Advanced Topics in Computer Graphics II

Introduction and Parametric Curves

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October 08th, 2024

- Reinhard Klein (rk@cs.uni-bonn.de)
- Lecture (4h)
    - Tuesday: 12:15 pm - 13:45 pm
    - Thursday: 12:15 pm - 13.45 pm

- Slides/materials on ecampus:

https://ecampus.uni-bonn.de/ilias.php?baseClass=ilrepositorygui&ref_id=3889436

- There you will find
    - **Slides**
    - **Exercises**
    - **auxiliary materials**

Organisation of the tutorials

- Domenic Zingsheim (zingsheim@cs.uni-bonn.de)
- **Exercises (2h)**
  - Timeslot is organized via ecampus until Friday, 17th, October, 23:59h

- Organisation
  - Practical part: Programming tasks (50%)
  - Theoretical part: list of questions (50%)

- 50% correct practial as well as 50% theoretical solutions is a prerequisite for the final examination!
- 70% of the sheets need to be passed (>50%)

Tutorials

- **Content of the practical part:**
  - C++ Framework, Geometry processing using OpenMesh and Eigen
  - Required software: editor, C++ compiler (current versions of Visual Studio, GCC, Clang), CMake
  - You can program at home
  - Exercises are presented as unfinished code
  - Code must be completed and well documented

- **Content of the theoretical part:**
  - Theoretical problems, enquiry (with literature)
  - Mathematical concepts
  - **Meant to deepen lecture content**
  - **Relevant for the exam**
  - **Only PDF-Files written using LaTex are accepted**

Examination

- Oral test
- Successful participation on the exercises required
- In addition to the content of the lecture the content of the exercises is included
- Schedule (preliminary)
    - First possible date: 2025-05-02, (or later by appointment)
    - Resit: End of March (or by appointment)

Contents of the lecture

Content

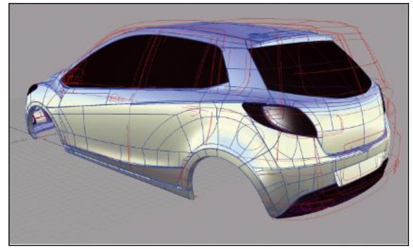
I. Single shape modeling
  i. Curves, Surfaces and Classical Differential Geometry
  ii. The Laplacian and ist applications in Geometry Processing
  iii. Parametrizations and Deformations

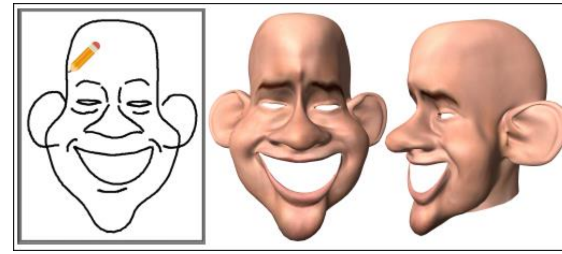
II. Capturing shapes
  i. Point cloud registration (ICP, Coherent Point Drift)
  ii. Surface Reconstruction from Depth Maps
  iii. Surface Reconstruction from Oriented Points

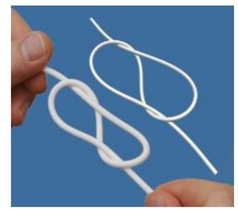
III. Shape ensembles
  i. Similarity and symmetries of shapes
  ii. Shape spaces
  iii. Geodesics in Shape space (morphable models)
  iv. Example: the space of human body shapes

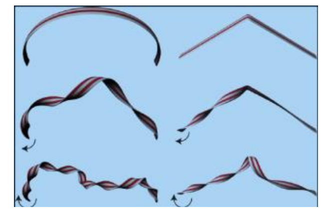
Parametric curves: Motivation

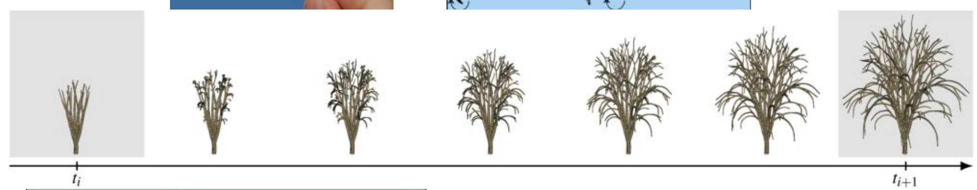
- Applications
  - Design, sketching
  - Animating knots
  - Plants
  - Skeleton(s), structure
  - ...

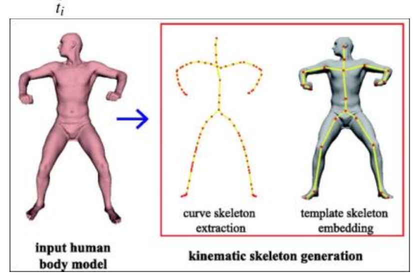


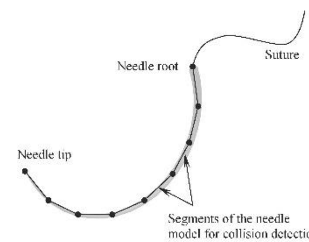












Bergou, Miklós, Max Wardetzky, Stephen Robinson, Basile Audoly, and Eitan Grinspun. "Discrete elastic rods." In *ACM SIGGRAPH 2008 papers*, pp. 1-12. 2008.

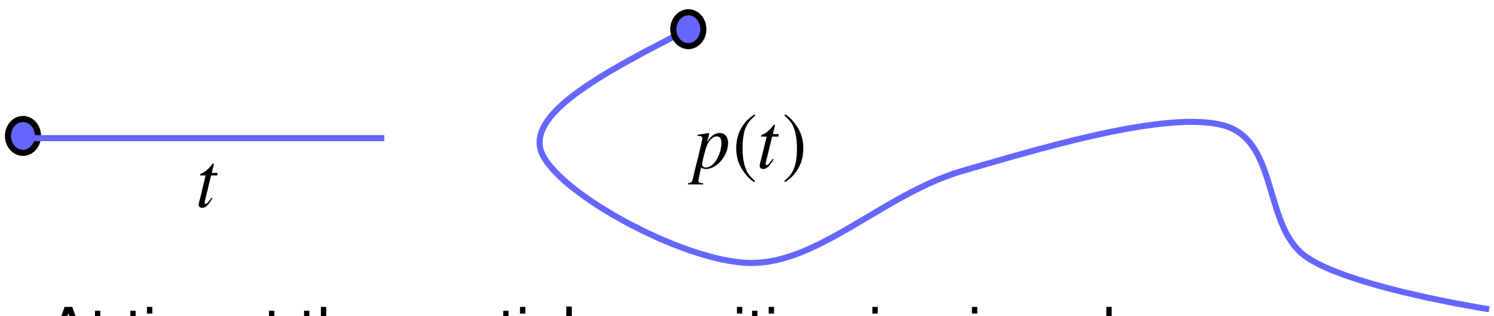
Han, Xiaoguang, Chang Gao, and Yizhou Yu. "DeepSketch2Face: a deep learning based sketching system for 3D face and caricature modeling." *ACM Transactions on graphics (TOG)* 36, no. 4 (2017): 1-12..

Golla, Tim, Tom Kneiphof, Heiner Kuhlmann, Michael Weinmann, and Reinhard Klein. 2020. "Temporal Upsampling of Point Cloud Sequences by Optimal Transport for Plant Growth Visualization." *Computer Graphics Forum* 39 (6): 167–79.

Luo, S., Feng, J. Symmetry-aware kinematic skeleton generation of a 3D human body model. Multimed Tools Appl 79, 20579–20602 (2020)

Parameteric Curves

- Intuition
  - A particle is moving in space



  - At time t the particle position is given by

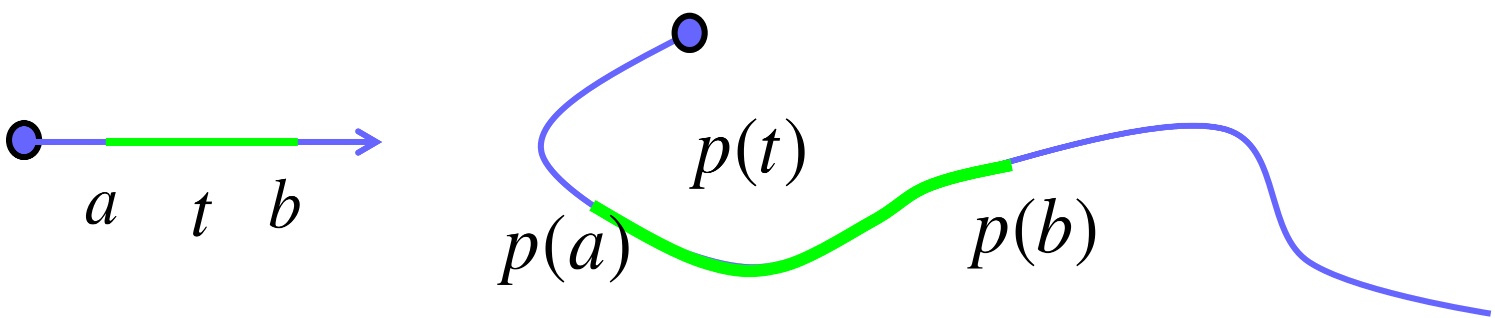
$$p:\mathbb{R}\to\mathbb{R}^d, d=1,2,3,\ldots$$

$$t\mapsto p(t)=\big(x(t),y(t),z(t)\big)^t$$

Parameteric Curves

- Definition
  - A parameterized differentiable curve is a differentiable map $p: \quad \supset I \to \quad {}^{d}, d = 1, 2, 3, \ldots$

  of an interval $I = [a, b]$ of the real line into ^{d}



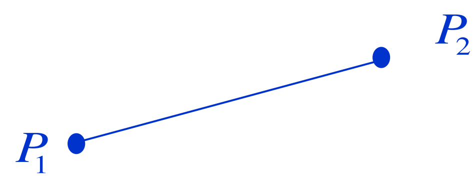
  - At time t its position is given by $t \mapsto p(t) = \left(x(t), y(t), z(t)\right)^{t}$

  - $p(I) \subset \quad {}^{d}$ is the *trace* of p

- Different curves can have the same trace:
  - The same curve segment can be parametrised differently:

$$p_1 : [0,1] \to \mathbb{R}^3,\ p(t) = tP_1 + (1-t)P_2$$

$$p_2 : [0,1] \to \mathbb{R}^3,\ p(t) = t^2P_1 + (1-t^2)P_2$$



- Definition:
  - A parametric curve is *n* times continuously differentiable, when the mapping *p* is *n* times continuously differentiable.
  - The derivative of *p*, *p´(t)* at point *t* is a vector in $\mathbb{R}^3$ that determines the tangent direction of the curve at the given point $u \in \mathbb{R}^3$. A curve is regular, when *p* is once continuously differentiable ist and *p´(t)*≠0 for all $t \in [a,b]$.

More examples

- More examples:

$$p : \mathbb{R} \to \mathbb{R}^d, d = 1, 2, 3, \ldots$$

$$t \mapsto p(t) = \left(x(t), y(t), z(t)\right)^t$$

  - Circle

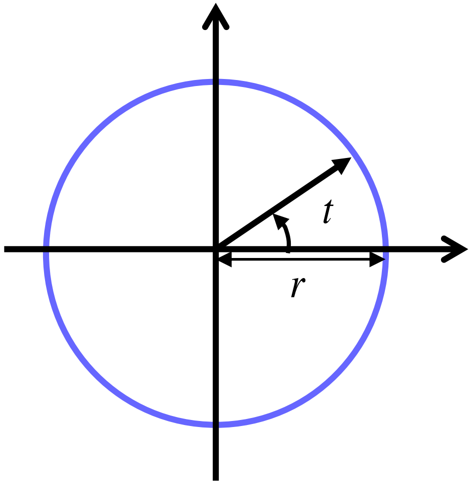
$$p(t) = r\left(\cos(t), \sin(t), 0\right)^t$$

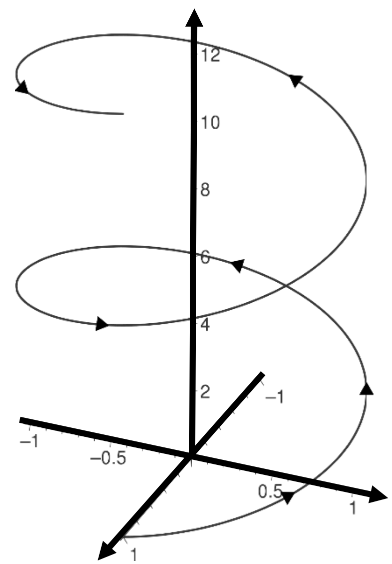
$$t \in [0, 2\pi]$$

  - Helix

$$p(t) = \left(r\cos(t), r\sin(t), bt\right)^t$$

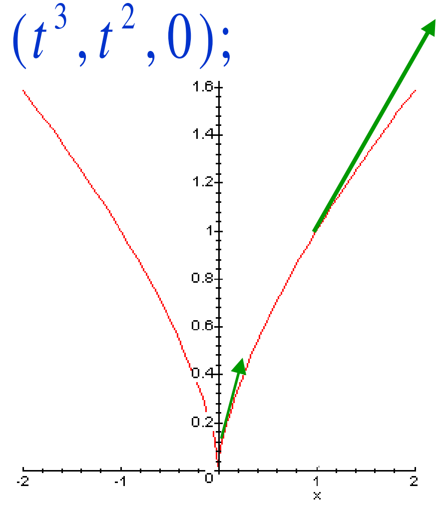
$$t \in [0, 2 \cdot 2\pi]$$





- More examples $p:[-2,2] \to \mathbb{R}^3,\ p(t) = (t^3, t^2, 0);$

  $p'(t) = (t^2, t, 0),$ so $p'(t)$ is continuously differentiable, but $p'(0) = 0,$ so p is not regular at 0.



