

# MATH1063: Introduction to Statistics

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# Chapter 1

## Introduction to Statistics

### 1.1 What is statistics?

#### 1.1.1 Early and modern definitions

The word *statistics* comes from the Latin *status*, meaning ‘the state’. In the mid-18th century, it referred to the *collection, processing, and use of data by governments*. During the rapid industrialisation of Europe in the early 19th century, statistics developed into a formal discipline. In 1834, the Royal Statistical Society was founded as the leading professional association for statisticians in the UK and globally. As the field grew, statistics came to mean the *interpretation of data and methods for extracting information to support decision making*. Today, statistics refers to the *collection, analysis, and interpretation of data*. Indeed, the Oxford English Dictionary defines *statistics* as

The practice or science of collecting and analysing numerical data in large quantities, especially for the purpose of inferring proportions in a whole from those in a representative sample.

Notice that the original connection to ‘the state’ is no longer present. Now, statistical methods are essential for anyone who wants to answer questions using data.

For example: Will it rain tomorrow? Is smoking harmful during pregnancy? What degree classification will I receive when I graduate? Will the stock market crash tomorrow?

#### 1.1.2 Statistics tames uncertainty

We often make decisions when outcomes are uncertain. Understanding the extent of this uncertainty helps us reduce the chance of making mistakes, or at least minimise the impact of those mistakes.

In other words:

$$\text{Uncertain knowledge} + \text{Understanding its uncertainty} = \text{Usable knowledge}$$

#### 1.1.3 Why should I study statistics as part of my degree?

Studying statistics will give you essential skills in data analysis and scientific reasoning. No matter which branch of mathematics, engineering, science, or social science you pursue, a solid understanding of statistics is important. Learning statistical theory also lets you apply your mathematical reasoning to real-world problems. This helps you improve both your mathematical and statistical skills.

All knowledge is, in final analysis, history.  
All sciences are, in the abstract, mathematics.

All judgements are, in their rationale, statistics.  
 – Prof. C. R. Rao

### 1.1.4 Lies, Damn Lies and Statistics?

People sometimes joke, ‘You can prove anything with statistics!’ Such comments reflect the fact that statistics and statistical methods are often misquoted or misused without proper verification or justification. This is even more important in recent years of the global pandemic, when every day we have been showered with a deluge of numbers.

Statistics can easily be misused or misinterpreted. As statisticians, it is our responsibility to apply statistical techniques correctly and develop strong, scientifically sound arguments. In particular, we must be careful about the assumptions we make.

### 1.1.5 What’s in this module?

- **Chapter 1:** We begin with basic statistics used in everyday life, such as the mean, median, mode, and standard deviation. We will discuss statistical analysis, report writing, and how to explore data using graphs. The R programming language will be introduced to assist with these tasks.
- **Chapter 2: Introduction to Probability.** We will define and interpret probability as a way to measure uncertainty, learn the rules of probability, and work through examples.
- **Chapter 3: Probability Distributions.** We will study various probability distributions that can be used to model the outcomes of different types of events.
- **Chapter 4: Statistical Inference.** We will cover the basics of statistical inference, including point and interval estimation and hypothesis testing.
- **Chapter 5: Simple Linear Regression.** We will introduce simple linear regressions models, which are used to model the relationships between two variables.

## 1.2 Example data sets

In this module, we will assume that we have data from  $n$  randomly selected sampling units, which we will conveniently denote by  $x_1, x_2, \dots, x_n$ . We will assume that these values are numeric, either discrete like counts, e.g. number of road accidents, or continuous, e.g. heights of 4-year-olds, marks obtained in an examination.

We will use the following example datasets to illustrate concepts throughout the module:

**Example 1.1** (Fast food service time). This dataset records the service times (in seconds) for customers at a fast-food restaurant. The first row shows customers served from 9–10am, and the second row shows those served from 2–3pm on the same day.

AM	38	100	64	43	63	59	107	52	86	77
PM	45	62	52	72	81	88	64	75	59	70

We would like to compare these AM and PM service times.

**Example 1.2** (Computer failures). This dataset lists the number of weekly failures in a university computer system over two years.

4 0 0 0 3 2 0 0 6 7 6 2 1 11 6 1 2 1 1 2 0 2 2 1 0 12 8 4 5 0 5 4 1 0 8 2 5 2 1  
 12 8 9 10 17 2 3 4 8 1 2 5 1 2 2 3 1 2 0 2 1 6 3 3 6 11 10 4 3 0 2 4 2 1 5 3 3  
 2 5 3 4 1 3 6 4 4 5 2 10 4 1 5 6 9 7 3 1 3 0 2 2 1 4 2 13

We would like to summarise this data and make predictions about future failures.

**Example 1.3** (Weight gain). Do students tend to gain weight during their first year at college? Professor David Levitsky at Cornell recruited students from two large introductory health classes. Although participation was voluntary, the students were similar to the rest of the class in terms of demographics such as sex and ethnicity. Sixty-eight students were weighed in the first week of the semester and again 12 weeks later. The first 10 rows of the data are:

Student number	Initial weight (kg)	Final weight (kg)
1	77.56423	76.20346
2	49.89512	50.34871
3	60.78133	61.68851
4	52.16308	53.97745
5	68.03880	70.30676
6	47.17357	48.08075
7	64.41006	67.13162
8	54.43104	56.24541
9	65.31725	67.13162
10	70.76035	69.85317

We would like to explore the data graphically and to test whether the data support the hypothesis that students gain weight during their first year in college.

**Example 1.4** (Billionaires). Each year, Fortune magazine publishes a list of the world's billionaires. The 1992 list includes 225 individuals, reporting their wealth, age, and geographic region (Asia, Europe, Middle East, United States, and Other).

The variables in the data are:

- **wealth**: Wealth of family or individual in billions of dollars
- **age**: Age in years (for families it is the maximum age of family members)
- **region**: Region of the World (Asia, Europe, Middle East, United States and Other).

The first 10 rows of the data are:

wealth	age	region
37.0	50	Middle East
24.0	88	United States
14.0	64	Asia
13.0	63	United States
13.0	66	United States
11.7	72	Europe
10.0	71	Middle East
8.2	77	United States
8.1	68	United States
7.2	66	Europe

We will investigate differences in wealth of billionaires by age and region using many exploratory graphical tools and statistical methods.

## 1.3 Introduction to R

R is a programming language designed for statistics. We will use R to perform calculations, summarise data, create exploratory plots, carry out statistical analyses, illustrate theorems, and calculate probabilities. You will gain hands-on experience with R during the computer practical sessions.

R is free to download—just search for ‘download R’ or visit <https://cran.r-project.org/>. We will use R through the RStudio integrated development environment, which you can download from <https://posit.co/download/rstudio-desktop/>. RStudio provides a user-friendly editor for R code and a console. Instructions for using R and RStudio will be provided in the practical sessions. If you plan to work on the practicals on your own computer, please install R and RStudio beforehand.

These notes do not cover the details of using R—that will be addressed in the practical sessions. Here, we provide simple R commands for calculating useful quantities and show some example graphical outputs.

## 1.4 Summarising data sets

### 1.4.1 Summarising categorical data

Categorical (non-numeric) data can be summarised using tables. For example, the results of 20 coin tosses might be summarised as 12 heads and 8 tails.

For the computer failure data (see Example 1.2), we can summarise the number of failures per week as follows:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	17
12	16	21	12	11	8	7	2	4	2	3	2	2	1	1

### 1.4.2 Measures of location

#### 1.4.2.1 Choosing a representative value for the data

We want a representative value for the data  $x_1, x_2, \dots, x_n$  that is a function of the data. If  $a$  is our chosen value, how much error does it have? We could measure total error as the sum of squared errors,

$$\text{SSE}(a) = \sum_{i=1}^n (x_i - a)^2$$

or as the sum of absolute errors,

$$\text{SAE}(a) = \sum_{i=1}^n |x_i - a|.$$

What value of  $a$  will minimise the SSE or the SAE? For SSE the answer is the sample mean and for SAE the answer is the sample median.

#### 1.4.2.2 The sample mean

The *sample mean* is

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \dots + x_n) = \frac{1}{n} \sum_{i=1}^n x_i.$$

For the service time data from Example 1.1, the AM mean time is 68.9 seconds and the PM mean time is 66.8 seconds.



**Theorem 1.1.** *The sample mean  $\bar{x}$  minimises the SSE.*

*Proof.* We have

$$\begin{aligned}
 \text{SSE}(a) &= \sum_{i=1}^n (x_i - a)^2 \\
 &= \sum_{i=1}^n (x_i - \bar{x} + \bar{x} - a)^2 \quad (\text{Add and subtract } \bar{x}) \\
 &= \sum_{i=1}^n \left\{ (x_i - \bar{x})^2 + 2(x_i - \bar{x})(\bar{x} - a) + (\bar{x} - a)^2 \right\} \\
 &= \sum_{i=1}^n (x_i - \bar{x})^2 + 2(\bar{x} - a) \sum_{i=1}^n (x_i - \bar{x}) + \sum_{i=1}^n (\bar{x} - a)^2 \\
 &= \sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - a)^2,
 \end{aligned}$$

since  $\sum_{i=1}^n (x_i - \bar{x}) = n\bar{x} - n\bar{x} = 0$ .

The first term is free of  $a$  and the second term is non-negative for any value of  $a$ . Hence the minimum occurs when the second term is zero, i.e. when  $a = \bar{x}$ .  $\square$

We have established the fact that the sum of (or mean) squares of the deviations from any number  $a$  is minimised when  $a$  is the mean. This justifies why we often use the mean as a representative value.

#### 1.4.2.3 The sample median

The *sample median* is the middle value in the ordered list of observations  $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$ . For the AM service time data

$$38 < 43 < 52 < 59 < 63 < 64 < 77 < 86 < 100 < 107.$$

If  $n$  is odd, there is a unique middle value. If  $n$  is even, there are two middle values (63 and 64 for the AM service time data). Any value between those two middle values is a sample median. By convention, we often use the mean of these values (63.5 for the AM service time data).

**Theorem 1.2.** *The sample median minimises the SAE.*

*Proof.* If  $a < x_{(1)}$ , then

$$\text{SAE}(a) = \sum_{i=1}^n (x_i - a). \quad (1.1)$$

As  $a$  increases, each term of (1.1) decreases until  $a$  reaches  $x_{(1)}$ , so  $\text{SAE}(x_{(1)}) < \text{SAE}(a)$  for all  $a < x_{(1)}$ . We conclude the minimiser of the SAE is at least  $x_{(1)}$ .

Now suppose  $x_{(k)} \leq a < x_{(k+1)}$ . Then

$$\begin{aligned}
 \text{SAE}(a) &= \sum_{i=1}^k (a - x_{(i)}) + \sum_{i=k+1}^n (x_{(i)} - a) \\
 &= (2k - n)a - \sum_{i=1}^k x_{(i)} + \sum_{i=k+1}^n x_{(i)}.
 \end{aligned}$$

The term  $-\sum_{i=1}^k x_{(i)} + \sum_{i=k+1}^n x_{(i)}$  is constant for each any  $a$  in the interval  $[x_{(k)}, x_{(k+1)}]$ . So the SAE in this interval is a straight line, with slope  $2k - n$ . This slope is negative if  $k < \frac{n}{2}$ , zero if  $k = \frac{n}{2}$ , and positive if  $k > \frac{n}{2}$ .

For each  $k < \frac{n}{2}$ , the SAE is decreasing in the interval  $[x_{(k)}, x_{(k+1)}]$ , and we conclude the minimiser of the SAE is at least  $x_{(k+1)}$ . Starting at  $k = 1$ , we continue increasing  $k$  by one and concluding the the minimiser of the SAE is at least  $x_{(k+1)}$ , until we reach a  $k$  such that  $k \geq \frac{n}{2}$ :

- If  $n$  is odd, this happens at  $k = \frac{n+1}{2}$ . In that case, since  $k > \frac{n}{2}$ , the SAE is increasing in the interval  $[x_{(k)}, x_{(k+1)}]$ , so we conclude the SAE is minimised at  $x_{(k)} = x_{(\frac{n+1}{2})}$ , the median point.
- If  $n$  is even, this happens at  $k = \frac{n}{2}$ . In that case the SAE is constant in the interval  $[x_{(k)}, x_{(k+1)}]$ , so we conclude that the SAE is minimised at any  $a$  between the two middle points  $x_{(\frac{n}{2})}$  and  $x_{(\frac{n}{2}+1)}$ , i.e. any median point.

□

We have established the fact that the sum of (or mean) of the absolute deviations from any number  $a$  is minimised when  $a$  is the median. This justifies why median is also often used as a representative value.

The mean is more affected by extreme values than the median. For example, for the AM service times, if the next observation is 190, the median will be 64 instead of 63.5, but the mean will jump to 79.9.

#### 1.4.2.4 The sample mode

The mode, or most frequent value in the data, is the best representative if we use a 0-1 error function instead of SAE or SSE. Here, the error is 0 if our guess  $a$  is correct and 1 if it is not. In this case, the best guess  $a$  is the mode of the data.

### 1.4.3 Measures of spread

A quick way to measure spread is the *range*, which is the difference between the maximum and minimum values. For the AM service times, the range is 69 ( $107 - 38$ ) seconds.

The *variance* is

$$\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2.$$

We have

$$\sum_{i=1}^n (x_i - \bar{x})^2 = \sum_{i=1}^n (x_i^2 - 2x_i\bar{x} + \bar{x}^2) = \sum_{i=1}^n x_i^2 - 2\bar{x}(n\bar{x}) + n\bar{x}^2 = \sum_{i=1}^n x_i^2 - n\bar{x}^2,$$

so we calculate variance as

$$\text{Var}(x) = \frac{1}{n-1} \left( \sum_{i=1}^n x_i^2 - n\bar{x}^2 \right).$$

Sometimes variance is defined with  $n$  instead of  $n-1$  in the denominator. We use  $n-1$  because this is the default in R. We will discuss this more in Chapter 4.

The *standard deviation* (sd) is the square root of the variance:

$$s = \text{sd}(x) = \sqrt{\text{Var}(x)}.$$

The standard deviation for the AM service times is 23.2 seconds. It has the same unit as the data.

The *interquartile range* (IQR) is the difference between the third quartile,  $Q_3$ , and the first quartile,  $Q_1$ . These are the observations ranked  $\frac{1}{4}(3n+1)$  and  $\frac{1}{4}(n+3)$  in the ordered list. The median is the

second quartile,  $Q_2$ . When  $n$  is even, the definitions of  $Q_3$  and  $Q_1$  are similar to that of the median,  $Q_2$ . The IQR for the AM service times is  $83.75 - 53.75 = 30$  seconds.

### 1.4.4 Summarising data in R

We can easily calculate the mean, median, variance, and standard deviation in R.

For example, suppose the weekly counts of computer failures from Example 1.2 are stored in an object called `compfail` in R. You will learn how to read the data into R during the practical sessions.

We can calculate these quantities with:

```
mean(compfail)
```

```
## [1] 3.75
```

```
median(compfail)
```

```
## [1] 3
```

```
var(compfail)
```

```
## [1] 11.43204
```

```
sd(compfail)
```

```
## [1] 3.38113
```

R saves us a lot of effort compared to calculating these by hand.

The `summary` command provides several useful summary statistics:

```
summary(compfail)
```

```
##      Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##      0.00   1.00   3.00   3.75   5.00   17.00
```

We can create a table summarising the counts by typing:

```
table(compfail)
```

```
## compfail
##  0  1  2  3  4  5  6  7  8  9 10 11 12 13 17
## 12 16 21 12 11  8  7  2  4  2  3  2  2  1  1
```

## 1.5 Exploratory data plots

### 1.5.1 Introduction

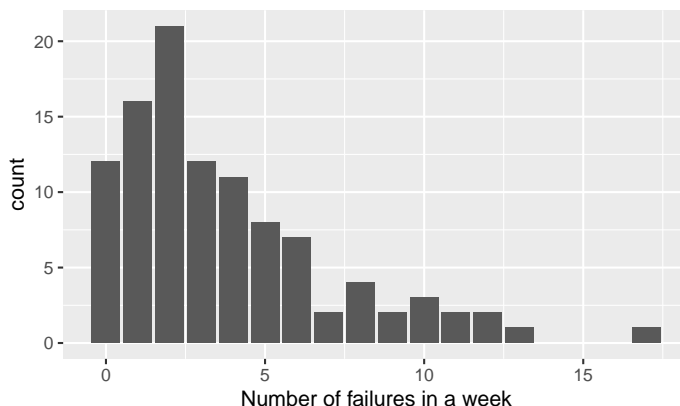
Often, the best way to gain insight is to plot the data. Plots help us check the shape of a variable's distribution and explore relationships between variables in a dataset.

The best type of plot depends on whether we are looking at a single variable or relationships between variables, and whether the variables are discrete or continuous.

Here, we show some plots produced by R. You will learn how to make these plots in the practical sessions.

### 1.5.2 Distribution of a single discrete variable

To explore the distribution of a single discrete variable, we can use a bar plot. For example, we can plot the weekly counts of computer failures from Example 1.2:

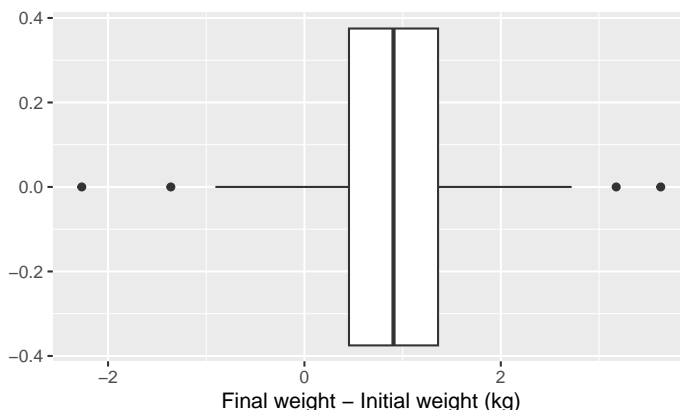


This plot shows the same information as the summary table of counts, but it is easier to understand visually.

### 1.5.3 Distribution of a single continuous variable

There are several types of plots we can use to summarise the shape of a single continuous variable's distribution. For example, suppose we are interested in the weight gain in Example 1.3.

We can use a box-and-whiskers plot to summarise the distribution of the weight difference:

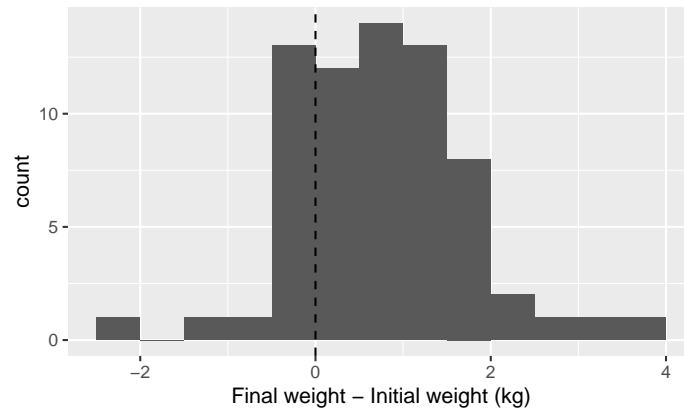


The line in the middle of the box is the median weight difference, and the box runs from the first quartile to the third quartile. The whiskers extend from the smallest value no more than  $1.5 \times \text{IQR}$  below the first quartile to the largest value no more than  $1.5 \times \text{IQR}$  above the third quartile.

Any outliers outside the whiskers are shown as separate points.

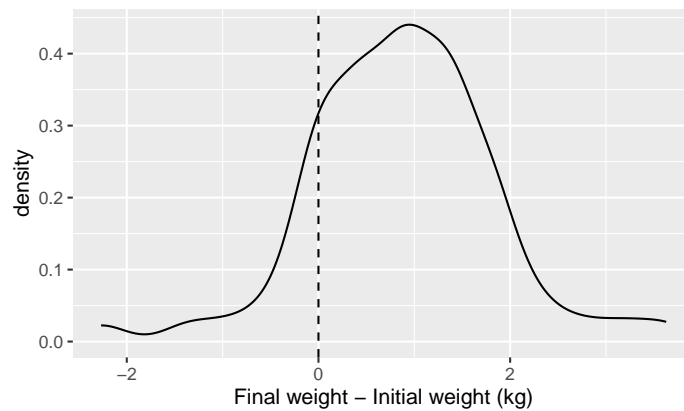
We see that most students in the sample gained weight during the study.

To get more information about the shape of the distribution, we could plot a histogram of the weight difference:



The histogram divides the  $x$ -axis into small intervals or “bins” (here, each of width 0.5) and counts how many times the variable (here, difference in weight) falls in each bin. Histograms can look very different depending on the bin width.

