Advanced Topics in Computer Graphics II
Geometry Processing

Differential Geometry

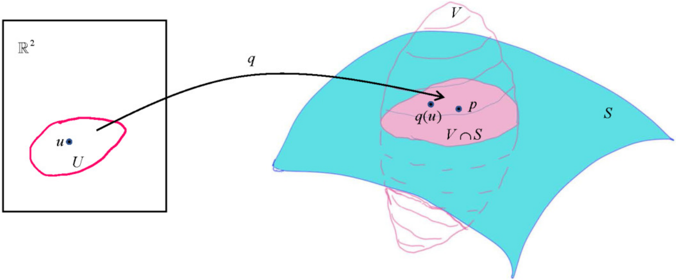
November 7, 2024

Differential Operators

Definition (Regular surface)

Let $S \subset \mathbb{R}^3$. S is called regular surface, if any point $p \in S$ has an open neighborhood $V \subset \mathbb{R}^3$ such that there is a open subset $U \subset \mathbb{R}^2$ and a smooth mapping $\boldsymbol{q} : U \to \mathbb{R}^3$ such that

- $q(U) = S \cap V$ and $q : U \to S \cap V$ is a homeomorphism.
- The vectors $d\boldsymbol{q}(e_1), d\boldsymbol{q}(e_2)$ are linear independent, i.e the Jacobian $\boldsymbol{J} = d\boldsymbol{q}$ has rank 2.



Definition (Parametric Surface)

A mapping

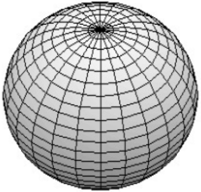
$$\begin{aligned} \boldsymbol{q}\colon \mathbb{R}^2 \supset \mathcal{D} &\to \mathbb{R}^3 \\ (u,v)^T &\mapsto (x(u,v)\,,\, y(u,v)\,,\, z(u,v))^T \end{aligned}$$

is called *parametric surface*.

Definition (Regularity of surfaces)

A parametric surface is called *regular* if $\boldsymbol{q}$ is continuously differentiable at least once and if the vectors $\boldsymbol{q}_u(u,v)$ and $\boldsymbol{q}_v(u,v)$ are linear independent for all $(u,v) \in \mathcal{D}$.

Example:



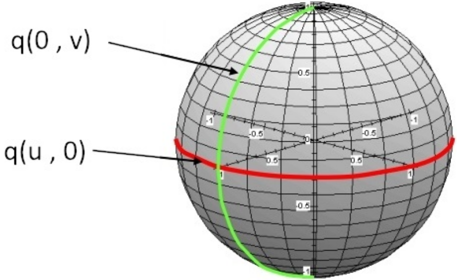
$$\boldsymbol{q}(u,v) = r(\cos(u)\cos(v), \sin(u)\cos(v), \sin(v))^T$$
$$(u,v)^T \in [-\pi,\pi] \times [-\pi/2, \pi/2]$$

Common choices are $\mathcal{D} = [a,b] \times [c,d]$ or $\mathcal{D} = [0,1]^2$.

Definition (Curves on surfaces)

The curves $\boldsymbol{p}(u) = \boldsymbol{q}(u, v_0)$ with fixed $v = v_0$ and $\boldsymbol{p}(v) = \boldsymbol{q}(u_0, v)$ with fixed $u = u_0$ are called *parameter curves* of the surface.

Example:



$$\boldsymbol{q}(u, v) = r(\cos(u)\cos(v), \sin(u)\cos(v), \sin(v))^T$$

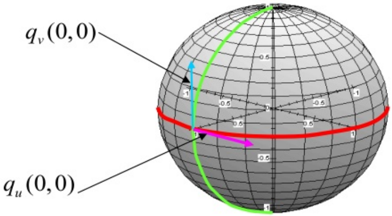
Definition (Derivative on surfaces)

A surface is said to be n times continuously differentiable, if the mapping $\boldsymbol{q}$ is n times continuously differentiable, i.e. $\boldsymbol{q}$ has continuous derivatives n times.
The vectors

$$\boldsymbol{q}_u(u,v) = \frac{\partial \boldsymbol{q}(u,v)}{\partial u} \quad , \quad \boldsymbol{q}_v(u,v) = \frac{\partial \boldsymbol{q}(u,v)}{\partial v}$$

are called u-*tangent and* v-*tangent at* (u,v).

Example:



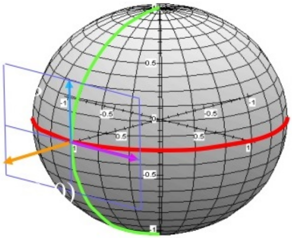
Definition (Tangent plane and Normals on surfaces)

If $\boldsymbol{q}$ is a regular parameterized surface, then the plane spanned by $\boldsymbol{q}_u(u,v)$ and $\boldsymbol{q}_v(u,v)$ is called the *tangent plane* $TP_{\boldsymbol{q}(u,v)}$ at surface point $\boldsymbol{q}(u,v)$.

The vector

$$\boldsymbol{n}(u,v) := \frac{\boldsymbol{q}_u(u,v) \times \boldsymbol{q}_v(u,v)}{\|\boldsymbol{q}_u(u,v) \times \boldsymbol{q}_v(u,v)\|_2}$$

is called the *normal vector* at position $\boldsymbol{q}(u,v)$



The normal vector is perpendicular to the tangent plane and independent from the parameterization.

Definition (Directional derivatives of surfaces)

Let $\boldsymbol{w} = (w_1, w_2)^T$ be a vector in the parameter domain.
Then the vector

$$d\boldsymbol{q}_{\boldsymbol{w}}(u,v) = \lim_{h\to 0} \frac{\boldsymbol{q}(u + h\cdot w_1, v + h\cdot w_2) - \boldsymbol{q}(u,v)}{h} = w_1 \cdot \boldsymbol{q}_u(u,v) + w_2 \cdot \boldsymbol{q}_v(u,v)$$

is called the *directional derivative* with respect to $\boldsymbol{w}$ or *differential* of $\boldsymbol{q}$.

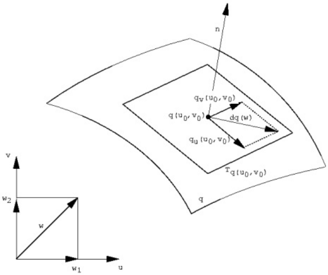
If we take a closer look at this definition, we realize that

$$d\boldsymbol{q}_{\boldsymbol{w}}(u,v) = \boldsymbol{J}(u,v)\,\boldsymbol{w}$$

where

$$\boldsymbol{J}(u,v) = \left(\begin{array}{c|c} \boldsymbol{q}_u(u,v) & \boldsymbol{q}_v(u,v) \end{array} \right) \in \mathbb{R}^{3\times 2}$$

is the Jacobian of $\boldsymbol{q}$.



The Jacobian of a parametric surface q

$$\boldsymbol{J}(u,v) = \left(\begin{array}{c|c} \boldsymbol{q}_u(u,v) & \boldsymbol{q}_v(u,v) \end{array} \right) \in \mathbb{R}^{3\times 2}$$

transforms vectors in the parameter domain into vectors on the tangent plane. Note, that angles and lengths of the vectors might be changed by this mapping.

