DCG, Wintersemester 2023/24 Lecture 13 (5 pages)

Arrangements of hyperplanes

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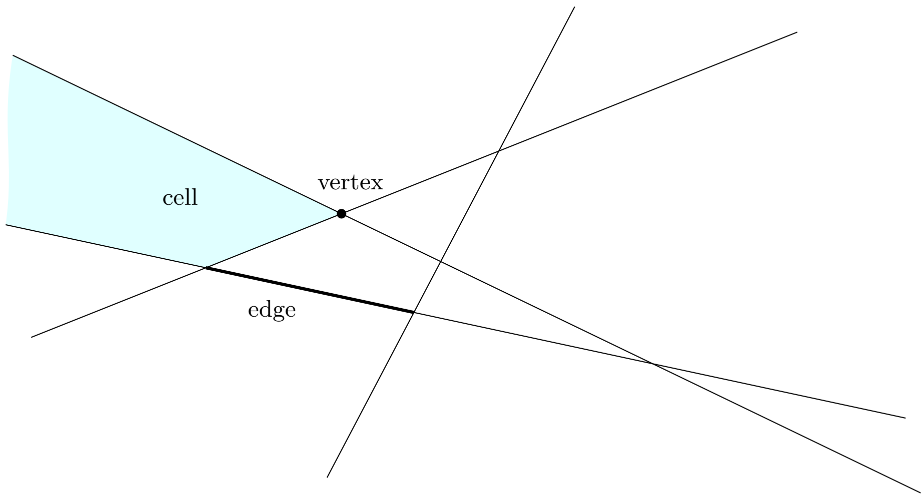
In the previous lecture we have seen that it can be useful to look at the structures formed by a set of hyperplanes. The subdivision that is induced by a finite set of hyperplanes is called an arrangement. In this lecture we want to have a closer look at the combinatorial structure of such arrangement of hyperplanes and analyze the complexity.

1 Arrangements of lines

We first consider the case in the plane, where hyperplanes are lines. Consider a set of n lines L in the plane. They divide the plane into convex subsets: The intersections of the lines are the *vertices* of the arrangement. By removing all the vertices lying on a line $\ell \in L$, the line is split into two unbounded rays and several segments. These are the *edges*. By removing all vertices and lines, the plane is divided into open convex polygons called the *cells*.



Lemma 13.1. *An arrangement of n lines in general position (no three lines intersect in a common point, no two lines are parallel) has $\binom{n}{2}$ vertices, n^2 edges, and $\binom{n}{2} + n + 1$ cells.*

Proof. By our general position assumption, we have that any two lines intersect in a unique point, so we have $\binom{n}{2}$ vertices. Any line is subdivided into n connected components by the intersections with the other $(n-1)$ lines, of these are $n-2$ bounded and 2 unbounded. For analyzing the number of cells (2-dimensional faces), consider removing a line from an arrangement of n lines. The removal decreases the number of cells by n, since every edge on the line is incident to exactly two cells, which are now merged into one. Therefore, we get for the total number of cells the recurrence

$$f(n) = f(n-1) + n, \quad f(0) = 1$$

which we can solve by iteration to $f(n) = 1 + 1 + 2 + 3 + \cdots + n$. Thus, the number of cells is equal to $\frac{n(n+1)}{2} + 1$, which is equal to the number in the theorem. $\square$

An alternative proof for the number of cells uses Steinitz theorem and Euler's formula for polytopes. Assume we connect all infinite edges to an additional vertex placed at infinity. Euler's formula states for number of edges e, faces f and vertices v, that

$$v + f - e = 2$$

Using $v = \binom{n}{2} + 1$ and $e = n^2$, as derived above, we get for the number of cells

$$f = 2 - \left(\binom{n}{2} + 1\right) + n^2 = 1 - \left(\frac{n^2 - n}{2}\right) + n^2 = \left(\frac{n^2 + n}{2}\right) + 1$$

which yields the same bound.

2 Arrangements of hyperplanes

Let H be a set of n hyperplanes in $\mathbb{R}^d$. The arrangement of H is a partition of $\mathbb{R}^d$ into convex subsets of dimensions 0 through d. We call 0-faces *vertices*, 1-faces *edges*, and d-faces *cells*. The cells are the connected components of

