Complexity of Convex Polytopes

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In this lecture we will have a closer look at the combinatorial structure of convex polytopes. In particular, we are interested in the number of faces of all dimensions.

In the plane it is clear that the number of faces of the convex hull of n points is bounded by $O(n)$, but what about higher dimensions. We will first study the case of $\mathbb{R}^3$ and then the case of general d.

1 Dual Polytope

We start by discussing H-polytopes and V-polytopes again to follow up on our previous claim that the set of H-polytopes in $\mathbb{R}^d$ is equal to the set of V-polytopes in $\mathbb{R}^d$.

Theorem 11.1 (Dual polytope)**.** *For any V-polytope $P \subseteq \mathbb{R}^d$ containing the origin in the interior the dual set P^* is an H-polytope in $\mathbb{R}^d$.*

Proof. The proof has two parts:
(i) P^* is a H-polyhedron in $\mathbb{R}^d$.
(ii) P^* is bounded.
We first prove (i). Since P is a convex polytope that contains 0 in its interior, there exists a finite set $V \subseteq \mathbb{R}^d \setminus \{0\}$, such that $P = \mathrm{conv}(V)$. By the definition of the dual set applied to P,

$$\begin{aligned} P^* &= \left\{ y \in \mathbb{R}^d \;\middle|\; \forall x \in \mathrm{conv}(V) : \langle x, y \rangle \leq 1 \right\} \\ &= \{0\} \cup \left\{ y \in \mathbb{R}^d \setminus \{0\} \;\middle|\; \mathrm{conv}(V) \subseteq \mathcal{D}(y)^- \right\} \end{aligned}$$

For any halfspace h it holds that

$$\mathrm{conv}(V) \subseteq h \quad \Leftrightarrow \quad V \subseteq h$$

This implies for any $y \in \mathbb{R}^d \setminus \{0\}$

$$\mathrm{conv}(V) \subseteq \mathcal{D}(y)^- \quad \Leftrightarrow \quad \forall x \in V : x \in \mathcal{D}(y)^-$$

Now, Observation 4.4 implies

$$P^* = \{0\} \cup \left\{ y \in \mathbb{R}^d \setminus \{0\} \;\middle|\; \forall x \in V : y \in \mathcal{D}(x)^- \right\}$$

and this is equal to $\bigcap_{x \in V} \mathcal{D}(x)^-$

The second part of the proof is to show (ii). Since by our assumptions P contains the origin in its interior, there exists an $\epsilon > 0$ such that

$$\left\{ p \in \mathbb{R}^d \;\middle|\; \|p\| \leq \epsilon \right\} \subseteq P$$

We claim that

$$P^* \subseteq \left\{ p \in \mathbb{R}^d \;\middle|\; \|p\| \leq \frac{1}{\epsilon} \right\}$$

which would imply that P^* is bounded. So assume for the sake of contradiction, that P^* contains a point a with $\|a\| > \frac{1}{\epsilon}$. Let $x = \frac{\epsilon}{\|a\|}a$. Note that $x \in P$, since $\|x\| = \epsilon$. We get

$$\langle a, x\rangle = \left\langle a, \frac{\epsilon}{\|a\|}a \right\rangle = \frac{\epsilon}{\|a\|}\langle a, a\rangle = \epsilon\|a\| > 1$$

However, this contradicts the fact that $a \in P^*$, since by definition of the dual set, it must be that $\langle a, x\rangle \leq 1$ for all $x \in P$. □

Using the arguments in the above proof, we can also show the claim from Lecture 4, that any V-polytope is an H-polytope and vice versa. Below is a sketch of a proof under the assumption that the polytope contains the origin, which can be ensured by using an appropriate translation.

