DCG, Wintersemester 2023/24 Lecture 1 (4 pages)

Basic notions and Radon's lemma

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1 Basic notions

We start by introducing some basic notions that help us to mathematically describe basic geometric objects, such as points, lines and planes. Let $\mathbb{R}^d$ denote the d-dimensional Euclidean space. A point is a d-tuple of real numbers $x = (x_1, \dots, x_d)$.

Definition 1.1 (Linear hull). *For points $p_1, \dots, p_n \in \mathbb{R}^d$ and values $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ the point*

$$\sum_{i=1}^{n} \alpha_i p_i$$

is a linear combination. The set of points $p_1, \dots, p_n$ is linearly dependent if there exists an index $i \in \{1, \dots, n\}$ such that p_i can be written as a linear combination of the other points in the set. We call the set linearly independent otherwise. The set of all linear combinations of a set of points

$$\operatorname{lin}(p_1, \dots, p_n) = \left\{ \sum_{i=1}^{n} \alpha_i p_i \;\middle|\; \alpha_i \in \mathbb{R} \right\}$$

is called the linear hull (also called the linear span).

Definition 1.2 (Affine hull). *For points $p_1, \dots, p_n \in \mathbb{R}^d$ and values $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ with $\sum_{i=1}^{n} \alpha_i = 1$ the point*

$$\sum_{i=1}^{n} \alpha_i p_i$$

is an affine combination. The set of points $p_1, \dots, p_n$ is affinely dependent if there exists an index $i \in \{1, \dots, n\}$ such that p_i can be written as an affine combination of the other points in the set. We call the set affinely independent otherwise. The set of all affine combinations of a set of points

$$\operatorname{aff}(p_1, \dots, p_n) = \left\{ \sum_{i=1}^{n} \alpha_i p_i \;\middle|\; \alpha_i \in \mathbb{R}, \sum_{i=1}^{n} \alpha_i = 1, \right\}$$

is called the affine hull.

An affine combination is a linear combination with the additional constraint that the sum of coefficients is 1. Using affine combinations we can describe lines, planes and hyperplanes.

Example 1.3. *The affine hull of two points $p_1, p_2 \in \mathbb{R}^2$ is*

$$\{\ \alpha_1 p_1 + \alpha_2 p_2 \ \mid\ \alpha_1, \alpha_2 \in \mathbb{R}, \alpha_1 + \alpha_2 = 1\ \}$$

which is equivalent to

$$\{\ t p_1 + (1-t) p_2 \ \mid\ t \in \mathbb{R}\ \}$$

This is the set of all points that lie on the line that contains p_1 and p_2.

Example 1.4. *The affine hull of three points $p_1, p_2, p_3 \in \mathbb{R}^2$, such that the three points do not lie on a common line, is equal to $\mathbb{R}^2$.*

Lemma 1.5. *Any set of $d+2$ points in $\mathbb{R}^d$ is affinely dependent.*

Proof. We derive the statement from the fact that $d+1$ points in $\mathbb{R}^d$ are linearly dependent.

Let $A = \{p_1, \ldots, p_{d+2}\}$. Consider an arbitrary index $i \in \{1, \ldots, d+2\}$ fixed and consider the $d+1$ points $q_j = p_j - p_i$ for $i \neq j$. Since $d+1$ points are linearly dependent, we can rewrite one of them as a linear combination of the others. That is, there exist values β_j and an index $r \in \{1, \ldots, d+2\}$ with $r \neq i$, such that

$$q_r = \sum_{\substack{j \in \{1,\ldots,d+2\} \\ j \neq i \text{ and } j \neq r}} \beta_j q_j .$$

Therefore

$$p_r = q_r + p_i = \left(\sum_{j \neq i \text{ and } j \neq r} \beta_j q_j \right) + p_i = \sum_{j \neq i \text{ and } j \neq r} \beta_j p_j - \left(\sum_{i \neq j \text{ and } j \neq r} \beta_j \right) p_i + p_i$$

Now we define $\beta_i = -\left(\sum_{j \neq i \text{ and } j \neq r} \beta_j\right) + 1$ and we have $\sum_{j \neq r} \beta_j = 1$ which satisfies the conditions of an affine combination. □

