Algorithmic Game Theory, Summer 2025		Lecture 25 (5 pages)
	Cost Sharing	
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Today, we will consider a problem that could also be called "fair division" but usually goes by *cost sharing*. Suppose you and your friends want to take the train. You could all buy separate tickets but, as it turns out, a group ticket is cheaper. How would you share the cost? Indeed, the individual tickets could have different prices because some people in your group have discount cards, don't go the full distance, and so on. Clearly, they do not want to contribute more to the group ticket than their single ticket.

1 Model

We assume that there are n agents N. Furthermore, there is a cost function $c\colon 2^N \to \mathbb{R}$. Given a subset of the agent $S \subseteq N$, the value $c(S)$ states the cost that the set of agents S (coalition) would have on their own. We assume $c(\emptyset) = 0$. In principle, costs can be negative.

Example 25.1. *In the example for train tickets, we might have five agents. A single ticket costs 2 Euros, whereas a group ticket for all five agents costs 5 Euros. Then,* $c(S) = \min\{2|S|, 5\}$.

Our goal is to come up with a cost-sharing vector $\psi(c) = (\psi_1(c), \ldots, \psi_n(c))$ that splits up the cost. So, $\psi_i(c) \in \mathbb{R}$ and $\sum_{i \in N} \psi_i(c) = c(N)$.

Clearly, there are different ways to split up the cost. In the following, we will get to know different such solution concepts, which are each defined by a number of constraints that the vector has to fulfill.

2 Core

Given a cost function c, a cost-sharing vector $\psi(c)$ is in the *core* of c if it fulfills the following requirements.

- Efficiency: $\sum_{i \in N} \psi_i(c) = c(N)$. That is, the individual costs of all agents exactly recover the social cost.
- Stability: $\sum_{i \in S} \psi_i(c) \leq c(S)$ for all $S \subseteq N$. That is, no coalition S pays more than what they would pay if they were on their own.

Example 25.2. *Let us come back to our ticket example. We might choose* $\psi_i(c) = 1$ *for each agent. This is clearly a core solution. Another core solution would be* $\psi_1(c) = \psi_2(c) = 2$, $\psi_3(c) = 1$, $\psi_4(c) = \psi_5(c) = 0$. *However,* $\psi_1(c) = 3$, $\psi_2(c) = 2$, $\psi_3(c) = \ldots = \psi_5(c) = 0$ *is not in the core because stability is violated for* $\{1\}$ *and* $\{1, 2\}$.

While these conditions are desirable generally, the core is sometimes empty, depending on the cost function. For example, consider $n = 3$, $c(S) = \left\lceil \frac{|S|}{2} \right\rceil$. In the language of tickets, this means that the only available tickets are for groups of up to two, which each cost 1 Euro.

Observe that now for $\psi(c)$ to be in the core, we would need

$$
\begin{aligned}
&\psi_1(c) + \psi_2(c) + \psi_3(c) = 2 \\
&\psi_1(c) + \psi_2(c) \leq 1 \\
&\psi_1(c) + \psi_3(c) \leq 1 \\
&\psi_2(c) + \psi_3(c) \leq 1
\end{aligned}
$$

The inequalities add up to $2\psi_1(c) + 2\psi_2(c) + 2\psi_3(c) \leq 3$, so there is no solution to this system of inequalities.

3 Shapley Value

The Shapley value is a different approach to cost sharing. Again, there are a couple of conditions that the vector $\psi(c)$ should fulfill, respectively different vectors for different cost functions.

- Symmetry: If for two agents $i \in N$ and $j \in N$ for all $S \subseteq N$ with $i, j \notin S$ we have $c(S \cup \{i\}) = c(S \cup \{j\}))$, then $\psi_i(c) = \psi_j(c)$. That is, if nothing changes if i and j swap their roles, then they have to get the same cost share.
- Dummy: If for an agent $i \in N$ for all $S \subseteq N$ we have $c(S \cup \{i\}) = c(S)$, then $\psi_i(c) = 0$.
- Efficiency: $\sum_{i=1}^n \psi_i(c) = c(N)$.
- Additivity: For any two cost functions c and c', let $c + c'$ be their pointwise sum. For all c and c' and all $i \in N$, we have $\psi_i(c + c') = \psi_i(c) + \psi_i(c')$.