To show that any H-polytope is a V-polytope, we proceed by induction on d. For $d = 1$ the claim is trivial, a polytope is an interval that can be described either as the intersection of halfspaces or as the convex hull of points. So consider $d > 1$. Consider the non-empty faces of P. Each face is a H-polytope in a dimension $d' < d$, so by induction each face can be described as a V-polytope. The convex hull of these faces must be contained in P, since P is convex. It remains to argue that P is contained in the convex hull of the faces. Consider any point x in P and let l be any line through x. The intersection of l with P is a line segment (since P is convex) that contains x. The endpoints lie each in a face of P, therefore x must be contained in the convex hull of the faces.

To show that any V-polytope P is an H-polytope we first apply duality to obtain an H-polytope P^* (by the above lemma). Now, we apply the arguments above to show that P^* is a V-polytope. Finally, we can apply duality again to obtain a H-polytope P^{**} which is equal to P.

2 Polytopes and planar graphs

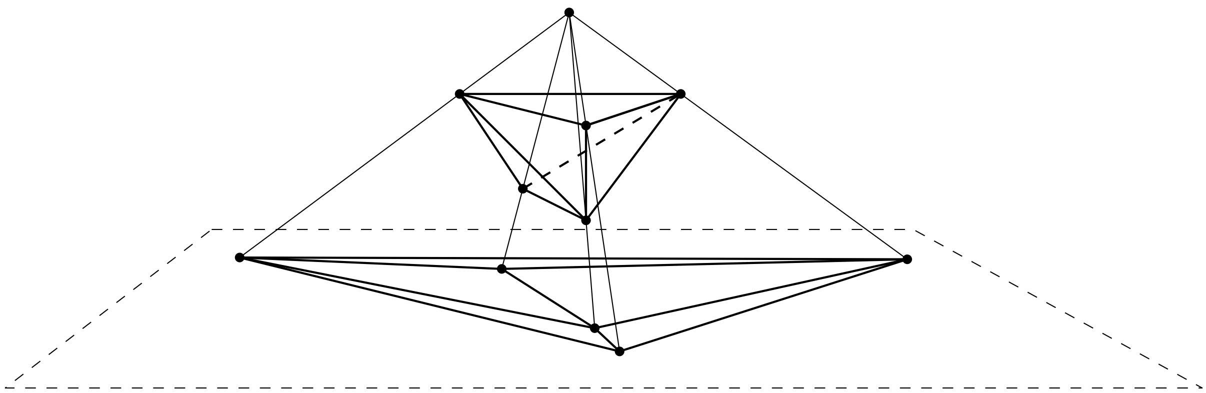
Interestingly, the incidence relations of the 1-faces and 2-faces of a convex polytope in $\mathbb{R}^3$ can be represented as a planar graph. A planar graph is a graph that can be drawn without crossings of edges. This is in fact, where the two terms *vertex* and *edge* which are used to refer to nodes and the relations of a graph originate—from their counterparts in polytopes.

To construct a planar graph from a 3-dimensional polytope, imagine the edges of the polytope to be constructed by wires and imagine rotating the polytope so that the top facet is horizontal. Now, imagine placing a light source above the top facet and a horizontal plane underneath the polytope. The shadow of the edges of the polytope which is cast onto the plane forms a edges of a planar graph. The light source can always be placed close enough to the surface that no two edges cross in the shadow. The resulting embedded planar graph is also called a Schlegel diagram.

Conversely, we can construct a polytope in $\mathbb{R}^3$ from every planar graph of a certain class. In particular, this is always possible if the graph is planar and vertex 3-connected, which means that the graph has at least 4 vertices and deleting fewer than 3 vertices leaves the graph connected. The exact relationship is formalized in a theorem by Steinitz, which we do not prove here.