  - Intuitively, the degree of differentiability of a regular curve is represented by its smoothness.
    - direction of movement $p'(t)$
    - speed of movement $\|p'(t)\|$

- $p:[a,b] \to \mathbb{R}^d,\ p(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_n t^n, c_i \in \mathbb{R}^d$

  is a polynomial of degree *n* in $\mathbb{R}^d$

- The set of all degree *n* polynomials form a vector space of dimension *n+1*.

$$(\alpha\, p + \beta\, q)(t) = \alpha\, p(t) + \beta\, q(t))$$

- The monomials *1,t,*t^2*,...*t^n form a base over this vector space.

  Computing polynomials using Horner‘s method:

$$p(t) = c_n t^n + \ldots + c_1 t \ \ + c_0$$

$$= (\ldots((c_n t + c_{n-1})t + c_{n-2})t + \ldots + c_1)t + c_0 \qquad \text{n Mult. + n Add.}$$

$$p(t) = c_3 t^3 + c_2 t^2 + c_1 t \ \ + c_0$$

$$= ((c_3 t + c_2)t + c_1)t + c_0$$

- Geometric meaning of the coefficients?

$$p(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_n t^n, c_i \in \mathbb{R}^d$$

$$c_0 = p(0),\ c_1 = p'(0),\ c_2 = \frac{1}{2} p''(0), \ldots, c_k = \frac{1}{k!} p^{(k)}(0)$$

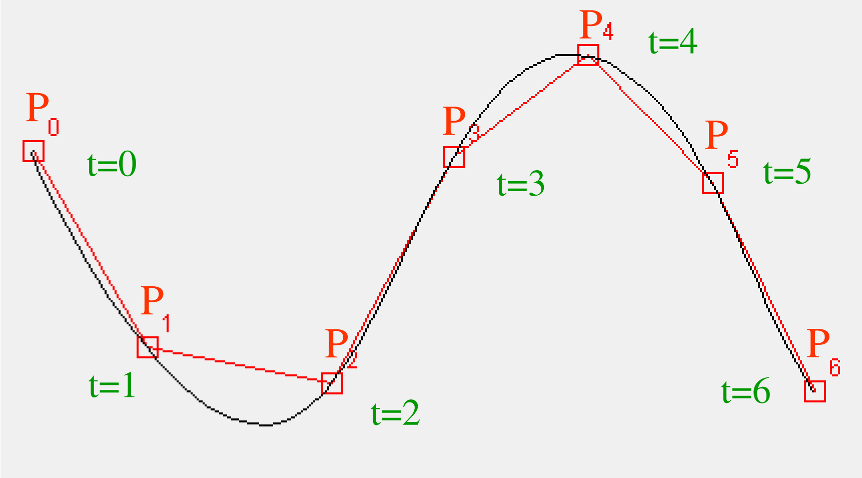
- Coefficients affect the derivatives of the curve at point *0*. Modeling of curves using such coefficients is practically impossible.

- Given: $P_i \in \mathbb{R}^d, t_i \in \mathbb{R}, i = 0,\ldots,n$

  (P_i control points, t_i knots or parameter values)

- Want: Polynomial curve for which

$$p(t_i) = P_i,\ i = 0,\ldots,n$$



- Given: $\text{knots } t_0 < t_1 < \ldots < t_n$

- Consider the degree *n* Lagrange-Polynomial

$$L_i^n(t) = \frac{(t-t_0)(t-t_1)\ldots(t-t_{i-1})(t-t_{i+1})\ldots(t-t_n)}{(t_i-t_0)(t_i-t_1)\ldots(t_i-t_{i-1})(t_i-t_{i+1})\ldots(t_i-t_n)}$$

- This gives: ,

$$L_i^n(t_k) = \delta_{ik} = \begin{cases} 1 & \text{for } i = k \\ 0 & \text{otherwise} \end{cases}$$

- Therefore, a linear combination of these polymonials solves the interpolation problem:

$$p(t) = \sum_{i=0}^{n} P_i L_i^n(t) = \sum_{i=0}^{n} P_i \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{t-t_j}{t_i-t_j}$$

- Using the cubic Hermite-Polynomials the derivatives (tangent-vectors) are also interpolated besides control points. In addition we have a useful new polynomial basis, the Hermite-Basis. In the cubic case we take the following four basis functions over the interval [0,1]:

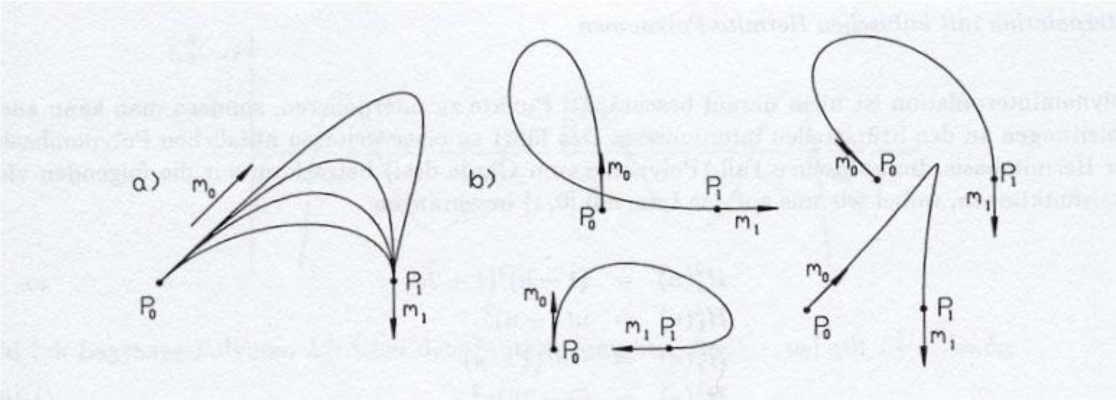
$$H_0^3(t) = (1-t)^2(1+2t) \qquad H_1^3(t) = t(1-t)^2$$

$$H_2^3(t) = -t^2(1-t) \qquad H_3^3(t) = (3-2t)t^2$$

- The curve

$$p(t) = p_0 H_0^3(t) + m_1 H_1^3(t) + m_2 H_2^3(t) + p_1 H_3^3(t)$$

is called Hermite curve. The coefficients 0 and 3 are points, while the coefficients 1 and 2 are the derivatives at these points.



- The polynomial $$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, \; t \in [0,1]$$

is the Bernstein polynomial of degree *n* over the interval [0,1]. They form a basis for the *n+1* dimensional polynom space.

- Properties of Bernstein polynomials:

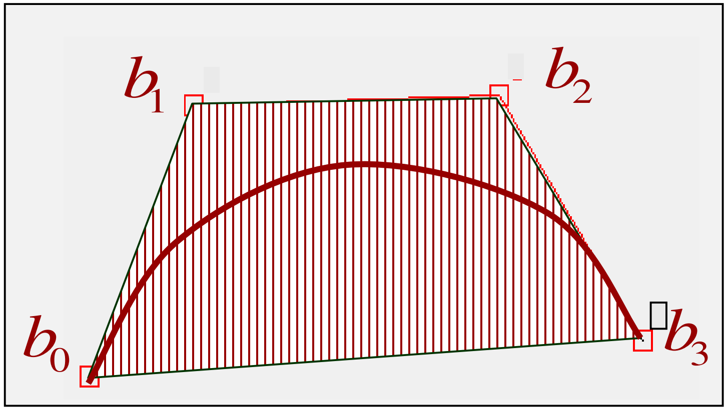
$\sum_{i=0}^{n} B_i^n(t) = 1$	partition of unity
$B_i^n(t) \geq 0 \;\; \mathrm{t} \in [0,1]$	nonnegativity
$B_i^n(t) = tB_{i-1}^{n-1}(t) + (1-t)B_i^{n-1}(t)$	recursive definition
$B_i^n(t) = B_{n-i}^n(1-t)$	symmetry

- The curve $$p(t)=\sum_{i=0}^{n} b_i B_i^n(t),\ t\in[0,1],\ b_i\in\mathbb{R}^d$$

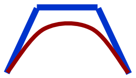
is called Bézier curve of degree *n* over the interval *[0,1]. The* points b_i, *i=0,...,n* are the Bézier points or control points and give the Bézier polygon or control polygon.

- The Bézier curve approximates the control polygon.
- Since $\sum_{i=0}^{n} B_i^n(t)=1$ holds Bézier curves are invariant under affine transformations.
- As $B_i^n(t)\geq 0\ \mathrm{t}\in[0,1]$ also holds, the curve is contained in the convex hull of its defining control points.

- Since $B_i^n(t) \geq 0 \ \mathrm{t} \in [0,1]$ holds and $\sum_{i=0}^{n} B_i^n(t) = 1$, the curve is contained in the convex hull of its control points.



- The derivatives of a Bézier curve are also polynomials and can be expressed as Bézier curves:
- $$p^{(k)}(t) = \frac{n!}{(n-k)!} \sum_{i=0}^{n-k} \Delta^k b_i B_i^{n-k}(t), \; t \in [0,1]$$



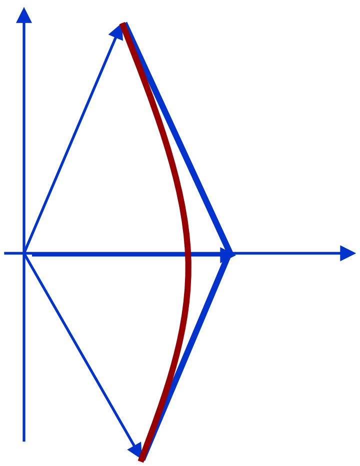
Where $\Delta^k b_i$ is defined recursively:

$$\Delta^0 b_i = b_i$$

$$\Delta^k b_i = \Delta^{k-1} b_{i+1} - \Delta^{k-1} b_i$$

- For example:

$$\Delta^1 b_i = b_{i+1} - b_i$$



- **Example: Bernstein polynomials**
- Quadratic:

$$B_0^2(t) = (1-t)^2$$
$$B_1^2(t) = 2(1-t)t$$
$$B_2^2(t) = t^2$$

$$\begin{pmatrix} B_0^2(t) \\ B_1^2(t) \\ B_2^2(t) \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}$$

- Cubic:

$$B_0^3(t) = (1-t)^3$$
$$B_1^3(t) = 3(1-t)^2 t$$
$$B_2^3(t) = 3(1-t)t^2$$
$$B_3^3(t) = t^3$$

$$\begin{pmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{pmatrix} = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

- **Example: Bezier Curves**
- Cubic:

$$p(t) = b_0 B_0^3(t) + b_1 B_1^3(t) + b_2 B_2^3(t) + b_3 B_3^3(t)$$

$$p(t) = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{pmatrix} = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{S_3^{Bernstein}} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

$$\begin{pmatrix} p_x(t) \\ p_y(t) \\ p_z(t) \end{pmatrix} = \begin{pmatrix} b_{0x} & b_{1x} & b_{2x} & b_{3x} \\ b_{0y} & b_{1y} & b_{2y} & b_{3y} \\ b_{0z} & b_{1z} & b_{2z} & b_{3z} \end{pmatrix} \begin{pmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{pmatrix} = \begin{pmatrix} b_{0x} & b_{1x} & b_{2x} & b_{3x} \\ b_{0y} & b_{1y} & b_{2y} & b_{3y} \\ b_{0z} & b_{1z} & b_{2z} & b_{3z} \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{S_3^{Bernstein}} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

- **Definition:**
  - Arc length

    $p:[a,b]\rightarrow \mathbb{R}^d, d=1,2,3,\ldots$

    $s:[a,b]\rightarrow[0,s(b)],\ u\mapsto s(u)=\int_a^u \|p'(t)\|dt$

- **Examples**:
  - $p(t)=r\left(\cos(t),\sin(t),0\right)^t,\ t\in[0,2\pi]$

    $\|p'(t)\|=r\left(-\sin(t),\cos(t),0\right)^t, \|p'(t)\|=r\sqrt{\left(\sin^2(t)+\cos^2(t)\right)}=r$

