DCG, Wintersemester 2023/24	Lecture 14 (6 pages)

The number of vertices of level at most k

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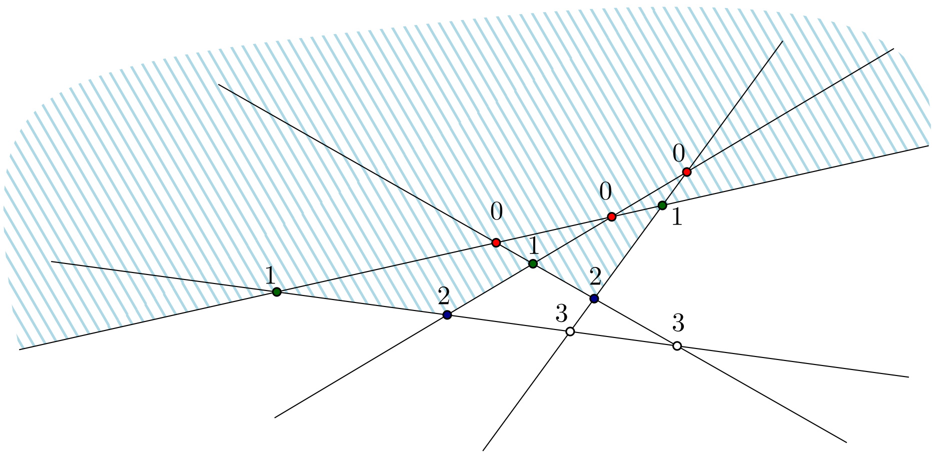
Recall the definition of the level of a point in an hyperplane arrangement, which we encountered when discussing higher-order Voronoi diagrams.

Definition 14.1 (Level)**.** *Let H be a finite set of hyperplanes in $\mathbb{R}^d$ and assume that none of them is vertical (parallel to the x_d-axis). The level of a point $x \in \mathbb{R}^d$ is the number of hyperplanes lying strictly above x.*



Example 14.2. *Example of the set of points of level at most 2 in a simple arrangement of 5 lines. In the figure above, the vertices of level 0 are colored in red, the vertices of level 1 are colored in green, and the vertices of level 2 are colored in blue. The remaining vertices are colored in white. The points in the shaded area are of level at most 2.*

1 Clarkson's theorem on levels

In this lecture, we want to show an upper bound for the number of vertices of level at most k. In particular, we want to show the following theorem by Clarkson.

Theorem 14.3 (Clarkson)**.** *The total number of vertices of level at most k in an arrangement of n hyperplanes in $\mathbb{R}^d$ for any fixed d is at most*

$$O(n^{\lfloor \frac{d}{2} \rfloor}(k+1)^{\lceil \frac{d}{2} \rceil})$$

Before we prove the theorem, we want to consider how tight this bound would be. For $k = 0$, the number of vertices of level at most k corresponds to the complexity of the intersection of n hyperplanes in $\mathbb{R}^d$. By the upper bound theorem we know that this is in $O(n^{\lfloor d/2 \rfloor})$ and this bound is tight, so for $k = 0$ the bound in Theorem 14.3 is tight. Now, consider the case that k is large, and in particular assume $k > d$ and assume that k is a multiple of d and that n is a multiple of $\frac{k}{d}$. We can construct an arrangement with many vertices of level at most k as follows. We start from an arrangement A of $\frac{nd}{k}$ hyperplanes with $\Omega((\frac{nd}{k})^{\lfloor d/2 \rfloor})$ vertices of level 0. We then replace each hyperplane with a group of $\frac{k}{d}$ hyperplanes that are parallel and very close to each other, such that each vertex of A gives rise to a group of $(\frac{k}{d})^d$ vertices. Consider a vertex v in such a group, where the generating vertex u of A was at level 0. There are most k hyperplanes that lie above it in the new arrangement, namely the hyperplanes of the d groups of $\frac{k}{d}$ hyperplanes that determined u in A. Therefore, we get $\Omega((\frac{nd}{k})^{\lfloor d/2 \rfloor}(\frac{k}{d})^d)$ vertices of level at most k, which simplifies to $\Omega(n^{\lfloor d/2 \rfloor}(\frac{k}{d})^{\lceil d/2 \rceil})$.

Next, we want to prove Theorem 14.3 for $d = 2$. We will use the following basic lemma.