Definition (First Fundamental Form)

Let $\boldsymbol{p} = \boldsymbol{q}(u, v)$ be a point on the surface and T_p be the tangent plane at $\boldsymbol{p}$. The bilinear map

$$\boldsymbol{I} : TP_{\boldsymbol{p}} \times TP_{\boldsymbol{p}} \to \mathbb{R} : \boldsymbol{I}(\boldsymbol{w_1}, \boldsymbol{w_2}) := \langle\, \boldsymbol{w_1} \,|\, \boldsymbol{w_2} \,\rangle_{\mathbb{R}^3}$$

is called the *First Fundamental Form* at the point $\boldsymbol{p}$.

It allows to measure angles and lengths in the tangent space without explicitly referring to the embedding space $\mathbb{R}^3$. Such a variable scalar product depending on (u, v) is also called a Riemannian metric on the parameter domain.

Lemma (First Fundamental Form)

The matrix

$$\boldsymbol{M}_1 = \boldsymbol{J}^T \boldsymbol{J} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

with

$$E = \langle\, \boldsymbol{q}_u \,|\, \boldsymbol{q}_u \,\rangle_{\mathbb{R}^3} \quad , \quad F = \langle\, \boldsymbol{q}_u \,|\, \boldsymbol{q}_v \,\rangle_{\mathbb{R}^3} \quad , \quad G = \langle\, \boldsymbol{q}_v \,|\, \boldsymbol{q}_v \,\rangle_{\mathbb{R}^3}$$

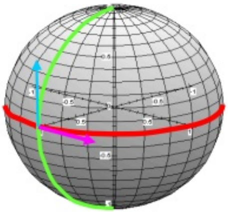
is the the matrix of the First Fundamental Form on the basis $\{\boldsymbol{q}_u, \boldsymbol{q}_v\}$.

Proof: Let $p_1(t) = q(u_1(t), v_1(t)), t \in (-\epsilon, \epsilon), p_2(t) = q(u_2(t), v_2(t)), t \in (-\epsilon, \epsilon)$, with $\boldsymbol{p} = p_1(0) = p_2(0)$ and $w_1 = p_1'(0), w_2 = p_2'(0) \in TP_{\boldsymbol{p}}$ be two tangent vectors. Then

$$\begin{aligned}
\boldsymbol{I}(w_1, w_2) &= \langle\, \boldsymbol{w_1} \,|\, \boldsymbol{w_2} \,\rangle_{\mathbb{R}^3} = \left\langle\, q_u u_1' + q_v v_1' \,\middle|\, q_u u_2' + q_v v_2' \,\right\rangle_{\mathbb{R}^3} \\
&= u_1' u_2' \langle\, q_u \,|\, q_u \,\rangle_{\mathbb{R}^3} + u_1' v_2' \langle\, q_u \,|\, q_v \,\rangle_{\mathbb{R}^3} + v_1' u_2' \langle\, q_v \,|\, q_u \,\rangle_{\mathbb{R}^3} + v_2' v_2' \langle\, q_v \,|\, q_v \,\rangle_{\mathbb{R}^3} \\
&= \begin{pmatrix} u_1' & v_1' \end{pmatrix} \begin{pmatrix} \langle\, q_u \,|\, q_u \,\rangle_{\mathbb{R}^3} & \langle\, q_u \,|\, q_v \,\rangle_{\mathbb{R}^3} \\ \langle\, q_v \,|\, q_u \,\rangle_{\mathbb{R}^3} & \langle\, q_v \,|\, q_v \,\rangle_{\mathbb{R}^3} \end{pmatrix} \begin{pmatrix} u_2' \\ v_2' \end{pmatrix} \\
&= \begin{pmatrix} u_1' & v_1' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u_2' \\ v_2' \end{pmatrix} = w_1^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} w_2
\end{aligned}$$

Example:

$$\boldsymbol{q}(u,v) = r\,(\cos(u)\cos(v), \sin(u)\cos(v), \sin(v))^T$$
$$\boldsymbol{q}_u(u,v) = r\,(-\sin(u)\cos(v), \cos(u)\cos(v), 0)^T$$
$$\boldsymbol{q}_v(u,v) = r\,(-\cos(u)\sin(v), -\sin(u)\sin(v), \cos(v))^T$$



$$\boldsymbol{M}_1 = \begin{pmatrix} \langle\, \boldsymbol{q}_u \mid \boldsymbol{q}_u \,\rangle & \langle\, \boldsymbol{q}_u \mid \boldsymbol{q}_v \,\rangle \\ \langle\, \boldsymbol{q}_u \mid \boldsymbol{q}_v \,\rangle & \langle\, \boldsymbol{q}_v \mid \boldsymbol{q}_v \,\rangle \end{pmatrix} = \begin{pmatrix} r^2\cos(v)^2 & 0 \\ 0 & r^2 \end{pmatrix}$$

Definition (Arc-length function of a curve on the surface)

Let $\boldsymbol{q}(u(t), v(t))$ be a curve on the surface. The *arc-length function* of this curve is defined as

$$s(t) = \int_a^t \left\| \frac{d\boldsymbol{q}(u(\tau), v(\tau))}{d\tau} \right\| d\tau$$

With this function, any regular curve can be made arc-length parameterized according to

$$\boldsymbol{q}(u(s), v(s)) := \boldsymbol{q}(u(t(s)), v(t(s))) \quad , \quad \forall s\colon \left\| \frac{d\boldsymbol{q}(u(s), v(s))}{ds} \right\| = 1$$

where $t(s)$ is the inverse function of $s(t)$. Consequently, the parameter t is often replaced by s to indicate that property!

$\rightarrow$ This function will now help us to define the arc-length of a curve in a specified interval.

The curve $\boldsymbol{q}(u(t), v(t))$ has the derivative

$$\frac{d\boldsymbol{q}(u(t), v(t))}{dt} = \frac{\partial \boldsymbol{q}}{\partial u}\frac{du}{dt} + \frac{\partial \boldsymbol{q}}{\partial v}\frac{dv}{dt} = \boldsymbol{q}_u u' + \boldsymbol{q}_v v'$$

So we can define its arc-length as

$$\begin{aligned} l(a,b) &= \int_a^b \left\| \frac{d\boldsymbol{q}(u(t), v(t))}{dt} \right\| dt \\ &= \int_a^b \sqrt{\left\langle \frac{d\boldsymbol{q}(u(t), v(t))}{dt} \;\middle|\; \frac{d\boldsymbol{q}(u(t), v(t))}{dt} \right\rangle} dt \\ &= \int_a^b \sqrt{\langle \boldsymbol{q}_u u' + \boldsymbol{q}_v v' \mid \boldsymbol{q}_u u' + \boldsymbol{q}_v v' \rangle} dt \\ &= \int_a^b \sqrt{(u_t, v_t)\, \boldsymbol{M}_1\, (u', v')^T} dt \end{aligned}$$

and also get the very useful relation

$$\left\| \frac{d\boldsymbol{q}(u(t), v(t))}{dt} \right\| = \sqrt{(u', v')\, \boldsymbol{M}_1\, (u', v')^T}$$

