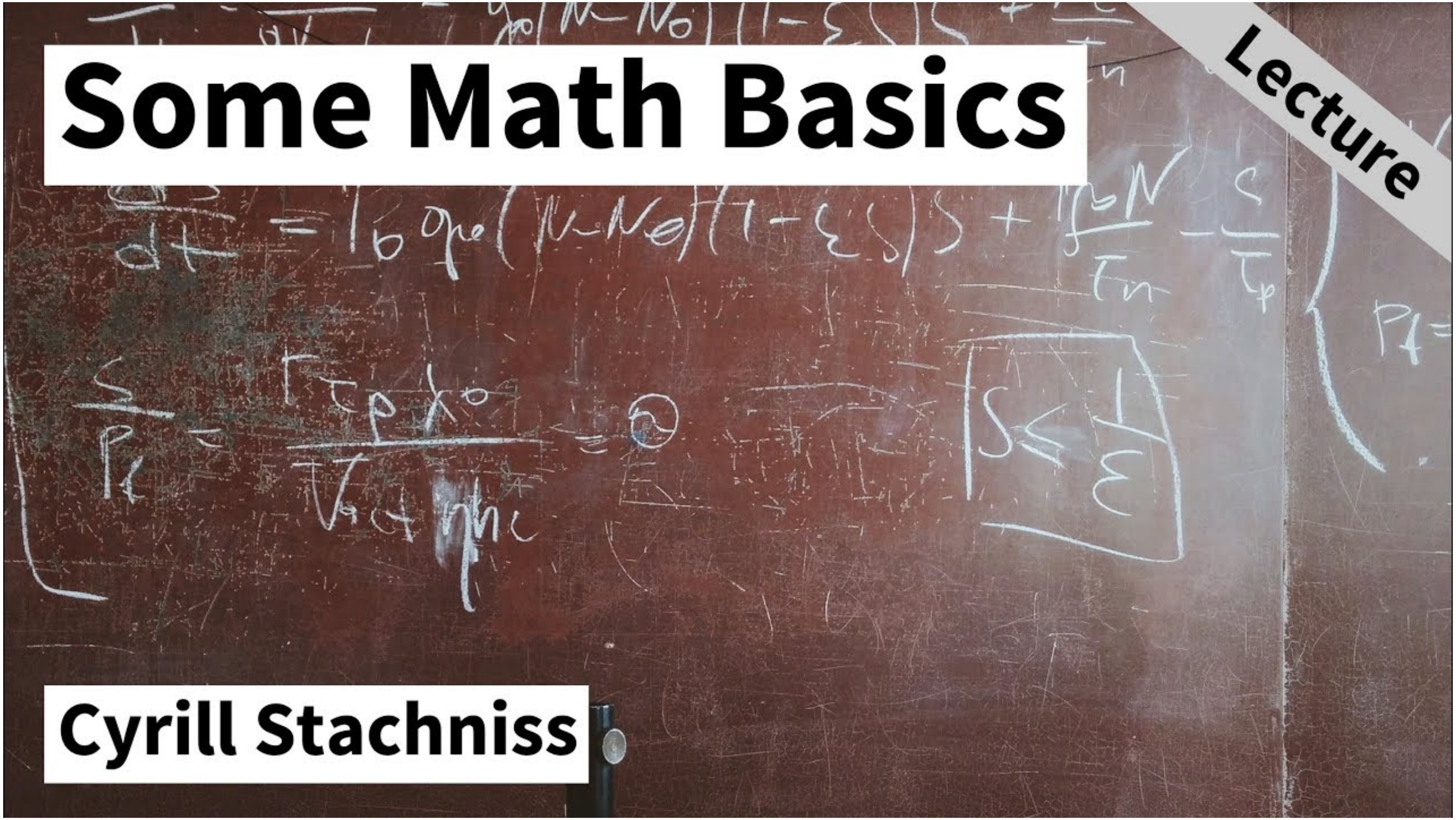


Summer term 2024 – Cyrill Stachniss

5 Minute Preparation for Today

https://www.youtube.com/watch?v=giOpcCPHitY

Photogrammetry & Robotics Lab

Some Math Basics

Cyrill Stachniss

The slides have been created by Cyrill Stachniss.

