Bayes-Nash Equilibria

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In this lecture we will continue our exploration of non-truthful mechanisms. In the last lecture, we analyzed pure Nash equilibria and alike. This has one major weakness. It requires the players to have full information. Essentially, the bidders have to know the other values. Today, we will get to know an equilibrium concept for *incomplete information*. The players know their own values but only have a *prior belief* about the other players' values.

1 Bayes-Nash Equilibria

We will assume that bidder i's value $v_i \in V_i$ is drawn independently from some distribution $\mathcal{D}_i$. These distributions are known to all bidders. A bidder chooses a bid b_i depending on the own valuation v_i, not knowing v_{-i} but only the distributions. We model this by saying that bidder i chooses a *bidding function* $\beta_i\colon V_i \to B_i$, mapping valuations to bids. Whenever the valuation is v_i, the bidder bids $\beta_i(v_i)$. For example, truthful bidding is represented by $\beta_i(v_i) = v_i$.

Definition 16.1 (Bayes-Nash equilibrium)**.** *A* (pure) Bayes-Nash equilibrium *(BNE) is a profile of bidding functions $(\beta_i)_{i\in N}$, $\beta_i\colon V_i \to B_i$, such that for all $i \in N$, all $v_i \in V_i$, and all $b'_i \in B_i$*

$$\mathbf{E}_{v_{-i}\sim\mathcal{D}_{-i}}\left[u_i(\beta(v), v_i)\right] \geq \mathbf{E}_{v_{-i}\sim\mathcal{D}_{-i}}\left[u_i((b'_i, \beta_{-i}(v)), v_i)\right] \quad ,$$

where $\beta(v) = (\beta_1(v_1), \ldots, \beta_n(v_n))$.

So, we take the perspective of a single bidder. She knows her own v_i. The other values $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ are drawn from $\mathcal{D}_1, \ldots, \mathcal{D}_{i-1}, \mathcal{D}_{i+1}, \ldots, \mathcal{D}_n$ respectively. The bidding function now tells her to bid $\beta_i(v_i)$. In an equilibrium, no other bid should give a higher utility. The other bidders keep playing according to the respective bidding functions. This, in particular, means that no other bidding function yields a higher expected utility when also taking the expectation over v_i.

Example 16.2. *In a truthful mechanism, $(\beta_i)_{i\in N}$ with $\beta_i(v_i) = v_i$ for all $i \in N$ and all $v_i \in V_i$ is a Bayes-Nash equilibrium. It is not necessarily the only one.*

Example 16.3. *Consider a first-price auction with two bidders, in which $\mathcal{D}_i$ is the uniform distribution on $[0, 1]$ for $i = 1, 2$. Let us show that $(\beta_i)_{i\in N}$ with $\beta_i(v_i) = \frac{1}{2}v_i$ for all $i \in N$ is a Bayes-Nash equilibrium.*

Observe that for symmetry reasons, it is enough to only consider bidder 1. Fix any $v_1 \in V_1$ and let us write out the expected utility when bidding some arbitrary $b'_1 \in B_1$. The expectation is over bidder 2's value, respectively the bid.

$$\mathbf{E}_{v_2\sim\mathcal{D}_2}\left[u_1((b'_1, \beta_2(v_2)), v_1)\right] = \int_0^1 u_1((b'_1, \beta_2(v_2)), v_1)\mathrm{d}v_2 = \int_0^1 u_1\left(\left(b'_1, \frac{v_2}{2}\right), v_1\right)\mathrm{d}v_2 \ .$$

Here, we used that $\beta_2(v_2) = \frac{v_2}{2}$. Now, what is the value of $u_1\left(\left(b'_1, \frac{v_2}{2}\right), v_1\right)$? If $b'_1 < \frac{v_2}{2}$, then it is 0, if $b'_1 > \frac{v_2}{2}$, then it is $v_1 - b'_1$. Therefore if $b'_1 \leq \frac{1}{2}$ then

$$\mathbf{E}_{v_2\sim\mathcal{D}_2}\left[u_1((b'_1, \beta_2(v_2)), v_1)\right] = \int_0^{2b'_1}(v_1 - b'_1)\mathrm{d}v_2 + \int_{2b'_1}^1 0\mathrm{d}v_2 = 2b'_1(v_1 - b'_1) = \frac{v_1^2}{2} - 2\left(b'_1 - \frac{v_1}{2}\right)^2 \ .$$

