DCG, Wintersemester 2024/24 Lecture 4 (5 pages)

Convex Polytopes

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In order to compute convex hulls for points in $\mathbb{R}^2$ and higher dimensions, we need to look at the combinatorial structure of convex hulls and how to represent such sets efficiently.

1 Duality

We first discuss a very useful concept that allows us to visualize the set of lines intersecting a convex polygon and which will help us understanding the combinatorial structure of convex hulls in higher dimensions.

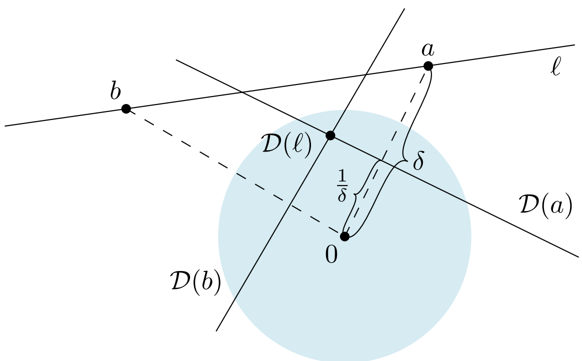
Definition 4.1 (Duality transform)**.** *Let $\mathcal{D}$ be a mapping that assigns to a point $a \in \mathbb{R}^d \setminus \{0\}$ the hyperplane*

$$h_a = \left\{ x \in \mathbb{R}^d \;\middle|\; \langle a, x \rangle = 1 \right\}$$

and to a hyperplane $h_a = \left\{ x \in \mathbb{R}^d \;\middle|\; \langle a, x \rangle = 1 \right\}$ not passing through the origin, it assigns the point

$$a \in \mathbb{R}^d \setminus \{0\}$$

What is the geometric interpretation of the duality transform? If $a \in \mathbb{R}^d$ is a point that lies at distance $\delta > 0$ from the origin, then $\mathcal{D}(a)$ is a hyperplane perpendicular to the line from 0 to a and intersecting that line at distance $1/\delta$ from the origin. Moreover, we can observe that the duality transform preserves incidences between points and hyperplanes, as $\langle a, p \rangle = 1$ ("the point p lies on the hyperplane h_a") is equivalent to $\langle p, a \rangle = 1$ ("the point $\mathcal{D}(h_a) = a$ lies on the hyperplane $\mathcal{D}(p) = h_p$"). As a consequence, for two different points $a, b \in \mathbb{R}^2$ contained in a line ℓ that does not pass through the origin, the dual point $\mathcal{D}(\ell)$ is the intersection point of the two dual lines $\mathcal{D}(a)$ and $\mathcal{D}(b)$. This is illustrated in the figure below, where the unit circle is shown for reference.



Definition 4.2 (Dual set)**.** *For a hyperplane $h = \left\{ x \in \mathbb{R}^d \;\middle|\; \langle a, x \rangle = 1 \right\}$ let h^- denote the closed halfspace bounded by h and containing the origin. That is,*

$$h^- = \left\{ x \in \mathbb{R}^d \;\middle|\; \langle a, x \rangle \leq 1 \right\}$$

For a set $X \subseteq \mathbb{R}^d$, we define the set dual to X, denoted by X^*, as follows:

$$X^* = \left\{ y \in \mathbb{R}^d \;\middle|\; \forall a \in X : \langle a, y \rangle \leq 1 \right\}$$

Note that for any $X \subseteq \mathbb{R}^d$ the dual set X^* trivially contains 0.

