Foundations of Audio Signal Processing

§4 Fourier Transform

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Preliminary Remarks

In her book **The World According to Wavelets** Barbara Burke Hubbard gives the following nice characterization of the Fourier transform:

- The Fourier transform is the mathematical procedure that breaks up a function into the frequencies that compose it, as a prism breaks up light into colors. It transforms a function f that depends on time into a new function, $\widehat{f}$, which depends on frequency. This new function is called the **Fourier transform** of the original function (or, when the original function is periodic, its **Fourier series**).

A function f and its Fourier transform $\widehat{f}$ constitute two facets of the same information:

- The function f shows the time information and hides the frequency information. In other words: By recording the air pressure fluctuations of a musical signal over time, the function f indicates **when** a note is played, but not **which** note.
- The Fourier transform $\widehat{f}$ shows the frequency information, but hides the time information. In other words: The Fourier transform $\widehat{f}$ shows **which** notes appear, but it is extremely difficult to find out **when** these are played.

Fourier Series of Periodic CT Signals

In the Hilbert space $L^2([0,1])$, two Hilbert bases are of particular importance for us. A proof of the following result can be found in most textbooks on Functional Analysis. For $k \in \mathbb{N}$ let

$$A_k := ([0,1] \ni t \mapsto \sqrt{2}\cos(2\pi kt)) \quad \text{and} \quad B_k := ([0,1] \ni t \mapsto \sqrt{2}\sin(2\pi kt)).$$

Theorem. The Hilbert space $L^2([0,1])$ possesses the following Hilbert bases (among others):

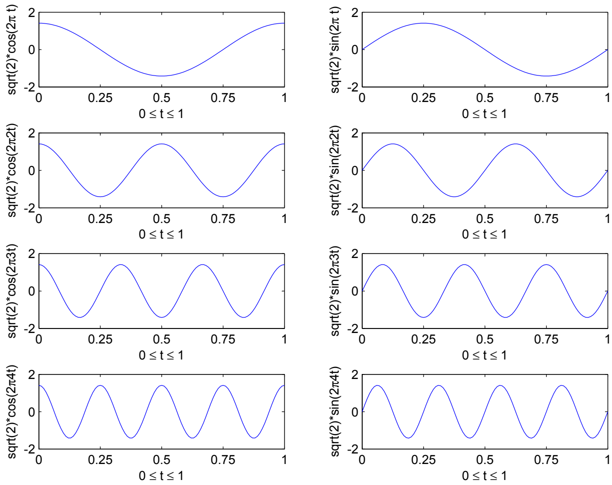
(1) $\{1, A_k, B_k \mid k \in \mathbb{N}\}$.
(2) $\{\mathbf{e}_k \mid k \in \mathbb{Z}\}$ with $\mathbf{e}_k(t) := e^{2\pi ikt}$ for $t \in [0,1]$.

By (1) & (2), every $f \in L^2([0,1])$ has the following **Fourier series expansions**:

$$\begin{aligned} f &= \langle f|1\rangle + \sum_{k\in\mathbb{N}}\langle f|A_k\rangle A_k + \sum_{k\in\mathbb{N}}\langle f|B_k\rangle B_k, \\ f &= \sum_{k\in\mathbb{Z}}\langle f|\mathbf{e}_k\rangle \mathbf{e}_k. \end{aligned}$$

Illustration of a Hilbert basis

The following figure shows the graphs of $A_1, \ldots, A_4$ (left) and $B_1, \ldots, B_4$ (right).



Fourier Coefficients

The scalar products

$$\begin{aligned}
\langle f|1\rangle &= \int_0^1 f(t)dt \\
\langle f|A_k\rangle &= \sqrt{2}\int_0^1 f(t)\cos(2\pi kt)dt \\
\langle f|B_k\rangle &= \sqrt{2}\int_0^1 f(t)\sin(2\pi kt)dt, \\
\langle f|\mathbf{e}_k\rangle &= \int_0^1 f(t)e^{-2\pi ikt}dt
\end{aligned}$$

are called the **Fourier coefficients** of f with respect to the above mentioned Hilbert bases (1) and (2), respectively. The coefficients corresponding to A_k, B_k, and $\mathbf{e}_k$ indicate how strong the frequency of k Hertz is involved in the signal f.