Theorem 11.2 (Steinitz theorem)**.** *A finite graph is isomorphic to the graph of a 3-dimensional convex polytope if and only if it is planar and vertex 3-connected.*

Theorem 11.3. *Let P be a convex polytope in $\mathbb{R}^3$ with $n \geq 4$ vertices, e edges and f facets. It holds that $e \leq 3(n-2)$ and $f \leq 2(n-2)$.*



Proof. Euler's formula for polytopes gives us an exact relationship between the number of facets, edges and vertices of a polytope in $\mathbb{R}^3$:

$$n - e + f = 2 \tag{1}$$

Let d_i the number of edges on the boundary of the ith facet of the polytope. We can estimate the total sum of the values d_i in two different ways, as follows. Since every edge is incident to exactly two facets and every facet has at least three edges on its boundary, we can write

$$2e = \sum_{i=1}^{f} d_i \geq 3f$$

This implies $f \leq \frac{2}{3}e$, plugging into Eulers formula (1) we get

$$e = f + n - 2 \leq \frac{2}{3}e + n - 2$$

And this implies the two bounds claimed in the theorem. $\square$

If all facets of the polytope are triangles, which is the case if the polytope is the convex hull of a point set in general position, then the inequalities in the above theorem become equalities.

3 Upper bound theorem

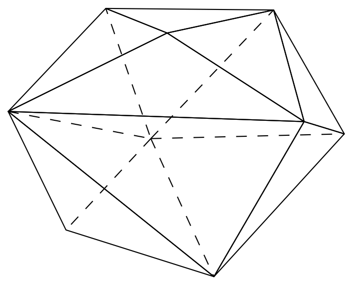
We now proceed to show an asymptotic upper bound on the number of faces of polytopes in higher dimensions. We want to simplify the exposition of the proof by assuming general position. This leads to the following two definitions.

Definition 11.4 (Simplicial polytope)**.** *A d-dimensional polytope P is called simplicial if each of its facets ($(d-1)$-faces) is a simplex (it contains exactly d vertices of P).*

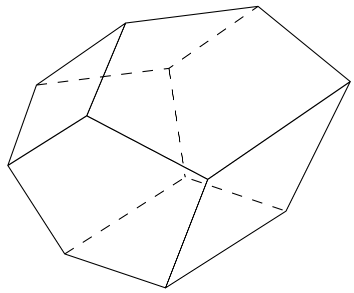
Definition 11.5 (Simple polytope)**.** *A d-dimensional polytope P is called simple if each of its vertices is contained in exactly d facets ($(d-1)$-faces).*

We recall the definition of general position and generalize it to $\mathbb{R}^d$.

Assumption 11.6. *(General position) For a given set of more than d points in $\mathbb{R}^d$ we may assume that no $d+1$ points lie on a common hyperplane. For a given set of more than d hyperplanes in $\mathbb{R}^d$ we may assume that no $d+1$ of them share a common point.*



(a) simplicial polytope



(b) simple polytope

A simplicial polytope can be obtained by taking the convex hull of a finite set of points in general position. A simple polytope can be obtained by taking the intersection of a finite number of halfspaces, where the bounding hyperplanes lie in general position.

However, in general, we cannot assume a polytope to be simple and simplicial at the same time (unless we only consider very restriced polytopes). The dual of a simple polytope is a simplicial polytope and vice versa.

Before we state the upper bound theorem, we want to state some observations regarding the combinatorial structure of the face lattices of simplicial and simple polytopes.

Observation 11.7. *Each $(d-1)$-face f of a d-dimensional simplicial polytope P contains exactly $\binom{d}{k}$ $(k-1)$-faces for $k = 0, 1, \ldots, d$. This follows from the fact that the faces of a simplex are again simplices, so every face (except for the (-1)-face and d-face) of a simplicial polytope is a simplex. Therefore, f contains d vertices of P and each subset of size k of these vertices forms a $(k-1)$-face, which is again a simplex, and which is contained in f.*

Observation 11.8. *Each vertex v of a d-dimensional simple polytope P is contained in exactly $\binom{d}{k}$ k-faces for $k = 0, \ldots, d$. This follows from the fact that v is incident to exactly d facets. Every subset of size $d-k$ of these facets uniquely determines a k-face incident to v, and there are $\binom{d}{d-k} = \binom{d}{k}$ subsets of size $d-k$.*

