DCG, Wintersemester 2024/25	Lecture 22 (5 pages)

The Art-Gallery Problem

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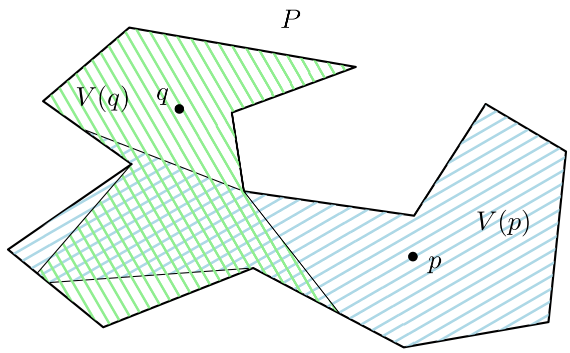
In this lecture we will show that the ε-net theorem can be applied to a classical problem in computational geometry, the art-gallery problem. In this problem we are given a polygon, which we think of as the floorplan of an art gallery, and we are asked to compute a small number of points, which we think of as guards, such that every point of the polygon is visible from at least one of the guards.

1 Definitions

Definition 22.1 (Simple polygon)**.** *Let $p_1, \ldots, p_n$ be a sequence of points. For convenience, we define $p_0 := p_n$ and $p_{n+1} = p_1$. Consider the closed chain of edges (p_i, p_{i+1}) for $i \in \{1, \ldots, n\}$. We say that this chain of edges is a simple polygon if no two edges intersect, except that each pair of consecutive edges (p_{i-1}, p_i) and (p_i, p_{i+1}) intersects in their common endpoint p_i. A simple polygon partitions the plane into two sets (interior and exterior). The set of points that lie inside the polygon or on an edge of the polygon form the interior and the remaining points in $\mathbb{R}^2$ form the exterior.*

Definition 22.2 (Visibility region)**.** *Let P be a simple polygon in the plane and let X be its interior. We say a point $q \in X$ is visible from another point $p \in X$ if the line segment connecting p and q is contained in X. The visibility region of a point p, denoted by $V(p)$, is the set of points that are visible from q.*



Definition 22.3 (The art gallery problem)**.** *The art gallery problem is to compute for a given simple polygon P with interior X a minimum number of points $G \subseteq X$ such that*

$$X = \bigcup_{p \in G} V(p)$$

In the example above, the set $\{p, q\}$ fulfills the condition of the set of guards G seeing everything inside the polygon. Equivalently, the visibility region of any point in the set X contains either p or q. It is a reasonable assumption that in an art gallery, the visibility region of any relevant point that needs to be guarded should be relatively large. If we can we assume that any visibility region covers at least a fraction of ε of the entire polygon, then the ε-net theorem gives us an upper bound on the minimum number of guards needed for this art gallery. But what is the VC-dimension of the corresponding set system?

2 The VC-dimension of visibility regions

We now consider the set system defined by the visibility regions of a fixed polygon P, and we show that its VC-dimension is constant.

Theorem 22.4. *Let X be the interior of a simple polygon. Consider the set system*

$$\mathcal{R} = \{V(p) \mid p \in X\}$$

The ground set is X and every set in $\mathcal{R}$ is defined as the visibility region of a point in X. The VC-dimension of $\mathcal{R}$ is upper-bounded by a constant independent of X.

The idea of the proof is the following. Assume for the sake of contradiction that there exist aribitrarily large subsets of X that are shattered by $\mathcal{R}$. Then, let $A \subset X$ be such a large set. We derive from A that there must be two sets (not too small) A_3 and S_3, such that A_3 is shattered by the visibility regions of points in S_3 and such that $\operatorname{conv}(A_3) \cap \operatorname{conv}(S_3) = \emptyset$, and moreover, no line determined by two points in A_3 intersects $\operatorname{conv}(S_3)$ and, vice versa, no line determined by two points of S_3 intersects $\operatorname{conv}(A_3)$. We then argue that such a configuration cannot exist, which then concludes the proof by contradiction.

The proof uses the notion of a dual set system, which is defined as follows.

