Helly's theorem and center-points

Anne Driemel updated: October 14, 2024

In the previous two lectures we discussed fundamental properties of convex sets in $\mathbb{R}^d$ and properties of the convex hull of a finite set of points in $\mathbb{R}^d$. Today we will discuss Helly's fundamental theorem on convex sets and an application of this theorem to prove the existence of a center-point of a finite set of points in $\mathbb{R}^d$.

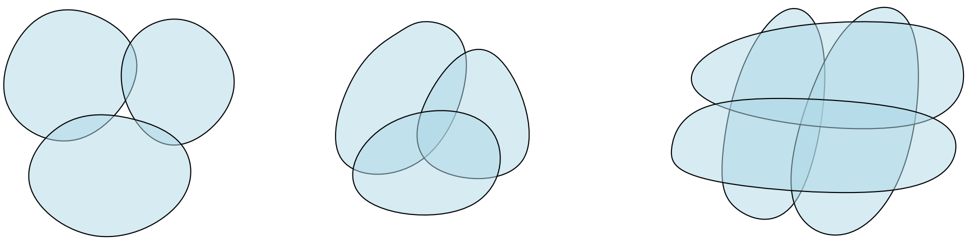
We will need the following fundamental fact, which is implied by the separation theorem.

Claim 3.1. *Any convex set C that contains a finite set $A \subseteq \mathbb{R}^d$ also contains* $\operatorname{conv}(A)$.

1 Helly's theorem

Theorem 3.2 (Helly's theorem)**.** *Let $C_1, C_2, \ldots, C_n$ be a finite collection of convex sets in $\mathbb{R}^d$ with $n \geq d+1$. Suppose that the intersection of every $d+1$ of these sets is non-empty, then the common intersection of the C_i is non-empty.*

Example 3.3. *Three examples of n convex sets in $\mathbb{R}^2$. In (a), every d sets have non-empty intersection, but the common intersection is empty. Example (b) shows the trivial case that $n = d+1$ and the $d+1$ sets have non-empty intersection. In (c), every $d+1$ sets have non-empty intersection and the common intersection is non-empty, as is assured by Helly's theorem.*



(a) (b) (c)

Proof. The proof is by induction on n for any fixed d. For the base case of $n = d+1$ the theorem is trivially satisfied. So assume $n > d+1$ and assume we have sets $C_1, \ldots, C_n$ satisfying the assumptions. By induction, we can assume that for each C_i it holds

$$\bigcap_{i \neq j} C_j \neq \emptyset$$

Now, for each C_i, fix a point $a_i \in \bigcap_{i \neq j} C_j$ and consider the set $A = \{a_1, \ldots, a_n\}$. Since n is at least $d+2$, we have by Radon's lemma that there exist disjoint index sets $I_1, I_2 \subset \{1, \ldots, n\}$ such that

$$\operatorname{conv}(A_1) \cap \operatorname{conv}(A_2) \neq \emptyset$$

for $A_1 = \{\ a_i \in A \ \mid \ i \in I_1 \ \}$ and $A_2 = \{\ a_i \in A \ \mid \ i \in I_2 \ \}$. Now, let q be a Radon point of this partition. That is, $q \in \operatorname{conv}(A_1) \cap \operatorname{conv}(A_2)$.

We claim that $q \in C_i$ for any $i \in \{1, \ldots, n\}$. This would prove the theorem as q would then be a point contained in the common intersection of these sets. We prove that this claim holds true. Since $I_1 \cap I_2 = \emptyset$, we have two cases

(i) $i \notin I_1$: In this case, $A_1 \subseteq C_i$ and therefore also $\operatorname{conv}(A_1) \subseteq C_i$.
(ii) $i \notin I_2$: In this case, $A_2 \subseteq C_i$ and therefore also $\operatorname{conv}(A_2) \subseteq C_i$.