First Fundamental Form

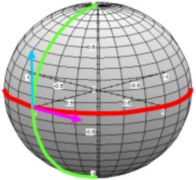
Similarly, we can compute the surface area by using Langrange's identity $\left\|\boldsymbol{q}_u \times \boldsymbol{q}_v\right\|^2 = \left\|q_u\right\|^2 \left\|q_v\right\|^2 - \langle\, \boldsymbol{q}_u \mid \boldsymbol{q}_v \,\rangle^2$:

$$
\begin{aligned}
A &= \iint_{\mathcal{D}} \left\|\boldsymbol{q}_u \times \boldsymbol{q}_v\right\| du\, dv = \iint_{\mathcal{D}} \sqrt{\left\|q_u\right\|^2 \left\|q_v\right\|^2 - \langle\, \boldsymbol{q}_u \mid \boldsymbol{q}_v \,\rangle^2}\, du\, dv \\
&= \iint_{\mathcal{D}} \sqrt{\det(\boldsymbol{M}_1)}\, du\, dv = \iint_{\mathcal{D}} \sqrt{EG - F^2}\, du\, dv
\end{aligned}
$$

Example:

$$
\boldsymbol{M}_1 = \begin{pmatrix} \langle\, \boldsymbol{q}_u \mid \boldsymbol{q}_u \,\rangle & \langle\, \boldsymbol{q}_u \mid \boldsymbol{q}_v \,\rangle \\ \langle\, \boldsymbol{q}_u \mid \boldsymbol{q}_v \,\rangle & \langle\, \boldsymbol{q}_v \mid \boldsymbol{q}_v \,\rangle \end{pmatrix} = \begin{pmatrix} r^2 \cos(v)^2 & 0 \\ 0 & r^2 \end{pmatrix}
$$

$$
\begin{aligned}
A &= \iint_{\mathcal{D}} \sqrt{EG - F^2}\, du\, dv = \int_{-\pi}^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^4 \cos(v)^2}\, du\, dv \\
&= 2\pi r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(v)\, dv = 2\pi r^2 \cdot 2 = 4\pi r^2
\end{aligned}
$$



Definition (Normal field)

Let $S \subset \mathbb{R}^3$ be a regular surface. A *normal field* on S is a mapping

$$n : S \to \mathbb{R}^3,$$

such that $n(p) \perp T_pS$ for all $p \in S$. If $||n(p)|| = 1$ for all $p \in S$ the normal field is called *unit normal field*

Note, that with n, also $-n$ is a normal field on S.

Definition (Orientability)

A regular surface $S \subset \mathbb{R}^3$ is called orientable, if there is a smooth unit normal field.

Definition (Normal Mapping)

The mapping

$$\boldsymbol{n} \colon \mathbb{R}^2 \supset \mathcal{D} \to \mathbb{R}^3$$
$$(u,v)^T \mapsto \boldsymbol{n}(u,v) := \frac{\boldsymbol{q}_u(u,v) \times \boldsymbol{q}_v(u,v)}{\|\boldsymbol{q}_u(u,v) \times \boldsymbol{q}_v(u,v)\|_2}$$

is a regular two-times continuously differentiable surface and called the *normal mapping* of $\boldsymbol{q}$.

Now, we can define a differential for this mapping:

Definition (Directional derivatives of normals)

Let $\boldsymbol{w} = (w_1, w_2)^T$ be a vector in the parameter domain.
Then the vector

$$d\boldsymbol{n}_{\boldsymbol{w}}(u,v) = \lim_{h \to 0} \frac{\boldsymbol{n}(u + h \cdot w_1, v + h \cdot w_2) - \boldsymbol{n}(u,v)}{h} = w_1 \cdot \boldsymbol{n}_u(u,v) + w_2 \cdot \boldsymbol{n}_v(u,v)$$

is called *directional derivative* with respect to $\boldsymbol{w}$ or *differential* of $\boldsymbol{n}$.

For any curve $\boldsymbol{q}(u(t), v(t)), t \in (-\epsilon, \epsilon)$ with $\boldsymbol{p} = \boldsymbol{q}(u(0), v(0))^T$ on the surface $\boldsymbol{S}$ the directional derivative $d\boldsymbol{n}(0,0)(u'(0), v'(0))$ in direction $(u'(0), v'(0))$ is perpendicular to $\boldsymbol{n}(0,0)$. Therefore,

$$d_{\boldsymbol{p}}n : TP_{\boldsymbol{p}} \rightarrow TP_{\boldsymbol{p}},$$

i.e. the directional derivative is a vector in the tangent plane $TP_{\boldsymbol{p}}$ at the surface point $\boldsymbol{p}$.

Definition (Shape operator)

Let $S \subset \mathbb{R}^3$ be a regular oriented surface. The mapping

$$W_{\boldsymbol{p}} : TP_{\boldsymbol{p}} \rightarrow TP_{\boldsymbol{p}} \tag{1}$$

$$W_{\boldsymbol{p}}(\boldsymbol{v}) = -d_{\boldsymbol{p}}n(\boldsymbol{v}) \tag{2}$$

is called *Weingarten-map* or *shape operator*.

Theorem

Let $S \subset \mathbb{R}^3$ be a regular oriented surface with shape operator $W_{\boldsymbol{p}} : TP_{\boldsymbol{p}} \to TP_{\boldsymbol{p}}, \boldsymbol{p} \in S$. Then $W_{\boldsymbol{p}}$ is selfadjoint with respect to the first Fundamental Form $\boldsymbol{I_p}$, i.e.

$$\boldsymbol{I_p}(W_{\boldsymbol{p}}(\boldsymbol{w}_1), \boldsymbol{w}_2) = \boldsymbol{I_p}(\boldsymbol{w}_1, W_{\boldsymbol{p}}(\boldsymbol{w}_2))$$

Sketch of proof: Let us choose a local parametrization q, U, V around $\boldsymbol{p}$. Let us consider the following relation:

$$\forall u, v \in \mathcal{U} \colon \langle\, \boldsymbol{n} \,|\, \boldsymbol{q}_u \,\rangle = 0$$

Now take the derivate with respect to u on both sides and rearrange the terms:

$$0 = \frac{\partial}{\partial u} \langle\, \boldsymbol{n} \,|\, \boldsymbol{q}_u \,\rangle = \left\langle\, \frac{\partial}{\partial u} \boldsymbol{n} \,\middle|\, \boldsymbol{q}_u \,\right\rangle + \left\langle\, \boldsymbol{n} \,\middle|\, \frac{\partial}{\partial u} \boldsymbol{q}_u \,\right\rangle$$

$$\langle\, \boldsymbol{n}_u \,|\, \boldsymbol{q}_u \,\rangle = -\langle\, \boldsymbol{n} \,|\, \boldsymbol{q}_{uu} \,\rangle$$

In the same way, this can be shown for the other three combinations of derivatives (with respect to v).