Equality in Lebesgue Spaces

- The equalities in the above mentioned theorem, like

$$f = \sum_{k\in\mathbb{Z}} \langle f|\mathbf{e}_k\rangle \mathbf{e}_k,$$

  are equalities in the L^2 sense, i.e., the functions on both sides of the equality are pointwise equal up to a subset of the domain $[0, 1]$, which has measure zero.
- Only additional conditions on f guarantee pointwise equality.
- If, e.g., f is continuously differentiable, then the Fourier series converges uniformly on $[0, 1]$ to f.

A Fundamental Isomorphism of Hilbert Spaces

In the last chapter, we discussed prototypes of Hilbert spaces. Here we illustrate this theorem for the special case of $L^2([0, 1])$. This theorem states that the Hilbert spaces $L^2([0, 1])$ and $\ell^2(\mathbb{Z})$ are isomorphic. More precisely:

Theorem. The function

$$f \mapsto \widehat{f} := (\langle f|\mathbf{e}_k\rangle)_{k\in\mathbb{Z}},$$

which assigns to every signal $f \in L^2([0, 1])$ the sequence of its Fourier coefficients, is an isomorphism of Hilbert spaces:

$$L^2([0, 1]) \xrightarrow{\simeq} \ell^2(\mathbb{Z}).$$

This mapping is not only a linear isomorphism, but it also keeps the scalar product invariant, i.e., for $f, g \in L^2([0, 1])$:

$$\langle f|g\rangle_{L^2([0,1])} = \langle \widehat{f}|\widehat{g}\rangle_{\ell^2(\mathbb{Z})}.$$

As $\|f\| = \sqrt{\langle f|f\rangle}$, this isomorphism is also a length-preserving transform.

The Unnormalized Case $L^2([a,b])$

We generalize the above results to Hilbert spaces of the form $L^2([a,b])$, where $a < b$ are real. This space can be regarded as the space of square-integrable $(b-a)$-periodic signals. For $k \in \mathbb{Z}$ we define the functions $e_k^{a,b} : [a,b] \to \mathbb{C}$ by

$$\mathbf{e}_k^{a,b}(t) := \tfrac{1}{\sqrt{b-a}} e^{2\pi i k t/(b-a)}.$$

Theorem. $\{\mathbf{e}_k^{a,b} \mid k \in \mathbb{Z}\}$ is a Hilbert basis of $L^2([a,b])$.
Thus every $f \in L^2([a,b])$ has the following **Fourier series expansion**:

$$f = \sum_{k \in \mathbb{Z}} \langle f | \mathbf{e}_k^{a,b} \rangle \mathbf{e}_k^{a,b}.$$

The normalized version is the case $a = 0$ and $b = 1$.

Fourier Transform for Non-periodic CT Signals

- The idea of Fourier series expansion can be generalized to non-periodic CT signals.
- For non-periodic CT signals, the integral frequencies $k \in \mathbb{Z}$ are not sufficient to describe an arbitrary signal completely.
- Here, one has to take into account all frequencies $\omega \in \mathbb{R}$. Furthermore, one has to replace summation by integration.
- This leads to the CT analogue of Fourier series expansions.

Theorem. For every signal $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\omega \in \mathbb{R}$ put

$$\widehat{f}(\omega) := \int_{\mathbb{R}} f(t) e^{-2\pi i \omega t} dt.$$

Then

$$f(t) = \int_{\mathbb{R}} \widehat{f}(\omega) e^{2\pi i \omega t} d\omega.$$

Fourier Coefficients for Non-periodic CT Signals

In the following let $\mathbf{e}_\omega \colon \mathbb{R} \to \mathbb{C}$ denote the CT exponential- or frequency signal $t \mapsto e^{2\pi i\omega t}$ of frequency $\omega \in \mathbb{R}$.

Remarks.