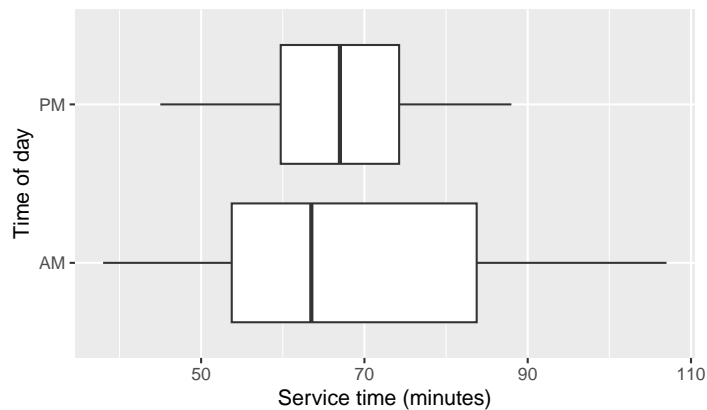
Alternatively, we could draw a density plot:



This shows an estimate of the probability density function, which we will learn about in Chapter 3.

#### 1.5.4 Relationship between continuous and discrete variables

We can also use a box-and-whiskers plot to show how a continuous variable changes with a discrete one. For example, for the service time data from Example 1.1, we are interested in how AM and PM service times compare:

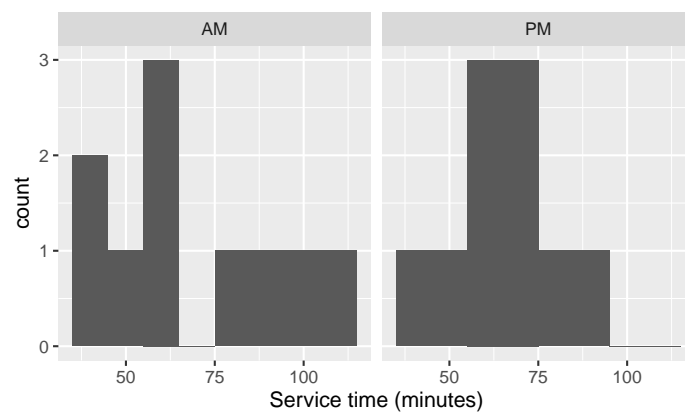


Since no outliers are shown in this case, the whiskers show the range of the data.

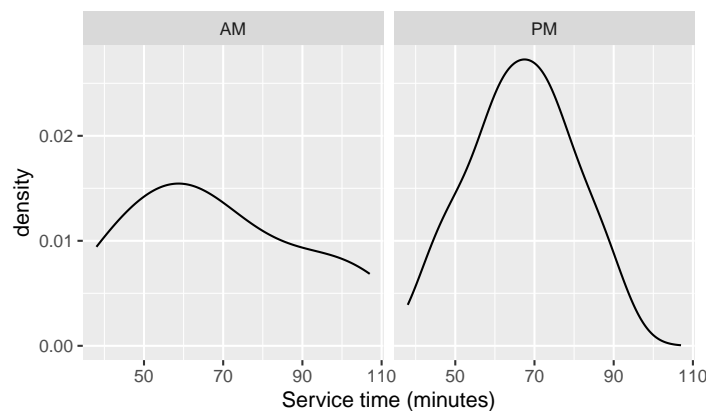
We see that AM service times are, on average, shorter but also more spread out than PM service times.

We can also use any of the other methods for plotting a single continuous variable and draw separate plots for each group in the discrete variable.

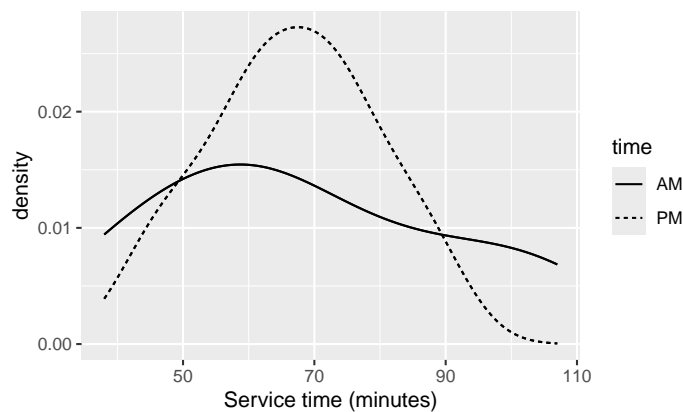
For example, we can use histograms:



Or density plots:

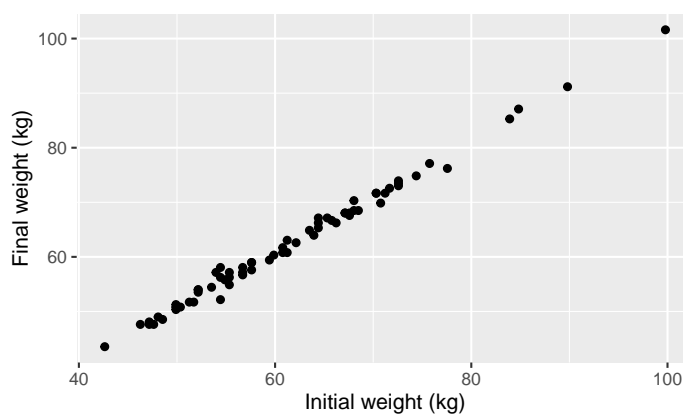


We could also plot the densities for the different groups on a single plot, using line type to distinguish between the groups:

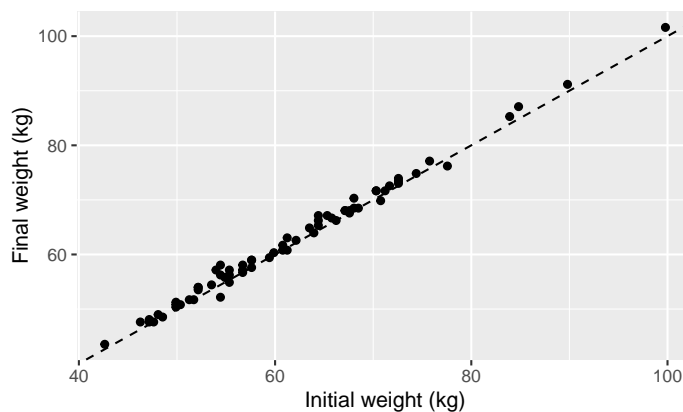


### 1.5.5 Relationship between two continuous variables

A scatter plot is a good way to look at the relationship between two continuous variables. For example, for the weight data from Example 1.3, we can plot final weights against initial weights:



We can add the straight line  $y = x$  to the plot:

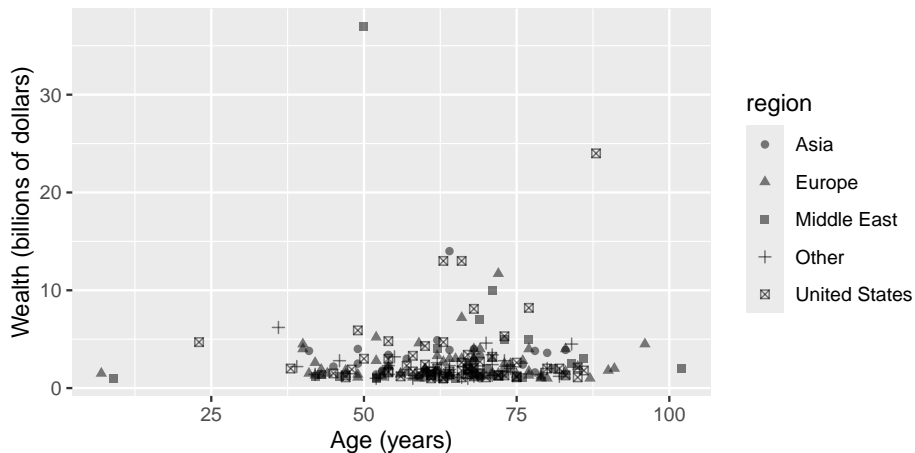


We see that most points lie above the  $y = x$  line, so most students in the sample gained weight during the study, as we saw before. This seems to be true regardless of the student's initial weight.

### 1.5.6 Relationships between more than two variables

We can combine the methods we've seen to plot relationships between more than two variables.

For example, in the billionaires data in Example 1.4, we can plot wealth against age as a scatter plot, using the shape of the points to show different regions:



This plot lets us quickly see some interesting features of the data. For example, there were a few very young billionaires, the richest person in the world at this time was age 50 and from the Middle East, and the richest few people had much more wealth than most others on the list.



## Chapter 2

# Introduction to Probability

### 2.1 Definitions of probability

We use probabilities to express how uncertain we are about whether certain events will happen. For example: (i) I believe there is more than an 60% chance that Manchester City will win the premiership title this season; (ii) the probability of a fair coin landing heads is 0.5.

As discussed in Chapter 1, uncertainty is everywhere, and probability is the language we use to measure it.

The football and coin examples show two different ways to interpret probability. The football example is based on the commentator's personal belief, not on repeated experiments. This is called subjective probability.

Subjective probability measures how plausible someone thinks an event is, based on their experience and available evidence (for example, knowing Manchester City are the current champions and have invested in new players).

Other examples include: "I think there is a 70% chance the FTSE 100 will rise tomorrow," or "The Met Office says there is a 40% chance of a white Christmas in Southampton this year."

The second way to define probability is by looking at the long-term relative frequency of an event in a random experiment that can be repeated many times under similar conditions (like tossing a coin).

Suppose we repeat a random experiment many times under the same conditions and count how often event  $A$  happens. The relative frequency,

$$\frac{\text{number of times } A \text{ occurs}}{\text{total number of repetitions}},$$

approaches a fixed value as the number of repetitions increases. This value is defined as  $P\{A\}$ .

For example, if we toss a coin 1000 times and count the number of heads, the proportion of heads is the relative frequency of event  $A$  (getting a head) in those 1000 tosses.

Sometimes, we can use basic facts about probability to find  $P\{A\}$  more easily. For example, if a coin is fair, then  $P\{\text{head}\} = P\{\text{tail}\}$  and  $P\{\text{head}\} + P\{\text{tail}\} = 1$ , so both probabilities are 0.5.

These basic facts are formalised as the axioms of probability, which are the foundation of probability theory.

## 2.2 Some definitions

Before we can state and use the axioms of probability, we need to introduce some terminology.

A *random experiment* is an experiment where we cannot predict the exact outcome, but we know all the possible outcomes.

The *sample space* ( $S$ ) is the set of all possible outcomes. For example, for a single coin toss,  $S = \{\text{head, tail}\}$ . For two coins,  $S = \{\text{HH, HT, TH, TT}\}$ , where H and T are head and tail.

An *event* is a specific result or set of results from the experiment. For example, getting ‘HH’ (two heads) is an event, and so is getting at least one head:  $\{\text{HH, HT, TH}\}$ .

We use capital letters like  $A, B, C$  to represent events. A single outcome is called an *elementary event* (e.g., ‘HH’), while a set of outcomes is a *composite event* (e.g., at least one head).

Probability theory is about finding the probability  $P\{A\}$  of a given event  $A$ .

**Example 2.1** (Die throw). Roll a six-faced die and observe the score on the uppermost face. Here  $S = \{1, 2, 3, 4, 5, 6\}$ , which is composed of six elementary events.

The *union* of two events  $A$  and  $B$ , written as  $A \cup B$ , is the set of outcomes that are in  $A$ ,  $B$ , or both. “ $A \cup B$  occurs” means “either  $A$  or  $B$  or both occur”.

For example, in Example 2.1, let  $A$  be the event “an even number is observed”:  $A = \{2, 4, 6\}$ . Let  $B$  be “a number larger than 3 is observed”:  $B = \{4, 5, 6\}$ . Then  $A \cup B = \{2, 4, 5, 6\}$ . If a 6 is observed, both  $A$  and  $B$  occur.

The *intersection* of two events  $A$  and  $B$ , written as  $A \cap B$ , is the set of outcomes common to both. “ $A \cap B$  occurs” means “both  $A$  and  $B$  occur”. In Example 2.1,  $A \cap B = \{4, 6\}$ . If  $C = \{1, 2, 3, 4, 5\}$  (“a number less than 6”), then  $A \cap C = \{2, 4\}$  (“an even number less than 6”).

We can generalize union and intersection to more than two events.

Two events  $A$  and  $D$  are *mutually exclusive* if  $A \cap D = \emptyset$ , meaning they have no outcomes in common. In other words,  $A$  and  $D$  cannot happen at the same time.

In Example 2.1, if  $D = \{1, 3, 5\}$  (“an odd number”), then  $A \cap D = \emptyset$ , so  $A$  and  $D$  are mutually exclusive.

The *complement* of an event  $A$ , written  $A'$ , is the set of all outcomes not in  $A$ . Note that  $A \cup A' = S$  and  $A \cap A' = \emptyset$ .

## 2.3 Axioms of probability

Probability is based on three main axioms:

**A1**  $P\{S\} = 1$ .

**A2**  $0 \leq P\{A\} \leq 1$  for any event  $A$ .

**A3**  $P\{A \cup B\} = P\{A\} + P\{B\}$  if  $A$  and  $B$  are mutually exclusive.

These axioms have several important consequences:

1. For any event  $A$ ,  $P\{A\} = 1 - P\{A'\}$ .
2. From (1) and Axiom A1,  $P\{\emptyset\} = 1 - P\{S\} = 0$ . So if  $A$  and  $B$  are mutually exclusive,  $P\{A \cap B\} = 0$ .
3. If  $D$  is a subset of  $E$ ,  $D \subset E$ , then  $P\{E \cap D'\} = P\{E\} - P\{D\}$ . For any events  $A$  and  $B$ ,  $P\{A \cap B'\} = P\{A\} - P\{A \cap B\}$ .
4. Axiom A3 extends to more than two mutually exclusive events:

$$P\{A_1 \cup A_2 \cup \dots \cup A_k\} = P\{A_1\} + P\{A_2\} + \dots + P\{A_k\}$$

if  $A_1, \dots, A_k$  are mutually exclusive. So, the probability of an event  $A$  is the sum of the probabilities of the individual outcomes in  $A$ .

5. For any two events  $A$  and  $B$ , the general addition rule is:

$$P\{A \cup B\} = P\{A\} + P\{B\} - P\{A \cap B\}.$$

*Proof.* We can write  $A \cup B = (A \cap B') \cup (A \cap B) \cup (A' \cap B)$ . All three of these are mutually exclusive events. Hence,

$$\begin{aligned} P\{A \cup B\} &= P\{A \cap B'\} + P\{A \cap B\} + P\{A' \cap B\} \\ &= P\{A\} - P\{A \cap B\} + P\{A \cap B\} + P\{B\} - P\{A \cap B\} \\ &= P\{A\} + P\{B\} - P\{A \cap B\}. \end{aligned}$$

□

(6) The sum of the probabilities of all the outcomes in the sample space  $S$  is 1.

## 2.4 Using combinatorics to find probabilities

### 2.4.1 Experiments with equally likely outcomes

If all outcomes are equally likely, we can find the probability of an event by counting how many outcomes are in the event and dividing by the total number of possible outcomes.

For an experiment with  $N$  equally likely outcomes, each outcome has probability  $1/N$ .

For any event  $A$ ,

$$P\{A\} = \frac{\text{number of outcomes in } A}{\text{total number of possible outcomes of the experiment}}.$$

So, to calculate the probability of an event, we just count the number of outcomes in the event and the total number of possible outcomes. In the next sections, we use combinatorics (the mathematics of counting) to help with this.

Return to Example 2.1 where a six-faced die is rolled. Suppose that one wins a bet if a 6 is rolled. Then the probability of winning the bet is  $1/6$  as there are six possible outcomes in the sample space and exactly one of those, 6, wins the bet. Suppose  $A$  denotes the event that an even-numbered face is rolled. Then  $P\{A\} = 3/6 = 1/2$  as we can expect.

**Example 2.2** (Dice throw). Roll 2 distinguishable dice and observe the scores. Here

$$S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), \dots, (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$$

which consists of 36 possible outcomes or elementary events,  $A_1, \dots, A_{36}$ . What is the probability of the outcome 6 in both the dice? The required probability is  $1/36$ . What is the probability that the sum of the two dice is greater than 6? How about the probability that the sum is less than any number, e.g. 8?

**Hint:** Write down the sum for each of the 36 outcomes and then find the probabilities asked just by inspection. Remember, each of the 36 outcomes has equal probability  $1/36$ .

### 2.4.2 Multiplication rule of counting

Suppose you have to complete  $k$  tasks in sequence. If there are  $n_i$  ways to do the  $i$ th task, then there are  $n_1 \times n_2 \times \cdots \times n_k$  ways to do all  $k$  tasks.

**Example 2.3** (Bus routes). Suppose there are 7 routes to London from Southampton and then there are 5 routes to Cambridge out of London. How many ways can I travel to Cambridge from Southampton via London? The answer is obviously 35.

### 2.4.3 The number of permutations of $k$ from $n$ : $P(n, k)$

Suppose we want to select  $k$  people from  $n$  and arrange them in  $k$  different chairs. How many ways can we do this?

Think of the  $i$ th task as choosing who sits in the  $i$ th chair. By the multiplication rule, there are  $n(n-1)\cdots(n-[k-1])$  ways.

This number is called the number of permutations of  $k$  from  $n$ , written  $P(n, k)$ :

$$P(n, k) = n(n-1)\cdots(n-[k-1]).$$

When  $k = n$ ,  $P(n, n) = n!$  ( $n$  factorial). By definition,  $0! = 1$ . We can also write

$$P(n, k) = \frac{n!}{(n-k)!}.$$

**Example 2.4** (Football). How many possible rankings are there for the 20 football teams in the premier league at the end of a season? This number is given by  $P(20, 20) = 20!$ , which is a huge number! How many possible permutations are there for the top 4 positions who will qualify to play in Europe in the next season? This number is given by  $P(20, 4) = 20 \times 19 \times 18 \times 17$ .

#### 2.4.3.1 The number of combinations of $k$ from $n$ : $\binom{n}{k}$

Now suppose we want to select  $k$  people from  $n$ , but we do not care about the order. How many ways can we do this? This is called a combination, written  $\binom{n}{k}$ .

To see this, recall that arranging  $k$  people in  $k$  chairs is  $P(n, k)$  ways. But for each group of  $k$  people, there are  $k!$  ways to arrange them. So,

$$\binom{n}{k} = \frac{P(n, k)}{k!} = \frac{n!}{(n-k)!k!}.$$

**Example 2.5** (Football). How many possible ways are there to choose 3 teams for the bottom positions of the premier league table at the end of a season? This number is given by  $\binom{20}{3} = 20 \times 19 \times 18 / 3!$ , which does not take into consideration the rankings of the three bottom teams.

**Example 2.6** (Microchips). A box contains 12 microchips of which 4 are faulty. A sample of size 3 is drawn from the box without replacement.

- How many selections of 3 can be made?  $\binom{12}{3}$
- How many samples have all 3 chips faulty?  $\binom{4}{3}$ .
- How many selections have exactly 2 faulty chips?  $\binom{8}{1}\binom{4}{2}$
- How many samples of 3 have 2 or more faulty chips?  $\binom{8}{1}\binom{4}{2} + \binom{4}{3}$

### 2.4.4 Calculation of probabilities of events under sampling ‘at random’

When we select a sample of size  $n$  from a box of  $N$  items without replacement, we say the sample is chosen “at random” if every possible sample of size  $n$  is equally likely. In this case, each possible sample is an equally likely outcome, and we assign equal probabilities.

**Example 2.7** (Microchips continued). In Example 2.6 assume that 3 microchips are selected at random without replacement. Then

- each outcome (sample of size 3) has probability  $1/\binom{12}{3}$ .
- $P\{\text{all 3 selected microchips are faulty}\} = \binom{4}{3}/\binom{12}{3}$ .
- $P\{2 \text{ chips are faulty}\} = \binom{8}{1}\binom{4}{2}/\binom{12}{3}$ .
- $P\{2 \text{ or more chips are faulty}\} = (\binom{8}{1}\binom{4}{2} + \binom{4}{3})/\binom{12}{3}$ .

### 2.4.5 A general ‘urn problem’

Example 2.6 is one particular case of the following general urn problem which can be solved by the same technique. A sample of size  $n$  is drawn at random without replacement from a box of  $N$  items containing a proportion  $p$  of defective items.

- How many defective items are in the box?  $Np$ . How many good items are there?  $N(1-p)$ . Assume these to be integers.
- The probability of exactly  $x$  defective items in the sample of  $n$  items is

$$\frac{\binom{Np}{x}\binom{N(1-p)}{n-x}}{\binom{N}{n}}.$$

- Which values of  $x$  (in terms of  $N$ ,  $n$  and  $p$ ) make this expression well defined? We’ll see later that these values of  $x$  and the corresponding probabilities make up what is called the *hyper-geometric* distribution.

