Foundations of Audio Signal Processing

§3 Audio Signals and Signal Spaces

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Winter Term

Signals

- A **signal is a function** f assigning a uniquely determined element of the range Y of f to each element of the domain X of f Notation:
$$f : X \to Y, \quad X \ni x \mapsto f(x) \in Y.$$
- Example 1: **Analog audio signal** f.
X: real time-interval, Y: sound pressure level (SPL), f assigns the current SPL $f(t)$ to every point in time t.
- Example 2: **Digital Image** f.
$X = [0 : 255] \times [0 : 255]$ pixel domain, $Y = [0 : 255]$: gray scale value,
f assigns a gray scale value $f(i, j)$ to every pixel position (i, j) .
- An analog audio signal is an example of a **(time) continuous signal**.
- A digital image is an example of a **discrete signal**.

CT- and DT-Signals

- If X is an interval in $\mathbb{R}^n$, then
$$f : X \to \mathbb{R} \quad \text{resp.} \quad f : X \to \mathbb{C}$$
is a **continuous** real- or complex-valued **signal** (in n variables).
- If X is an interval in $\mathbb{Z}^n$, then
$$f : X \to \mathbb{R} \quad \text{resp.} \quad f : X \to \mathbb{C}$$
is a **discrete** real- or complex-valued **signal** (in n variables).
- We mostly consider the cases $n = 1$ or $n = 2$. As X is often a time interval ($n = 1$), we will talk of **continuous time** and **discrete time signals**, respectively.
- **CT-signals** and **DT-signals** are the shorthands for continuous time and discrete time signals, respectively.
- In case of CT-signals and DT-signals, the variable is typically denoted by t and n, respectively.

Even, Odd & Periodic Signals

Let $X \in \{\mathbb{R}, \mathbb{C}\}$. A function $f : X \to \mathbb{C}$ is called

- **even**, if $f(-x) = f(x)$ for all $x \in X$,
- **odd**, if $f(-x) = -f(x)$ for all $x \in X$.
- T-**periodic**, $T > 0$, if $f(x) = f(x + T)$ for all $x \in X$.
  The smallest such T (if it exists!) is called the **fundamental period** of f.

Theorem. Every function $f : X \to \mathbb{C}$ can be written as the sum of its **even and odd part**,

$$f^+(x) := \tfrac{1}{2}(f(x) + f(-x)) \quad \text{resp.} \quad f^-(x) := \tfrac{1}{2}(f(x) - f(-x))$$

$$f(x) = f^+(x) + f^-(x).$$

Proof. It is easy to see that f^+ is an even and f^- is an odd function. Furthermore, it is easy to prove that $f = f^+ + f^-$. □

Sine- & Exponential Signals

- Let A, f and φ be real numbers. Then the mapping

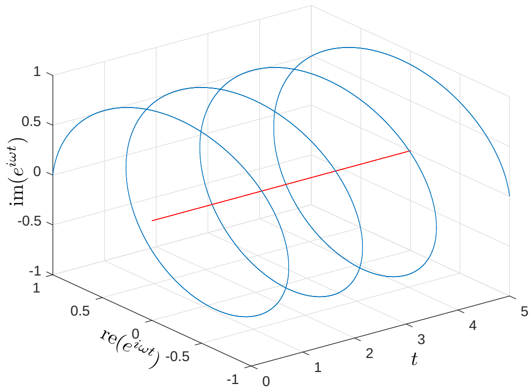
$$t \mapsto A \cdot \sin(2\pi f t + \varphi)$$

  defines the **sine signal** with **amplitude** A, **frequency** f and **phase** φ.
- Musical point of view: amplitude $\equiv$ sound level; frequency $\equiv$ pitch
- zero-phase sine signals are odd functions.
- If C and a are complex, $t \mapsto C \cdot e^{at}$ defines a complex **exponential signal**.
- Important special case: $C = 1$ und a purely imaginary, e.g., $a = i\omega$ with $0 \neq \omega \in \mathbb{R}$. The signal

$$t \mapsto e^{i\omega t}$$

  is a periodic signal with **wavelength** $T = 2\pi/|\omega|$.

Exponential Signals



Matlab Example

The sinc Function

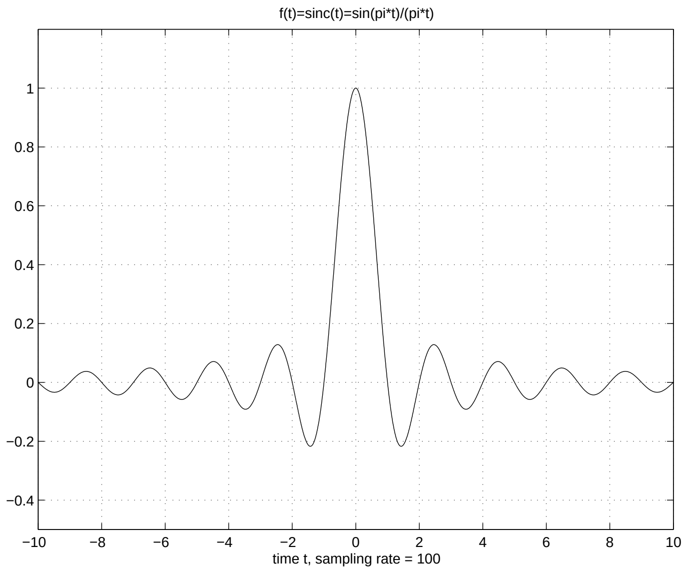
- In the context of the sampling theorem the following sinc function will play a fundamental role:

$$\mathrm{sinc}(t) := \begin{cases} \frac{\sin(\pi t)}{\pi t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases}$$