- The assumption $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is of purely technical nature and guarantees that all occuring integrals are finite. There are weaker conditions on f, still guaranteeing the finiteness of the integrals.
- The equation $f(t) = \int_{\mathbb{R}} \widehat{f}(\omega) e^{2\pi i\omega t} d\omega$ shows that every signal f, which satisfies certain integrability conditions, can be written as a continuous linear combination of frequency signals $\mathbf{e}_\omega$.
- The value $\widehat{f}(\omega)$ specifies the **intensity**, with which the frequency signal $\mathbf{e}_\omega$ is involved in the signal f. These values $\widehat{f}(\omega)$ now play a role analogous to the Fourier coefficients in the Fourier series representation.
- Note that the frequency signals $\mathbf{e}_\omega$ as $\frac{1}{\omega}$-periodic functions are not contained in any of the spaces $L^p(\mathbb{R})$ for $1 \leq p < \infty$. Hence the right hand side in $\widehat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i\omega t} dt$ is not a scalar product!

The Plancherel Theorem

Definition. Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then

$$\omega \mapsto \widehat{f}(\omega) := \int_{-\infty}^{\infty} f(t) e^{-2\pi i\omega t} dt, \quad \omega \in \mathbb{R},$$

defines a function $\widehat{f} \colon \mathbb{R} \to \mathbb{C}$, called the **Fourier integral** or the **Fourier transform** of f. Sometimes, we will use the notation $F(f)$ instead of $\widehat{f}$.

One can show that the definition of the Fourier transform of functions $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ can be extended to all signals $f \in L^2(\mathbb{R})$. The following celebrated result states that the continuation of the Fourier transform to all of $L^2(\mathbb{R})$ keeps the scalar product invariant and hence it is energy preserving.

Theorem (Plancherel). The Fourier transform $f \mapsto \widehat{f}$ defines a unitary transform $L^2(\mathbb{R}) \to L^2(\mathbb{R})$. Hence for all $f, g \in L^2(\mathbb{R})$:

- $\widehat{f} \in L^2(\mathbb{R})$,
- $\langle f | g \rangle = \langle \widehat{f} | \widehat{g} \rangle$,
- $\| f \| = \| \widehat{f} \|$.

Properties of the Fourier Transform (1)

Theorem. Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ (or more generally $f \in L^2(\mathbb{R})$). The Fourier transform has the following properties:

(1) For $t_0 \in \mathbb{R}$ define the t_0-**translate** of f by $f_{t_0}(t) := f(t - t_0)$. Then

$$\widehat{f_{t_0}}(\omega) = e^{-2\pi i \omega t_0} \widehat{f}(\omega).$$

(2) For $\omega_0 \in \mathbb{R}$ define the ω_0-**modulation** of f by $f^{\omega_0}(t) := e^{-2\pi i \omega_0 t} f(t)$. Then

$$\widehat{f^{\omega_0}}(\omega) = \widehat{f}(\omega + \omega_0).$$

(3) Let f be differentiable with $f' \in L^2(\mathbb{R})$. Then

$$\widehat{f'}(\omega) = 2\pi i \omega \widehat{f}(\omega).$$

Properties of the Fourier Transform (2)

Theorem. Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ (or more generally $f \in L^2(\mathbb{R})$). The Fourier transform has the following properties:

(4) If $\widehat{f}$ is differentiable and $g(t) := tf(t)$, then

$$\widehat{f}'(\omega) = -2\pi i \widehat{g}(\omega).$$

(5) For $s \in \mathbb{R} \setminus \{0\}$ the s-**scaled version** $t \mapsto f(t/s)$ is also in $L^2(\mathbb{R})$ and

$$\widehat{f(\tfrac{\cdot}{s})}(\omega) = |s| \cdot \widehat{f}(\omega s).$$

Proof. We will only prove (1), the other proofs are highly recommended as exercises! $\square$

Translation & Modulation

We are going to prove that the Fourier transform of a translated signal is a modulated version of the Fourier transform of the original signal:

$$\widehat{f_{t_0}}(\omega) = e^{-2\pi i\omega t_0}\widehat{f}(\omega).$$

For the proof we need the following.

