Signal processing and inverse problems

Computational Photography

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5-minute Fourier recap

Frequency domain

$$\hat{f}(\nu) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \nu x}\, dx$$

Fourier transform

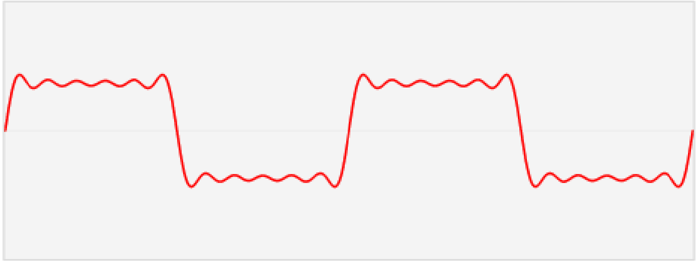
$\mathcal{F}$

Inverse Fourier transform

$\mathcal{F}^{-1}$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\nu) e^{2\pi i \nu x}\, d\nu$$

Primal domain
(spatial, temporal)



Animation by Lucas V. Barbosa
https://commons.wikimedia.org/wiki/File:Fourier_transform_time_and_frequency_domains.gif

Important properties of Fourier transform

Primal domain	**Fourier domain**	**Name**
$\alpha f + \beta g$	$\alpha \hat{f} + \beta \hat{g}$	Linearity
$(f \otimes g)(x)$	$\left(\hat{f} \cdot \hat{g}\right)(\nu)$	Convolution theorem
$f(\alpha x)$	$^1/_{\lvert\alpha\rvert}\, \hat{f}(^\nu/_\alpha)$	Scaling
$f(x, y)$	$\mathcal{F}_y\left(\mathcal{F}_x(f)\right)$	Separability in n-dim.
$^{d^n f(x)}/_{dx^n}$	$(2\pi i\nu)^n \hat{f}(\nu)$	Derivatives

Prominent Fourier pairs:

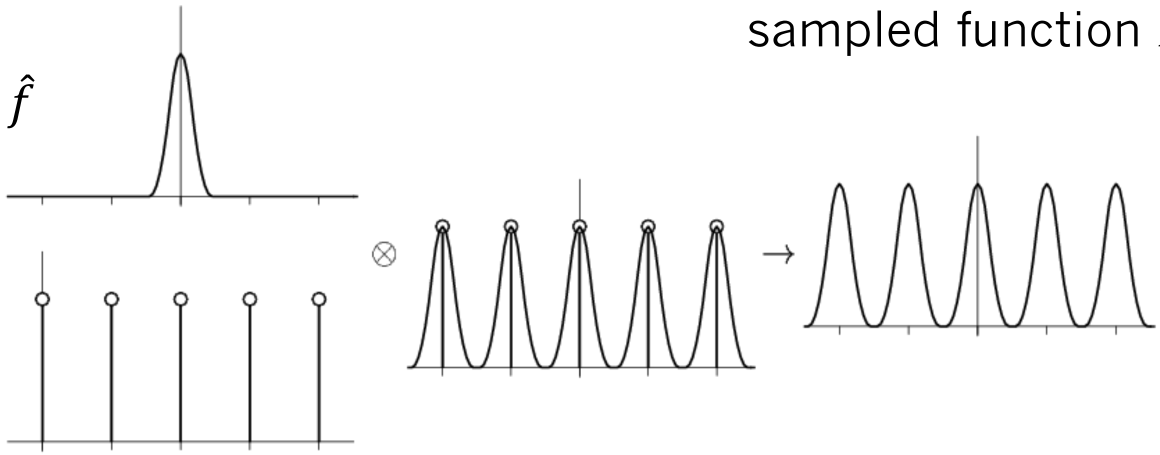
1	$\delta(\nu)$	Constant – Dirac pulse
$\sum_{n=-\infty}^{\infty} \delta(x - nT)$	$^1/_T \sum_{k=-\infty}^{\infty} \delta(\nu - {}^k/_T)$	Dirac comb
$\mathrm{rect}(x)$	$\sin(\nu)/\nu$	Box function – sinc
$e^{-\alpha x^2}$	$\sqrt{^\pi/_\alpha}\, e^{-^{(\pi\nu)^2}/_\alpha}$	Gaussian

Signal Processing Basics – Sampling

Spectrum of
continuous function $\hat{f}$

Spectrum of
sampling function $\hat{s}$

Spectrum of
sampled function F_s



Sampling in the Frequency Domain

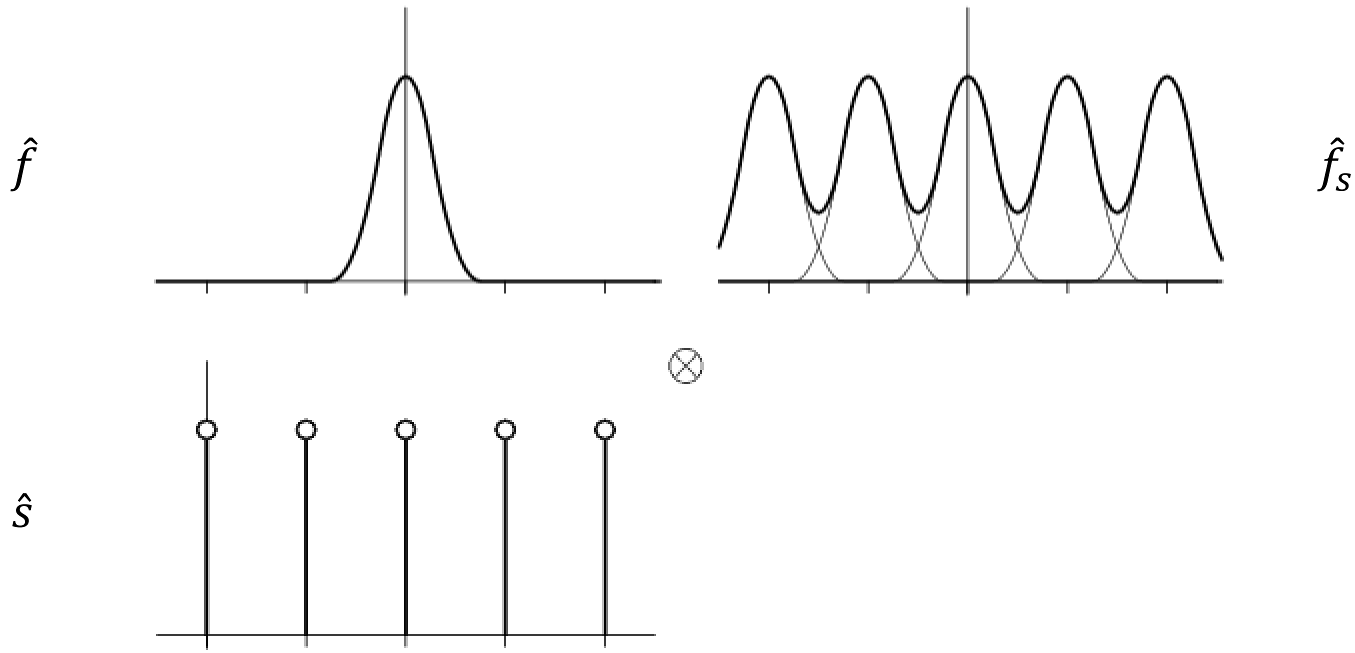
Convolution theorem

=> in frequency domain: $\hat{f} \otimes \hat{s} = \hat{f}_s$

Frequency spectrum of original function is copied multiple times!

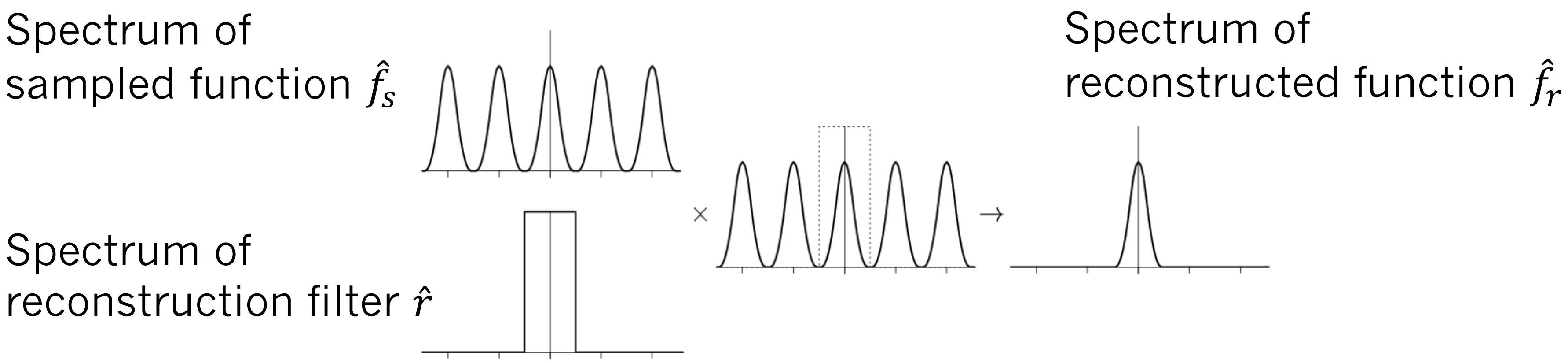
Signal Processing Basics – Aliasing

- If f has too high frequency content, aliasing occurs
- Undersampling: Overlap of copies in spectrum $\hat{f}_s$