- Exercise: The sinc function is continuous at $t = 0$.
- Note: the set of roots of the sinc function is just the set of all non-zero integers:

$$\{t \in \mathbb{R} | \mathrm{sinc}(t) = 0\} = \mathbb{Z} \setminus \{0\}.$$

The Graph of the sinc Function



Chirp Signals

- Sine, exponential- and sinc function are signals of constant frequency.
- A class of signals with persistent frequency changes are the (linear) chirp signals. These chirps are defined by

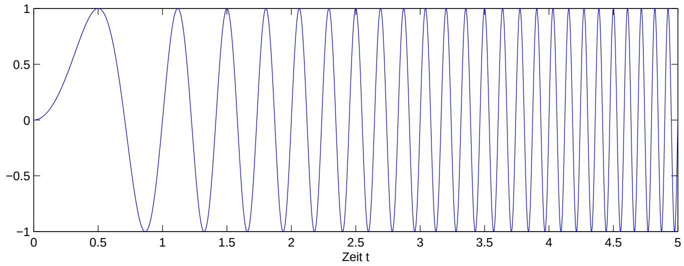
$$t \mapsto \sin(2\pi(f_0 + \tfrac{k}{2}t)t)$$

  with real constants f_0 and $k > 0$.
- For such signals, the frequency increases linearly with k.
- Remark: Bats locate objects with the help of chirp signals!
- Technical applications: remote sensing (SAR), modulation process (CSS)

Chirp-Signals

$$t \mapsto \sin(2\pi(f_0 + \tfrac{k}{2}t)t)$$

with $f_0 = 0$ and $k = 2$

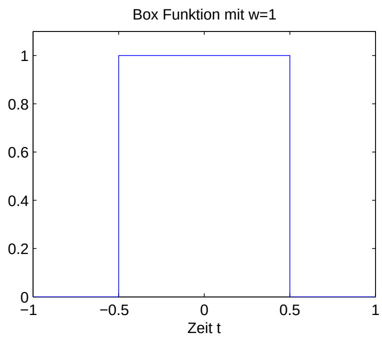


Audio example ($f_0 = 100$ und $k = 30$):

Box Function

- The **box function** of width $w > 0$ is defined by

$$b_w(t) := \begin{cases} 1 & \text{if } |t| \leq w/2 \\ 0 & \text{otherwise.} \end{cases}$$

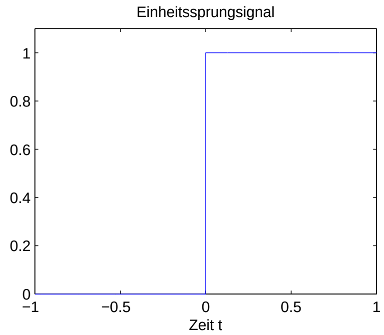


- Note: b_w is a CT-signal, but it has two discontinuities (jumps) at $t = \pm\frac{w}{2}$.

Heaviside or Unit Step Function

- The **Heaviside** or **unit step function** is defined by

$$u(t) := \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0. \end{cases}$$



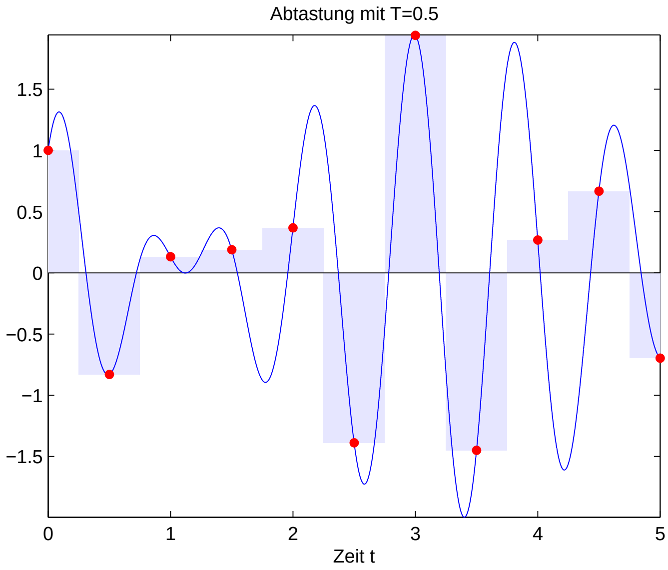
- Note: u is a CT-signal, but it has a jump discontinuity at $t = 0$.

Sampling

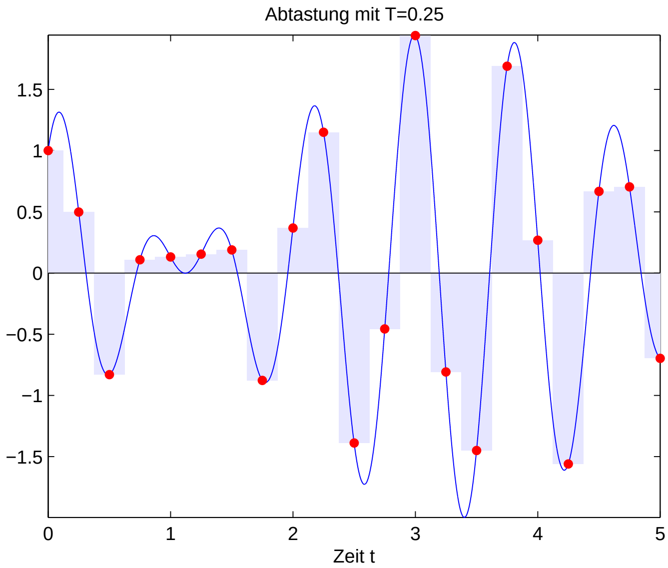
DT-signals are often derived from CT-signals via sampling.