**Example 2.8** (Selecting a committee). There are 10 students available for a committee of which 4 are men and 6 are women. A random sample of 3 students are chosen to form the committee — what is the probability that exactly one is a man? The total number of possible outcomes of the experiment is equal to the number of ways of selecting 3 students from 10 and is given by  $\binom{10}{3}$ . The number of outcomes in the event ‘exactly one is a man’ is equal to the number of ways of selecting 3 students from 10 with exactly one man, and given by  $\binom{4}{1}\binom{6}{2}$ . Hence

$$\begin{aligned} P\{\text{exactly one man}\} &= \frac{\text{number of ways of selecting one man and two women}}{\text{number of ways of selecting 3 students}} \\ &= \frac{\binom{4}{1}\binom{6}{2}}{\binom{10}{3}} = \frac{4 \times 15}{120} = \frac{1}{2} \end{aligned}$$

**Example 2.9** (The National Lottery). In Lotto, a winning ticket has six numbers from 1 to 59 matching those on the balls drawn on a Wednesday or Saturday evening. The ‘experiment’ consists of drawing the balls from a box containing 59 balls. The ‘randomness’, the equal chance of any set of six numbers being drawn, is ensured by the spinning machine, which rotates the balls during the selection process. What is the probability of winning the jackpot? The total number of possible selections of six balls/numbers is  $\binom{59}{6}$ . There is only 1 selection for winning the jackpot. Hence

$$P\{\text{jackpot}\} = \frac{1}{\binom{59}{6}} = 2.22 \times 10^{-8},$$

which is roughly 1 in 45 million.

There is one other way of win a very large prize, of £1 million, by using the bonus ball — matching 5 of the selected 6 balls plus matching the bonus ball. The probability of this is given by

$$P\{5 \text{ matches} + \text{bonus}\} = \frac{6}{\binom{59}{6}} = 1.33 \times 10^{-7}.$$

Other smaller prizes are given for fewer matches. The corresponding probabilities are:

$$P\{5 \text{ matches}\} = \frac{\binom{6}{5} \binom{53}{1}}{\binom{59}{6}} = 7.06 \times 10^{-6}.$$

$$P\{4 \text{ matches}\} = \frac{\binom{6}{4} \binom{53}{2}}{\binom{59}{6}} = 0.000459.$$

$$P\{3 \text{ matches}\} = \frac{\binom{6}{3} \binom{53}{3}}{\binom{59}{6}} = 0.0104.$$

## 2.5 Conditional probability and Bayes' Theorem

How do we update probabilities when we have new information? For example, a person may have a certain disease, whether or not they show symptoms. If we know a randomly selected person has the symptom, what is the probability they have the disease? Having the symptom does not guarantee the disease.

Conditional probability is common in fields like actuarial science and medicine. For example, “What is the probability a person will survive another 20 years, given they are alive at age 40?” In many real problems, we want to find the probability of event  $A$  when we already know that event  $B$  has happened. For this, we use conditional probability.

**Example 2.10** (Die throw continued). Return to the rolling of a fair die (Example 2.1). Let

$$A = \{\text{a number greater than 3}\} = \{4, 5, 6\}, B = \{\text{an even number}\} = \{2, 4, 6\}.$$

It is clear that  $P\{B\} = 3/6 = 1/2$ . This is the unconditional probability of the event  $B$ . It is sometimes called the *prior* probability of  $B$ .

However, suppose that we are told that the event  $A$  has already occurred. What is the probability of  $B$  now given that  $A$  has already happened?

The sample space of the experiment is  $S = \{1, 2, 3, 4, 5, 6\}$ , which contains  $n = 6$  equally likely outcomes.

Given the partial knowledge that event  $A$  has occurred, only the  $n_A = 3$  outcomes in  $A = \{4, 5, 6\}$  could have occurred. However, only some of the outcomes in  $B$  among these  $n_A$  outcomes in  $A$  will make event  $B$  occur; the number of such outcomes is given by the number of outcomes  $n_{A \cap B}$  in both  $A$  and  $B$ , i.e.,  $A \cap B$ , and equal to 2. Hence the probability of  $B$ , given the partial knowledge that event  $A$  has occurred, is equal to

$$\frac{2}{3} = \frac{n_{A \cap B}}{n_A} = \frac{n_{A \cap B}/n}{n_A/n} = \frac{P\{A \cap B\}}{P\{A\}}$$

Hence we say that  $P\{B|A\} = 2/3$ , which is often interpreted as the *posterior* probability of  $B$  given  $A$ . The additional knowledge that  $A$  has already occurred has helped us to revise the prior probability of  $1/2$  to  $2/3$ .

This simple example leads to the following general definition of conditional probability.

### 2.5.1 Definition of conditional probability

For events  $A$  and  $B$  with  $P\{A\} > 0$ , the conditional probability of event  $B$ , given that event  $A$  has occurred, is

$$P\{B|A\} = \frac{P\{A \cap B\}}{P\{A\}}.$$

**Example 2.11.** Of all individuals buying a mobile phone, 60% include a 64 GB hard disk in their purchase, 40% include a 16 MP camera and 30% include both. If a randomly selected purchase includes a 16 MP camera, what is the probability that a 64GB hard disk is also included? The conditional probability is given by

$$P\{64 \text{ GB}|16 \text{ MP}\} = \frac{P\{64 \text{ GB} \cap 16 \text{ MP}\}}{P\{16 \text{ MP}\}} = \frac{0.3}{0.4} = 0.75.$$

### 2.5.2 Multiplication rule of conditional probability

By rearranging the conditional probability definition, we obtain the multiplication rule of conditional probability:

$$P\{A \cap B\} = P\{A\}P\{B|A\}.$$

Clearly the roles of  $A$  and  $B$  could be interchanged:

$$P\{A \cap B\} = P\{B\}P\{A|B\}.$$

Hence the multiplication rule of conditional probability for two events is

$$P\{A \cap B\} = P\{B\}P\{A|B\} = P\{A\}P\{B|A\}.$$

It is straightforward to show by mathematical induction the following multiplication rule of conditional probability for  $k(\geq 2)$  events  $A_1, A_2, \dots, A_k$ :

$$P\{A_1 \cap A_2 \cap \dots \cap A_k\} = P\{A_1\}P\{A_2|A_1\}P\{A_3|A_1 \cap A_2\} \dots P\{A_k|A_1 \cap A_2 \cap \dots \cap A_{k-1}\}.$$

**Example 2.12** (Selecting a committee continued). Return to the committee selection example (Example 2.8), where there are 4 men (M) and 6 women (W). We want to select a 2-person committee. Find:

- (i) the probability that both are men,
- (ii) the probability that one is a man and the other is a woman.

We have already dealt with this type of urn problem by using the combinatorial method. Here, the multiplication rule is used instead. Let  $M_i$  be the event that the  $i$ th person is a man, and  $W_i$  be the event that the  $i$ th person is a woman,  $i = 1, 2$ . Then

$$\text{Prob in (i)} = P\{M_1 \cap M_2\} = P\{M_1\}P\{M_2|M_1\} = \frac{4}{10} \times \frac{3}{9},$$

$$\begin{aligned} \text{Prob in (ii)} &= P\{M_1 \cap W_2\} + P\{W_1 \cap M_2\} \\ &= P\{M_1\}P\{W_2|M_1\} + P\{W_1\}P\{M_2|W_1\} \\ &= \frac{4}{10} \times \frac{6}{9} + \frac{6}{10} \times \frac{4}{9} \end{aligned}$$

You can find the probability that 'both are women' in a similar way.

### 2.5.3 Total probability formula

**Example 2.13** (Phones). Suppose that in our world there are only three phone manufacturing companies: A Pale, B Sung and C Windows, and their market shares are respectively 30, 40 and 30 percent. Suppose also that respectively 5, 8, and 10 percent of their phones become faulty within one year. If I buy a phone randomly (ignoring the manufacturer), what is the probability that my phone will develop a fault within one year? After finding the probability, suppose that my phone developed a fault in the first year — what is the probability that it was made by A Pale?

Company	Market share	Percent defective
A Pale	30%	5%
B Sung	40%	8%
C Windows	30%	10%

To answer this type of question, we derive two of the most useful results in probability theory: the total probability formula and Bayes' theorem. First, let us derive the total probability formula.

Let  $B_1, B_2, \dots, B_k$  be a set of mutually exclusive, i.e.

$$B_i \cap B_j = \emptyset, \text{ for all } 1 \leq i \neq j \leq k.$$

and exhaustive events, i.e.:

$$B_1 \cup B_2 \cup \dots \cup B_k = S.$$

Now any event  $A$  can be represented by

$$A = A \cap S = (A \cap B_1) \cup (A \cap B_2) \cup \dots \cup (A \cap B_k)$$

where  $(A \cap B_1), (A \cap B_2), \dots, (A \cap B_k)$  are mutually exclusive events. Hence the Axiom A3 of probability gives

$$\begin{aligned} P\{A\} &= P\{A \cap B_1\} + P\{A \cap B_2\} + \dots + P\{A \cap B_k\} \\ &= P\{B_1\}P\{A|B_1\} + P\{B_2\}P\{A|B_2\} + \dots + P\{B_k\}P\{A|B_k\}. \end{aligned}$$

This last expression is called the *total probability formula* for  $P\{A\}$ .

**Example 2.14** (Phones continued). We can now find the probability of the event, say  $A$ , that a randomly selected phone develops a fault within one year. Let  $B_1, B_2, B_3$  be the events that the phone is manufactured respectively by companies A Pale, B Sung and C Windows. Then we have:

$$\begin{aligned} P\{A\} &= P\{B_1\}P\{A|B_1\} + P\{B_2\}P\{A|B_2\} + P\{B_3\}P\{A|B_3\} \\ &= 0.30 \times 0.05 + 0.40 \times 0.08 + 0.30 \times 0.10 \\ &= 0.077. \end{aligned}$$

Now suppose that my phone has developed a fault within one year. What is the probability that it was manufactured by A Pale? To answer this we need to introduce Bayes' Theorem.

### 2.5.4 Bayes' theorem

**Theorem 2.1** (Bayes' Theorem). *Let  $A$  and  $B$  be events. Then*

$$P\{B|A\} = \frac{P\{B\}P\{A|B\}}{P\{A\}}.$$



*Proof.* From the definition of conditional probability, we have

$$P\{B|A\} = \frac{P\{B \cap A\}}{P\{A\}} = \frac{P\{B\}P\{A|B\}}{P\{A\}}.$$

□

The probability  $P\{B|A\}$  is called the posterior probability of  $B$  given  $A$  and  $P\{B\}$  is called the prior probability. Bayes' theorem is the rule that converts the prior probability into the posterior probability by using the additional information that some other event,  $A$  above, has already occurred.

**Example 2.15** (Phones continued). The probability that my faulty phone was manufactured by A Pale is

$$P\{B_1|A\} = \frac{P\{B_1\}P\{A|B_1\}}{P\{A\}} = \frac{0.30 \times 0.05}{0.077} = 0.1948.$$

Similarly, the probability that the faulty phone was manufactured by B Sung is 0.4156, and the probability that it was manufactured by C Windows is  $1 - 0.1948 - 0.4156 = 0.3896$ .

As in this example, we usually need to use the total probability formula to calculate the denominator  $P(A)$  in Bayes' theorem.

## 2.6 Independent events

### 2.6.1 Introduction and definition of independence

Sometimes, knowing that one event has happened changes the probability of another event. But in many cases, it does not. When the probability does not change, we say the events are independent.

Intuitively, events  $A$  and  $B$  are independent if knowing that one occurs does not affect the probability of the other. In other words,  $P\{B|A\} = P\{B\}$  (if  $P\{A\} > 0$ ), and  $P\{A|B\} = P\{A\}$  (if  $P\{B\} > 0$ ).

Formally,  $A$  and  $B$  are *independent* if  $P\{A \cap B\} = P\{A\}P\{B\}$ .

**Example 2.16** (Die throw). Throw a fair die. Let  $A$  be the event that “the result is even” and  $B$  be the event that “the result is greater than 3”. We want to show that  $A$  and  $B$  are not independent.

For this, we have  $P\{A \cap B\} = P\{\text{either a 4 or 6 thrown}\} = 1/3$ , but  $P\{A\} = 1/2$  and  $P\{B\} = 1/2$ , so that  $P\{A\}P\{B\} = 1/4 \neq 1/3 = P\{A \cap B\}$ . Therefore  $A$  and  $B$  are not independent events.

Independence is often assumed for physical reasons, but sometimes incorrectly. Wrongly assuming independence can have serious consequences (for example, in the 2008 financial crisis). If events are independent, we can use the simple product formula for joint probability.

**Example 2.17** (Dice throw). Two fair dice when thrown together are assumed to behave independently. Hence the probability of two sixes is  $1/6 \times 1/6 = 1/36$ .

**Example 2.18** (Assessing risk in legal cases). There have been some disastrous miscarriages of justice as a result of incorrect assumption of independence. Please read “Incorrect use of independence — Sally Clark Case” on Blackboard.

**Theorem 2.2** (Independence of complementary events). *If  $A$  and  $B$  are independent, so are  $A'$  and  $B'$ .*

*Proof.* Given that  $P\{A \cap B\} = P\{A\}P\{B\}$ , we need to show that  $P\{A' \cap B'\} = P\{A'\}P\{B'\}$ . We

have

$$\begin{aligned}
 P\{A' \cap B'\} &= 1 - P\{A \cup B\} \\
 &= 1 - [P\{A\} + P\{B\} - P\{A \cap B\}] \\
 &= 1 - [P\{A\} + P\{B\} - P\{A\}P\{B\}] \\
 &= [1 - P\{A\}] - P\{B\}[1 - P\{A\}] \\
 &= [1 - P\{A\}][1 - P\{B\}] \\
 &= P\{A'\}P\{B'\}
 \end{aligned}$$

□

### 2.6.2 Independence with three events

We can extend the ideas of conditional probability and independence to more than two events.

Three events  $A$ ,  $B$ , and  $C$  are independent if:

$$P\{A \cap B\} = P\{A\}P\{B\}, \quad P\{A \cap C\} = P\{A\}P\{C\}, \quad P\{B \cap C\} = P\{B\}P\{C\}, \quad (2.1)$$

$$P\{A \cap B \cap C\} = P\{A\}P\{B\}P\{C\}. \quad (2.2)$$

Note: Pairwise independence ((2.1)) does NOT imply three-way independence ((2.2)). To show  $A$ ,  $B$ , and  $C$  are independent, both conditions must hold.

**Example 2.19.** A box contains eight tickets, each labelled with a binary number. Two are labelled with the binary number 111, two are labelled with 100, two with 010 and two with 001. An experiment consists of drawing one ticket at random from the box. Let  $A$  be the event “the first digit is 1”,  $B$  the event “the second digit is 1” and  $C$  be the event “the third digit is 1”. It is clear that  $P\{A\} = P\{B\} = P\{C\} = 4/8 = 1/2$  and  $P\{A \cap B\} = P\{A \cap C\} = P\{B \cap C\} = 1/4$ , so the events are pairwise independent, i.e. (2.1) holds. However  $P\{A \cap B \cap C\} = 2/8 \neq P\{A\}P\{B\}P\{C\} = 1/8$ . So (2.2) does not hold and  $A$ ,  $B$  and  $C$  are not independent.

## Chapter 3

# Probability Distributions

### 3.1 Introduction

Combinatorial probabilities from the previous chapter are often problem-specific and tedious to compute. This chapter introduces structured approaches that make probability calculations easier. We represent outcomes of random experiments as values of a random variable, which lets us compute probabilities efficiently in many realistic, stylised settings.

### 3.2 Random variables

#### 3.2.1 Introduction

This section introduces probability distributions of random variables via their probability functions. For discrete variables the probability function is the probability mass function (pmf); for continuous variables it is the probability density function (pdf).

A random variable maps each outcome in the sample space to a real number. For example, consider a fair coin toss with outcomes head or tail. Define the random variable that maps head to 1 and tail to 0:

$$\text{Head} \rightarrow 1, \text{Tail} \rightarrow 0.$$

We can conveniently denote the random variable by  $X$  which is the number of heads obtained by tossing a single coin. The possible values of  $X$  are 0 and 1.

This is a simple example, and the idea generalises directly: define a function from outcomes to real numbers. For instance, toss a coin  $n$  times and let  $X$  be the number of heads; then  $X$  takes integer values from 0 to  $n$ . As another example, select a student at random and measure their height. The resulting value, in metres, lies on a continuum (e.g., 1.432 m), and we cannot know it in advance because the selected student is random.

We use uppercase letters (e.g.,  $X$ ,  $Y$ ,  $Z$ ) for random variables and the corresponding lowercase letters (e.g.,  $x$ ,  $y$ ,  $z$ ) for particular values (e.g.,  $x = 1.432$  m). We also write probabilities as  $P(X \in A)$ , read “the probability that  $X$  lies in  $A$ ,” instead of  $P\{A\}$ .

#### 3.2.2 Discrete and continuous random variables

A random variable is discrete if it takes a finite or countably infinite set of values (e.g., the number of Apple users among 20 students; the number of credit cards a person carries). It is continuous if it can

take any real value (e.g., a student's height). Some variables mix discrete and continuous parts (e.g., daily rainfall: zero on some days, a positive continuous amount on others).

### 3.2.3 Probability distribution of a random variable

By the first axiom of probability, total probability is 1. Because a random variable maps outcomes to the real line, the probabilities assigned to all its possible values must sum (or integrate) to 1. A probability distribution allocates this total probability across those values.

**Example 3.1.** Returning to the coin-tossing experiment, if the probability of getting a head with a coin is  $p$  (and therefore the probability of getting a tail is  $1 - p$ ), then the probability that  $Y = 0$  is  $1 - p$  and the probability that  $Y = 1$  is  $p$ . This gives us the probability distribution of  $Y$ , and we say that  $Y$  has the probability function

$$P(Y = y) = \begin{cases} 1 - p & \text{for } y = 0 \\ p & \text{for } y = 1. \end{cases}$$

This is an example of the **Bernoulli distribution** with parameter  $p$ , the simplest discrete distribution.

**Example 3.2.** Suppose we consider tossing the coin twice and again defining the random variable  $X$  to be the number of heads obtained. The values that  $X$  can take are 0, 1 and 2 with probabilities  $(1 - p)^2$ ,  $2p(1 - p)$  and  $p^2$ , respectively. Here the probability function is

$$P(X = x) = \begin{cases} (1 - p)^2 & \text{for } x = 0 \\ 2p(1 - p) & \text{for } x = 1 \\ p^2 & \text{for } x = 2. \end{cases}$$

This is a particular case of the Binomial distribution. We will learn about it soon.

In general, for a discrete random variable we define a function  $f(x)$  to denote  $P(X = x)$  (or  $f(y)$  to denote  $P(Y = y)$ ) and call the function  $f(x)$  the probability mass function (pmf) of  $X$ . Not every function qualifies: probabilities are nonnegative and sum to 1. Thus, for  $f(x)$  to be a pmf of  $X$ , we require:

1.  $f(x) \geq 0$  for all possible values of  $x$ .
2.  $\sum_{\text{all } x} f(x) = 1$

In Example 3.2, we may rewrite the probability function in the general form

$$f(x) = \binom{2}{x} p^x (1 - p)^{2-x}, \text{ for } x = 0, 1, 2,$$

where  $f(x) = 0$  for any other value of  $x$ .

### 3.2.4 Continuous random variables

In many situations (both theoretical and practical) we often encounter random variables that are inherently continuous because they are measured on a continuum (such as time, length, weight) or can be conveniently well-approximated by considering them as continuous (such as the annual income of adults in a population, closing share prices).

For a continuous random variable,  $P(X = x) = 0$  for any specific  $x$  (e.g., exactly 1.2), because measurements lie on a continuum. Instead, we assign probabilities to intervals with positive length, such as  $P(1.2 < X < 1.9)$ .

For a continuous random variable  $X$ , probabilities derive from a nonnegative function  $f(x)$ , the probability density function (pdf). We compute probabilities via integrals, e.g.,

$$P(a < X < b) = \int_a^b f(u)du,$$

which is naturally interpreted as the area under the curve  $f(x)$  inside the interval  $(a, b)$ . This is demonstrated in Figure 3.1. Recall that we do not use  $f(x) = P(X = x)$  for any  $x$  as by convention we set  $P(X = x) = 0$ .

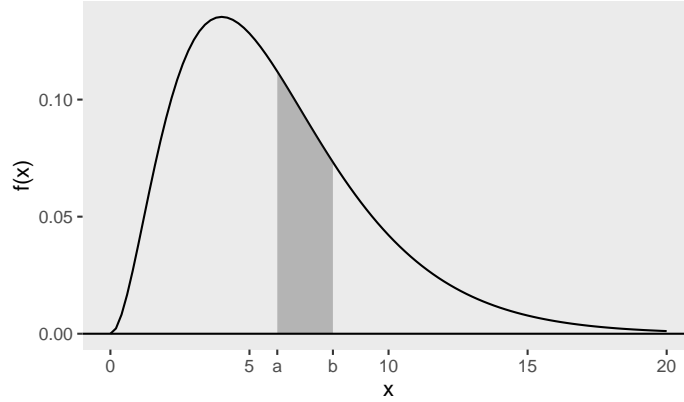


Figure 3.1: The shaded area is  $P(a < X < b)$  if the pdf of  $X$  is the drawn curve.

Since we are dealing with probabilities which are always between 0 and 1, just any arbitrary function  $f(x)$  cannot be a pdf of some random variable. For  $f(x)$  to be a pdf, as in the discrete case, we must have

1.  $f(x) \geq 0$  for all possible values of  $x$ , i.e.  $-\infty < x < \infty$ ,
2.  $\int_{-\infty}^{\infty} f(u)du = 1$ .

### 3.2.5 Cumulative distribution function (cdf)

We also use the cumulative distribution function (cdf), which gives the probability that the random variable is less than or equal to a given value.