Here we used Claim 3.1 applied to the set A_1 (respectively, A_2) and the convex set C_i. In both cases, it follows that $q \in C_i$, since by construction q is contained in the convex hull of both sets A_1 and A_2. □

2 Center-points

Definition 3.4 (Center-point)**.** *Let A be a finite set of n points in $\mathbb{R}^d$. A point $p \in \mathbb{R}^d$ is called a center-point of A if any closed halfspace containing p contains at least $\frac{n}{d+1}$ points of A.*

Example 3.5. *An example of a center-point for $d = 1$ is the median of a set of numbers.*

Theorem 3.6 (Center-point theorem)**.** *Any finite set $A \subseteq \mathbb{R}^d$ has at least one center-point.*

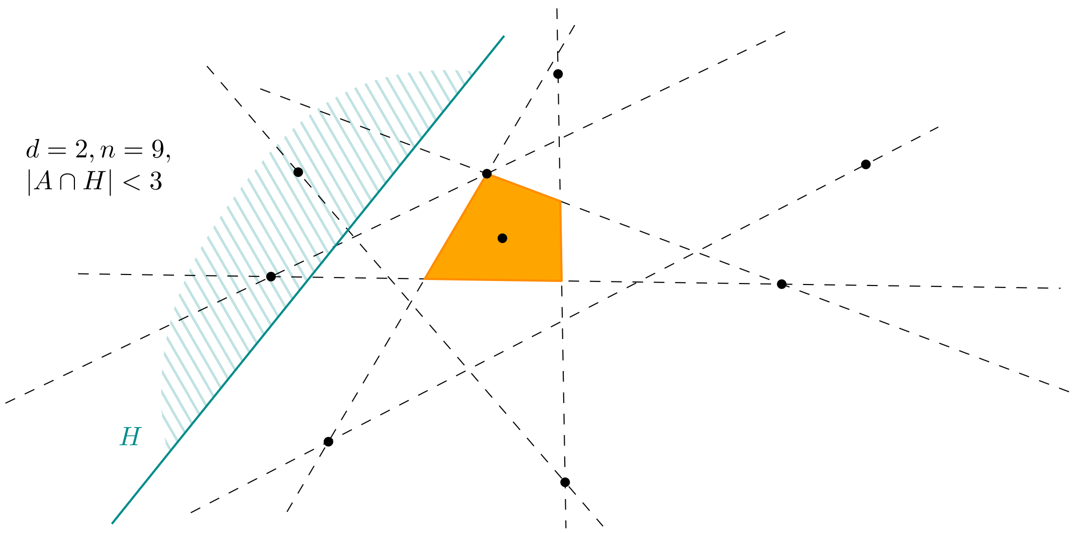
Before we prove the theorem, we want to illustrate the concept of the center-point. We can first rephrase the definition. In other words, $p \in \mathbb{R}^d$ is not a center-point if and only if p is contained in a closed halfspace $H^{\leq}_{a,b}$

$$H^{\leq}_{a,b} = \left\{ \, x \in \mathbb{R}^d \;\middle|\; \langle a, x \rangle \leq b \, \right\}$$

such that $|A \cap H^{\leq}_{a,b}| < \frac{n}{d+1}$. Thus, we can write the set of center-points of A as follows

$$S_A = \mathbb{R}^d \setminus \bigcup_{\substack{a \in \mathbb{R}^d, b \in \mathbb{R} \\ |A \cap H^{\leq}_{a,b}| < \frac{n}{d+1}}} H^{\leq}_{a,b}$$

Example 3.7. *Using the equation above, we can visualize the set of center-points in an example of $n = 9$ points in $\mathbb{R}^2$ by constructing the union of halfspaces that contain at most 2 points.*



Proof of Theorem 3.6. In order to apply Helly's theorem to show the existence of a center-point, we want to rewrite the union as an intersection. We can rewrite the set of center-points using

De Morgan's law for sets as follows

$$S_A = \bigcap_{\substack{a\in\mathbb{R}^d, b\in\mathbb{R} \\ |A\cap H^{\leq}_{a,b}|<\frac{n}{d+1}}} \mathbb{R}^d \setminus H^{\leq}_{a,b}$$