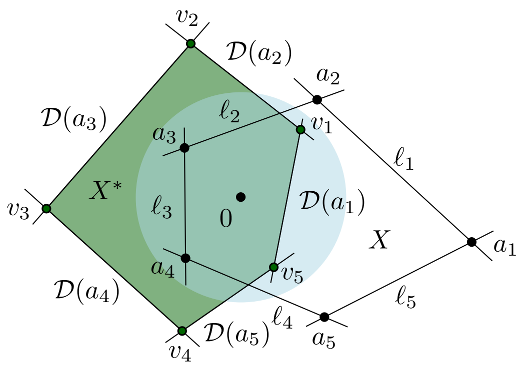
Example 4.3. *For a single point $a \in \mathbb{R}^d$, the dual set $\{a\}^*$ is the closed halfspace which is bounded by the dual hyperplane $\mathcal{D}(a)$ and which contains the origin. We call $\{a\}^*$ the halfspace dual to a and we denote it with $\mathcal{D}(a)^-$.*

Observation 4.4. *Let $p \in \mathbb{R}^d \setminus \{0\}$ and let h be a hyperplane not passing through the origin.*

$$p \in h^- \quad \Leftrightarrow \quad \mathcal{D}(h) \in \mathcal{D}(p)^-$$

In other words, for a point p that is contained in a halfspace h^-, we have that the dual point $\mathcal{D}(h)$ is contained in the dual halfspace $\mathcal{D}(p)^-$.

Example 4.5. *Consider a concrete example of points $a_1, \ldots, a_5 \in \mathbb{R}^2$ (see figure below). Let $X = \operatorname{conv}(a_1, \ldots, a_5) \subseteq \mathbb{R}^2$. Let $\ell_1, \ldots, \ell_5$ be the lines supporting the edges of the convex hull and let $v_i = \mathcal{D}(\ell_i)$. The dual set X^* is the intersection of dual halfspaces $\mathcal{D}(a_i)^-$ and this set is equal to $\operatorname{conv}(v_1, \ldots, v_5)$.*



2 Convex polytopes

There are two equivalent ways to define a convex polytope. We will not be concerned with non-convex polytopes. Therefore, we will often drop the adjective "convex" and simply refer to them as polytopes.

Definition 4.6 (H-polytope and V-polytope)**.** *An H-polyhedron is an intersection of finitely many closed halfspaces in $\mathbb{R}^d$, it is a H-polytope if it is bounded. A V-polytope is the convex hull of a finite point set in $\mathbb{R}^d$.*

Claim 4.7. *Each V-polytope is an H-polytope and each H-polytope is a V-polytope.*

Before we look at a proof of this claim, we give examples of convex polytopes and their representation as V-polytopes and as H-polytopes.

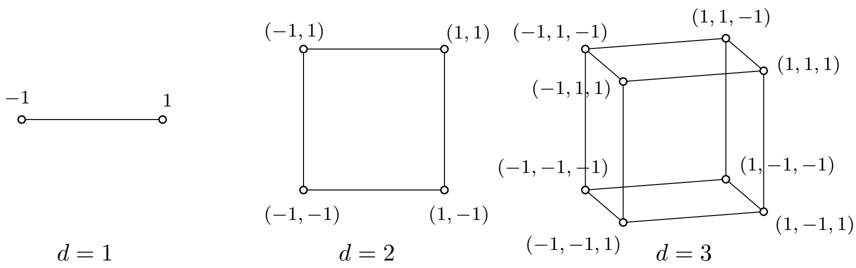
Example 4.8 (Hypercubes)**.** *The d-dimensional hypercube is the Cartesian product $[-1,1]^d$. We can represent this set as a V-polytope as the convex hull of 2^d points, as follows*

$$\operatorname{conv}(\{-1,1\}^d)$$

We can represent this set also as an H-polytope using the $2d$ halfspaces

$$h_i^- = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \;\middle|\; -1 \leq x_i \right\} \quad h_i^+ = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \;\middle|\; x_i \leq 1 \right\}$$

for all $i = 1, \ldots, d$.