Signal Processing Basics – Reconstruction

- Frequency-domain reconstruction is simple:
  - Suppress copies of the frequency spectrum
  - Multiply by box function (reconstruction filter)

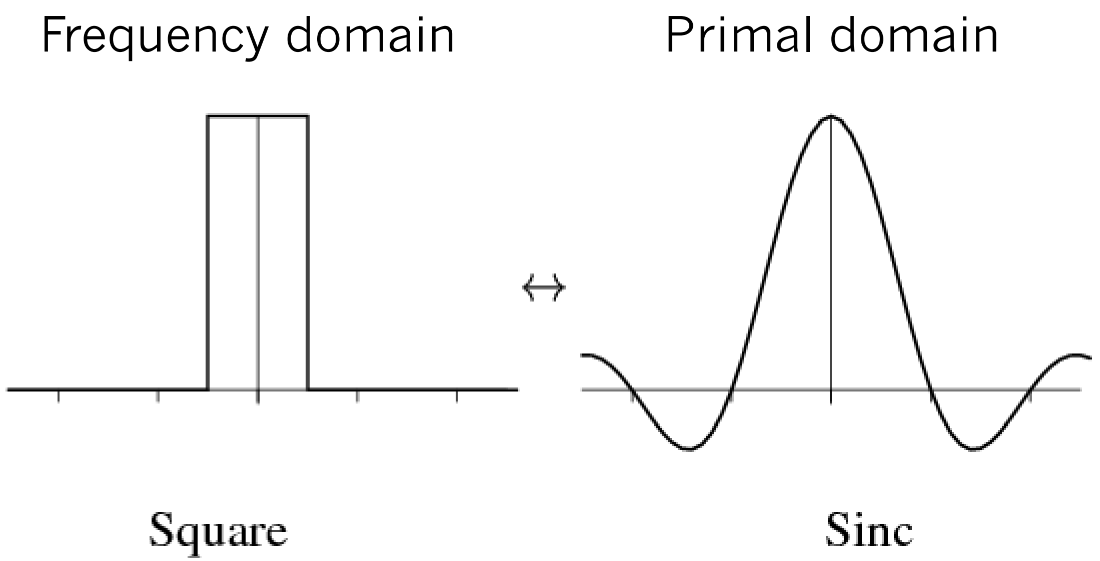


Reconstruction in the Frequency Domain

- Reconstruction in frequency domain: $\hat{f}_s \cdot \hat{r} = \hat{f}_r$

Signal Processing Basics – Reconstruction

- Reconstruction in primal domain = convolution
  Convolution theorem: $f_s \otimes r = f_r$

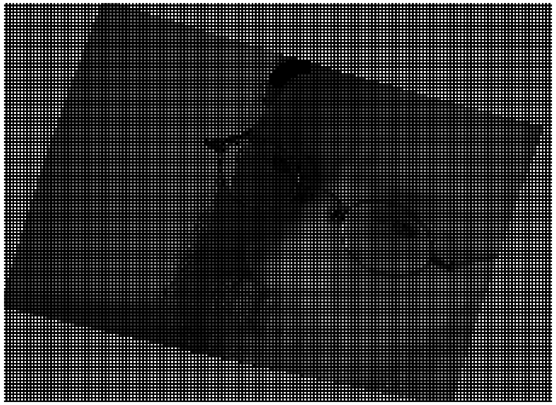


- Fourier transform of box is $\mathrm{sinc}(x) = \frac{\sin(x)}{x}$
- Infinite support: need *all* samples even for local reconstruction

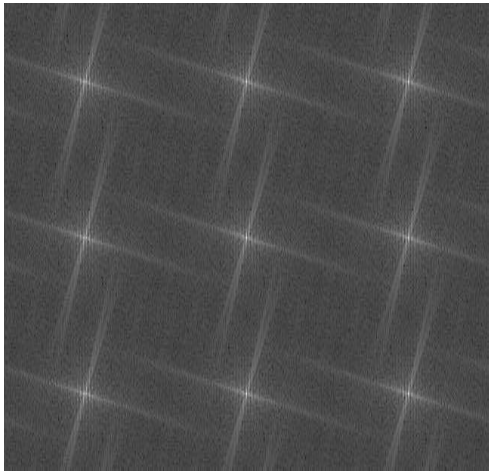
Multi-dimensional sampling

- 2D function – sampling function is "bed-of-nails"
- Spectral copies are spread in two dimensions

"Continuous" image



Sampled image



Fourier spectrum

Signal processing – Lessons learned

- Function f needs to be band-limited
  - e.g., low-pass filtered to narrow support in frequency domain
- Sampling rate must be sufficient
  - Increases spacing between copies of spectrum
- Reconstruction filter with local support in frequency domain
  - Ideal: box filter – sinc

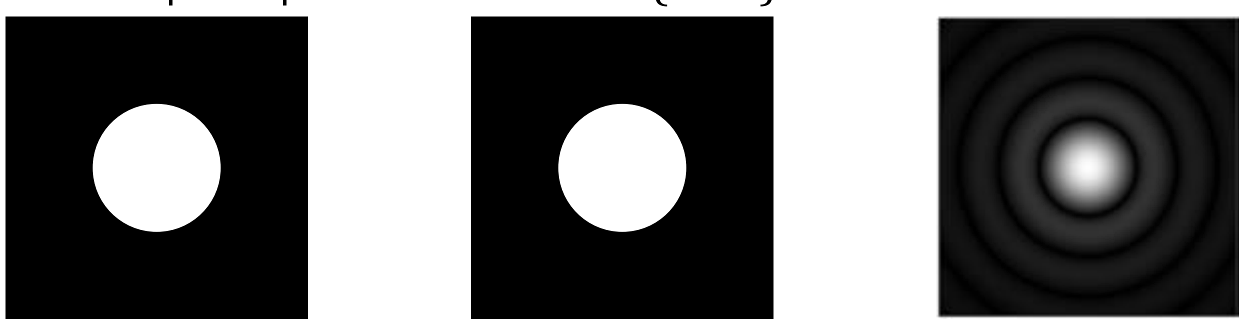
Deconvolution

MTF, OTF, PSF

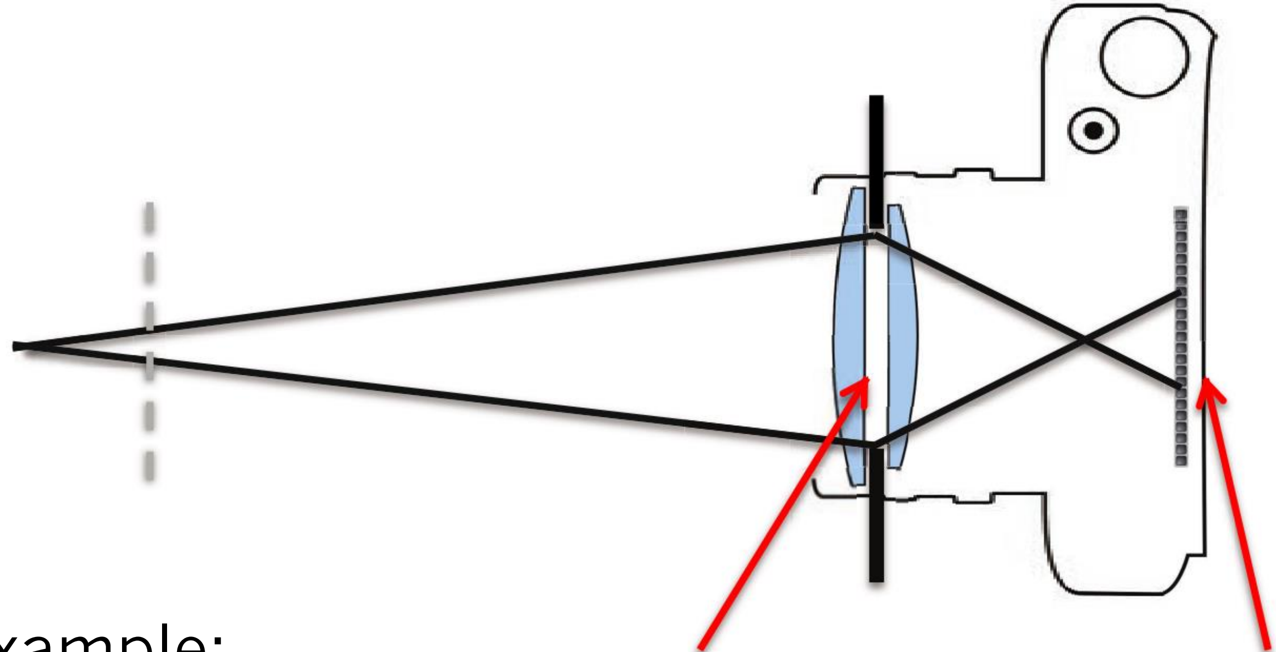
- Point spread function (PSF) –
  fundamental concept in optics: how does a single scene point spread out in the image?
- Optical transfer function (OTF) –
  (complex) Fourier transform of PSF
- Modulation transfer function (MTF) –
  magnitude of OTF

