Foundations of Audio Signal Processing

§2 Complex Numbers

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Lecture at Institut für Informatik, Universität Bonn

Winter Term

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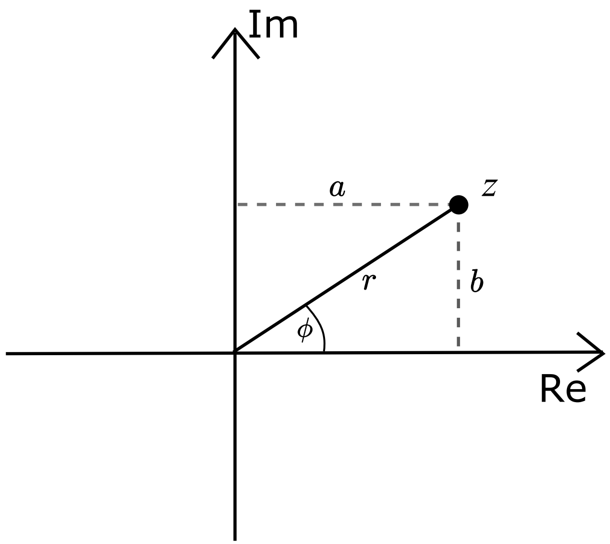

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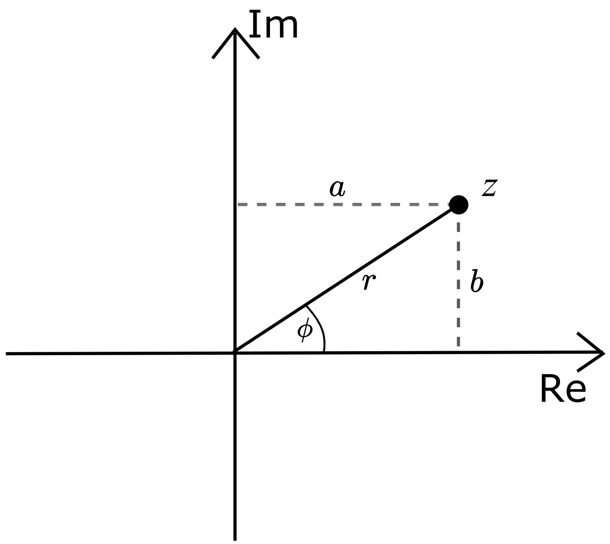

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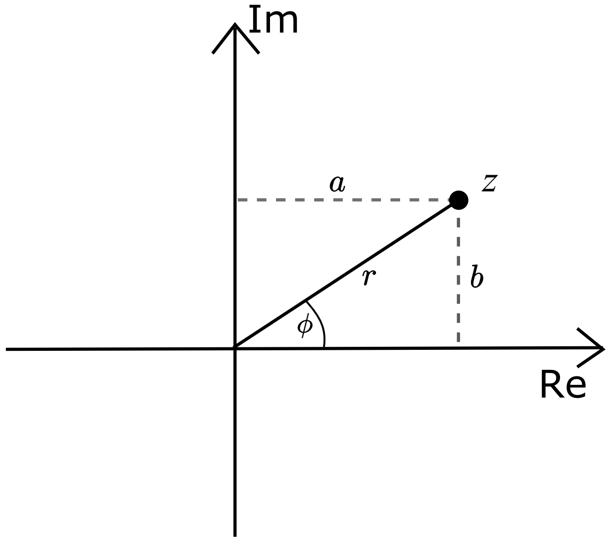

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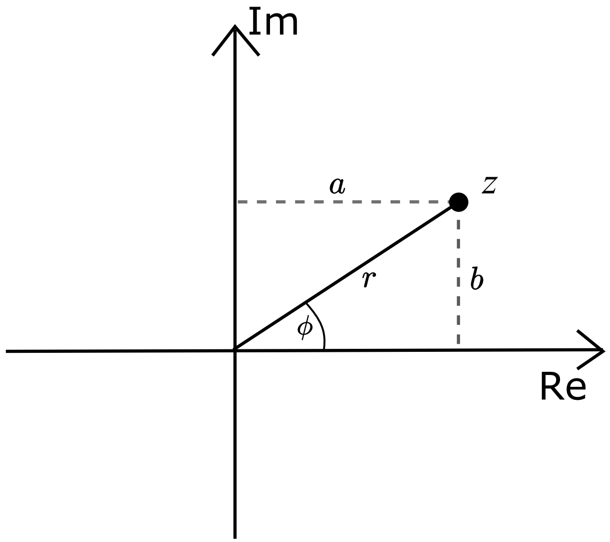

