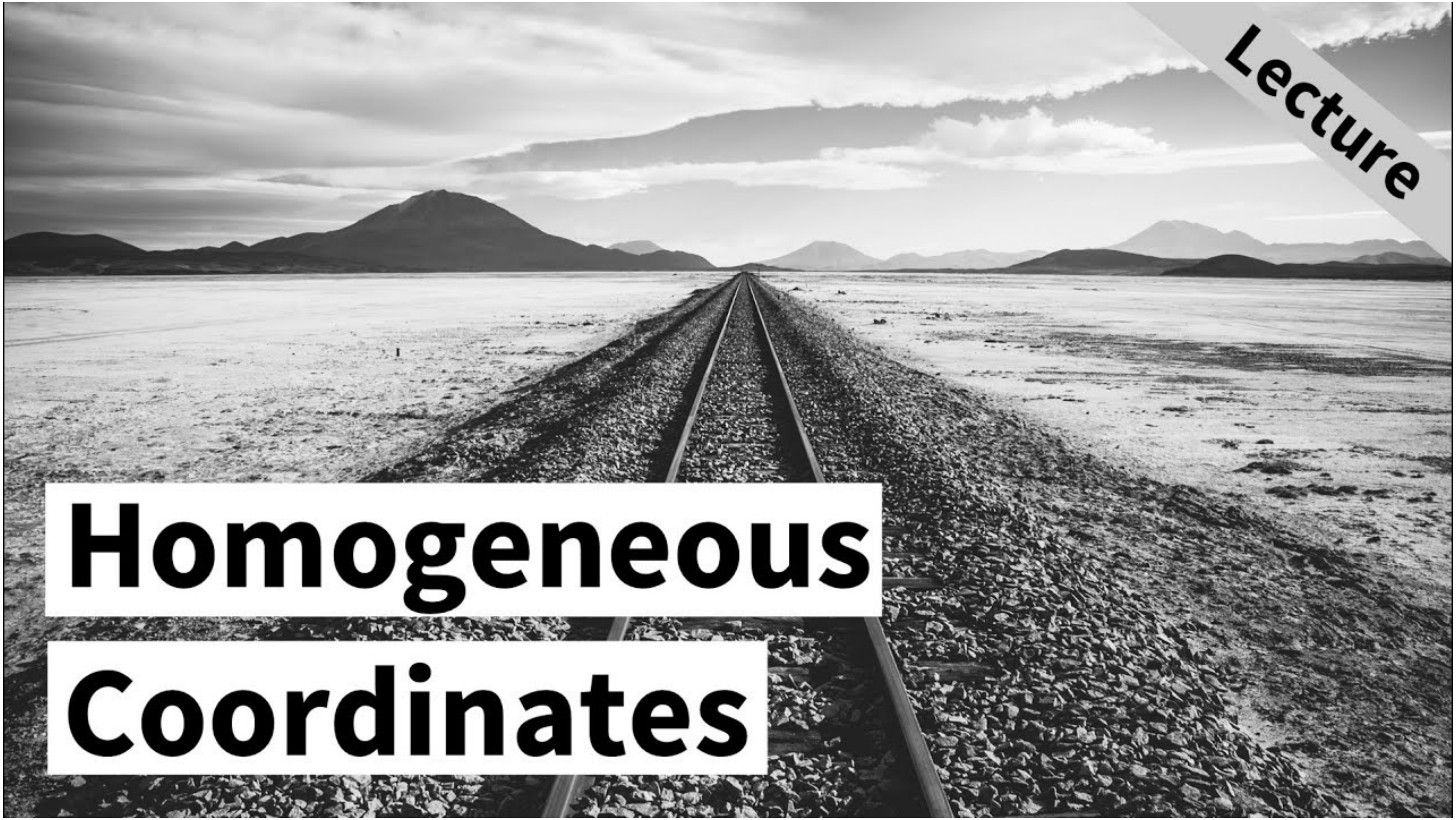


Summer term 2024 – Cyrill Stachniss

5 Minute Preparation for Today

https://www.ipb.uni-bonn.de/5min/

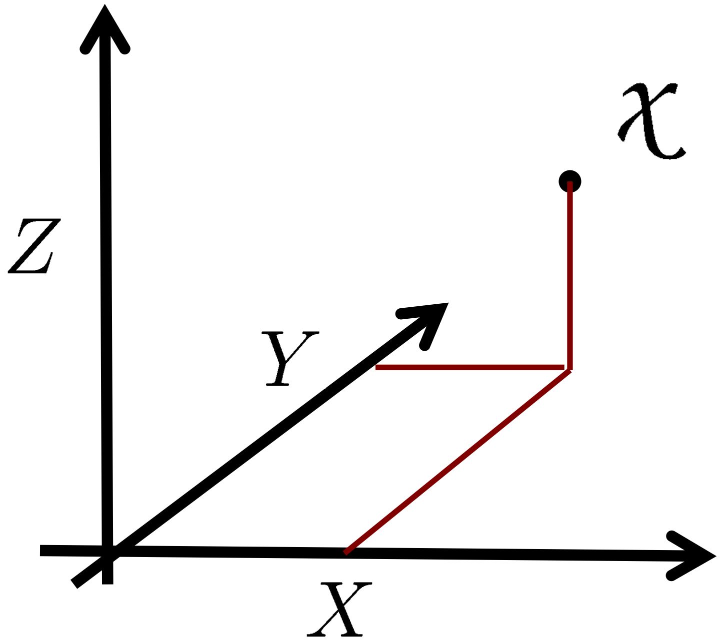
Photogrammetry & Robotics Lab

Homogeneous Coordinates

Cyrill Stachniss

The slides have been created by Cyrill Stachniss.