Example:

MTF=|OTF| OTF= $\mathcal{F}\{\text{PSF}\}$ PSF

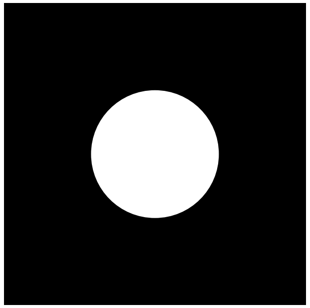


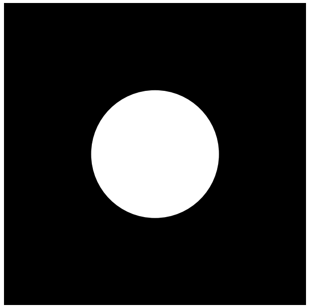
MTF, OTF, PSF

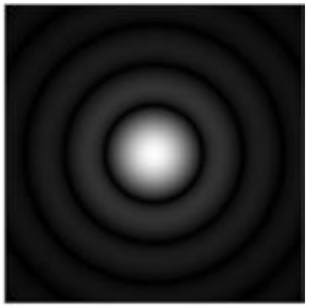


Example:

MTF=|OTF| OTF= $\mathcal{F}\{PSF\}$ PSF

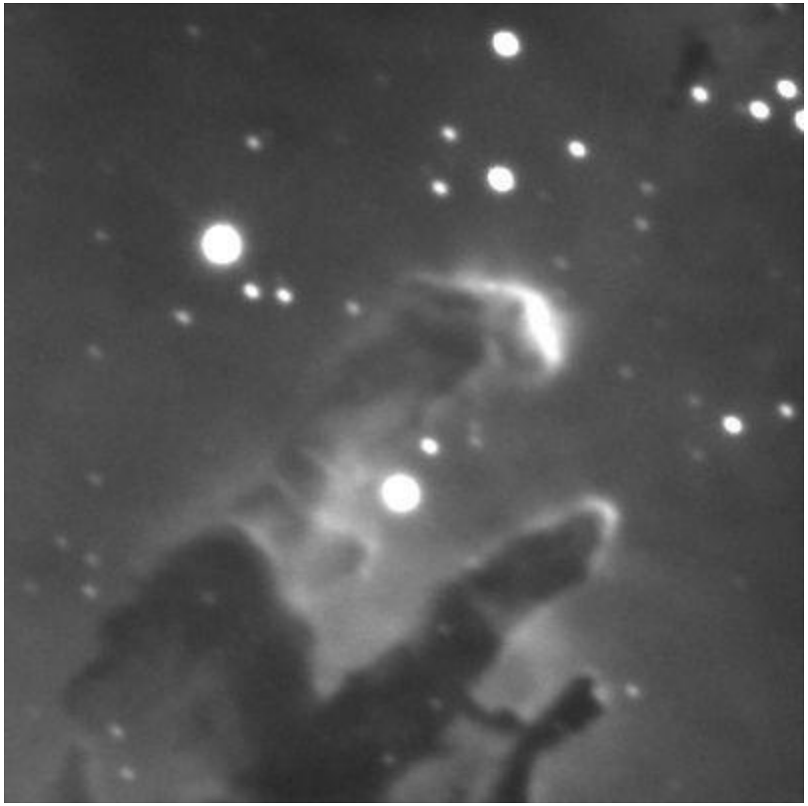






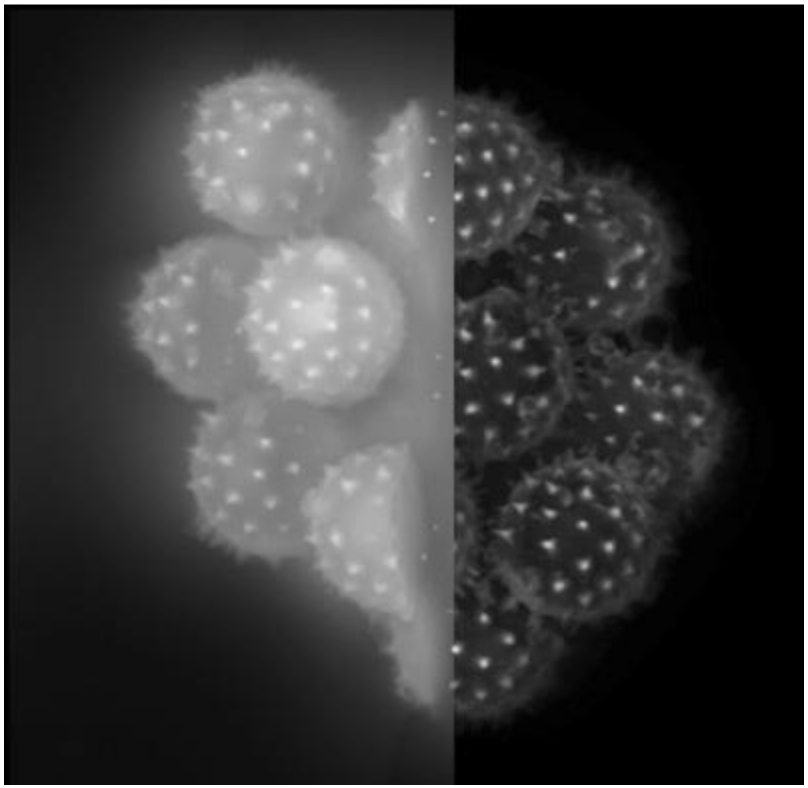
Applications – Astronomy

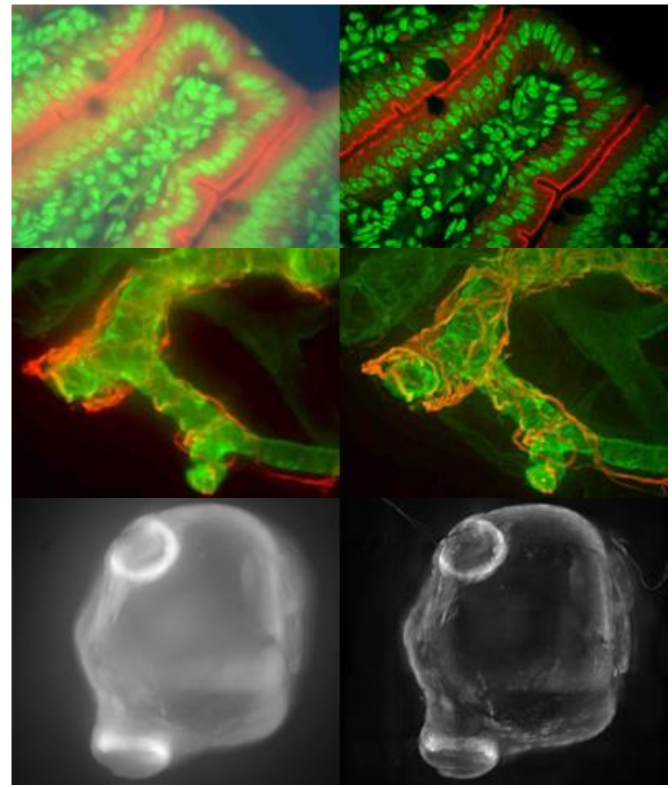
- Before
- After



Images courtesy of Robert Vanderbei

Applications – Microscopy





Images courtesy of Meyer Instruments

Applications – Photography

Images taken with simple lens – own work
[Heide et al. 2013]

Inverse Problems – Definition

- Forward problem:
  - Given a mathematical model M and its parameters m, compute (predict) observations o

$$o = M(m)$$

- Inverse problem:
  - Given observations o and a mathematical model M, compute the model's parameters m

$$m = M^{-1}(o)$$

  - If M is unknown, we call the problem “blind”

Inverse Problems – Deconvolution

- Forward problem – Convolution
  - Example: Blur filter
  - Given an image m and a filter kernel k, compute the blurred image

$$o = m \otimes k$$

Inverse Problems – Deconvolution

- Inverse problem – *Deconvolution*
  - Example: Blur filter
  - Given a blurred image o and a filter kernel k, compute the sharp image
  - Need to invert

$$o = m \otimes k + n$$

  - $n =$ noise

Def. signal-to-noise ratio (SNR)

$$\mathrm{SNR} = \frac{\text{mean signal} \;=\; 0.5}{\text{noise stdev.} \;=\; \sigma}$$

$\xrightarrow{?}$

Deconvolution in Fourier space

- Convolution theorem

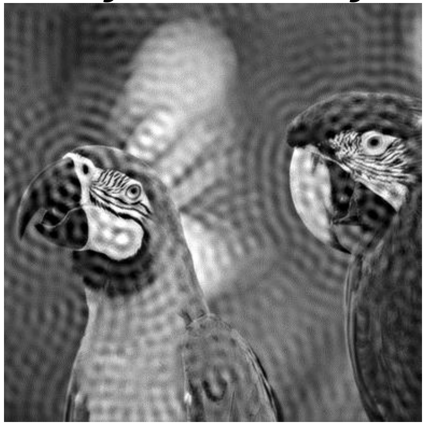
$$o = m \otimes \mathrm{k} \Leftrightarrow \mathcal{F}(o) = \mathcal{F}(m) \cdot \mathcal{F}(k)$$