Motivation

- We use several concepts from math
- **Goal:** Provide a short reminder for few things that we will use on our way

Brief, informal, incomplete, and unordered set of explanations

Motivation

- We use several concepts from math
- **Goal:** Provide a short reminder
- **Topics**
  - Solving $Ax=b$
  - Solving $Ax=0$ using SVD
  - Least squares with Gauss Newton
  - Skew-symmetric matrix
  - Derivative of rotation matrices
  - Homogenous coordinates (own lecture)

System of Linear Equations

$$A\boldsymbol{x} = \boldsymbol{b}$$

Linear Equation System: Ax=b

Three cases:

- A is squared and has full rank
- A is **over**determined
- A is **under**determined

Solving Ax=b, w/ Exact Solution

- **A is a square matrix with full rank**
- Best-case situation, unique solution
- Can be solved in many ways...

Solving Ax=b, w/ Exact Solution

- **A is a square matrix with full rank**
- Best-case situation, unique solution
- Can be solved through
  - Gauss elimination
  - Inversion of A : $\boldsymbol{x} = A^{-1}\boldsymbol{b}$
  - Cholesky decomposition $\mathrm{chol}(A) = LL^{\mathsf{T}}$ with lower triangular matrix L and then solving $L\boldsymbol{y} = \boldsymbol{b}$ and $L^{\mathsf{T}}\boldsymbol{x} = \boldsymbol{y}$
  - QR decomposition
  - Conjugate gradients

Solving Ax=b, A overdetermined

- **Common real-world situation**
- No exact solution exists
- We aim at finding minimizing $\|A\boldsymbol{x} - \boldsymbol{b}\|$ instead of solving $A\boldsymbol{x} = \boldsymbol{b}$:

$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} \|A\boldsymbol{x} - \boldsymbol{b}\|$$

- Ordinary least squares approach
- Solution can be obtained through

$$\boldsymbol{x} = (A^\mathsf{T} A)^{-1} A^\mathsf{T} \boldsymbol{b}$$

Solving Ax=b, A underdetermined

- **Infinitively many solutions exist** (or no solution if inconsistent)
- Not enough information available
- Approach: Find $\boldsymbol{x}$ which solves $A\boldsymbol{x} = \boldsymbol{b}$ and minimizes $\|\boldsymbol{x}\|$
- Solution

$$\boldsymbol{x} = A^{\mathsf{T}}(AA^{\mathsf{T}})^{-1}\boldsymbol{b}$$

Homogenous System

$$A\boldsymbol{x} = \mathbf{0}$$

Homogenous System: Ax=0

- Find a solution $\mathbf{x} \neq \mathbf{0}$ fulfilling $A\mathbf{x} = \mathbf{0}$
- Means system is underdetermined
- There exists a null space of *A* called $\operatorname{null}(A)$ and all $\mathbf{x}$ fulfilling $A\mathbf{x} = \mathbf{0}$ are elements of it
- *A*'s rank deficiency defines the dimensionality of the null space

Eigenvalues

- For a squared matrix, we have

$$\dim(A) = \dim(\mathrm{null}(A)) + \mathrm{rank}(A)$$

- Which impact does this have on the Eigenvalues of A ?

Eigenvalues

- For a squared matrix, we have

$$\dim(A) = \dim(\mathrm{null}(A)) + \mathrm{rank}(A)$$

- Which impact does this have on the Eigenvalues of A ?
- There are $\mathrm{rank}(A)$ non-zero Eigenvalues
- There are $\dim(\mathrm{null}(A))$ Eigenvalues that are zero

Eigenvector

- For each Eigenvector $\boldsymbol{\nu}$ holds $A\boldsymbol{\nu} = \lambda\boldsymbol{\nu}$
- Thus, for those with Eigenvalue 0 we have $A\boldsymbol{\nu} = 0\boldsymbol{\nu} = 0$