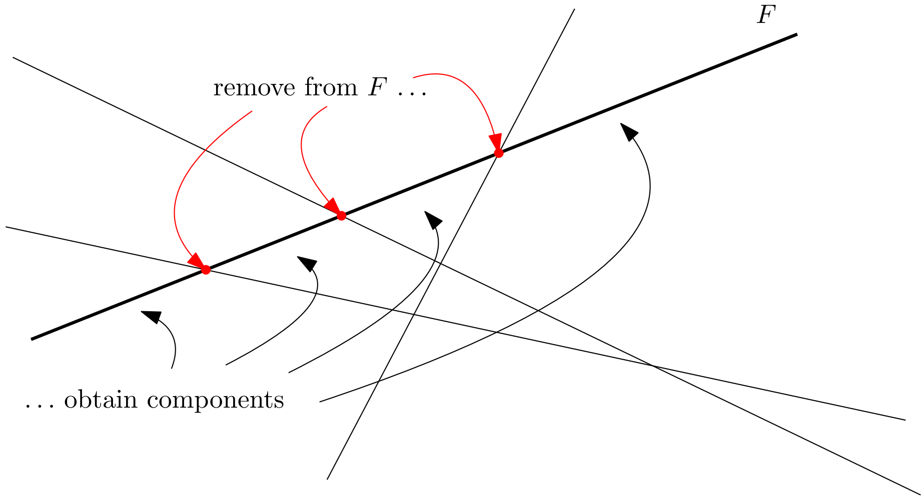
$$\mathbb{R}^d \setminus \bigcup_{h \in H} h$$

To obtain the k-faces of the arrangement, for $0 \leq k < d$, we consider every possible k-flat F defined as the intersection of some $(d - k)$ hyperplanes of H. The k-faces of the arrangement lying within F are the connected components of

$$F \setminus \bigcup_{h \in H \setminus H_F} h$$

where

$$H_F = \{\ h \in H \ \mid\ F \subseteq h\ \}$$



3 Sign vectors

Let H be a set of n hyperplanes in $\mathbb{R}^d$. We can describe the faces of the arrangement of H using sign vectors, which can be defined as follows. Any hyperplane h is of the form

$$h = \left\{ x \in \mathbb{R}^d \;\middle|\; \langle a, x \rangle = b \right\}$$

for some $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$, and partitions $\mathbb{R}^d$ into three regions:

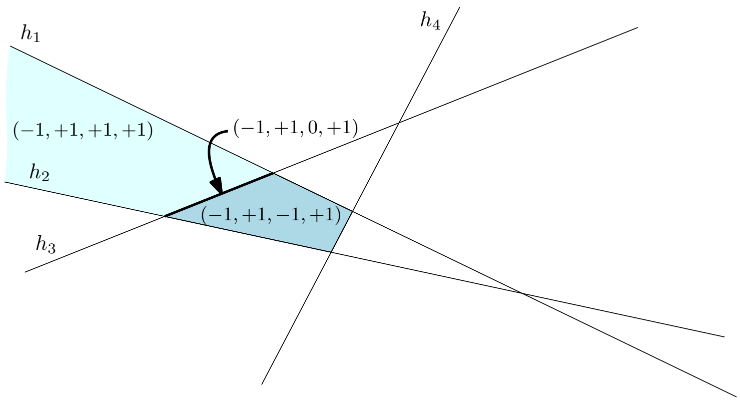
(i) h itself,
(ii) the halfspace $h_- = \left\{ x \in \mathbb{R}^d \;\middle|\; \langle a, x \rangle < b \right\}$, and
(iii) the halfspace $h_+ = \left\{ x \in \mathbb{R}^d \;\middle|\; \langle a, x \rangle > b \right\}$.

A face f the arrangement can be described by its sign vector, denoted with $\sigma(f)$, which is an element of $\{-1, 0, +1\}^n$, where every component is associated with a hyperplane $h \in H$ and is defined as

$$\sigma_h = \begin{cases} +1 & \text{if } F \subseteq h_+ \\ 0 & \text{if } F \subseteq h \\ -1 & \text{if } F \subseteq h_- \end{cases}$$

The sign vector determines the face f as follows, it holds $f = \bigcap_{h \in H} h^{\sigma_h}$, where we denote $h^0 = h, h^{+1} = h_+$ and $h^{-1} = h_-$.

Note that not every sign vector corresponds to a nonempty face. There are 3^n sign vectors for an arrangment of n hyperplanes, but the number of faces of an arrangement of lines is usually smaller, for example in the plane there are $O(n^2)$ faces.



4 Number of cells

We first specify our general position assumption for a set of hyperplanes in $\mathbb{R}^d$.

Definition 13.2 (Simple Arrangement). *A set of hyperplanes is in general position if*
(i) the intersection of every k hyperplanes is $(d-k)$-dimensional, for $k = 2, \ldots, d$, and
(ii) the intersection of every $d+1$ hyperplanes is empty.
We call such an arrangement simple.

Every d-tuple of hyperplanes in a simple arrangement determines exactly one vertex, and so a simple arrangement of n hyperplanes has exactly $\binom{n}{d}$ vertices. We are now interested in the number of cells (d-faces), the following theorem gives the exact answer.