- If $T > 0$, then the CT-signal $f : \mathbb{R} \to \mathbb{C}$ is transformed by T-**sampling** into the DT-Signal $x : \mathbb{Z} \to \mathbb{C}$: $x(n) := f(T \cdot n)$.
- The **sampling rate** is the number of samples per second. unit: Hertz (Hz).
- The sampling rate of x is $\frac{1}{T}$ Hz.
- The sampling rate is important for the signal's quality. Side effects caused by sampling are called **aliasing**.
- Common sampling rates for speech and music signals (1 kHz = 1000 Hz):
  - Telephone: 8 kHz
  - Digital Radio: 32 kHz
  - Compact Disk: 44,1 kHz
  - Studio applications 48, 96 kHz

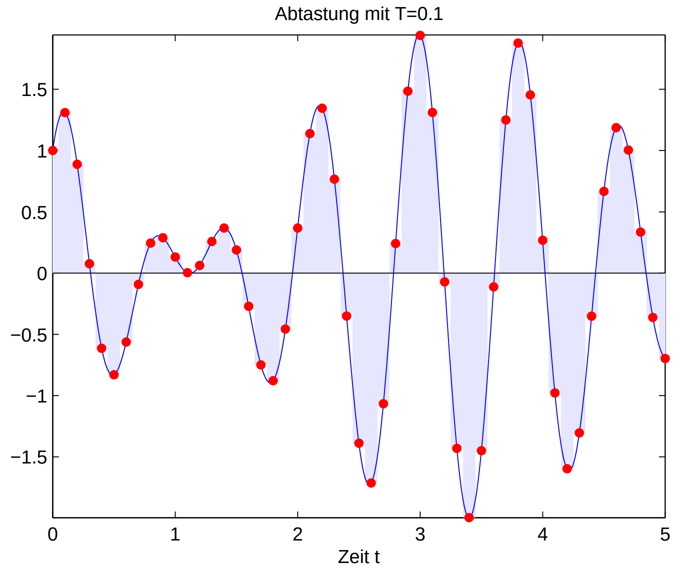
Sampling



Sampling



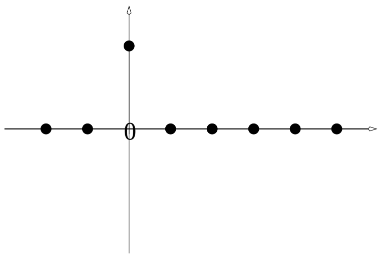
Sampling



Discrete Unit Impulse

- The time discrete **unit impulse function** $\delta : \mathbb{Z} \to \{0, 1\}$ is equal to 1 at $n = 0$ and vanishes everywhere else:

$$\delta(n) := \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases}$$

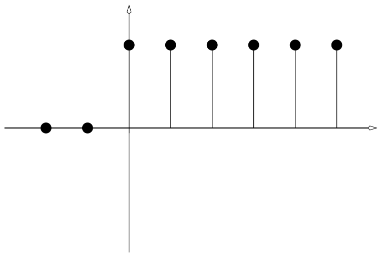


- Note: The sinc function restricted to $\mathbb{Z}$ coincides with the unit impulse!

Discrete Unit Step Function

- The time discrete **unit step function** $u : \mathbb{Z} \to \{0, 1\}$ is equal to zero for all negative integers and equal to 1 for all non-negative integers:

$$u(n) := \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n \geq 0. \end{cases}$$



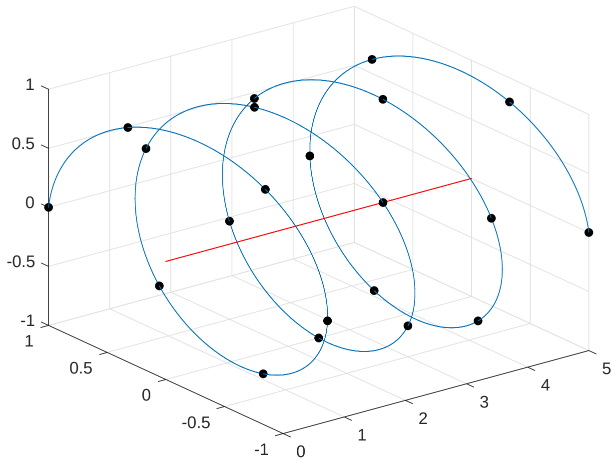
Discrete Frequency Signals

- Discrete time **frequency signals** of frequency $\omega \in [0, 1)$ are signals of the form

$$(c \cdot e^{2\pi i \omega n})_{n \in \mathbb{Z}}$$

  with $0 \neq c \in \mathbb{C}$.
- A frequency signal of frequency ω is periodic iff ω is rational, i.e., there exist $k, \ell \in \mathbb{Z}$, $\ell \neq 0$, with $\omega = k/\ell$.