Eigenvector

- For each Eigenvector $\boldsymbol{\nu}$ holds $A\boldsymbol{\nu} = \lambda\boldsymbol{\nu}$
- Thus, for those with Eigenvalue 0 we have $A\boldsymbol{\nu} = 0\boldsymbol{\nu} = 0$
- **Result:** all Eigenvectors corresponding to an Eigenvalue of 0 solve $A\boldsymbol{x} = \mathbf{0}$
- The same holds for all linear combinations of these Eigenvectors
- These Eigenvectors form $\mathrm{null}(A)$

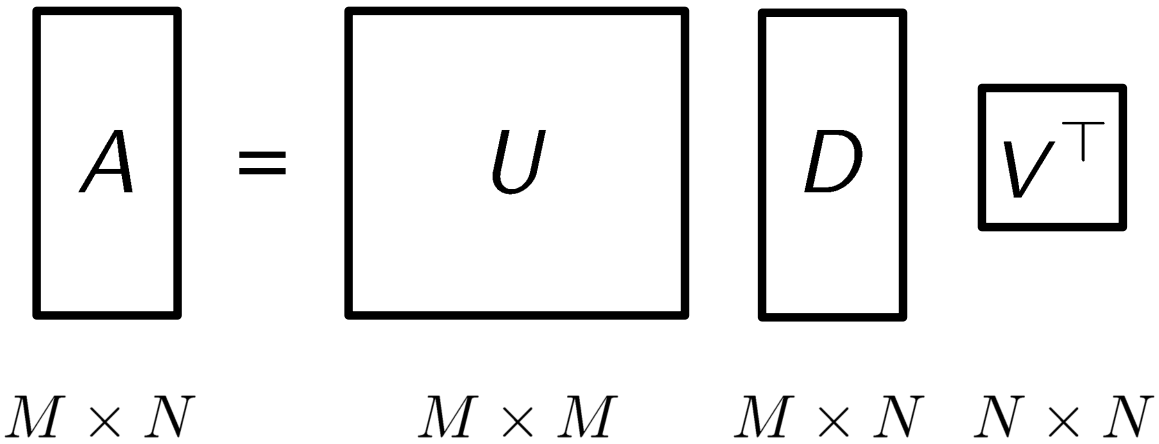
Eigenvector & Singular Vectors

- If A is square, real, symmetric and has non-negative Eigenvalues, then Eigenvalues equal to singular values
- Singular vectors and values also defined for non-square matrices
- We can use SVD to compute the singular values and vectors