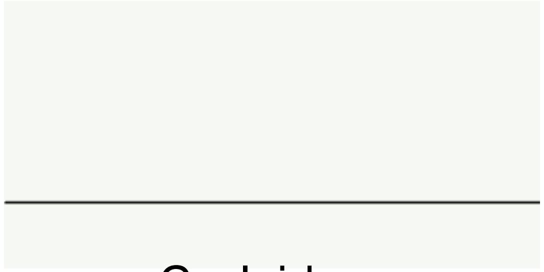
    $s:[0,2\pi]\rightarrow[0,s(2\pi)],\ u\mapsto s(u)=\int_0^u r dt=ru$

    $p(t)=(t^3,t^2,0),\ t\in[-2,2]$

    $p'(t)=(3t^2,2t,0),\ \|p'(t)\|=t\sqrt{9t^2+4}$

  - $s:[a,b]\rightarrow[0,s(b)],\ u\mapsto s(u)=\int_a^u t\sqrt{9t^2+4}dt$

- Closed-Form Arc Lenght Gallery (taken from Mirela Ben-Chen)



Cycloid

$$p(t)=\left(rt-r\sin(t),r-r\cos(t),0\right)^t,\ t\in[0,2\pi]$$

$$\|p'(t)\|=r\left(1-\cos(t),\sin(t),0\right)^t,\|p'(t)\|=r\sqrt{2\left(1-\cos(t)\right)}$$

$$s:[0,2\pi]\to[0,s(2\pi)],\ s(2\pi)=\int_0^{2\pi} r\sqrt{2\left(1-\cos(t)\right)}dt=8r$$

Logarithmic Spiral

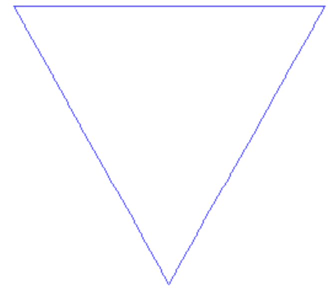
$$p(t)=\left(ae^{bt}\cos(t),ae^{bt}\sin(t),0\right)^t$$

Catenary

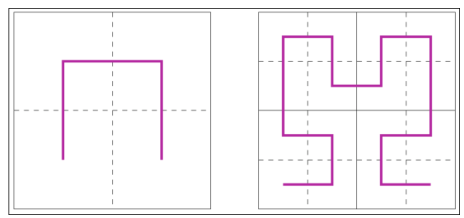
$$p(t)=\left(t,\frac{a}{2}\left(e^{t/a}+e^{t/a}\right),0\right)^t$$

Curves with infinite Length

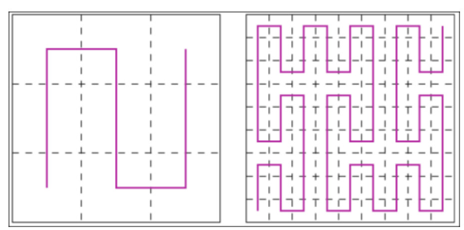
- Some curves have infinite length:
  - Koch snowflake:
  - Space filling curves



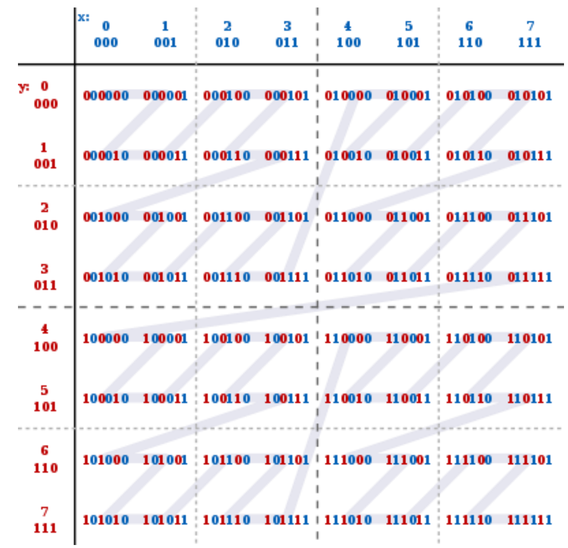
- **Hilbert curve**
- **Peano curve**
- Gosper curve
- Moore curve
- Sierpiński curve
- **Z-order curve**



Hilbert Curve



Peano Curve



Z-order curve

Bader, Michael, and Space-Filling Curves. "An Introduction with Applications in Scientific Computing." *Texts in computational science and engineering* (2012).

- Spline: Given a set of knots $t_0 \le t_1 \le \ldots \le t_n$ and corresponding intervals $Ij := [t_{j-1}, t_j], j = 1, \ldots, n$ a spline is a mapping $q : [t_0, t_n] \rightarrow \;^d, d = 2, 3, \ldots$ such that for each interval $Ij, j = 1, \ldots, n$ the spline segment

$$q_j : [t_{j-1}, t_j] \rightarrow \;^d, d = 2, 3, \ldots$$

is a polynomial.

- The intervals $[t_i, t_{i+1}]$, $i = 0, . . ., n-1$ define the knot vector $\mathbf{T} = (t_0, t_1, \ldots, t_n)$
- The spline segments join at the knots. At these parameters the properties of the derivatives are very important:

    - The tangents can differ both in length and in direction. If the directions are the same, but the lengths are different, then the curve is smooth at the point, but not differentiable $\Rightarrow$ therefore we have to differentiate the terms geometric, resp. parametric continuity.

• **Definition:** (C^n continuity):

Let

$$q_1 : [a_1, b_1] \to \mathbb{R}^3,$$
$$q_2 : [a_2, b_2] \to \mathbb{R}^3$$

be two n times continuously differentiable regular curves.

q_1, q_2 are C^n continuous at the points b_1, a_2 if

$$q_1^{(k)}(b_1) = q_2^{(k)}(a_2) \quad \text{for all } k = 0, \dots, n.$$

Geometric continuity

- **Definition: (**G^n continuity)

Let $q_1 : [a_1, b_1] \rightarrow \ {}^3$, $q_2 : [a_2, b_2] \rightarrow \ {}^3$ be two n times continuously differentiable regular curves. q_1, q_2 are G^n continuous at the points b_1, a_2 if there exists a reparametrization $r_1 = q_1 \circ \varphi$ of q_1 with a bijective, differentiable mapping $\varphi : [a_0, b_0] \rightarrow [a_1, b_1],\ \varphi'(u) > 0 \ \forall\ u \in [a_0, b_0]$ such that

$r_1 = q_1 \circ \varphi$ and q_2 are C^n continuous at b_1, $a_{2.}$ Differentiating r_1 according to the chain rule leads to:

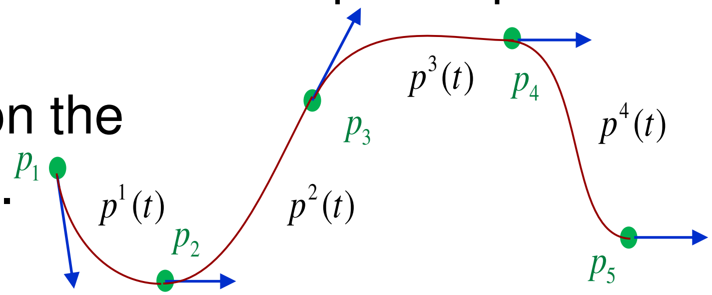
$$q_2(a_2) = r_1(b_0) = q_1\left(\varphi(b_0)\right)$$

$$q_2'(a_2) = r_1'(b_0) = q_1'\left(\varphi(b_0)\right) \cdot \varphi'(b_0)$$

$$q_2''(a_2) = r_1''(b_0) = q_1''\left(\varphi(b_0)\right) \cdot \varphi'(b_0)^2 + q_1'\left(\varphi(b_0)\right) \cdot \varphi''(b_0)$$

The coefficients $\beta_i := \varphi^{(i)}(b_0)$ are called β constraints.

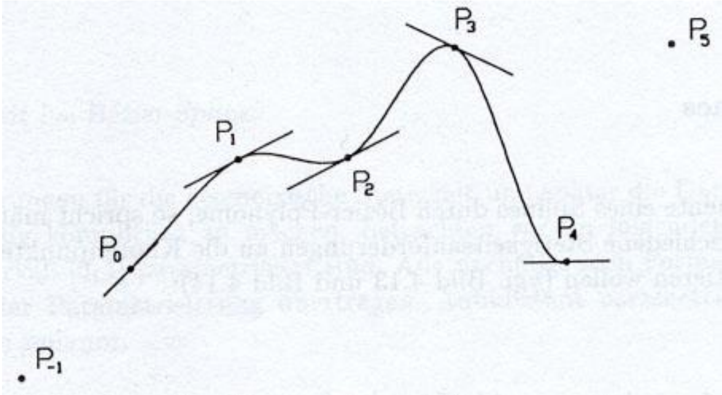
- Hermite Splines use cubic Hermite Polynomials to interpolate points and tangents at the control points.
- The form of the curve strongly depends on the direction and length of the tangent vectors.



- The FMILL method formulate the tangents using the control points:

  At P_i the tangent direction m_i is given by half of the chord $P_{i-1}P_{i+1}$. The resulting interpolation is called Catmull-Rom spline. It is a C^1 continuous spline.

$$p^i(t) = \begin{pmatrix} p_{i-1} & p_i & p_{i+1} & p_{i+1} \end{pmatrix} \underbrace{\begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ 1 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}}_{FMILL} \underbrace{\begin{pmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 3 & -2 \end{pmatrix}}_{S_3^{Hermite}} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$



- Are there basis functions for splines?
- Let $n \leq m$ and $T = (t_0 = \ldots = t_n, t_{n+1}, \ldots, t_m, t_{m+1} = \ldots = t_{m+n+1})$

be a weakly monotonic sequence of knots with $t_i < t_{i+n+1}, 0 \leq i \leq m$.

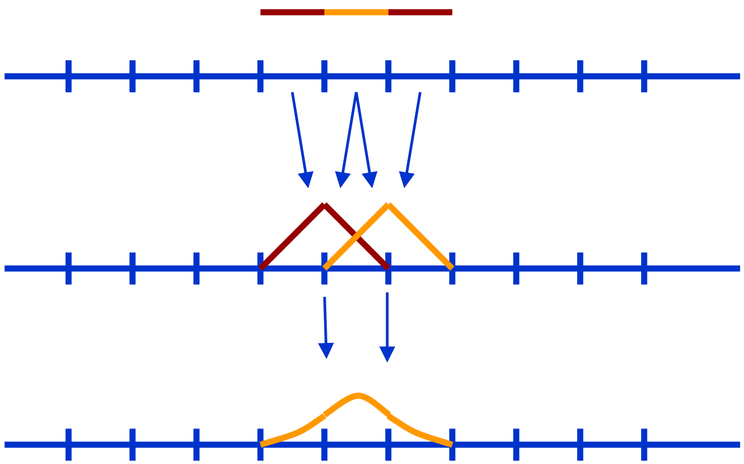
The recursively defined functions

$$N_i^0(t) := \begin{cases} 1 \text{ if } t_i \leq t < t_{i+1} \\ 0 \text{ otherwise} \end{cases}$$

$$N_i^r(t) := \frac{t - t_i}{t_{i+r} - t_i} N_i^{r-1}(t) + \frac{t_{i+1+r} - t}{t_{i+1+r} - t_{i+1}} N_{i+1}^{r-1}(t) \text{ for } 1 \leq r \leq n.$$

- are called normalized B-splines of degree *n* over *T*. Since the distance of consequtive knots is not constant they are also refered to as non uniform normalized B-splines.

- $N_i^n(t)$ piecewisely consists of polynomials of degree n over *T*:



$$N_i^0(t) := \begin{cases} 1 \text{ if } t_i \le t < t_{i+1} \\ 0 \text{ otherwise} \end{cases}$$

$$N_i^r(t) := \frac{t - t_i}{t_{i+r} - t_i} N_i^{r-1}(t) + \frac{t_{i+1+r} - t}{t_{i+1+r} - t_{i+1}} N_{i+1}^{r-1}(t)$$

The functions $N_i^n(t)$ have local support, i.e. $N_i^n(t) = 0$ for $t \notin [t_i, t_{i+n+1}]$.

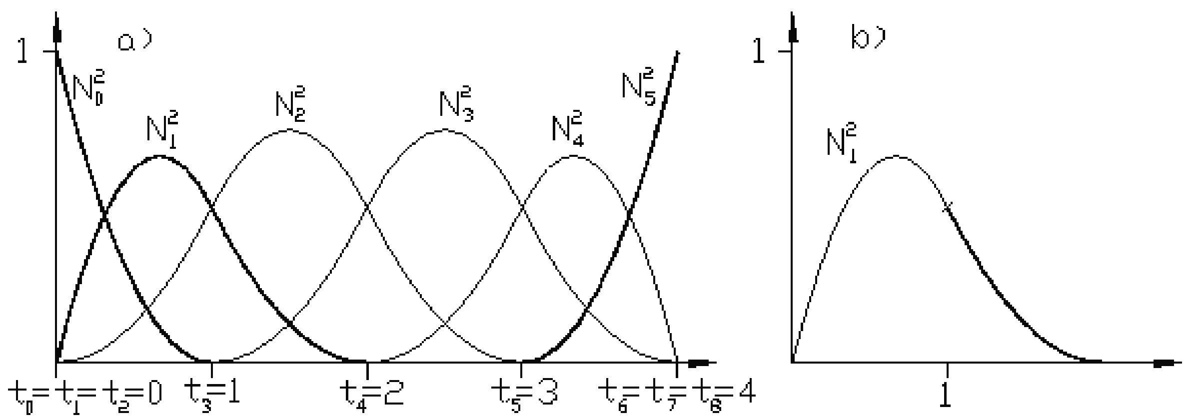
$N_i^n(t) \ge 0$ holds for all $t \in [t_0, t_{m+n+1}]$.