Lemma 14.4. $1 - x \geq \mathrm{e}^{-2x}$ *for* $x \in [0, \frac{1}{2}]$

Proof. Let $f(x) = 1 - x - \mathrm{e}^{-2x}$. We need to show that $f(x) \geq 0$ for $x \in [0, \frac{1}{2}]$. It holds that $f(0) = 0$, $f(1/2) = 1/2 - 1/\mathrm{e}$. In order to check the values in between, we take the derivative

$$f'(x) = 2\mathrm{e}^{-2x} - 1.$$

For $x \leq \frac{\ln 2}{2}$ we have $f'(x) \geq 0$ and for $x \geq \frac{\ln 2}{2}$ we have $f'(x) \leq 0$. Therefore, the function f stays non-negative in the interval $[0, \frac{1}{2}]$. □

2 Simple arrangements of lines in the plane

We start by showing the theorem in the simplified setting of lines in the plane.

Let H be a set of n lines in the plane, such that no three lines intersect in the same point (see also Definition 13.2), and none of them is vertical. Let $p \in (0, 1)$ be a real value (we use p as a parameter of the construction). Choose a subset $R \subseteq H$ by sampling each line in H indepently at random with probability p.

Let X denote the number of vertices of level 0 of the arrangement of lines in the sample R. The set of points of level 0 is the top face of the arrangement (the only face that contains a vertical ray that approaches $+\infty$ in the y-coordinate). Any line of R can contribute at most one edge to the boundary of the top face. Therefore, the number of vertices of level 0 is smaller than $|R|$, for any R. Since R is random, X is a random variable and we can consider its expectation. We estimate the expectation of X in two ways. Since $X \leq \left|R\right|$ for any R, we have

$$\mathbf{E}\left[X\right] \leq \mathbf{E}\left[\left|R\right|\right] = p \cdot n \tag{1}$$

Now, let V be the vertices of the arrangement of the full set H. For each $v \in V$, let A_v denote the event that v is a vertex of level 0 in the arrangement of R. We can determine the number of vertices of level 0 by counting the vertices for which the event occurs. For this we define an indicator function I. Let $I(A_v)$ be 1 if the event A_v occurs and 0 otherwise. It holds that

$$X = \sum_{v \in V} I(A_v) \tag{2}$$

Now, we analyze the probability that the event A_v happens. A_v occurs if and only if the following conditions are satisfied.

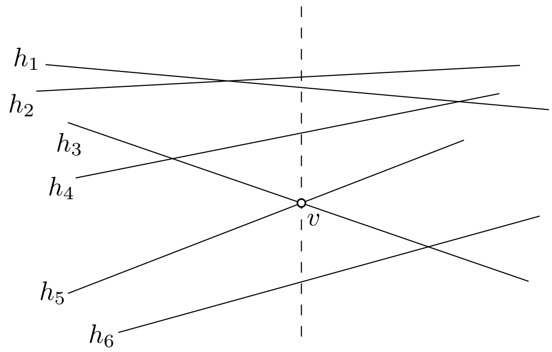
(i) Both lines determining the vertex v are included in the sample R.

(ii) None of the lines of H passing above v are included in the sample R.

Example 14.5. *The figure above illustrates the conditions for a vertex v to be at level 0 of the random sample R: (i) Lines h_3 and h_5 have to be included in the sample R; (ii) Lines h_1, h_2, h_4 are not included in the sample R.*

Let $\ell(v)$ denote the level of v in the arrangement of the full set H. By the above,

$$\mathbf{Pr}\left[A_v\right] = p^2 \cdot (1 - p)^{\ell(v)}$$



Let $V_{\leq k} \subseteq V$ be the set of vertices of level at most k in the arrangement of the full set H.

$$\begin{aligned}\mathbf{E}[X] &= \sum_{v \in V} \mathbf{Pr}[A_v] \\ &\geq \sum_{v \in V_{\leq k}} \mathbf{Pr}[A_v] \\ &= \sum_{v \in V_{\leq k}} p^2 \cdot (1-p)^{\ell(v)} \\ &\geq \sum_{v \in V_{\leq k}} p^2 \cdot (1-p)^k \\ &= \left|V_{\leq k}\right| \left(p^2 \cdot (1-p)^k\right)\end{aligned}$$

Combining this with (1), we obtain

$$\left|V_{\leq k}\right| \leq \frac{n \cdot p}{p^2 \cdot (1-p)^k} = \frac{n}{p \cdot (1-p)^k}$$

Now, we use Lemma 14.4 and obtain, for $p \leq \frac{1}{2}$:

$$\left|V_{\leq k}\right| \leq \frac{n}{p \cdot \mathrm{e}^{-2pk}} = \frac{1}{p} \cdot \mathrm{e}^{2pk} \cdot n$$

Now, choose $p = \frac{1}{k+1}$ and obtain (for $k \geq 1$)

$$\left|V_{\leq k}\right| \leq (k+1) \cdot \mathrm{e}^{2\frac{k}{k+1}} \cdot n < (k+1) \cdot \mathrm{e}^2 \cdot n < 9(k+1)n$$

We have proven the following theorem for $k \geq 1$. (For $k = 0$ we already knew that the number of vertices is in $O(n)$.)