We see that that the last term is maximized exactly for $b'_1 = \frac{v_1}{2}$, so for all v_1 and b'_1

$$\mathbf{E}_{v_2\sim\mathcal{D}_2}\left[u_1\left(\left(\frac{v_1}{2}, \beta_2(v_2)\right), v_1\right)\right] \geq \mathbf{E}_{v_2\sim\mathcal{D}_2}\left[u_1((b'_1, \beta_2(v_2)), v_1)\right] \quad ,$$

which is exactly the equilibrium condition.

More generally, one can show that in a first-price auction with n identically distributed bidders with $F(x) = \mathbf{Pr}[v_i \leq x]$, there is a symmetric Bayes-Nash equilibrium with with $\beta_i(v_i) = \frac{n-1}{(F(v_i))^{n-1}} \int_0^{v_i} tF'(t)(F(t))^{n-2}\mathrm{d}t$ (see Section 4). If the distributions are not identical, there is no such simple formula for the equilibrium.

Also for this setting, we can define the Price of Anarchy

$$PoA_{\mathsf{BNE}} = \max_{\text{distribution } \mathcal{D}} \max_{\beta \text{ is BNE for } \mathcal{D}} \frac{\mathbf{E}_{v\sim\mathcal{D}}[OPT(v)]}{\mathbf{E}_{v\sim\mathcal{D}}[SW_v(\beta(v))]}.$$

So, we now consider the worst choice of distributions, and, again, the worst equilibrium. The value $OPT(v)$ is now a random variable, therefore we take its expectation.

2 Bound for First-Price Auction

Our goal is to bound the Price of Anarchy for Bayes-Nash equilibria. Let's first consider a first-price auction.

Theorem 16.4. *In a first-price auction, $PoA_{\mathsf{BNE}} \leq 2$.*

However, we can still derive a guarantee. This is in the spirit of a Price-of-Anarchy bound.

Before we come to the proof for Bayes-Nash equilibria, let us first recap the steps in the full-information setting for pure Nash equilibria. That is, the valuations v and the bids b are fixed now.

It is important to observe that we can write the social welfare $\sum_{i\in N} v_i(f(b))$ also as the sum of utilities and payments: $\sum_{i\in N} v_i(f(b)) = \sum_{i\in N} u_i(b, v_i) + \sum_{i\in N} p_i(b)$.

If bidder i bids $\frac{1}{2}v_i$, then her utility is $\frac{1}{2}v_i$ if she wins the item with this bid, meaning that $\max_{i'\neq i} b_{i'} < \frac{1}{2}v_i$. Otherwise it is 0. So, always the utility is at least $\frac{1}{2}v_i - \max_{i\neq i'} b_{i'}$. Furthermore, the utility is always non-negative under bid $\frac{1}{2}v_i$.

As we are in an equilibrium, $u_i(b, v_i) \geq u_i((\frac{1}{2}v_i, b_{-i}), v_i) \geq \max\{0, \frac{1}{2}v_i - \max_{i'} b_{i'}\}$. Therefore

$$\sum_{i\in N} u_i(b, v_i) + \sum_{i\in N} p_i(b) \geq \max_i \frac{1}{2}v_i - \max_i b_i + \sum_{i\in N} p_i(b) = \frac{1}{2}\max_i v_i \ .$$

Proof of Theorem 16.4. We bound $\mathbf{E}_{v\sim\mathcal{D}}\left[\sum_{i\in N} u_i(\beta(v), v_i)\right]$. To this end, we use that for each bidder for each v_i

$$\mathbf{E}_{v_{-i}\sim\mathcal{D}_{-i}}[u_i(\beta(v), v_i)] \geq \mathbf{E}_{v_{-i}\sim\mathcal{D}_{-i}}\left[u_i\left(\left(\frac{v_i}{2}, \beta_{-i}(v)\right), v_i\right)\right] \ .$$

This holds for every v_i, so it also holds if we draw v_i from $\mathcal{D}_i$ and take this expectation:

$$\mathbf{E}_{v\sim\mathcal{D}}[u_i(\beta(v), v_i)] \geq \mathbf{E}_{v\sim\mathcal{D}}\left[u_i\left(\left(\frac{v_i}{2}, \beta_{-i}(v)\right), v_i\right)\right] \ .$$

