Price of Anarchy

Thomas Kesselheim

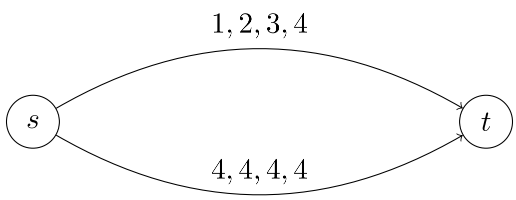
Last Update: May 3, 2025

One of the main goals of algorithmic game theory is to quantify the performance of a system of selfish agents. Usually the "social cost" incurred by all players is higher than if there is a central authority taking charge to minimize social cost. We will develop tools that will allow us to (upper and lower) bound the potential increase.

Here we will define *social cost* as the sum of all players' cost; formally, for a state s let $SC(s) = \sum_{i \in N} c_i(s)$ denote the social cost of s. Sometimes it makes more sense to consider the *maximum* cost incurred by any player.

1 Motivating Example

Example 8.1 (Pigou's Example, Discrete Version)**.** *Consider the following symmetric network congestion game with four players.*



There are five kinds of states:

(a) all players use the top edge, social cost: 16

(b) three players use the top edge, one player uses the bottom edge, social cost: 13

(c) two players use the top edge, two players use the bottom edge, social cost: 12

(d) one player uses the top edge, three players use the bottom edge, social cost: 13

(e) all players use the bottom edge, social cost: 16

Observe that only states of kind (a) and (b) can be pure Nash equilibria. The social cost, however, is minimized by states of kind (c). Therefore, when considering pure Nash equilibria, due to selfish behavior, we lose up to a factor of $\frac{16}{12}$ *and at least a factor of* $\frac{13}{12}$.

More generally, we refer to the worst-case ratio between the social cost at equilibrium and the optimal social cost as the price of anarchy.

Definition 8.2. *Given a cost-minimization game, let* $\mathsf{PNE} \subseteq S$ *be the set of all states that are pure Nash equilibria. The* price of anarchy for pure Nash equilibria *is defined as*

$$PoA_{\mathsf{PNE}} = \frac{\max_{s \in \mathsf{PNE}} SC(s)}{\min_{s \in S} SC(s)} \enspace .$$

2 Tight Bound for Affine Delay Functions

We next provide a tight bound on the price of anarchy for (non-decreasing) *affine delay functions* of the form $d_r(k) = a_r \cdot k + b_r$, where $a_r, b_r \in \mathbb{Z}_{\geq 0}$.

Theorem 8.3. *In every congestion game with affine delay functions, the price of anarchy for pure Nash equilibria is upper bounded by* $\frac{5}{2} = 2.5$*.*

Proof. Let $s \in \mathsf{PNE}$ be a pure Nash equilibrium and let s^* be a state that minimizes social cost. We have to show $SC(s) \leq \frac{5}{2} SC(s^*)$.

Note that, as s is a pure Nash equilibrium, we have $c_i(s) \leq c_i(s_i^*, s_{-i})$. This gives us

$$SC(s) = \sum_{i \in N} c_i(s) \leq \sum_{i \in N} c_i(s_i^*, s_{-i}) \enspace .$$

In the remainder, we will show that

$$\sum_{i \in N} c_i(s_i^*, s_{-i}) \leq \frac{5}{3} \cdot SC(s^*) + \frac{1}{3} SC(s) \enspace . \tag{1}$$

This then implies the desired bound.

By definition, we have

$$c_i(s_i^*, s_{-i}) = \sum_{r \in s_i^*} d_r(n_r(s_i^*, s_{-i})) \enspace .$$

Furthermore, as all d_r are non-decreasing, we have $d_r(n_r(s_i^*, s_{-i})) \leq d_r(n_r(s) + 1)$. This way, we get

$$\sum_{i \in N} c_i(s_i^*, s_{-i}) \leq \sum_{i \in N} \sum_{r \in s_i^*} d_r(n_r(s) + 1) \enspace .$$