Singular Value Decomposition

- SVD decomposes a matrix A into

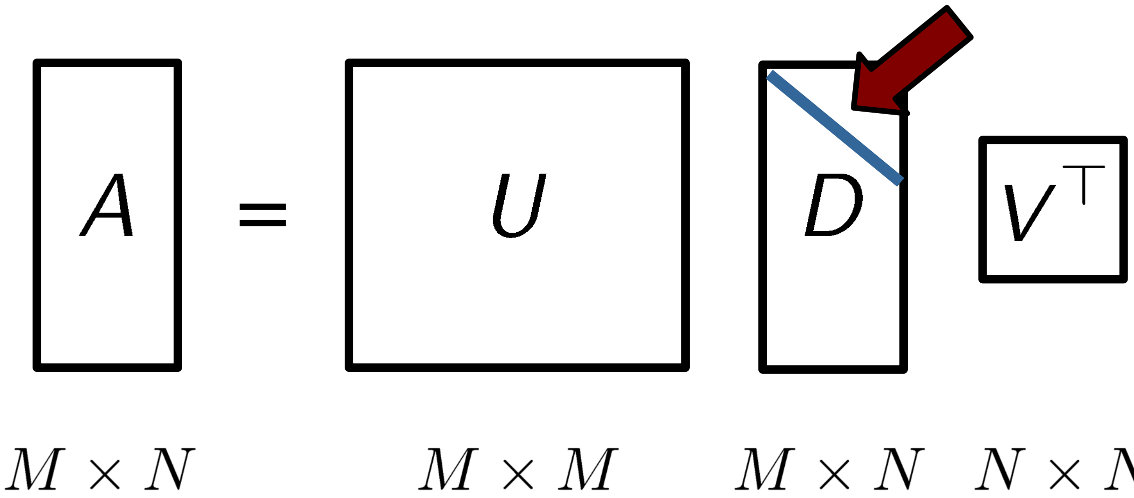
$$A = UDV^{\mathsf{T}}$$



Singular Values

- SVD decomposes a matrix A into

$$A = UDV^\top$$

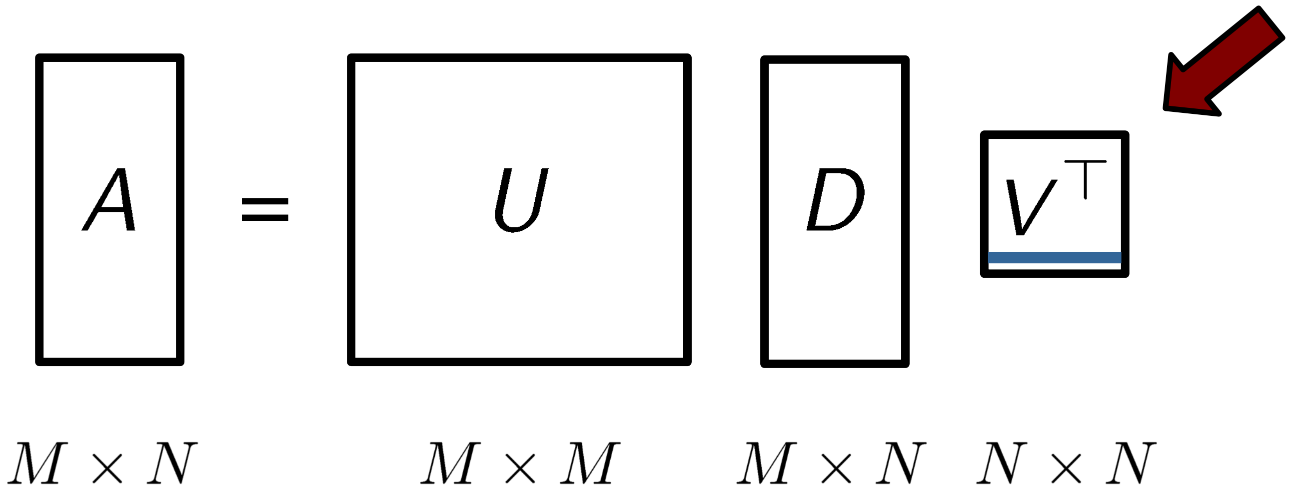


- D is a diagonal matrix of singular values sorted from large to small
- U, V are orthogonal matrices