Therefore, we get

$$\boldsymbol{I_p}(\boldsymbol{w}_1, W_{\boldsymbol{p}}(\boldsymbol{w}_2)) = \left\langle\, \frac{\partial^2}{\partial v \partial u} \boldsymbol{q} \,\middle|\, \boldsymbol{n} \,\right\rangle = \left\langle\, \frac{\partial^2}{\partial u \partial v} \boldsymbol{q} \,\middle|\, \boldsymbol{n} \,\right\rangle = \boldsymbol{I_p}(W_{\boldsymbol{p}}(\boldsymbol{w}_1), \boldsymbol{w}_2)$$

Definition (Second Fundamental Form)

Let $S \subset \mathbb{R}^3$ be a regular oriented surface. The bilinear form belonging to the Weingarten Map is called *Second Fundamental Form* of the surface S at the point $p \in S$:

$$\boldsymbol{II}_{\boldsymbol{p}}(w_1, w_2) = \boldsymbol{I}_{\boldsymbol{p}}(W_{\boldsymbol{p}}(w_1), w_2), w_1, w_2 \in T_{\boldsymbol{p}}S$$

Lemma (Second Fundamental Form)

The matrix

$$\boldsymbol{M}_2 = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

with

$$L = \langle\, \boldsymbol{q}_{uu} \,|\, \boldsymbol{n} \,\rangle \quad , \quad M = \langle\, \boldsymbol{q}_{uv} \,|\, \boldsymbol{n} \,\rangle \quad , \quad N = \langle\, \boldsymbol{q}_{vv} \,|\, \boldsymbol{n} \,\rangle$$

is the the matrix of the Second Fundamental Form on the basis $\{\boldsymbol{q}_u, \boldsymbol{q}_v\}$*.*

Sketch of proof: Let $w_1 = w_{11}\boldsymbol{q}_u + w_{12}\boldsymbol{q}_v$. Then

$$\begin{aligned}
\boldsymbol{I_p}(W_{\boldsymbol{p}}(w_1), w_2) &= \langle\, w_{11}W(\boldsymbol{q}_u) + w_{12}W(\boldsymbol{q}_v) \,|\, w_{21}\boldsymbol{q}_u + w_{22}\boldsymbol{q}_v \,\rangle \\
&= (w_{11}, w_{12}) \begin{pmatrix} \langle\, W(\boldsymbol{q}_u) \,|\, \boldsymbol{q}_u \,\rangle & \langle\, W(\boldsymbol{q}_u) \,|\, \boldsymbol{q}_v \,\rangle \\ \langle\, W(\boldsymbol{q}_v) \,|\, \boldsymbol{q}_u \,\rangle & \langle\, W(\boldsymbol{q}_v) \,|\, \boldsymbol{q}_v \,\rangle \end{pmatrix} \begin{pmatrix} w_{21} \\ w_{22} \end{pmatrix} \\
&= (w_{11}, w_{12}) \begin{pmatrix} \langle\, \boldsymbol{q}_{uu} \,|\, \boldsymbol{q}_u \,\rangle & \langle\, \boldsymbol{q}_{uv} \,|\, \boldsymbol{q}_v \,\rangle \\ \langle\, \boldsymbol{q}_{vu} \,|\, \boldsymbol{q}_u \,\rangle & \langle\, \boldsymbol{q}_{vv} \,|\, \boldsymbol{q}_v \,\rangle \end{pmatrix} \begin{pmatrix} w_{21} \\ w_{22} \end{pmatrix}
\end{aligned}$$

How to measure the curvature of a surface?

Idea: In general, the curvature of an arc-length parameterized curve $\boldsymbol{q}(u(s), v(s))$ on the surface is the change of its tangent

$$T(s) = \frac{d\boldsymbol{q}(u(s), v(s))}{ds} = \boldsymbol{q}_u\, u_s + \boldsymbol{q}_v\, v_s, \quad \left\| \frac{d\boldsymbol{q}(u(s), v(s))}{ds} \right\| = 1$$

$$K(S) = \frac{d^2\boldsymbol{q}(u(s), v(s))}{ds^2} = \boldsymbol{q}_{uu}\, u_s^2 + 2\,\boldsymbol{q}_{uv}\, u_s\, v_s + \boldsymbol{q}_{vv}\, v_s^2 + \boldsymbol{q}_u\, u_{ss} + \boldsymbol{q}_v\, v_{ss}$$

Definition (Normal and Geodesic Curvature)

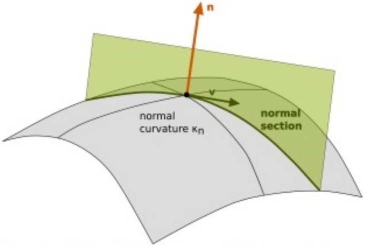
The curvature vector can be decomposed into two orthogonal parts:

$$K(s) = \frac{d^2\boldsymbol{q}(u(s), v(s))}{ds^2} = \kappa_n\, \boldsymbol{n} + \kappa_g\, \boldsymbol{g}$$

where $\boldsymbol{g}$ is a unit vector in the tangent plane.
κ_n is called *Normal Curvature*. κ_g is called *Geodesic Curvature*.

Decomposed curvature:

$$K(s) = \frac{d^2\boldsymbol{q}(u(s), v(s))}{ds^2} = \boldsymbol{q}_{uu}\, u_s^2 + 2\, \boldsymbol{q}_{uv}\, u_s\, v_s + \boldsymbol{q}_{vv}\, v_s^2 + \boldsymbol{q}_u u_{ss} + \boldsymbol{q}_v\, v_{ss}$$
$$= \kappa_n\, \boldsymbol{n} + \kappa_g\, \boldsymbol{g}$$



Example with $\kappa_n \neq 0, \kappa_g = 0$

Example with $\kappa_n = 0, \kappa_g \neq 0$

Note: $\boldsymbol{n}$ and $\boldsymbol{g}$ are varying, so they depend on s.

Now, we want to derive a simple formula for the normal curvature.

$$\left\langle \frac{d^2\boldsymbol{q}}{ds^2} \,\middle|\, \boldsymbol{n} \right\rangle = \left\langle \boldsymbol{q}_{uu}\, u_s^2 + 2\, \boldsymbol{q}_{uv}\, u_s\, v_s + \boldsymbol{q}_{vv}\, v_s^2 + \boldsymbol{q}_u\, u_{ss} + \boldsymbol{q}_v\, v_{ss} \,\middle|\, \boldsymbol{n} \right\rangle$$

$$= \left\langle \boldsymbol{q}_{uu}\, u_s^2 + 2\, \boldsymbol{q}_{uv}\, u_s\, v_s + \boldsymbol{q}_{vv}\, v_s^2 \,\middle|\, \boldsymbol{n} \right\rangle$$

$$= \boldsymbol{w}^T \cdot \begin{pmatrix} \langle\, \boldsymbol{q}_{uu} \,|\, \boldsymbol{n} \,\rangle & \langle\, \boldsymbol{q}_{uv} \,|\, \boldsymbol{n} \,\rangle \\ \langle\, \boldsymbol{q}_{uv} \,|\, \boldsymbol{n} \,\rangle & \langle\, \boldsymbol{q}_{vv} \,|\, \boldsymbol{n} \,\rangle \end{pmatrix} \cdot \boldsymbol{w} \qquad \text{with } \boldsymbol{w} = (u_s, v_s)^T$$