Discrete Frequency Signals



Matlab Example

Discrete Sine Signals

- Discrete time **sine signals** are of the form

$$\mathbb{Z} \ni n \mapsto A \cdot \cos(2\pi\omega n + \varphi)$$

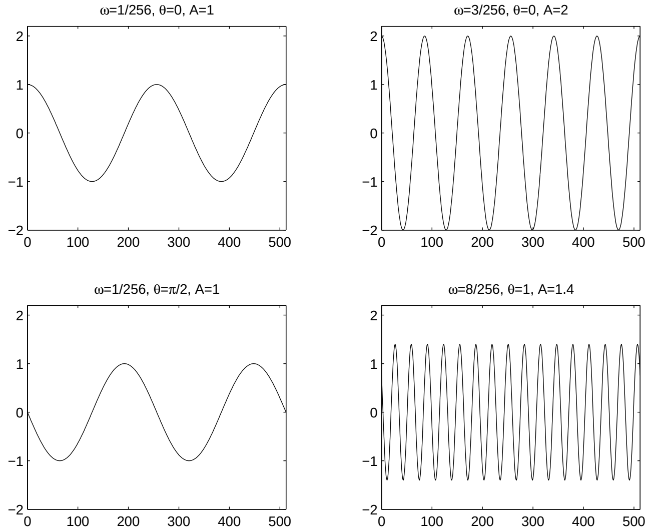
with a real φ.

- Such a signal contains the (complex) frequencies ω and $-\omega$, for

$$\cos(2\pi\omega n + \varphi) = \frac{1}{2}\left(e^{i(2\pi\omega n+\varphi)} + e^{-i(2\pi\omega n+\varphi)}\right).$$

Discrete Sine Signals

$$\mathbb{Z} \ni n \mapsto A \cdot \cos(2\pi\omega n + \theta)$$



Signal Spaces

- Within an orchestra, every musician generates an audio signal. The audience perceive a **superposition** of these signals. In addition, some subsignals are amplified (at least in an idealized szenario), think of all the first violinists.
- From a mathematical point of view, superposition and amplification can be viewed as addition and scalar multiplication, respectively. Notions from Linear Algebra form an adequate tool for modeling.
- In the following, we consider a signal to a lesser extent as a single object, but as a point in a high dimensional vector space. This point of view allows us to use **methods from Linear Algebra** for signal analysis.
- These methods will be combined with **methods from Analysis**. On this basis, we can design good tools for the analysis as well as the synthesis of signals.

Superposition & Amplification of Signals

Definition. Let $f : X \to \mathbb{C}$ and $g : X \to \mathbb{C}$ be signals with a common domain X, e.g., $X = \mathbb{R}$ or $X = \mathbb{Z}$. Furthermore, let $\lambda \in \mathbb{C}$.

- The **superposition** (mathematically: **sum**) $f + g$ of f and g is defined by

$$(f + g)(x) := f(x) + g(x) \quad \text{for } x \in X.$$

- The **amplification** (mathematically: **scalar multiple**) λf of f by the factor λ is defined by

$$(\lambda f)(x) := \lambda \cdot f(x) \quad \text{for } x \in X.$$

Theorem. The set $\mathbb{C}^X := \{f | f : X \to \mathbb{C}\}$ with the above addition and scalar multiplication defines a vector space over the complex numbers. For infinite X, this space is of infinite dimension.

Proof. A good exercise! □

Effects of Superpositions

We consider the superposition of two audio signals of (roughly) the same frequency.

(1) Let $f(t) := A \cdot e^{i\omega t}$ and $g(t) := B \cdot e^{i\omega t}$ with real constants A, B. Comparing $f + g$ with f we observe:

- $A > 0$, $B > 0$ yields an **amplification** of the signal;
- $A > 0$, $B < 0$ yields an **attenuation** of the signal;
- $A = -B$ yields a **cancellation** of the signal.

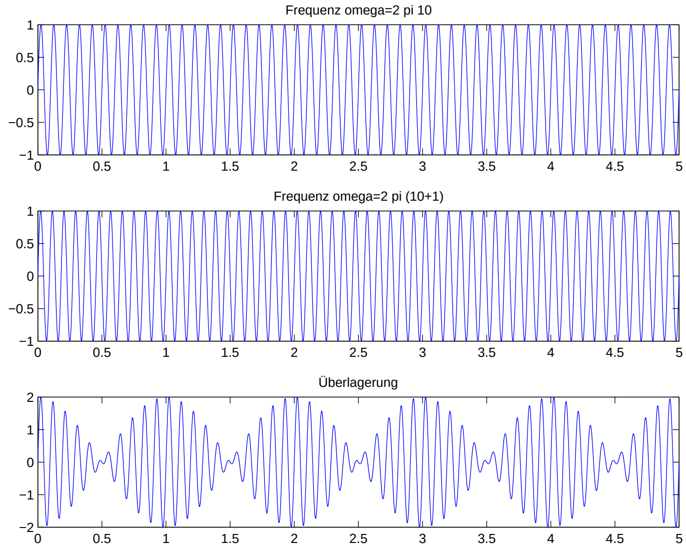
(2) Let $f(t) := A \cdot e^{i\omega t}$ and $g(t) := B \cdot e^{i(\omega+\epsilon)t}$ with real constants A, B, and with a small positiv ϵ. For the sum of f and g we obtain:

$$(f + g)(t) = (A + Be^{i\epsilon t})e^{i\omega t}.$$

The amplitude of the resulting function $f + g$ is subject to a constant change, which can be perceived as an unpleasant **amplitude vibrato**.

