Thomas Kesselheim
Rebecca Lehming

April 9, 2025
Due date: April 16, 2025 at 2:00 p.m.

Algorithmic Game Theory

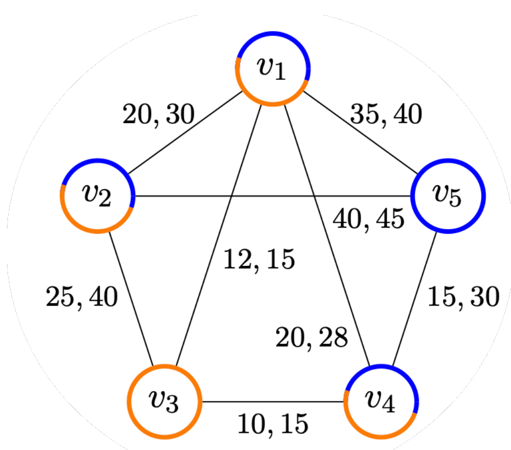
Summer Term 2025

Exercise Set 1

If you want to hand in your solutions for this problem set, please send them via email to rlehming@uni-bonn.de *by Tuesday evening – make sure to send a pdf-file which contains your name and email address. Of course, submitting solutions in groups is also possible.*

If you would like to present one of the solutions in class, please also send an email to rlehming@uni-bonn.de *containing the* ***task*** *which you would like to present and in* ***which of the tutorials*** *you would like to do so. Deadline for the email is Tuesday, 10:00 pm. Please note that the tasks will be alloecated on a first-come-first-served basis, so sending this email earlier than Tuesday evening is highly recommended.*

Exercise 1:
A connection game is a congestion game with n agents and an undirected graph $G = (V, E)$. Every agent i is associated with a subset of vertices $V_i \subseteq V$. The set of strategies Σ_i consists of all connected, acyclic subgraphs G'_i with $V'_i = V_i$ and $E'_i \subseteq (E \cap (V_i \times V_i))$, for every player i. Every edge e is assigned a delay function $d_e(n_e) : \{1, ..., n\} \to Z$, where n_e is the number of agents i selecting a subgraph G'_i with $e \in E'_i$.

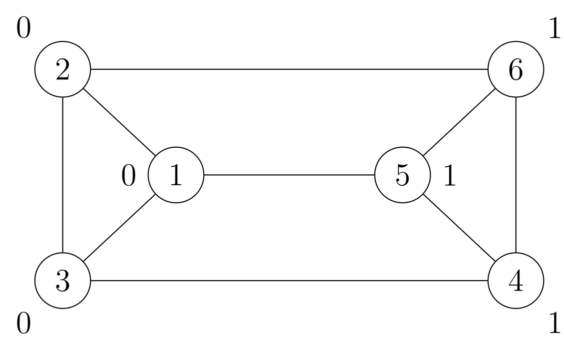


a) Consider the above instance of a connection game with two players. The vertices in V_1 are indicated in orange, while the vertices in V_2 are marked in blue.
Let the initial strategy of player 1 be given by the subgraph G'_1 with edges $E'_1 = \{\{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}\}$.
Player 2 chooses subgraph $G'_2$with $E'_2 = \{\{v_1, v_5\}, \{v_2, v_5\}, \{v_4, v_5\}\}$ as his strategy. Perform best-response improvement steps until a pure Nash equilibrium is reached. Player 1 should deviate first.

b) Prove that every sequence of best-response improvement steps in a connection game converges in $O(n^2 \cdot |E| \cdot |V|)$ many steps.
Hint: *You can use the following property without proving it.*
Let G' be the strategy of agent i in state S, and let G'' be a best response of i for S_{-i}. Then, there exists a transforming sequence from G' to G'', where in every step, one edge $e' \in (E' \setminus E'')$ is exchanged by an edge $e'' \in (E'' \setminus E')$. For each step, the resulting graph is a feasible strategy for agent i. In particular, the delay is (weakly) reduced in every step.

Exercise 2:
In a *consensus game*, we are given an undirected graph $G = (V, E)$ with vertex set $V = \{1, \ldots, n\}$. Each vertex $i \in V$ is a player and her action consists of choosing a bit $b_i \in \{0, 1\}$. Let $N(i) = \{j \in V \mid \{i, j\} \in E\}$ denote the set of neighbours of player i, i.e., all players j connected to i via an edge. Furthermore, let $\mathbf{b} = (b_1, \ldots, b_n)$ be the vector of players' choices. The loss $D_i(\mathbf{b})$ for player i is the number of neighbours that she disagrees with, i.e.,

$$D_i(\mathbf{b}) = \sum_{j \in N(i)} |b_j - b_i| \, .$$



a) Calculate the loss D_i of player 1 for the actions depicted in the graph above.

b) Show that a consensus game represented as an undirected Graph G can also be modeled as a congestion game Γ. To this end, specify the tuple $\Gamma = (N, R, (\Sigma_i)_{i \in N}, (d_r)_{r \in R})$ and show that the loss D_i coincides with the cost c_i.

c) Prove that in a congestion game modeling a consensus game with $|V| = n$ players all improvement sequences have length $O(n^2)$.