Now, we define the open halfspace $H^{>}_{a,b}$ as the complement of the closed halfspace $H^{\leq}_{a,b}$:

$$\mathbb{R}^d \setminus H^{\leq}_{a,b} = H^{>}_{a,b} = \left\{ x \in \mathbb{R}^d \;\middle|\; \langle a, x\rangle > b \right\}$$

and using $n - \frac{n}{d+1} = \frac{dn}{d+1}$ and the fact that $H^{>}_{a,b} \cap H^{\leq}_{a,b} = \emptyset$, so $|A \cap H^{>}_{a,b}| + |A \cap H^{\leq}_{a,b}| = n$ and $|A \cap H^{>}_{a,b}| > \frac{dn}{d+1}$ if and only if $|A\cap H^{\leq}_{a,b}| < \frac{n}{d+1}$, we can rewrite the set of center-points as follows

$$S_A = \bigcap_{\substack{a\in\mathbb{R}^d, b\in\mathbb{R} \\ |A\cap H^{>}_{a,b}|>\frac{dn}{d+1}}} H^{>}_{a,b} \tag{1}$$

We now want to apply Helly's theorem, but we cannot apply it directly since the above is not a finite intersection. We will consider the convex hulls of certain subsets of A as the convex sets on which the theorem can be applied. Let $A_1, \dots, A_m$ be the subsets of A for which it holds that

(i) $\left|A_i\right| > \frac{dn}{d+1}$

(ii) there exist $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$, such that $A_i = A \cap H^{>}_{a,b}$

Now, we can write the set of center-points using (1) as follows

$$S_A = \bigcap_{1\leq i\leq m} \bigcap_{\substack{a\in\mathbb{R}^d, b\in\mathbb{R} \\ A_i\subseteq H^{>}_{a,b}}} H^{>}_{a,b} = \bigcap_{1\leq i\leq m} \operatorname{conv}(A_i) \tag{2}$$

Here, the last equality follows from a variation of Theorem 2.8 from the previous lecture.[1] Clearly, $m \leq 2^n$ and thus the sequence of sets $C_1 = \operatorname{conv}(A_1), \dots, C_m = \operatorname{conv}(A_m)$ is a finite collection of convex sets. Now, we are ready to apply Helly's theorem.

Consider any index set $I \subseteq \{1, \dots, m\}$ of size at most $d+1$. We claim that $\bigcap_{i\in I} \operatorname{conv}(A_i) \neq \emptyset$ for any such index set I. We show this by showing that

$$\Big|A \setminus \bigcap_{i\in I} \operatorname{conv}(A_i)\Big| < n \tag{3}$$

By De Morgan's law for sets

$$\Big|A \setminus \bigcap_{i\in I} \operatorname{conv}(A_i)\Big| = \Big|\bigcup_{i\in I} A \setminus \operatorname{conv}(A_i)\Big| \leq \sum_{i\in I} \Big|A \setminus \operatorname{conv}(A_i)\Big| \tag{4}$$

By the properties[2] of the A_i we have that

$$\big|A \setminus \operatorname{conv}(A_i)\big| = \big|A \setminus A_i\big| < n - \frac{dn}{d+1} = \frac{n}{d+1}$$

[1] We showed that the convex hull of a finite set A is equal to the intersection of closed halfspaces that contain A. Using the same proof one can show that the convex hull of A is equal to the intersection of open halfspaces that contain A.

[2] A_i is chosen as a subset of A that can be obtained from the intersection of A with a halfspace, therefore A_i contains all points of A that lie in the convex hull of A_i. This can be shown using the separation theorem.