And by linearity of expectation, we also get

$$\begin{aligned}
\mathbf{E}_{v\sim\mathcal{D}}\left[\sum_{i\in N} u_i(\beta(v), v_i)\right] &= \sum_{i\in N} \mathbf{E}_{v\sim\mathcal{D}}[u_i(\beta(v), v_i)] \\
&\geq \sum_{i\in N} \mathbf{E}_{v\sim\mathcal{D}}\left[u_i\left(\left(\frac{v_i}{2}, \beta_{-i}(v)\right), v_i\right)\right] \\
&= \mathbf{E}_{v\sim\mathcal{D}}\left[\sum_{i\in N} u_i\left(\left(\frac{v_i}{2}, \beta_{-i}(v)\right), v_i\right)\right] \ .
\end{aligned}$$

For every fixed v, we also have

$$u_i\left(\left(\frac{v_i}{2}, \beta_{-i}(v)\right), v_i\right) \geq \frac{v_i}{2} - \max_{i'} \beta_{i'}(v_{i'}) \quad \text{and} \quad u_i\left(\left(\frac{v_i}{2}, \beta_{-i}(v)\right), v_i\right) \geq 0 \ .$$

This gives us

$$\sum_{i \in N} u_i\left(\left(\frac{v_i}{2}, \beta_{-i}(v)\right), v_i\right) \geq \max_{i \in N} u_i\left(\left(\frac{v_i}{2}, \beta_{-i}(v)\right), v_i\right) \geq \max_{i \in N} \frac{v_i}{2} - \max_{i \in N} \beta_i(v_i) \ .$$

As we are in a first-price auction, $\max_{i \in N} \beta_i(v_i) = \sum_{i \in N} p_i(\beta(v))$, so

$$\sum_{i \in N} u_i\left(\left(\frac{v_i}{2}, \beta_{-i}(v)\right), v_i\right) + \sum_{i \in N} p_i(\beta(v)) \geq \max_{i \in N} \frac{v_i}{2} \ .$$

The rest follows directly by linearity of expectation. □

3 Bound for Any Smooth Mechanism

Now, we will show a bound for any smooth mechanism.

Definition 16.5 (Smooth Mechanism, simplified version)**.** *Let $\lambda, \mu \geq 0$. A mechanism $\mathcal{M} = (f, p)$, $f\colon B \to X$, $p\colon B \to \mathbb{R}^n$, is (λ, μ)-*smooth *if for any valuation profile $v \in V$ for each player $i \in N$ there exists a bid b_i^* such that for any profile of bids $b \in B$ we have*

$$\sum_{i \in N} u_i(b_i^*, b_{-i}) \geq \lambda \cdot \max_{x \in X} \sum_{i \in N} v_i(x) - \mu \sum_{i \in N} p_i(b) \ .$$

Recall that we showed a first-price auction to be $(\frac{1}{2}, 1)$-smooth by setting $b_i^* = \frac{v_i}{2}$. Our proof above used exactly this property. The difficulty in extending the proof to any smooth mechanism is that generally the deviation bid b_i^* does not only depend on v_i but also on the other bidders' valuations v_{-i}. When we show smoothness of the all-pay auction, this is indeed crucial. Interestingly, using a very smart argument, this is not a problem and we can still derive the same bound.

Theorem 16.6. *If a mechanism $\mathcal{M}$ is (λ, μ)-smooth and players have the possibility to withdraw from the mechanism then*

$$PoA_{\mathsf{BNE}} \leq \frac{\max\{\mu, 1\}}{\lambda} \ .$$

Proof. We write out the dependence of b_i^* on v explicitly as $b_i^*(v)$ for this proof.