Definition 1.6 (Convex hull)**.** *For points $p_1, \ldots, p_n \in \mathbb{R}^d$ and values $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ with $\sum_{i=1}^n \alpha_i = 1$ and $\alpha_i \geq 0$ for all $1 \leq i \leq n$ the point*

$$\sum_{i=1}^{n} \alpha_i p_i$$

is a convex combination. The set of points $p_1, \ldots, p_n$ is convexly dependent if there exists an index $i \in \{1, \ldots, n\}$ such that p_i can be written as a convex combination of the other points in the set. Otherwise we say points lie in convex position. The set of all convex combinations of a set of points

$$\operatorname{conv}(p_1, \ldots, p_n) = \left\{ \sum_{i=1}^{n} \alpha_i p_i \;\middle|\; \alpha_i \in \mathbb{R}, \sum_{i=1}^{n} \alpha_i = 1, \alpha_i \geq 0 \right\}$$

is called the convex hull.

Example 1.7. *The convex hull of two points $p_1, p_2 \in \mathbb{R}^2$ is the set*

$$\{\, tp_1 + (1-t)p_2 \mid t \in [0,1] \,\}.$$

This is the line segment with endpoints p_1 and p_2.

Example 1.8. *The convex hull of three points $p_1, p_2, p_3 \in \mathbb{R}^2$ is the set*

$$\{\, \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 \mid \alpha_1, \alpha_2, \alpha_3 \geq 0 \textit{ and } \alpha_1 + \alpha_2 + \alpha_3 = 1 \,\}.$$

This is the triangle with corners p_1, p_2, and p_3.

2 Radon's lemma

Lemma 1.9 (Radon's lemma)**.** *For any set A of $d+2$ points in $\mathbb{R}^d$ there are subsets $A_1, A_2 \subseteq A$ with $A_1 \cap A_2 = \emptyset$ and a point q, such that $q \in \operatorname{conv}(A_1) \cap \operatorname{conv}(A_2)$. We call such a point q a Radon point.*

Proof. Let $A = \{p_1, \ldots, p_{d+2}\}$. Since we have $d+2$ points, these are affinely dependent. Thus, there exists a $p_i \in A$ which can be expressed as an affine combination of the other points. That is, there exist values α_j for $1 \leq j \leq n$ with $i \neq j$, such that

$$p_i = \sum_{i \neq j} \alpha_j p_j \quad \text{with} \quad \sum_{i \neq j} \alpha_j = 1$$

Now let $\alpha_i = -1$. We can add $\alpha_i p_i$ to both sides of the equality and obtain

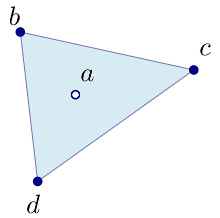
$$0 = \sum_{j=1}^{d+2} \alpha_j p_j \quad \text{mit} \quad \sum_{j=1}^{d+2} \alpha_j = 0.$$

We now define two index sets $I_1 = \{\ i \ \mid\ \alpha_i > 0\ \}$ and $I_2 = \{\ i \ \mid\ \alpha_i < 0\ \}$. We have

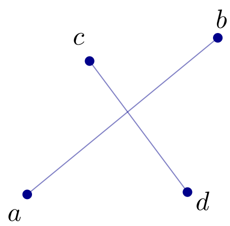
$$-\sum_{i \in I_2} \alpha_i p_i = \sum_{i \in I_1} \alpha_i p_i \quad \text{and} \quad -\sum_{i \in I_2} \alpha_i = \sum_{i \in I_1} \alpha_i$$

Now let $\gamma = \sum_{i \in I_1} \alpha_i$, and define $q_1 = \sum_{i \in I_1} \beta_i p_i$ with $\beta_i = \frac{\alpha_i}{\gamma}$. This is a convex combination of the points of A which have index in I_1. Similarly we can define $q_2 = \sum_{i \in I_2} \beta_i p_i$ with $\beta_i = -\frac{\alpha_i}{\gamma}$. This is a convex combination of the points in A which have index in I_2. At the same time we have $q_1 = q_2$ and $I_1 \cap I_2 = \emptyset$. This implies the lemma. $\square$

Example 1.10 (Radon point)**.** *For 4 distinct points a, b, c, d in the plane there are essentially two possibilities for the two subsets in the above lemma.*



(a) $A_1 = \{a\}, A_2 = \{b, c, d\}$.
A point a is contained in the convex hull of the points $\{b, c, d\}$. In this example a a Radon point.