For a discrete random variable  $X$ , the cdf is the cumulative sum of the pmf  $f(u)$  up to (and including)  $u = x$ . That is,

$$P(X \leq x) \equiv F(x) = \sum_{u \leq x} f(u).$$

**Example 3.3.** Let  $X$  be the number of heads in the experiment of tossing two fair coins. Then the probability function is

$$P(X = 0) = 1/4, P(X = 1) = 1/2, P(X = 2) = 1/4.$$

From the definition, the CDF is given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/4 & \text{if } 0 \leq x < 1 \\ 3/4 & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

The cdf for a discrete random variable is a step function. The jump-points are the possible values of the random variable, and the height of a jump gives the probability of the random variable taking that value. It is clear that the probability mass function is uniquely determined by the cdf.

For a continuous random variable  $X$ , the cdf is defined as

$$P(X \leq x) \equiv F(x) = \int_{-\infty}^x f(u)du.$$

The fundamental theorem of calculus then tells us that

$$f(x) = \frac{dF(x)}{dx},$$

so for a continuous random variable the pdf is the derivative of the cdf. Also for any random variable  $X$ ,  $P(c < X \leq d) = F(d) - F(c)$ . Let us consider an example.

**Example 3.4** (Uniform distribution). Suppose

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise.} \end{cases}$$

In this case we say  $X$  has *uniform distribution* on the interval  $(a, b)$ , which we will write as  $X \sim U(a, b)$ . We now have the cdf

$$F(x) = \int_a^x \frac{1}{b-a} du = \frac{x-a}{b-a}, \quad a < x < b.$$

A quick check confirms that  $F'(x) = f(x)$ . If  $a = 0$  and  $b = 1$ , then

$$P(0.5 < X < 0.75) = F(0.75) - F(0.5) = 0.25.$$

We will see more examples later.

## 3.3 Summaries of a random variable

### 3.3.1 Introduction

In Section 1.4, we defined various summaries of sample data  $x_1, \dots, x_n$ , such as the mean and variance. A random variable  $X$  with either a pmf  $f(x)$  or a pdf  $f(x)$  may be summarised using similar measures.

### 3.3.2 Expectation

The mean of  $X$ , or expectation, is defined as

$$E(X) = \begin{cases} \sum_{\text{all } x} xf(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} xf(x)dx & \text{if } X \text{ is continuous} \end{cases}$$

when the sum or integral exists. Informally, expectation is the sum or integral of “value  $\times$  probability.” We write  $E(\cdot)$  for expectation and often denote  $E(X)$  by  $\mu$ .

**Example 3.5.** Consider the fair-die tossing experiment, with each of the six sides having a probability of  $1/6$  of landing face up. Let  $X$  be the number on the up-face of the die. Then

$$E(X) = \sum_{x=1}^6 xP(X=x) = \sum_{x=1}^6 x/6 = 3.5.$$

**Example 3.6.** Suppose  $X \sim U(a, b)$ , with pdf  $f(x) = \frac{1}{b-a}$ ,  $a < x < b$ . Then

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_a^b \frac{x}{b-a}dx \\ &= \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}, \end{aligned}$$

the mid-point of the interval  $(a, b)$ .

If  $Y = g(X)$  for any function  $g(\cdot)$ , then  $Y$  is a random variable as well. To find  $E(Y)$  we simply use the value times probability rule, i.e. the expected value of  $Y$  is either sum or integral of its value,  $g(x)$ , times probability  $f(x)$ :

$$E(Y) = E(g(X)) = \begin{cases} \sum_{\text{all } x} g(x)f(x) & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{\infty} g(x)f(x)dx & \text{if } X \text{ is continuous.} \end{cases}$$

For example, if  $X$  is continuous, then  $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$ . We prove an important property of expectation, namely expectation is a linear operator.

**Theorem 3.1** (Linearity of expectation). *If  $Y = aX + b$ , then  $E(Y) = aE(X) + b$ .*

*Proof.* The proof is given for the continuous case. In the discrete case replace integral ( $\int$ ) by summation ( $\sum$ ).

$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} (ax + b)f(x)dx \\ &= a \int_{-\infty}^{\infty} xf(x)dx + b \int_{-\infty}^{\infty} f(x)dx \\ &= aE(X) + b, \end{aligned}$$

using the total probability is 1 property ( $\int_{-\infty}^{\infty} f(x)dx = 1$ ) in the last integral. □

This is very convenient, e.g. suppose  $E(X) = 5$  and  $Y = -2X + 549$  then  $E(Y) = 539$ .

We will also prove an important property of expectation for symmetric random variables.

**Theorem 3.2.** *If the pmf/pdf of  $X$  is symmetric about  $c$  (i.e.,  $f(c+x) = f(c-x)$  for all  $x > 0$ ), then  $E(X) = c$ .*

*Proof.* The proof is given for the continuous case. In the discrete case replace integral ( $\int$ ) by summation ( $\sum$ ).

First, let  $Y = X - c$ . Then  $Y$  is symmetric about 0, with probability density function  $f(y) = f(-y)$

for all  $y > 0$ . Then

$$\begin{aligned}
 E(Y) &= \int_{-\infty}^{\infty} yf(y)dy \\
 &= \int_{-\infty}^0 yf(y)dy + \int_0^{\infty} yf(y)dy \\
 &= \int_0^{\infty} -zf(-z)dz + \int_0^{\infty} yf(y)dy, \text{ substituting } z = -y \\
 &= -\int_0^{\infty} zf(z)dz + \int_0^{\infty} yf(y)dy, \text{ since } f(-z) = f(z) \\
 &= 0.
 \end{aligned}$$

So, by the linearity of expectation,  $E(X) = E(Y + c) = E(Y) + c = 0 + c = c$ .  $\square$

This result makes it very easy to find the expectation of any symmetric random variable. The two examples we saw before, of a fair die and of a uniform random variable, were both symmetric, and they have expectation equal to the point of symmetry.

### 3.3.3 Variance

Variance measures the spread of a random variable and is defined by

$$\text{Var}(X) = E(X - \mu)^2 = \begin{cases} \sum_{\text{all } x} (x - \mu)^2 f(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx & \text{if } X \text{ is continuous,} \end{cases}$$

where  $\mu = E(X)$ , and when the sum or integral exists. When the variance exists, it is the expectation of  $(X - \mu)^2$  where  $\mu$  is the mean of  $X$ . A useful identity is:

**Theorem 3.3.**

$$\text{Var}(X) = E(X - \mu)^2 = E(X^2) - \mu^2.$$

*Proof.* We have

$$\begin{aligned}
 \text{Var}(X) &= E(X - \mu)^2 \\
 &= E(X^2 - 2X\mu + \mu^2) \\
 &= E(X^2) - 2\mu E(X) + \mu^2 \\
 &= E(X^2) - 2\mu\mu + \mu^2 \\
 &= E(X^2) - \mu^2.
 \end{aligned}$$

$\square$

Thus, variance equals the expected square minus the square of the mean. We denote variance by  $\sigma^2$ , which is always nonnegative and equals zero only if  $X = \mu$  with probability 1 (no randomness). The square root,  $\sigma$ , is the standard deviation.

**Example 3.7** (Uniform distribution). Suppose  $X \sim U(a, b)$ , with pdf  $f(x) = \frac{1}{b-a}$ ,  $a < x < b$ . Then

$$\begin{aligned}
 E(X^2) &= \int_a^b \frac{x^2}{b-a} dx \\
 &= \frac{b^3 - a^3}{3(b-a)} \\
 &= \frac{b^2 + ab + a^2}{3}.
 \end{aligned}$$



Hence

$$\text{Var}(X) = \frac{b^2 + ab + a^2}{3} - \left(\frac{b+a}{2}\right)^2 = \frac{(b-a)^2}{12},$$

after simplification.

We prove one important property of the variance.

**Theorem 3.4.** Suppose  $Y = aX + b$  then  $\text{Var}(Y) = a^2 \text{Var}(X)$

*Proof.* Write  $\mu = E(X)$ . Then  $E(Y) = a\mu + b$  and

$$\begin{aligned} \text{Var}(Y) &= E[(Y - E(Y))^2] \\ &= E[(aX + b - a\mu - b)^2] \\ &= E[a^2(X - \mu)^2] \\ &= a^2 E[(X - \mu)^2] \\ &= a^2 \text{Var}(X) \end{aligned}$$

□

This is a very useful result, e.g. suppose  $\text{Var}(X) = 25$  and  $Y = -X + 5,000,000$ ; then  $\text{Var}(Y) = \text{Var}(X) = 25$  and the standard deviation,  $\sigma = 5$ . In words a location shift,  $b$ , does not change variance but a multiplicative constant,  $a$  say, gets squared in variance,  $a^2$ .

### 3.3.4 Quantiles

For  $0 < p < 1$ , a  $p$ th quantile (or 100 $p$ th percentile) of  $X$  with cdf  $F$  is any  $q$  such that  $F(q) = p$ . If  $F$  is invertible,  $q = F^{-1}(p)$ .

The 50th percentile is called the *median*. The 25th and 75th percentiles are called the *quartiles*.

**Example 3.8** (Uniform distribution). Suppose  $X \sim U(a, b)$ , with pdf  $f(x) = \frac{1}{b-a}$ ,  $a < x < b$ . We have shown in Example 3.4 that the cdf is

$$F(x) = \frac{x-a}{b-a}, \quad a < x < b.$$

So for a given  $p$ ,  $F(q) = p$  implies

$$q = a + p(b-a).$$

The median of  $X$  is  $\frac{b+a}{2}$  and the quartiles are  $\frac{b+3a}{4}$  and  $\frac{3b+a}{4}$ .

The median of a symmetric random variable is the point of symmetry: :: {theorem} Suppose  $X$  is a random variable with a probability function or probability density function which is symmetric about some value  $c$ , so

$$f(c+x) = f(c-x) \text{ for all } x > 0.$$

Then the median of  $X$  is  $c$ . ::

## 3.4 Standard discrete distributions

### 3.4.1 Bernoulli distribution

Bernoulli trials are independent trials with two outcomes (success S, failure F) and common success probability.

Suppose that we conduct one Bernoulli trial, where we get a success ( $S$ ) or failure ( $F$ ) with probabilities  $P\{S\} = p$  and  $P\{F\} = 1 - p$  respectively. Let  $X$  be an indicator of success:

$$X = \begin{cases} 1 & \text{if } S \\ 0 & \text{if } F. \end{cases}$$

Then  $X$  has Bernoulli distribution with parameter  $p$ , written as  $X \sim \text{Bernoulli}(p)$ .

The Bernoulli distribution has pmf

$$f(x) = p^x(1-p)^{1-x}, x = 0, 1.$$

Hence

$$\begin{aligned} E(X) &= 0 \cdot (1-p) + 1 \cdot p = p, \\ E(X^2) &= 0^2 \cdot (1-p) + 1^2 \cdot p = p \end{aligned}$$

and

$$\text{Var}(X) = E(X^2) - (E(X))^2 = p - p^2 = p(1-p).$$

### 3.4.2 Binomial distribution

#### 3.4.2.1 Introduction and definition

In  $n$  independent Bernoulli trials with success probability  $p$ , let  $X$  be the number of successes. Then  $X \sim \text{Bin}(n, p)$  with  $X \sim \text{Bin}(n, p)$ .

An outcome of the experiment (of carrying out  $n$  such independent trials) is represented by a sequence of  $S$ 's and  $F$ 's (such as  $SS \dots FS \dots SF$ ) that comprises  $x$   $S$ 's, and  $(n-x)$   $F$ 's. The probability associated with this outcome is

$$P\{SS \dots FS \dots SF\} = pp \dots (1-p)p \dots p(1-p) = p^x(1-p)^{n-x}.$$

For this sequence,  $X = x$ , but there are many other sequences which will also give  $X = x$ . In fact there are  $\binom{n}{x}$  such sequences. Hence

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, \dots, n.$$

This is the pmf of the binomial distribution with parameters  $n$  and  $p$

How can we guarantee that  $\sum_{x=0}^n P(X = x) = 1$ ? This guarantee is provided by the binomial theorem:

**Theorem 3.5** (Binomial theorem). *For any positive integer  $n$  and real numbers  $a$  and  $b$ ,*

$$(a+b)^n = b^n + \binom{n}{1} ab^{n-1} + \dots + \binom{n}{x} a^x b^{n-x} + \dots + a^n.$$

Setting  $a = p$  and  $b = 1 - p$  in the binomial theorem shows  $\sum_{x=0}^n \binom{n}{x} p^x (1-p)^{n-x} = 1$ .

**Example 3.9.** Suppose that widgets are manufactured in a mass production process with 1% defective. The widgets are packaged in bags of 10 with a money-back guarantee if more than 1 widget per bag is defective. For what proportion of bags would the company have to provide a refund?

First, we find the probability that a randomly selected bag has at most 1 defective widget. Let  $X$  be the number of defective widgets in a bag, then  $X \sim \text{Bin}(n = 10, p = 0.01)$ . So this probability is equal to

$$P(X = 0) + P(X = 1) = (0.99)^{10} + 10(0.01)^1(0.99)^9 = 0.9957.$$

Hence the probability that a refund is required is  $1 - 0.9957 = 0.0043$ , i.e. only just over 4 in 1000 bags will incur the refund on average.

**Example 3.10.** A binomial random variable can also be described using the urn model. Suppose we have an urn (population) containing  $N$  individuals, a proportion  $p$  of which are of type  $S$  and a proportion  $1 - p$  of type  $F$ . If we select a sample of  $n$  individuals at random with replacement, then the number,  $X$ , of type  $S$  individuals in the sample follows the binomial distribution with parameters  $n$  and  $p$ .

### 3.4.2.2 Using R to calculate probabilities

Probabilities under all the standard distributions have been calculated in R and will be used throughout MATH1063. You will not be required to use any tables. For the binomial distribution the command

```
dbinom(x=3, size=5, prob=0.34)
```

calculates the pmf of  $\text{Bin}(n = 5, p = 0.34)$  at  $x = 3$ , with value  $P(X = 3) = \binom{5}{3}(0.34)^3(1 - 0.34)^{5-3}$ . The function `pbinom` returns the cdf (probability up to and including the argument). Thus

```
pbinom(q=3, size=5, prob=0.34)
```

will return the value of  $P(X \leq 3)$  when  $X \sim \text{Bin}(n = 5, p = 0.34)$ . As a check, in Example 3.9, we may compute the probability that a randomly selected bag has at most 1 defective widget with the command

```
pbinom(q=1, size=10, prob=0.01)
```

```
## [1] 0.9957338
```

which matches our earlier calculations.

### 3.4.2.3 Expectation

Let  $X \sim \text{Bin}(n, p)$ . We have

$$E(X) = \sum_{x=0}^n xP(X = x) = \sum_{x=0}^n x \binom{n}{x} p^x (1-p)^{n-x}.$$

Below we prove that  $E(X) = np$ . Recall that  $k! = k(k-1)!$  for any  $k > 0$ .

$$\begin{aligned} E(X) &= \sum_{x=0}^n x \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=1}^n x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \\ &= \sum_{x=1}^n \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x} \\ &= np \sum_{x=1}^n \frac{(n-1)!}{(x-1)!(n-1-x+1)!} p^{x-1} (1-p)^{n-1-x+1} \\ &= np \sum_{y=0}^{n-1} \frac{(n-1)!}{y!(n-1-y)!} p^y (1-p)^{n-1-y} \\ &= np \sum_{y=0}^{n-1} \binom{n-1}{y} p^y (1-p)^{n-1-y} \\ &= np \end{aligned}$$

where we used the substitution  $y = x - 1$  and then conclude the last sum equals one as it is the sum of all probabilities in the  $\text{Bin}(n-1, p)$  distribution.

### 3.4.2.4 Variance

Let  $X \sim \text{Bin}(n, p)$ . Then  $\text{Var}(X) = np(1 - p)$ . It is difficult to find  $E(X^2)$  directly, but the factorial structure allows us to find  $E[X(X - 1)]$ . Recall that  $k! = k(k - 1)(k - 2)!$  for any  $k > 1$ .

$$\begin{aligned}
 E[(X(X - 1))] &= \sum_{x=0}^n x(x - 1) \binom{n}{x} p^x (1 - p)^{n-x} \\
 &= \sum_{x=2}^n x(x - 1) \frac{n!}{x!(n - x)!} p^x (1 - p)^{n-x} \\
 &= \sum_{x=2}^n \frac{n!}{(x - 2)!(n - x)!} p^x (1 - p)^{n-x} \\
 &= n(n - 1)p^2 \sum_{x=2}^n \frac{(n - 2)!}{(x - 2)!(n - 2 - x + 2)!} p^{x-2} (1 - p)^{n-2-x+2} \\
 &= n(n - 1)p^2 \sum_{y=0}^{n-2} \frac{(n - 2)!}{(y)!(n - 2 - y)!} p^y (1 - p)^{n-2-y} \\
 &= n(n - 1)p^2
 \end{aligned}$$

where we used the substitution  $y = x - 2$  and then conclude the last sum equals one as it is the sum of all probabilities in the  $\text{Bin}(n - 2, p)$  distribution. Now,  $E(X^2) = E[X(X - 1)] + E(X) = n(n - 1)p^2 + np$ . Hence,

$$\text{Var}(X) = E(X^2) - (E(X))^2 = n(n - 1)p^2 + np - (np)^2 = np(1 - p).$$

It is illuminating to see these direct proofs. Later on we shall apply statistical theory to directly prove these.

## 3.4.3 Geometric distribution

### 3.4.3.1 Introduction and definition

Suppose that we have the same situation as for the binomial distribution but we consider a different random variable  $X$ , which is defined as the number of trials that lead to the first success. The outcomes for this experiment are:

$$\begin{array}{ll}
 S & X = 1, \quad P(X = 1) = p \\
 FS & X = 2, \quad P(X = 2) = (1 - p)p \\
 FFS & X = 3, \quad P(X = 3) = (1 - p)^2 p \\
 FFFS & X = 4, \quad P(X = 4) = (1 - p)^3 p \\
 \vdots & \vdots
 \end{array}$$

In general,

$$P(X = x) = (1 - p)^{x-1} p, \quad x = 1, 2, \dots$$

This is the geometric distribution, supported on  $\{1, 2, \dots\}$ , written  $X \sim \text{Geo}(p)$ .

**Example 3.11.** In a board game that uses a single fair die, a player cannot start until they have rolled a six. Let  $X$  be the number of rolls needed until they get a six. Then  $X$  is a Geometric random variable with success probability  $p = 1/6$ .

In order to check the probability function sums to one, we will need to use the general result on the geometric series:

**Theorem 3.6** (Geometric series). *For any real numbers  $a$  and  $r$  such that  $|r| < 1$ ,*

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1 - r}.$$

We now check the probability function sums to one:

$$\begin{aligned}
 \sum_{x=1}^{\infty} P(X = x) &= \sum_{x=1}^{\infty} (1-p)^{x-1} p \\
 &= \sum_{y=0}^{\infty} (1-p)^y p \quad (\text{substitute } y = x - 1) \\
 &= \frac{p}{1 - (1-p)} \quad (\text{geometric series, } a = p, r = 1 - p) \\
 &= 1
 \end{aligned}$$

We can also find the probability that  $X > k$  for some given positive integer  $k$  :

$$\begin{aligned}
 \sum_{x=k+1}^{\infty} P(X = x) &= \sum_{x=k+1}^{\infty} (1-p)^{x-1} p \\
 &= p [(1-p)^{k+1-1} + (1-p)^{k+2-1} + (1-p)^{k+3-1} + \dots] \\
 &= p(1-p)^k \sum_{y=0}^{\infty} (1-p)^y \\
 &= (1-p)^k
 \end{aligned}$$

### 3.4.3.2 Memoryless property

Let  $X$  follow the geometric distribution and suppose that  $s$  and  $k$  are positive integers. We then have

$$P(X > s + k \mid X > k) = P(X > s).$$

This means the process “forgets” how long it has already waited: the remaining wait distribution is the same as at time zero. Using the definition of conditional probability,

$$P\{A \mid B\} = \frac{P\{A \cap B\}}{P\{B\}}.$$

Now the proof,

$$\begin{aligned}
 P(X > s + k \mid X > k) &= \frac{P(X > s + k, X > k)}{P(X > k)} \\
 &= \frac{P(X > s + k)}{P(X > k)} \\
 &= \frac{(1-p)^{s+k}}{(1-p)^k} \\
 &= (1-p)^s,
 \end{aligned}$$

which does not depend on  $k$ . Note that the event  $X > s + k$  and  $X > k$  implies and is implied by  $X > s + k$  since  $s > 0$ .

