DCG, Wintersemester 2024/25 Lecture 12 (6 pages)

Higher-dimensional Voronoi Diagrams

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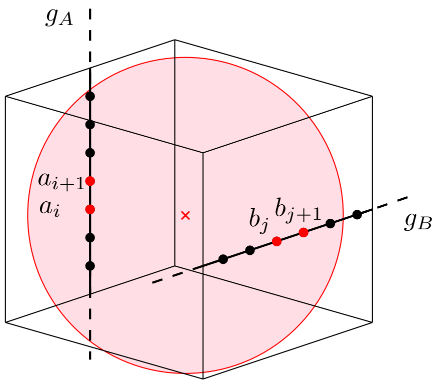
In this lecture we want to study the complexity of Voronoi diagrams for finite sets of points in $\mathbb{R}^d$. We will first look at the case $d = 2$ and $d = 3$. For higher dimensions we will see that there exists a surprising relationship of Voronoi diagrams in $\mathbb{R}^d$ and convex polytopes in $\mathbb{R}^{d+1}$, which allows us to bound their complexity.

1 Complexity of Voronoi diagrams in $\mathbb{R}^2$ and $\mathbb{R}^3$

What is the *complexity* of a Voronoi diagram of a set of n points in $\mathbb{R}^2$? For points that lie in the plane, we define this complexity as the sum of the total number of Voronoi edges and Voronoi vertices. We know that the Delaunay triangulation has a linear number of edges and faces. Since every Voronoi edge corresponds to a unique edge of the Delaunay triangulation, and every Voronoi vertex corresponds to a unique face of the Delaunay triangulation, it must be that the number of Voronoi edges and vertices is also linear in n.

So what about points in higher dimensions? Here, we define the complexity as the total number of faces (of all dimensions) of the Voronoi regions. We will first look at $d = 3$ and show that the complexity can be quadratic in n.

For this let g_A and g_B be two lines in $\mathbb{R}^3$ that do not lie in the same plane. Let $A = \{a_1, \ldots, a_m\}$ be a set of $m = \lceil \frac{n}{2} \rceil$ points on g_A and let $B = \{b_1, \ldots, b_{m'}\}$ be a set of $m' = \lfloor \frac{n}{2} \rfloor$ points on g_B. We assume that the points are ordered along the line that contains them. That is, we assume that between any two points a_i and a_{i+1} there is no other point of A on g_A and, similarly, between any two points b_i and b_{i+1} there is no other point of B on g_B.



Now, for any tuple $(i, j) \in \{1, \ldots, m-1\} \times \{1, \ldots, m'-1\}$ consider the ball that has a_i, a_{i+1}, b_j and b_{j+1} on its boundary. Since the four points are affinely independent, they determine a unique ball in R^3.

By our assumption on the ordering of the points the ball does not contain any other points of $A \cup B$. Therefore, the center of the ball lies at the intersection of the four Voronoi regions $\mathrm{reg}(a_i)$, $\mathrm{reg}(a_{i+1})$, $\mathrm{reg}(b_j)$, $\mathrm{reg}(b_{j+1})$ in the Voronoi diagram of the set $A \cup B$. It is thus a Voronoi vertex of this diagram. This implies that the Voronoi diagram has at least $(m-1)(m'-1) \in \Omega(n^2)$ different Voronoi vertices.

Note that the constructed set of points does not lie in general position, however, one can easily believe, that the construction can be slightly modified to ensure general position without decreasing the number of Voronoi vertices. This leads to the general question, if we can show an upper bound on the total number of faces of all dimensions of the Voronoi regions in a Voronoi diagram of n points.