Effects of Superpositions

$\omega = 2\pi 10$ and $\epsilon = 2\pi$



Effects of Superpositions

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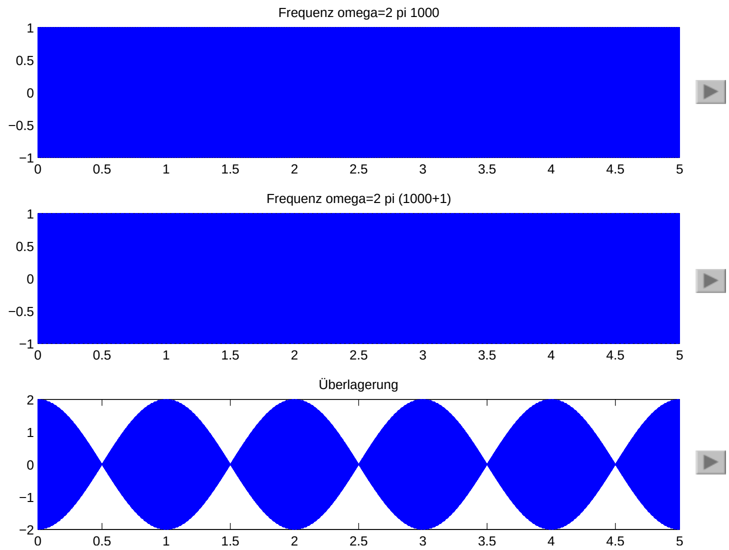
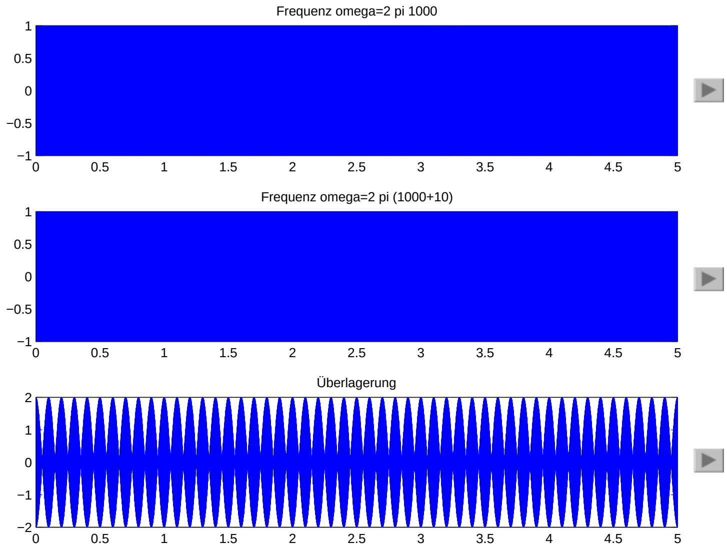


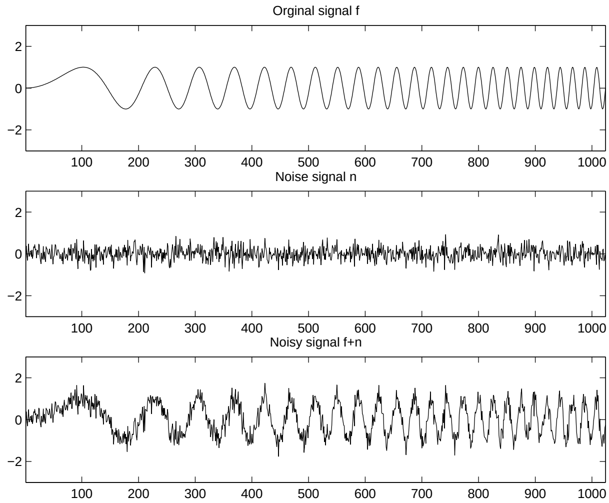
Effects of Superpositions

$\omega = 2\pi 1000$ and $\epsilon = 2\pi 10$



Superposition with a Noise Signal

The superposition of a signal with a noise signal is another typical example for the addition of signals. The next figure shows the superposition of a chirp signal with a noise signal:



Comments on Signal Spaces

- The signal spaces $\mathbb{C}^{\mathbb{Z}}$ and $\mathbb{C}^{\mathbb{R}}$ contain by far too many signals. Most signals contained in those spaces are irrelevant for applications as they have no physical realization.
- From a **physical point of view**, one is mainly interested in signals of finite energy.
- From a **mathematical point of view**, signal spaces are desirable, where
  - a distance between two signals may be defined (comparability)
  - a signal may be suitably transformed and analyzed, e.g., in order to determine its pitch .
- I the following we give some mathematical preparations.

Metric Spaces

- A metric allows to specify
  - the **distance** between two signals,
  - the **length** or the **energy** of an individual signal.

Definition. A **metric** on a set M is specified by a mapping

$$d : M \times M \to \mathbb{R},$$

such that for all $x, y, z \in M$ the following properties hold:

- $d(x, y) \geq 0$,
- $d(x, y) = 0$ iff. $x = y$,
- $d(x, y) = d(y, x)$,
- $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Normed vector spaces

- A metric may be defined for an arbitrary set M.
- Up to now, all signal spaces we considered are vector spaces, which are as such equipped with a linear structure.
- The following definition links metric and linear structures.

Definition. A **norm** on a $\mathbb{C}$-vector space V is defined by a mapping

$$\|\cdot\| : V \to \mathbb{R},$$

such that for all $x, y \in V$ and all $\lambda \in \mathbb{C}$ the following properties hold:

- $\|x\| = 0$ iff. $x = 0$,
- $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$,
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

Remark. A norm on V induces a metric d on V: $d(x, y) := \|x - y\|$.