### 3.4.3.3 Expectation and variance

Let  $X \sim \text{Geo}(p)$ . We can show that  $E(X) = 1/p$  using the negative binomial series:

**Theorem 3.7** (Negative binomial series). *For any positive integer  $n$  and real number  $x$  such that  $|x| < 1$*

$$(1-x)^{-n} = 1 + nx + \frac{1}{2}n(n+1)x^2 + \frac{1}{6}n(n+1)(n+2)x^3 + \dots + \frac{n(n+1)(n+2)\dots(n+k-1)}{k!}x^k + \dots$$

We have

$$\begin{aligned} E(X) &= \sum_{x=1}^{\infty} xP(X=x) \\ &= \sum_{x=1}^{\infty} xp(1-p)^{x-1} \\ &= p[1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + \dots] \end{aligned}$$

The series in the square brackets is the negative binomial series with  $n = 2$  and  $x = 1 - p$ . Thus  $E(X) = p(1 - 1 + p)^{-2} = 1/p$ . It can be shown that  $\text{Var}(X) = (1-p)/p^2$  using negative binomial series. But this is more complicated and is not required here. The second-year module MATH2011 will provide an alternative proof.

### 3.4.4 Hypergeometric distribution

Consider an urn with  $N$  individuals, of which a proportion  $p$  are type  $S$  and  $1-p$  are type  $F$ . Sampling  $n$  without replacement, let  $X$  be the number of type  $S$ ; then  $X$  has the hypergeometric distribution, with pmf

$$P(X=x) = \frac{\binom{Np}{x} \binom{N(1-p)}{n-x}}{\binom{N}{n}}, \quad x = 0, 1, \dots, n,$$

assuming that  $x \leq Np$  and  $n-x \leq N(1-p)$  so that the above combinations are well defined. The mean and variance of the hypergeometric distribution are given by

$$E(X) = np, \quad \text{Var}(X) = np(1-p) \frac{N-n}{N-1}.$$

### 3.4.5 Negative binomial distribution

In Bernoulli trials, let  $X$  be the total number of trials needed to observe the  $r$ th success, for a fixed positive integer  $r$ . Then  $X$  follows the negative binomial distribution with parameters  $r$  and  $p$ . [Note: if  $r = 1$ , the negative binomial distribution is just the geometric distribution.] Firstly we need to identify the possible values of  $X$ . Possible values for  $X$  are  $x = r, r+1, r+2, \dots$ . The probability mass function is

$$\begin{aligned} P(X=x) &= \binom{x-1}{r-1} p^{r-1} (1-p)^{(x-1)-(r-1)} \times p \\ &= \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x = r, r+1, \dots \end{aligned}$$

**Example 3.12.** A man plays roulette, betting on red each time. He decides to keep playing until he achieves his second win. The success probability for each game is  $18/37$  and the results of games are independent. Let  $X$  be the number of games played until he gets his second win. Then  $X$  is a negative binomial random variable with  $r = 2$  and  $p = 18/37$ .

What is the probability he plays more than 3 games? We have

$$P(X > 3) = 1 - P(X=2) - P(X=3) = 1 - p^2 - 2p^2(1-p) = 0.520.$$

Derivation of the mean and variance of the negative binomial distribution involves complicated negative binomial series and will be skipped for now, but will be proved in Section 3.7. For completeness we note down the mean and variance:

$$E(X) = \frac{r}{p}, \quad \text{Var}(X) = r \frac{1-p}{p^2}.$$

Thus when  $r = 1$ , the mean and variance of the negative binomial distribution are equal to those of the geometric distribution.

### 3.4.6 Poisson distribution

#### 3.4.6.1 Introduction and definition

The Poisson distribution arises as a limit of  $\text{Bin}(n, p)$  when  $n \rightarrow \infty$ ,  $p \rightarrow 0$ , and  $\lambda = np$  stays finite. It models counts of rare events. Its support is  $\{0, 1, 2, \dots\}$ . Examples include the number of cases in a region, the number of texts a student sends per day, or the number of credit cards a person carries.

Let us find the pmf of the Poisson distribution as the limit of the pmf of the binomial distribution.

Recall that if  $X \sim \text{Bin}(n, p)$  then  $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$ . Now:

$$\begin{aligned} P(X = x) &= \binom{n}{x} p^x (1-p)^{n-x} \\ &= \binom{n}{x} \frac{n^n}{n^n} p^x (1-p)^{n-x} \\ &= \frac{n(n-1) \cdots (n-x+1)}{n^x x!} (np)^x (n(1-p))^{n-x} \frac{1}{n^{n-x}} \\ &= \frac{n}{n} \frac{(n-1)}{n} \cdots \frac{(n-x+1)}{n} \frac{\lambda^x}{x!} \left(1 - \frac{\lambda}{n}\right)^{n-x} \\ &= \frac{n}{n} \frac{(n-1)}{n} \cdots \frac{(n-x+1)}{n} \frac{\lambda^x}{x!} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x}. \end{aligned}$$

Now it is easy to see that the above tends to

$$e^{-\lambda} \frac{\lambda^x}{x!}$$

as  $n \rightarrow \infty$  for any fixed value of  $x$  in the range  $0, 1, 2, \dots$ . Note that we have used the exponential limit:

$$e^{-\lambda} = \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n,$$

and

$$\lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^{-x} = 1$$

and

$$\lim_{n \rightarrow \infty} \frac{n}{n} \frac{(n-1)}{n} \cdots \frac{(n-x+1)}{n} = 1.$$

A random variable  $X$  has the Poisson distribution with parameter  $\lambda$  if it has the pmf:

$$P(X = x) = e^{-\lambda} \frac{\lambda^x}{x!}, \quad x = 0, 1, 2, \dots$$

We write  $X \sim \text{Poisson}(\lambda)$ . It is easy to show  $\sum_{x=0}^{\infty} P(X = x) = 1$ , i.e.  $\sum_{x=0}^{\infty} e^{-\lambda} \frac{\lambda^x}{x!} = 1$ . The identity you need is simply the expansion of  $e^\lambda$ .

**3.4.6.2 Expectation**

Let  $X \sim \text{Poisson}(\lambda)$ . Then

$$\begin{aligned}
 E(X) &= \sum_{x=0}^{\infty} xP(X=x) \\
 &= \sum_{x=0}^{\infty} xe^{-\lambda} \frac{\lambda^x}{x!} \\
 &= e^{-\lambda} \sum_{x=1}^{\infty} x \frac{\lambda^x}{x!} \\
 &= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda \cdot \lambda^{(x-1)}}{(x-1)!} \\
 &= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{(x-1)}}{(x-1)!} \\
 &= \lambda e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} \quad (y = x-1) \\
 &= \lambda e^{-\lambda} e^{\lambda} \quad \text{using the expansion of } e^{\lambda} \\
 &= \lambda.
 \end{aligned}$$

**3.4.6.3 Variance**

Let  $X \sim \text{Poisson}(\lambda)$ . Then

$$\begin{aligned}
 E[X(X-1)] &= \sum_{x=0}^{\infty} x(x-1)P(X=x) \\
 &= \sum_{x=0}^{\infty} x(x-1)e^{-\lambda} \frac{\lambda^x}{x!} \\
 &= e^{-\lambda} \sum_{x=2}^{\infty} x(x-1) \frac{\lambda^x}{x!} \\
 &= e^{-\lambda} \sum_{x=2}^{\infty} \lambda^2 \frac{\lambda^{x-2}}{(x-2)!} \\
 &= \lambda^2 e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} \quad (y = x-2) \\
 &= \lambda^2 e^{-\lambda} e^{\lambda} = \lambda^2 \quad \text{using the expansion of } e^{\lambda}.
 \end{aligned}$$

Now,  $E(X^2) = E[X(X-1)] + E(X) = \lambda^2 + \lambda$ . Hence,

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

Hence, the mean and variance are the same for the Poisson distribution.

**3.4.6.4 Using R to calculate probabilities**

For the Poisson distribution the command



```
dpois(x=3, lambda=5)
```

calculates the pmf of  $\text{Poisson}(\lambda = 5)$  at  $x = 3$ . That is, the command will return the value  $P(X = 3) = e^{-5} \frac{5^3}{3!}$ . The command `ppois` returns the cdf or the probability up to and including the argument. Thus

```
ppois(q=3, lambda=5)
```

will return the value of  $P(X \leq 3)$  when  $X \sim \text{Poisson}(\lambda = 5)$ .

## 3.5 Standard continuous distributions

### 3.5.1 Uniform distribution

#### 3.5.1.1 Definition and properties

A continuous random variable  $X$  is said to follow the uniform distribution if its pdf is of the form:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

where  $a < b$  are parameters. We write  $X \sim U(a, b)$ .

We have already derived various properties the uniform distribution:

- **cumulative distribution function:**

$$F(x) = \frac{x-a}{b-a}, \quad a < x < b$$

from Example 3.4.

- **expectation:**

$$E(X) = \frac{b+a}{2}$$

from Example 3.6.

- **variance:**

$$\text{Var}(x) = \frac{(b-a)^2}{12}$$

from Example 3.7.

- **quantiles:** The  $p$ th quantile is  $a + p(b-a)$  from Example 3.8. The median is  $\frac{b+a}{2}$ .

#### 3.5.1.2 Using R to calculate probabilities

For  $X \sim U(a = -1, b = 1)$ , the command

```
dunif(x = 0.5, min = -1, max = 1)
```

calculates the pdf at  $x = 0.5$ . We specify  $a$  with the `min` argument and  $b$  with `max` argument. The command `pnif` returns the cdf or the probability up to and including the argument. Thus

```
pnif(q = 0.5, min = -1, max = 1)
```

will return the value of  $P(X \leq 0.5)$ .

The command `qunif` can be used to calculate quantiles. Thus

```
qunif(p = 0.5, min = -1, max = 1)
```

finds the median (the 0.5 quantile).

### 3.5.2 Exponential distribution

#### 3.5.2.1 Introduction and definition

A continuous random variable  $X$  is said to follow the exponential distribution if its pdf is of the form:

$$f(x) = \begin{cases} \theta e^{-\theta x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

where  $\theta > 0$  is a parameter. We write  $X \sim \text{Exponential}(\theta)$ . The support is  $(0, \infty)$ , and the tail decays exponentially as  $x \rightarrow \infty$  at rate  $\theta$  (the rate parameter).

It is easy to prove that  $\int_0^\infty f(x)dx = 1$ . This is left as an exercise. To find the mean and variance of the distribution we need to introduce the *gamma function*:

The gamma function  $\Gamma(\cdot)$  is defined for any positive number  $a$  as

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$$

We have the following facts:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}; \quad \Gamma(1) = 1; \quad \Gamma(a) = (a-1)\Gamma(a-1) \text{ if } a > 1$$

These last two facts imply that  $\Gamma(k) = (k-1)!$  when  $k$  is a positive integer. Find  $\Gamma\left(\frac{3}{2}\right)$ .

#### 3.5.2.2 Expectation and variance

By definition,

$$\begin{aligned} E(X) &= \int_{-\infty}^\infty x f(x) dx \\ &= \int_0^\infty x \theta e^{-\theta x} dx \\ &= \int_0^\infty y e^{-y} \frac{dy}{\theta} \quad (\text{substitute } y = \theta x) \\ &= \frac{1}{\theta} \int_0^\infty y^{2-1} e^{-y} dy \\ &= \frac{1}{\theta} \Gamma(2) \\ &= \frac{1}{\theta} \quad \text{since } \Gamma(2) = 1! = 1. \end{aligned}$$

Now,

$$\begin{aligned} E(X^2) &= \int_{-\infty}^\infty x^2 f(x) dx \\ &= \int_0^\infty x^2 \theta e^{-\theta x} dx \\ &= \theta \int_0^\infty \left(\frac{y}{\theta}\right)^2 e^{-y} \frac{dy}{\theta} \quad (\text{substitute } y = \theta x) \\ &= \frac{1}{\theta^2} \int_0^\infty y^{3-1} e^{-y} dy \\ &= \frac{1}{\theta^2} \Gamma(3) \\ &= \frac{2}{\theta^2} \quad \text{since } \Gamma(3) = 2! = 2, \end{aligned}$$

and so

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 2/\theta^2 - 1/\theta^2 = 1/\theta^2.$$

For this random variable the mean is equal to the standard deviation.

### 3.5.2.3 Using R to calculate probabilities

For  $X \sim \text{Exponential}(\theta = 0.5)$ , the command

```
dexp(x = 3, rate = 1/2)
```

calculates the pdf at  $x = 3$ . The rate parameter to be supplied is the  $\theta$  parameter here. The command `pexp` returns the cdf or the probability up to and including the argument. Thus

```
pexp(q = 3, rate = 1/2)
```

will return the value of  $P(X \leq 3)$ .

### 3.5.2.4 Cumulative distribution function and quantiles

The cdf for  $x > 0$  is

$$F(x) = P(X \leq x) = \int_0^x \theta e^{-\theta u} du = 1 - e^{-\theta x}.$$

We have  $F(0) = 0$  and  $F(x) \rightarrow 1$  when  $x \rightarrow \infty$  and  $F(x)$  is non-decreasing in  $x$ . The cdf can be used to solve many problems. A few examples follow.

**Example 3.13** (Mobile phone). Suppose that the lifetime of a phone (e.g. the time until the phone does not function even after repairs), denoted by  $X$ , manufactured by the company A Pale, is exponentially distributed with mean 550 days. 1. Find the probability that a randomly selected phone will still function after two years, i.e.  $X > 730$ ? (Assume there is no leap year in the two years.) 2. What are the times by which we expect 25%, 50%, 75% and 90% of the manufactured phones to have failed?

Here the mean  $1/\theta = 550$ . Hence  $\theta = 1/550$  is the rate parameter. The solution to the first problem is

$$P(X > 730) = 1 - P(X \leq 730) = 1 - (1 - e^{-730/550}) = e^{-730/550} = 0.2652.$$

Alternatively, we can do the calculation in R:

```
1 - pexp(q = 730, rate = 1 / 550)
```

```
## [1] 0.2651995
```

For the second problem we are given the probabilities of failure (0.25, 0.50, etc.). We will have to invert the probabilities to find the value of the random variable. In other words, we will have to find a  $q$  such that  $F(q) = p$ , where  $p$  is the given probability: the  $p$ th quantile of  $X$ .

The cdf of the exponential distribution is  $F(q) = 1 - e^{-\theta q}$ , so to find the  $p$ th quantile we must solve  $p = 1 - e^{-\theta q}$  for  $q$ .

$$\begin{aligned} p &= 1 - e^{-\theta q} \\ \Rightarrow e^{-\theta q} &= 1 - p \\ \Rightarrow -\theta q &= \log(1 - p) \\ \Rightarrow q &= \frac{-\log(1 - p)}{\theta}. \end{aligned}$$

In Example 3.13,  $\theta = 1/550$ , so we have

$$q = -550 \times \log(1 - p) = \begin{cases} 158 & \text{for } p = 0.25 \\ 381 & \text{for } p = 0.50 \\ 762 & \text{for } p = 0.75 \\ 1266 & \text{for } p = 0.90. \end{cases}$$

which gives the time in days until we expect 25%, 50%, 75% and 90% of the manufactured phones to have failed.

In R you can find these values by

```
qexp(p = 0.25, rate = 1/550)
qexp(p = 0.50, rate = 1/550)
```

and so on. The function

```
qexp(p, rate)
```

calculates the  $p$ th quantile of the exponential distribution with parameter **rate**.

**Example 3.14** (Survival function). The exponential distribution often models survival times. If  $T$  is survival time, the survival function is  $S(t) = 1 - F(t) = e^{-\theta t}$ . See Figure 3.2 for an example plot.

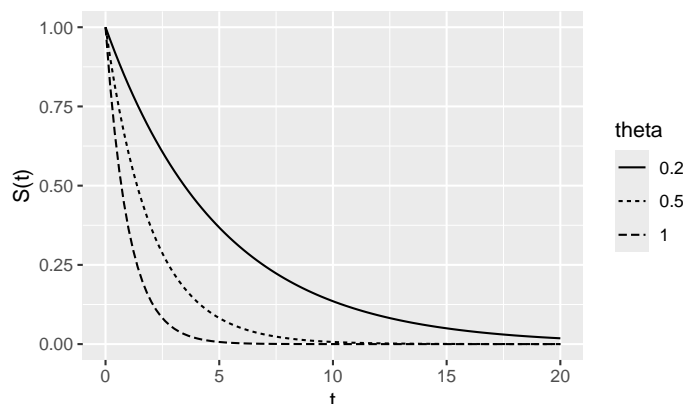


Figure 3.2:  $S(t)$  for  $\theta = 0.2, 0.5, 1$ .

Assuming the mean survival time to be 100 days for a fatal late detected cancer, we can expect that half of the patients survive 69.3 days after chemo since

```
qexp(0.50, rate=1/100)
```

```
## [1] 69.31472
```

You will learn more about this in a third-year module, MATH3085: Survival Models, which is important in the actuarial profession.

### 3.5.2.5 Memoryless property

Like the geometric distribution, the exponential distribution is memoryless: conditional on surviving to time  $t$ , the distribution of the additional lifetime is the same as at time 0.

The proof is exactly as in the case of the geometric distribution, reproduced below. Recall the definition of conditional probability:

$$P\{A \mid B\} = \frac{P\{A \cap B\}}{P\{B\}}.$$

Now the proof,

$$\begin{aligned} P(X > s+t \mid X > t) &= \frac{P(X > s+t, X > t)}{P(X > t)} \\ &= \frac{P(X > s+t)}{P(X > t)} \\ &= \frac{e^{-\theta(s+t)}}{e^{-\theta t}} \\ &= e^{-\theta s} \\ &= P(X > s). \end{aligned}$$

Note that the event  $X > s+t$  and  $X > t$  implies and is implied by  $X > s+t$  since  $s > 0$ .

**Example 3.15.** Suppose the time  $T$  between any two successive arrivals in a hospital emergency department has exponential distribution,  $T \sim \text{Exponential}(\lambda)$ . Historically, the mean of these inter-arrival times is 5 minutes. Estimate  $\lambda$ , and hence estimate

- (i)  $P(0 < T < 5)$ ,
- (ii)  $P(T < 10 \mid T > 5)$ .

An estimate of  $E(T)$  is 5. As  $E(T) = \frac{1}{\lambda}$  we take  $\frac{1}{5}$  as the estimate of  $\lambda$ .

- (i)  $P(0 < T < 5) = \int_0^5 \frac{1}{5} e^{-t/5} dt = [-e^{-t/5}]_0^5 = 1 - e^{-1} = 0.63212$ .
- (ii) We have

$$\begin{aligned} P(T < 10 \mid T > 5) &= \frac{P(5 < T < 10)}{P(T > 5)} \\ &= \frac{\int_5^{10} \frac{1}{5} e^{-t/5} dt}{\int_5^{\infty} \frac{1}{5} e^{-t/5} dt} = \frac{[-e^{-t/5}]_5^{10}}{[-e^{-t/5}]_5^{\infty}} \\ &= 1 - e^{-1} = 0.63212. \end{aligned}$$

### 3.5.3 Normal distribution

#### 3.5.3.1 Definition

A random variable  $X$  is said to have the normal distribution with parameters  $\mu$  and  $\sigma^2$  if it has pdf of the form

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}, \quad -\infty < x < \infty \quad (3.1)$$

where  $-\infty < \mu < \infty$  and  $\sigma > 0$  are two given constants. We write  $X \sim N(\mu, \sigma^2)$ .

Later we show  $E(X) = \mu$  and  $\text{Var}(X) = \sigma^2$ .

The density (pdf) is much easier to remember and work with when the mean  $\mu = 0$  and variance  $\sigma^2 = 1$ . This special case is called the *standard* normal distribution. In this case, we simply write:

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{x^2}{2} \right\}.$$

We often use  $Z$  to denote a random variable with standard normal distribution.

It is easy to see that  $f(x) > 0$  for all  $x$ . Next, we show  $\int_{-\infty}^{\infty} f(x)dx = 1$  or total probability equals 1, so that  $f(x)$  defines a valid pdf:

$$\begin{aligned}
 \int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{z^2}{2}\right\} dz \quad (\text{substitute } z = \frac{x-\mu}{\sigma} \text{ so that } dx = \sigma dz) \\
 &= \frac{1}{\sqrt{2\pi}} 2 \int_0^{\infty} \exp\left\{-\frac{z^2}{2}\right\} dz \quad (\text{since the integrand is an even function}) \\
 &= \frac{1}{\sqrt{2\pi}} 2 \int_0^{\infty} \exp\{-u\} \frac{du}{\sqrt{2u}} \quad (\text{substitute } u = \frac{z^2}{2} \text{ so that } z = \sqrt{2u} \text{ and } dz = \frac{du}{\sqrt{2u}}) \\
 &= \frac{1}{2\sqrt{\pi}} 2 \int_0^{\infty} u^{\frac{1}{2}-1} \exp\{-u\} du \quad (\text{rearrange the terms}) \\
 &= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) \quad (\text{recall the definition of the Gamma function}) \\
 &= \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1 \quad \text{as } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.
 \end{aligned}$$

### 3.5.3.2 Linear transformations

**Theorem 3.8.** If  $X \sim N(\mu, \sigma^2)$  and  $Y = aX + b$ , then  $Y \sim N(a\mu + b, a^2\sigma^2)$ .

*Proof.* The cumulative distribution function of  $Y$  is

$$\begin{aligned}
 F_Y(y) &= P(Y \leq y) = P(aX + b \leq y) = P\left(X \leq \frac{y-b}{a}\right) \\
 &= \int_{-\infty}^{\frac{y-b}{a}} f_X(x)dx, \text{ where } f_X(x) \text{ is the pdf of } X \\
 &= \int_{-\infty}^y \frac{1}{a} f_X\left(\frac{u-b}{a}\right) du, \text{ substituting } u = ax + b.
 \end{aligned}$$

So  $Y$  has probability density function

$$\begin{aligned}
 f_Y(y) &= \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \\
 &= \frac{1}{a} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{\left(\frac{y-b}{a} - \mu\right)^2}{2\sigma^2}\right\} \\
 &= \frac{1}{\sqrt{2\pi a^2\sigma^2}} \exp\left\{-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}\right\},
 \end{aligned}$$

which is the  $N(a\mu + b, a^2\sigma^2)$  probability density function. So  $Y \sim N(a\mu + b, a^2\sigma^2)$ . □

An important consequence is that if  $X \sim N(\mu, \sigma^2)$ , then  $Z = (X-\mu)/\sigma \sim N(0, 1)$ . We can “standardise” any normal random variable by subtracting the mean  $\mu$  then dividing by the standard deviation  $\sigma$ .