- All B-splines sum to one:

If t_j is a single knot, i.e. $t_{j-1} \neq t_j \neq t_{j+1}$, then $N_i^n(t_j)$

is at least C^{n-1}-continous.

- At a multiple knot $s=t_{j+1}=...=t_{j+\mu}$ of multiplicity μ the normalized B-splines N_i^n of degree n are at least $C^{n-\mu}$-continous.



- Let $n \le m$ and $T = (t_0 = \ldots = t_n, t_{n+1}, \ldots, t_m, t_{m+1} = \ldots = t_{m+n+1})$

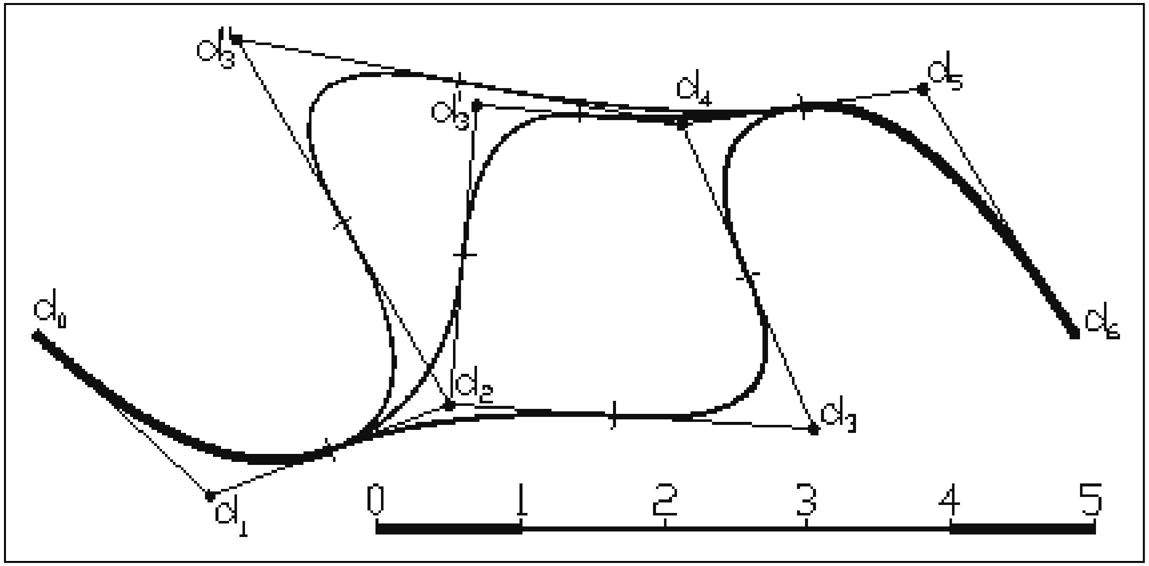
be a weakly monotonic sequence of knots with $t_0 < t_{i+n+1}, 0 \le i \le m$

and $d_0, \ldots, d_m \in \mathbb{R}^d,\ 0 \le i \le m$ a set of control points. The curve

$$p(t) = \sum_{i=0}^{m} d_i N_i^n(t),\ \mathrm{d}_i \in \mathbb{R}^d$$

is callled B-spline curve of degree *n* over *T*. The control points are also referred to as de Boor Points. They constitute the control-polygon.

- As $N_i^n = 0$, for $t \notin [t_i, t_{i+n+1}]$ the *i*-th de Boor-point $\mathbf{d_i}$ influences the curve only within the parameter interval $[t_i, t_{i+n+1}]$.
- Therefore, the shape of the curve within the parameter interval $[t_i, t_{i+1}]$ is completely determined by the de Boor-points $d_{i-n}, \ldots, d_i$



- **Definition:**
  - A curve *p* is called arc length parametrised, if

$$\left\| p'(t) \right\| = 1,\ u \in [a,b]$$

- **Remark:**

  In general curves are not arc length parametrized, e.g. Bezier Curves, Bsplines, Subdivision Curves are not arc length parametrized.

Arc Lenght Parametrization

- **Theorem:**

Let $p : I \rightarrow {}^{d}$ be a regular parametrized curve, and $s(t)$ its arc lenght. Then the inverse function $t(s)$ exists, and

$$q(s) = p\left(t(s)\right)$$

is parametrized by arc length.

- **Proof:**

$$p \text{ is regular } \Rightarrow s(t) = \left\| p'(t) \right\| > 0 \ \forall t$$

$$\Rightarrow s(t) \text{ is a monotonic increasing function}$$

$$\Rightarrow \text{the inverse function } t(s) \text{ exists}$$

$$\Rightarrow q'(s) = p'\left(t(s)\right) \cdot t'(s) = p'\left(t(s)\right) \cdot \frac{1}{s'\left(t(s)\right)} = p'\left(t(s)\right) \cdot \frac{1}{\left\| p'\left(t(s)\right) \right\|}$$

$$\Rightarrow \left\| q'(s) \right\| = 1$$

Tangent, Normal, Binomal, Curvature

- **Definition:**

  Tangent vector, curvature vector and curvature for **arc length** parametrised curves:

$T(s) := p'(s)$ Tangent vector

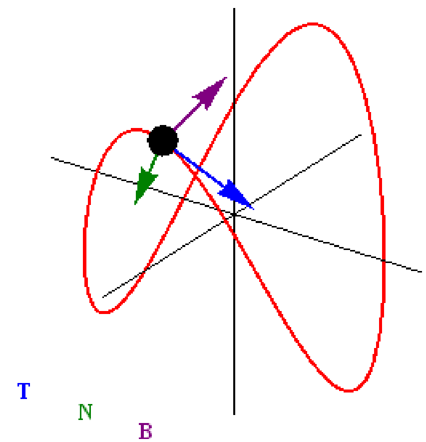
$K(s) := T'(s) = p''(s)$ Curvature vector

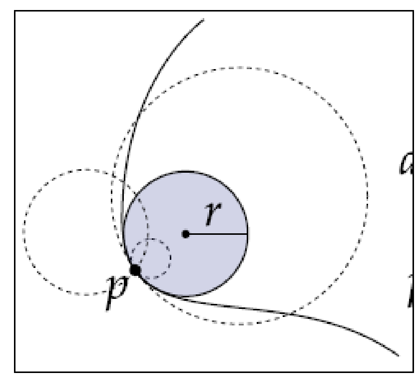
$N(s) := \dfrac{T'(s)}{\|T'(s)\|}$ Normal vector

$\kappa(s) := \|T'(s)\| = \|p''(s)\|$ Curvature

$r(s) := 1/\kappa(s)$ If $\|p''(s)\| \neq 0 : r(s)$ radius of curvature at s

$B(s) := T(s) \times N(s)$ Binormal

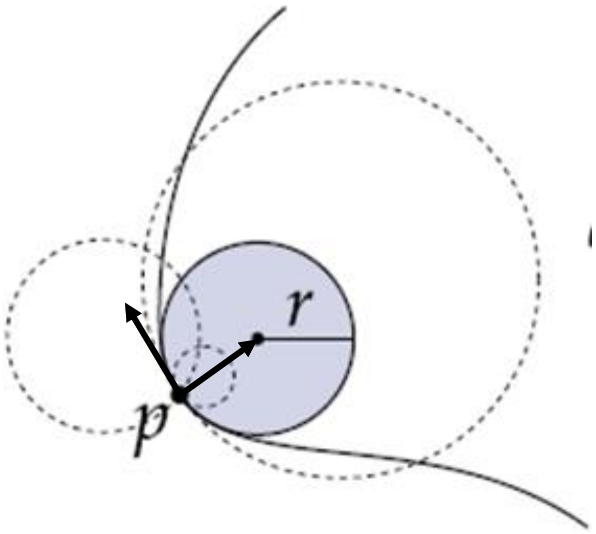




- **Definition:**
  - The plane determined by the unit tangent and normal vectors $T(s)$ and $N(s)$ is called the *osculating plane.* Suppose that $\kappa(s) \neq 0$ The corresponding center of curvature is the point M at distance

$$r(s) = {}^{1}\!/_{\kappa(s)}$$ along $N(s)$.

  - The circle with center at M and radius r is called the *osculating circle* to the curve at the point $p=p(s)$



Torsion

- For the binormal, we have

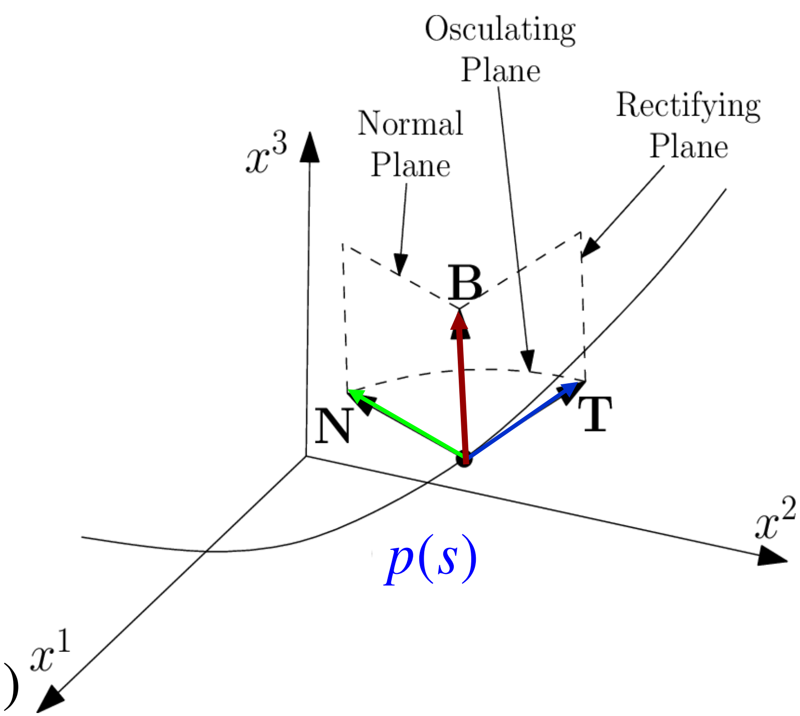
$$\begin{aligned} B'(s) &= T'(s)\times N(s) + T(s)\times N'(s) \\ &= \underbrace{\kappa(s)N(s)\times N(s)}_{0} + T(s)\times N'(s) \\ &= T(s)\times N'(s) \end{aligned}$$

and using the fact that

$$\begin{aligned} N'(s) &= B'(s)\times T(s) + B(s)\times T'(s) \\ &= \tau(s)N(s)\times T(s) + B(s)\times \kappa(s)N(s) \\ &= -\tau(s)B(s) - \kappa(s)T(s) \end{aligned}$$

it follows that

$$B'(s) = \tau(s)N(s)$$



$\tau(s)$ is called torsion. It measures how fast the curve leave the osculating plane.