Integration by Substitution: If I is a real interval, $F : I \to \mathbb{C}$ is in $L^1(I) \cap L^2(I)$, and $\varphi : [a,b] \to I$ is continuously differentiable, then

$$\int_a^b F(\varphi(t)) \cdot \varphi'(t)dt = \int_{\varphi(a)}^{\varphi(b)} F(x)dx.$$

Proof of (1). With $F(x) := f(x)e^{-2\pi i\omega(x+t_0)}$ and the substitution $\varphi(t) := t - t_0$ we obtain

$$\begin{aligned}\widehat{f_{t_0}}(\omega) &= \int_{\mathbb{R}} f_{t_0}(t)e^{-2\pi i\omega t}dt = \int_{\mathbb{R}} f(t-t_0)e^{-2\pi i\omega t}dt \\ &= \int_{\mathbb{R}} f(x)e^{-2\pi i\omega(x+t_0)}dx = e^{-2\pi i\omega t_0}\widehat{f}(\omega).\end{aligned}$$

This proves (1). □

Inverse Fourier Transform

By Plancherel's Theorem, $f \mapsto \widehat{f}$ is a unitary automorphism of $L^2(\mathbb{R})$. Our next goal is to describe the inverse mapping.

Definition/Theorem. Let $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. The **inverse Fourier transform** of g, denoted by $\check{g}$, has the form

$$\check{g}(t) = \int_{-\infty}^{\infty} g(\omega)e^{2\pi i\omega t}d\omega.$$

As $\check{g}(t) = \widehat{g}(-t)$, the definition of $\check{g}$ can be extended to all of $L^2(\mathbb{R})$.

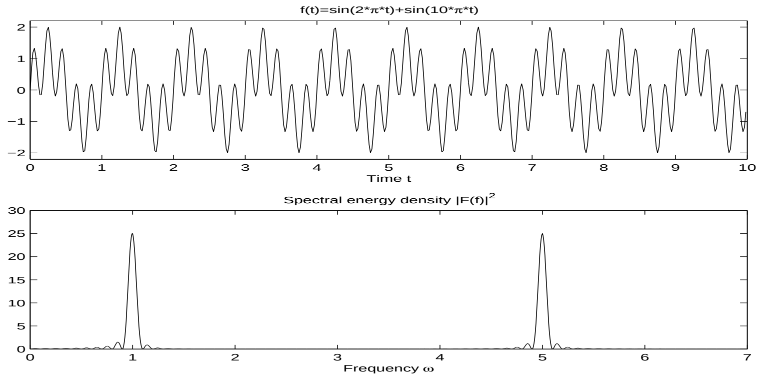
Theorem. Let $g = \widehat{f}$ denote the Fourier transform of a signal $f \in L^2(\mathbb{R})$. Then $g \in L^2(\mathbb{R})$ and $\check{g} = f$. In other words

$$(\widehat{f})^\vee = f = (\check{f})^\wedge.$$

We are going to present some typical pairs $(f, \widehat{f})$.

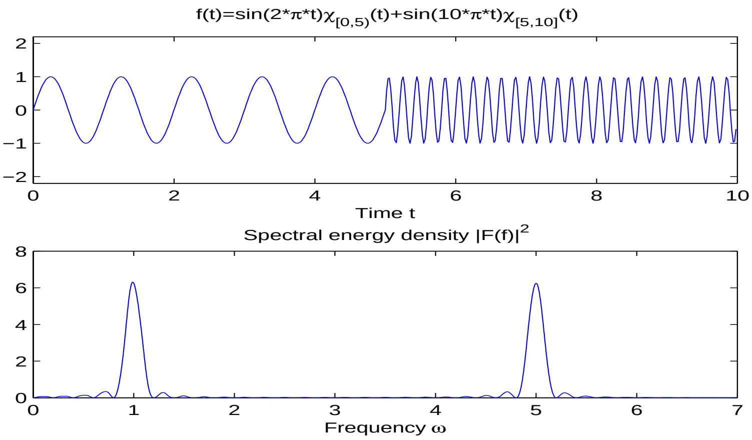
Superposition of two Sinusoids

Let the CT signal f be the superposition of two sinusoids of frequency 1 Hz and 5 Hz within the interval $[0, 10]$, outside this interval f is identically equal to zero. The following figure shows the pair $(f, |\widehat{f}|^2)$. The so-called **ripples** in the spectrum result from discontinuities of f at the borders of the interval, leading to **destructive interference**.