2 Lifting to paraboloid in $d+1$ dimensions

To show a relationship between Voronoi diagrams in $\mathbb{R}^d$ and polytopes $\mathbb{R}^{d+1}$, we define a lifting to the unit paraboloid in $\mathbb{R}^{d+1}$. Let

$$U = \left\{ x \in \mathbb{R}^{d+1} \;\middle|\; x_{d+1} = \sum_{i=1}^{d} x_i^2 \right\}.$$

Let $f : \mathbb{R}^d \mapsto \mathbb{R}$ be defined as

$$f(x) = x_1^2 + \cdots + x_d^2,$$

and define the mapping $u : \mathbb{R}^d \mapsto \mathbb{R}^{d+1}$ as $u(x) = (x_1, \ldots, x_d, f(x))$

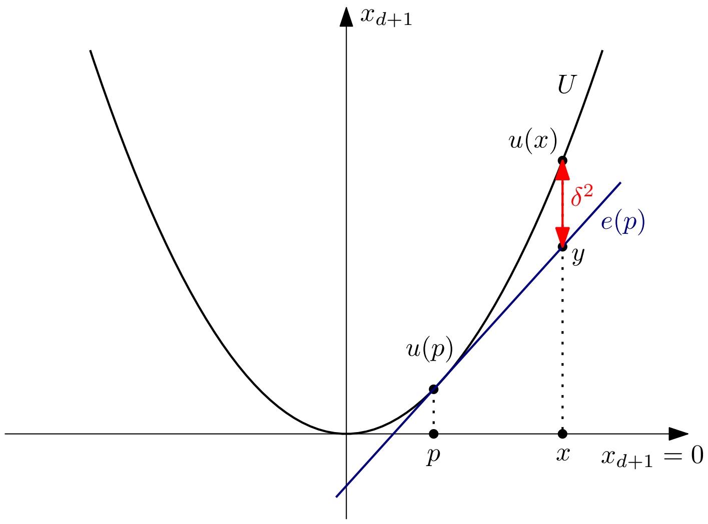
Intuitively, we can think of the space $\mathbb{R}^d$ as the hyperplane with equation $x_{d+1} = 0$ and we can think of the function u as a "lifting map" to the paraboloid in $\mathbb{R}^{d+1}$. Regard the axis corresponding to x_{d+1} as vertical.

For a point $p = (p_1, \ldots, p_d) \in \mathbb{R}^d$ let $e(p)$ denote the hyperplane in $\mathbb{R}^{d+1}$ with equation

$$x_{d+1} = 2p_1x_1 + \cdots + 2p_dx_d - p_1^2 - \cdots - p_d^2$$

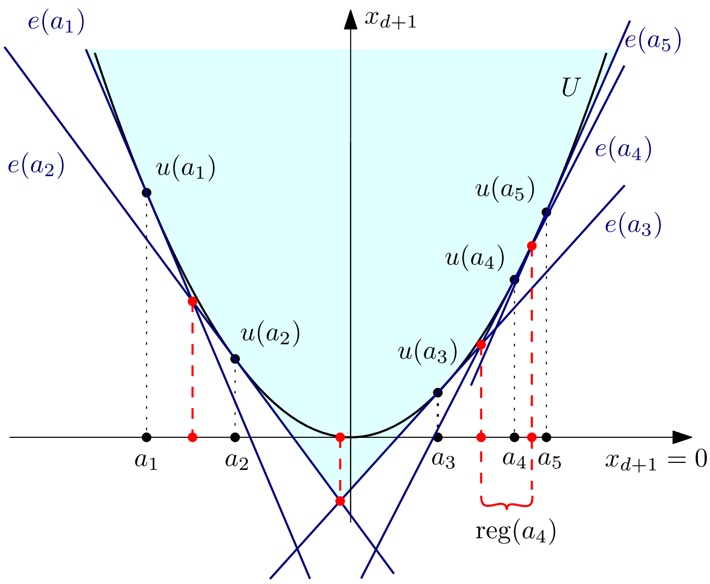
Geometrically, this is the hyperplane tangent to the paraboloid U at the point $u(p)$.