We can use this to bound the sum in (4) and obtain (3) as claimed, which implies

$$\bigcap_{i \in I} \mathrm{conv}(A_i) \neq \emptyset$$

Now, if $m < d+1$, this implies by (2) that S_A is non-empty. Otherwise, this implies that the intersection of any $d+1$ sets of the collection $C_1, \ldots, C_m$ is non-empty and from Helly's theorem we obtain that S_A is non-empty. □

3 Halfspace depth

For large values of d, the factor of $\frac{1}{d+1}$ in the definition of the center point becomes very small, which may be undesirable. On the other hand, choosing a larger factor in the definition may lead to the situation that no center point exists. In fact, it is known that, in this sense, the center-point theorem is tight. In particular, for any d and $n \geq d+1$, we can construct a set $A \subseteq \mathbb{R}^d$ of n points, such that for any $q \in \mathbb{R}^d$ there exists a halfspace H containing q, such that $|H \cap A| \leq \frac{n}{d+1}$. Figure 1 shows the construction using $n = 30$ and $d = 2$. This illustrates that we cannot guarantee that there exists a center point when we require a larger fraction in the definition of the center point.

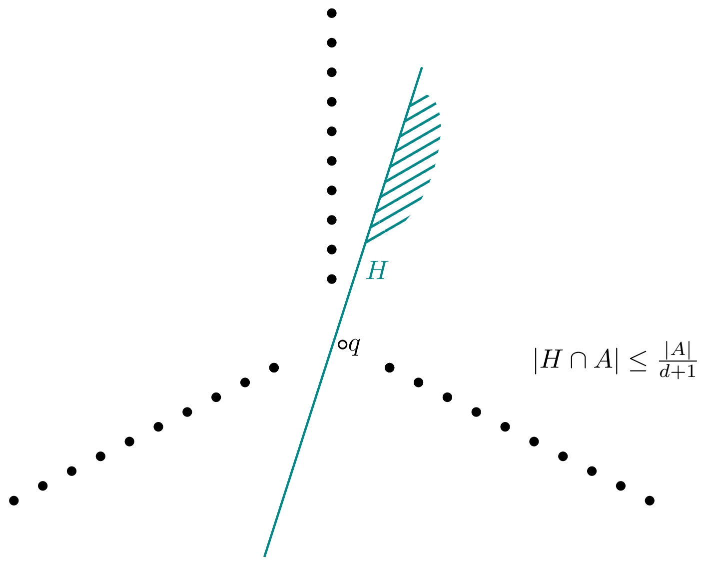


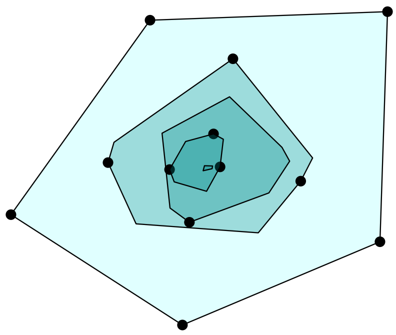
Figure 1: A set of points $A \subseteq \mathbb{R}^2$, such that for any $q \in \mathbb{R}^2$ there exists a halfspace H containing q which has the property that $|A \cap H| \leq \frac{|A|}{3}$. This shows that the factor of $\frac{1}{d+1}$ in the definition of the center-point is tight.

This leads to the following definition of depth in a point set, which connects the convex hull with the center point.

Definition 3.8. *Let $A \subseteq \mathbb{R}^d$ be a finite set of points and let $q \in \mathbb{R}^d$. Let*

$$\mathrm{depth}_A(q) = \min_{\substack{a \in \mathbb{R}^d, b \in \mathbb{R} \\ q \in H^{\leq}_{a,b}}} |A \cap H^{\leq}_{a,b}|$$

Example 3.9. *The figure below shows regions of different depth with respect to A indicated by different colors. The convex hull of A contains all points of depth at least 1. A center-point is a point with depth at least $\frac{n}{d+1}$; in this example, these are the points in the two innermost regions (including their boundary).*



References

- Jiří Matoušek, Chapter 1, Lectures on Discrete Geometry, Springer Graduate Texts in Mathematics.