We will show that there is a unique choice of cost-share vectors that fulfills these properties. Our first step is to define a vector that adds agents one after the other and measures the respective cost increase. We add the agents in an arbitrary but fixed order. To this end, we use permutations $\pi\colon N \to N$ of the agents and define

$$\varphi_i(c, \pi) = c(\{\pi(1), \ldots, \pi(k)\}) - c(\{\pi(1), \ldots, \pi(k-1)\}) \enspace , \text{ where } k = \pi^{-1}(i) \enspace .$$

Example 25.3. *We come back to our running example, in which a single ticket is 2 Euros and a group ticket is 5 Euros.*

For π with $\pi(i) = i$, we have $\varphi_1(c, \pi) = \varphi_2(c, \pi) = 2$, $\varphi_3(c, \pi) = 1$, $\varphi_4(c, \pi) = \varphi_5(c, \pi) = 0$. Clearly, this does not fulfill the symmetry condition yet.

To make the symmetry condition fulfilled, we take the *average over all permutations π*. To this end, let denote by S_n the set of all permutations $\pi\colon N \to N$. Recall that $|S_n| = n!$. Define agent i's Shapley value under cost function c as

$$\psi_i(c) = \frac{1}{n!} \sum_{\pi \in S_n} \varphi_i(c, \pi) \enspace . \tag{1}$$

Theorem 25.4. *The vector $\psi(c)$ defined by Equation* (1) *fulfills the symmetry, dummy, efficiency, and additivity condition.*

Proof. For the dummy, efficiency, and additivity condition, we can observe that for any permutation π, the vector $\varphi(c, \pi)$ already fulfills the respective property. This translates to the respective property for $\psi(c)$. For completeness, we provide the respective calculations below.

The more interesting question is the symmetry condition. We will use that for each permutation π, there is exactly one permutation $\pi^{(i,j)}$ that swaps i and j in the image of π. That is,

$$\pi^{(i,j)}(\ell) = \begin{cases} j & \text{if } \pi(\ell) = i \\ i & \text{if } \pi(\ell) = j \\ \pi(\ell) & \text{otherwise} \end{cases}$$

Now, let $i \neq j$ be chosen such that $c(S \cup \{i\}) = c(S \cup \{j\})$ for all S with $i, j \notin S$. Let $\pi(k) = i$ and $\pi(\ell) = j$. If $k < \ell$, let $S = \{\pi(1), \ldots, \pi(k-1)\} = \{\pi^{(i,j)}(1), \ldots, \pi^{(i,j)}(k-1)\}$. We have

$$\begin{aligned} \varphi_i(c, \pi) &= c(\{\pi(1), \ldots, \pi(k)\}) - c(\{\pi(1), \ldots, \pi(k-1)\}) \\ &= c(S \cup \{i\}) - c(S) \\ &= c(S \cup \{j\}) - c(S) \\ &= c(\{\pi^{(i,j)}(1), \ldots, \pi^{(i,j)}(k)\}) - c(\{\pi^{(i,j)}(1), \ldots, \pi^{(i,j)}(k-1)\}) \\ &= \varphi_j(c, \pi^{(i,j)}) \enspace . \end{aligned}$$

If $k > \ell$, let $S = \{\pi(1), \ldots, \pi(k-1)\} \setminus \{j\} = \{\pi^{(i,j)}(1), \ldots, \pi^{(i,j)}(k-1)\} \setminus \{i\}$. We have

$$\begin{aligned}\varphi_i(c,\pi) &= c(\{\pi(1), \ldots, \pi(k)\}) - c(\{\pi(1), \ldots, \pi(k-1)\}) \\ &= c(S \cup \{i,j\}) - c(S \cup \{j\}) \\ &= c(S \cup \{i,j\}) - c(S \cup \{i\}) \\ &= c(\{\pi^{(i,j)}(1), \ldots, \pi^{(i,j)}(k)\}) - c(\{\pi^{(i,j)}(1), \ldots, \pi^{(i,j)}(k-1)\}) \\ &= \varphi_j(c, \pi^{(i,j)}) \ .\end{aligned}$$