- Deconvolution:

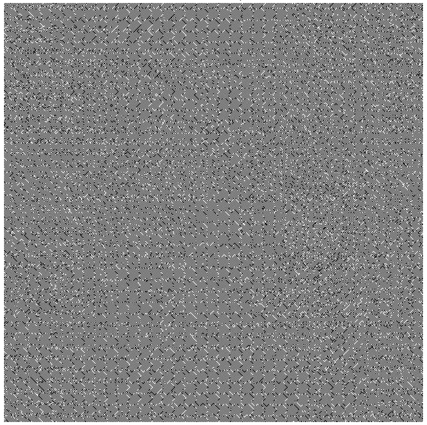
$$\mathcal{F}(m) = {}^{\mathcal{F}(o)}\!\big/_{\mathcal{F}(k)}$$

- Problems:
  - Division by zero
  - Gibbs phenomenon (ringing)

If you're lucky:

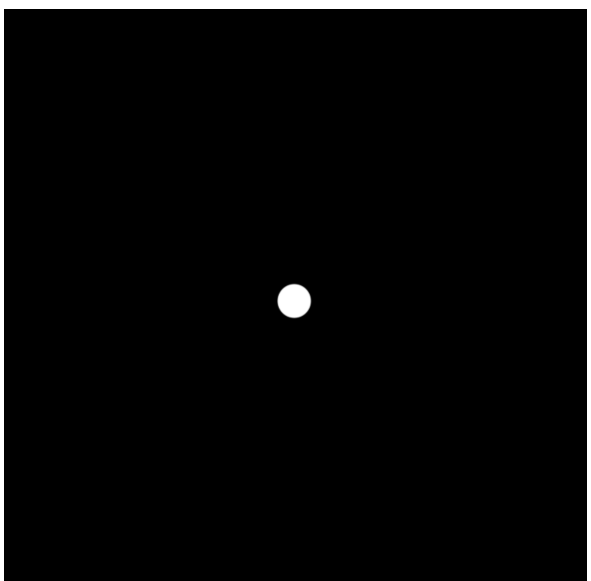


Otherwise ($\sigma = 0.05$):



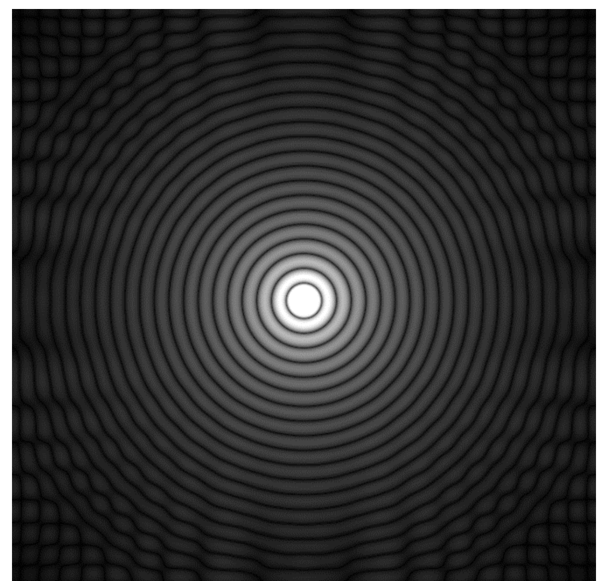
Deconvolution by inverse filtering

- Most common: $\mathcal{F}(k)$ is a low-pass filter
- $1/\mathcal{F}(k)$, the inverse filter, boosts high frequencies
- amplifies noise and numerical errors



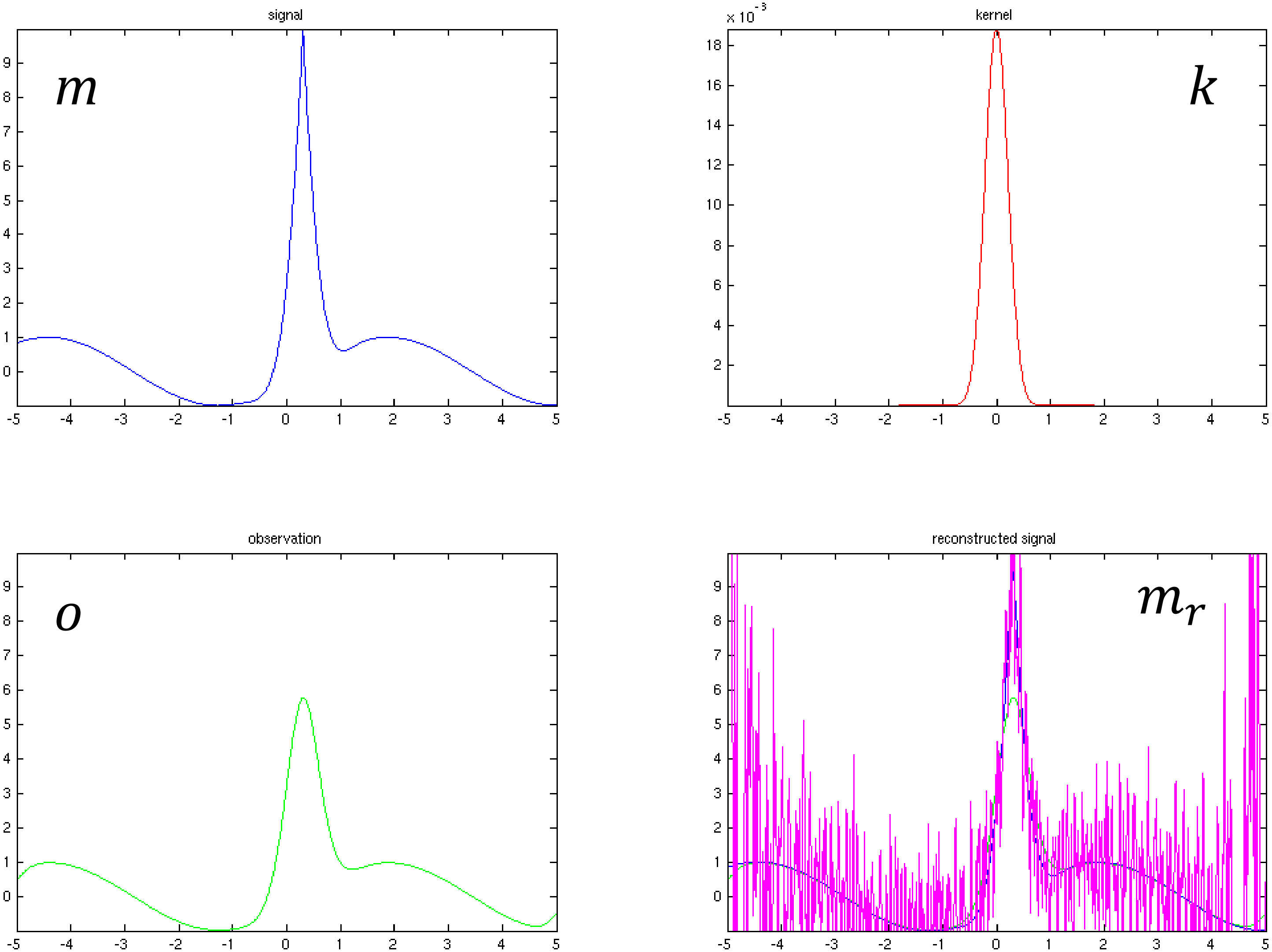
"Point spread function"

$$\xrightarrow{\mathcal{F}}$$



"Modulation transfer function"