**3.5.3.3 Expectation and variance**

We claimed that  $E(X) = \mu$  and  $\text{Var}(X) = \sigma^2$ , and now will prove these results.

$E(X) = \mu$  because  $f(x)$  is symmetric about  $\mu$ .

To prove  $\text{Var}(X) = \sigma^2$ , we show that  $\text{Var}(Z) = 1$  where  $Z = \frac{X-\mu}{\sigma}$ . Once we have shown that, we will have  $\text{Var}(X) = \sigma^2 \text{Var}(Z) = \sigma^2$ .

Since  $E(Z) = 0$ ,  $\text{Var}(Z) = E(Z^2)$ , which is calculated below:

$$\begin{aligned}
 E(Z^2) &= \int_{-\infty}^{\infty} z^2 f(z) dz \\
 &= \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz \\
 &= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} z^2 \exp\left\{-\frac{z^2}{2}\right\} dz \quad (\text{since the integrand is an even function}) \\
 &= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} 2u \exp\{-u\} \frac{du}{\sqrt{2u}} \quad (\text{substitute } u = \frac{z^2}{2} \text{ so that } z = \sqrt{2u} \text{ and } dz = \frac{du}{\sqrt{2u}}) \\
 &= \frac{4}{2\sqrt{\pi}} \int_0^{\infty} u^{\frac{1}{2}} \exp\{-u\} du \\
 &= \frac{2}{\sqrt{\pi}} \int_0^{\infty} u^{\frac{3}{2}-1} \exp\{-u\} du \\
 &= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \quad (\text{definition of the gamma function}) \\
 &= \frac{2}{\sqrt{\pi}} \left(\frac{3}{2}-1\right) \Gamma\left(\frac{3}{2}-1\right) \quad (\text{reduction property of the gamma function}) \\
 &= \frac{2}{\sqrt{\pi}} \frac{1}{2} \sqrt{\pi} \quad (\text{since } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}) \\
 &= 1,
 \end{aligned}$$

as we hoped for! This proves  $\text{Var}(X) = \sigma^2$ .

**3.5.3.4 Calculating probabilities**

Suppose  $X \sim N(\mu, \sigma^2)$  and we are interested in finding  $P(a \leq X \leq b)$  for two constants  $a$  and  $b$ . To do this, we can use the fact that  $Z = \frac{X-\mu}{\sigma} \sim N(0, 1)$  and rewrite the probability of interest in terms of standard normal probabilities:

$$\begin{aligned}
 P(a \leq X \leq b) &= P\left(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right) \\
 &= P\left(Z \leq \frac{b-\mu}{\sigma}\right) - P\left(Z \leq \frac{a-\mu}{\sigma}\right) \\
 &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right),
 \end{aligned}$$

where we use the notation  $\Phi(\cdot)$  to denote the cdf of the standard normal distribution, i.e.

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\} du.$$

This result allows us to find the probabilities about a normal random variable  $X$  of any mean  $\mu$  and variance  $\sigma^2$  through the probabilities of the standard normal random variable  $Z$ . For this reason,

only  $\Phi(z)$  is tabulated. Further more, due to the symmetry of the pdf of  $Z$ ,  $\Phi(z)$  is tabulated only for positive  $z$  values. Suppose  $a > 0$ , then

$$\begin{aligned}\Phi(-a) &= P(Z \leq -a) = P(Z > a) \\ &= 1 - P(Z \leq a) \\ &= 1 - \Phi(a).\end{aligned}$$

In R, `pnorm` computes normal cdf values with arguments `mean` (for  $\mu$ ) and `sd` (for  $\sigma$ ). Defaults are `mean = 0`, `sd = 1` (standard normal). So, we use the command

```
pnorm(1)
```

```
## [1] 0.8413447
```

to calculate  $\Phi(1) = P(Z \leq 1)$ . We can also use the command

```
pnorm(15, mean=10, sd=2)
```

```
## [1] 0.9937903
```

to calculate  $P(X \leq 15)$  when  $X \sim N(\mu = 10, \sigma^2 = 4)$  directly.

1.  $P(-1 < Z < 1) = \Phi(1) - \Phi(-1) = 0.6827$ . This means that 68.27% of the probability lies within 1 standard deviation of the mean.
2.  $P(-2 < Z < 2) = \Phi(2) - \Phi(-2) = 0.9545$ . This means that 95.45% of the probability lies within 2 standard deviations of the mean.
3.  $P(-3 < Z < 3) = \Phi(3) - \Phi(-3) = 0.9973$ . This means that 99.73% of the probability lies within 3 standard deviations of the mean.

We are often interested in the quantiles (inverse-cdf of probability),  $\Phi^{-1}(\cdot)$  of the normal distribution for various reasons. We find the  $p$ th quantile by issuing the R command `qnorm(p)`

1. `qnorm(0.95) =  $\Phi^{-1}(0.95) = 1.645$` . This means that the 95th percentile of the standard normal distribution is 1.645. This also means that  $P(-1.645 < Z < 1.645) = \Phi(1.645) - \Phi(-1.645) = 0.90$ .
2. `qnorm(0.975) =  $\Phi^{-1}(0.975) = 1.96$` . This means that the 97.5th percentile of the standard normal distribution is 1.96. This also means that  $P(-1.96 < Z < 1.96) = \Phi(1.96) - \Phi(-1.96) = 0.95$ .

**Example 3.16.** Suppose the marks in MATH1063 follow the normal distribution with mean 58 and standard deviation 32.25.

1. What percentage of students will fail (i.e. score less than 40) in MATH1063?  
Answer: `pnorm(40, mean=58, sd=32.25) = 28.84%`.
2. What percentage of students will get an A result (score greater than 70)?  
Answer: `1 - pnorm(70, mean=58, sd=32.25) = 35.49%`.
3. What is the probability that a randomly selected student will score more than 90?  
Answer: `1 - pnorm(90, mean=58, sd=32.25) = 0.1605`.
4. What is the probability that a randomly selected student will score less than 25?  
Answer: `pnorm(25, mean=58, sd=32.25) = 0.1531`.
5. What is the probability that a randomly selected student scores a 2:1 (i.e. a mark between 60 and 70)? Left as an exercise.

**Example 3.17.** A lecturer set and marked an examination and found that the distribution of marks was  $N(42, 14^2)$ . The school's policy is to present scaled marks whose distribution is  $N(50, 15^2)$ . What linear transformation should the lecturer apply to the raw marks to accomplish this and what would the raw mark of 40 be transformed to?



Let  $X$  be the raw mark and  $Y$  the scaled mark. We have  $X \sim N(\mu_x = 42, \sigma_x^2 = 14^2)$  and aim to define the scaling such that  $Y \sim N(\mu_y = 50, \sigma_y^2 = 15^2)$ . If we standardise both variables, they should each have standard normal distribution, so we choose  $Y$  such that

$$\frac{X - \mu_x}{\sigma_x} = \frac{Y - \mu_y}{\sigma_y}$$

giving

$$Y = \mu_y + \frac{\sigma_y}{\sigma_x}(X - \mu_x) = 50 + \frac{15}{14}(X - 42).$$

Now at raw mark  $X = 40$ , the transformed mark would be

$$Y = 50 + \frac{15}{14}(40 - 42) = 47.86.$$

### 3.5.3.5 Log-normal distribution

If  $X \sim N(\mu, \sigma^2)$  and  $Y = \exp(X)$ , then  $Y$  is log-normal with parameters  $\mu$  and  $\sigma^2$ .

The mean of the random variable  $Y$  is given by

$$\begin{aligned} E(Y) &= E[\exp(X)] \\ &= \int_{-\infty}^{\infty} \exp(x) \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx \\ &= \exp\left\{-\frac{\mu^2 - (\mu + \sigma^2)^2}{2\sigma^2}\right\} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2 - 2(\mu + \sigma^2)x + (\mu + \sigma^2)^2}{2\sigma^2}\right\} dx \\ &= \exp\left\{-\frac{\mu^2 - (\mu + \sigma^2)^2}{2\sigma^2}\right\} \quad (\text{integrating a } N(\mu + \sigma^2, \sigma^2) \text{ random variable over its domain}) \\ &= \exp\{\mu + \sigma^2/2\}. \end{aligned}$$

Similarly, one can show that

$$\begin{aligned} E(Y^2) &= E[\exp(2X)] \\ &= \int_{-\infty}^{\infty} \exp(2x) \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx \\ &= \dots \\ &= \exp\{2\mu + 2\sigma^2\}. \end{aligned}$$

Hence, the variance is given by

$$\text{Var}(Y) = E(Y^2) - (E(Y))^2 = \exp\{2\mu + 2\sigma^2\} - \exp\{2\mu + \sigma^2\}.$$

The log-normal distribution is often used in practice for modelling economic variables of interest in business and finance, e.g. volume of sales, income of individuals. You do not need to remember the mean and variance of the log-normal distribution.

## 3.6 Joint distributions

### 3.6.1 Introduction

We often study multiple random variables together (e.g., height and weight) to exploit relationships between them. Joint probability distributions provide the framework. We introduce covariance, correlation, and independence.

### 3.6.2 Joint distribution of discrete random variables

For discrete  $X$  and  $Y$ ,  $f(x, y) = P(X = x, Y = y)$  is the joint probability mass function (pmf). It must satisfy:

$$f(x, y) \geq 0 \quad \text{for all } x \text{ and } y$$

and

$$\sum_{\text{all } x} \sum_{\text{all } y} f(x, y) = 1.$$

The marginal probability mass functions (marginal pmfs) of  $X$  and  $Y$  are respectively

$$f_X(x) = \sum_y f(x, y), \quad f_Y(y) = \sum_x f(x, y)$$

Use the identity  $\sum_x \sum_y f(x, y) = 1$  to prove that  $f_X(x)$  and  $f_Y(y)$  are really pmfs.

**Example 3.18.** Suppose that two fair dice are tossed independently one after the other. Let

$$X = \begin{cases} -1 & \text{if the result from die 1 is larger} \\ 0 & \text{if the results are equal} \\ 1 & \text{if the result from die 1 is smaller} \end{cases}$$

and let  $Y = |\text{difference between the two dice}|$ . Find the joint probability pmf for  $X$  and  $Y$ .

There are 36 possible outcomes for the results of the dice rolls, and each gives a pair of values  $(x, y)$  for  $X$  and  $Y$ .

	1	2	3	4	5	6
1	(0, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)
2	(-1, 1)	(0, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)
3	(-1, 2)	(-1, 1)	(0, 0)	(1, 1)	(1, 2)	(1, 3)
4	(-1, 3)	(-1, 2)	(-1, 1)	(0, 0)	(1, 1)	(1, 2)
5	(-1, 4)	(-1, 3)	(-1, 2)	(-1, 1)	(0, 0)	(1, 1)
6	(-1, 5)	(-1, 4)	(-1, 3)	(-1, 2)	(-1, 1)	(0, 0)

The joint pmf is given in Table 3.1.

	$y$					
$x$	0	1	2	3	4	5
-1	$\frac{0}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
0	$\frac{6}{36}$	$\frac{0}{36}$	$\frac{0}{36}$	$\frac{0}{36}$	$\frac{0}{36}$	$\frac{0}{36}$
1	$\frac{0}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Table 3.1: The joint probabilities for  $X$  and  $Y$

The marginal pmf for  $X$  is

$$f_X(x) = \begin{cases} \frac{15}{36} & \text{if } x = -1 \\ \frac{6}{36} & \text{if } x = 0 \\ \frac{15}{36} & \text{if } x = 1. \end{cases}$$

**Exercise:** Write down the marginal distribution of  $Y$  and hence find the mean and variance of  $Y$ .

### 3.6.3 Joint distribution of continuous random variables

For continuous  $X$  and  $Y$ , a nonnegative function  $f(x, y)$  is a joint pdf if

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

The marginal pdfs of  $X$  and  $Y$  are respectively

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

**Example 3.19.** Define a joint pdf by

$$f(x, y) = \begin{cases} 6xy^2 & \text{if } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

How can we show that the above is a pdf? It is non-negative for all  $x$  and  $y$  values. But does it integrate to 1? We are going to use the following rule.

**Result:** Suppose that a real-valued function  $f(x, y)$  is continuous in a region  $A$ , where  $A = \{(x, y) \text{ such that } a < x < b \text{ and } c < y < d\}$ . Then

$$\int_A f(x, y) dx dy = \int_c^d \int_a^b f(x, y) dx dy.$$

The same result holds if  $a$  and  $b$  depend upon  $y$ , but  $c$  and  $d$  should be free of  $x$  and  $y$ . When we evaluate the inner integral  $\int_a^b f(x, y) dx$ , we treat  $y$  as constant.

**Notes:** To evaluate a bivariate integral over a region  $A$  we:

- Draw a picture of  $A$  whenever possible.
- Rewrite the region  $A$  as an intersection of two one-dimensional intervals. The first interval is obtained by treating one variable as constant.
- Perform two one-dimensional integrals.

**Example 3.20.** Continuing Example 3.19,

$$\begin{aligned} \int_0^1 \int_0^1 f(x, y) dx dy &= \int_0^1 \int_0^1 6xy^2 dx dy \\ &= 6 \int_0^1 y^2 dy \int_0^1 x dx \\ &= 3 \int_0^1 y^2 dy \left[ \text{as } \int_0^1 x dx = \frac{1}{2} \right] \\ &= 1. \left[ \text{as } \int_0^1 y^2 dy = \frac{1}{3} \right] \end{aligned}$$

Now we can find the marginal pdfs as well.

$$f_X(x) = 2x, \quad 0 < x < 1, \quad f_Y(y) = 3y^2, \quad 0 < y < 1.$$

The probability of any event in the two-dimensional space can be found by integration and again more details will be provided in the second-year module MATH2011, Statistical Distribution Theory. You will come across multivariate integrals in the second semester module MATH1060, Multivariable Calculus.

### 3.6.4 Covariance and correlation

We first define the expectation of a real-valued scalar function  $g(X, Y)$  of  $X$  and  $Y$  :

$$E[g(X, Y)] = \begin{cases} \sum_x \sum_y g(x, y) f(x, y) & \text{if } X \text{ and } Y \text{ are discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dx dy & \text{if } X \text{ and } Y \text{ are continuous.} \end{cases}$$

**Example 3.21.** Continuing Example 3.18, let  $g(x, y) = xy$ .

$$E(XY) = (-1)(0)0 + (-1)(1)\frac{5}{36} + \cdots + (1)(5)\frac{1}{36} = 0.$$

**Exercises:** Try  $g(x, y) = x$ . It will be the same thing as  $E(X) = \sum_x x f_X(x)$ .

We will not consider any continuous examples as the second-year module MATH2011 will study them in detail.

Suppose that two random variables  $X$  and  $Y$  have joint pmf or pdf  $f(x, y)$  and let  $E(X) = \mu_x$  and  $E(Y) = \mu_y$ . The covariance between  $X$  and  $Y$  is defined by

$$\text{Cov}(X, Y) = E[(X - \mu_x)(Y - \mu_y)] = E(XY) - \mu_x \mu_y.$$

Let  $\sigma_x^2 = \text{Var}(X) = E(X^2) - \mu_x^2$  and  $\sigma_y^2 = \text{Var}(Y) = E(Y^2) - \mu_y^2$ . The correlation coefficient between  $X$  and  $Y$  is defined by:

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}} = \frac{E(XY) - \mu_x \mu_y}{\sigma_x \sigma_y}.$$

It can be proved that for any two random variables,  $-1 \leq \text{Corr}(X, Y) \leq 1$ . The correlation  $\text{Corr}(X, Y)$  is a measure of linear dependency between two random variables  $X$  and  $Y$ , and it is free of the measuring units of  $X$  and  $Y$  as the units cancel in the ratio.

### 3.6.5 Independence

Two events  $A$  and  $B$  are independent if  $P(A \cap B) = P(A)P(B)$ . Analogously, random variables  $X$  and  $Y$  with joint pmf/pdf  $f(x, y)$  are independent iff  $f(x, y) = f_X(x)f_Y(y)$  for all  $x, y$ .

In the discrete case  $X$  and  $Y$  are independent if each cell probability,  $f(x, y)$ , is the product of the corresponding row and column totals. In Example 3.18  $X$  and  $Y$  are not independent.

**Example 3.22.** Suppose  $X$  and  $Y$  have joint pdf given by the probability table:

		$y$			
$x$	0	$\frac{1}{6}$	$\frac{2}{12}$	$\frac{3}{12}$	Total $\frac{1}{3}$
	1	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{2}$
	2	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{6}$
	Total	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	1

Verify that in the following example  $X$  and  $Y$  are independent. We need to check all 9 cells.

**Example 3.23.** Let  $f(x, y) = 6xy^2, 0 < x < 1, 0 < y < 1$ . Check that  $X$  and  $Y$  are independent.

**Example 3.24.** Let  $f(x, y) = 2x, 0 \leq x \leq 1, 0 \leq y \leq 1$ . Check that  $X$  and  $Y$  are independent.

Sometimes the joint pdf may look like something you can factorise, but  $X$  and  $Y$  may not be independent because they may be related in the domain. For instance,

1.  $f(x, y) = \frac{21}{4}x^2y, x^2 \leq y \leq 1$ . Not independent!
2.  $f(x, y) = e^{-y}, 0 < x < y < \infty$ . Not independent!

Here are some useful consequences of independence:

- Suppose that  $X$  and  $Y$  are independent random variables. Then

$$P(X \in A, Y \in B) = P(X \in A) \times P(Y \in B)$$

for any events  $A$  and  $B$ . That is, the joint probability can be obtained as the product of the marginal probabilities. We will use this result in the next section. For example, suppose Jack and Jess are two randomly selected students. Let  $X$  denote the height of Jack and  $Y$  denote the height of Jess. Then

$$P(X < 182 \text{ and } Y > 165) = P(X < 182) \times P(Y > 165).$$

This holds for any thresholds and inequalities.

- Let  $g(x)$  be a function of  $x$  only and  $h(y)$  be a function of  $y$  only. Then, if  $X$  and  $Y$  are independent,

$$E[g(X)h(Y)] = E[g(X)] \times E[h(Y)].$$

As a special case, let  $g(x) = x$  and  $h(y) = y$ . Then we have

$$E(XY) = E(X) \times E(Y).$$

Consequently, for independent random variables  $X$  and  $Y$ ,  $\text{Cov}(X, Y) = 0$  and  $\text{Corr}(X, Y) = 0$ . But the converse is not true in general. That is, merely having  $\text{Corr}(X, Y) = 0$  does not imply that  $X$  and  $Y$  are independent random variables.

## 3.7 Sums of random variables

Sums of random variables appear frequently in practice and theory. For example, an exam mark is the sum of question scores, and the sample mean is proportional to a sum.

Suppose we have obtained a random sample from a distribution with pmf or pdf  $f(x)$ , so that  $X$  can either be a discrete or a continuous random variable. We will learn more about random sampling in the next chapter. Let  $X_1, \dots, X_n$  denote the random sample of size  $n$  where  $n$  is a positive integer. We use upper case letters since each member of the random sample is a random variable. For example, I toss a fair coin  $n$  times and let  $X_i$  take the value 1 if a head appears in the  $i$ th trial and 0 otherwise. Now I have a random sample  $X_1, \dots, X_n$  from the Bernoulli distribution with probability of success equal to 0.5 since the coin is assumed to be fair.

We can get a random sample from a continuous random variable as well. Suppose it is known that the distribution of the heights of first-year students is normal with mean 175 centimetres and standard deviation 8 centimetres. I can randomly select a number of first-year students and record each student's height.

Suppose  $X_1, \dots, X_n$  is a random sample from a population with distribution  $f(x)$ . Then it can be shown that the random variables  $X_1, \dots, X_n$  are mutually independent, i.e.

$$P(X_1 \in A_1, X_2 \in A_2, \dots, X_n \in A_n) = P(X_1 \in A_1) \times P(X_2 \in A_2) \times \dots \times P(X_n \in A_n)$$

for any set of events,  $A_1, A_2, \dots, A_n$ . That is, the joint probability can be obtained as the product of individual probabilities. An example of this for  $n = 2$  was given in Section 3.6.5.