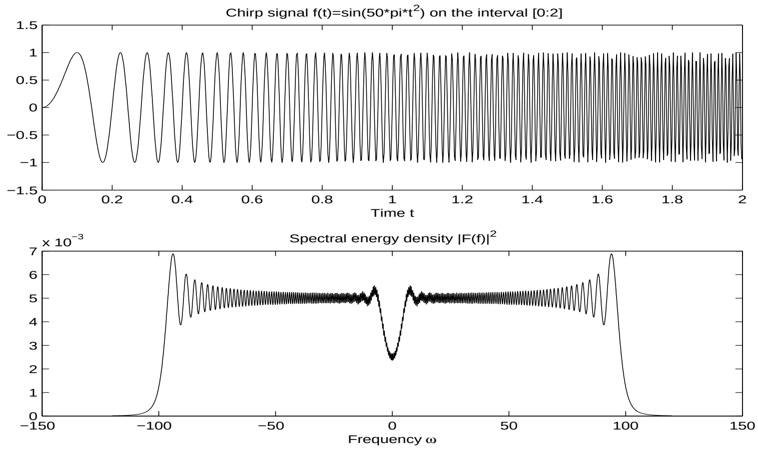
Concatenation of two Sinusoids

In our next example, the CT signal f is a 1 Hz test signal (pure sine) of 5 seconds length followed by a 5 Hz test signal (pure sine) of 5 seconds length. Outside the interval $[0, 10]$, f is identically equal to zero. The following figure shows the pair $(f, |\widehat{f}|^2)$. Note that the magnitude spectrum looks quite similar to the graph in the last example, although in the first example both sinoids are active simultaneously, while in our second example the sinoids sound one after the other. Thus with the help of the Fourier transform it is hard to figure out when which pitch is played.



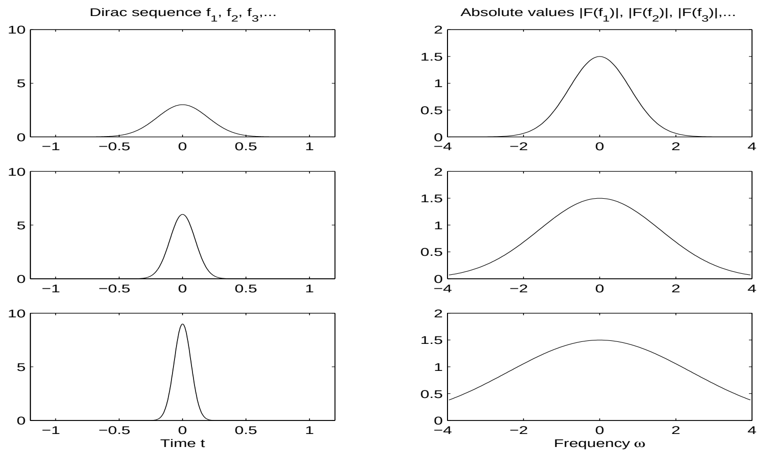
Spectral Energy Density of a Chirp

Let f denote a chirp signal restricted to the interval $[0, 2]$: $f(t) := \sin(50 \cdot \pi t^2)$. Roughly speaking, its instantaneous frequency ω_0 at time $t = t_0$ is given by the derivative of its phase divided by 2π, i.e., $\omega_0 = 50 \cdot t_0$. As f is identically equal to zero outside $[0, 2]$, we have a frequency band $[-100, 100]$ modulo ripples.



The Spectrum of Dirac Sequences

A **Dirac sequence** is a sequence of functions $(f_n)_{n\in\mathbb{N}}$ of 2-norm equal to 1, i.e., $\|f_n\| = 1$, such that the functions f_n for growing n are more and more concentrated around 0. The limit of such sequences is the so-called **Dirac δ distribution**. The following figure indicates that the δ distribution is a superposition of all frequencies.

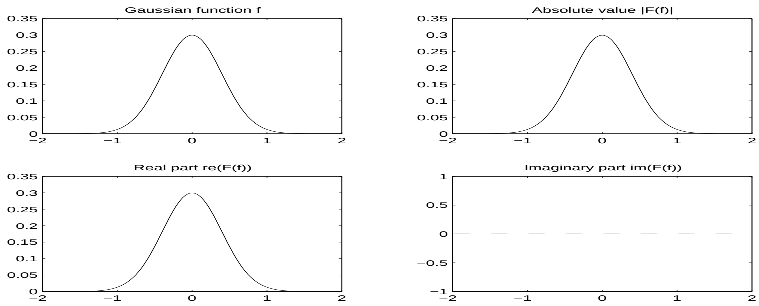


The Spectrum of the Gauß Function

The **Gauß function**, defined by

$$f(t) := \frac{1}{\sqrt{2\pi}} e^{-t^2/2},$$

is the probability density function of the standard normal distribution. The Gauß function has the remarkable property that it coincides with its Fourier transform. It is optimal in the sense of the Heisenberg uncertainty principle, to be discussed below.