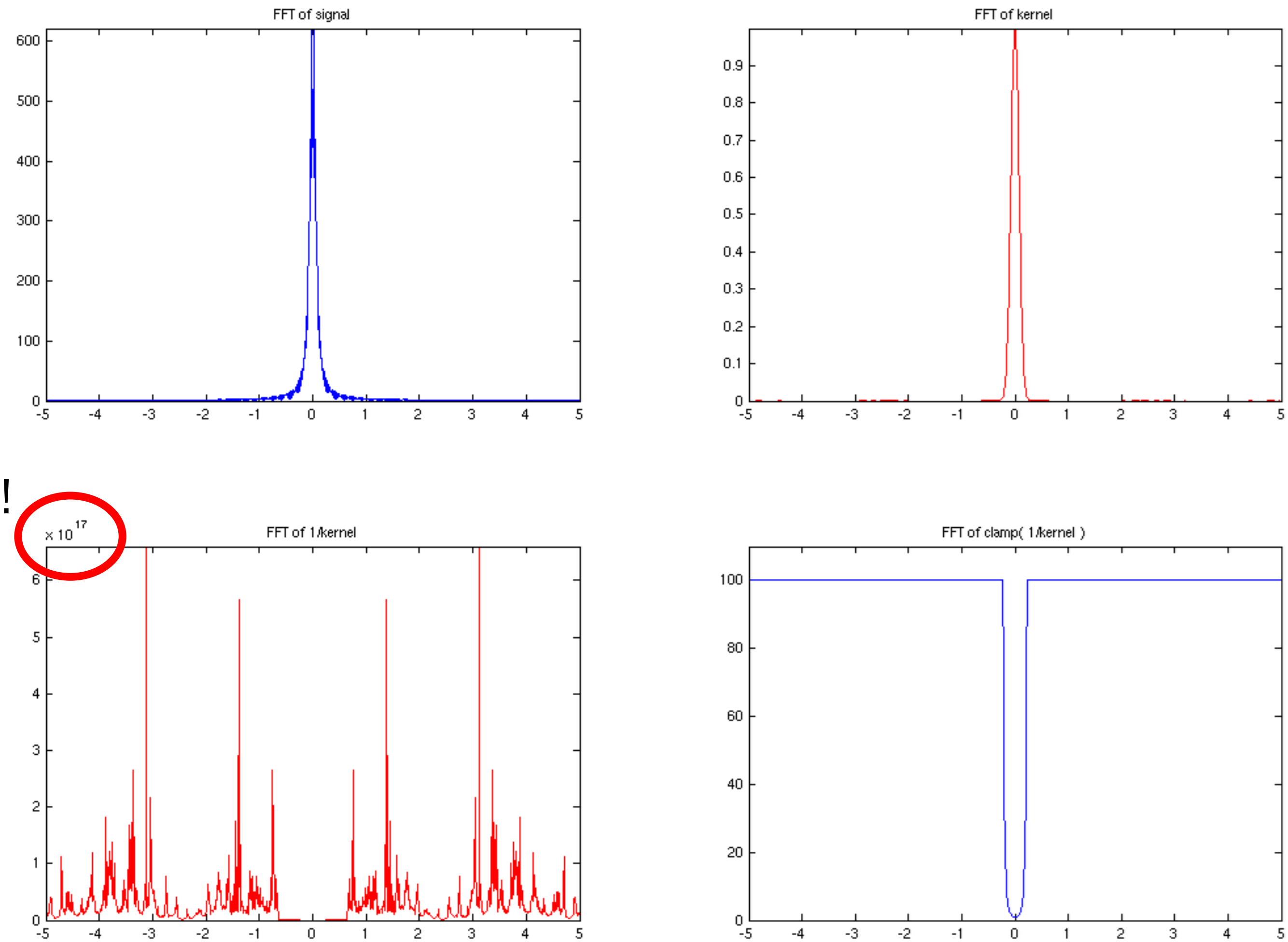
Inverse filtering – 1D example



- Even for perfect data, noisy reconstruction

Inverse filtering – 1D example

6×10^{17} !!!

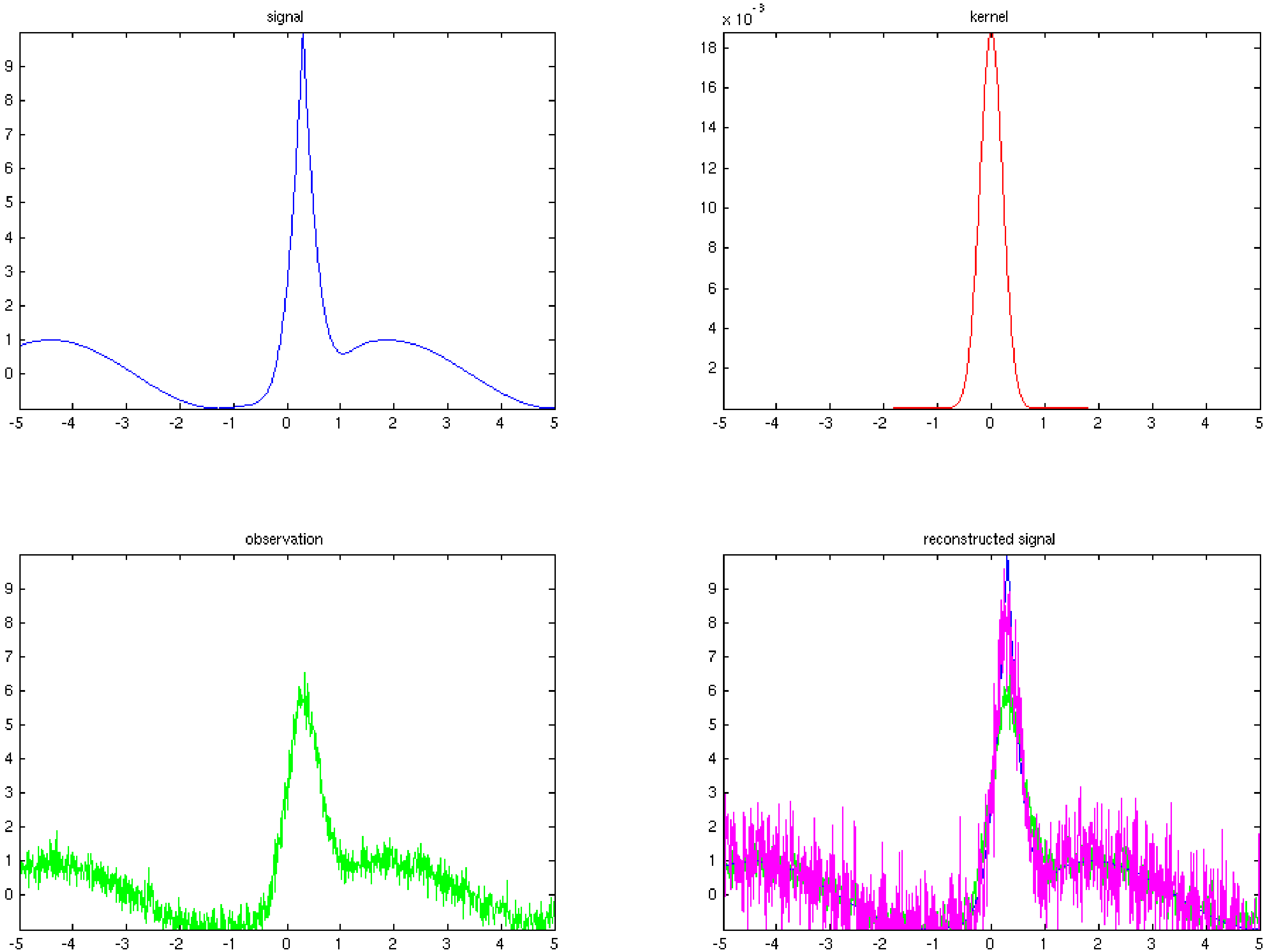


- Spectra of signal, filter and inverse filter

Deconvolution – 1D example

- Solution: Restrict frequency response of high-boost filter (clamping)
- $m_r = \mathcal{F}^{-1}\{\mathcal{F}(o) \cdot G\}$
- with $G = \begin{cases} {}^{1}/_{\mathcal{F}(k)}, & \text{if } {}^{1}/_{\mathcal{F}(k)} < \gamma \\ \gamma\, {}^{\mathcal{F}(k)}/_{|\mathcal{F}(k)|}, & \text{else} \end{cases}$

Deconvolution – 1D example



- Reconstruction with clamped inverse filter

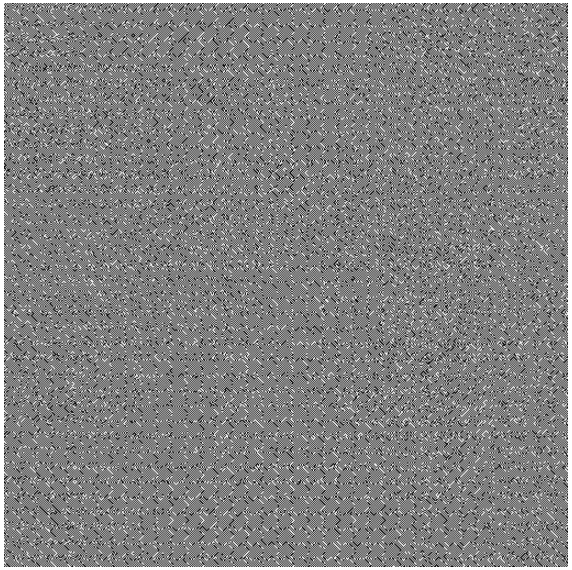
“Informed” clamping: Wiener filter

$$m_r = \mathcal{F}^{-1}\left\{\frac{|\mathcal{F}\{k\}|^2}{|\mathcal{F}\{k\}|^2 + 1/\mathrm{SNR}} \cdot \frac{\mathcal{F}\{o\}}{\mathcal{F}\{k\}}\right\}$$

For frequencies where $\mathcal{F}\{k\}$ small, this **damping term** goes to 0.

