\ 35 & 40 & 42 & 48 \\ 45 & 0 & 54 & 0 \end{pmatrix}$$

Correspondence Test

- In reality, $w = (\mathbf{x}'' \otimes \mathbf{x}')^\mathsf{T} \mathbf{f}$ is seldom =0
- w has the variance

$$\sigma_w^2 = \left(\frac{\partial w}{\partial \mathbf{x}'}\right) \Sigma_{x'x'} \left(\frac{\partial w}{\partial \mathbf{x}'}\right)^\mathsf{T} + \left(\frac{\partial w}{\partial \mathbf{x}''}\right) \Sigma_{x''x''} \left(\frac{\partial w}{\partial \mathbf{x}''}\right)^\mathsf{T} + \left(\frac{\partial w}{\partial \mathbf{f}}\right) \Sigma_{ff} \left(\frac{\partial w}{\partial \mathbf{f}}\right)^\mathsf{T}$$

- Direct result from the variance propagation (DE: Varianzfortpflanzung)
- Assumes known errors on $\mathbf{x}'$ and $\mathbf{x}''$ and in the elements of F

Correspondence Test

- In reality, $w = (\mathbf{x}'' \otimes \mathbf{x}')^\mathsf{T} \mathbf{f}$ is seldom =0
- w has the variance

$$\sigma_w^2 = \left(\frac{\partial w}{\partial \mathbf{x}'}\right) \Sigma_{x'x'} \left(\frac{\partial w}{\partial \mathbf{x}'}\right)^\mathsf{T} + \left(\frac{\partial w}{\partial \mathbf{x}''}\right) \Sigma_{x''x''} \left(\frac{\partial w}{\partial \mathbf{x}''}\right)^\mathsf{T} + \left(\frac{\partial w}{\partial \mathbf{f}}\right) \Sigma_{ff} \left(\frac{\partial w}{\partial \mathbf{f}}\right)^\mathsf{T}$$

- where

$$\left(\frac{\partial w}{\partial \mathbf{x}'}\right) = \mathbf{x}''^\mathsf{T} \mathsf{F}^\mathsf{T} \qquad \left(\frac{\partial w}{\partial \mathbf{x}''}\right) = \mathbf{x}'^\mathsf{T} \mathsf{F} \qquad \left(\frac{\partial w}{\partial \mathbf{f}}\right) = (\mathbf{x}'' \otimes \mathbf{x}')^\mathsf{T}$$

See: Förstner, Wrobel: Photogrammetric Computer Vision, Chapter 12.2.3 ("The Geometry of the Image Pair")

Correspondence Test

- Given the variance, we can formulate a significance test with

$$z = \frac{w_i}{\sigma_{w_i}} \sim N(0, 1)$$

- where

$$z = \frac{(\mathbf{x}'' \otimes \mathbf{x}')^\mathsf{T} \mathbf{f}}{\sqrt{\mathbf{x}''^\mathsf{T} \mathsf{F}^\mathsf{T} \Sigma_{x'x'} \mathsf{F} \mathbf{x}'' + \mathbf{x}'^\mathsf{T} \mathsf{F} \Sigma_{x''x''} \mathsf{F}^\mathsf{T} \mathbf{x}' + (\mathbf{x}'' \otimes \mathbf{x}')^\mathsf{T} \Sigma_{ff} (\mathbf{x}'' \otimes \mathbf{x}')}}$$

- Note: test value is point-dependent!

Correspondence Test

- Given the variance, we can formulate a significance test with

$$z = \frac{w_i}{\sigma_{w_i}} \sim N(0, 1)$$

- The test allows us to discard a hypothesis if $|z| > k_\alpha$ where k_α defines the threshold for the confidence level, e.g., $k_\alpha = 1.96$ for $\alpha = 5\%$

Summary

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  (F for the calibrated camera pair)
- Coplanarity constraint ${}^{k}\mathbf{x}'^{\mathsf{T}} \mathsf{E}\, {}^{k}\mathbf{x}'' = 0$
- Parameterization of the relative orientation

Literature

- Förstner, Wrobel: Photogrammetric Computer Vision, Ch. 12.2

Slide Information

- The slides have been created by Cyrill Stachniss as part of the photogrammetry and robotics courses.
- I tried to acknowledge all people from whom I used images or videos. In case I made a mistake or missed someone, please let me know.
- The photogrammetry material heavily relies on the very well written lecture notes by Wolfgang Förstner and the Photogrammetric Computer Vision book by Förstner & Wrobel.
- Parts of the robotics material stems from the great Probabilistic Robotics book by Thrun, Burgard and Fox.
- If you are a university lecturer, feel free to use the course material. If you adapt the course material, please make sure that you keep the acknowledgements to others and please acknowledge me as well. To satisfy my own curiosity, please send me email notice if you use my slides.

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