By exchanging the sums, we have

$$\sum_{i \in N} \sum_{r \in s_i^*} d_r(n_r(s) + 1) = \sum_{r \in R} \sum_{i : r \in s_i^*} d_r(n_r(s) + 1) = \sum_{r \in R} n_r(s^*) d_r(n_r(s) + 1) \enspace .$$

To simplify notation, we write n_r for $n_r(s)$ and n_r^* for $n_r(s^*)$. Recall that delays are $d_r(n_r) = a_r n_r + b_r$. In combination, we get

$$\sum_{i \in N} c_i(s_i^*, s_{-i}) \leq \sum_{r \in R} n_r^* (a_r(n_r + 1) + b_r) \enspace . \tag{2}$$

This we will have to bound in terms of

$$SC(s^*) = \sum_{r \in R} n_r^* (a_r n_r^* + b_r) \quad \text{and} \quad SC(s) = \sum_{r \in R} n_r (a_r n_r + b_r) \enspace .$$

The following lemma comes to our rescue.

Lemma 8.4 (Christodoulou, Koutsoupias, 2005)**.** *For all integers* $y, z \in \mathbb{Z}_{\geq 0}$ *we have*

$$y(z+1) \leq \frac{5}{3} \cdot y^2 + \frac{1}{3} \cdot z^2 \enspace .$$

Proof. The case $y = 0$ is trivial.

Next, we turn to the case $y = 1$. Note that, as z is an integer, we have $(z-1)(z-2) \geq 0$. Therefore, we have

$$z^2 - 3z + 2 = (z-1)(z-2) \geq 0 \enspace ,$$

which implies

$$z \leq \frac{2}{3} + \frac{1}{3} z^2 \enspace ,$$

and therefore

$$y(z+1) = z + 1 \leq \frac{5}{3} + \frac{1}{3}z^2 = \frac{5}{3}y^2 + \frac{1}{3}z^2 \ .$$

Finally, consider the case $y > 1$. We now use

$$0 \leq \left(\sqrt{\frac{3}{4}}y - \sqrt{\frac{1}{3}}z\right)^2 = \frac{3}{4}y^2 + \frac{1}{3}z^2 - yz \ .$$

Using $y \leq \frac{y^2}{2}$, we get

$$y(z+1) = yz + y \leq \frac{3}{4}y^2 + \frac{1}{3}z^2 + \frac{1}{2}y^2 \leq \frac{5}{3}y^2 + \frac{1}{3}z^2 \ .$$

□

Let us consider the term in Equation (2) for a fixed $r \in R$. We have

$$n_r^*(a_r(n_r+1) + b_r) = a_r n_r^*(n_r+1) + b_r n_r^* \ .$$

Lemma 8.4 implies that

$$n_r^*(n_r+1) \leq \frac{5}{3}(n_r^*)^2 + \frac{1}{3}n_r^2 \ .$$

Thus, we get

$$\begin{aligned} n_r^*(a_r(n_r+1) + b_r) &\leq \frac{5}{3}a_r(n_r^*)^2 + \frac{1}{3}a_r n_r^2 + b_r n_r^* \\ &\leq \frac{5}{3}a_r(n_r^*)^2 + \frac{5}{3}b_r n_r^* + \frac{1}{3}a_r n_r^2 + \frac{1}{3}b_r n_r \\ &= \frac{5}{3}\left(a_r n_r^* + b_r\right) n_r^* + \frac{1}{3}\left(a_r n_r + b_r\right) n_r \ , \end{aligned}$$

where in the second step we used that $b_r \geq 0$. Summing up these inequalities for all resources $r \in R$, we get

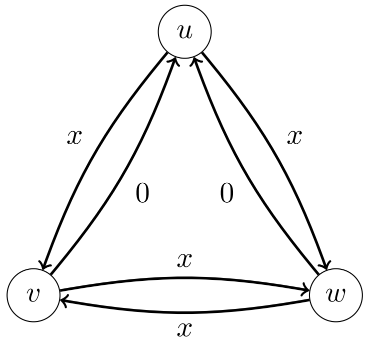
$$\begin{aligned} \sum_{r \in R} n_r^*(a_r(n_r+1) + b_r) &\leq \frac{5}{3}\sum_{r \in R} n_r^*(a_r n_r^* + b_r) + \frac{1}{3}\sum_{r \in R} n_r(a_r n_r + b_r) \\ &= \frac{5}{3} \cdot SC(s^*) + \frac{1}{3} \cdot SC(s) \ , \end{aligned}$$

which shows Equation (1). □

3 Lower Bound

Theorem 8.5. *There are congestion games with affine delay functions whose price of anarchy for pure Nash equilibria is $\frac{5}{2}$.*

Proof sketch. We consider the following (asymmetric) network congestion game. Notation 0 or x on an edge means that $d_r(x) = 0$ or $d_r(x) = x$ for this edge.