... to keep this one from ruining reconstruction

Naïve inverse filtering



Wiener filtering

Algebraic reconstruction

- Convolution

$$o(x) = \int_{-\infty}^{\infty} m(x')k(x - x')dx'$$

- Discretization: Linear combination of basis functions

$$m(x) = \sum_{i=1}^{N} m_i \phi_i(x)$$

Algebraic deconvolution

- Discretization:
- Observations are linear combinations of convolved basis functions
- Linear system with unknowns m_i

$$o(x) = m(x) \otimes k(x)$$
$$= \int_{-\infty}^{\infty} m(x')k(x-x')dx'$$
$$\approx \int_{-\infty}^{\infty} \sum_{i=1}^{N} m_i \phi_i(x')\, k(x-x')dx'$$
$$= \sum_{i=1}^{N} m_i \int_{-\infty}^{\infty} \phi_i(x')k(x-x')dx'$$
$$= \sum_{i=1}^{N} m_i (\phi_i(x) \otimes k(x))$$

$$\mathbf{o} = \mathbf{M}\mathbf{m}$$

Convolution as matrix-vector product

- Discrete Laplacian: convolution with 2D kernel

$$\begin{bmatrix} & -1 & \\ -1 & +4 & -1 \\ & -1 & \end{bmatrix}$$

- Recall corresponding 2D→2D convolution matrix:

$$M_\Delta = \begin{bmatrix} \begin{bmatrix} +4 & -1 & \\ -1 & \ddots & -1 \\ & -1 & +4 \end{bmatrix} & \begin{bmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{bmatrix} & \\ \begin{bmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{bmatrix} & \ddots & \begin{bmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{bmatrix} \\ & \begin{bmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{bmatrix} & \begin{bmatrix} +4 & -1 & \\ -1 & \ddots & -1 \\ & -1 & +4 \end{bmatrix} \end{bmatrix}$$

read, “output pixel $(i, j) = 4 \cdot$ input pixel $(i, j) - \sum$ input pixels $(i \pm 1, j \pm 1)$”

Algebraic deconvolution

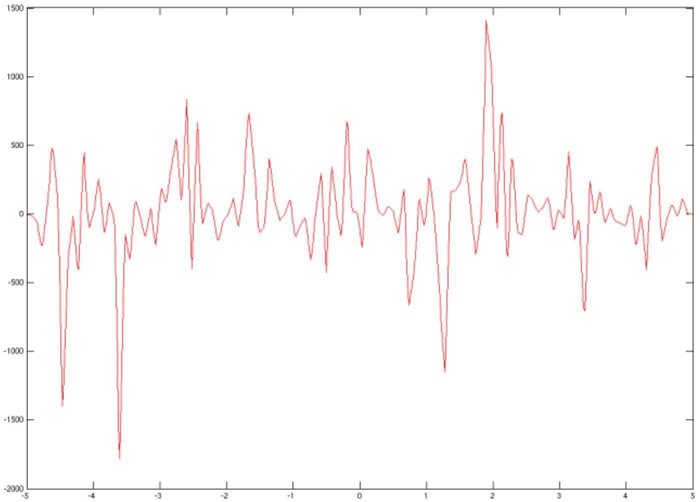
- Normal equations

$$\min_{\mathbf{m}} \|\mathbf{Mm} - \mathbf{o}\|_2^2 = \min_{\mathbf{m}} (\mathbf{Mm} - \mathbf{o})^T (\mathbf{Mm} - \mathbf{o}) = \min_{\mathbf{m}} f(\mathbf{m})$$

$$\nabla f = 2(\mathbf{M}^T\mathbf{M})\mathbf{m} - 2\mathbf{M}^T\mathbf{o} = 0$$

Solve $(\mathbf{M}^T\mathbf{M})\mathbf{m} = \mathbf{M}^T\mathbf{o}$ to obtain solution in least-squares sense

Apply to deconvolution problem



Algebraic deconvolution

- Why?

- Analyze distribution of eigenvalues

$$\det(\mathbf{M}) = \prod_{i=0}^{N} \lambda_i \,, \qquad \det(\mathbf{M}) = 0 \Rightarrow$$

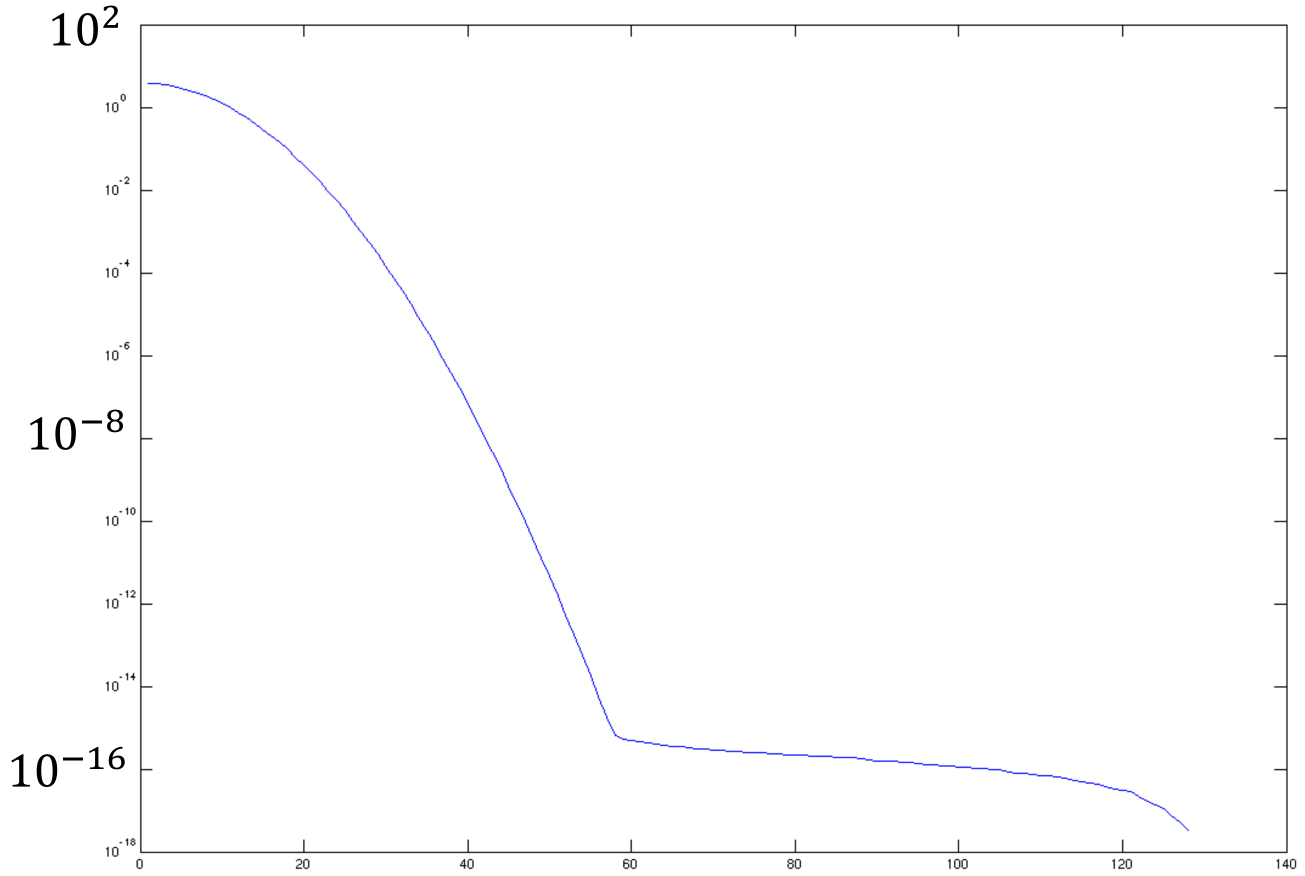
- matrix $\mathbf{M}$ is under-determined

- Check singular values (square root of eigenvalues of $\mathbf{M}^T\mathbf{M}$)

Algebraic deconvolution

- Singular values of $\mathbf{M}^T\mathbf{M}$ – more than half are below machine epsilon $\approx 10^{-16}$ (double precision)

Log plot!



Algebraic deconvolution

- Why is this bad?
- Singular value decomposition: $\mathbf{U}$, $\mathbf{V}$ orthonormal, $\mathbf{D}$ diagonal