So, always

$$\varphi_i(c,\pi) = \varphi_j(c, \pi^{(i,j)}) \ .$$

As every permutation appears exactly once as $\pi^{(i,j)}$ in S_n, we have

$$\psi_j(c) = \frac{1}{n!} \sum_{\pi \in S_n} \varphi_j(c, \pi^{(i,j)}) = \frac{1}{n!} \sum_{\pi \in S_n} \varphi_i(c,\pi) = \psi_i(c) \ .$$

In more detail, for the dummy condition, we observe

$$\varphi_i(c,\pi) = c(\{\pi(1), \ldots, \pi(k)\}) - c(\{\pi(1), \ldots, \pi(k-1)\}) = 0$$

and therefore

$$\psi_i(c) = \frac{1}{n!} \sum_{\pi \in S_n} \varphi_i(c,\pi) = 0 \ .$$

For the efficiency condition, we have

$$\sum_{i=1}^n \varphi_i(c,\pi) = \sum_{k=1}^n c(\{\pi(1), \ldots, \pi(k)\}) - \sum_{k=1}^n c(\{\pi(1), \ldots, \pi(k-1)\}) = c(N) - c(\emptyset) = c(N)$$

by a telescoping sum and therefore

$$\sum_{i=1}^n \psi_i(c) = \frac{1}{n!} \sum_{\pi \in S_n} \sum_{i=1}^n \varphi_i(c,\pi) = \frac{1}{n!} \sum_{\pi \in S_n} c(N) = c(N) \ .$$

For the additivity condition, we have

$$\begin{aligned}\varphi_i(c+c',\pi) &= c(\{\pi(1), \ldots, \pi(k)\}) - c(\{\pi(1), \ldots, \pi(k-1)\}) \\ &\quad + c'(\{\pi(1), \ldots, \pi(k)\}) - c'(\{\pi(1), \ldots, \pi(k-1)\}) \\ &= \varphi_i(c,\pi) + \varphi_i(c',\pi) \ .\end{aligned}$$

This gives

$$\psi_i(c+c') = \frac{1}{n!} \sum_{\pi \in S_n} \varphi_i(c+c',\pi) = \frac{1}{n!} \sum_{\pi \in S_n} \varphi_i(c,\pi) + \frac{1}{n!} \sum_{\pi \in S_n} \varphi_i(c',\pi) = \psi_i(c) + \psi_i(c') \ . \quad \square$$

4 Uniqueness of the Shapley Value

Interestingly, this is the only choice of cost vectors that fulfills these conditions. So, our conditions fully characterize the way of sharing the cost. Recall that this is different in the definition of the core, where there might be multiple solutions (or none).

Theorem 25.5. *The vectors $\psi(c)$ defined by Equation (1) are the only ones that fulfill the symmetry, dummy, efficiency, and additivity condition.*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ ? & 1 & 0 & 0 & 0 \\ ? & ? & 1 & 0 & 0 \\ ? & ? & ? & 1 & 0 \\ ? & ? & ? & ? & 1 \end{pmatrix}$$

Figure 1: Structure of constraint matrices in proof of Lemma 25.7.

Lemma 25.6. *For $T \subseteq N$, let c_T be the cost function such that $c_T(S) = 1$ if $S \supseteq T$, 0 otherwise. That is, for $\alpha \in \mathbb{R}$, $\alpha \cdot c_T$ is the function that has value α if all of T appear in S, otherwise it is 0. Then, for any fixed α and T, the unique $\psi(\alpha c_T)$ that fulfills the symmetry, dummy, and efficiency condition is*

$$\psi_i(\alpha \cdot c_T) = \begin{cases} \frac{\alpha}{|T|} & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases} .$$

Proof. It is easy to see that the ψ function defined this way fulfills the mentioned conditions. So, we only have to show it is the only one.

For $i \notin T$, we have to have $\psi_i(\alpha c_T) = 0$ by the dummy condition.