- Straight line:

$$p(t) = p_1 + t\,\frac{p_2 - p_1}{\left\| p_2 - p_1 \right\|}$$

$$p'(t) = \frac{p_2 - p_1}{\left\| p_2 - p_1 \right\|}$$

$$p''(t) = 0 \;\Rightarrow\; \left\| p''(t) \right\| = 0$$

- Circle:

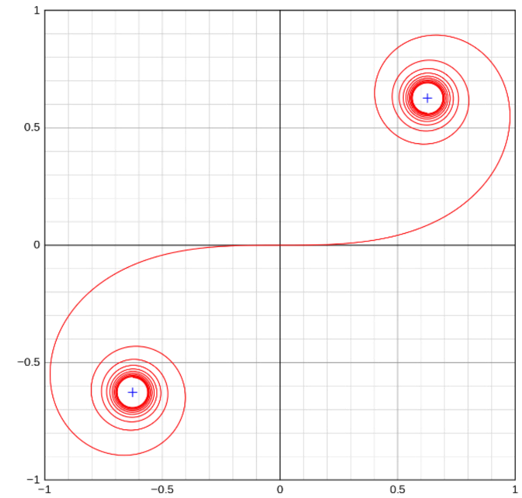
$$p(t) = r\left(\cos\left(s/r\right), \sin\left(s/r\right), 0\right), s \in [0, 2\pi r]$$

$$p'(t) = \left(-\sin\left(s/r\right), \cos\left(s/r\right), 0\right)$$

$$p''(t) = 1/r\left(-\cos\left(s/r\right), -\sin\left(s/r\right), 0\right) \;\Rightarrow\; \left\| p''(t) \right\| = \frac{1}{r}$$

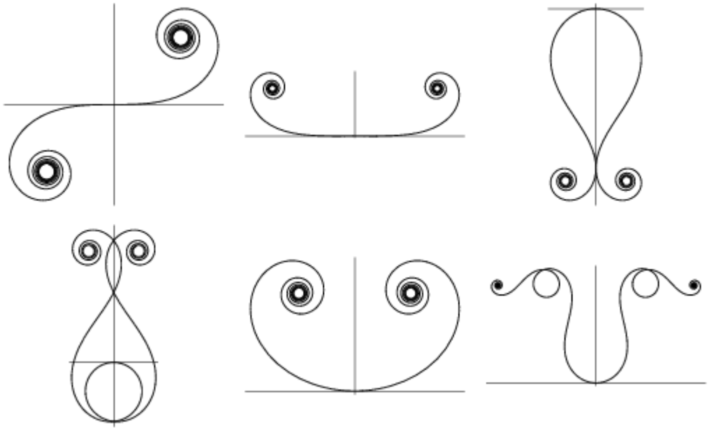
- Cornu Spiral
  - A curve for which

$$\kappa(s) = s$$



- Generalized Cornu Spiral
  - A curve for which

$$\kappa(s) = q(s),\ \text{where } q \text{ polynomial}$$



$$\kappa(s) = s,\ \kappa(s) = s^2,\ \kappa(s) = s^2 - 2{,}19$$

$$\kappa(s) = s^2 - 4,\ \kappa(s) = s^2 + 1,\ \kappa(s) = 5s^4 - 18s^2 + 5$$

- **Lemma:**

  Let $f, g : I \rightarrow {}^{d}$ be differentiable maps which satisfy

  $$f \cdot g = const \ \forall \ t$$

  Then $$f'(t) \cdot g(t) = -f(t) g'(t)$$

  In particular:

  $$\left\| f(t) \right\| = const \text{ if and only if } f'(t) \cdot f(t) = 0 \ \forall \ t$$

- **Proof:** $f \cdot g = const \ \forall t \Rightarrow \left(f \cdot g \right)'(t) = 0.$

  $$\left(f \cdot g \right)'(t) = f'(t) \cdot g(t) + f(t) g'(t) = 0 \Leftrightarrow f'(t) \cdot g(t) = -f(t) g'(t)$$

  $$f = g : f'(t) \cdot f(t) = -f(t) f'(t) \Rightarrow f'(t) \cdot f(t) = 0$$

Normal Vector

- Let p be parameterized by arc length. Then

$$\left\| p'(t) \right\|^2 = p'(t) \cdot p'(t) = 1 \ \forall \ t$$

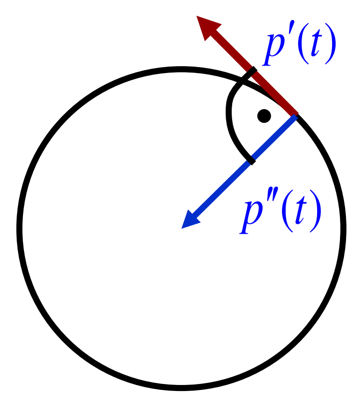
Applying the Lemma we get $p'(t) \cdot p''(t) = 0$, i.e. the tangent vector $p'(t)$ is orthogonal to $p''(t)$, i.e. on arc lenght parametrized curves tangent and curvature vector are perpendicular.

- Example:

$$p(t) = r\left(\cos(t), \sin(t), 0\right)^t$$

$$p'(t) = r\left(\sin(t), \cos(t), 0\right)^t$$

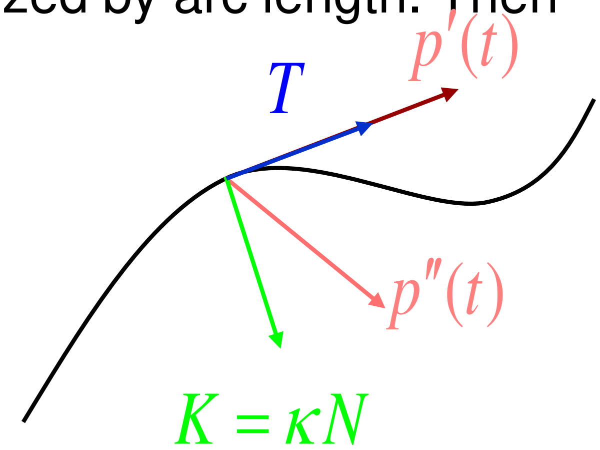
$$p''(t) = -r\left(\cos(t), \sin(t), 0\right)^t$$



- **Lemma:**

Let $p : I \to {}^{d}$ be a cuve not necessarily parametrized by arc length. Then

$$\kappa(t) := \frac{\left\| p''(t) \times p'(t) \right\|}{\left\| p'(t) \right\|^{3}}$$



- **Proof:**

$$p'(t) = \left\| p'(t) \right\| T(t)$$

$$p''(t) = \left\| p'(t) \right\|' T(t) + \left\| p'(t) \right\| T'(t)$$

$$p'(t) \times p''(t) = \underbrace{\left\| p'(t) \right\| T(t) \times \left\| p'(t) \right\|' T(t)}_{0} + \left\| p'(t) \right\| T(t) \times \left\| p'(t) \right\| T'(t)$$

$$= \left\| p'(t) \right\|^{2} T(t) \times T'(t)$$

$$\left\| p'(t) \times p''(t) \right\| = \left\| p'(t) \right\|^{2} \left\| T(t) \times T'(t) \right\| = \left\| p'(t) \right\|^{2} \left\| T'(t) \right\| = \left\| p'(t) \right\|^{2} \left\| \frac{dT}{ds}\big(s(t)\big) \underbrace{\frac{ds}{dt}(t)}_{\left\| p'(t) \right\|} \right\|$$

$$\kappa(t) := \frac{\left\| p''(t) \times p'(t) \right\|}{\left\| p'(t) \right\|^{3}}$$

- Suppose that the curve is given by **r**(*t*), where the parameter *t* need no longer be arclength. Then the unit tangent vector **T** may be written as

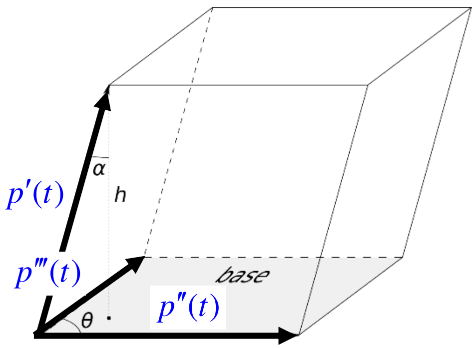
$$T(t) = \frac{p'(t)}{\|p'(t)\|}$$

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{p'(t)\times\left(p''(t)\times p'(t)\right)}{\left\|p'(t)\times\left(p''(t)\times p'(t)\right)\right\|}$$

$$B(t) = T(t)\times N(t) = \frac{p'(t)\times p''(t)}{\|p'(t)\times p''(t)\|}$$

- For the derivatives we get

$$\begin{pmatrix} T'(t) \\ N'(t) \\ B'(t) \end{pmatrix} = \|p'(t)\| \begin{pmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{pmatrix} \begin{pmatrix} T(t) \\ N(t) \\ B(t) \end{pmatrix}$$



with

$$\kappa(t) := \frac{\|p'(t)\times p''(t)\|}{\|p'(t)\|^3} \qquad \tau(t) := \frac{p'(t)\cdot\left(p''(t)\times p'''(t)\right)}{\|p'(t)\times p''(t)\|^2} = \frac{\det\left(p'(t) \quad p''(t) \quad p'''(t)\right)}{\|p'(t)\times p''(t)\|^2}$$

- **Definition:**

A Frenet frame is a moving reference frame of *n* orthonormal vectors $\mathbf{e}_i(t)$ which are used to describe a curve locally at each point $p(t)$. Given a C^n curve p which is regular of order *n* the Frenet frame for the curve is the set of orthonormal vectors

$$\mathbf{e}_1(t),\ldots,\mathbf{e}_n(t)$$

constructed using the Gram-Schmidt orthogonalization algorithm with

$$\mathbf{e}_1(t)=\frac{p'(t)}{\|p'(t)\|} \qquad \mathbf{e}_j(t)=\frac{\hat{\mathbf{e}}_j(t)}{\|\hat{\mathbf{e}}_j(t)\|},\ \hat{\mathbf{e}}_j(t)=p^{(j)}(t)-\sum_{i=1}^{n-1}\left\langle p^{(j)}(t),\mathbf{e}_i(t)\right\rangle\cdot\mathbf{e}_i(t)$$

The generalized curvatures and are defined as

$$\chi_i(t)=\frac{\left\langle e_i'(t),e_{i+1}(t)\right\rangle}{\|p'(t)\|}$$

- Remark:

The bending energy of a rod is proportional to the rod curvature. The shape of a rod is the solution of the following variation problem:

$$E = c\int_0^l \kappa(t)dt \rightarrow \min,$$

where c is a constant and l is the length of the curve. This energy can be approximated by

$$E \approx c\int_0^l p''(t)dt$$

- How to find the minimum?

$\Longrightarrow$ Euler Lagrange Equation

- **Lemma** (Euler Lagrange Equation): Let g be twice continuously differentiable real valued function and $\hat{u} \in D$ a local extremal function of the functional E with

$$E = \int_a^b g\left(u(t), u'(t), u''(t); t\right) dt \qquad \text{on} \qquad D = \left\{ u \in C^2\left[a,b\right] \middle| u(a) = u_a, u(b) = u_b \right\}$$

for given values of u_a, u_b, then the Euler Lagrange Equation holds:

$$g_p(\hat{u}, \hat{u}', \hat{u}''; t) - \frac{d}{dt} g_{p'}(\hat{u}, \hat{u}', \hat{u}''; t) + \frac{d^2}{dt^2} g_{p''}(\hat{u}, \hat{u}', \hat{u}''; t) = 0 \;\forall\; t \in \left[a,b\right]$$

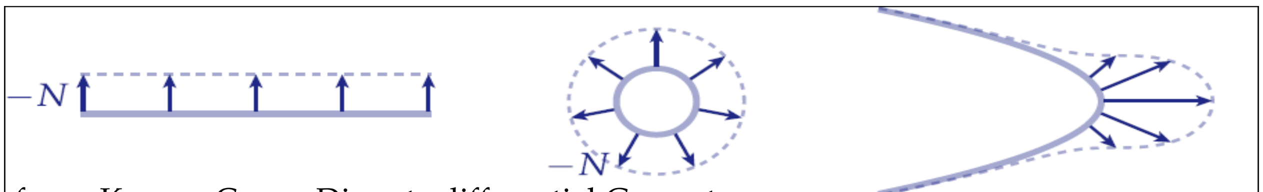
- Applying this to $E = \int_0^l p''(t)^2 dt$ leads to the following Euler-Lagrange equation:

$$2 \frac{d^2}{dt^2} \hat{u}'' = 0 \;\forall\; t \in \left[a,b\right]$$

- Therefore, $\hat{u} \in D$ is a cubic polynomial!

Length variation

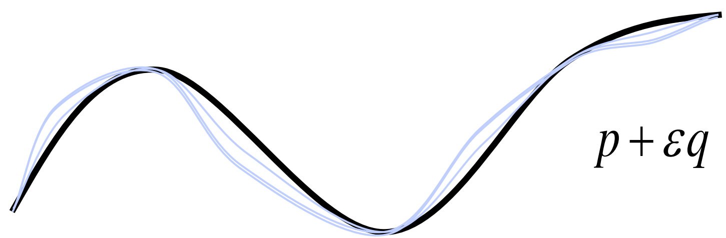
- The fastest way to decrease the length of a curve is to move it in the normal direction, with speed proportional to curvature.
  - Intuition: in flat regions, moving the curve doesn't change its length; in curved regions, the change in length (*per unit length*) is large:



from: Keenan Crane, Discrete differential Geometry

- **Theorem** (length variation) Let $p:[0,L]\rightarrow\ ^2$ be an arbitrary curve and suppose that we have another curve $q:[0,L]\rightarrow\ ^2$ with $q(0)=q(L)=\mathbf{0}$ Then

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} length(p+\varepsilon q) = -\int_0^L \langle q(t), \kappa(t)N(t)\rangle dt$$



- Therfore, the motion that most quickly decreases length is $q(t)=\kappa(t)N(t)$

- **Proof:** $length(p(t)+\varepsilon q(t)) = \int_0^L \left\| p'(t)+\varepsilon q'(t) \right\| dt$

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_0^L \left\| p'(t)+\varepsilon q'(t) \right\| dt = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_0^L \sqrt{\left\langle p'(t)+\varepsilon q'(t), p'(t)+\varepsilon q'(t) \right\rangle} dt$$