$$\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{V}^T$$

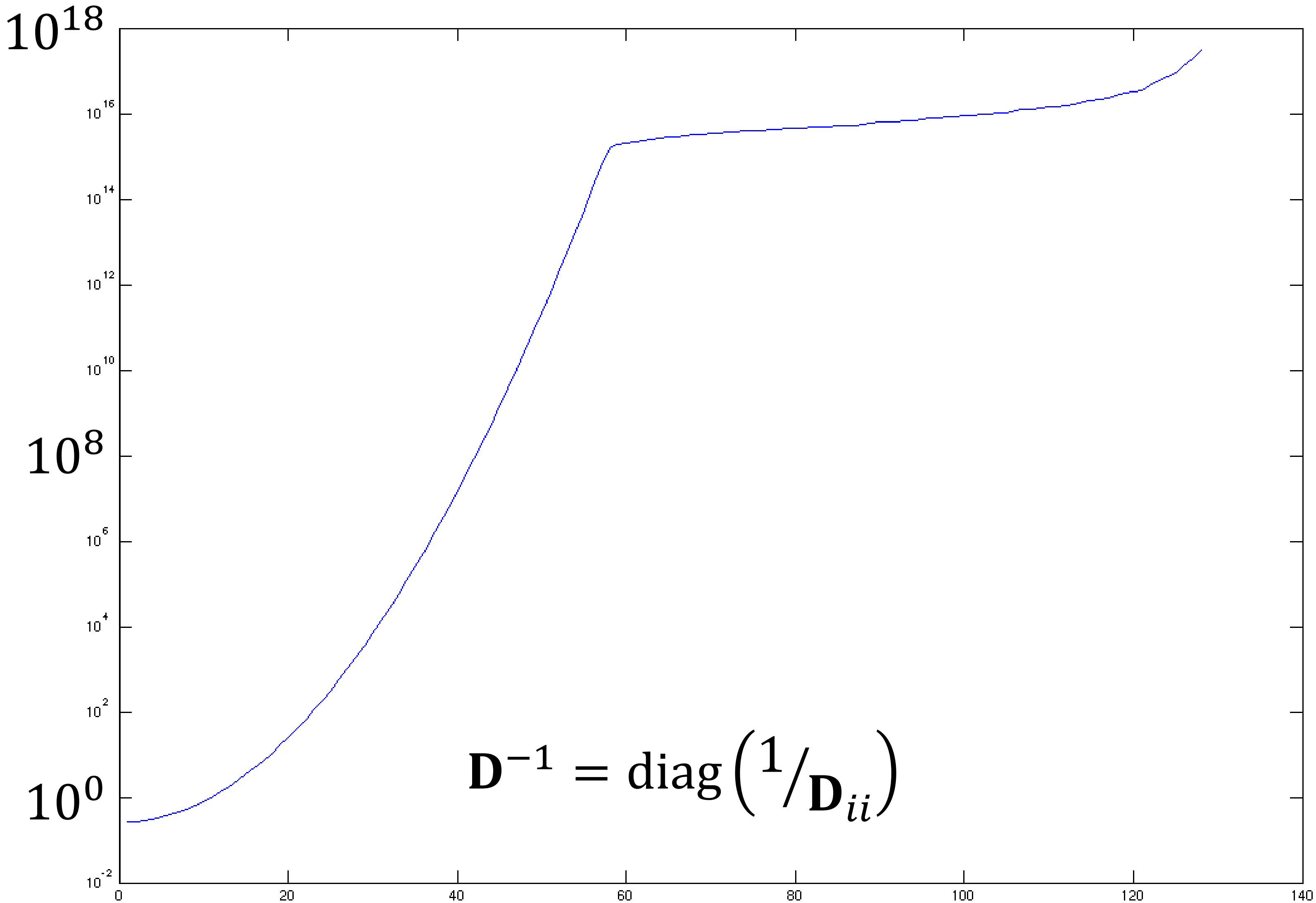
- Inverse of M:

$$\begin{aligned}\mathbf{M}^{-1} &= (\mathbf{U}\mathbf{D}\mathbf{V}^T)^{-1} \\ &= (\mathbf{V}^T)^{-1}\mathbf{D}^{-1}\mathbf{U}^{-1} \\ &= \mathbf{V}\mathbf{D}^{-1}\mathbf{U}^T\end{aligned}$$

- Singular values are diagonal elements of $\mathbf{D}$

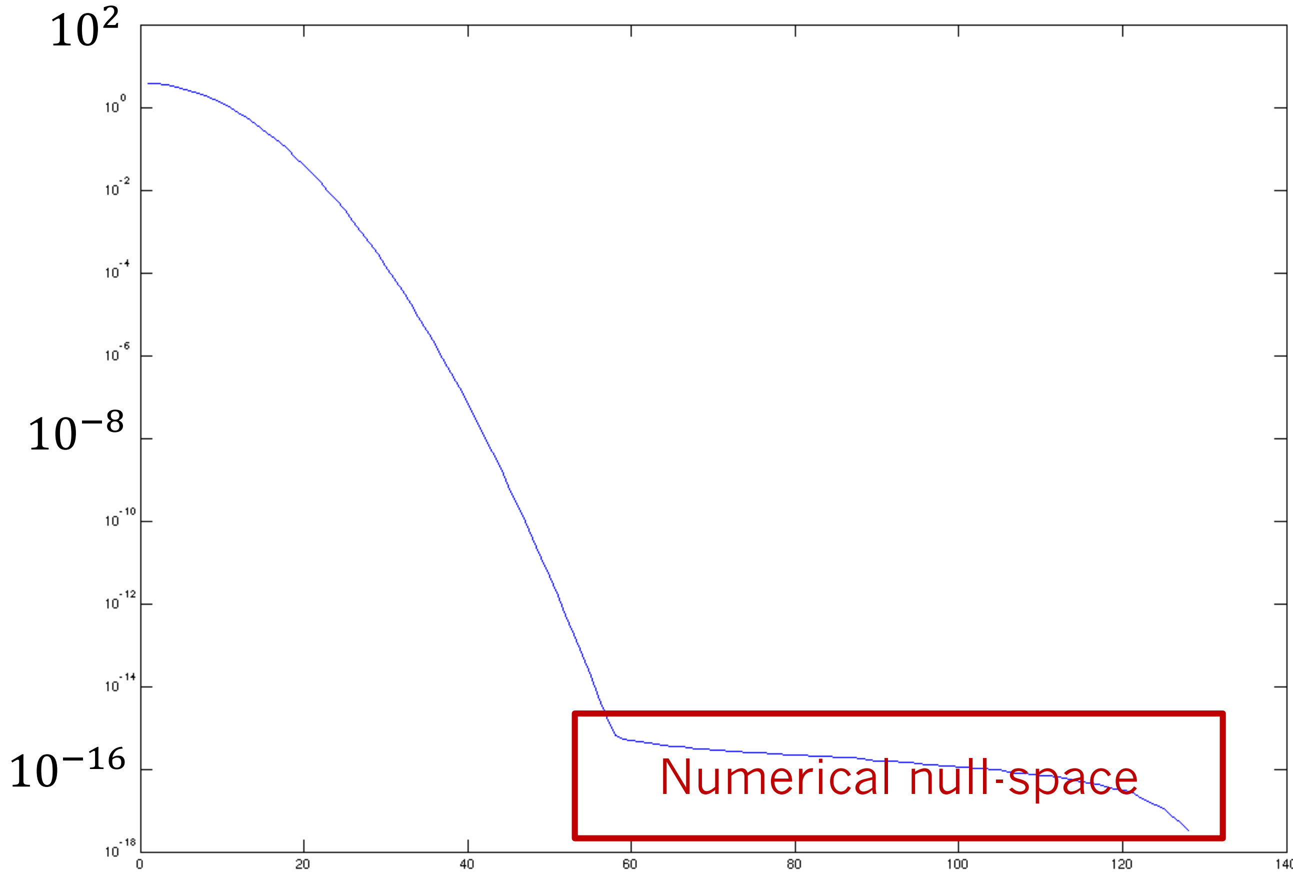
Inversion: $$\mathbf{D}^{-1} = \mathrm{diag}\left({}^{1}/_{\mathbf{D}_{ii}}\right)$$

Algebraic deconvolution



Algebraic deconvolution

- Inverse problems are often ill-conditioned
- Inversion causes amplification of noise



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Well-posed and ill-posed problems

Definition [Hadamard 1902]:

- A problem is **well-**posed if
  1. a solution exists
  2. the solution is unique
  3. the solution continually depends on the data

- A problem is **ill-**posed if it is not well-posed
  - Most often conditions 2 and 3 are violated
  - If model has a (numerical) null space, slight change in data causes large change in solution
  - Noise is amplified when inverting the model

Condition number

- Condition number as measure of stability for numerical inversion
- Ratio between largest and smallest singular value
- $\rho(\mathbf{M}) = \frac{\sigma_1}{\sigma_N}, \quad \sigma_1 > \cdots > \sigma_N$ singular values of $\mathbf{M}$
- Smaller condition number $\Rightarrow$ less problems

Truncated SVD

- Solution to stability problems: avoid dividing by near-zero values

- Truncated Singular Value Decomposition (TSVD):

$$\mathbf{d}^{+} = \begin{cases} \frac{1}{\mathbf{D}_{ii}}, & \mathbf{D}_{ii} > \epsilon \\ 0, & \text{else} \end{cases}$$