Definition 22.5 (Dual set system)**.** *Let $\mathcal{R}$ be a finite set system with ground set X. The dual set system $\mathcal{R}^*$ is defined as follows. The ground set is the set*

$$Y = \{y_r \mid r \in \mathcal{R}\}$$

and the set system $\mathcal{R}^*$ consists of the following sets. For each $x \in X$, there is a $r_x \in \mathcal{R}^*$, such that

$$r_x = \{y_r \mid x \in r\}$$

Example 22.6. *We give an example of a finite set system and its dual. Any set system can be specified by a binary matrix with $|X|$ columns and $|\mathcal{R}|$ rows. The entry at a_{ij} is 1 if and only if the jth element of X is included in the ith set of $\mathcal{R}$. We call this representation an incidence matrix. For example, a set system with $X = \{1, 2, 3, 4\}$ and sets $\mathcal{R} = \{\{2, 3, 4\}, \{1, 3, 4\}, \{2, 4\}\}$ is specified by the following matrix.*

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

In this representation, the dual set system is simply the transpose of this matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

The ground set is a set $Y = \{1, 2, 3\}$ and the sets are $\mathcal{R}^* = \{\{2\}, \{1, 3\}, \{1, 2\}, \{1, 2, 3\}\}$

The following lemma can be shown by a careful analysis of the incidence matrix of a set system and of its dual. We omit the proof here.

Lemma 22.7. *Let $\mathcal{R}$ be a set system with VC-dimension δ. The dual set system $\mathcal{R}^*$ has VC-dimension at least $\lfloor \log_2 \delta \rfloor$ and at most $2^{\delta+1} - 1$.*

Proof of Theorem 22.4. Let d be a parameter value. Assume there exists a set $A \subset X$ shattered by $\mathcal{R}$ with $|A| = \mathrm{d}$. If A is shattered, then there exists a set $S \subset X$ of size 2^{d} such that for each $B \subseteq A$ there is an $s_B \in S$ with $V(s_B) \cap A = B$.

Consider the arrangement $\mathcal{A}$ of the set of lines through pairs of points of A. Let m be the number of lines and let f be the number of faces of the arrangement. It holds for the number of lines that $m = \binom{\mathrm{d}}{2}$. By the analysis from Lecture 13, we have for the total number of faces

$$f \leq \underbrace{\binom{m}{2} + m + 1}_{\text{cells}} + \overbrace{m^2}^{\text{edges}} + \underbrace{\binom{m}{2}}_{\text{vertices}} = 2m^2 + 1 \leq \mathrm{d}^4$$

By the pigeonhole principle, there must be a face containing a subset $S' \subseteq S$ of at least $\frac{|S|}{f}$ points of S. Thus, we have

$$|S'| \geq \frac{|S|}{f} \geq \frac{|S|}{\mathrm{d}^4} = \frac{2^{\mathrm{d}}}{\mathrm{d}^4}$$

Now consider the set system $\mathcal{R}'$ on ground set A where

$$\mathcal{R}' = \{V(p) \cap A \mid p \in S'\}$$

Note that each $p \in S'$ defines a distinct set in $\mathcal{R}'$ and therefore, $|\mathcal{R}'| = |S'|$. Let d_1 be the VC-dimension of $\mathcal{R}'$ (so there is a set $A_1 \subseteq A$ such that $\left|A_1\right| = \mathrm{d}_1$ and $\left|A_1\right|$ is shattered by $\mathcal{R}'$).

By the growth lemma (Lecture 23) and by the above, it holds that

$$\frac{2^{\mathrm{d}}}{\mathrm{d}^4} \leq \left|S'\right| = \left|\mathcal{R}'\right| = \left|\mathcal{R}'|_A\right| = \max_{\substack{A'' \subseteq A \\ |A''| = \mathrm{d}}} \left|\mathcal{R}'|_{A''}\right| \leq \left(\frac{\mathrm{ed}}{\mathrm{d}_1}\right)^{\mathrm{d}_1} \leq (\mathrm{ed})^{\mathrm{d}_1}$$

From this we can follow that

$$\mathrm{d}_1 \geq \frac{\mathrm{d} - \log_2\left(\mathrm{d}^4\right)}{\log_2(\mathrm{ed})} \in \Omega(\mathrm{d}/\log_2 \mathrm{d})$$

This implies that for a large set A we can always find a large set $A_1 \subseteq A$ such that A_1 is shattered by the visibility regions of a set $S_1 \subseteq S'$ that is contained in a single face of the arrangement determined by the points of A. We have derived the following set system with ground set A_1.

$$\mathcal{R}_1 = \{V(p) \cap A_1 \mid p \in S_1\}$$

Note that by construction its VC-dimension is d_1.