There are four players with different source sink pairs. Refer to this table for a socially optimal state of social cost 4 and a pure Nash equilibrium of social cost 10.

player	source	sink	strategy in OPT	cost in OPT	strategy in PNE	cost in PNE
1	u	v	$u \to v$	1	$u \to w \to v$	3
2	u	w	$u \to w$	1	$u \to v \to w$	3
3	v	w	$v \to w$	1	$v \to u \to w$	2
4	w	v	$w \to v$	1	$w \to u \to v$	2

□

4 Other Equilibrium Concepts

To extend the notion of price of anarchy to other equilibrium concepts, we assume that there is a set Eq of probability distributions over the set of states S, which correspond to equilibria. In the case of pure Nash equilibria, each of these distributions concentrates all its mass on a single point. Again we set $SC(s) = \sum_{i \in N} c_i(s)$ but depending on the application it may also make sense to replace the sum by a maximum.

Definition 8.6. *Given a cost-minimization game, let* Eq *be a set of probability distributions over the set of states* S*. For some probability distribution* p*, let* $SC(p) = \sum_{s \in S} p(s) SC(s)$ *be the expected social cost. The* price of anarchy for Eq *is defined as*

$$PoA_{\mathsf{Eq}} = \frac{\max_{p \in \mathsf{Eq}} SC(p)}{\min_{s \in S} SC(s)} \enspace .$$

Given the respective equilibria exist, we have

$$1 \leq PoA_{\mathsf{PNE}} \leq PoA_{\mathsf{MNE}} \leq PoA_{\mathsf{CE}} \leq PoA_{\mathsf{CCE}} \enspace .$$

Example 8.7. *Recall the game* Chicken*, in which two drivers are driving towards an intersection. They can either cross (C) or stop (S). If both cross, they crash and have a high cost.*

	C(ross)	S(top)
C(ross)	100, 100	0, 1
S(top)	1, 0	1, 1

The only two pure Nash equilibria are (C, S) and (S, C), which are both socially optimal. So $PoA_{\mathsf{PNE}} = 1$*. However, there is another, symmetric mixed Nash equilibrium, in which both players cross with* $\frac{1}{100}$ *probability and stop with* $\frac{99}{100}$ *probability. For this probability distribution* p*, we have* $SC(p) = \frac{1}{100} \cdot \frac{1}{100} \cdot (100 + 100) + 2 \cdot \frac{1}{100} \cdot \frac{99}{100} \cdot (1 + 0) + \frac{99}{100} \cdot \frac{99}{100} \cdot (1 + 1) = 1.9801$*. So* $PoA_{\mathsf{MNE}} = 1.9801$*.*

5 Smooth Games

A very helpful technique to derive upper bounds on the price of anarchy in all these equilibrium concepts is *smoothness*.

Definition 8.8. *A game is called* (λ, μ)-smooth *for* $\lambda > 0$ *and* $\mu < 1$ *if, for every pair of states* $s, s^* \in S$*, we have*

$$\sum_{i \in N} c_i(s^*_i, s_{-i}) \leq \lambda \cdot SC(s^*) + \mu \cdot SC(s) \enspace .$$

Observe that this condition needs to hold for *all* states $s, s^* \in S$, as opposed to only pure Nash equilibria or only social optima. We consider the cost that each player incurs when unilaterally deviating from s to his strategy in s^*. If the game is smooth, then we can upper-bound the sum of these costs in terms of the social cost of s and s^*.

Effectively, we already proved the following theorem when we were bounding the price of anarchy for pure Nash equilibria.