ϵ: Regularization parameter

$$\mathbf{D}^{+} = \operatorname{diag}(\mathbf{d}^{+})$$

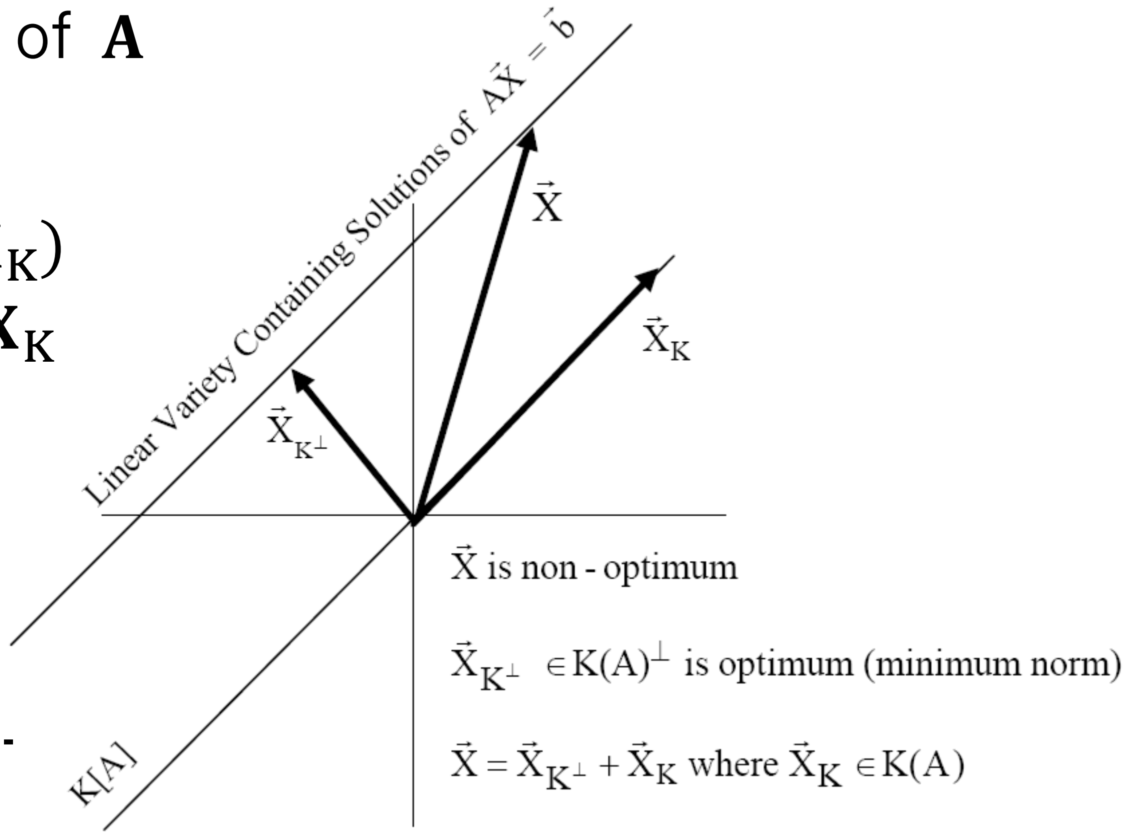
$$\mathbf{M}^{+} = \mathbf{V}\mathbf{D}^{+}\mathbf{U}^{T}$$

Minimum-norm solution to $\mathbf{Ax} = \mathbf{b}$

- K is the null-space of $\mathbf{A}$

$$
\begin{aligned}
\mathbf{AX}_{\mathrm{K}} &= 0 \\
\mathbf{AX} &= \mathbf{A}(\mathbf{X}_{\mathrm{K}^\perp} + \mathbf{X}_{\mathrm{K}}) \\
&= \mathbf{AX}_{\mathrm{K}^\perp} + \mathbf{AX}_{\mathrm{K}} \\
&= \mathbf{AX}_{\mathrm{K}^\perp} + 0 \\
&= \mathbf{AX}_{\mathrm{K}^\perp} \\
&= \mathbf{b}
\end{aligned}
$$

$\mathbf{X}_{\mathrm{K}^\perp}$ is the minimum-norm solution



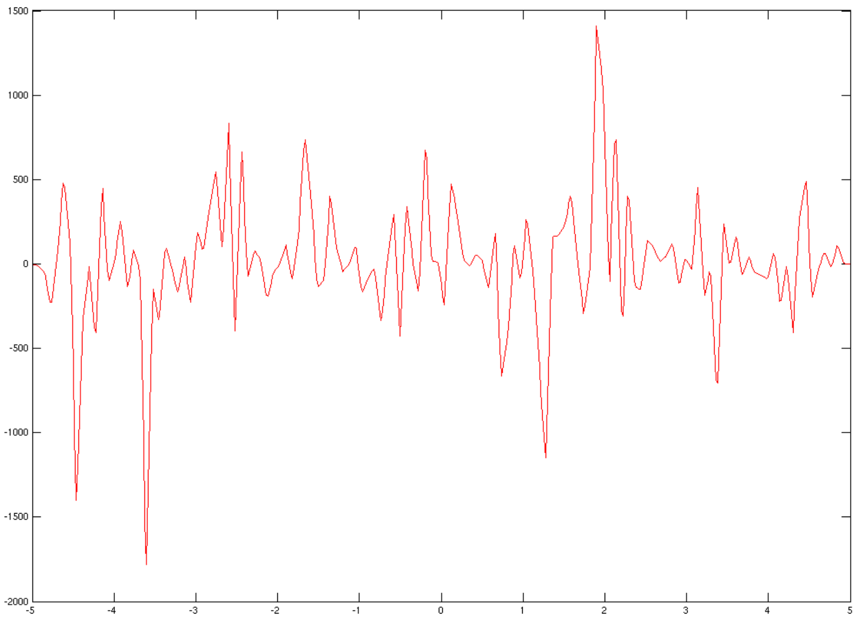
Regularization

- Countering the effect of ill-conditioned problems is called **regularization**
- An ill-conditioned problem behaves like a singular (under-constrained) system
- Family of solutions exist
  - Impose additional knowledge to pick a favorable solution
  - TSVD results in minimum-norm solution

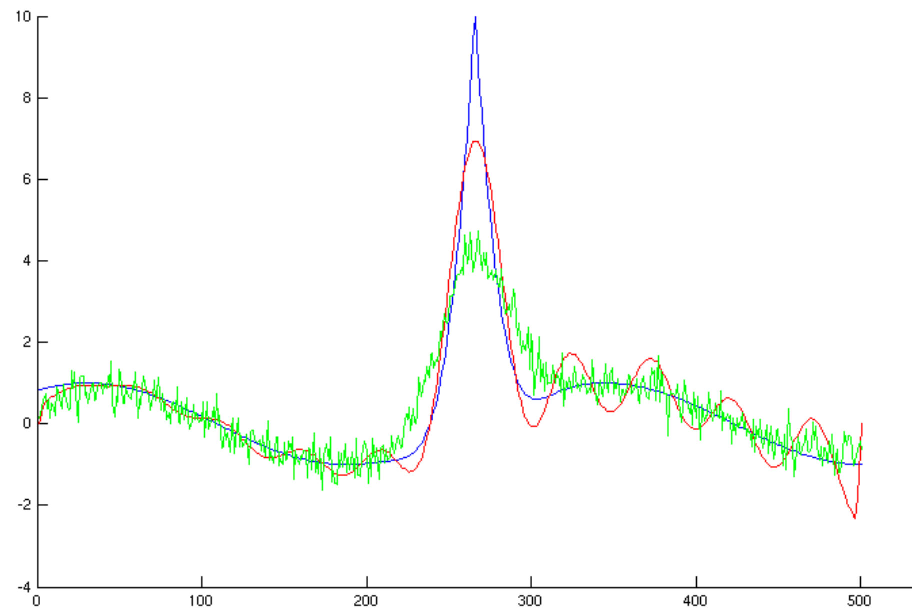
Example – 1D deconvolution

- Our example deconvolved using TSVD
- Much smoother than Fourier deconvolution

Unregularized solution



TSVD regularized solution $\epsilon = 10^{-6}$



Regularized least-squares problem

$$\mathbf{x}_{\text{opt}} = \arg\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \Gamma(\mathbf{x})$$

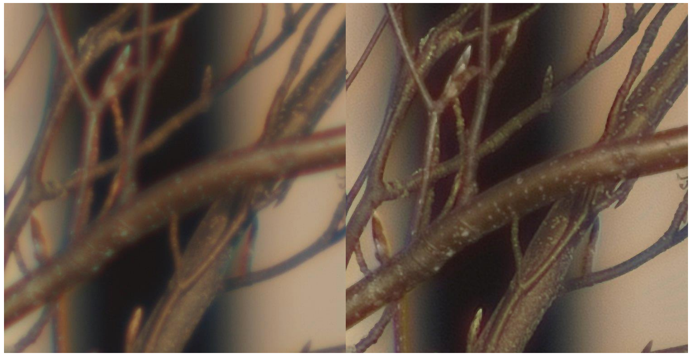
- Tikhonov regularization: $\Gamma(\mathbf{x}) = \|\mathbf{\Gamma}\mathbf{x}\|_2^2$

  leads to normal equations:

$$\left(\mathbf{A}^T\mathbf{A} + \mathbf{\Gamma}^T\mathbf{\Gamma}\right)\mathbf{x} = \mathbf{A}^T\mathbf{b}$$

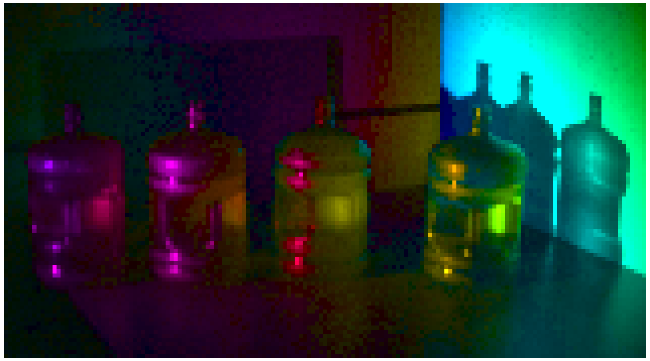
Regularization is the key to (almost) all inverse problems!

- Deblurring
  Gradient sparsity
  Cross-channel coherence



- Computed tomography
  Image-space total variation prior

- Transient imaging
  Spatio-temporal gradient prior



Total Variation (TV)

- Next lecture: Learn about a state-of-the-art prior + numerical method for image reconstruction