Box and Sinc Function (1)

Theorem. The Fourier transform of the normalized box function is the sinc function.

Proof. Let f denote the normalized box function. Thus f is identically equal to 1 on the interval $[-\frac{1}{2}, \frac{1}{2}]$ and identically equal to 0 outside this interval. For $\omega \neq 0$ we get

$$\begin{aligned} \widehat{f}(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-2\pi i\omega t} dt \quad = \quad \int_{-1/2}^{1/2} 1 \cdot e^{-2\pi i\omega t} dt \\ &= \left[\frac{1}{-2\pi i\omega} e^{-2\pi i\omega t}\right]_{-1/2}^{1/2} \quad = \quad \frac{1}{-2\pi i\omega}\left(e^{-\pi i\omega} - e^{\pi i\omega}\right) \\ &= \frac{\sin(\pi\omega)}{\pi\omega} \;= \mathrm{sinc}\,(\omega). \end{aligned}$$

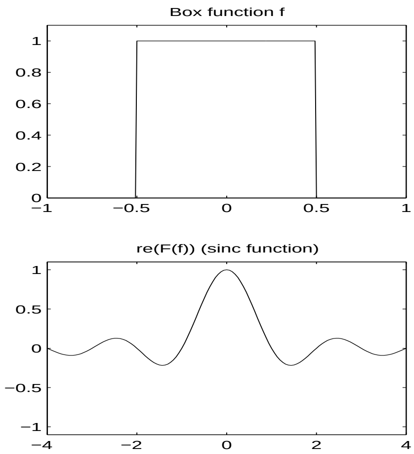
For $\omega = 0$ we obtain

$$\widehat{f}(0) = \int_{-\infty}^{\infty} f(t) dt \;= \int_{-1/2}^{1/2} 1\; dt \;= 1 \;= \mathrm{sinc}\,(0).$$

This proves the theorem. $\square$

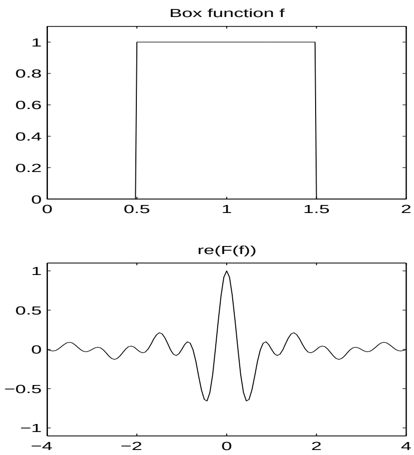
Box and Sinc Function (2)

The sinc function as the Fourier transform of the normalized box function is purely real-valued. This is illustrated by the following figure.



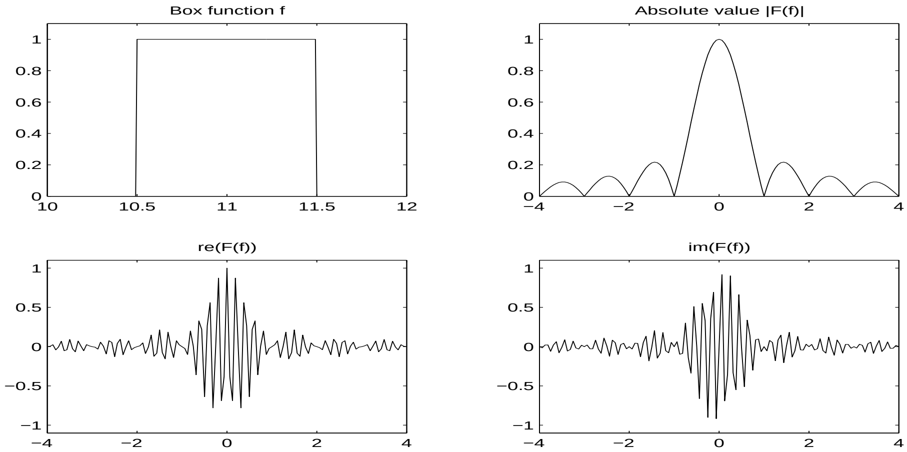
Translation in Time Induces Spectral Modulation (1)

We illustrate the fact that translation in time induces a spectral modulation. The following figure shows the 1-translated box function and its Fourier transform.