Let $\tilde{v}$ be any valuation profile. Because $(\beta_i)_{i \in N}$ is a Bayes-Nash equilibrium, bidder i would not prefer to unilaterally switch to strategy $b_i^*(v_i, \tilde{v}_{-i})$, that is

$$\mathbf{E}_{v_{-i}}\left[u_i(\beta(v), v_i)\right] \geq \mathbf{E}_{v_{-i}}\left[u_i((b_i^*(v_i, \tilde{v}_{-i}), \beta_{-i}(v)), v_i)\right]$$

for every v_i and every $\tilde{v}$. This is, in particular, true if $\tilde{v}$ is a random valuation profile, drawn independently from the distributions that v is drawn from. We also take the expectation over v_i to get

$$\mathbf{E}_{v}\left[u_i(\beta(v), v_i)\right] \geq \mathbf{E}_{v,\tilde{v}}\left[u_i((b_i^*(v_i, \tilde{v}_{-i}), \beta_{-i}(v)), v_i)\right] \ . \tag{1}$$

This inequality is not surprising at all: Bidder i would not prefer to bid some other bid, which is only based on some valuation for the other bidders that he imagined.

But in the following step, the magic happens. Note that v_{-i} and $\tilde{v}_{-i}$ are independent and identically distributed. Therefore, the expectation is the same if we swap them

$$\mathbf{E}_{v,\tilde{v}}\left[u_i((b_i^*(v_i, \tilde{v}_{-i}), \beta_{-i}(v)), v_i)\right] = \mathbf{E}_{v,\tilde{v}}\left[u_i((b_i^*(v), \beta_{-i}(\tilde{v})), v_i)\right] \ .$$

Let us have a closer look at what this means. On the left-hand side, we assumed that the bidders except i just follow whatever the Bayes-Nash equilibrium tells them to do with respect to their actual valuations. Bidder i just "hallucinates" some valuations $\tilde{v}_{-i}$ and chooses a bid that is good against this hallucination—which does not have any meaning with respect to the real values v. On the right-hand side, things have flipped. Now, bidder i actually does the right thing against v_{-i} but the other bidders potentially do something strange: They are bidding what the Bayes-Nash equilibrium tells them to do for a *different* valuation profile. The reason why this is true is that u_i does not depend on v_i directly.

Fixing v and $\tilde{v}$, $\beta(\tilde{v})$ is just some bid profile. By the smoothness inequality, we therefore have

$$\sum_{i \in N} u_i((b_i^*(v), \beta_{-i}(\tilde{v})), v_i) \geq \lambda OPT(v) - \mu \sum_{i \in N} p_i(\beta(\tilde{v})) \ .$$

By linearity of expectation, this implies

$$\sum_{i \in N} \mathbf{E}_{v,\tilde{v}} \left[u_i((b_i^*(v), \beta_{-i}(\tilde{v})), v_i) \right] \geq \lambda \mathbf{E}_v \left[OPT(v) \right] - \mu \mathbf{E}_{\tilde{v}} \left[\sum_{i \in N} p_i(\beta(\tilde{v})) \right]$$

and in combination with Equation (1)

$$\sum_{i \in N} \mathbf{E}_v \left[u_i(\beta(v), v_i) \right] \geq \lambda \mathbf{E}_v \left[OPT(v) \right] - \mu \mathbf{E}_{\tilde{v}} \left[\sum_{i \in N} p_i(\beta(\tilde{v})) \right] \ .$$

Now, we use again that v and $\tilde{v}$ are identically distributed, which means that $\mathbf{E}_{\tilde{v}} \left[\sum_{i \in N} p_i(\beta(\tilde{v})) \right] = \mathbf{E}_v \left[\sum_{i \in N} p_i(\beta(v)) \right]$. So

$$\sum_{i \in N} \mathbf{E}_v \left[u_i(\beta(v), v_i) \right] \geq \lambda \mathbf{E}_v \left[OPT(v) \right] - \mu \mathbf{E}_v \left[\sum_{i \in N} p_i(\beta(v)) \right] \ ,$$

which implies because $u_i(b, v_i) = v_i(f(b)) - p_i(b)$

$$\sum_{i \in N} \mathbf{E}_v \left[v_i(f(\beta(v))) \right] \geq \lambda \mathbf{E}_v \left[OPT(v) \right] - (\mu - 1) \mathbf{E}_v \left[\sum_{i \in N} p_i(\beta(v)) \right] \ .$$