**Example 3.25** (Distribution of the sum of independent binomial random variables). Suppose  $X \sim \text{Bin}(m, p)$  and  $Y \sim \text{Bin}(n, p)$  independently. Note that  $p$  is the same in both distributions. Using the above fact that joint probability is the multiplication of individual probabilities, we can conclude that

$Z = X + Y$  has the binomial distribution. It is intuitively clear that this should happen since  $X$  comes from  $m$  Bernoulli trials and  $Y$  comes from  $n$  Bernoulli trials independently, so  $Z$  comes from  $m + n$  Bernoulli trials with common success probability  $p$ .

Next we will prove the result mathematically, by finding the probability mass function of  $Z = X + Y$  directly and observing that it is of the appropriate form. In our proof, we will need to use the fact that

$$\sum_{x+y=z} \binom{m}{x} \binom{n}{y} = \binom{m+n}{z}$$

where the above sum is also over all possible integer values of  $x$  and  $y$  such that  $0 \leq x \leq m$  and  $0 \leq y \leq n$ . This fact may be proved by using the binomial theorem, but we state it here without proof.

Note that

$$P(Z = z) = P(X = x, Y = y)$$

subject to the constraint that  $x + y = z$ ,  $0 \leq x \leq m$ ,  $0 \leq y \leq n$ . Thus,

$$\begin{aligned} P(Z = z) &= \sum_{x+y=z} P(X = x, Y = y) \\ &= \sum_{x+y=z} \binom{m}{x} p^x (1-p)^{m-x} \binom{n}{y} p^y (1-p)^{n-y} \\ &= \sum_{x+y=z} \binom{m}{x} \binom{n}{y} p^z (1-p)^{m+n-z} \\ &= p^z (1-p)^{m+n-z} \sum_{x+y=z} \binom{m}{x} \binom{n}{y} \\ &= \binom{m+n}{z} p^z (1-p)^{m+n-z}, \text{ using the fact above.} \end{aligned}$$

Thus, we have proved that the sum of independent binomial random variables with common probability is binomial as well. This is called the reproductive property of random variables.

Now we will state two main results without proof. The proofs will be presented in the second-year distribution theory module MATH2011. Suppose that  $X_1, \dots, X_n$  is a random sample from a population distribution with finite variance, and suppose that  $E(X_i) = \mu_i$  and  $\text{Var}(X_i) = \sigma_i^2$ . Define a new random variable

$$Y = X_1 + X_2 + \dots + X_n.$$

Then:

1.  $E(Y) = \mu_1 + \mu_2 + \dots + \mu_n$ .
2.  $\text{Var}(Y) = \sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2$ .

That is:

- The expectation of the sum of independent random variables is the sum of the expectations of the individual random variables
- the variance of the sum of independent random variables is the sum of the variances of the individual random variables.

The second result is *only* true for independent random variables, e.g. random samples. Now we will consider many examples.

**Example 3.26** (Mean and variance of binomial distribution). Suppose  $Y \sim \text{Bin}(n, p)$ . Then we can write:

$$Y = X_1 + X_2 + \dots + X_n$$

where each  $X_i$  is an independent Bernoulli trial with success probability  $p$ . We have shown before that,  $E(X_i) = p$  and  $\text{Var}(X_i) = p(1-p)$  by direct calculation. Now the above two results imply that:

$$E(Y) = E\left(\sum_{i=1}^n X_i\right) = p + p + \dots + p = np. \text{Var}(Y) = \text{Var}(X_1) + \dots + \text{Var}(X_n) = p(1-p) + \dots + p(1-p) = np(1-p).$$

Thus we avoided the complicated sums used to derive  $E(X)$  and  $\text{Var}(X)$  in Sections 3.4.2.3 and 3.4.2.4.

**Example 3.27** (Mean and variance of negative binomial distribution). Recall that the negative binomial random variable  $Y$  is the number of trials needed to obtain the  $r$ th success in a sequence of independent Bernoulli trials, each with success probability  $p$ . Let  $X_i$  be the number of trials needed after the  $(i-1)$ th success to obtain the  $i$ th success. Each  $X_i$  is a geometric random variable and  $Y = X_1 + \dots + X_r$ . Hence

$$E(Y) = E(X_1) + \dots + E(X_r) = 1/p + \dots + 1/p = r/p$$

and

$$\text{Var}(Y) = \text{Var}(X_1) + \dots + \text{Var}(X_r) = (1-p)/p^2 + \dots + (1-p)/p^2 = r(1-p)/p^2.$$

**Example 3.28** (Sum of independent normal random variables). Suppose that  $X_i \sim N(\mu_i, \sigma_i^2)$ ,  $i = 1, 2, \dots, k$  are independent random variables. Suppose that

$$Y = a_1 X_1 + \dots + a_k X_k.$$

Then we can prove that:

$$Y \sim N\left(\sum_{i=1}^k \mu_i, \sum_{i=1}^k \sigma_i^2\right).$$

It is clear that  $E(Y) = \sum_{i=1}^k \mu_i$  and  $\text{Var}(Y) = \sum_{i=1}^k \sigma_i^2$ . But that  $Y$  has the normal distribution cannot yet be proved with the theory we know. This proof will be provided in the second-year distribution theory module MATH2011.

As a consequence of the stated result we can easily see the following. Suppose  $X_1$  and  $X_2$  are independent  $N(\mu, \sigma^2)$  random variables. Then  $2X_1 \sim N(2\mu, 4\sigma^2)$ ,  $X_1 + X_2 \sim N(2\mu, 2\sigma^2)$ , and  $X_1 - X_2 \sim N(0, 2\sigma^2)$ . Note that  $2X_1$  and  $X_1 + X_2$  have different distributions. Suppose that  $X_i \sim N(\mu, \sigma^2)$ ,  $i = 1, \dots, n$  are independent. Then

$$X_1 + \dots + X_n \sim N(n\mu, n\sigma^2),$$

and consequently

$$\bar{X} = \frac{1}{n}(X_1 + \dots + X_n) \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

## 3.8 The Central Limit Theorem

### 3.8.1 Introduction

The Central Limit Theorem (CLT) describes the behaviour of sums (and averages) of independent random variables. It applies broadly: independence and finite means and variances suffice. A version follows.

### 3.8.2 Statement of the Central Limit Theorem (CLT)

Let  $X_1, \dots, X_n$  be independent random variables with finite  $E(X_i) = \mu_i$  and finite  $\text{Var}(X_i) = \sigma_i^2$ . Define  $Y = \sum_{i=1}^n X_i$ . Then, for a sufficiently large  $n$ , the central limit theorem states that  $Y$  is approximately normally distributed with

$$E(Y) = \sum_{i=1}^n \mu_i, \quad \text{Var}(Y) = \sum_{i=1}^n \sigma_i^2.$$

This also implies that  $\bar{X} = \frac{1}{n}Y$  also follows the normal distribution approximately, as the sample size  $n \rightarrow \infty$ . In particular, if  $\mu_i = \mu$  and  $\sigma_i^2 = \sigma^2$ , i.e. all means are equal and all variances are equal, then the CLT states that, as  $n \rightarrow \infty$ ,

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

Equivalently,

$$\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim N(0, 1)$$

as  $n \rightarrow \infty$ . The notion of convergence is explained by the convergence of distribution of  $\bar{X}$  to that of the normal distribution with the appropriate mean and variance. It means that the cdf of the left hand side,  $\sqrt{n}(\bar{X} - \mu)/\sigma$ , converges to the cdf of the standard normal random variable,  $\Phi(\cdot)$ . In other words,

$$\lim_{n \rightarrow \infty} P\left(\sqrt{n} \frac{(\bar{X} - \mu)}{\sigma} \leq z\right) = \Phi(z), \quad -\infty < z < \infty.$$

So for “large samples”, we can use  $N(0, 1)$  as an approximation to the sampling distribution of  $\sqrt{n}(\bar{X} - \mu)/\sigma$ . This result is ‘exact’, i.e. no approximation is required, if the distribution of the  $X_i$ ’s are normal in the first place — this was discussed in the previous lecture.

How large must  $n$  be? It depends on how close the  $X_i$  distributions are to normal. The approximation is often reasonable for  $n \approx 20$  (sometimes smaller), but heavy skewness typically requires larger  $n$ . We investigate in one example in Figure 3.3.

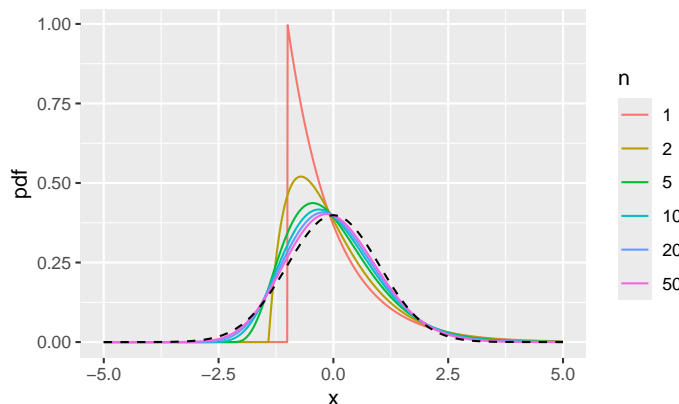


Figure 3.3: Distribution of normalised sample means for samples of different sizes. Initially very skew (original distribution,  $n = 1$ ) becoming rapidly closer to standard normal (dashed line) with increasing  $n$ .



### 3.8.3 Application of CLT to binomial distribution

We know that a binomial random variable  $Y$  with parameters  $n$  and  $p$  is the number of successes in a set of  $n$  independent Bernoulli trials, each with success probability  $p$ . We may write

$$Y = X_1 + X_2 + \cdots + X_n,$$

where  $X_1, \dots, X_n$  are independent Bernoulli random variables with success probability  $p$ . It follows from the CLT that, for a sufficiently large  $n$ ,  $Y$  is approximately normally distributed with expectation  $E(Y) = np$  and variance  $\text{Var}(Y) = np(1-p)$ .

Hence, for given integers  $y_1$  and  $y_2$  between 0 and  $n$  and a suitably large  $n$ , we have

$$\begin{aligned} P(y_1 \leq Y \leq y_2) &= P\left\{ \frac{y_1 - np}{\sqrt{np(1-p)}} \leq \frac{Y - np}{\sqrt{np(1-p)}} \leq \frac{y_2 - np}{\sqrt{np(1-p)}} \right\} \\ &\approx P\left\{ \frac{y_1 - np}{\sqrt{np(1-p)}} \leq Z \leq \frac{y_2 - np}{\sqrt{np(1-p)}} \right\} \end{aligned}$$

where  $Z \sim N(0, 1)$ .

Because  $Y$  is integer-valued, apply a continuity correction: replace  $[y_1, y_2]$  by  $[y_1 - 0.5, y_2 + 0.5]$  in the normal approximation.

$$\begin{aligned} P(y_1 \leq Y \leq y_2) &= P(y_1 - 0.5 \leq Y \leq y_2 + 0.5) \\ &= P\left\{ \frac{y_1 - 0.5 - np}{\sqrt{np(1-p)}} \leq \frac{Y - np}{\sqrt{np(1-p)}} \leq \frac{y_2 + 0.5 - np}{\sqrt{np(1-p)}} \right\} \\ &\approx P\left\{ \frac{y_1 - 0.5 - np}{\sqrt{np(1-p)}} \leq Z \leq \frac{y_2 + 0.5 - np}{\sqrt{np(1-p)}} \right\}. \end{aligned}$$

What do we mean by a suitably large  $n$ ? A commonly-used guideline is that the approximation is adequate if  $np \geq 5$  and  $n(1-p) \geq 5$ .

**Example 3.29.** A producer of natural yoghurt believed that the market share of their brand was 10%. To investigate this, a survey of 2500 yoghurt consumers was carried out. It was observed that only 205 of the people surveyed expressed a preference for their brand. Should the producer be concerned that they might be losing market share?

Assume that the conjecture about market share is true. Then the number of people  $Y$  who prefer this product follows a binomial distribution with  $p = 0.1$  and  $n = 2500$ . So the mean is  $np = 250$ , the variance is  $np(1-p) = 225$ , and the standard deviation is 15. The exact probability of observing ( $Y \leq 205$ ) is given by the sum of the binomial probabilities up to and including 205, which is difficult to compute. However, this can be approximated by using the CLT:

$$\begin{aligned} P(Y \leq 205) &= P(Y \leq 205.5) \\ &= P\left\{ \frac{Y - np}{\sqrt{np(1-p)}} \leq \frac{205.5 - np}{\sqrt{np(1-p)}} \right\} \\ &\approx P\left\{ Z \leq \frac{205.5 - np}{\sqrt{np(1-p)}} \right\} \\ &= P\left\{ Z \leq \frac{205.5 - 250}{15} \right\} \\ &= \Phi(-2.967) = 0.0015. \end{aligned}$$

This probability is so small that it casts doubt on the validity of the assumption that the market share is 10%.

Although the exact binomial probabilities are difficult to compute by hand, in this case we may compute them in R. Recall  $Y \sim \text{Bin}(n = 2500, p = 0.1)$ , so  $P(Y \leq 205)$  is

```
pbinom(205, size = 2500, prob = 0.1)
```

```
## [1] 0.001173725
```

In this case the normal approximation was good enough to correctly conclude that this probability is very small (of the order of 0.1%), which was all we needed to answer the question of interest here.

## Chapter 4

# Statistical Inference

### 4.1 Statistical modelling

#### 4.1.1 Introduction

Statistical inference uses observed data to draw conclusions, make predictions, and support decisions. We do this by positing probability distributions (statistical models) for how the data are generated. For example, suppose the true proportion of mature students is  $p$  with  $0 < p < 1$ . For a random sample of  $n$  students, each observation can be modelled as  $\text{Bernoulli}(p)$ , taking value 1 for a mature student and 0 otherwise. Here the model is Bernoulli with success probability  $p$ .

To illustrate with another example, suppose we have observed fast food waiting times in the morning and afternoon, as in Example 1.1. If time is treated as continuous, each customer's waiting time can be modelled as a normal random variable.

In general:

- The model's form helps us understand the data-generating process.
- If a model explains the observed data well, it can inform predictions and decisions about future (unobserved) data.
- Models, together with appropriate methodology, allow us to quantify uncertainty in conclusions, predictions, and decisions.

We use  $x_1, x_2, \dots, x_n$  for  $n$  observations of random variables  $X_1, X_2, \dots, X_n$  (uppercase for random variables, lowercase for observed values). For the fast food waiting time example, we have  $n = 20$ ,  $x_1 = 38, x_2 = 100, \dots, x_{20} = 70$ , and  $X_i$  is the waiting time for the  $i$ th person in the sample.

#### 4.1.2 Statistical models

Suppose we denote the complete data by the vector  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  and use  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  for the corresponding random variables. A statistical model specifies a probability distribution for the random variables  $\mathbf{X}$  corresponding to the data observations  $\mathbf{x}$ . Providing a specification for the distribution of  $n$  jointly varying random variables can be a daunting task, particularly if  $n$  is large. However, this task is made much easier if we can make some simplifying assumptions, such as

1.  $X_1, X_2, \dots, X_n$  are independent random variables,
2.  $X_1, X_2, \dots, X_n$  have the same probability distribution (so  $x_1, x_2, \dots, x_n$  are observations of a single random variable  $X$ ).

Assumption 1 depends on the sampling mechanism and is common in practice. For the Southampton student example, it requires random selection from all students. It is violated when samples are

temporally or spatially correlated (e.g., daily air pollution over a year, or measurements from nearby locations). In this module we consider only data sets for which Assumption 1 is reasonable.

Assumption 2 is not always appropriate, but is often reasonable when modelling a single variable. In the fast food waiting time example, it requires no systematic AM–PM differences.

If Assumptions 1 and 2 both hold, then  $X_1, \dots, X_n$  are independent and identically distributed (i.i.d.).

### 4.1.3 A fully specified model

Sometimes a model fully specifies the distribution of  $X_1, \dots, X_n$ . For example,  $X \sim N(\mu, \sigma^2)$  with  $\mu = 100$  and  $\sigma^2 = 100$  is fully specified: there are no unknowns to estimate (though we may still use data to assess plausibility). Such cases are uncommon unless external theory justifies the parameter values.

### 4.1.4 A parametric statistical model

A parametric statistical model specifies a family of distributions indexed by a finite set of parameters. The model's form is fixed; only the parameter values are unknown. For example, the Poisson distribution is parameterised by the parameter  $\lambda$  which happens to be the mean of the distribution; the normal distribution is parameterised by two parameters, the mean  $\mu$  and the variance  $\sigma^2$ . When parameters are unknown (e.g.,  $\lambda, \mu, \sigma^2$ ), we estimate them from data. We discuss estimation next.

### 4.1.5 A nonparametric statistical model

Sometimes we avoid specifying a precise parametric form for the distribution of  $X_1, \dots, X_n$  (e.g., when a histogram is clearly non-normal). We may still assume  $X_1, \dots, X_n$  are i.i.d., but model them nonparametrically (distribution-free methods), without committing to a finite-dimensional parametric family.

**Example 4.1** (Computer failures). Let  $X$  denote the count of computer failures per week from Example 1.2, summarised in the following table:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	17
12	16	21	12	11	8	7	2	4	2	3	2	2	1	1

We want to estimate how many weeks in the next year the system will fail at least once. The answer is  $52 \times (1 - P(X = 0))$ . But how would you estimate  $P(X = 0)$ ? Consider two approaches.

1. **Nonparametric.** Estimate  $P(X = 0)$  by the relative frequency of number of zeros in the above sample, which is 12 out of 104. Thus our estimate of  $P(X = 0)$  is  $12/104$ . Hence, our estimate of the number of weeks when there will be at least one computer failure is  $52 \times (1 - 12/104) = 46$ .
2. **Parametric.** Suppose we assume that  $X$  follows the Poisson distribution with parameter  $\lambda$ . Then the answer to the above question is

$$52 \times (1 - P(X = 0)) = 52 \times \left(1 - e^{-\lambda} \frac{\lambda^0}{0!}\right) = 52 \times (1 - e^{-\lambda})$$

which involves the unknown parameter  $\lambda$ . For the Poisson distribution we know that  $E(X) = \lambda$ . Hence we could use the sample mean  $\bar{X}$  to estimate  $E(X) = \lambda$ . Thus we estimate  $\hat{\lambda} = \bar{x} = 3.75$ . This type of estimator is called a moment estimator. Now our estimate of the number of weeks when there will be at least one computer failure is  $52 \times (1 - e^{-3.75}) = 50.78 \approx 51$ , which is very different from our answer of 46 from the nonparametric approach.

### 4.1.6 Should we prefer parametric or nonparametric and why?

Prefer the parametric approach when its assumptions (e.g., Poisson for counts) are justified, assessed via plots or by comparing fitted probabilities

$$\hat{P}(X = x) = e^{-\hat{\lambda}} \frac{\hat{\lambda}^x}{x!}$$

to sample frequencies. Model-based analyses are often more precise and provide principled uncertainty quantification. Use nonparametric methods when the parametric model is not credible.

## 4.2 Estimation

### 4.2.1 Introduction

After collecting data and proposing a model, the first task is typically estimation.

- For a parametric model, estimate the unknown parameters (e.g., for i.i.d. Poisson data, estimate  $\lambda = E(X)$ ).
- For a nonparametric model, estimate population features of  $X$  (e.g.,  $\mu = E(X)$ ,  $\sigma^2 = \text{Var}(X)$ ).

We use  $\theta$  for the estimand (the quantity to be estimated). In the examples above,  $\theta$  may be  $\lambda$ , or  $\mu$ , or  $\sigma^2$ .

### 4.2.2 Population and sample

Recall: a statistical model specifies a probability distribution for  $\mathbf{X}$  corresponding to observations  $\mathbf{x}$ .

- The observations  $\mathbf{x} = (x_1, \dots, x_n)$  are called the sample, and quantities derived from the sample are sample quantities. For example, as in Chapter 1, we call

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

the sample mean.

- The probability distribution for  $X$  specified in our model represents all possible observations which might have been observed in our sample, and is therefore sometimes referred to as the population. Quantities derived from this distribution are population quantities. For example, if our model is that  $X_1, \dots, X_n$  are i.i.d., following the common distribution of a random variable  $X$ , then we call  $E(X)$  the *population mean*.

### 4.2.3 Statistic and estimator

A statistic  $T(\mathbf{x})$  is any function of the observed data  $x_1, \dots, x_n$  (and does not depend on unknowns). An estimate of  $\theta$  is a statistic used to estimate  $\theta$ , denoted  $\hat{\theta}(\mathbf{x})$ . The corresponding random variable  $\tilde{\theta}(\mathbf{X})$  is the estimator. An estimate is a realised number (e.g.,  $\bar{x}$ ); an estimator is a random variable (e.g.,  $\bar{X}$ ).

The probability distribution of  $\tilde{\theta}(\mathbf{X})$  is its sampling distribution. The estimate  $\hat{\theta}(\mathbf{x})$  is one realisation from that distribution.

**Example 4.2.** Suppose that we have a random sample  $X_1, \dots, X_n$  from the uniform distribution on the interval  $(0, \theta)$  where  $\theta > 0$  is unknown. Suppose that  $n = 5$  and we have the sample observations  $x_1 = 2.3, x_2 = 3.6, x_3 = 20.2, x_4 = 0.9, x_5 = 17.2$ . We aim to estimate  $\theta$ . How can we proceed?