Theorem 13.3. *The number of cells (d-faces) in a simple arrangement of n hyperplanes in* $\mathbb{R}^d$ *equals*

$$\Phi_d(n) = \sum_{i=0}^{d} \binom{n}{i}$$

Proof. We use an induction on $n + d$. We have two base cases:
 (i) $d = 1, n \geq 1$
 (ii) $n = 1, d \geq 1$

In case (i), an arrangement of n hyperplanes corresponds to n points on a line. We have $n + 1$ cells, and we can write

$$\Phi_1(n) = \binom{n}{0} + \binom{n}{1} = n + 1$$

In case (ii), we have only one $(d-1)$-dimensional hyperplane in $\mathbb{R}^d$ which subdivides the space into two cells, and we can write

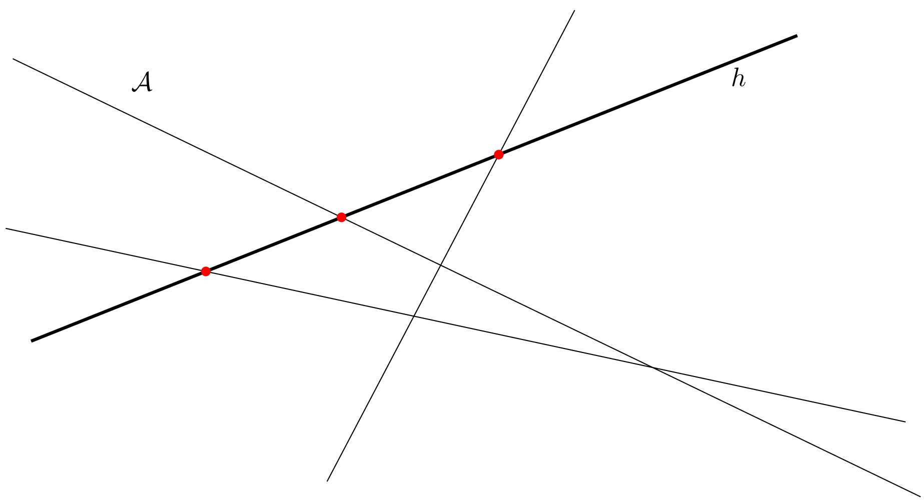
$$\Phi_d(1) = \binom{1}{0} + \binom{1}{1} + \dots \binom{1}{d} = 2$$

This proves the statement for these two base cases.

Now assume we have a set H of n hyperplanes in $\mathbb{R}^d$. Fix one hyperplane $h \in H$ and consider the arrangement $\mathcal{A}$ of $H \setminus \{h\}$. By induction, the number of (d-dimensional) cells is

$$\Phi_d(n-1) = \binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{d}$$

Now consider the intersection of h with $\mathcal{A}$.



The $(n-1)$ hyperplanes in $H \setminus \{h\}$ divide h into a set of $(d-1)$-faces. Let this set of faces be denoted with $\mathcal{F}$.

By induction, we have

$$|\mathcal{F}| = \Phi_{d-1}(n-1) = \binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{d-1}$$

Every $(d-1)$-face in $\mathcal{F}$ partitions one d-face of $\mathcal{A}$ into two d-faces (cells). Therefore, the total increase in the number of cells in the arrangement when adding the nth hyperplane is $|\mathcal{F}|$. So, the number of cells in the resulting arrangement is equal to

$$\Phi_d(n-1) + \Phi_{d-1}(n-1)$$

Now we use the recurrence for binomial coefficients:

$$\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1}$$

We simplify the expression, as follows

$$\begin{aligned}
\Phi_d(n-1) + \Phi_{d-1}(n-1) &= \sum_{i=0}^{d} \binom{n-1}{i} + \sum_{i=0}^{d-1} \binom{n-1}{i} \\
&= \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{d} \\
&\qquad\qquad + \binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{d-1} \\
&= \binom{n-1}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{d} \\
&= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{d} = \Phi_d(n)
\end{aligned}$$

□

5 Number of k-faces

What is the number of k-faces in a simple arrangment of n hyperplanes in $\mathbb{R}^d$? For $k = d-1$ we can consider each hyperplane $h \in H$ separately and we get $\Phi_{d-1}(n-1)$ faces of dimension $d-1$, which are induced by the $n-1$ remaining hyperplanes. In total, we get

$$n \cdot \Phi_{d-1}(n-1) \in O(n^d)$$

In general, we can look at the $(d-k)$-faces. Each such face lies at the intersection of k hyperplanes of H, if we fix a k-tuple of hyperplanes of H, we get $\Phi_{d-k}(n-k)$ faces of dimension $(d-k)$, which are induced by the $n-k$ remaining hyperplanes. In total, we get

$$\binom{n}{k} \cdot \Phi_{d-k}(n-k) \in O(n^d)$$

Therefore, the total number of faces of an arrangment of hyperplanes in $\mathbb{R}^d$ is in $O(n^d)$, for any fixed d.

References

- Jiří Matoušek, Chapter 6.1, Lectures on Discrete Geometry, Springer Graduate Texts in Mathematics.