If $\mu \leq 1$, then we are done. Otherwise, we use again $u_i(\beta(v_i)) = v_i(f(\beta(v))) - p_i(\beta(v)) \geq 0$ because bidders can withdraw from the mechanism and so $p_i(\beta(v)) \leq v_i(f(\beta(v)))$. This again implies

$$\sum_{i \in N} \mathbf{E}_v \left[v_i(f(\beta(v))) \right] \geq \frac{\lambda}{\mu} \mathbf{E}_v \left[OPT(v) \right] \ ,$$

which is what we claimed. □

References and Further Reading

- Vijay Krishna, Auction Theory, Academic Press. (Book on many aspects of auction theory, including the symmetric Bayes-Nash equilibria of the first-price auction.)
- Vasilis Syrgkanis and Éva Tardos. Composable and Efficient Mechanisms. STOC'13. (Smoothness for mechanisms)

4 Bonus: Deriving Symmetric Bayes-Nash Equilibria of First-Price Auctions

We will derive a generalization of this equilibrium for arbitrary numbers of players n and arbitrary continuous, identical distributions $\mathcal{D}_1, \ldots, \mathcal{D}_n$.

We will assume that for all $i \in N$ and all $x \in \mathbb{R}_{\geq 0}$

$$\mathbf{Pr}\left[v_i \leq x\right] = F(x) = \int_0^x f(t)\mathrm{d}t \ .$$

We also write $G(x)$ for $(F(x))^{n-1}$.

Let us assume that there is a Bayes-Nash equilibrium $(\beta_i)_{i\in N}$ in which all functions are identical and differentiable as well as invertible. Then we have for all $y \in \mathbb{R}_{\geq 0}$

$$\mathbf{E}_{v_{-i}\sim\mathcal{D}_{-i}}\left[u_i((y, \beta_{-i}(v)), v_i)\right] = (v_i - y)\mathbf{Pr}\left[\bigwedge_{j\neq i} \beta_j(v_j) < y\right] = (v_i - y)\prod_{j\neq i}\mathbf{Pr}\left[\beta_j(v_j) < y\right]$$

If we let ϕ denote the inverse of β_i, then, $\mathbf{Pr}\left[\beta_j(v_j) < y\right] = \mathbf{Pr}\left[v_j < \phi(y)\right] = F(\phi(y))$ as $\beta_j = \beta_i$. So we get

$$\mathbf{E}_{v_{-i}\sim\mathcal{D}_{-i}}\left[u_i((y, \beta_{-i}(v)), v_i)\right] = (v_i - y)\prod_{j\neq i} F(\phi(y)) = (v_i - y)G(\phi(y)) \ .$$

If $\beta_i(v_i) = y$, then y has to be a local maximum of the above function. That is

$$\frac{\mathrm{d}}{\mathrm{d}y}(v_i - y)G(\phi(y)) = 0 \ .$$

The derivative can be calculated by standard rules

$$\frac{\mathrm{d}}{\mathrm{d}y}(v_i - y)G(\phi(y)) = -G(\phi(y)) + (v_i - y)G'(\phi(y))\phi'(y) \ .$$

By the inverse function theorem, we have $\phi'(y) = \frac{1}{\beta_i'(\phi(y))}$. That is, if $\beta_i(v_i) = y$ then

$$-G(\phi(y)) + (v_i - y)G'(\phi(y))\frac{1}{\beta_i'(\phi(y))} = 0 \ .$$

Replacing all occurrences of y by $\beta_i(v_i)$ (so $\phi(y) = v_i$), we get

$$-G(v_i) + (v_i - \beta_i(v_i))G'(v_i)\frac{1}{\beta_i'(v_i)} = 0 \ ,$$

or equivalently

$$\beta_i'(v_i)G(v_i) + \beta_i(v_i)G'(v_i) = v_iG'(v_i) \ .$$

This has to hold for all $v_i \in \mathbb{R}_{>0}$. Observe that the left-hand side is exactly the derivative of $\beta_i G$. So, all solutions to this equation have the form

$$\beta_i(v_i)G(v_i) = \int v_iG'(v_i)\mathrm{d}v_i + \text{constant} \ .$$

As $\beta_i(0) = 0$, we have

$$\beta_i(v_i) = \frac{1}{G(v_i)}\int_0^{v_i} tG'(t)\mathrm{d}t \ .$$

One can verify that this is indeed an equilibrium the same way we did this in Example 16.3. And, as we have seen, it is necessarily the only symmetric equilibrium.