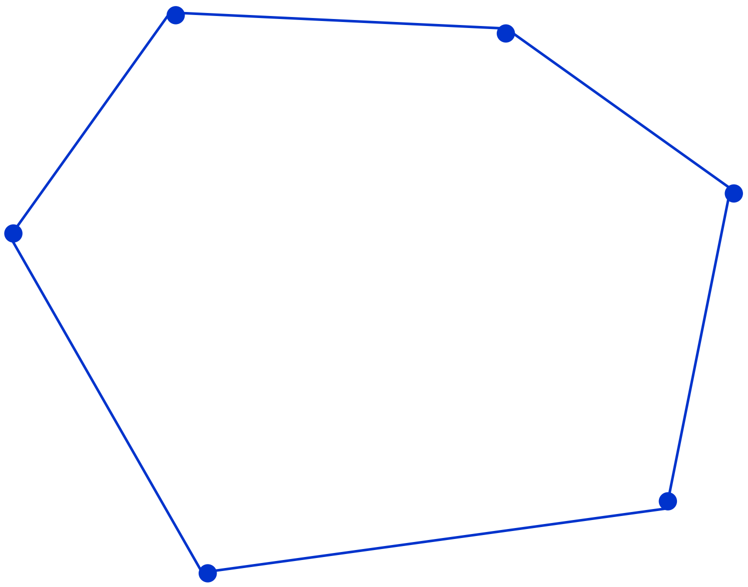
$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_0^L \sqrt{\left\langle p'(t), p'(t) \right\rangle + 2\varepsilon \left\langle p'(t), q'(t) \right\rangle + \varepsilon^2 \left\langle q'(t), q'(t) \right\rangle} dt$$

$$= \int_0^L \frac{1}{2\sqrt{\left\langle p'(t), p'(t) \right\rangle + 2\varepsilon \left\langle p'(t), q'(t) \right\rangle + \varepsilon \left\langle q'(t), q'(t) \right\rangle}} \cdot \left(2\left\langle p'(t), q'(t) \right\rangle + 2\varepsilon \left\langle q'(t), q'(t) \right\rangle \right)\Big|_{\varepsilon=0} dt$$

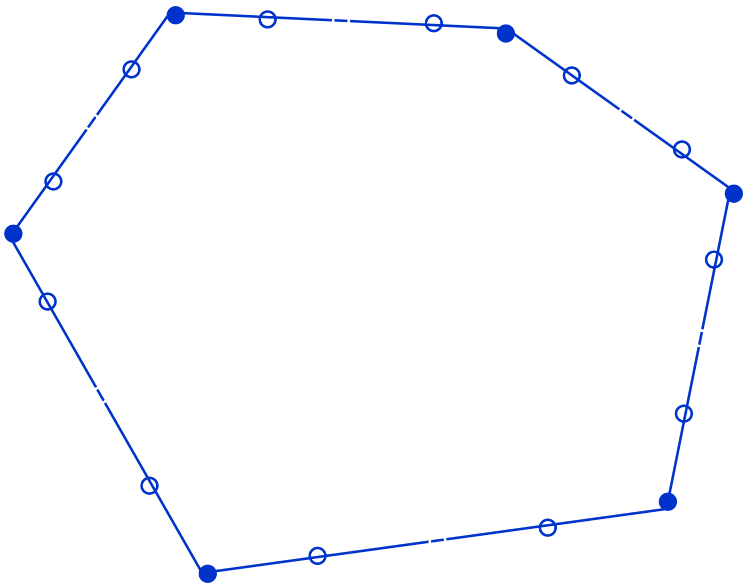
$$= \int_0^L \frac{1}{\sqrt{\left\langle p'(t), p'(t) \right\rangle}} \left\langle p'(t), q'(t) \right\rangle dt = \int_0^L \left\langle \frac{p'(t)}{\left\| p'(t) \right\|}, q'(t) \right\rangle dt$$

$$= \underbrace{\left[\left\langle \frac{p'(t)}{\left\| p'(t) \right\|}, q(t) \right\rangle \right]_0^L}_{=0} - \int_0^L \left\langle \kappa(t) N(t), q' \right\rangle dt$$

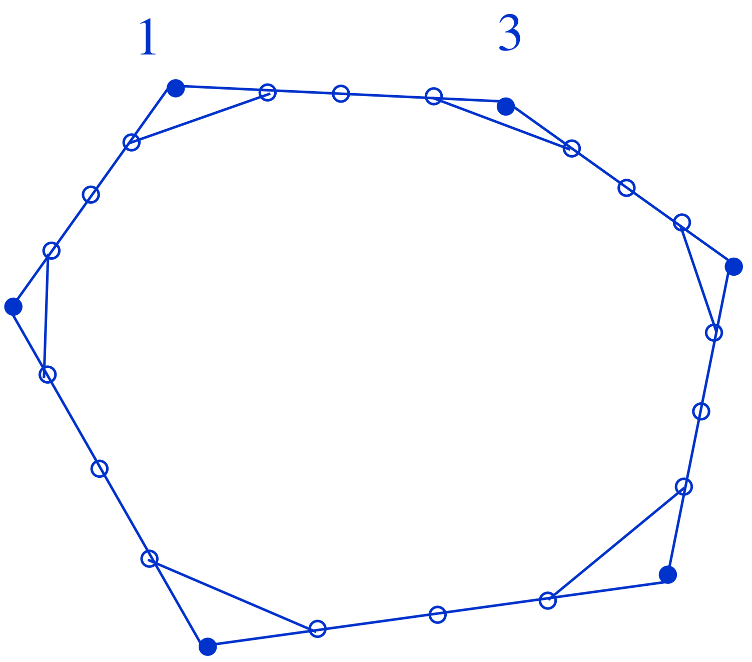
Corner Cutting: (de Rham in the late forties)



Corner Cutting: (de Rham in the late forties)



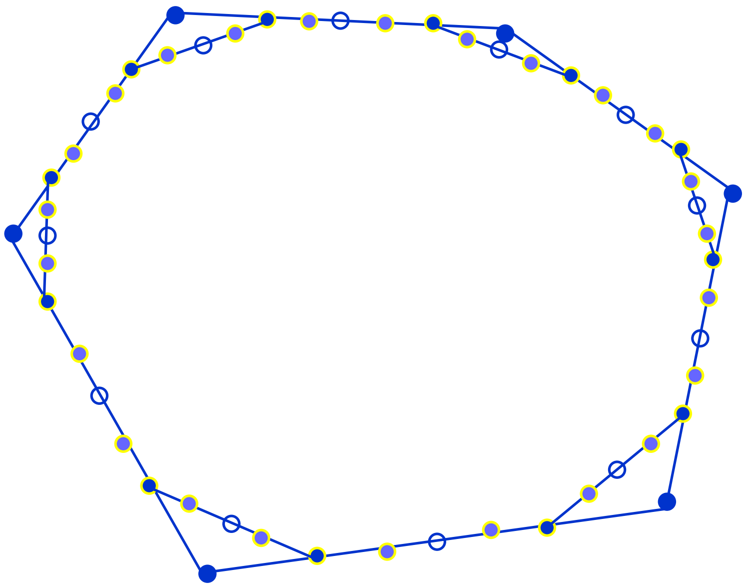
Corner Cutting: (de Rham in the late forties)



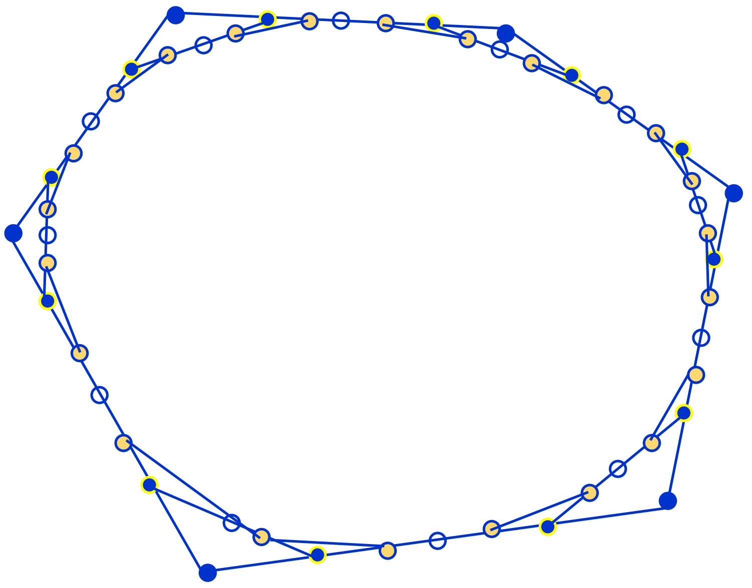
$$\left(\frac{3}{4},\frac{1}{4}\right)$$

$$\left(\frac{1}{4},\frac{3}{4}\right)$$

Corner Cutting: (de Rham in the late forties)



Corner Cutting: (de Rham in the late forties)



Theorem (Riesenfeld 75): Using recursive Corner Cutting the polygon converges towards a quadratic B-spline!

- How to choose subdivision rules ?

1. Efficiency (few operations)
2. Compact support (region of influence of a point should be small).
3. Local definition (far away points should not influence the computation)
4. Affine Invariance (if the polygon is affinely transformed, the curve should transform in the same way.)
5. Simplicity of the rules
6. Differentiablility of the resulting curves

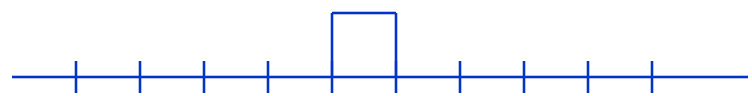
Let us consider a B-spline over a uniform knot vector (can also be generalized to non uniform knot vectors, e.g. Sederberg, Siggraph 98)

1. Piecewise constant functions

$$x(t)=\sum_i d_i N_i^0(t)$$

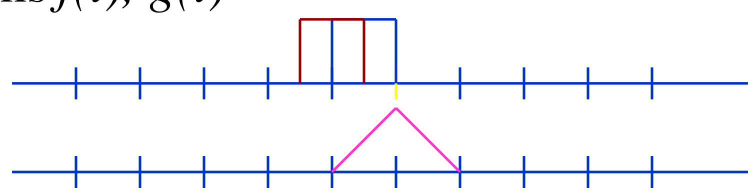
$$N_i^0(t)=N^0(t-i)$$

$$N^0(t)=\begin{cases} 1, & 0\le t<1 \\ 0, & otherwise \end{cases}$$



2. Consider convolution of two functions *f(t), g(t)*

$$f\otimes g(t)=\int f(s)\,g(t-s)\,ds$$

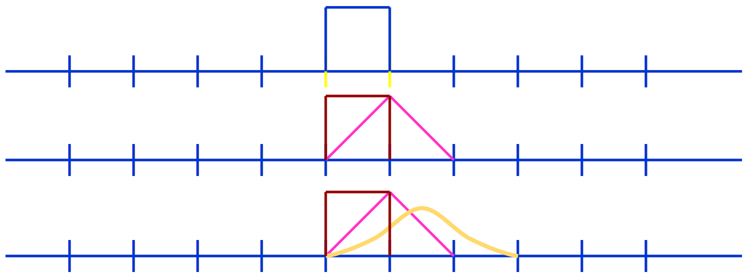


$$g(t-s)=g(-(-t+s)=h(-(t-s))$$

with $h(t)=g(-t)$

Theorem: The B-spline basis function $N^k(t)$ are obtained by convolving the box function $N^0(t)$ with $N^{k-1}(t)$.

Example: $N^1(t) = N^0(t) \otimes N^0(t) = \int N^0(s) N^0(t-s)\,ds$



Theorem: If $f(t)$ is C^k-continuous, then $N^0 \otimes f(t)$ is C^{k+1}-continuous.

Corollary: N^k is C^{k-1}-continuous.

B-spline refinement (key for subdivision splines)

Theorem:

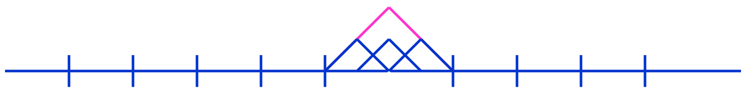
$$N^{l}(t)=\frac{1}{2^{l}}\sum_{k=0}^{l}\binom{l+1}{k}N^{l}(2t-k)$$

i.e. B-splines of degree l can be written as a linear combination of translates *(k)* and dilates *(2t)* of it self.