Lemma 12.1. *Let $p \in \mathbb{R}^d$, for any point $x \in \mathbb{R}^d$ the point $u(x)$ lies above the hyperplane $e(p)$ or on it, and the vertical distance of $u(x)$ to $e(p)$ is δ^2 with $\delta = \|x - p\|$.*



Proof. The statement follows directly from the definitions. Let $y = \mathbb{R}^{d+1}$ be the point of the hyperplane $e(p)$ which lies vertically above x. That is, y has coordinates $y_1 = x_1, \ldots y_d = x_d$, and

$$y_{d+1} = 2p_1x_1 + \cdots + 2p_dx_d - p_1^2 - \cdots - p_d^2$$



The $(d+1)$th coordinate of $u(x)$ is $f(x)$. We have

$$f(x) - y_{d+1} = (x_1 - p_1)^2 + \cdots + (x_d - p_d)^2 = \delta^2 \geq 0$$

□

Theorem 12.2. *Let $\mathcal{E}(p)$ denote the halfspace lying above the hyperplane $e(p)$. Consider a set of n points $P \subset \mathbb{R}^d$. The Voronoi diagram of P is the vertical projection of the facets of the polyhedron*

$$Q = \bigcap_{p \in P} \mathcal{E}(p)$$

onto the hyperplane

$$h = \left\{ x \in \mathbb{R}^{d+1} \;\middle|\; x_{d+1} = 0 \right\}$$

Proof. Consider a point $a_i \in P$. We claim that for any $x \in \mathbb{R}^d$ it holds that $x \in \mathrm{reg}(a_i)$ if and only if x lies in the vertical projection of the face of Q defined by $e(a_i) \cap Q$.

In particular, we claim that $x \in \mathrm{reg}(a_i)$ if and only if $e(a_i)$ is highest among all $e(a_j)$ for $1 \leq j \leq n$ at the vertical line at x.

This is implied by Lemma 12.1 above. Since Q is the intersection of upper halfspaces bounded by the hyperplanes $e(a_1), \ldots, e(a_n)$, this implies the theorem.

□

3 Complexity of Voronoi diagrams in $\mathbb{R}^d$

Theorem 12.3. *The maximum total number of faces of all regions of the Voronoi diagram of a set P of n points in $\mathbb{R}^d$ is $O(n^{\lceil \frac{d}{2} \rceil})$.*

Proof. We prove the theorem for sets of points in general position: not all points lie in the same $(d-1)$-dimensional plane, and no $d+2$ points lie on the boundary of a common sphere[1].

[1]For a more general proof, we would need a more general version of the upper bound theorem, which can be found in Matoušek's book.

By Theorem 12.2 the total number of faces of the Voronoi diagram equals the total number of faces of a H-polyhedron with at most n facets in $\mathbb{R}^{d+1}$. Let this polyhedron be P. We can intersect P with a sufficiently large simplex such that the complexity does not decrease (choose the simplex large enough so that it contains all bounded faces of P) and it is defined by $n+d+2$ halfspaces (a simplex in $\mathbb{R}^{d+1}$ has $d+2$ facets). The dual polytope of P is a simplicial polytope of at most $N = n+d+2$ vertices. By the upper bound theorem the total number of faces is

$$O\left(N^{\lfloor \frac{d+1}{2} \rfloor}\right) \in O\left(N^{\lceil \frac{d}{2} \rceil}\right) \in O\left(n^{\lceil \frac{d}{2} \rceil}\right)$$

for any fixed d, since indeed $\lfloor \frac{d+1}{2} \rfloor \leq \lceil \frac{d}{2} \rceil$. □