Norms: Examples

- $\mathbb{C}^n$ equipped with the **euclidean norm** $\|x\|_2 := (|x_1|^2 + \ldots + |x_n|^2)^{1/2}$.
- $\mathbb{C}^n$ equipped with the **sum norm** $\|x\|_1 := (|x_1| + \ldots + |x_n|)$. The corresponding metric is also called **Manhattan metric**. Why?
- $\mathbb{C}^n$ equipped with the **maximum norm** $\|x\|_\infty := \max\{|x_1|, \ldots, |x_n|\}$.
- More generally: For a real $p \in [1, \infty)$, $\mathbb{C}^n$ equipped with the p-**norm**

$$\|x\|_p := (|x_1|^p + \ldots + |x_n|^p)^{1/p}.$$

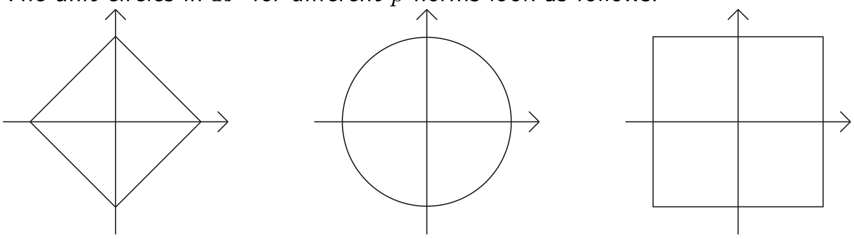
Replacing $\mathbb{C}$ by $\mathbb{R}$ in the above, one obtains normed $\mathbb{R}$-vector spaces.
With this, we have already introduced the most important signal spaces for finite signals!

Unit circles in $\mathbb{R}^2$

It is important to understand the notion of "length" in $\mathbb{R}^n$ for the different p-norms. For the case of $n = 2$ we consider the corresponding unit circle

$$\{z \in \mathbb{R}^2 : \|z\|_p = 1\}.$$

The unit circles in $\mathbb{R}^2$ for different p-norms look as follows:



Scalar products in real spaces

- Besides the notion of length, angles - and particularly right angles - are of fundamental importance.
- The scalar product of real-valued vectors gives a connection between the euclidean length and the angle via

$$\langle x|y\rangle := \sum_{i=1}^{n} x_i y_i = \|x\|_2 \cdot \|y\|_2 \cdot \cos\alpha,$$

  where α denotes the angle enclosed by $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$.
- In the above formula, $\|y\|_2 \cdot \cos\alpha$ denotes the scalar projection of y onto x.
- Geometrical description of the scalar product in the $\mathbb{R}^2$:
- Particularly interesting is the case $\cos\alpha = 0$ where the enclosed angle is $90°$. In this case, the vectors are perpendicular, x and y are then called **orthogonal**.

Scalar products in the complex domain

Definition. A **scalar product** or **inner product** on a $\mathbb{C}$-vector space V is defined by a mapping

$$\langle\cdot|\cdot\rangle : V \times V \to \mathbb{C},$$

such that for all $x, y \in V$ the following properties hold:

- $\langle x|x\rangle \geq 0$,
- $\langle x|x\rangle = 0$ iff. $x = 0$,
- $\langle\cdot|\cdot\rangle$ is $\mathbb{C}$-linear in the first component,
- $\langle x|y\rangle = \langle y|x\rangle^* = \overline{\langle y|x\rangle}$.

Remark. A scalar product on V induces a norm on V by

$$\|x\| := \sqrt{\langle x|x\rangle}.$$

Hilbert spaces

Definition. A complete (i.e., all Cauchy sequences converge) vector space with a norm induced by a scalar product is called **Hilbert space**.

Definition. Two vectors of a Hilbert space are called **orthogonal**, if their scalar product is zero: $\langle x|y\rangle = 0$.

Theorem.

(1) $\mathbb{C}^n$ together with the p-norm is a Hilbert space if and only if $p = 2$.

(2) For $p = 2$, the induced scalar product is given by

$$\langle x|y\rangle := \sum_{i=1}^{n} x_i \overline{y_i},$$

for $x, y \in \mathbb{C}^n$.

(3) The unit vectors $e_1, \ldots, e_n$ constitute an **orthonormal basis** of $\mathbb{C}^n$, this is, $\langle e_i|e_j\rangle = \delta_{ij}$.

We will now considerably generalize statement (3).

Hilbert bases

Theorem. Let I denote a non-empty set of indexes and let $(e_i)_{i\in I}$ denote an **ON-system** of the Hilbert space V, i.e. $\langle e_i|e_j\rangle = \delta_{ij}$ for $i,j \in I$. Then the following statements are equivalent:

(1) **Completeness**: If $x \in V$ is orthogonal to all e_i, then $x = 0$.
(2) **Parseval equality**: For each $x \in V$ $\|x\|^2 = \sum_{i\in I} |\langle x|e_i\rangle|^2$ holds.
(3) **Fourier series**: Every $x \in V$ has a representation

$$x = \sum_{i\in I} \langle x|e_i\rangle e_i,$$

where the number of non-zero scalar products $\langle x|e_i\rangle$ is at most countably infinite and the series unconditionally (i.e., regardless of the arrangement of the coefficients) converges towards x.

Definition. A complete ON-system is called **Hilbert basis** of V.