Theorem 8.9. *Every congestion game with affine delay functions is* $\left(\frac{5}{3}, \frac{1}{3}\right)$*-smooth.*

From such a bound, getting a bound on the Price of Anarchy is easy: If s is a pure Nash equilibrium and s^* is socially optimal, then

$$\begin{aligned} SC(s) &\leq \sum_{i \in N} c_i(s_i^*, s_{-i}) && \text{(as } s \text{ is a pure Nash equilibrium)} \\ &\leq \lambda \cdot SC(s^*) + \mu \cdot SC(s) && \text{(by smoothness)} \end{aligned}$$

On both sides subtract $\mu \cdot SC(s)$, this gives

$$(1 - \mu) \cdot SC(s) \leq \lambda \cdot SC(s^*)$$

and rearranging yields

$$\frac{SC(s)}{SC(s^*)} \leq \frac{\lambda}{1 - \mu} \ .$$

But the argument does not stop here: Smoothness directly gives a bound even for coarse correlated equilibria.

Theorem 8.10. *In a* (λ, μ)*-smooth game, the PoA for coarse correlated equilibria is at most*

$$\frac{\lambda}{1 - \mu} \ .$$

Proof. Let s be distributed according to a coarse correlated equilibrium p, and let s^* be an optimum solution, which minimizes social cost. Note that $SC(p) = \mathbf{E}_{s \sim p}[SC(s)]$. Then:

$$\begin{aligned} \mathbf{E}_{s \sim p}[SC(s)] &= \sum_{i \in N} \mathbf{E}_{s \sim p}[c_i(s)] && \text{(by linearity of expectation)} \\ &\leq \sum_{i \in N} \mathbf{E}_{s \sim p}[c_i(s_i^*, s_{-i})] && \text{(as } p \text{ is a CCE)} \\ &= \mathbf{E}_{s \sim p}\left[\sum_{i \in N} c_i(s_i^*, s_{-i})\right] && \text{(by linearity of expectation)} \\ &\leq \mathbf{E}_{s \sim p}[\lambda \cdot SC(s^*) + \mu \cdot SC(s)] && \text{(by smoothness)} \end{aligned}$$

On both sides subtract $\mu \cdot \mathbf{E}_{s \sim p}[SC(s)]$, this gives

$$(1 - \mu) \cdot \mathbf{E}_{s \sim p}[SC(s)] \leq \lambda \cdot SC(s^*)$$

and rearranging yields

$$\frac{\mathbf{E}_{s \sim p}[SC(s)]}{SC(s^*)} \leq \frac{\lambda}{1 - \mu} \ .$$ □

That is, in a (λ, μ)-smooth game, we have

$$PoA_{\mathsf{PNE}} \leq PoA_{\mathsf{MNE}} \leq PoA_{\mathsf{CE}} \leq PoA_{\mathsf{CCE}} \leq \frac{\lambda}{1 - \mu} \ .$$

For many classes of games, there are choices of λ and μ such that all relations become equalities. These games are referred to as *tight*. We have already seen one such example: All congestion games with affine cost functions are $(\frac{5}{3}, \frac{1}{3})$-smooth, which implies $PoA_{\mathsf{CCE}} \leq \frac{5}{2}$ but there is an example in which $PoA_{\mathsf{PNE}} = \frac{5}{2}$.

Acknowledgments

Parts of these lecture notes are based on earlier versions by Berthold Vöcking, Martin Hoefer and Paul Dütting.

References and Further Reading

- B. Awerbuch, Y. Azar, A. Epstein. The Price of Routing Unsplittable Flow. STOC 2005. (PoA for pure NE in congestion games)
- G. Christodoulou, E. Koutsoupias. The Price of Anarchy of finite Congestion Games. STOC 2005. (PoA for pure NE in congestion games)
- A. Blum, M. Hajiaghayi, K. Ligett, A. Roth. Regret Minimization and the Price of Total Anarchy. STOC 2008 (PoA for No-Regret Sequences)
- T. Roughgarden. Intrinsic Robustness of the Price of Anarchy. STOC 2009. (Smoothness Framework and Unification of Previous Results)