Now, we want to apply the same procedure to obtain a similar condition on the reverse direction. To this end, consider the set system dual to $\mathcal{R}_1$.

$$\mathcal{R}_2 = \{V(a) \cap S_1 \mid a \in A_1\}$$

Let d_2 be the VC-dimension of this set system. By Lemma 22.7 its VC-dimension is at least $\lfloor \log_2(\mathrm{d}_1) \rfloor$.

Now, select a (large) set $A_2 \subseteq S_1$ and a set $S_2 \subseteq A_1$, such that A_2 is shattered by the visibility regions of the points of S_2 and $|A_2| = \mathrm{d}_2$.

As before, consider the arrangement $\mathcal{A}'$ of lines through pairs of points of A_2 and choose a (large) subset $S_3 \subseteq S_2$ that lie in a single face of $\mathcal{A}'$, using the same construction as before. We obtain a set A_3 of size d_3 (sufficiently large) such that A_3 is shattered by the visibility regions

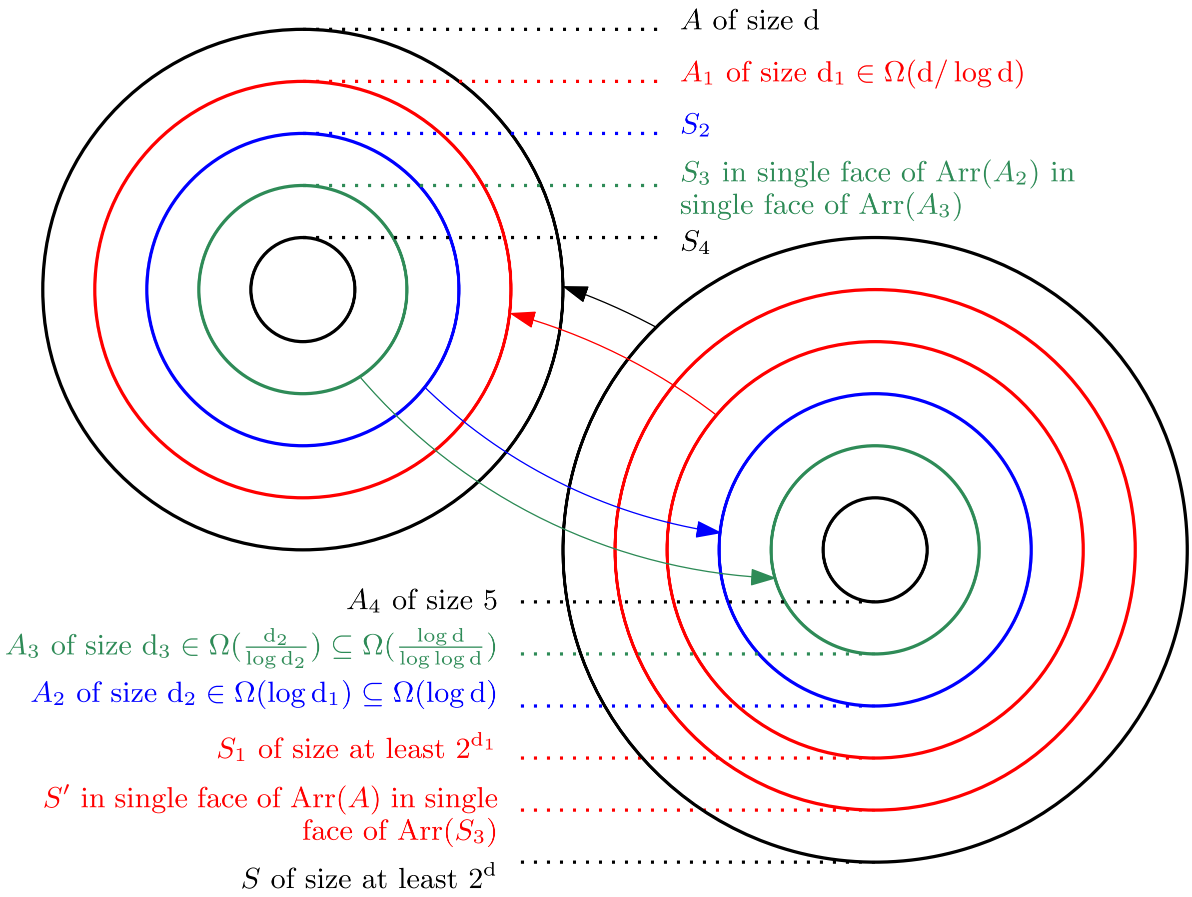


Figure 1: Relations between the sets in the proof of Theorem 22.4. An arrow from a set Y to a set Z indicates that the visibility regions of the points in Y shatter Z.