Theorem 14.6. *The total number of vertices of level at most k in an arrangement of n hyperplanes in $\mathbb{R}^2$ for any fixed d is at most $O(n(k+1))$.*

3 Simple arrangements of hyperplanes in $\mathbb{R}^d$

Let H be a set of hyperplanes in $\mathbb{R}^d$ in general position (Definition 13.2), and none of them is vertical (parallel to the x_d-axis). The proof is the same as for $d = 2$ in the previous section, but we use a different probability distribution. Let $r \leq n$ be a natural number. We sample $R \subseteq H$ as a subset of size r with all $\binom{n}{r}$ subsets being equally probable. By the asymptotic upper bound theorem from Lecture 7 we have for the number of vertices of level 0 in the arrangement of R, that

$$X \in O(r^{\lfloor \frac{d}{2} \rfloor}) \tag{3}$$

The conditions for A_v occurring are

(i) The hyperplanes defining the vertex v are all part of the sample R.

(ii) None of the hyperplanes lying above v are in the sample R.

Now, we can determine the probability of A_v by counting the number of r-element subsets that lead to A_v occurring.

We first analyze the number of such subsets. For simplicity of notation, let $\ell := \ell(v)$. Consider a fixed r-element subset of H that satisfies the conditions (i) and (ii) above. The d hyperplanes defining v must be contained. There are $n - d - \ell$ hyperplanes that can be chosen as the remaining $r - d$ elements. Thus, the number of such subsets that satisfy the above conditions is $\binom{n-d-\ell}{r-d}$.

Any fixed subset of size r has the probability of $\frac{1}{\binom{n}{r}}$ to be chosen. Therefore, the probability of choosing a subset that satisfies the conditions is $\mathbf{Pr}\left[A_v\right] = P(\ell)$, where

$$P(\ell) := \frac{\binom{n-d-\ell}{r-d}}{\binom{n}{r}}. \tag{4}$$

Note that P is a decreasing function in ℓ. Therefore,

$$\mathbf{E}\left[X\right] = \sum_{v \in V} \mathbf{Pr}\left[A_v\right] \geq \sum_{v \in V_{\leq k}} P(k) = \left|V_{\leq k}\right| \cdot P(k)$$

Thus,

$$\left|V_{\leq k}\right| \leq \frac{\mathbf{E}\left[X\right]}{P(k)} \tag{5}$$

We now make the following claim, which we will prove later.

Claim 14.7. *For $1 \leq k \leq \frac{n}{2d} - 1$ and $r = \lfloor \frac{n}{k+1} \rfloor$ and assuming $d > 2$ and $n \geq 4$, we have*

$$P(k) \geq c_d (k+1)^{-d}$$

for some value of c_d depending only on d.

Now, choose $r = \lfloor \frac{n}{k+1} \rfloor$, and assume that $1 \leq k \leq \frac{n}{2d} - 1$. (Otherwise, if $k > \frac{n}{2d}$, the claimed bound is $O(n^d)$, or if $k = 0$, then the bound follows from the upper bound theorem.)

We also have that $\mathbf{E}\left[X\right] \in O\left(r^{\lfloor \frac{d}{2} \rfloor}\right)$ from (3) (since the bound holds for any sample R, it also holds in expectation). Combining this with (5) and the above claim, and using our choice of r, we get that

$$\left|V_{\leq k}\right| \in O\left(r^{\lfloor \frac{d}{2} \rfloor}(k+1)^d\right) \in O\left(\left(\lfloor \frac{n}{k+1} \rfloor\right)^{\lfloor \frac{d}{2} \rfloor} (k+1)^d\right)$$

We simplify the bound as follows.

$$\begin{aligned}
\left(\lfloor\frac{n}{k+1}\rfloor\right)^{\lfloor\frac{d}{2}\rfloor}\cdot(k+1)^d &\leq \left(\frac{n}{k+1}\right)^{\lfloor\frac{d}{2}\rfloor}\cdot(k+1)^d \\
&= \left(\frac{n}{k+1}\right)^{\lfloor\frac{d}{2}\rfloor}\cdot(k+1)^{\lfloor\frac{d}{2}\rfloor}\cdot(k+1)^{\lceil\frac{d}{2}\rceil} \\
&= n^{\lfloor\frac{d}{2}\rfloor}\cdot(k+1)^{\lceil\frac{d}{2}\rceil}
\end{aligned}$$

This implies, that

$$\left|V_{\leq k}\right| \in O\left(n^{\lfloor\frac{d}{2}\rfloor}\cdot(k+1)^{\lceil\frac{d}{2}\rceil}\right) \tag{6}$$

This proves Theorem 14.3 for simple arrangements under the assumption that Claim 14.7. For non-simple arrangements, it is easy to see that the bound also holds, as simple arrangements maximize the number of vertices.