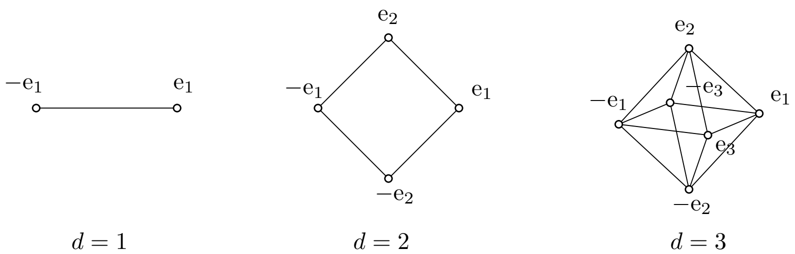
Example 4.9 (Crosspolytopes)**.** *The d-dimensional crosspolytope is the convex hull of the "coordinate cross". Let $\mathrm{e}_1, \ldots, \mathrm{e}_d$ denote the vectors of the standard orthonormal basis. We can represent the crosspolytope as a V-polytope:*

$$\operatorname{conv}(\mathrm{e}_1, -\mathrm{e}_1, \mathrm{e}_2, -\mathrm{e}_2, \ldots, \mathrm{e}_d, -\mathrm{e}_d)$$

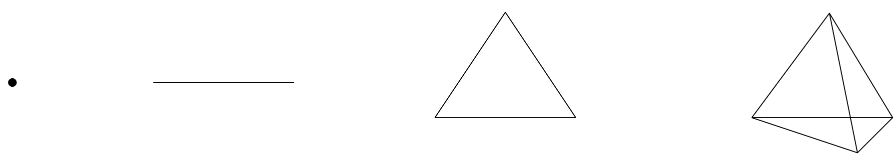
We can represent it as a H-polytope using the set $X = \{-1, 1\}^d$ using 2^d halfspaces of the form

$$h = \left\{ \; x \in \mathbb{R}^d \;\middle|\; \langle a, x \rangle \leq 1 \; \right\}$$

for all $a \in X$.



Example 4.10 (Simplices)**.** *A simplex is the convex hull of an affinely independent point set in some $\mathbb{R}^d$. Therefore, it is a V-polytope. We can also represent any d-dimensional simplex as a H-polytope. ($\rightarrow$ Exercise)*



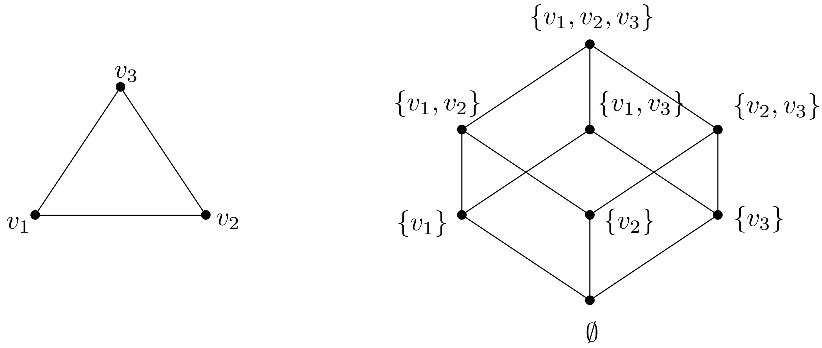
Definition 4.11 (Faces of a polytope)**.** *A face of a convex polytope P is defined as*

(i) either P itself, or

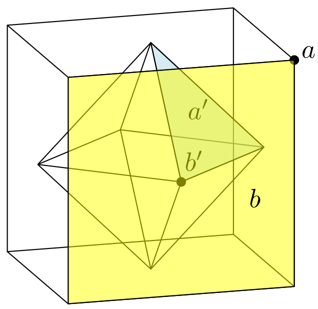
(ii) a subset of P of the form $P \cap h$ where h is a hyperplane such that P is fully contained in one of the closed halfspaces determined by h.

The dimension of a face is the dimension of the affine hull of this face. A face of dimension j is called a j-face. Some faces have special names: We call the 0-faces the vertices, the 1-faces the edges and the $(d-1)$-faces of a d-dimensional polytope the facets of the polytope. For technical reasons, we also define for each polytope, a (-1)-dimensional face, which we define as the empty set.