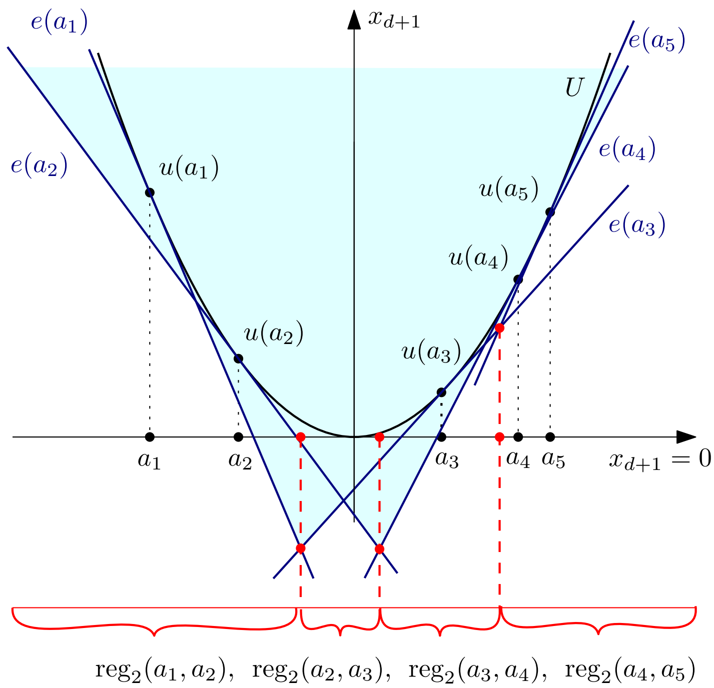
One can show that this upper bound can also be realized by constructing a Voronoi diagram with many faces. (→ Exercise)

4 Higher-order Voronoi diagrams

In the last part of this lecture we want to discuss a generalization of Voronoi diagrams to Voronoi diagrams of higher order. Consider a finite set $P \subset \mathbb{R}^d$ and let k be a natural number. For each subset $S \subseteq P$ with $|S| = k$ we define a region $\mathrm{reg}_k(S)$ which consists of the points $x \in \mathbb{R}^d$ for which the points in S are the k nearest points among points in P. Formally,

$$\mathrm{reg}_k(S) = \left\{ x \in \mathbb{R}^d \;\middle|\; \forall p \in S\ \forall q \in P \setminus S : \ \|x-p\| \leq \|x-q\| \right\}$$

By Lemma 12.1, we can derive the regions of the Voronoi diagram of higher order from the hyperplanes obtained by lifting the points to the paraboloid. This leads to the definition of levels in hyperplane arrangments. Below is an example for $k=2$.



Definition 12.4 (Level)**.** *Let H be a finite set of hyperplanes in $\mathbb{R}^d$ and assume that none of them is vertical (parallel to the x_d-axis). The level of a point $x \in \mathbb{R}^d$ is the number of hyperplanes lying strictly above x.*

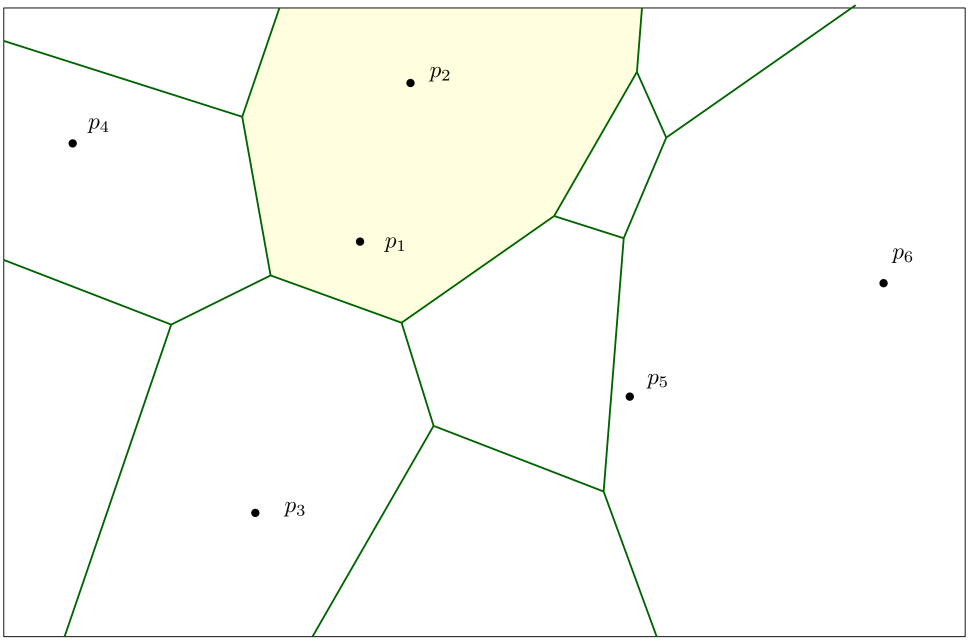


Figure 1: Voronoi diagram of order 2 for a set of points in $\mathbb{R}^2$. The shaded region is the Voronoi region of p_1 and p_2.