Fundamental Theorems

Theorem of Pythagoras: If $x_1, \dots x_n$ are pairwise orthogonal elements of the Hilbert space V, then the energy of the sum of the x_i is the same as the sum of the single energies:

$$\|\sum_{j=1}^{n} x_j\|^2 = \sum_{j=1}^{n} \|x_j\|^2.$$

Cauchy-Schwarz inequality: For elements x, y of a Hilbert space V the following holds:

$$|\langle x|y\rangle| \le \|x\| \cdot \|y\|.$$

Parseval identity: A unitary isomorphism $T : V \to W$ between Hilbert spaces conserves both the norm and the scalar product:

$$\|x\| = \|Tx\| \quad \text{and} \quad \langle x|y\rangle = \langle Tx|Ty\rangle.$$

The L^p- and ℓ^p-Spaces

- We are now sufficiently prepared to introduce signal spaces of practical relevance.
- These spaces were named after the French mathematician Henri Léon **Lebesgue**.
- The short-hands for these spaces begin with
  - a lower case ℓ (for DT signals) and
  - an upper case L (for CT signals).
- On the one hand, we will introduce ℓ^p spaces like $\ell^p(\mathbb{Z})$ or $\ell^p(\mathbb{N})$,
- on the other hand, we will discuss L^p spaces like $L^p(\mathbb{R})$ or $L^p([0,1])$.
- We will start with the ℓ^p spaces for DT signals.

Lebesgue Spaces for DT Signals

Definition. Let $p \in [1, \infty)$. The **Lebesgue space** $\ell^p(\mathbb{Z})$ consists of all sequences $x : \mathbb{Z} \to \mathbb{C}$, which are of finite p-length. More formally:

$$\ell^p(\mathbb{Z}) := \{x : \mathbb{Z} \to \mathbb{C} \mid \sum_{n\in\mathbb{Z}} |x_n|^p < \infty\}.$$

If $p = \infty$, the space $\ell^\infty(\mathbb{Z})$ is defined as the space of all bounded signals with common domain $\mathbb{Z}$:

$$\ell^\infty(\mathbb{Z}) := \{x : \mathbb{Z} \to \mathbb{C} \mid \exists B > 0 \, \forall n \in \mathbb{Z} : |x_n| \leq B\}.$$

Remark. If the parameter p is large, then small values are attenuated ($|x_n| < 1 \Rightarrow |x_n|^p \ll 1$) while large values are amplified ($|x_n| > 1 \Rightarrow |x_n|^p \gg 1$).

Theorem. For $p \in [1, \infty]$ the set $\ell^p(\mathbb{Z})$ is a linear subspace of $\mathbb{C}^{\mathbb{Z}}$.

Norms for the $\ell^p(\mathbb{Z})$-Spaces

Theorem. The mappings

$$\begin{aligned} \ell^p(\mathbb{Z}) \ni x \mapsto \|x\|_p &:= \left(\sum_{n\in\mathbb{Z}} |x(n)|^p\right)^{1/p} \quad \text{for } 1 \le p < \infty \text{ and} \\ \ell^\infty(\mathbb{Z}) \ni x \mapsto \|x\|_\infty &:= \sup\{|x(n)| \colon n \in \mathbb{Z}\} \end{aligned}$$

define a norm on $\ell^p(\mathbb{Z})$ and $\ell^\infty(\mathbb{Z})$, respectively.
Remark. Replacing in $\ell^p(\mathbb{Z})$ the common domain $\mathbb{Z}$ of the signals by an arbitrary finite or countable infinite set I, we obtain corresponding spaces $\ell^p(I)$. In case $I = [1 : n]$, the space $\ell^p(I)$ is just $\mathbb{C}^n$ equipped with the p-norm.

Relationships between the ℓ^p Spaces

Theorem. **(Jensen's Inequality)** For $1 \le p < q \le \infty$ the following holds:

(1) $\ell^p(\mathbb{Z})$ is a proper linear subspace of $\ell^q(\mathbb{Z})$. In particular

$$\ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z}) \subset \ell^3(\mathbb{Z}) \subset \ldots \subset \ell^\infty(\mathbb{Z}).$$

(2) For all $x \in \ell^p(\mathbb{Z})$ we have $\|x\|_q \le \|x\|_p$. In particular, for $x \in \ell^1(\mathbb{Z})$

$$\|x\|_1 \ge \|x\|_2 \ge \|x\|_3 \ge \ldots \ge \|x\|_\infty.$$

Three important ℓ^p Spaces

For us, the ℓ^p spaces $\ell^1(\mathbb{Z})$, $\ell^2(\mathbb{Z})$, and $\ell^\infty(\mathbb{Z})$ are of particular interest. Here,

$$\begin{aligned} \ell^1(\mathbb{Z}) &= \text{space of all absolute summable sequences} \\ \ell^2(\mathbb{Z}) &= \text{space of all square-summable sequences} \\ \ell^\infty(\mathbb{Z}) &= \text{space of all bounded sequences.} \end{aligned}$$

Theorem. The space $\ell^2(\mathbb{Z})$ is the only Hilbert space among the Banach spaces $\ell^p(\mathbb{Z})$, $p \in [1, \infty]$. The 2-norm is induced from the scalar product

$$\langle x|y\rangle := \sum_{n\in\mathbb{Z}} x(n)\overline{y(n)}.$$

The indicator functions $(\delta_n)_{n\in\mathbb{Z}}$ corresponding to the elements of $\mathbb{Z}$ form a Hilbert basis of $\ell^2(\mathbb{Z})$.

In a little while, we will discuss other Hilbert bases of $\ell^2(\mathbb{Z})$, which will play a crucial role in signal analysis.

Lebesgue Spaces for CT Signals: Preliminaries

- We are going to discuss the $L^p(\mathbb{R})$ spaces.
- $L^p(\mathbb{R})$ is the CT analogon to the DT space $\ell^p(\mathbb{Z})$.
- Memory hook: $L^p(\mathbb{R})$ evolves from $\ell^p(\mathbb{Z})$, by replacing $\mathbb{Z}$ with $\mathbb{R}$ and summation with integration.
- Some of the results for DT signals carry over to CT signals.
- However, there are CT specific phenomena! One reason for this is the fact that $|\mathbb{Z}| \ll |\mathbb{R}|$.