A Point in the 3D Euclidean World



$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Is this always the best representation of geometric object in the 3D world?

Pinhole Camera

- Popular model to approximate the imaging process of a perspective camera

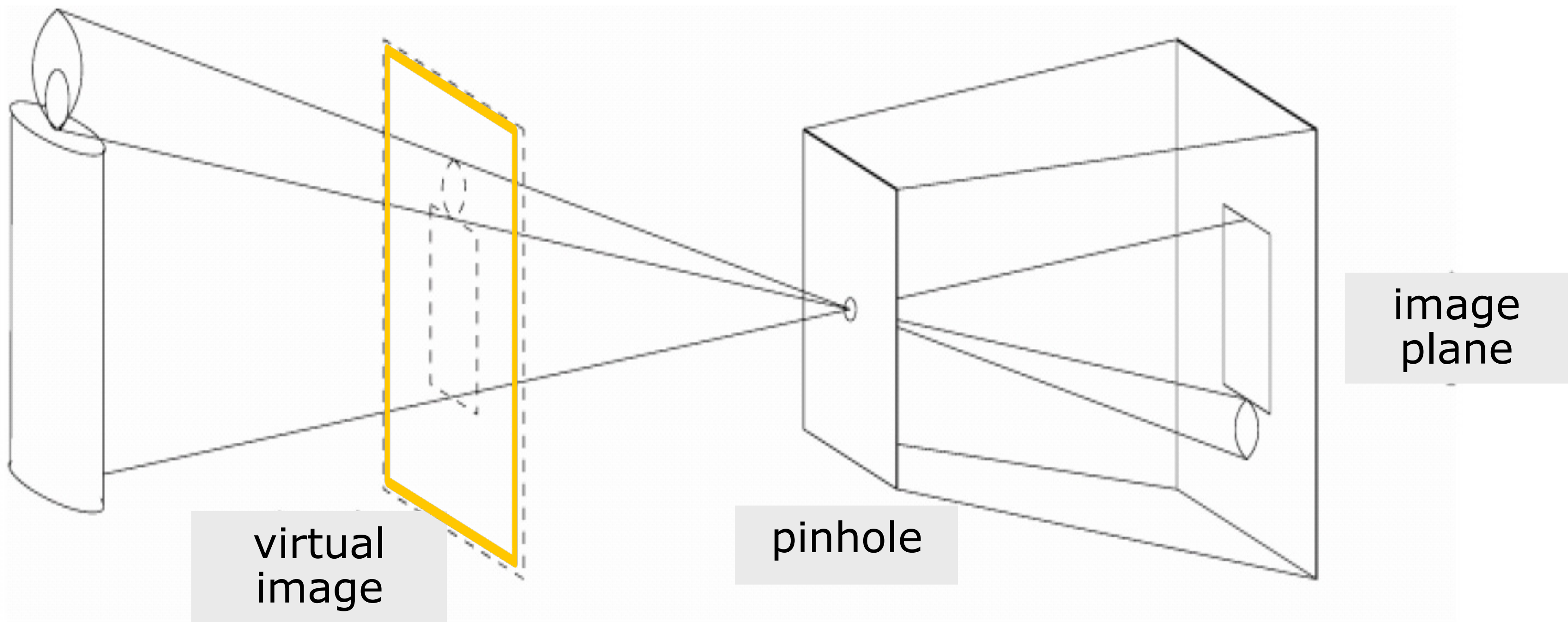


Image Courtesy: Forsyth and Ponce

Pinhole Camera Model

- A box with an **infinitesimal small hole**
- **Camera center** is the intersection point of the rays
- The back wall is the **image plane**
- The distance between the camera center and image plane is the **camera constant**

Geometry and Images

What can we say about the geometry?

Image courtesy: Förstner

Pinhole Camera Properties

- **Line-preserving:** straight lines are mapped to straight lines
- **Not length-preserving:** size of objects is inverse proportional to the distance
- **Not angle-preserving:** Angles between lines change

Perspective Projection

- Straight lines stay straight
- Parallel lines may not remain parallel

Image courtesy: Förstner

Vanishing Point
(DE: Fluchtpunkt)

Image Courtesy: J. Jannene

Vanishing Points

- Parallel lines are not parallel anymore
- All mapped parallel lines intersect in a vanishing point
- The vanishing point is the “point at infinity” for the parallel lines
- Every direction has exactly one vanishing point

How to describe “points at infinity”?