Singular Vectors

- SVD decomposes a matrix A into

$$A = UDV^\top$$

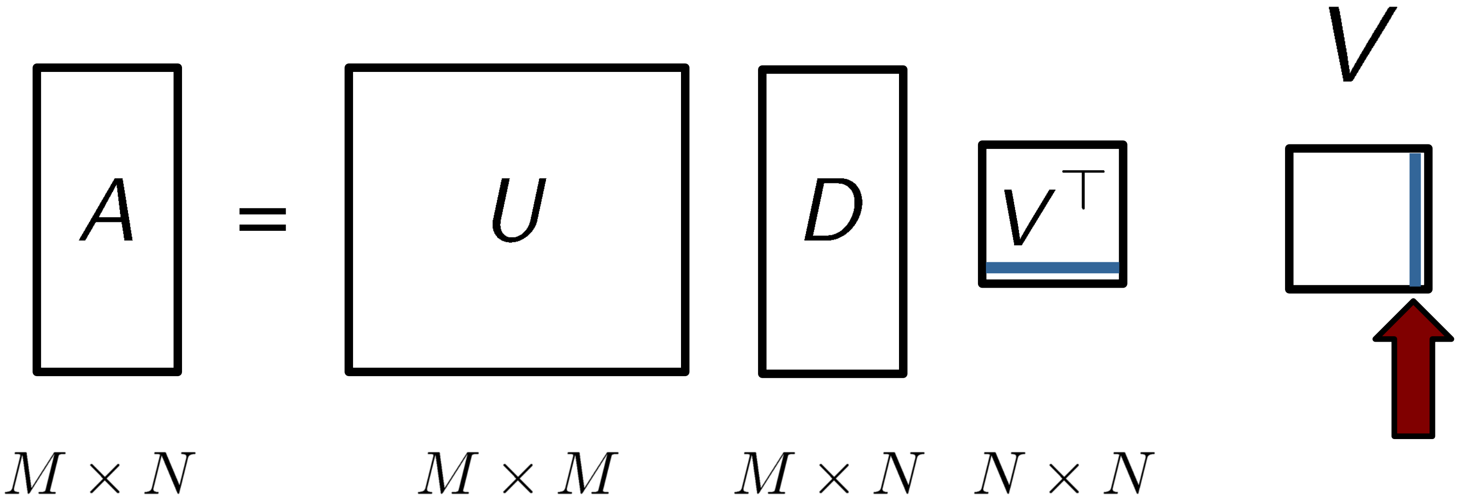


- $V^\top$ stores the corresponding singular vectors to the values

Singular Vectors

- SVD decomposes a matrix A into

$$A = UDV^\top$$



- Math libraries often returns V not $V^\top$
- The last column of V stores the vector corresponding to the smallest value

Solution to Ax=0 via SVD

- Decompose A using SVD: $A = UDV^{\mathsf{T}}$
- Check of the smallest singular value in D is zero: $D_{NN} \stackrel{?}{=} 0$
- **If so, the last column of V is a non-trivial solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \mathbf{0}$**

Solution to Ax=0 via SVD

- Decompose A using SVD: $A = UDV^{\mathsf{T}}$
- Check of the smallest singular value in D is zero: $D_{NN} \stackrel{?}{=} 0$
- If so, the last column of V is a non-trivial solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \mathbf{0}$
- **If not**, there is **no non-trivial solution** (i.e., only the trivial exists)
- **However,** the last column of V represents the vector that minimizes $\|A\boldsymbol{x}\|$ under the constraint $\|\boldsymbol{x}\| = 1$

Least Squares
(an non-Geodetic view)

Least Squares in 5 Minutes

https://www.youtube.com/watch?v=87S82fh4rI4

Graphical Explanation

$$\mathbf{x}$$

$$f_1(\mathbf{x}) = \hat{\mathbf{z}}_1 \qquad \mathbf{z}_1$$

$$f_2(\mathbf{x}) = \hat{\mathbf{z}}_2 \qquad \mathbf{z}_2$$

$$\ldots$$

$$f_n(\mathbf{x}) = \hat{\mathbf{z}}_n \qquad \mathbf{z}_n$$

state (unknown)	predicted measurements	real measurements

Error Function

- Error $\mathbf{e}_i$ is typically the **difference** between the **predicted and actual** measurement

$$\mathbf{e}_i(\mathbf{x}) = \mathbf{z}_i - f_i(\mathbf{x})$$

- We assume that the error has **zero mean** and is **normally distributed**
- Gaussian error with information matrix $\mathbf{\Lambda}_i$
- The squared error of a measurement depends only on the state and is a scalar

$$e_i(\mathbf{x}) = \mathbf{e}_i(\mathbf{x})^T \mathbf{\Lambda}_i \mathbf{e}_i(\mathbf{x})$$

Linearizing the Error Function

- Approximate the error functions around an initial guess $\mathbf{x}$ via Taylor expansion

$$\mathbf{e}_i(\mathbf{x} + \Delta\mathbf{x}) \simeq \underbrace{\mathbf{e}_i(\mathbf{x})}_{\mathbf{e}_i} + \mathbf{J}_i(\mathbf{x})\Delta\mathbf{x}$$