For proofs of the next non-trivial results, we refer to the literature on Functional Analysis.

Lebesgue Spaces for CT Signals

Definition. Let $p \in [1, \infty)$. The **Lebesgue space** $L^p(\mathbb{R})$ consists of all measurable functions $f : \mathbb{R} \to \mathbb{C}$, which are of finite p-length. More formally:

$$L^p(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C} | f \text{ measurable and } \int_{\mathbb{R}} |f(t)|^p dt < \infty\}.$$

For $p = \infty$ the space $L^\infty(\mathbb{R})$ is defined as the space of all measurable functions $f : \mathbb{R} \to \mathbb{C}$, which are essentially bounded:

$$L^\infty(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C} | f \text{ measurable and } \text{ess sup}_{t\in\mathbb{R}} |f(t)| < \infty\}.$$

Here,

$$\text{ess sup}_{t\in\mathbb{R}} |f(t)| := \inf\{a \geq 0 | \mu(\{t : |f(t)| > a\}) = 0\},$$

and μ denotes the so-called **Borel measure** on $\mathbb{R}$. This measure generalizes the notion of the length of an intervall to more general subsets of $\mathbb{R}$.

Theorem. For $p \in [1, \infty]$ the space $L^p(\mathbb{R})$ is a linear subspace of $\mathbb{C}^{\mathbb{R}}$.

Norms on the $L^p(\mathbb{R})$-Spaces

Theorem. The mappings

$$\begin{aligned} L^p(\mathbb{R}) \ni f \mapsto \|f\|_p &:= \left(\int_{\mathbb{R}} |f(t)|^p dt\right)^{1/p} \quad \text{for } 1 \leq p < \infty \text{ and} \\ L^\infty(\mathbb{R}) \ni f \mapsto \|f\|_\infty &:= \text{ess sup}\{|f(t)| : t \in \mathbb{R}\} \end{aligned}$$

define a norm on $L^p(\mathbb{R})$ and $L^\infty(\mathbb{R})$, respectively. With respect to these norms, the spaces $L^p(\mathbb{R})$ are complete, hence are Banach spaces.

Remark. Strictly speaking: $L^p(\mathbb{R})$ consists of equivalence classes of functions: two functions $f, g \in L^p(\mathbb{R})$ are equivalent iff $\|f - g\|_p = 0$.

Warning. Between the L^p spaces there is no analogon to $p < q \Rightarrow \ell^p(\mathbb{Z}) \subset \ell^q(\mathbb{Z})$. E.g., define

$$f(t) := \begin{cases} t^{-1/2} & \text{for } t \in (0, 1] \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad g(t) := \begin{cases} t^{-1} & \text{for } t \in [1, \infty) \\ 0 & \text{otherwise} \end{cases}.$$

Then $f \in L^1(\mathbb{R}) \setminus L^2(\mathbb{R})$ and $g \in L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$.

$L^2(\mathbb{R})$ is a Hilbert Space

Theorem. The space $L^2(\mathbb{R})$ is a Hilbert space with respect to the scalar product

$$\langle f|g\rangle := \int_{\mathbb{R}} f(t)\overline{g(t)}dt,$$

for $f, g \in L^2(\mathbb{R})$. In particular

$$\langle f|f\rangle = \|f\|_2^2.$$

Remark. $L^2(\mathbb{R})$ is the only Hilbert space among the $L^p(\mathbb{R})$ spaces.

Lebesgue Spaces for Periodic CT Signals

- The Lebesgue spaces $L^p(\mathbb{R})$ are an accumulation of CT Signals $f : \mathbb{R} \to \mathbb{C}$ that satisfy certain integrability conditions.
- In the next slides we will discuss another class of CT signals: **periodic signals**.

Definition. Let $\lambda \in \mathbb{R}_{>0}$. A signal $f : \mathbb{R} \to \mathbb{C}$ is called λ-**periodic**, if $f(t) = f(t + \lambda)$, for all $t \in \mathbb{R}$.

Remarks.

- Every λ-periodic function f is completely specified, if f is known on the interval $[0, \lambda)$.
- On the other hand, every function $g\colon [0, \lambda) \to \mathbb{C}$ can be uniquely extended to a λ-periodic function $f\colon \mathbb{R} \to \mathbb{C}$.
- If f is λ-periodic, then $t \mapsto f(\lambda \cdot t)$ defines a 1-periodic function. By this transformation we can w.l.o.g. restrict our study to 1-periodic functions.

$L^2([0,1])$ is a Hilbert Space

Theorem. The space $L^2([0,1])$ is a Hilbert space with respect to the scalar product

$$\langle f|g\rangle := \int_0^1 f(t)\overline{g(t)}dt,$$

for $f, g \in L^2([0,1])$. In particular

$$\langle f|f\rangle = \|f\|_2^2.$$

Remarks.

- Among the Banach spaces $L^p([0,1])$, $p \in [1,\infty]$, $L^2([0,1])$ is the only one, whose norm is induced from a scalar product, i.e., $L^2([0,1])$ is the only Hilbert space among the $L^p([0,1])$ spaces.
- If $a < b$ are real, one can analogously define the Hilbert space $L^2([a,b])$ of all measurable functions $f : [a,b] \to \mathbb{C}$ which are square-integrable.