Interpretation: The second fundamental form applied to the pair of identical vectors $(\boldsymbol{w}_{TP}, \boldsymbol{w}_{TP})$ can be interpreted as the change of the normal in direction of $\boldsymbol{w}_{TP}$ projected to the plane spanned by $\boldsymbol{w}_{TP}$ and the normal $\boldsymbol{n}$ at point $\boldsymbol{p}$.[1]

Let us consider the following relation:

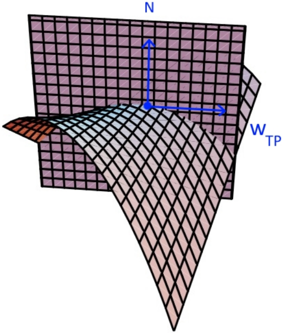
$$\forall u, v \in \mathcal{D} \colon \langle \, \boldsymbol{n} \, | \, \boldsymbol{q}_u \, \rangle = 0$$

Now take the derivate with respect to u on both sides and rearrange the terms:

$$0 = \frac{\partial}{\partial u} \langle \, \boldsymbol{n} \, | \, \boldsymbol{q}_u \, \rangle = \left\langle \, \frac{\partial}{\partial u} \boldsymbol{n} \, \middle| \, \boldsymbol{q}_u \, \right\rangle + \left\langle \, \boldsymbol{n} \, \middle| \, \frac{\partial}{\partial u} \boldsymbol{q}_u \, \right\rangle$$

$$\langle \, \boldsymbol{n}_u \, | \, \boldsymbol{q}_u \, \rangle = -\langle \, \boldsymbol{n} \, | \, \boldsymbol{q}_{uu} \, \rangle$$

In the same way, this can be shown for the other three combinations of derivatives (with respect to v).



[1] L. Vas. *The Second Fundamental Form. Geodesics. The Curvature Tensor. The Fundamental Theorem of Surfaces. Manifolds*. Department of Mathematics, Physics and Statistics, University of the Sciences, Philadelphia, 2015.

Now, compute the change of the normal projected to the plane:

$$\begin{aligned}
-\langle\, d\boldsymbol{n}_{\boldsymbol{w}} \,|\, \boldsymbol{w}_{TP} \,\rangle &= -\langle\, w_1 \cdot \boldsymbol{n}_u + w_2 \cdot \boldsymbol{n}_v \,|\, \boldsymbol{w}_{TP} \,\rangle \\
&= -\langle\, w_1 \cdot \boldsymbol{n}_u + w_2 \cdot \boldsymbol{n}_v \,|\, w_1 \cdot \boldsymbol{q}_u + w_2 \cdot \boldsymbol{q}_v \,\rangle \\
&= -\boldsymbol{w}^T \cdot \begin{pmatrix} \langle\, \boldsymbol{n}_u \,|\, \boldsymbol{q}_u \,\rangle & \langle\, \boldsymbol{n}_v \,|\, \boldsymbol{q}_u \,\rangle \\ \langle\, \boldsymbol{n}_u \,|\, \boldsymbol{q}_v \,\rangle & \langle\, \boldsymbol{n}_v \,|\, \boldsymbol{q}_v \,\rangle \end{pmatrix} \cdot \boldsymbol{w}
\end{aligned}$$

If we use the relations from above, we get:

$$\begin{aligned}
-\boldsymbol{w}^T \cdot \begin{pmatrix} \langle\, \boldsymbol{n}_u \,|\, \boldsymbol{q}_u \,\rangle & \langle\, \boldsymbol{n}_v \,|\, \boldsymbol{q}_u \,\rangle \\ \langle\, \boldsymbol{n}_u \,|\, \boldsymbol{q}_v \,\rangle & \langle\, \boldsymbol{n}_v \,|\, \boldsymbol{q}_v \,\rangle \end{pmatrix} \cdot \boldsymbol{w} &= \boldsymbol{w}^T \cdot \begin{pmatrix} \langle\, \boldsymbol{n} \,|\, \boldsymbol{q}_{uu} \,\rangle & \langle\, \boldsymbol{n} \,|\, \boldsymbol{q}_{uv} \,\rangle \\ \langle\, \boldsymbol{n} \,|\, \boldsymbol{q}_{uv} \,\rangle & \langle\, \boldsymbol{n} \,|\, \boldsymbol{q}_{vv} \,\rangle \end{pmatrix} \cdot \boldsymbol{w} \\
&= \boldsymbol{w}^T \cdot \begin{pmatrix} \langle\, \boldsymbol{q}_{uu} \,|\, \boldsymbol{n} \,\rangle & \langle\, \boldsymbol{q}_{uv} \,|\, \boldsymbol{n} \,\rangle \\ \langle\, \boldsymbol{q}_{uv} \,|\, \boldsymbol{n} \,\rangle & \langle\, \boldsymbol{q}_{vv} \,|\, \boldsymbol{n} \,\rangle \end{pmatrix} \cdot \boldsymbol{w} \\
&= \boldsymbol{w}^T \cdot \boldsymbol{M}_2 \cdot \boldsymbol{w}
\end{aligned}$$

So the second fundamental form expressed this change exactly.

So far, we have considered the normal curvature for arc-length parameterized curves on the surface. But, we can infer a more general result:

Lemma (Normal curvature of regular curves)

Let $\boldsymbol{q}(u(t), v(t))$ *with* $\boldsymbol{p}(t) = (u(t), v(t))^T$ *be a regular parameterized curve. Then the normal curvature is invariant under different parameterizations and can be computed as*

$$\kappa_n = \frac{\boldsymbol{w}_t^T \begin{pmatrix} L & M \\ M & N \end{pmatrix} \boldsymbol{w}_t}{\boldsymbol{w}_t^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} \boldsymbol{w}_t}$$

where $\boldsymbol{w}_t = (u_t, v_t)^T$ *is a vector in the parameter domain.*

Note: As a consequence, the normal curvature does not only depend on the second fundamental form $\boldsymbol{M}_2$ but also on the first fundamental form $\boldsymbol{M}_1$ for an arbitrary regular curve.