- $\mathbf{J}$ is the Jacobian

$$\mathbf{J}_f(x) = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \cdots & \frac{\partial f_2(x)}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_m(x)}{\partial x_1} & \frac{\partial f_m(x)}{\partial x_2} & \cdots & \frac{\partial f_m(x)}{\partial x_n} \end{pmatrix}$$

Gauss-Newton

Iterate the following steps:

- Linearize around $\mathbf{x}$ and compute for each measurement

$$\mathbf{e}_i(\mathbf{x} + \mathbf{\Delta x}) \simeq \mathbf{e}_i(\mathbf{x}) + \mathbf{J}_i \mathbf{\Delta x}$$

- Compute the terms for the linear system $\mathbf{b}^\top = \sum_i \mathbf{e}_i^\top \mathbf{\Lambda}_i \mathbf{J}_i \quad \mathbf{H} = \sum_i \mathbf{J}_i^\top \mathbf{\Lambda}_i \mathbf{J}_i$

- Solve the linear system

$$\mathbf{\Delta x}^* \;=\; -\mathbf{H}^{-1}\mathbf{b}$$

- Updating state $\mathbf{x} \leftarrow \mathbf{x} + \mathbf{\Delta x}^*$

Skew-Symmetric Matrices

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- A skew-symmetric matrix is a matrix S for which holds $S^\top = -S$

Skew-Symmetric Matrices

- A skew-symmetric matrix is a matrix S for which holds $S^\top = -S$
- S has zeros on the main diagonal
- $\forall S \in \mathbb{R}^{3\times 3} : \det(S) = 0$
- $\det(S) = 0 \quad \text{if } \dim(S) \text{ odd.}$

Skew-Symmetric Matrices in 3D

- In $\mathbb{R}^3$ we can express the cross product through a skew-symmetric matrix

$$\boldsymbol{a} \times \boldsymbol{b} = [\boldsymbol{a}]_\times \boldsymbol{b} = S_a \boldsymbol{b}$$

$$[\boldsymbol{a}]_\times = S_a = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

Skew-Symmetric Matrices in 3D

- In $\mathbb{R}^3$ we can express the cross product through a skew-symmetric matrix

$$\boldsymbol{a} \times \boldsymbol{b} = [\boldsymbol{a}]_\times \boldsymbol{b} = S_a \boldsymbol{b}$$

$$[\boldsymbol{a}]_\times = S_a = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}}_{\boldsymbol{a}} \times \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\boldsymbol{b}} = \begin{bmatrix} -a_3 b_2 + a_2 b_3 \\ a_3 b_1 - a_1 b_3 \\ -a_2 b_1 + a_1 b_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}}_{S_a} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\boldsymbol{b}}$$

Derivative of a Rotation Matrix

Derivative of a Rotation Matrix

- Skew-symmetric matrices are useful to formulate the derivative of a rotation matrix
- For any rotation matrix R holds

$$RR^\top = I$$

- Consider a rotation by θ around x-axis

$$R_x(\theta)$$

- Then, we have $R_x(\theta)R_x^\top(\theta) = I$

Derivative of a Rotation Matrix

- Compute derivative (chain rule)

$$R_x(\theta) R_x^\top(\theta) = I$$

$$\frac{d}{d\theta}\left(R_x(\theta) R_x^\top(\theta)\right) = \frac{d}{d\theta} I$$

$$\frac{d}{d\theta} R_x(\theta) R_x^\top(\theta) + R_x(\theta) \frac{d}{d\theta} R_x^\top(\theta) = 0$$

Derivative of a Rotation Matrix

- Compute derivative (chain rule)

$$R_x(\theta) R_x^\top(\theta) = I$$

$$\frac{d}{d\theta}\left(R_x(\theta) R_x^\top(\theta)\right) = \frac{d}{d\theta} I$$

$$\frac{d}{d\theta} R_x(\theta) R_x^\top(\theta) + R_x(\theta) \frac{d}{d\theta} R_x^\top(\theta) = 0$$