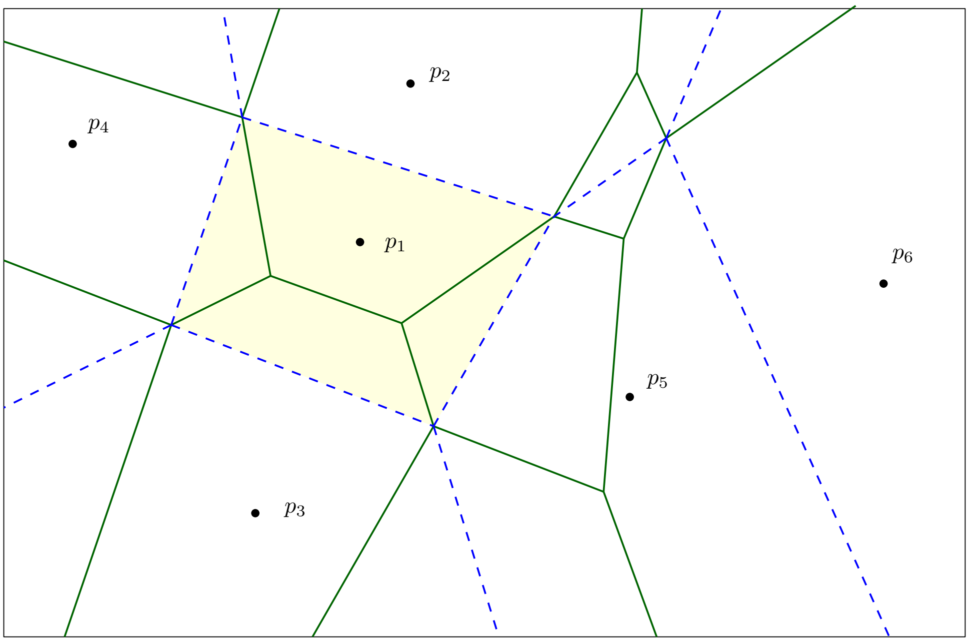


Figure 2: Overlay of Voronoi diagrams of order 1 (dashed) and 2 (solid) for a set of points in $\mathbb{R}^2$. The shaded region is the Voronoi region of p_1. If we remove p_1 and compute the Voronoi diagram of the remaining points and intersect it with this region, we locally obtain the order-2 Voronoi diagram.

Consider the arrangement of all hyperplanes $e(p)$ for $p \in P$. By Lemma 12.1, the k points of P that are closest to any point x are exactly those that correspond to the topmost k hyperplanes below $u(x)$. If we move x around, we cross the boundary of a region in the k-th order Voronoi diagram exactly at the places where the set of the topmost k hyperplanes below $u(x)$ changes, that is, where the k-th hyperplane below $u(x)$ crosses the $(k+1)$-th hyperplane below $u(x)$. Thus, the k-th order Voronoi diagram can be derived from edges at level $k-1$ in the arrangement of hyperplanes.

Another way to generate a Voronoi diagram of higher order, is by computing the diagram recursively. Consider the Voronoi-diagram of order 1 for a set of points P. For each Voronoi region, we can remove the point that is associated with this region and compute the Voronoi diagram of the remaining points and intersect it with this region. This way, we obtain the Voronoi-diagram of order 2. This can be iterated for computing Voronoi diagrams of higher order.

References

- Jiří Matoušek, Chapter 5.7, Lectures on Discrete Geometry, Springer Graduate Texts in Mathematics.