Definition 4.12 (Face lattice)**.** *Let $\mathcal{F}(P)$ be the set of all faces of a convex polytope P (including the empty face $\emptyset$ of dimension -1). The graph representing the partial ordering of $\mathcal{F}(P)$ by inclusion is called the face lattice.*



Claim 4.13. *For each $j = -1, 0, \ldots, d$ the j-faces of P are in bijective correspondence with the $(d - j - 1)$-faces of P^*. This correspondence also reverses inclusion; in particular, the face lattice of P^* arises by turning the face lattice of P upside down.*



Example 4.14. *Let P be the hypercube given as a V-polytope of the vertex set $V = \{-1, 1\}^3$. Its dual is the crosspolytope given by the intersection of halfspaces*

$$h = \left\{ \ y \in \mathbb{R}^3 \ \middle| \ \langle y, v\rangle \leq 1 \ \right\}$$

for $v \in V$. Consider the vertex $a = (1, 1, 1)$ of P. The corresponding face in P^* is

$$a' = \left\{ \ y \in \mathbb{R}^3 \ \middle| \ \langle y, a\rangle = 1 \ \right\} \cap P^*$$

The vertex a is contained in the face b given by

$$b = \left\{ \ (x_1, x_2, x_3) \in \mathbb{R}^3 \ \middle| \ x_2 = 1 \ \right\} \cap P$$

The face of P^* corresponding to b is the vertex $b' = (0, 1, 0)$. Clearly, b' is contained in a', and thus the inclusion relation is reversed.

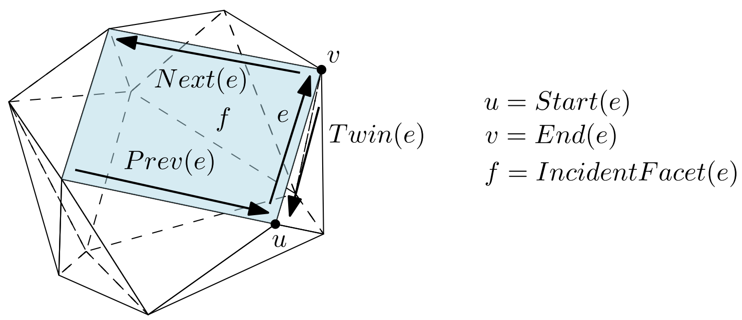
3 Doubly-connected edge list (DCEL)

We will also have to store the incidence relationships between vertices and edges when computing convex hulls in $\mathbb{R}^3$. In principle, it would be possible to use the graph of the face lattice. However, this would be very inefficient, since the graph contains a lot of redundancy. For the special case of storing polytopes in $\mathbb{R}^3$, we can instead use the so-called doubly-connected edge list (DCEL). This data structure is especially useful for performing local operations on the polytope during the update operation. The name derives from the simpler doubly-linked list, which is what we used to store the convex hull in $\mathbb{R}^2$.

The doubly-connected edge list stores the incidence relations between the vertices, edges and facets of the polytope in the following way. The main idea of the data structure is to store for every edge two so-called *half-edges*, which are two copies of the same edge, one for each direction. We associate a half-edge with the incident facet that lies to the left of the directed edge, when viewed from the outside of the polytope. Half-edges are connected in counter-clockwise order along the boundary of their incident facets.

Concretely, we store the following attribute information and pointers with every edge, vertex, and facet.

- For every edge connecting two vertices u and v, we store two half-edges, one for each direction. A half-edge e from u to v stores the following pointers
  - $IncidentFacet(e)$: the incident facet f,
  - $Prev(e)$: the previous half-edge on the boundary of f,
  - $Next(e)$: the next half-edge on the boundary of f,
  - $Twin(e)$: the half-edge from v to u,
  - $Start(e)$, $End(e)$: pointing to the vertex u, and the vertex v.
- For every vertex v, we store
  - $Coordinates(v)$: the coordinates of the vertex v, and
  - $IncidentEdge(v)$: a pointer to an arbitrary half-edge that has v as its origin.
- For every facet f, we store
  - $Normal(f)$: a vector that is orthogonal to the supporting hyperplane of f, pointing to the outside of the polytope, and
  - $IncidentEdge(f)$ a pointer to an arbitrary half-edge on the boundary of f.



References

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- Mark de Berg, Otfried Cheong, Marc van Kreveld, Mark Overmars. Computational Geometry— Algorithms and Applications. Third Edition. Springer. Chapter 2.2