- Exploiting $(AB)^\top = B^\top A^\top$ leads us to

$$\frac{d}{d\theta} R_x(\theta) R_x^\top(\theta) + \left(\frac{d}{d\theta} R_x(\theta) R_x^\top(\theta)\right)^\top = 0$$

Derivative of a Rotation Matrix

- Rewrite $\underline{\frac{d}{d\theta} R_x(\theta) R_x^\top(\theta)} + \left(\frac{d}{d\theta} R_x(\theta) R_x^\top(\theta) \right)^\top = 0$
- as $S + S^\top = 0$
- This directly leads to $S^\top = -S$, which is a skew-symmetric matrix
- We can now exploit the fact that $\frac{d}{d\theta} R_x(\theta) R_x^\top(\theta)$ is a skew-symmetric matrix

Derivative of a Rotation Matrix

- We have $S = \frac{d}{d\theta} R_x(\theta) R_x^\top(\theta)$

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Remember:

$$\begin{aligned} \frac{d}{d\theta}\sin(\theta) &= \cos(\theta) \\ \frac{d}{d\theta}\cos(\theta) &= -\sin(\theta) \end{aligned}$$

- So

$$S = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta \\ 0 & \cos\theta & -\sin\theta \end{bmatrix}}_{\frac{d}{d\theta} R_x(\theta)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}}_{R_x^\top(\theta)}$$

Derivative of a Rotation Matrix

$$
\begin{aligned}
S &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta \\ 0 & \cos\theta & -\sin\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \\
&= S_{\boldsymbol{e}_x}
\end{aligned}
$$

with the unit vector $\boldsymbol{e}_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Derivative of a Rotation Matrix

- This means $\frac{d}{d\theta} R_x(\theta) R_x^\top(\theta) = S_{\boldsymbol{e}_x}$
- and thus

$$\frac{d}{d\theta} R_x(\theta) = \frac{d}{d\theta} R_x(\theta) \underbrace{R_x^\top(\theta) R_x(\theta)}_{I} = S_{\boldsymbol{e}_x} R_x(\theta)$$

- The derivative of a rotation matrix $R_x(\theta)$ is the skew-symmetric matrix $S_{\boldsymbol{e}_x}$ times the rotation matrix itself

$$\frac{d}{d\theta} R_x(\theta) = S_{\boldsymbol{e}_x} R_x(\theta)$$

The Same for x,y,z Axes

- We can repeat the same to x, y, z and obtain

$$\frac{d}{d\theta} R_x(\theta) = S_{\boldsymbol{e}_x} R_x(\theta)$$

$$\frac{d}{d\theta} R_y(\theta) = S_{\boldsymbol{e}_y} R_y(\theta)$$

$$\frac{d}{d\theta} R_z(\theta) = S_{\boldsymbol{e}_z} R_z(\theta)$$

- and even for an arbitrary rot. axis $\boldsymbol{r}$

$$\frac{d}{d\theta} R_r(\theta) = S_r R_r(\theta)$$

Infinitesimal Small Rotations

- Similarly, we can also approximate an infinitesimally small rotation by

$$R \approx I + \mathrm{d}R = I + S_{\mathrm{d}r} = I + \begin{bmatrix} 0 & -\mathrm{d}\kappa & \mathrm{d}\phi \\ \mathrm{d}\kappa & 0 & -\mathrm{d}\omega \\ -\mathrm{d}\phi & \mathrm{d}\omega & 0 \end{bmatrix}$$

- Thus,

$$\mathrm{d}R = S_{\mathrm{d}r} = \begin{bmatrix} 0 & -\mathrm{d}\kappa & \mathrm{d}\phi \\ \mathrm{d}\kappa & 0 & -\mathrm{d}\omega \\ -\mathrm{d}\phi & \mathrm{d}\omega & 0 \end{bmatrix}$$

Summary

This lecture was a **brief and informal reminder** of concepts we will need

- Solving Ax=b
- Solving Ax=0 using SVD
- Least squares with Gauss Newton
- Skew-symmetric matrix
- Derivative of a rotation matrix
- Own lecture: Homogenous coordinates