DCG, Wintersemester 2024/25 — Lecture 23 (5 pages)

Metric Embeddings

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1 Definitions

Definition 23.1 (Metric space)**.** *A metric space is a pair (X, μ) where X is a set and*

$$\mu : X \times X \to [0, \infty)$$

is a function satisfying the following conditions for any $x, y, z \in X$:

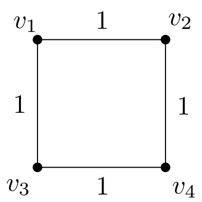
(i) $\mu(x, y) = 0$ *if and only if* $x = y$ *(identity)*

(ii) $\mu(x, y) = \mu(y, x)$ *(symmetry)*

(iii) $\mu(x, y) \leq \mu(x, z) + \mu(z, y)$ *(triangle inequality)*

A finite metric is a metric space (X, μ), where X is finite. Let $|X| = n$, then the function μ can be given by specifying an $n \times n$ matrix of the $\binom{n}{2}$ function values of μ. Another way to specify a finite metric space is by defining an edge-weighted graph with vertex set X, where μ corresponds to the shortest-path metric.

Example 23.2. *Example of a metric space defined by a graph on vertex set $V = \{v_1, v_2, v_3, v_4\}$ (left). The same metric space can be given by specifying all pairwise distances (right).*



$$\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

In this lecture we will study the following question. For a given metric space (X, μ), can we find a mapping $f : X \to \mathbb{R}^d$, such that for any $x, y \in X$:

$$\mu(x, y) = \|f(x) - f(y)\|_2$$

where $\| \cdot \|_2$ denotes the Euclidean norm. We denote with ℓ_2^d the metric space defined by $\mathbb{R}^d$ with the Euclidean distance. Often, such a mapping does not exist. Therefore we will relax the requirement on the metric embedding, as follows.

Definition 23.3 (Metric embedding)**.** *Let (X, μ) and (Y, σ) be finite metric spaces. A metric embedding is a mapping $f : X \to Y$. We define*

$$\textit{expansion}(f) = \max_{\substack{x,y \in X \\ x \neq y}} \frac{\sigma(f(x), f(y))}{\mu(x, y)}$$

and

$$\textit{contraction}(f) = \max_{\substack{x,y \in X \\ x \neq y}} \frac{\mu(x, y)}{\sigma(f(x), f(y))}$$

We call the product $\textit{expansion}(f) \cdot \textit{contraction}(f)$ the distortion D of f. If $D = 1$ we say f is isometric.

Example 23.4. *An example for a metric embedding $f : V \to \mathbb{R}^2$ of the metric space given in Example 23.2 can be specified as follows*

$$f(v_1) = (0,1), \quad f(v_2) = (1,1), \quad f(v_3) = (0,0), \quad f(v_4) = f(1,0)$$

The expansion of f is 1, the contraction of f is $\sqrt{2}$, and the distortion is therefore $\sqrt{2}$.

2 Embedding the Hamming cube

Definition 23.5 (Hamming cube)**.** *Consider the finite metric space (X, μ), where $X = \{0,1\}^m$ and*

$$\mu(x,y) = \sum_{i=1}^{m} |x_i - y_i|$$

In other words, the set X are all vertices of the unit hypercube in $\mathbb{R}^m$ and the distance defined by μ between two vectors is the number of positions where the two vectors differ. This is also called the Hamming distance. The metric space can be represented as a graph which has the vertex set $V = X$ and the edge set

$$E = \{(u,v) \in X \times X \mid \mu(u,v) = 1\}$$

We call this metric space the Hamming cube and we denote it with C_m.

Theorem 23.6. *Let $m \geq 2$ and $n = 2^m$. There is no metric embedding with distortion D of the Hamming cube C_m into ℓ_2^d, for any dimension d, with*

$$D < \sqrt{m} = \sqrt{\log_2 n}$$

Proof. Consider any mapping $f : \{0,1\}^m \to \mathbb{R}^d$, for any d, and assume the Euclidean distance in the target space. We assume that *contraction*$(f) = 1$. (Otherwise, let *contraction*$(f) = s$, then we can scale f uniformly by factor s and get a mapping f' with contraction equal to 1, which has the same distortion as f.)