To prove the lemma, we first have a look at the definition of the normal curvature that is

$$\kappa_n = \left\langle \frac{d^2\boldsymbol{q}}{ds^2} \,\middle|\, \boldsymbol{n} \right\rangle = \boldsymbol{w}^T \begin{pmatrix} L & M \\ M & N \end{pmatrix} \boldsymbol{w} \qquad \text{with } \boldsymbol{w} = (u_s, v_s)^T$$

Since $\boldsymbol{q}$ is regular, the component u_s (and v_s respectively) of $\boldsymbol{w}$ can be written as

$$u_s = \frac{du}{ds} = \frac{du}{dt} \cdot \frac{dt}{ds} = \frac{du}{dt} \Big/ \frac{ds}{dt} = u_t \Big/ \frac{ds}{dt} \qquad , \qquad v_s = v_t \Big/ \frac{ds}{dt}$$

and directly leads to the relation

$$\boldsymbol{w} = \boldsymbol{w}_t \Big/ \frac{ds}{dt}$$

This means that we need to divide by the derivative of the arc-length function $s(t)$ which is defined as

$$s(t) = \int_a^t \left\| \frac{d\boldsymbol{q}(u(\tau), v(\tau))}{d\tau} \right\| d\tau$$

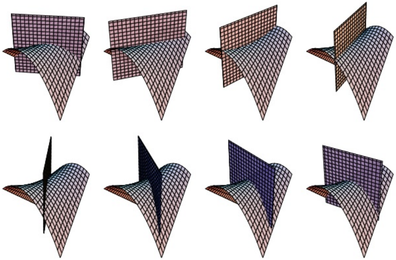
Its derivative is given by

$$\begin{aligned}
\frac{ds}{dt} &= \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\int_a^{t+\Delta t} \left\| \frac{d\boldsymbol{q}(u(\tau), v(\tau))}{d\tau} \right\| d\tau - \int_a^t \left\| \frac{d\boldsymbol{q}(u(\tau), v(\tau))}{d\tau} \right\| d\tau}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\int_t^{t+\Delta t} \left\| \frac{d\boldsymbol{q}(u(\tau), v(\tau))}{d\tau} \right\| d\tau}{\Delta t} \\
&= \left\| \frac{d\boldsymbol{q}(u(t), v(t))}{dt} \right\| \\
&= \sqrt{(u_t, v_t)\, \boldsymbol{M}_1\, (u_t, v_t)^T}
\end{aligned}$$

We have already shown the last equality previously. This gives us the final result

$$\kappa_n = \boldsymbol{w}^T \begin{pmatrix} L & M \\ M & N \end{pmatrix} \boldsymbol{w} = \frac{\boldsymbol{w}_t^T \begin{pmatrix} L & M \\ M & N \end{pmatrix} \boldsymbol{w}_t}{\left(\frac{ds}{dt} \right)^2} = \frac{\boldsymbol{w}_t^T \begin{pmatrix} L & M \\ M & N \end{pmatrix} \boldsymbol{w}_t}{\boldsymbol{w}_t^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} \boldsymbol{w}_t}$$

We successfully computed the normal curvature κ_n in surface tangent direction $\boldsymbol{w}_{TP}$.
Now we can analyze what happens when changing the tangent (rotating the plane):



Definition (Principal Curvatures)

κ_n is periodic and has at most two extrema called the *Principal Curvatures*:

$$\kappa_1 = \max \kappa_n \quad , \quad \kappa_2 = \min \kappa_n$$

The corresponding tangents $\boldsymbol{w}_1, \boldsymbol{w}_2$ are called *Principal Curvature Directions*.

How to compute the principal curvatures?

$$\frac{\partial \kappa_n}{\partial \boldsymbol{w}} = 0 \qquad , \qquad \kappa_n = \frac{\boldsymbol{w}^T \boldsymbol{M}_2 \boldsymbol{w}}{\boldsymbol{w}^T \boldsymbol{M}_1 \boldsymbol{w}}$$

$$\frac{2 \boldsymbol{M}_2 \boldsymbol{w} (\boldsymbol{w}^T \boldsymbol{M}_1 \boldsymbol{w}) - (\boldsymbol{w}^T \boldsymbol{M}_2 \boldsymbol{w}) 2 \boldsymbol{M}_1 \boldsymbol{w}}{(\boldsymbol{w}^T \boldsymbol{M}_1 \boldsymbol{w})^2} = 0$$

$$\boldsymbol{M}_2 \boldsymbol{w} (\boldsymbol{w}^T \boldsymbol{M}_1 \boldsymbol{w}) - (\boldsymbol{w}^T \boldsymbol{M}_2 \boldsymbol{w}) \boldsymbol{M}_1 \boldsymbol{w} = 0$$

$$\boldsymbol{M}_2 \boldsymbol{w} - \frac{\boldsymbol{w}^T \boldsymbol{M}_2 \boldsymbol{w}}{\boldsymbol{w}^T \boldsymbol{M}_1 \boldsymbol{w}} \boldsymbol{M}_1 \boldsymbol{w} = 0$$

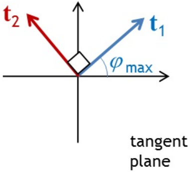
$$\boldsymbol{M}_2 \boldsymbol{w} - \kappa_n \boldsymbol{M}_1 \boldsymbol{w} = 0$$

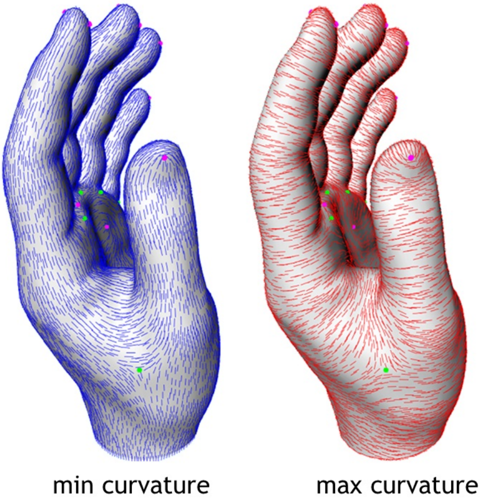
$$\boldsymbol{M}_1^{-1} \boldsymbol{M}_2 \boldsymbol{w} = \kappa_n \boldsymbol{w}$$

So $\boldsymbol{w}$ is an eigenvector to the eigenvalue κ_n. Thus, we have to solve a standard eigenvalue problem.

- Eigenvalues: Solve characteristic polynomial $\det(\boldsymbol{M}_2 - \kappa_n \boldsymbol{M}_1) = 0$
- Eigenvectors: Linear equation system with eigenvalue κ_n

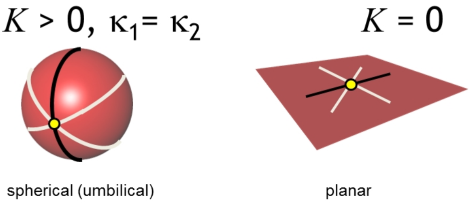
Example:



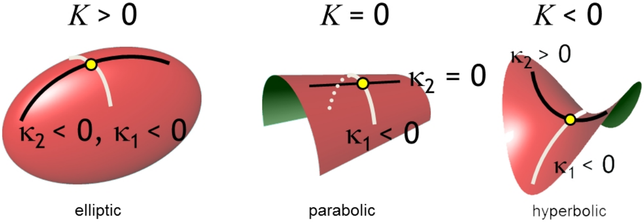


Using the principal curvatures, one can classify surfaces:

Isotropic:
all directions are
principal directions



Anisotropic:
2 distinct
principal
directions



Theorem (Euler Relationship)

The principal curvatures κ_1 *and* κ_2 *and the normal curvature* κ_n *at an arbitrary point* $\boldsymbol{q}(u, v)$ *are related via Euler's formula*

$$\kappa_n(\boldsymbol{w}) = \kappa_1 \cdot \cos(\phi)^2 + \kappa_2 \cdot \sin(\phi)^2$$

where ϕ *denotes the angle between the rotated normal plane and the first principal curvature direction* $\boldsymbol{w}_1$*.*

So the principal curvatures are closely related to the normal curvature. We can even go further and define:

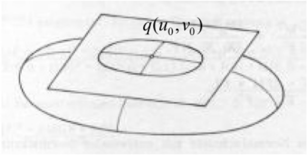
Definition (Dupin Indicatrix)

Let $\xi = \frac{\cos(\phi)}{\sqrt{|\kappa_n|}}$ and $\eta = \frac{\sin(\phi)}{\sqrt{|\kappa_n|}}$. Then the *Dupin Indicatrix* is given by

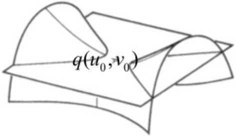
$$\pm 1 = \kappa_1 \, \xi^2 + \kappa_2 \, \eta^2$$

The Dupin Indicatrix characterizes the surface:

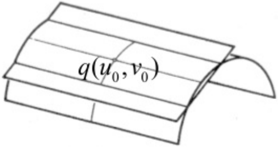
Elliptic Point: Both principal curvatures have the same sign.