(b) $A_1 = \{a, b\}, A_2 = \{c, d\}$.
Two line segments $\overline{ab}$ and $\overline{cd}$ intersect in one point. The point of intersection is a Radon point.

3 More affine notions

Definition 1.11 (Linear subspace)**.** *A set $A \subseteq \mathbb{R}^d$ is a linear subspace of $\mathbb{R}^d$ if and only if*

(i) $\forall x, y \in A : x + y \in A$

(ii) $\forall \alpha \in \mathbb{R}, x \in A : \alpha x \in A$

Definition 1.12 (Affine subspace)**.** *Let L be a linear subspace of $\mathbb{R}^d$ and let $x \in \mathbb{R}^d$. Then,*

$$A = \{\ x + y \ \mid \ y \in L\ \}$$

is an affine subspace of $\mathbb{R}^d$.

We can interpret these sets geometrically. Any line that passes through the origin is a linear subspace. Any line (not necessarily passing through the origin) is an affine subspace.

Definition 1.13 (Affine mapping)**.** *An affine mapping has the form $f : \mathbb{R}^k \to \mathbb{R}^d$ with $f : y \mapsto By + c$, where B is a $d \times k$ matrix and $c \in \mathbb{R}^d$. It is a composition of a linear map and a translation.*

Definition 1.14 (Hyperplane)**.** *Let $a \in \mathbb{R}^d \setminus \{0\}$ and $b \in \mathbb{R}$. The set*

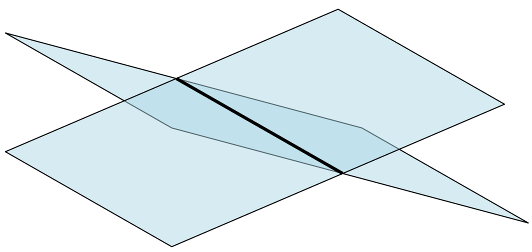
$$\left\{\ x \in \mathbb{R}^d \ \middle|\ \langle a, x\rangle = b\ \right\}$$

is a hyperplane. It is a $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$. Using the definition of the inner product

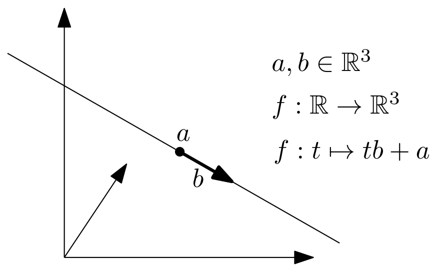
$$\langle a, x\rangle = a_1x_1 + a_2x_2 + \cdots + a_dx_d$$

we can see that a hyperplane is the set of solutions to a linear equality. A hyperplane can also be given as the image of an affine mapping.

Definition 1.15 (k-flat)**.** *A k-flat is a k-dimensional affine subspace of $\mathbb{R}^d$. It can be given either as an intersection of hyperplanes, or as the image of an affine mapping.*



(a) k-flat given as intersection of two hyperplanes for $k = 1$ and $d = 3$.



(b) k-flat given as the image of an affine mapping for $k = 1$ and $d = 3$.

Definition 1.16 (Halfspace)**.** *A halfspace is a set bounded by a hyperplane. Given $a \in \mathbb{R}^d \setminus \{0\}$ and $b \in \mathbb{R}$, the set*

$$\left\{\ x \in \mathbb{R}^d \ \middle|\ \langle a, x\rangle \geq b\ \right\}$$

is a halfspace bounded by the hyperplane given by a and b.

References

- Jiří Matoušek, Chapter 1, Lectures on Discrete Geometry, Springer Graduate Texts in Mathematics.