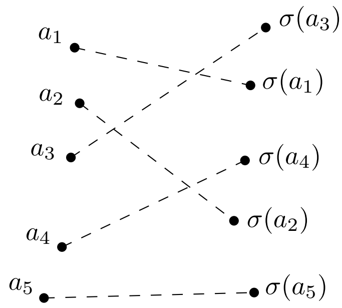
of a set $S_3 \subseteq S_2$ and all of S_3 lies in a single face of the arrangement $\mathcal{A}'$ determined by the points of A_2; see Figure 1 for a sketch of the relations between the sets defined so far.

Note that we have $S_3 \subseteq S_2 \subseteq A_1 \subseteq A$ and $A_3 \subseteq A_2 \subseteq S_1 \subseteq S$. By construction, S_1 lies in a single cell of the arrangement of lines through pairs of points of A, and thus, $A_3 \subseteq S_1$ also lies in a single cell of the coarser arrangement that is defined by lines through pairs of points of $S_3 \subseteq A$. Likewise, S_3 lies in a single cell of the arrangement of lines through pairs of points of A_2, and thus, S_3 also lies in a single cell of the coarser arrangement that is defined by lines through pairs of points of $A_3 \subseteq A_2$.

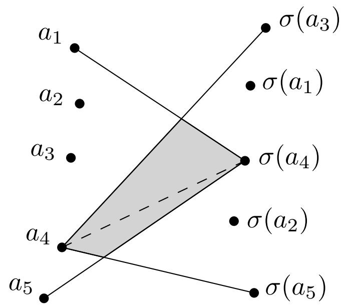
Thus, we have now derived two sets A_3 and S_3 with the conditions mentioned in the proof sketch above: A_3 is shattered by the visibility regions of points in S_3, and the convex hull of either set is not cut by any line through a pair of points of the other set. We now want to argue that such a configuration cannot exist for arbitrarily large values of d, which then concludes the proof by contradiction.

Recall that d_3 is bounded from below by a growing function of d_2, which is bounded from below by a growing function of d_1, which is bounded from below by a growing function of d. Suppose that the initial d was so large that $d_3 = |A_3| \geq 5$. Now, choose a subset A_4 of 5 points from A_3. For each $a \in A_4$ consider a point $\sigma(a) \in S_3$ that sees all points of A_3 except a. Let $S_4 = \{\sigma(a) \mid a \in A_4\}$.

We have the following situation. (Dashed lines correspond to *invisibility* and they form a matching.)



Moreover, because no line through a pair of points from A_3 intersects $\text{conv}(S_3)$, the points $\sigma(a_i)$ and $\sigma(a_j)$ must see the three points $A_4 \setminus \{a_i, a_j\}$ in the same clockwise angular order; combining these three points' orderings for all pairs i, j defines a unique clockwise angular order for the entire set A_4. By a symmetric argument, we obtain a unique counterclockwise angular ordering of S_4. Now, choose an $a_i \in A_4$ such that a_i is neither the first nor the last and such that $\sigma(a_i)$ is neither the first nor the last. There must be such a point, since we have 5 points to choose from. Then we have a situation such as the following (dashed lines correspond to invisibility and solid lines correspond to visibility):



By definition, a_i (a_4 in the example) can see the first and the last point of S_4. Moreover, $\sigma(a_i)$ can see the first and the last point of A_4. This defines four solid segments (uninterrupted lines of sight) that are part of X, the interior of a simple polygon, so the quadrilateral in the middle (shaded in the figure) must also be part of X. Therefore, it is not possible that a_i and $\sigma(a_i)$ are invisible to each other. This contradiction proves the theorem. □

References

- Jiří Matoušek, Chapter 10.3 in *Lectures on Discrete Geometry*, Springer Graduate Texts in Mathematics.