Translation in Time Induces Spectral Modulation (2)

Again, we illustrate the fact that translation in time induces a spectral modulation. The following figure shows the 11-translated box function and its Fourier transform.



Fourier Transform of DT Signals: Preliminaries

We turn to the Fourier transform and the Fourier representation of DT signals. Recall that the Fourier transform assigns to every periodic function $f \in L^2([0,1])$ its sequence of Fourier coefficients

$$\widehat{f} := (\langle f|\mathbf{e}_k\rangle)_{k\in\mathbb{Z}},$$

where $\mathbf{e}_k := ([0,1] \ni t \mapsto e^{2\pi ikt})$. This sequence is square-summable: $\widehat{f} \in \ell^2(\mathbb{Z})$. In addition, these Fourier coefficients allow the reconstruction of f (in the L^2 sense):

$$f = \sum_{k\in\mathbb{Z}} \langle f|\mathbf{e}_k\rangle \mathbf{e}_k.$$

In the following, we will go the reverse direction: we will start with a DT signal $x \in \ell^2(\mathbb{Z})$ and define its Fourier transform $\widehat{x} \in L^2([0,1])$. Through $\widehat{x}$ one can reconstruct x.

Fourier Transform of DT Signals

Definition. The **Fourier transform** $\widehat{x}$ of the DT signal $x \in \ell^2(\mathbb{Z})$ is defined by

$$\widehat{x}(\omega) := \sum_{k\in\mathbb{Z}} x(k)e^{-2\pi i\omega k}, \quad \text{for } \omega \in [0,1].$$

Theorem.

- $\ell^2(\mathbb{Z}) \ni x \mapsto \widehat{x} \in L^2([0,1])$ defines a unitary isomorphism.
- $x \in \ell^2(\mathbb{Z})$ has the **Fourier representation**:

$$x(n) = \int_0^1 \widehat{x}(\omega)e^{2\pi i\omega n}d\omega = \int_0^1 \widehat{x}(\omega)\mathbf{e}_\omega(n)d\omega,$$

where $\mathbf{e}_\omega(t) := e^{2\pi i\omega t}$.

Properties of DT Fourier Transform

Theorem. Let $x, y \in \ell^2(\mathbb{Z})$, $\lambda \in \mathbb{C}$, $k \in \mathbb{Z}$, $\omega_0 \in [0,1]$. Then:

(1) Linearity: $\widehat{x+y} = \widehat{x} + \widehat{y}$ and $\widehat{\lambda x} = \lambda\widehat{x}$.
(2) Translation in time induces a spectral modulation:
$\widehat{x_k}(\omega) = e^{-2\pi i\omega k}\widehat{x}(\omega)$, where $x_k(n) := x(n-k)$.
(3) A modulation in time induces a frequency shift:
$\widehat{x^{\omega_0}}(\omega) = \widehat{x}(\omega+\omega_0)$, where $x^{\omega_0}(n) := e^{-2\pi i\omega_0 n}x(n)$.
(4) Complex conjugation in time induces frequency reversal and complex conjugation in the frequency domain: $y = \overline{x} \Rightarrow \widehat{y}(\omega) = \overline{\widehat{x}(-\omega)}$,
important special case: $x = \overline{x}$, i.e. x is real-valued.
(5) Time reversal induces frequency reversal:
$\forall n \in \mathbb{Z}$: $y(n) := x(-n) \Rightarrow \widehat{y}(\omega) = \widehat{x}(-\omega)$.

Proof. The proofs are highly recommended as exercises. □

Fourier Transform: DT versus CT

Let $f \in L^2(\mathbb{R})$ be a piecewise continuous CT signal and $x \in \ell^2(\mathbb{Z})$ the 1-sampled version of f, i.e., $x(k) := f(k)$, $k \in \mathbb{Z}$. By definition,

$$\widehat{f}(\omega) := \int_{-\infty}^{\infty} f(t)e^{-2\pi i\omega t}dt$$

and

$$\widehat{x}(\omega) := \sum_{k=-\infty}^{\infty} f(k)e^{-2\pi i\omega k}.$$

Hence, for every $\omega \in \mathbb{R}$, $\widehat{x}(\omega)$ is a Riemann sum for $\widehat{f}(\omega)$.