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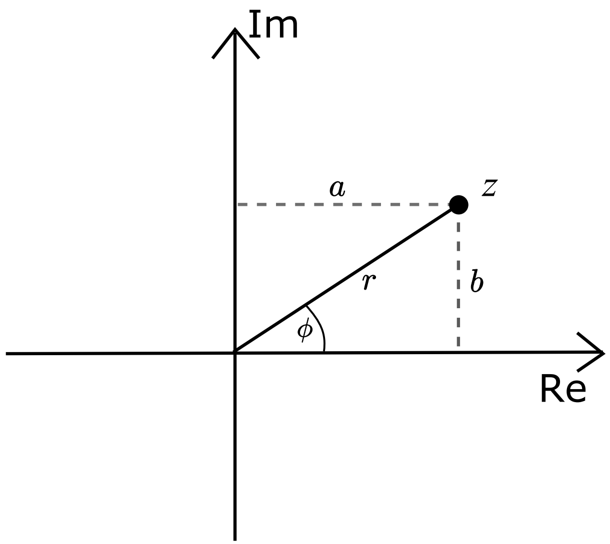

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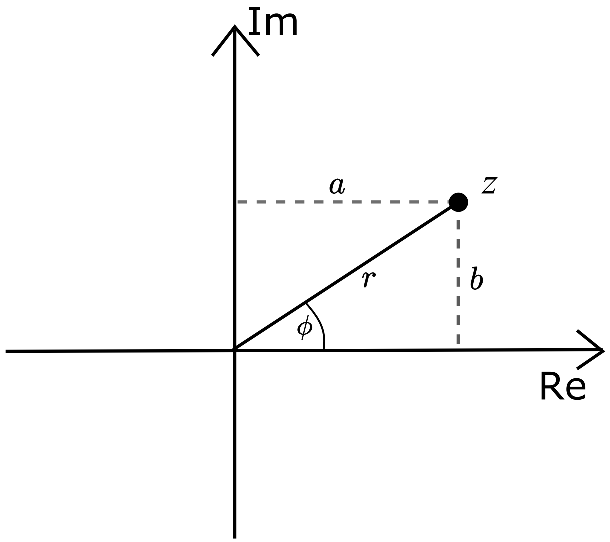

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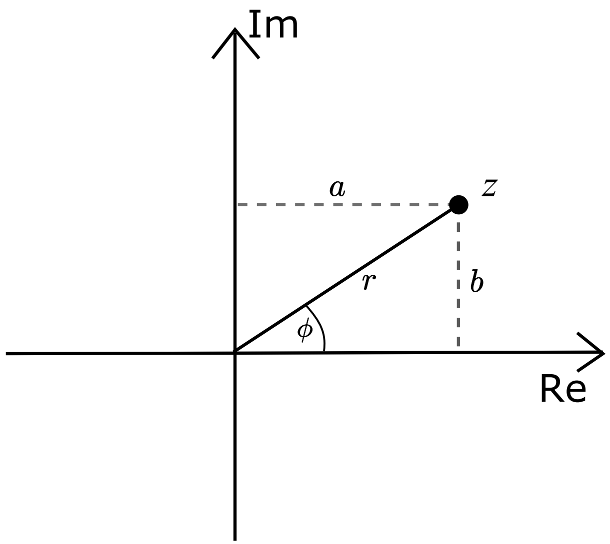

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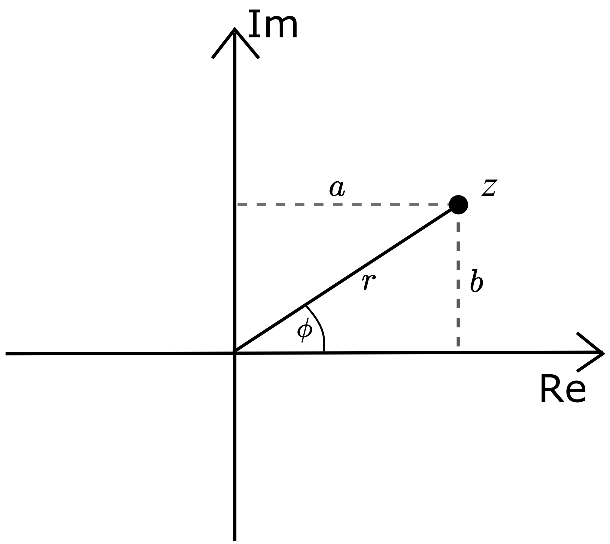

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$$w = \cos\phi + i\sin\phi.$$
- This number ϕ, the **argument** of w and of z, is denoted by $\arg(z) = \arg(w) := \phi$.
- In general, $(|z|, \arg(z))$ is the representation of a non-zero complex number z in **polar coordinates**.