Hyperbolic Point: Principal curvatures have different signs.



Parabolic Point: One principal curvature equals 0. If both principal curvatures equal 0, the point is said to be a planar point.



Definition (Gaussian Curvature)

The product between both principal curvatures is called *Gaussian Curvature*:

$$\kappa_G := \kappa_1 \cdot \kappa_2 = \frac{LN - M^2}{EG - F^2}$$

Definition (Mean Curvature)

The mean of both principal curvatures is called *Mean Curvature*:

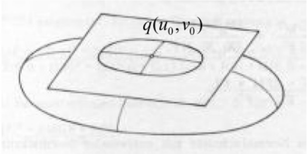
$$\kappa_M := \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\frac{EN + GL - 2FM}{EG - F^2}$$

Mean Curvature can be regarded as the average normal curvature:

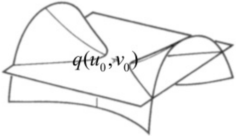
$$\kappa_M = \frac{1}{\pi}\int_0^{\pi} \kappa_n(\boldsymbol{w}(\theta))\, d\theta$$

The surface patches can be locally classified according to their Gaussian Curvature:

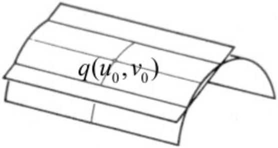
$\kappa_G > 0$: Elliptic point, local surface shaped like an ellipsoid.



$\kappa_G < 0$: Hyperbolic point, local surface shaped like a saddle.



$\kappa_G = 0$: Parabolic point, local surface shaped like a cylinder.



Lemma (Curvature Relations)

The Principal Curvatures κ_1, κ_2 can be expressed in terms of the Gaussian Curvature κ_G and the Mean Curvature κ_M:

$$\kappa_1 = \kappa_M + \sqrt{\kappa_M^2 - \kappa_G} \quad , \quad \kappa_2 = \kappa_M - \sqrt{\kappa_M^2 - \kappa_G}$$

Proof.

The definitions of Mean and Gaussian Curvature are

$$\kappa_G = \kappa_1 \cdot \kappa_2 \quad , \quad \kappa_M := \frac{1}{2}(\kappa_1 + \kappa_2)$$

Putting these two equations into one leads to

$$\kappa^2 - 2\,\kappa_M\,\kappa + \kappa_G = 0$$

The solution of this quadratic function are the principal curvatures. □

More about Differential Geometry can be found in the book of Do Carmo, 1976.

$$M = \{p, f(p) | p \in \Omega\}$$

$$F\colon \Omega \to M$$
$$p \mapsto (p, f(p))$$

Normal:

$$n(F) = \frac{(-\nabla f(p), 1)^T}{\|(-\nabla f(p), 1)^T\|}$$

Tangent Space:

$$T_F M = \mathsf{span}\left\{ \left(e_i, \frac{\partial f}{\partial p_i} \right) \middle| i = 1, \ldots, d \right\}$$

First Fundamental Form:

$$M_1 = \mathbf{I} + \nabla f \otimes \nabla f, \quad \text{in particular if} \quad \nabla f(p) = 0 \Rightarrow g = \mathbf{I}$$

Second Fundamental Form:

$$M_2 = \alpha D^2 f \quad \text{Hessian matrix of } f$$

$$M = \{p \in \Omega \ f(p) = c\}$$
$$f\colon \mathbb{R}^{d+1} \supseteq \Omega \to \mathbb{R}$$

Normal:

$$n(F) = \frac{\nabla f(p)}{\|\nabla f(p)\|}$$

Tangent Space:

$$T_pM = \mathsf{span}\left\{v \in \mathbb{R}^{d+1} | v \cdot n = 0\right\}$$

Projection onto Tangent Space:

$$P = (\mathbf{I} - n \otimes n)$$

with

$$Pw = w - (n \cdot w)n, \qquad P^2 = P = P^T$$

$$Dn = \frac{1}{\|\nabla f\|}\left(f_{ij} - \frac{f_i}{\|\nabla f\|}\frac{f_k}{\|\nabla f\|}f_{kj}\right)_{ij} = \frac{1}{\|\nabla f\|}PD^2 f,$$

where $f_i = \frac{\partial f}{\partial x_i}$, and $\frac{f_i}{\|\nabla f\|}\frac{f_k}{\|\nabla f\|}f_{,kj} = \sum_k \frac{f_i}{\|\nabla f\|}\frac{f_k}{\|\nabla f\|}f_{kj}$

Einstein summation convention:
"I have made a great discovery in mathematics:
I have suppressed the summation sign every time that the summation must be made over an index which occurs twice..."

Example: Implicit Surfaces

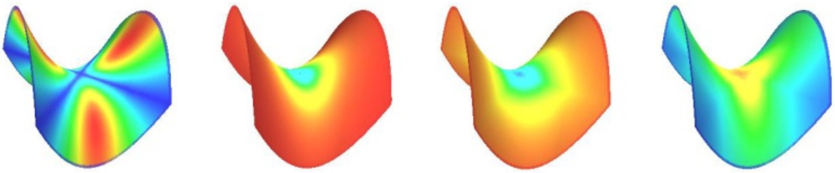
$$Dn = \frac{1}{\|\nabla f\|}\left(f_{ij} - \frac{f_i}{\|\nabla f\|}\frac{f_k}{\|\nabla f\|}f_{kj}\right)_{ij} = \frac{1}{\|\nabla f\|}PD^2 f$$

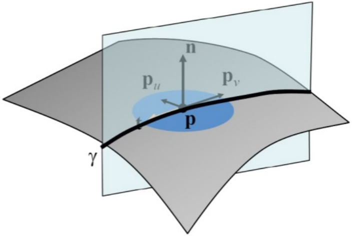
$$v_{\mathsf{min}} = \arg\min_{v,\|v=1\|}\left\|\underbrace{DnP}_{A}v\right\|^2, \qquad v_{\mathsf{max}} = \arg\max_{v,\|v=1\|}\left\|\underbrace{DnP}_{A}v\right\|^2$$

Eigenvectors of A are principal curvature directions corresponding eigenvalues are principal curvatures

Goal: We want to define curvature on discrete polygon meshes:

- Normal Curvature
- Mean Curvature
- Gaussian Curvature
- Principal Curvature



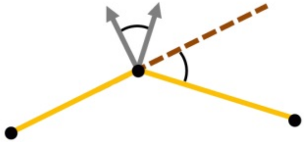


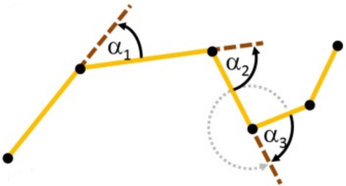
The curve γ is the intersection of the surface with the plane through $\boldsymbol{n}$ and $\boldsymbol{t}$.