Projective Geometry Motivation

- **Euclidian geometry is suboptimal to describe the central projection**
- In Euclidian geometry, the math can get difficult
- **Projective geometry** is an alternative algebraic representation of geometric objects and transformations

Homogeneous Coordinates

- H.C. are a system of coordinates used in projective geometry
- Formulas involving H.C. are often simpler than in the Cartesian world
- Points at infinity can be represented using finite coordinates
- A single matrix can represent affine and projective transformations

Notation

Point $\mathcal{X}$ (or y or p)

- in homogeneous coordinates $\mathbf{x}$
- in Euclidian coordinates $\boldsymbol{x}$

Line l (or m)

- in homogeneous coordinates $\mathbf{l}$

Plane $\mathcal{A}$

- in homogeneous coordinates $\mathbf{A}$

2D vs. **3D** space

- lowercase = 2D; capitalized = 3D

Homogeneous Coordinates

Definition

The representation $\mathbf{x}$ of a geometric object is **homogeneous** if $\mathbf{x}$ and $\lambda\mathbf{x}$ represent the same object for $\lambda \neq 0$

Example

$$\mathbf{x} = \lambda\,\mathbf{x}$$

homogeneous

$$\boldsymbol{x} \neq \lambda\,\boldsymbol{x}$$

Euclidian

Homogeneous Coordinates

- H.C. use a n+1 dimensional vector to represent the n-dimensional Euclidian point
- Set dimension n+1 to the value 1
- Example for $\mathbb{R}^2/\mathbb{P}^2$

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

Definition

The representation $\mathbf{x}$ of a geometric object is **homogeneous** if $\mathbf{x}$ and $\lambda\mathbf{x}$ represent the same object for $\lambda \neq 0$

Example

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 2\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2 \end{bmatrix} = \begin{bmatrix} wx \\ wy \\ w \end{bmatrix}$$

Euclidian homogeneous

Definition

- Homogeneous Coordinates of a point χ in the plane $\mathbb{R}^2$ is a 3-dim. vector

$$\chi: \quad \mathbf{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \text{ with } |\mathbf{x}|^2 = u^2 + v^2 + w^2 \neq 0$$

- it corresponds to Euclidian coordinates

$$\chi: \quad \boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u/w \\ v/w \end{bmatrix} \text{ with } w \neq 0$$

Example: Projective Plane

The projective plane $\mathbb{P}^2(\mathbb{R})$ or $\mathbb{P}^2$ contains

- All points $\mathcal{X}$ of the Euclidian plane $\mathbb{R}^2$ with $\boldsymbol{x} = [x, y]^\top$ expressed through the 3-valued vector (e.g., $\mathbf{x} = [x, y, 1]^\top$)
- and all points at infinity, i.e.,

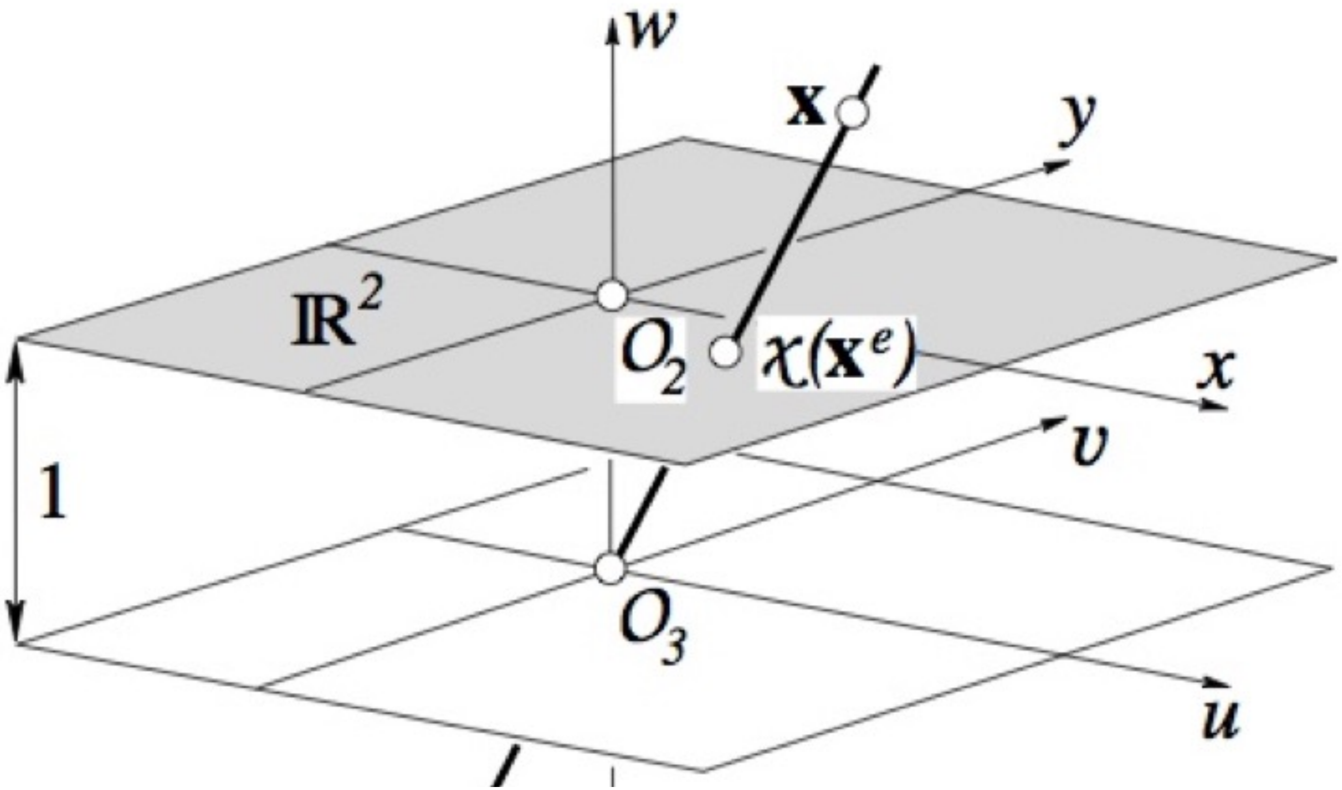
  $\mathbf{x} = [x, y, 0]^\top$
- except $[0, 0, 0]^\top$