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$$z + w = (a, b) + (c, d) := (a + c, b + d).$$

Similarly, if $z = a + ib$ and $w = c + id$, then

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The unit circle is of great importance in Signal Processing.

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One often uses the shorthand $f : D \to R$ and $f : x \mapsto y$ or $f(x) = y$ in case $(x, y) \in G$.

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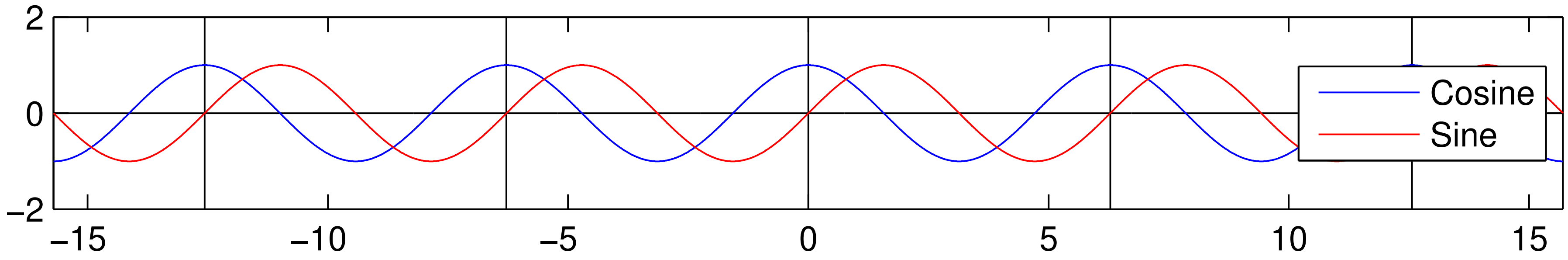

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$$\frac{1}{2i}(e^{iz} - e^{-iz}) = \underbrace{\tfrac{1}{2i}\sum_{n\geq 0}\frac{(iz)^{2n}}{(2n)!}}_{=:A} + \underbrace{\tfrac{1}{2i}\sum_{n\geq 0}\frac{(iz)^{2n+1}}{(2n+1)!}}_{=:B} - \underbrace{\tfrac{1}{2i}\sum_{n\geq 0}\frac{(-iz)^{2n}}{(2n)!}}_{=:C} - \underbrace{\tfrac{1}{2i}\sum_{n\geq 0}\frac{(-iz)^{2n+1}}{(2n+1)!}}_{=:D}.$$

Hence (1) follows from $A = C$ and $B = -D = i \cdot \sum_{n\geq 0} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = i\sin(z)$.

The proof of (2) is similar to that of (1).

Relationships between exp, sin, cos

Theorem. For all $z \in \mathbb{C}$ the following holds:

(1) $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$.
(2) $\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$.
(3) $e^{iz} = \cos(z) + i\sin(z)$ (Euler's formula).
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Euler’s Formula at the Unit Circle

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If z is real, $\sin(z)$ and $\cos(z)$ are real as well. Hence Euler's Formula $e^{iz} = \cos(z) + i\sin(z)$ tells us that $\cos(z)$ is the real part of e^{iz}, whereas $\sin(z)$ is the imaginary part of e^{iz}.

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$$z = |z| \cdot e^{i\arg(z)} = re^{i\phi} = r \cdot (\cos(\phi) + i\sin(\phi)).$$

Euler’s Formula: Addition Theorem

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An alternative formula is $e^{i(\alpha\pm\beta)} = \cos(\alpha \pm \beta) + i\sin(\alpha \pm \beta)$. Comparing real and imaginary parts, our claims follow. □

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Theorem. If $z = |z| \cdot e^{i\arg(z)}$ and $w = |w| \cdot e^{i\arg(w)}$ are two complex numbers, then their product zw has absolute value $|z| \cdot |w|$ and argument $(\arg(z) + \arg(w)) \bmod 2\pi$:

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In other words: If (r, ϕ) and (s, ψ) are the polar representations of two non-zero complex numbers , then their product has the polar representation

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Complex Numbers: Roots of Unity

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- a **primitive** n-**th root of unity** iff $\Omega^n = 1$ and $\Omega^m \neq 1$ for all $m \in [1 : n-1]$.

If C_n denotes the set of all n-th root of unity, then the following is true:

- C_n contains a primitive n-th root of unity Ω_n, e.g., $\Omega_n = e^{\pm 2\pi i/n}$.
- $C_n = \{\Omega_n^m | m \in [0 : n-1]\}$, i.e., C_n consists of all powers of a primitive n-th root of unity.
- Ω_n^k is a primitive n-th root of unity iff $\mathrm{GCD}(k, n) = 1$.

Examples:

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- $C_2 = \{1, -1\}$, $\Omega_2 = -1$,
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Complex Numbers: Roots of Unity

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