Note that for growing $|\omega|$, the functions $t \mapsto f(t)e^{-2\pi i\omega t}, t \in \mathbb{R}$, oscillate more and more. On the other hand, the sampled versions $k \mapsto f(k)e^{-2\pi i\omega k}$, $k \in \mathbb{Z}$, are not able to detect oscillations of frequencies greater than 1. The reason is that such oscillations take place between two adjacent sampling points. This effect is known as **aliasing**. We will study this effect below in more detail.

Discrete Fourier Transform: Preliminary Remarks

The computation of the Fourier transform of a CT signal f or the computation of the Fourier coefficients of a periodic CT signal g requires for every (!) $\omega \in \mathbb{R}$ and for every (!) $k \in \mathbb{Z}$, respectively, the evaluation of integrals:

$$\widehat{f}(\omega) = \int_{\mathbb{R}} f(t)e^{-2\pi i\omega t}dt \quad \text{resp.} \quad \langle g|\mathbf{e}_k\rangle = \int_0^1 g(t)e^{-2\pi ikt}dt.$$

This is computationally impossible. The approximative computation of such signals via Riemann sums can be computationally impossible, as well.

It is therefore imperative, to design fast algorithms for the computation of reasonable approximations of Fourier coefficients of significant frequencies, possibly at the expense of precision. In practice one often has to deal with finite DT signals. Now, we will look at this case and we will also discuss the connection with the Fourier transforms of (periodic) CT signals.

Discrete Fourier Transform (1)

Theorem. Let $N \in \mathbb{N}$ and let $\Omega_N := e^{2\pi i/N}$ be a primitive N-th root of unity. For $k \in [0 : N-1]$ let

$$f_k := \frac{1}{\sqrt{N}}(\Omega_N^{0\cdot k}, \Omega_N^{1\cdot k}, \ldots, \Omega_N^{(N-1)\cdot k})^\top.$$

Then the following holds.

- The vectors $f_0, \ldots, f_{N-1}$ form an ON-basis ($\equiv$ Hilbert basis) of $\mathbb{C}^N$.
- If $v = (v_0, \ldots, v_{N-1})^\top$ is a finite DT signal, then the N Fourier coefficients $\widehat{v}_k := \langle v | f_k \rangle$, $k \in [0 : N-1]$, can be obtained simultaneously by matrix-vector multiplication: $\widehat{v} = \mathrm{DFT}_N \cdot v$, where

$$\mathrm{DFT}_N := \frac{1}{\sqrt{N}}\left(\Omega_N^{-k\cdot j}\right)_{0\le k,j<N}$$

  is the $N \times N$ **DFT matrix**.

Discrete Fourier Transform (2)

Proof of the Theorem. First we prove that $f_0, \ldots, f_{N-1}$ form an ON-basis of $\mathbb{C}^N$:

$$\|f_k\| = \sum_{j=0}^{N-1} \frac{1}{\sqrt{N}}\Omega_N^{j\cdot k} \cdot \overline{\frac{1}{\sqrt{N}}\Omega_N^{j\cdot k}} = \frac{1}{N}\sum_{j=0}^{N-1} \Omega_N^{j\cdot k} \cdot \Omega_N^{-j\cdot k} = 1.$$

Hence all f_k are of length 1. For $0 \le k < j < N$, the orthogonality of f_k and f_j results from the formula

$$1 + q + q^2 + \ldots + q^{N-1} = \frac{1-q^N}{1-q},$$

which is valid for all $q \in \mathbb{C} \setminus \{1\}$. By specializing this formula to $q = \Omega_N^{k-j} \neq 1$ and using $q^N = 1$ for such a q, the orthogonality of f_k and f_j follows.

Furthermore,

$$\widehat{v}_k = \langle v | f_k \rangle = \sum_{j=0}^{N-1} v_j \overline{\frac{1}{\sqrt{N}}\Omega_N^{j\cdot k}} = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} \Omega_N^{-j\cdot k} v_j.$$

The remaining claims follow easily. □