From Homogeneous to Euclidian Coordinates

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} u/w \\ v/w \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

homogeneous Euclidian

From Homogeneous to Euclidian Coordinates



$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} u/w \\ v/w \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Image courtesy: Förstner

3D Points

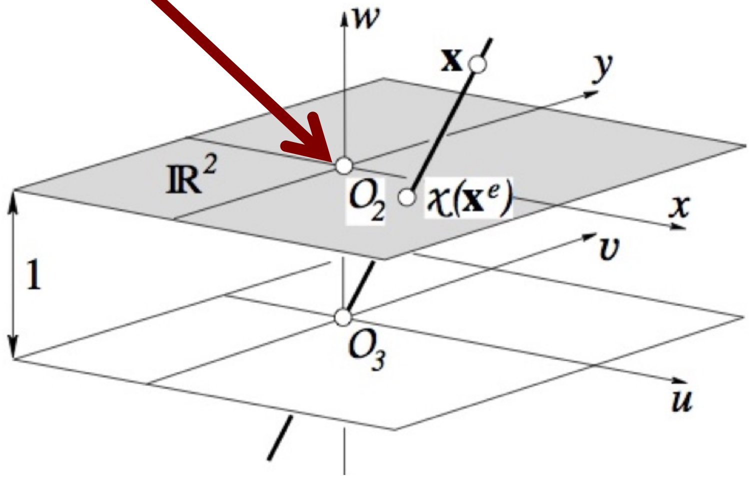
Analogous for points in 3D Euclidian space $\mathbb{R}^3$

homogeneous Euclidian

$$\mathbf{X} = \begin{bmatrix} U \\ V \\ W \\ T \end{bmatrix} = \begin{bmatrix} U/T \\ V/T \\ W/T \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} U/T \\ V/T \\ W/T \end{bmatrix}$$

Origin of the Euclidian Coordinate System in H.C.

$$\mathbf{O}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{O}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$



Transformations

Transformations

- A projective transformation is an invertible linear mapping

$$\mathbf{x}' = \mathrm{H}\mathbf{x}$$

homogeneous vector

homogeneous matrix

homogeneous vector

3D Transformations

- General projective mapping

$$\mathbf{X}' = \mathrm{H}\mathbf{X}$$

- Question: How should H look like to realize relevant transformation?
- Eg, translation, rotation, scale change, rigid-body, similarity, affine, projective

Important 3D Transformations

- General projective mapping $\mathbf{X}' = \mathsf{H}\mathbf{X}$
- Translation: 3 parameters
  (3 translations)

$$\mathsf{H} = \lambda \begin{bmatrix} I & \boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{t} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

homogeneous property

Important 3D Transformations

- Rotation: 3 parameters
  (3 rotation)

rotation
matrix

$$\mathsf{H} = \lambda \begin{bmatrix} \underline{R} & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

Recap – Rotation Matrices

- 2D:

$$R^{2D}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

- 3D:

$$R_x^{3D}(\omega) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega) & -\sin(\omega) \\ 0 & \sin(\omega) & \cos(\omega) \end{bmatrix} \quad R_y^{3D}(\phi) = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$$

$$R_z^{3D}(\kappa) = \begin{bmatrix} \cos(\kappa) & -\sin(\kappa) & 0 \\ \sin(\kappa) & \cos(\kappa) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^{3D}(\omega, \phi, \kappa) = R_z^{3D}(\kappa) R_y^{3D}(\phi) R_x^{3D}(\omega)$$

Important 3D Transformations

- Rotation: 3 parameters
  (3 rotation)

$$\mathrm{H} = \lambda \begin{bmatrix} \underline{R} & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

- Rigid body transformation: 6 params
  (3 translation + 3 rotation)

$$\mathrm{H} = \lambda \begin{bmatrix} \underline{R} & \underline{\boldsymbol{t}} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

Important 3D Transformations

- Similarity transformation: 7 params
  (3 trans + 3 rot + 1 scale)

$$\mathsf{H} = \lambda \begin{bmatrix} \underline{m}R & \boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

(angle-preserving)

- Affine transformation: 12 parameters
  (3 trans + 3 rot + 3 scale + 3 sheer)

$$\mathsf{H} = \lambda \begin{bmatrix} \underline{A} & \boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

(not angle-preserving but parallel lines remain parallel)

Important 3D Transformations

- Projective transformation: 15 params.