Example:

$$N^{l}(t)=\frac{1}{2^{1}}\sum_{k=0}^{l+1}\binom{1+1}{k}N^{1}(2t-k)$$

$$=\frac{1}{2}\left(N^{1}(2t)+2N^{1}(2t-1)+N^{1}(2t-2)\right)$$



Proof of the theorem: By induction over degree *(l)*:

$$N^0(t) = N^0(2t) + N^0(2t-1)$$

$$N^l(t) = \otimes_{i=0}^{l} N^0(t) = \otimes_{i=0}^{l} (N^0(2t) + N^0(2t-1))$$

By multiplying using the following rules we get:

$$\mathrm{f}(\mathrm{t}) \otimes (\mathrm{g}(\mathrm{t}) + \mathrm{h}(\mathrm{t})) = \mathrm{f}(\mathrm{t}) \otimes \mathrm{g}(\mathrm{t}) + \mathrm{f}(\mathrm{t}) \otimes \mathrm{h}(\mathrm{t}) \qquad \text{(Linearity)}$$

$$\mathrm{f}(\mathrm{t}-\mathrm{i}) \otimes \mathrm{g}(\mathrm{t}-\mathrm{k}) = \mathrm{m}(\mathrm{t}-\mathrm{i}-\mathrm{k}) \qquad \text{(Time shift)}$$

$$\mathrm{f}(2\mathrm{t}) \otimes \mathrm{g}(2\mathrm{t}) = \frac{1}{2}\mathrm{m}(2\mathrm{t}) \qquad \text{(Time scale)}$$

Let $\gamma(t) = \sum_i d_i N_i^l(t)$
be a B-spline curve of degree *l* over a uniform knot vector. Let

$$\mathbf{d} = \begin{pmatrix} . \\ d_{-2} \\ d_{-1} \\ d_0 \\ d_1 \\ d_2 \\ . \end{pmatrix}$$

An (infinite) vector of control points
(we do not consider start and end here)

And $\mathbf{N^l(t)} = (\dots N^l(t+2)\ N^l(t+1)\ N^l(t)\ N^l(t-1)\ N^l(t-2)\ \dots)$
the vector of basis functions. We then have

$$\gamma(t) = \mathbf{N^l(t)} \times \mathbf{d}$$

Let $\mathbf{N(t)} = (\ldots,\ N(t+2), N(t+1), N(t), N(t-1), N(t-2),\ \ldots)$

$= (\ldots,\quad N_{-2}(t),\quad N_{-1}(t),\quad N_0(t),\quad N_1(t),\quad N_2(t),\quad \ldots)$

the vector of dilates of the basis functions.

we then have $\mathbf{N(t)} = \mathbf{N}(2\mathbf{t}) \times \mathbf{S}$

where S is a matrix whose entries can be computed according to the above theorem (l denotes the degree of the B-spline):

$$S_{2n+k,n} = S_k = \frac{1}{2^l}\binom{l+1}{k},\ k = 0,\ldots,l+1$$

The sequence $s=(\ldots,s_{-1},s_0,s_1,\ldots)$ is called subdivision mask

Example l=1:

$$\mathbf{N(t)} = (\dots N(t+2)\ N(t+1)\ N(t)\ N(t-1)\ N(t-2)\ \dots)$$

$$= (\dots N(2t+2)\ N(2t+1)\ N(2t)\ N(2t-1)\ N(2t-2)\ \dots)\frac{1}{2}\begin{array}{c}\left(\begin{array}{ccccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2 & 0 & 0 & 0 & 0 & \cdot \\ \cdot & 1 & 1 & 0 & 0 & 0 & \cdot \\ \cdot & 0 & 2 & 0 & 0 & 0 & \cdot \\ \cdot & 0 & 1 & 1 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & 2 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & 1 & 1 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & 2 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right) \\ \begin{array}{ccccccc} & & & \uparrow & & & \\ & & & j=0 & & & \end{array}\end{array}\begin{array}{l} \\ \\ \\ \\ \leftarrow\ i=0 \\ \\ \\ \\ \\ \\ \\ \end{array}$$

There are twice as many rows as columns!

Therefore we have:

$$\gamma(t) = \mathbf{N(t)} \times \mathbf{d} = \mathbf{N(2t)} \times \mathbf{S} \times \mathbf{d}$$

Remark:

- The same curve can be written using twice as many B-splines with half the support.
- W.r.t. **N(t)** the curve has the control points **d**.
- W.r.t. **N(2t)** the control points **Sd**.

Proceeding in this way we get:

$$\begin{aligned}\gamma(t) &= \mathbf{N(t)} \times \mathbf{d}^0 \\ &= \mathbf{N(2t)} \times \mathbf{S} \times \mathbf{d}^0 = \mathbf{N(2t)} \times \mathbf{d}^1 \\ &= \mathbf{N(2^j t)} \times \mathbf{S}^j \times \mathbf{d}^0 = \mathbf{N(2^j t)} \times \mathbf{d}^j\end{aligned}$$

with $\mathbf{d}^{j+1} = \mathbf{S}\mathbf{d}^j$,

where S denotes the (infinite) subdivision matrix.

Example l=1:

$$\gamma(t) = \mathbf{N(t)} \times \mathbf{d} = \mathbf{N}(2\mathbf{t}) \times \mathbf{S} \times \mathbf{d}$$

$$= (\ldots N(2t+2)\ N(2t+1)\ N(2t)\ N(2t-1)\ N(2t-2)\ \ldots)\frac{1}{2}\begin{array}{c} \begin{array}{ccccccc} & & & j=0 & & & \\ & & & \downarrow & & & \end{array} \\ \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2 & 0 & 0 & 0 & 0 & \cdot \\ \cdot & 1 & 1 & 0 & 0 & 0 & \cdot \\ \cdot & 0 & 2 & 0 & 0 & 0 & \cdot \\ \cdot & 0 & 1 & 1 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & 2 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & 1 & 1 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & 2 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \end{array} \begin{pmatrix} \cdot \\ d^0_{-2} \\ d^0_{-1} \\ d^0_0 \\ d^0_1 \\ d^0_2 \\ \cdot \end{pmatrix}$$

$$= (\ldots N(2t+2)\ N(2t+1)\ N(2t)\ N(2t-1)\ N(2t-2)\ \ldots)\frac{1}{2}\begin{pmatrix} \cdot \\ d^0_{-2} + d^0_{-1} \\ 2d_{-1} \\ d^0_{-1} + d^0_0 \\ 2d_0 \\ d^0_0 + d^0_1 \\ \cdot \end{pmatrix}$$

Matrix multiplication yields:

$$d_i^{j+1} = \sum_m s_{i,m} d_m^j$$

$$S_{2n+k,n} = S_k = \frac{1}{2^l}\binom{l+1}{k}, \; k = 0, \ldots, l+1$$

Therefore we have for the even and odd entries:

odd:

$$d_{2i+1}^{j+1} = \sum_m s_{2i+1,m} d_m^j = \sum_m s_{2(i-m)+1} d_m^j$$

even:

$$d_{2i}^{j+1} = \sum_m s_{2i,m} d_m^j = \sum_m s_{2(i-m)} d_m^j$$

$$\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & s_{-1-1}^{1} & s_{-10}^{-1} & s_{-11}^{-3} & s_{-12}^{-5} & s_{-13}^{-7} & \cdot \\
\cdot & s_{0-1}^{2} & s_{00}^{0} & s_{01}^{-2} & s_{02}^{-4} & s_{03}^{-6} & \cdot \\
\cdot & s_{1-1}^{3} & s_{10}^{1} & s_{11}^{-1} & s_{12}^{-3} & s_{13}^{-5} & \cdot \\
\cdot & s_{2-1}^{4} & s_{20}^{2} & s_{21}^{0} & s_{22}^{-2} & s_{23}^{-4} & \cdot \\
\cdot & s_{3-1}^{5} & s_{30}^{3} & s_{31}^{1} & s_{32}^{-1} & s_{33}^{-3} & \cdot \\
\cdot & s_{4-1}^{6} & s_{40}^{4} & s_{41}^{2} & s_{42}^{0} & s_{43}^{-2} & \cdot \\
\cdot & s_{5-1}^{7} & s_{50}^{5} & s_{51}^{3} & s_{52}^{1} & s_{53}^{-1} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}$$

Spline Refinement

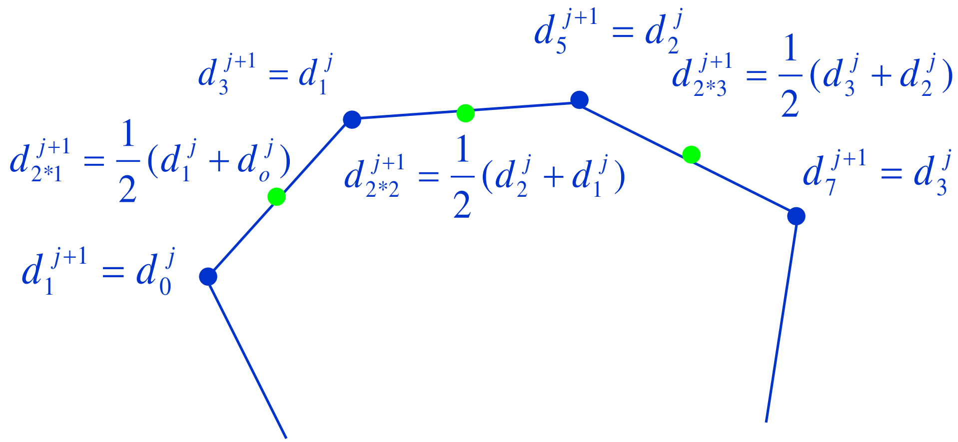
i.e. we have different subdivision rules for even and odd numbered control points!

Example: piecewise linear, l=1, k=0,1,2

odd:

$$d_{2i+1}^{j+1} = \sum_m s_{2i+1,m} d_m^j = \sum_m s_{2(i-m)+1} d_m^j$$

$$(i = m \Rightarrow k = 1,\ \text{d.h.}\, d_{2i+1}^{j+1} = d_i^j)$$



$$d_{2i}^{j+1} = \sum_m s_{2i,m} d_m^j = \sum_m s_{2(i-m)} d_m^j$$

$$(m = i \Rightarrow k = 0, m = (i-1) \Rightarrow k = 2,\ \text{d.h.}\ d_{2i}^{j+1} = \frac{1}{2} d_i^j + \frac{1}{2} d_{i-1}^j$$

Even numbered points on the $j+1$th level are newly generated,
Odd numbered points already exist on the jth level.

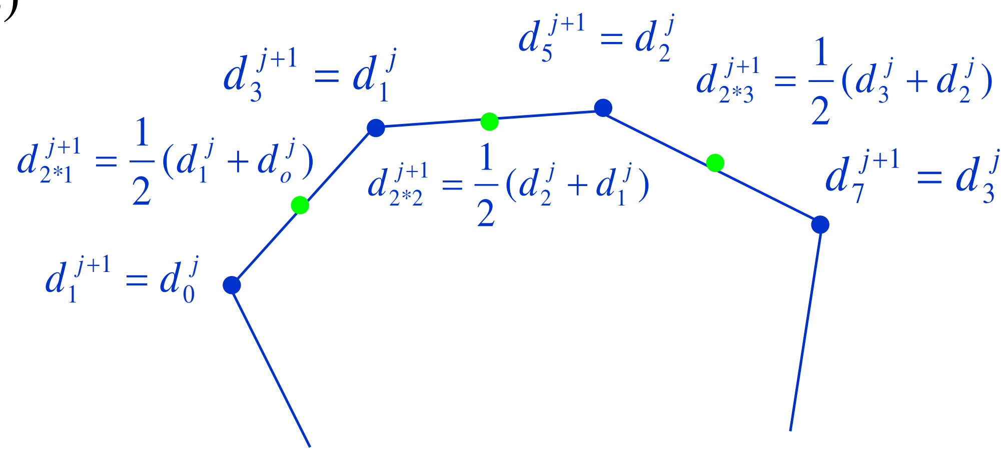
Subdivision schemes with this property are called interpolating.
Schemes that change all points in each step are called approximating.

1. Splitting (Insertion of mid-points)

$$(\ldots,{}^0 d_{-1}^{j+1},{}^0 d_0^{j+1},{}^0 d_1^{j+1},\ldots)$$

$${}^0 d_{2i+1}^{j+1} := d_i^j$$

$${}^0 d_{2i}^{j+1} := \frac{1}{2}(d_i^j + d_{i-1}^j)$$



2. Averaging

$$d_i^{j+1} := \sum_k s_k' \, {}^0 d_{i+k}^{j+1}$$

Averaging Mask

Example: Corner Cutting (quadratic B-splines)