Let E be the edges of the Hamming cube C_m. Let F be the set of "long diagonals" defined as follows

$$F = \{(u,v) \in X \times X \mid \mu(u,v) = m\}$$

We make the following claim.

Claim 23.7.

$$\sum_{(u,v) \in F} \|f(u) - f(v)\|^2 \leq \sum_{(u,v) \in E} \|f(u) - f(v)\|^2$$

Assuming the claim holds true, we finish the proof. Consider the average of the squared length of the embedded edges of E. We have

$$\frac{1}{|E|} \sum_{(u,v) \in E} \|f(u) - f(v)\|^2 \geq \frac{1}{|E|} \sum_{(u,v) \in F} \|f(u) - f(v)\|^2 \geq \frac{|F|}{|E|} m^2$$

Where the second inequality follows from *contraction*$(f) = 1$ and $\mu(u,v) = m$ for $(u,v) \in F$.

What is the size of the sets E and F? We have that $|E| = m|X|/2 = m2^{m-1}$, since every vertex of X is connected to m other vertices, by flipping each one of the m bits individually.

For F we have that $|F| = |X|/2 = 2^{m-1}$, since every vertex of X is connected to 1 other vertex via an edge in F, by flipping all of its bits at once. It follows that

$$\frac{|F|}{|E|}m^2 = \frac{2^{m-1} \cdot m^2}{m \cdot 2^{m-1}} = m$$

Now, putting the above together we get the following lower bound on the average

$$\frac{1}{|E|} \sum_{(u,v)\in E} \|f(u) - f(v)\|^2 \geq m$$

which implies the following lower bound on the maximum

$$\max_{(u,v)\in E} \|f(u) - f(v)\| \geq \sqrt{m}$$

Since any edge in E has length equal to 1, we get that the expansion of f is at least $\sqrt{m}$. Now, since the contraction is equal to 1 we get that the distortion is at least $\sqrt{m}$. □

It remains to prove the claim. We will first prove it in the simple case $m = 2$ (the 4-cycle).

Lemma 23.8. *For any $x_1, x_2, x_3, x_4 \in \mathbb{R}^d$ it holds that*

$$\|x_1 - x_3\|^2 + \|x_2 - x_4\|^2 \leq \|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + \|x_3 - x_4\|^2 + \|x_4 - x_1\|^2$$

Proof. Assume $d = 1$. For any $x_1, x_2, x_3, x_4 \in \mathbb{R}$ it holds that

$$(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2 + (x_4 - x_1)^2 - (x_1 - x_3)^2 - (x_2 - x_4)^2$$

is equal to

$$(x_1 - x_2)^2 - 2x_2x_3 + (x_3 - x_4)^2 - 2x_4x_1 + 2x_1x_3 + 2x_2x_4$$

which is equal to

$$((x_1 - x_2) + (x_3 - x_4))^2$$

and this is at least 0, which proves the claim for $d = 1$. For $d > 1$ we can apply the above inequality for each coordinate and we can sum these inequalities to derive the statement. □

Proof of Claim 23.7. We prove the claim by induction on m. The base case ($m = 2$) is given by the lemma above. So consider $m > 2$. We divide the vertex set X of the Hamming cube into two sets. Recall that $X = \{0, 1\}^m$. Let X_0 be the vertices where the last coordinate is 0 and let X_1 be the vertices where the last coordinate is 1. That is,

$$X_0 = \{u0 \mid u \in \{0, 1\}^{m-1}\}$$

$$X_1 = \{u1 \mid u \in \{0, 1\}^{m-1}\}$$

The set X_0 induces an $(m - 1)$-dimensional subcube, let E_0 be its edge set and let

$$F_0 = \{(u, v) \mid u, v \in X_0, \mu(u, v) = m - 1\}$$

be the set of "long diagonals". Let E_1 and F_1 be the edges defined in the same way for X_1.

Let $E_{01} = E \setminus (E_0 \cup E_1)$. Note that the edges of E_{01} are the edges connecting the two subcubes.