Proof of Claim 14.7. By the assumption on k and r we have that $2d \leq r \leq \frac{n}{2}$. We use the formula for the bionmial coefficient which states that

$$\binom{a}{b} = \frac{a!}{b!(a-b)!}$$

We apply this to the definition of $P(k)$ and obtain

$$\begin{aligned}
P(k) &= \frac{\binom{n-d-k}{r-d}}{\binom{n}{r}} \\
&= \frac{(n-d-k)!}{(r-d)!(n-k-r)!}\cdot\frac{r!(n-r)!}{n!}
\end{aligned}$$

We can regroup the terms as follows

$$\begin{aligned}
P(k) &= \frac{(n-d-k)!}{(n-k-r)!}\cdot\frac{r!}{(r-d)!}\cdot\frac{(n-r)!}{n!} \\
&= (n-d-k)\,(n-d-k-1)\cdot\dots\cdot(n-k-r+1) \\
&\quad\cdot\ r(r-1)\cdot\dots\cdot(r-d+1) \\
&\quad\cdot\ \frac{1}{n(n-1)\cdot\dots\cdot(n-r+1)} \\
&= \underbrace{\frac{r(r-1)\cdot\dots\cdot(r-d+1)}{n(n-1)\cdot\dots\cdot(n-d+1)}}_{I_1}\cdot\underbrace{\frac{n-d-k}{n-d}\cdot\frac{n-d-k-1}{n-d-1}\cdots\frac{n-k-r+1}{n-r+1}}_{I_2}
\end{aligned}$$

Now, in I_1 we have d terms in the enumerator with the smallest term being $(r-d+1)$ and we have d terms in the denominator with the largest term being n. Moreover, by our assumptions on k, r and d we have $(r-d+1) \geq \frac{r}{2}$ and therefore

$$I_1 \geq \left(\frac{r-d+1}{n}\right)^d \geq \left(\frac{r}{2n}\right)^d$$

In I_2 we have $(r-d)$ terms and, since by our assumptions $(r-d) \geq \frac{r}{2}$, we have

$$I_2 \geq \left(1-\frac{k}{n-d}\right)\cdot\left(1-\frac{k}{n-d-1}\right)\cdots\left(1-\frac{k}{n-r+1}\right) \geq \left(1-\frac{k}{n-r+1}\right)^{\frac{r}{2}}$$

Since $r \leq \frac{n}{2} + 1$ we have

$$1 - \frac{k}{n - r + 1} \geq 1 - \frac{2k}{n}$$

and since

$$k \leq \frac{n}{2d} - 1 \leq \frac{n}{2d} \leq \frac{n}{4}$$

we have $\frac{2k}{n} \leq \frac{1}{2}$ and therefore we can use Lemma 14.4 setting $x = \frac{2k}{n}$ and obtain

$$P(k) \geq I_1 \cdot I_2 \geq \left(\frac{r}{2n}\right)^d \cdot \left(1 - \frac{2k}{n}\right)^{\frac{r}{2}} \geq \left(\frac{r}{2n}\right)^d \cdot \mathrm{e}^{-\frac{2kr}{n}}$$

Now, since $r = \lfloor \frac{n}{k+1} \rfloor$ and $k \leq \frac{n}{4}$ we have (for $n \geq 4$)

$$\frac{r}{n} \geq \frac{\left(\frac{n}{k+1} - 1\right)}{n} \geq \frac{n - k - 1}{(k+1)n} \geq \frac{n - \frac{n}{4} - 1}{(k+1)n} \geq \frac{\frac{n}{2} + \left(\frac{n}{4} - 1\right)}{(k+1)n} \geq \frac{1}{2(k+1)}$$

Therefore

$$P(k) \geq \left(\frac{1}{4(k+1)}\right)^d \mathrm{e}^{-\frac{2k}{2(k+1)}} \geq c_d (k+1)^{-d}$$

for $c_d = 4^{-d}\mathrm{e}^{-1}$ □

References

- Jiří Matoušek, Chapter 6.3, Lectures on Discrete Geometry, Springer Graduate Texts in Mathematics.