$$S_{2i-k,i} = S_k = \frac{1}{2^l}\binom{l+1}{k}, \quad k = 0,\dots,l+1$$

$$s = (\dots,0,s_0,s_1,s_2,s_3,0,\dots)$$

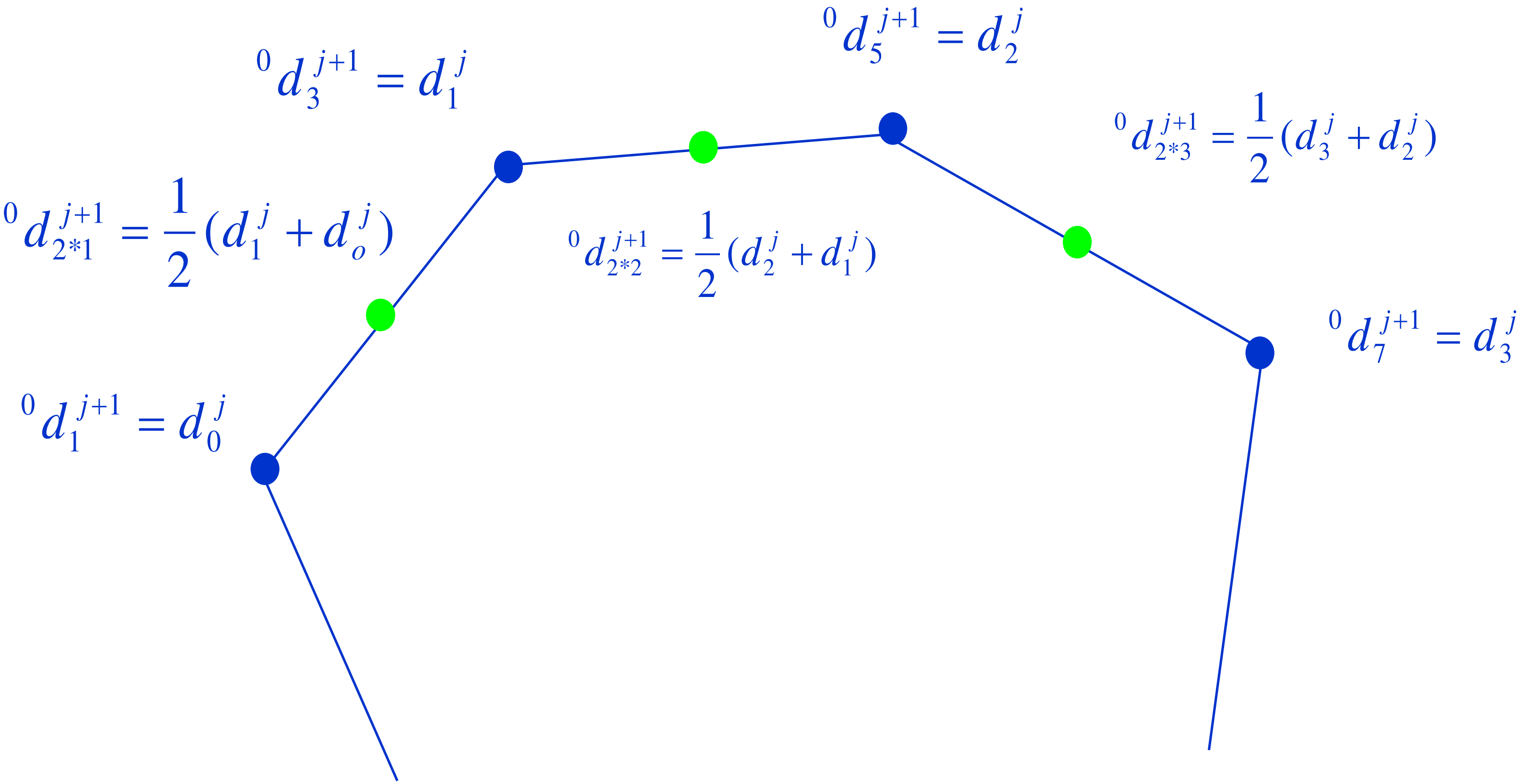
$$= \frac{1}{4}(\dots,0,1,3,3,1,0,\dots)$$

$$S = \frac{1}{4}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 3 & 1 & 0 & 0 & 0 & \cdot \\ \cdot & 1 & 3 & 0 & 0 & 0 & \cdot \\ \cdot & 0 & 3 & 1 & 0 & 0 & \cdot \\ \cdot & 0 & 1 & 3 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & 3 & 1 & 0 & \cdot \\ \cdot & 0 & 0 & 1 & 3 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & 3 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$d_i^{j+1} := \frac{1}{4}(3d_{i-1}^j + d_i^j) \qquad d_{i+1}^{j+1} := \frac{1}{4}(d_{i-1}^j + 3d_i^j)$$

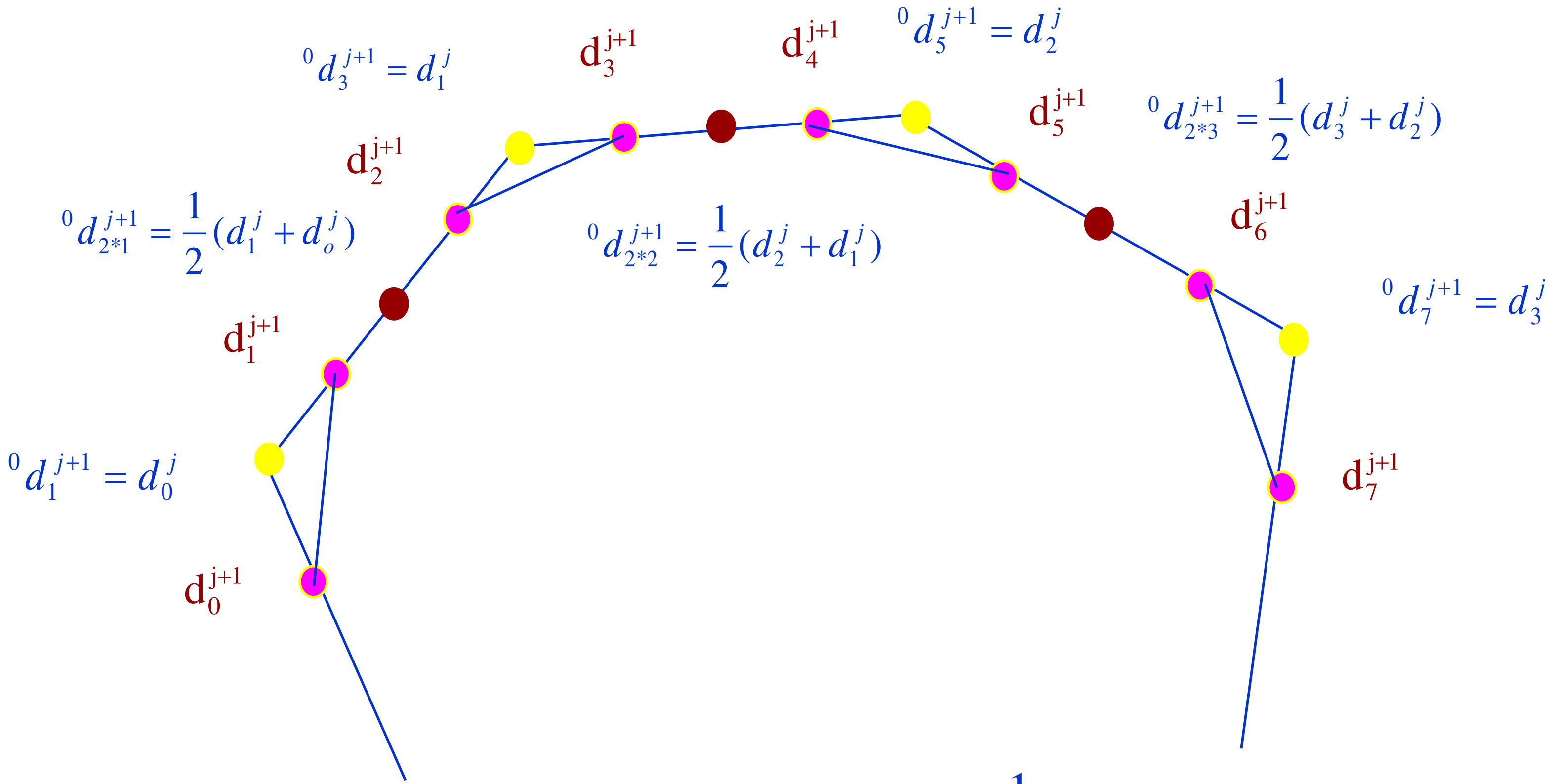
Mask for even points $\frac{1}{4}(\dots,0,3,1,0,\dots)$ Mask for odd points $\frac{1}{4}(\dots,0,1,3,0,\dots)$

In order to indicate the index, the mask is written as (s_{-1},s_0)



${}^{0}d_{3}^{j+1} = d_{1}^{j}$
${}^{0}d_{5}^{j+1} = d_{2}^{j}$
${}^{0}d_{2*3}^{j+1} = \frac{1}{2}(d_{3}^{j} + d_{2}^{j})$
${}^{0}d_{2*1}^{j+1} = \frac{1}{2}(d_{1}^{j} + d_{o}^{j})$
${}^{0}d_{2*2}^{j+1} = \frac{1}{2}(d_{2}^{j} + d_{1}^{j})$
${}^{0}d_{7}^{j+1} = d_{3}^{j}$
${}^{0}d_{1}^{j+1} = d_{0}^{j}$

Spline Refinement



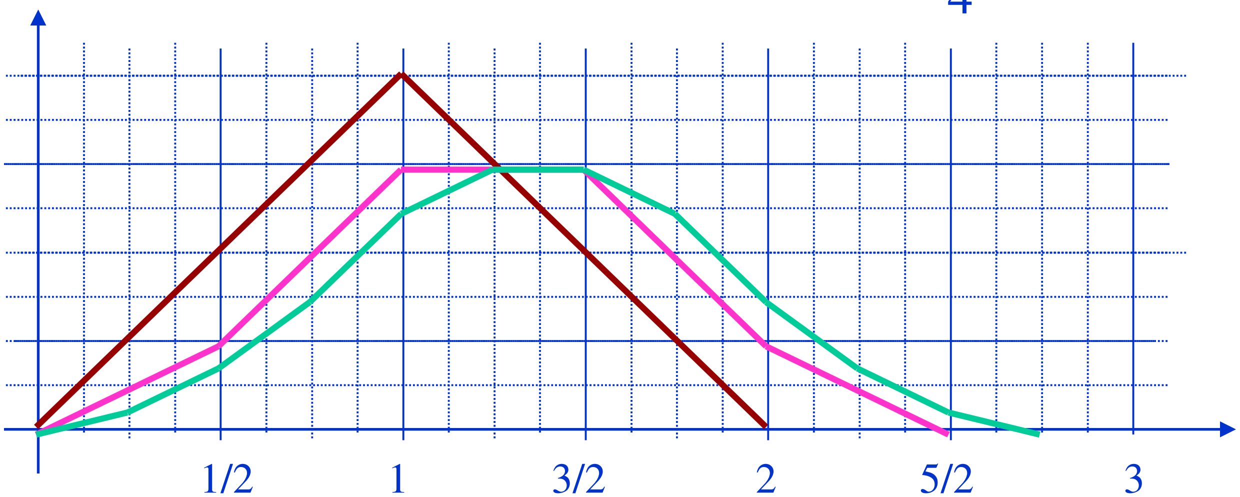
Averaging Maske for even and odd points $d_i^{j+1} = \frac{1}{2}({}^0d_i^{j+1} + {}^0d_{i-1}^{j+1})$

$(\ldots,0,1,1,0,\ldots)$

Fundamental solution: 1D

- Example: quadratic B-Spline

$$d_i^{j+1} := \frac{1}{4}(3d_{i-1}^j + d_i^j) \qquad d_{i+1}^{j+1} := \frac{1}{4}(d_{i-1}^j + 3d_i^j)$$



$$P^0(t) = N^1(t)S^0d^0$$

$$P^1(t) = N^1(2t)S^1d^0$$

$$P^2(t) = N^1(4t)S^2d^0$$

$$\mathrm{d}^0 = (\cdots, \underset{0}{1}, \cdots)$$

$$\mathrm{S}^1\mathrm{d}^0 = \mathrm{d}^1 = \left(\cdots, \underbrace{1/4}_{0}, \underbrace{3/4}_{1}, \underbrace{3/4}_{2}, \underbrace{1/4}_{3}, \cdots\right)$$

$$\mathrm{S}^2\mathrm{d}^0 = \mathrm{Sd}^1 = \left(\cdots, \underbrace{1/16}_{0}, 3/16, 6/16, 10/16, 12/16, 12/16, 10/16, 6/16, 3/16, \underbrace{1/16}_{9}, \cdots\right)$$

- For a convergent subdivision scheme there is the limit

$$\lim_{j\to\infty} P^j(t) = \sum_i d_i^0 \lim_{j\to\infty} \Phi_i^j(t) = \sum_i d_i^0 \Phi_i(t)$$

- The limit curve is a linear combination of points d_i^0 with weights
- The functions $\Phi_i(t) := \lim_{j\to\infty} \Phi_i^j(t)$ fulfill the relation $\Phi_i(t) = \Phi(t-i)$
- Therefore, there is a function $\Phi_i(t)$, such that all subdivision curves with initial points d_i^0 are linear combinations of the points d_i^0 with weights $\Phi(t-i)$
- The function $\Phi_i(t)$ is called fundamental solution of the subdivision scheme.

- Remark: There are subdivision schemes that do not convert towards a limit function. For example using the Averaging-Maske

$$(s_0, s_1) = \frac{1}{2}(1+\sqrt{3}, 1-\sqrt{3})$$

results in fractal-like curves, which are nowhere differentiable.

- Questions:
  - How can we build suitable subdivision masks? (for example from known schemes)
  - Which subdivision masks result in continuous or differentiable curves?

Warren, Joe, and Henrik Weimer. *Subdivision methods for geometric design: A constructive approach*. Elsevier, 2001.
Peters, Jörg, and Ulrich Reif. *Subdivision surfaces*. Springer Berlin Heidelberg, 2008.
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Andersson, Lars-Erik, and Neil F. Stewart. *Introduction to the mathematics of subdivision surfaces*. Society for Industrial and Applied Mathematics, 2010.

- Determing local properties-invariant neighborhoods
  - Which control points influence the curve around $t=t_0$?
  - What are the basis functions of a given, arbitrary subdivision scheme
  - Which basis functions influence the curve at a certain parameter value?
  - How can we ceck if a tangent exists for a certain point on the limit curve?
  - How can we compute tangents at a certain point?