So, using the efficiency condition, we have to have $\sum_{i \in T} \psi_i(\alpha c_T) = \alpha$.

Finally, by the symmetry condition, we have to have $\psi_i(\alpha c_T) = \psi_j(\alpha c_T)$ for all $i, j \in T$. Therefore, $\psi_i(\alpha c_T) = \frac{\alpha}{|T|}$ for all $i \in T$. □

Lemma 25.7. *For every cost function c, there is a unique choice of values $(\alpha_T)_{T \subseteq N, T \neq \emptyset}$, $\alpha_T \in \mathbb{R}$ for all T such that $c = \sum_{T \subseteq N, T \neq \emptyset} \alpha_T c_T$.*

Proof. We have to choose $(\alpha_T)_{T \subseteq N, T \neq \emptyset}$ such that for all $S \subseteq N$, we have $c(S) = \sum_{T \subseteq N, T \neq \emptyset} \alpha_T c_T(S)$. Recall that $c_T(S) = 1$ if and only if $S \supseteq T$. So, we can rewrite the right-hand side as $\sum_{T \subseteq N, T \neq \emptyset} \alpha_T c_T(S) = \sum_{T \subseteq S, T \neq \emptyset} \alpha_T$.

That is, we get system of linear equations stating that

$$\sum_{T \subseteq S, T \neq \emptyset} \alpha_T = c(S) \quad \text{for all } S \subseteq N,\ S \neq \emptyset .$$

Note that we have $2^n - 1$ variables and just as many constraints. So, it suffices to show that the system always has a unique solution, which proves the lemma.

To see this most clearly, sort the variables and constraints by increasing $|T|$ or $|S|$ respectively (breaking ties arbitrarily but the same way in both dimensions). On the diagonal of the constraint matrix, every entry is 1 because these are the entries for which $S = T$. Above the diagonal, the set T has either smaller size than S, so the entry is 0, or they have equal size but are distinct, in which case the entry is 0 again. That is, we have a matrix with 1 on the diagonal, 0 above it, and arbitrary entries below it (see Figure 1). Following the Gaussian algorithm, we see that there is always a unique solution. □

Now we are ready to prove Theorem 25.5 by decomposing the function c to a sum of $\alpha_T c_T$ and using the additivity condition.

Proof of Theorem 25.5. Note that we only have to show that the solution is unique. To this end, we use the unique decomposition $c = \sum_{T \subseteq N, T \neq \emptyset} \alpha_T c_T$ according to Lemma 25.7.

By additivity, we have to have

$$\psi_i(c) = \psi_i\left(\sum_{T \subseteq N, T \neq \emptyset} \alpha_T c_T\right) = \sum_{T \subseteq N, T \neq \emptyset} \psi_i(\alpha_T c_T) .$$

We have $\psi_i(\alpha_T c_T) = \frac{\alpha_T}{|T|}$ if $i \in T$ and 0 otherwise by Lemma 25.6. That is,

$$\psi_i(c) = \sum_{T \subseteq N, T \neq \emptyset} \psi_i(\alpha_T c_T) = \sum_{T \subseteq N, i \in T} \frac{\alpha_T}{|T|} \ .$$

As the vector $(\alpha_T)_{T \subseteq N, T \neq \emptyset}$ is unique by Lemma 25.7, this is the unique solution. As the definition in Equation (1) also fulfills the same conditions, they have to coincide. □

5 Outlook

The game we considered today is a *cooperative game with transferable utilities*. Usually, you will see it defined by values rather than costs but as they are arbitrary reals one translates to the other.

An interesting question is when a core solution exists. Under some conditions regarding the cost functions c, the Shapley value is in the core. Also, we did not discuss how to compute any of this. The way we defined the Shapley value, it takes $\Theta(n!)$ time to compute. By exploiting symmetries, it can be reduced to $\Theta(2^n)$.

Finally, one should remark that questions such as house allocation and stable matching are cooperative games with *non-transferable utilities*. The question is always to find a solution that is stable against possible coalitions.

6 Further Reading

- Chapter 12 in the Karlin/Peres book
- Chapter 15 in the AGT book