$$\mathrm{H} = \lambda \begin{bmatrix} A & \boldsymbol{t} \\ \underline{\boldsymbol{a}^\top} & 1 \end{bmatrix}$$

affine transformation + 3 parameters

- These 3 parameters are the projective part and they are the reason that **parallel lines may not stay parallel**

Transformations for 2D

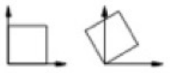
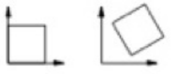
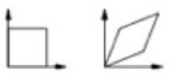
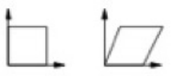
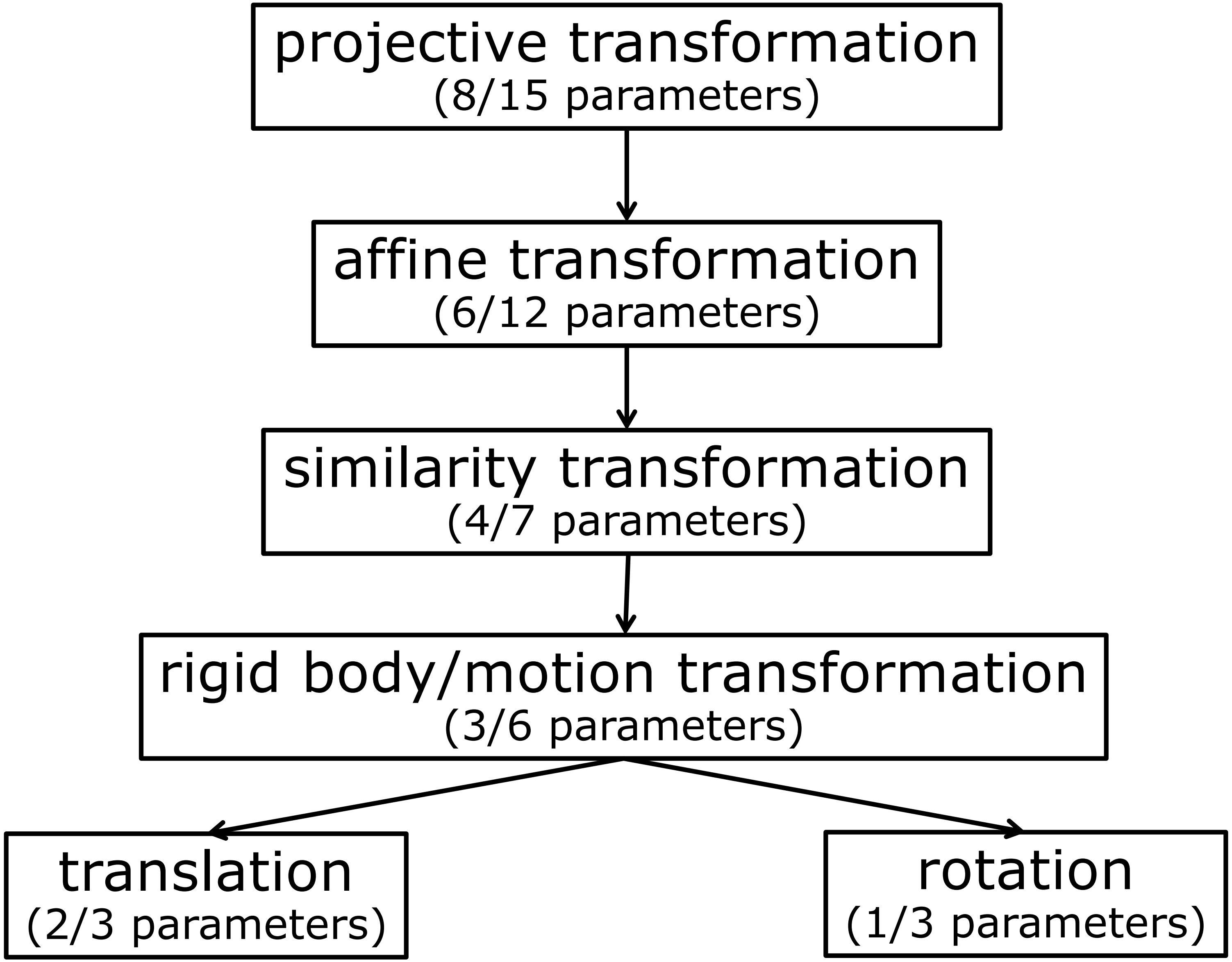
2D Transformation	Figure	d. o. f.	H	H
Translation		2	$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} I & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Mirroring at y-axis		1	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} Z & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Rotation		1	$\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Motion		3	$\begin{bmatrix} \cos\varphi & -\sin\varphi & t_x \\ \sin\varphi & \cos\varphi & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} R & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Similarity		4	$\begin{bmatrix} a & -b & t_x \\ b & a & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \lambda R & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Scale difference		1	$\begin{bmatrix} 1+m/2 & 0 & 0 \\ 0 & 1-m/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} D & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Shear		1	$\begin{bmatrix} 1 & s/2 & 0 \\ s/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} S & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Asym. shear		1	$\begin{bmatrix} 1 & s' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} S' & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Affinity		6	$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} A & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Projectivity		8	$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$	$\begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{p}^\mathsf{T} & 1/\lambda \end{bmatrix}$