So the normal curvature is

$$\kappa_n(\gamma(\boldsymbol{p}))$$

and describes the rate of change in normal.
$\rightarrow$ Discrete curvature: turning angles





Let $\boldsymbol{t}$ be the tangent vector to $\boldsymbol{q}$ at point $\boldsymbol{p}$ and let $\boldsymbol{\alpha}(s)$ be an arc-length parameterized curve on $\boldsymbol{q}$ with

$$\boldsymbol{\alpha}(0) = \boldsymbol{p} \quad , \quad \boldsymbol{\alpha}^{'}(0) = \boldsymbol{t} \quad , \quad \boldsymbol{\alpha}^{''}(0) \text{ colinear with } \boldsymbol{n}$$

Then $\boldsymbol{\alpha}^{''}(0) = \kappa_n \, \boldsymbol{n}$.

Now expand the curve in a Taylor series around the point $\boldsymbol{p}$:

$$\begin{aligned} \boldsymbol{\alpha}(s) &= \boldsymbol{\alpha}(0) + \boldsymbol{\alpha}^{'}(0)\, s + \frac{1}{2}\boldsymbol{\alpha}^{''}(0)\, s^2 + \mathcal{O}\big(s^3\big) \\ &= \boldsymbol{p} + s\, \boldsymbol{t} + \frac{1}{2}\, \kappa_n\, s^2\, \boldsymbol{n} + \mathcal{O}\big(s^3\big) \end{aligned}$$

Consider the following terms

$$\begin{aligned} 2\, \boldsymbol{n}^T \left(\boldsymbol{\alpha}(s) - \boldsymbol{p}\right) &= \kappa_n\, s^2 + \mathcal{O}\big(s^3\big) \\ \|\boldsymbol{\alpha}(s) - \boldsymbol{p}\|^2 &= s^2 + \mathcal{O}\big(s^3\big) \end{aligned}$$

So we get

$$\frac{2\,\boldsymbol{n}^T\left(\boldsymbol{\alpha}(s)-\boldsymbol{p}\right)}{\left\|\boldsymbol{\alpha}(s)-\boldsymbol{p}\right\|^2}=\kappa_n+\mathcal{O}(s)$$

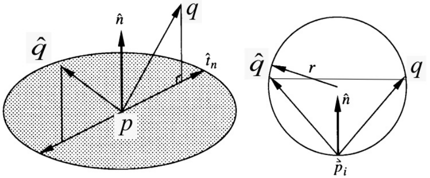
which leads to

$$\lim_{s\to 0}\frac{2\,\boldsymbol{n}^T\left(\boldsymbol{\alpha}(s)-\boldsymbol{p}\right)}{\left\|\boldsymbol{\alpha}(s)-\boldsymbol{p}\right\|^2}=\kappa_n$$

So our approximation of the normal curvature is given as

$$\kappa_n\approx\frac{2\,\boldsymbol{n}^T\left(\hat{\boldsymbol{q}}-\boldsymbol{p}\right)}{\left\|\hat{\boldsymbol{q}}-\boldsymbol{p}\right\|^2}$$

This approximation can be interpreted geometrically as curvature ($\frac{1}{r}$) of the following circle:



Curvature can be well approximated on a Mesh. Could these values be averaged?[2]

Consider the unit tangent vector

$$\boldsymbol{t}(\theta) = \cos(\theta)\,\boldsymbol{e}_1 + \sin(\theta)\,\boldsymbol{e}_2$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2$ denote the principal curvature directions. Then

$$\kappa_n(\boldsymbol{t}(\theta)) = \kappa_1 \cdot \cos(\theta)^2 + \kappa_2 \cdot \sin(\theta)^2$$

Now define the matrix

$$\boldsymbol{M} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \kappa_n(\boldsymbol{t}(\theta))\,\boldsymbol{t}(\theta)\,\boldsymbol{t}(\theta)^T\,d\theta$$

This matrix has the following properties:

- $\boldsymbol{n}$ is an eigenvector to eigenvalue 0
- $\boldsymbol{e}_1$ is an eigenvector to eigenvalue $m_1 = \frac{3}{8}\kappa_1 + \frac{1}{8}\kappa_2$
- $\boldsymbol{e}_2$ is an eigenvector to eigenvalue $m_2 = \frac{1}{8}\kappa_1 + \frac{3}{8}\kappa_2$

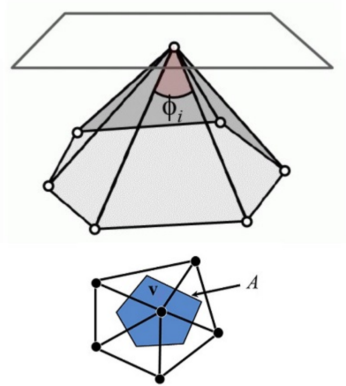
[2] G. Taubin. "Estimating the tensor of curvature of a surface from a polyhedral approximation". In: *Computer Vision, 1995. Proceedings., Fifth International Conference on*. IEEE. 1995, pp. 902–907.

Once these eigenvalues are known, the principal curvatures can be computed by

$$\kappa_1 = 3m_1 - m_2 \quad , \quad \kappa_2 = 3m_2 - m_1$$

These observations lead to the following algorithm:

- Estimate normal at vertex $\boldsymbol{v}_i$ (weighted by adjacent polygon areas)
- Compute normal curvatures κ_{ij} in the direction of all neighboring vertices $\boldsymbol{v}_j$ (see above)
- Approximate $\boldsymbol{M}$ at $\boldsymbol{v}_i$ by $\boldsymbol{M} = \sum_{\boldsymbol{v}_j} w_{ij}\, \kappa_{ij}\, \boldsymbol{t}_{ij}\, \boldsymbol{t}_{ij}^T$ where $\boldsymbol{t}_{ij}$ denotes the projection of $\boldsymbol{v}_j - \boldsymbol{v}_i$ onto the tangent plane. Weights w_{ij} are chosen according to the normalized surface areas of the neighboring triangles with $\sum_{\boldsymbol{v}_j} w_{ij} = 1$
- Compute eigenvectors and eigenvalues of $\boldsymbol{M}$
- Finally, compute principal curvatures



The discrete Gaussian Curvature can be computed using the angle deficit:

$$\kappa_G = \frac{1}{A(\boldsymbol{v})}\left(2\pi - \sum_{i=1}^{|\mathcal{N}(\boldsymbol{v})|} \phi_i\right)$$