Theorem 11.9. *For a d-dimensional convex polytope P with n vertices the total number of faces is in $O(n^{\lfloor d/2 \rfloor})$, assuming d is constant.*

Proof for simplicial polytopes. Let $f_k(P)$ denote the number of k-faces of P for $k = 1, \ldots, d-1$. Assume the following holds:
(*a*) $f_0(P) + f_1(P) + \cdots + f_{d-1}(P) \leq 2^d \cdot f_{d-1}(P)$, and
(*b*) $f_{d-1}(P) \leq 2 \cdot f_{\lfloor d/2 \rfloor - 1}(P)$
This would imply the bound directly, since by Observation 11.7 every k-face of P is determined by $k+1$ vertices of P. For $k = \lfloor d/2 \rfloor - 1$ we get that every k-face is determined by a subset of $\lfloor d/2 \rfloor$ vertices of P. The total number of k-faces is thus bounded by the number of subsets of vertices of this size. Therefore,

$$f_{\lfloor d/2 \rfloor - 1}(P) \leq \binom{n}{\lfloor d/2 \rfloor}$$

and this would imply the bound in the theorem. It remains to prove (a) and (b). For this we consider the dual polytope P^*. By Claim 6.13 from the previous lecture there is a bijective correspondence between the j-faces of P and the $(d-j-1)$-faces of P^* for each j. Therefore, the following statements are equivalent to the above
(a') $f_{d-1}(P^*) + f_{d-2}(P^*) + \cdots + f_0(P^*) \leq 2^d \cdot f_0(P^*)$, and

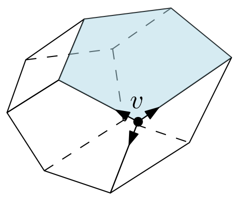
(b') $f_0(P^*) \leq 2 \cdot f_{\lceil d/2 \rceil}(P^*)$

Now, we can use Observation 11.8 to show the first statement. Fix a vertex v, this vertex is incident to

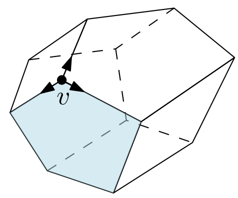
$$\sum_{i=0}^{d} \binom{d}{i} = 2^d$$

faces of dimension $i = 0, \ldots, d$. Since $f_0(P^*)$ denotes the number of vertices of P^*, the statement in (a') directly follows.

In order to prove (b'), we rotate the polytope P^* so that no two vertices have the same x_d-coordinate. Now, consider again a vertex v of P^* and let m_1 be the number of edges going upwards and let m_2 be the number of edges going downwards (considering the x_d-coordinate as vertical). By Observation 11.8, we have $m_1 + m_2 = d$ and therefore it must be that either (i) $m_1 \geq \lceil d/2 \rceil$, or (ii) $m_2 \geq \lceil d/2 \rceil$



(i) v is the lowest vertex on a face



(ii) v is the highest vertex on a face

In case (i), every subset of $\lceil d/2 \rceil$ edges going *upwards* from v determines a $\lceil d/2 \rceil$-face for which v is the *lowest* vertex. In case (ii), every subset of $\lceil d/2 \rceil$ edges going *downwards* from v determines a $\lceil d/2 \rceil$-face for which v is the *highest* vertex.

The lowest and highest vertices of each face are unique, therefore, the total number of vertices that are lowest or highest to a $\lceil d/2 \rceil$-face is at most twice the number of such $\lceil d/2 \rceil$-faces. At the same time, we just showed that every vertex v of P^* is lowest or highest vertex to at least one $\lceil d/2 \rceil$-face, therefore (b') follows. $\square$

References

- Jiří Matouŝek, Chapter 5, Lectures on Discrete Geometry, Springer Graduate Texts in Mathematics.
- Mark de Berg, Otfried Cheong, Marc van Kreveld, Mark Overmars. Computational Geometry— Algorithms and Applications. Third Edition. Springer. Chapter 2.2