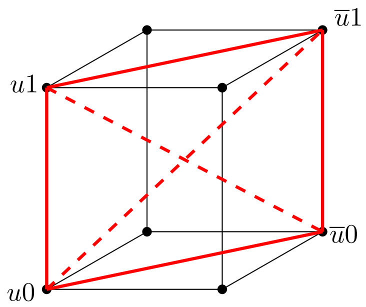


Figure 1: The 4-cycle connecting the vertices $u0, \overline{u}0, \overline{u}1, u1$ (solid red lines). The dashed lines show long diagonals in C_m between these points.

Let $\overline{u}$ denote the vector u with all bits flipped. Now, fix a vector $u \in \{0,1\}^{m-1}$ and consider the 4-cycle spanning the vertices $u0, \overline{u}0, \overline{u}1, u1$ (Note that u and $\overline{u}$ yield the same 4-cycle.)

For any $S \subset X^2$ we define the shorthand

$$\sum_S = \sum_{(u,v)\in S} \|f(u) - f(v)\|^2$$

We have by Lemma 23.8

$$\begin{aligned}
2 \cdot \sum_F &= \sum_{u\in\{0,1\}^{m-1}} \|f(u0) - f(\overline{u}1)\|^2 + \|f(\overline{u}0) - f(u1)\|^2 \\
&\leq \sum_{u\in\{0,1\}^{m-1}} \|f(u0) - f(\overline{u}0)\|^2 + \|f(u1) - f(\overline{u}1)\|^2 \\
&\qquad\qquad + \|f(u0) - f(u1)\|^2 + \|f(\overline{u}0) - f(\overline{u}1)\|^2 \\
&= 2\left(\sum_{F_0} + \sum_{F_1} + \sum_{E_{01}}\right)
\end{aligned}$$

By the inductive hypothesis we have

$$\sum_{F_0} \leq \sum_{E_0} \quad \text{and} \quad \sum_{F_1} \leq \sum_{E_1}$$

for the two subcubes. Putting everything together we get

$$2\sum_F \leq 2\left(\sum_{E_0} + \sum_{E_1} + \sum_{E_{01}}\right)$$

Dividing both sides by 2 proves the claim. □

3 An isometric embedding

Denote with ℓ_∞^d the metric space $\mathbb{R}^d$ equipped with the metric μ defined by

$$\mu(x,y) = \|x - y\|_\infty = \max_{1\leq i\leq d} |x_i - y_i|$$

for any $x, y \in \mathbb{R}^d$.

Theorem 23.9. *For any finite metric space (X, μ) with $|X| = n$, there is an isometric embedding into the metric space ℓ_∞^n.*

Proof. Define a metric embedding f from (X, μ) to ℓ_∞^n, by defining n functions $f_1, \ldots, f_n : X \to [0, \infty)$ where f_i gives the ith coordinate of the embedded point, and is defined as

$$f_i(x) = \mu(x, x_i)$$

where x_i denotes the ith element of X.

We show that for any $x, y \in X$

$$\|f(x) - f(y)\|_\infty = \mu(x, y)$$

First, we show that f has contraction at most 1. That is,

$$\|f(x) - f(y)\|_\infty \geq \mu(x, y)$$

Suppose that $y = x_j$, then $f_j(x) = \mu(x, y)$ and $f_j(y) = \mu(y, y) = 0$. Therefore,

$$\max_i |f_i(x) - f_i(y)| \geq |f_j(x) - f_j(y)| = \mu(x, y)$$

We now show that also the expansion is at most 1. That is,

$$\|f(x) - f(y)\|_\infty \leq \mu(x, y)$$

This is equivalent to stating that for all $x, y \in X$ and for all $x_i \in X$ it holds that

$$|f_i(x) - f_i(y)| = |\mu(x, x_i) - \mu(y, x_i)| \leq \mu(x, y)$$

Assume without loss of generality that $\mu(x, x_i) \geq \mu(y, x_i)$. (Otherwise swap the roles of x and y) By the triangle inequality, we have

$$\mu(x, x_i) \leq \mu(x, y) + \mu(y, x_i)$$

and equivalently,

$$\mu(x, x_i) - \mu(y, x_i) \leq \mu(x, y)$$

This implies that expansion and contraction are both at most 1. Therefore, the distortion is at most 1. □

References

- Jiří Matoušek, Chapter 15.1 and 15.4, Lectures on Discrete Geometry, Springer Graduate Texts in Mathematics.