Image courtesy: Schindler

Transformations Hierarchy



Inverting and Chaining

- Inverting a transformation

$$
\begin{aligned}
\mathbf{X}' &= \mathsf{H}\mathbf{X} \\
\mathbf{X} &= \mathsf{H}^{-1}\mathbf{X}'
\end{aligned}
$$

- Chaining transformations via matrix products (not commutative)

$$
\begin{aligned}
\mathbf{X}' &= \mathsf{H}_1\mathsf{H}_2\mathbf{X} \\
&\neq \mathsf{H}_2\mathsf{H}_1\mathbf{X}
\end{aligned}
$$

Chaining and Inverting Transformations

- Chaining transformations via matrix products (not commutative)

$$\begin{aligned} \mathbf{X}' &= \mathsf{H}_1 \mathsf{H}_2 \mathbf{X} \\ &\neq \mathsf{H}_2 \mathsf{H}_1 \mathbf{X} \end{aligned}$$

- Inverting a transformation

$$\begin{aligned} \mathbf{X}' &= \mathsf{H} \mathbf{X} \\ \mathbf{X} &= \mathsf{H}^{-1} \mathbf{X}' \end{aligned}$$

Homogeneous Lines
(Images, 2D)

Representations of Lines

- Hesse normal form
  (angle ϕ, distance d)

$$x \cos \phi + y \sin \phi - d = 0$$

- Intercept form

$$\frac{x}{x_0} + \frac{y}{y_0} = 1 \qquad \text{or} \qquad \frac{x}{x_0} + \frac{y}{y_0} - 1 = 0$$

- Standard form

$$ax + by + c = 0$$

Representations of Lines

- Hesse normal form

$$x \cos\phi + y \sin\phi - d = 0 \Rightarrow (\cos\phi)x + (\sin\phi)y - d = 0$$

- Intercept form

$$\frac{x}{x_0} + \frac{y}{y_0} - 1 = 0 \Rightarrow \left(\frac{1}{x_0}\right)x + \left(\frac{1}{y_0}\right)y - 1 = 0$$

- Standard form

$$ax + by + c = 0 \Rightarrow ax + by + c = 0$$

All form linear equations that are equal to zero

Representations of Lines

$$\underset{\text{standard}}{\mathbf{l}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\underset{\text{Hesse}}{\mathbf{l}} = \begin{bmatrix} \cos\phi \\ \sin\phi \\ -d \end{bmatrix}$$

$$\underset{\text{intercept}}{\mathbf{l}} = \begin{bmatrix} \frac{1}{x_0} \\ \frac{1}{y_0} \\ -1 \end{bmatrix}$$

Line Equation Can be Expressed by the Dot-Product

$$\underset{\text{standard}}{\mathbf{l}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\underset{\text{Hesse}}{\mathbf{l}} = \begin{bmatrix} \cos\phi \\ \sin\phi \\ -d \end{bmatrix}$$

$$\underset{\text{intercept}}{\mathbf{l}} = \begin{bmatrix} \frac{1}{x_0} \\ \frac{1}{y_0} \\ -1 \end{bmatrix}$$

$$\underset{\text{point}}{\mathbf{x}} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{x} \cdot \mathbf{l} = 0$$

Definition

- Homogeneous Coordinates of a line ℓ in the plane is a 3-dim. vector

$$\ell : \quad \mathbf{l} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \text{ with } |\mathbf{l}|^2 = l_1^2 + l_2^2 + l_3^2 \neq 0$$

- Corresponds to the Euclidian representation

$$l_1 x + l_2 y + l_3 = 0$$

Test If a Point Lies on a Line

- A point

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- lies on a line

$$\mathbf{l} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$$

- if $\mathbf{x} \cdot \mathbf{l} = 0$

Intersecting Lines

- Given two lines l, m expressed in H.C., we look for the intersection $x = l \cap m$

How to find the intersection of two lines?

Intersecting Lines

- Given two lines ℓ, m expressed in H.C., we look for the intersection $\chi = \ell \cap m$
- Find the point $\boldsymbol{x} = [x, y]^T$ through the following system linear equations

$$\begin{bmatrix} \mathbf{l} \cdot \mathbf{x} \\ \mathbf{m} \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -l_3 \\ -m_3 \end{bmatrix}.$$

Reminder: Cramer's rule

- A system of linear equations can be solved via Cramer's rule

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \qquad\qquad x_i = \frac{\det(A_i)}{\det(A)}$$

- with A_i being the matrix in which the i[th] column is replaced by $\boldsymbol{b}$
- Easily applicable for 2 by 2 systems

Intersecting Lines

- Solution of

$$\begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -l_3 \\ -m_3 \end{bmatrix}$$

- through Cramer's rule

$$x = \frac{D_1}{D_3} \qquad y = \frac{D_2}{D_3}$$

$$\begin{aligned} D_1 &= \det(A_1) = l_2 m_3 - l_3 m_2 \\ D_2 &= \det(A_2) = l_3 m_1 - l_1 m_3 \\ D_3 &= \det(A) = l_1 m_2 - l_2 m_1 \end{aligned}$$

Intersecting Lines

- Solution from Cramer's rule

$$x = \frac{D_1}{D_3} \qquad y = \frac{D_2}{D_3}$$

$$\begin{aligned} D_1 &= \det(A_1) = l_2 m_3 - l_3 m_2 \\ D_2 &= \det(A_2) = l_3 m_1 - l_1 m_3 \\ D_3 &= \det(A) = l_1 m_2 - l_2 m_1 \end{aligned}$$

- can be homogenously rewritten as

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} D_1/D_3 \\ D_2/D_3 \\ 1 \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}$$

Intersecting Lines

- Thus, the solution of

$$\begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -l_3 \\ -m_3 \end{bmatrix}$$

- can be expressed in vector form as

$$\mathbf{x} = \frac{1}{D_3}\boldsymbol{D} = \mathbf{l} \times \mathbf{m}$$

- This is the cross product of the lines!

Intersecting Lines

- The intersection of two lines in H.C. is

$$\boxed{\chi = \ell \cap m : \quad \mathbf{x} = \mathbf{l} \times \mathbf{m}}$$

- **Simple way for computing the intersection of two lines using H.C.**

Line Between Two Points

- H.C. also offer a simple way for computing a line through two points
- Given two points $\mathcal{x} = [x_i], \mathcal{y} = [y_i]$, find the line $\ell = [l_i]$ connecting both points

How to find a line that connects two given points?

Line Between Two Points

- H.C. also offer a simple way for computing a line through two points
- Given two points $\chi = [x_i], y = [y_i]$, find the line $\ell = [l_i]$ connecting both points
- We write that as $\ell = \chi \wedge y$ ("wedge")
- Solution via a system of linear eqns.

$$\begin{bmatrix} \mathbf{x} \cdot \mathbf{l} \\ \mathbf{y} \cdot \mathbf{l} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} -x_3\, l_3 \\ -y_3\, l_3 \end{bmatrix}$$

Line Between Two Points

- Cramer's rule again solves

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} -x_3\, l_3 \\ -y_3\, l_3 \end{bmatrix}$$

- by

$$l_1 = \frac{D_1}{D_3} \qquad l_2 = \frac{D_2}{D_3}$$

- with

$$\begin{aligned} D_1 &= \det(A_1) = l_3(x_2 y_3 - y_2 x_3) \\ D_2 &= \det(A_2) = l_3(x_3 y_1 - y_3 x_1) \\ D_3 &= \det(A) = x_1 y_2 - x_2 y_1 \end{aligned}$$

Line Between Two Points

- Cramer's leads to

$$l_1 = \frac{D_1}{D_3} \qquad l_2 = \frac{D_2}{D_3}$$

$$\begin{aligned} D_1 &= l_3(x_2 y_3 - y_2 x_3) \\ D_2 &= l_3(x_3 y_1 - y_3 x_1) \\ D_3 &= x_1 y_2 - x_2 y_1 \end{aligned}$$

- and we use

$$l_3 = l_3 \frac{D_3}{D_3}$$

- which results in

$$\mathbf{l} = \left[\frac{D_1}{D_3}, \frac{D_2}{D_3}, l_3 \frac{D_3}{D_3}\right]^\top \longrightarrow \mathbf{l} = \frac{l_3}{D_3} \begin{bmatrix} x_2 y_3 - y_2 x_3 \\ x_3 y_1 - y_3 x_1 \\ x_1 y_2 - x_2 y_1 \end{bmatrix}$$

Line Between Two Points

- We again exploit the cross product and the homogeneous property

$$\mathbf{l} = \frac{l_3}{D_3} \begin{bmatrix} x_2 y_3 - y_2 x_3 \\ x_3 y_1 - y_3 x_1 \\ x_1 y_2 - x_2 y_1 \end{bmatrix} = \begin{bmatrix} x_2 y_3 - y_2 x_3 \\ x_3 y_1 - y_3 x_1 \\ x_1 y_2 - x_2 y_1 \end{bmatrix} = \mathbf{x} \times \mathbf{y}$$

- Thus we obtain

$$\boxed{\mathcal{l} = \mathcal{x} \wedge \mathcal{y}: \quad \mathbf{l} = \mathbf{x} \times \mathbf{y}}$$

Typical Line Operations

- A point lies on a line if

$$\mathbf{x} \cdot \mathbf{l} = 0$$

- Intersection of two lines

$$\mathcal{X} = \ell \cap m : \quad \mathbf{x} = \mathbf{l} \times \mathbf{m}$$

- A line through two given points

$$\ell = \mathcal{X} \wedge y : \quad \mathbf{l} = \mathbf{x} \times \mathbf{y}$$

Points and Lines at Infinity

Points at Infinity

- It is possible to **explicitly** model infinitively distant points **with finite coordinates**

$$\mathcal{X}_\infty : \quad \mathbf{x}_\infty = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$$

- We can **maintain the direction** to that infinitively distant point
- Great tool when working with cameras as they are bearing-only sensors

Intersection at Infinity

- All lines ℓ with $\ell \cdot x_\infty = 0$ pass through x_∞
- We can interpret ℓ as a line in Hesse form

$$\mathbf{l}_{\text{Hesse}} = \begin{bmatrix} \cos\phi \\ \sin\phi \\ -d \end{bmatrix}$$

- **First two dimensions** determine the **direction** of the line ℓ
- This means $[u, v] \cdot [\cos\phi, \sin\phi] = 0$

Intersection at Infinity

- All lines ℓ with $\ell \cdot x_\infty = 0$ pass through x_∞
- This means $[u, v] \cdot [\cos\phi, \sin\phi] = 0$
- This hold for any line parallel to ℓ ,
  i.e. for any line $\mathbf{m} = [\cos\phi, \sin\phi, *]^T$

All parallel lines meet at one point at infinity!

Intersection at Infinity

- All lines ℓ with $\ell \cdot x_\infty = 0$ pass through x_∞
- This means $[u, v] \cdot [\cos\phi, \sin\phi] = 0$
- This hold for any line parallel to ℓ , i.e. for any line $\mathbf{m} = [\cos\phi, \sin\phi, *]^T$
- This can also be seen by

$$\mathbf{l} \times \mathbf{m} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} a \\ b \\ d \end{bmatrix} = \begin{bmatrix} bd - bc \\ ac - ad \\ ab - ab \end{bmatrix} = \begin{bmatrix} bd - bc \\ ac - ad \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$$

All parallel lines meet at one point at infinity!

Parallel Lines Meet at Infinity

Image Courtesy: J. Jannene

Infinitively Distant Objects

- Infinitively distant point

$$\mathcal{X}_\infty : \quad \mathbf{x}_\infty = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$$

- The infinitively distant line is the **ideal line**

$$\mathcal{l}_\infty : \quad \mathbf{l}_\infty = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- $\mathcal{l}_\infty$ can be interpreted as the horizon

Infinitively Distant Objects

- All points at infinity lie on the line at infinity called the **ideal line** given by

$$\mathbf{x}_\infty \cdot \mathbf{l}_\infty = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

- The ideal line can be seen as the horizon

Analogous for 3D Objects

- 3D point

$$\mathbf{X} = \begin{bmatrix} U \\ V \\ W \\ T \end{bmatrix} = \begin{bmatrix} U/T \\ V/T \\ W/T \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} U/T \\ V/T \\ W/T \end{bmatrix}$$

- Plane

$$\mathbf{A} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

Point on a Plane

- Via the scalar product, we can again test if a point lies on a plane

$$\mathbf{A} \cdot \mathbf{X} = \mathbf{A}^\mathsf{T}\mathbf{X} = \mathbf{X}^\mathsf{T}\mathbf{A} = 0$$

- which is based on

$$AX + BY + CZ + D = 0 \quad \text{or} \quad N \cdot X - S = 0$$

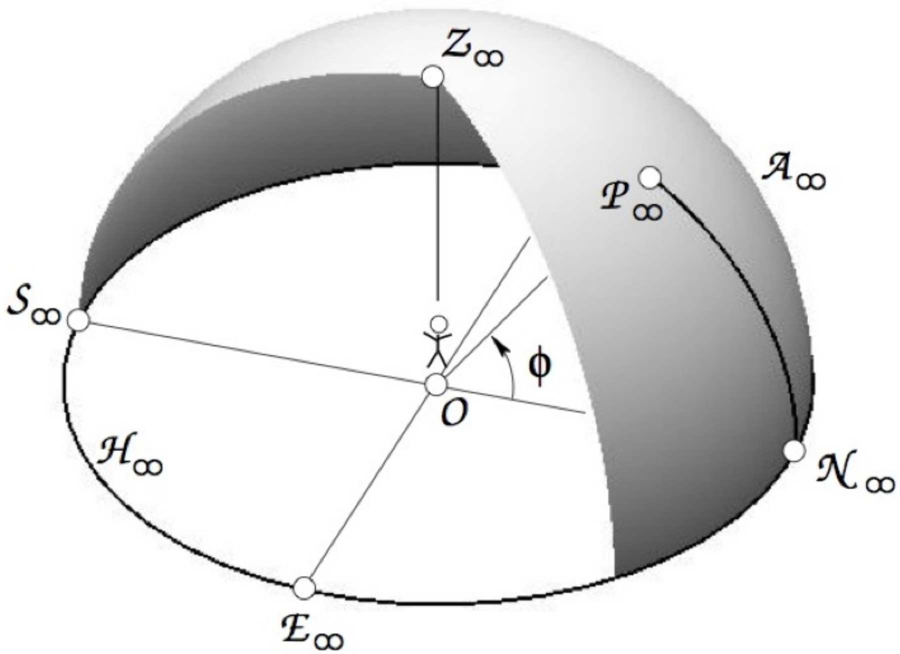
3D Objects at Infinity

- 3D point

$$\mathbf{P}_\infty = \begin{bmatrix} U \\ V \\ W \\ 0 \end{bmatrix}$$

- Plane

$$\mathbf{A}_\infty = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$



Conclusion

- Homogeneous coordinates are an alternative representation for geometric objects
- They can simplify mathematical expressions
- They can model points at infinity
- Easy chaining and inversion of transformations
- Uses an extra dimension (n+1)
- Equivalence up to scale

Being Familiar with Homogeneous Coordinates is Key for the Remaining Course

Literature

- Förstner & Wrobel: Photogrammetric Computer Vision, Springer, 2016
  - Chapter 5.1 – 5.3: H.C., points & lines
  - Chapter 6.1 – 6.4: transformations