Here the pdf  $f(x) = \frac{1}{\theta}$  for  $0 \leq x \leq \theta$  and 0 otherwise. Hence  $E(X) = \int_0^\theta \frac{1}{\theta} x dx = \frac{\theta}{2}$ . There are many possible estimators for  $\theta$ , e.g.  $\hat{\theta}_1(\mathbf{X}) = 2\bar{X}$ , which is motivated by the method of moments because

$\theta = 2E(X)$ . A second estimator is  $\hat{\theta}_2(\mathbf{X}) = \max\{X_1, X_2, \dots, X_n\}$ , which is intuitive since  $\theta$  must be greater than or equal to all observed values and thus the maximum of the sample value will be closest to  $\theta$ .

How should we choose between  $\hat{\theta}_1$  and  $\hat{\theta}_2$ ? We need the sampling distributions to assess bias and variability (defined next). First, derive the distribution of  $\hat{\theta}_2$ . Let  $Y = \hat{\theta}_2(\mathbf{X}) = \max\{X_1, \dots, X_n\}$ . For  $0 < y < \theta$ , the cdf is

$$\begin{aligned} P(Y \leq y) &= P(\max\{X_1, X_2, \dots, X_n\} \leq y) \\ &= P(X_1 \leq y, X_2 \leq y, \dots, X_n \leq y) \quad \text{since } \max \leq y \text{ if and only if each } \leq y \\ &= P(X_1 \leq y) P(X_2 \leq y) \cdots P(X_n \leq y) \quad \text{since the } X_i \text{ are independent} \\ &= \frac{y}{\theta} \times \frac{y}{\theta} \times \cdots \times \frac{y}{\theta} \\ &= \left(\frac{y}{\theta}\right)^n. \end{aligned}$$

Now the pdf of  $Y$  is

$$f(y) = \frac{dF(y)}{dy} = n \frac{y^{n-1}}{\theta^n}, \quad 0 \leq y \leq \theta.$$

Using this pdf, it can be shown that  $E(\hat{\theta}_2) = E(Y) = \frac{n}{n+1}\theta$ , and

$$\text{Var}(\hat{\theta}_2) = \frac{n\theta^2}{(n+2)(n+1)^2}.$$

#### 4.2.4 Bias and mean square error

In the uniform distribution example we saw that the estimator  $\hat{\theta}_2 = Y = \max\{X_1, X_2, \dots, X_n\}$  is a random variable and its pdf is given by  $f(y) = n \frac{y^{n-1}}{\theta^n}$  for  $0 \leq y \leq \theta$ . This probability distribution is called the sampling distribution of  $\hat{\theta}_2$ . From this we have seen that  $E(\hat{\theta}_2) = \frac{n}{n+1}\theta$ .

In general, we define the bias of an estimator  $\tilde{\theta}(\mathbf{X})$  of  $\theta$  to be

$$\text{bias}(\tilde{\theta}) = E(\tilde{\theta}) - \theta.$$

An estimator  $\tilde{\theta}(\mathbf{X})$  is said to be unbiased if

$$\text{bias}(\tilde{\theta}) = 0, \text{ i.e. if } E(\tilde{\theta}) = \theta.$$

Thus an estimator is unbiased if its expected value equals the target: “right on average” under repeated sampling.

Thus for the uniform distribution example,  $\hat{\theta}_2$  is a biased estimator of  $\theta$  and

$$\text{bias}(\hat{\theta}_2) = E(\hat{\theta}_2) - \theta = \frac{n}{n+1}\theta - \theta = -\frac{1}{n+1}\theta,$$

which goes to zero as  $n \rightarrow \infty$ . However,  $\hat{\theta}_1 = 2\bar{X}$  is unbiased since  $E(\hat{\theta}_1) = 2E(\bar{X}) = 2\frac{\theta}{2} = \theta$ .

Unbiasedness does not guarantee estimates close to  $\theta$ . A more informative measure is the mean squared error (MSE), defined as

$$\text{MSE}(\tilde{\theta}) = E[(\tilde{\theta} - \theta)^2]$$

If  $\tilde{\theta}$  is unbiased ( $E(\tilde{\theta}) = \theta$ ), then  $\text{MSE}(\tilde{\theta}) = \text{Var}(\tilde{\theta})$ . In general:

**Theorem 4.1.**

$$\text{MSE}(\tilde{\theta}) = \text{Var}(\tilde{\theta}) + \text{bias}(\tilde{\theta})^2$$

The proof is similar to the proof of Theorem 1.1.

*Proof.*

$$\begin{aligned} \text{MSE}(\tilde{\theta}) &= E[(\tilde{\theta} - \theta)^2] \\ &= E[(\tilde{\theta} - E(\tilde{\theta}) + E(\tilde{\theta}) - \theta)^2] \\ &= E[(\tilde{\theta} - E(\tilde{\theta}))^2 + (E(\tilde{\theta}) - \theta)^2 + 2(\tilde{\theta} - E(\tilde{\theta}))(E(\tilde{\theta}) - \theta)] \\ &= E[(\tilde{\theta} - E(\tilde{\theta}))^2] + E[(E(\tilde{\theta}) - \theta)^2] + 2E[(\tilde{\theta} - E(\tilde{\theta}))(E(\tilde{\theta}) - \theta)] \\ &= \text{Var}(\tilde{\theta}) + [E(\tilde{\theta}) - \theta]^2 + 2(E(\tilde{\theta}) - \theta)E[(\tilde{\theta} - E(\tilde{\theta}))] \\ &= \text{Var}(\tilde{\theta}) + \text{bias}(\tilde{\theta})^2 + 2(E(\tilde{\theta}) - \theta)[E(\tilde{\theta}) - E(\tilde{\theta})] \\ &= \text{Var}(\tilde{\theta}) + \text{bias}(\tilde{\theta})^2. \end{aligned}$$

□

**Example 4.3.** Continuing the  $U(0, \theta)$  example:  $\hat{\theta}_1 = 2\bar{X}$  is unbiased, whereas  $\text{bias}(\hat{\theta}_2) = -\frac{1}{n+1}\theta$ . How do they compare via MSE? Since  $\hat{\theta}_1$  is unbiased, its MSE equals its variance. Later we prove that for a random sample from any population

$$\text{Var}(\bar{X}) = \frac{\text{Var}(X)}{n},$$

where  $\text{Var}(X)$  is the variance of the population sampled from. Returning to our example, we know that if  $X \sim U(0, \theta)$  then  $\text{Var}(X) = \frac{\theta^2}{12}$ . Therefore we have

$$\text{MSE}(\hat{\theta}_1) = \text{Var}(\hat{\theta}_1) = \text{Var}(2\bar{X}) = 4 \text{Var}(\bar{X}) = 4 \frac{\theta^2}{12n} = \frac{\theta^2}{3n}.$$

Now, for  $\hat{\theta}_2$  we know that:

1.  $\text{Var}(\hat{\theta}_2) = \frac{n\theta^2}{(n+2)(n+1)^2}$
2.  $\text{bias}(\hat{\theta}_2) = -\frac{1}{n+1}\theta$ .

Now

$$\begin{aligned} \text{MSE}(\hat{\theta}_2) &= \text{Var}(\hat{\theta}_2) + \text{bias}(\hat{\theta}_2)^2 \\ &= \frac{n\theta^2}{(n+2)(n+1)^2} + \frac{\theta^2}{(n+1)^2} \\ &= \frac{\theta^2}{(n+1)^2} \left( \frac{n}{n+2} + 1 \right) \\ &= \frac{\theta^2}{(n+1)^2} \frac{2n+2}{n+2}. \end{aligned}$$

The MSE of  $\hat{\theta}_2$  is an order of magnitude smaller than the MSE of  $\hat{\theta}_1$  (of order  $1/n^2$  rather than  $1/n$ ), providing justification for the preference of  $\hat{\theta}_2 = \max\{X_1, X_2, \dots, X_n\}$  as an estimator of  $\theta$ .

### 4.3 Estimating the population mean

#### 4.3.1 Introduction

A common task is to estimate a population mean, together with its uncertainty. Point estimates without uncertainty are often uninformative. For example, “ice-free within 20–30 years” conveys both a central value and a range. Here we introduce the standard error of an estimator to quantify uncertainty in estimating a population mean.

#### 4.3.2 Estimation of a population mean

Suppose that  $X_1, \dots, X_n$  is a random sample from any probability distribution  $f(x)$ , which may be discrete or continuous. Suppose that we want to estimate the unknown population mean  $E(X) = \mu$  and variance,  $\text{Var}(X) = \sigma^2$ . In order to do this, it is not necessary to make any assumptions about  $f(x)$ , so this may be thought of as nonparametric inference.

We have the following results:

**Theorem 4.2.** *Suppose  $X_1, \dots, X_n$  is a random sample, with  $E(X) = \mu$ . The sample mean*

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

*is an unbiased estimator of  $\mu = E(X)$ , i.e.  $E(\bar{X}) = \mu$ .*

In other words, the sample mean is an unbiased estimator of the population mean.

*Proof.* We have

$$E[\bar{X}] = \frac{1}{n} \sum_{i=1}^n E(X_i) = \frac{1}{n} \sum_{i=1}^n E(X) = E(X)$$

so  $\bar{X}$  is an unbiased estimator of  $E(X)$ . □

**Theorem 4.3.** *Suppose  $X_1, \dots, X_n$  is a random sample, with  $\text{Var}(X) = \sigma^2$ . Then*

$$\text{Var}(\bar{X}) = \frac{\sigma^2}{n}.$$

*Proof.* Using that for independent variables the variance of a sum equals the sum of variances (Section 3.7), we have

$$\text{Var}[\bar{X}] = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X) = \frac{n}{n^2} \text{Var}(X) = \frac{\sigma^2}{n},$$

□

Together, Theorems 4.2 and 4.3 imply that the MSE of  $\bar{X}$  is  $\text{Var}(X)/n$ .

**Theorem 4.4.** *Suppose  $X_1, \dots, X_n$  is a random sample, with  $\text{Var}(X) = \sigma^2$ . The sample variance with divisor  $n - 1$*

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

*is an unbiased estimator of  $\sigma^2$ , i.e.  $E(S^2) = \sigma^2$ .*

In other words, the sample variance is an unbiased estimator of the population variance.



*Proof.* We need to show  $E(S^2) = \sigma^2$ . We have

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 = \frac{1}{n-1} \left[ \sum_{i=1}^n X_i^2 - n\bar{X}^2 \right].$$

To evaluate the expectation of the above, we need  $E(X_i^2)$  and  $E(\bar{X}^2)$ . In general, we know for any random variable,

$$\text{Var}(Y) = E(Y^2) - (E(Y))^2 \Rightarrow E(Y^2) = \text{Var}(Y) + (E(Y))^2.$$

Thus, we have

$$E(X_i^2) = \text{Var}(X_i) + (E(X_i))^2 = \sigma^2 + \mu^2,$$

and

$$E(\bar{X}^2) = \text{Var}(\bar{X}) + (E(\bar{X}))^2 = \sigma^2/n + \mu^2,$$

from Theorems 4.2 and 4.3. So

$$\begin{aligned} E(S^2) &= E \left\{ \frac{1}{n-1} \left[ \sum_{i=1}^n X_i^2 - n\bar{X}^2 \right] \right\} \\ &= \frac{1}{n-1} \left[ \sum_{i=1}^n E(X_i^2) - nE(\bar{X}^2) \right] \\ &= \frac{1}{n-1} \left[ \sum_{i=1}^n (\sigma^2 + \mu^2) - n(\sigma^2/n + \mu^2) \right] \\ &= \frac{1}{n-1} [n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2] \\ &= \sigma^2 \equiv \text{Var}(X). \end{aligned}$$

□

### 4.3.3 Standard deviation and standard error

For an unbiased estimator  $\tilde{\theta}$ ,

$$\text{MSE}(\tilde{\theta}) = \text{Var}(\tilde{\theta})$$

and therefore the sampling variance of the estimator is an important summary of its quality.

We usually prefer to focus on the standard deviation of the sampling distribution of  $\tilde{\theta}$ ,

$$\text{s.d.}(\tilde{\theta}) = \sqrt{\text{Var}(\tilde{\theta})}.$$

In practice  $\text{s.d.}(\tilde{\theta})$  is unknown, but we can estimate it from the sample. The standard error,  $\text{s.e.}(\tilde{\theta})$ , is an estimate of the standard deviation of the sampling distribution of  $\tilde{\theta}$ .

We proved that

$$\text{Var}[\bar{X}] = \frac{\sigma^2}{n} \Rightarrow \text{s.d.}(\bar{X}) = \frac{\sigma}{\sqrt{n}}.$$

As  $\sigma$  is unknown, we cannot calculate this standard deviation. However, we know that  $E(S^2) = \sigma^2$ , i.e. that the sample variance is an unbiased estimator of the population variance. Hence  $S^2/n$  is an unbiased estimator for  $\text{Var}(\bar{X})$ . Therefore we obtain the standard error of the mean,  $\text{s.e.}(\bar{X})$ , by plugging in the estimate

$$s = \left( \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 \right)^{1/2}$$

of  $\sigma$  into  $\text{s.d.}(\bar{X})$  to obtain

$$\text{s.e.}(\bar{X}) = \frac{s}{\sqrt{n}}.$$

For the computer failure data, the estimate  $\bar{x} = 3.75$  has standard error

$$\text{s.e.}(\bar{X}) = \frac{3.381}{\sqrt{104}} = 0.332.$$

This is one choice of standard error. Under parametric assumptions about  $f(x)$ , alternative (often smaller) standard errors may be available. For example, if  $X_1, \dots, X_n$  are i.i.d.  $\text{Poisson}(\lambda)$  random variables,  $E(X) = \lambda$ , so  $\bar{X}$  is an unbiased estimator of  $\lambda$ .  $\text{Var}(X) = \lambda$ , so another  $\text{s.e.}(\bar{X}) = \sqrt{\hat{\lambda}/n} = \sqrt{\bar{x}/n}$ . In the computer failure data example, this is  $\sqrt{3.75/104} = 0.19$ .

## 4.4 Confidence intervals

### 4.4.1 Introduction

An estimate  $\tilde{\theta}$  of a parameter  $\theta$  is sometimes referred to as a point estimate. The usefulness of a point estimate is enhanced if some kind of measure of its precision can also be provided. Usually, for an unbiased estimator, this will be a standard error, an estimate of the standard deviation of the associated estimator, as we have discussed previously. An alternative summary of the information provided by the observed data about the location of a parameter  $\theta$  and the associated precision is a confidence interval.

Suppose that  $x_1, \dots, x_n$  are observations of random variables  $X_1, \dots, X_n$  whose joint pdf is specified apart from a single parameter  $\theta$ . To construct a confidence interval for  $\theta$ , we need to find a random variable  $T(\mathbf{X}, \theta)$  whose distribution does not depend on  $\theta$  and is therefore known. This random variable  $T(\mathbf{X}, \theta)$  is called a *pivot* for  $\theta$ . Hence we can find numbers  $h_1$  and  $h_2$  such that

$$P(h_1 \leq T(\mathbf{X}, \theta) \leq h_2) = 1 - \alpha \quad (4.1)$$

where  $1 - \alpha$  is any specified probability. If (4.1) can be ‘inverted’, we can write it as

$$P(g_1(\mathbf{X}) \leq \theta \leq g_2(\mathbf{X})) = 1 - \alpha.$$

Thus, with probability  $1 - \alpha$ , the random interval  $(g_1(\mathbf{X}), g_2(\mathbf{X}))$  contains  $\theta$ . After observing  $x_1, \dots, x_n$  we obtain the realised interval  $(g_1(\mathbf{x}), g_2(\mathbf{x}))$ , a  $100(1 - \alpha)\%$  confidence interval for  $\theta$ . Over repeated samples,  $100(1 - \alpha)\%$  of such intervals would contain  $\theta$ . Common levels are 90%, 95%, and 99%.

### 4.4.2 Confidence interval for a normal mean

Let  $X_1, \dots, X_n$  be i.i.d.  $N(\mu, \sigma^2)$  random variables. From the Central Limit Theorem (Section 3.8), we know that for large  $n$ ,

$$\bar{X} \sim N(\mu, \sigma^2/n) \quad \Rightarrow \quad \sqrt{n} \frac{(\bar{X} - \mu)}{\sigma} \sim N(0, 1).$$

Suppose we know that  $\sigma = 10$ , so  $\sqrt{n}(\bar{X} - \mu)/\sigma$  is a pivot for  $\mu$ . Then we can use the distribution function of the standard normal distribution to find values  $h_1$  and  $h_2$  such that

$$P\left(h_1 \leq \sqrt{n} \frac{(\bar{X} - \mu)}{\sigma} \leq h_2\right) = 1 - \alpha$$



Figure 4.1:  $h_1$  and  $h_2$  are chosen to make the shaded area equal to the confidence level  $1 - \alpha$

for a chosen value of  $1 - \alpha$  which is called the confidence level. So  $h_1$  and  $h_2$  are chosen so that the shaded area in Figure 4.1 is equal to the confidence level  $1 - \alpha$ .

It is common to choose a symmetric interval, allocating  $\alpha/2$  to each tail, so

$$-h_1 = h_2 \equiv h \quad \text{and} \quad \Phi(h) = 1 - \frac{\alpha}{2}.$$

The most common choice of confidence level is  $1 - \alpha = 0.95$ , in which case  $h = 1.96 = \text{qnorm}(0.975)$ . You may also occasionally see 90% ( $h = 1.645 = \text{qnorm}(0.95)$ ) or 99% ( $h = 2.58 = \text{qnorm}(0.995)$ ) intervals.

Therefore we have

$$\begin{aligned} P\left(-1.96 \leq \sqrt{n} \frac{(\bar{X} - \mu)}{\sigma} \leq 1.96\right) &= 0.95 \\ \Rightarrow P\left(\bar{X} - 1.96 \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + 1.96 \frac{\sigma}{\sqrt{n}}\right) &= 0.95. \end{aligned}$$

Hence,  $\bar{X} - 1.96 \frac{\sigma}{\sqrt{n}}$  and  $\bar{X} + 1.96 \frac{\sigma}{\sqrt{n}}$  are the endpoints of a random interval which includes  $\mu$  with probability 0.95. The observed value of this interval,  $(\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}})$ , is called a 95% confidence interval for  $\mu$ .

**Example 4.4** (Fast food waiting times). For the fast food waiting time data, we have  $n = 20$  data points combined from the morning and afternoon data sets. We have  $\bar{x} = 67.85$  and  $n = 20$ . Hence, under the normal model assuming (just for the sake of illustration)  $\sigma = 18$ , a 95% confidence interval for  $\mu$  is

$$\begin{aligned} 67.85 - 1.96(18/\sqrt{20}) &\leq \mu \leq 67.85 + 1.96(18/\sqrt{20}) \\ \Rightarrow 59.96 &\leq \mu \leq 75.74 \end{aligned}$$

The R command is

```
mean(fastfood) + c(-1, 1) * qnorm(0.975) * 18 / sqrt(20)
```

assuming `fastfood` is the vector containing 20 waiting times.

In practice  $\sigma$  is usually unknown, so we use alternative methods (next sections).

### 4.4.3 Some remarks about confidence intervals

1. Notice that  $\bar{x}$  is an unbiased estimate of  $\mu$ ,  $\sigma/\sqrt{n}$  is the standard error of the estimate and 1.96 (in general  $h$  in the above discussion) is a critical value from the associated known sampling distribution. The formula  $(\bar{x} \pm 1.96\sigma/\sqrt{n})$  for the confidence interval is then generalised as:

$$\text{Estimate} \pm \text{Critical value} \times \text{Standard error}$$

where the estimate is  $\bar{x}$ , the critical value is 1.96 and the standard error is  $\sigma/\sqrt{n}$ . We will see that this formula holds in many examples, but not all.

2. A  $100(1-\alpha)\%$  confidence interval is one realisation from a procedure that contains  $\theta$  in  $100(1-\alpha)\%$  of repeated samples.
3. A confidence interval is not a probability statement about  $\theta$  (a fixed constant in classical inference). Avoid statements like  $P(1.3 < \theta < 2.2) = 0.95$ .
4. Misinterpreting CIs as probability intervals can mislead. For example, if  $X_1, X_2 \sim U(\theta - 1/2, \theta + 1/2)$  i.i.d., then  $(\min(x_1, x_2), \max(x_1, x_2))$  is a 50% CI for  $\theta$ , but after observing  $(0.3, 0.9)$  it is not correct to assert  $P(0.3 < \theta < 0.9) = 0.5$ .

### 4.4.4 Confidence intervals using the CLT

#### 4.4.4.1 Introduction

Confidence intervals are generally difficult to find. The difficulty lies in finding a pivot, i.e. a statistic  $T(\mathbf{X}, \theta)$  such that

$$P(h_1 \leq T(\mathbf{X}, \theta) \leq h_2) = 1 - \alpha$$

for two suitable numbers  $h_1$  and  $h_2$ , and also that the above can be inverted to put the unknown  $\theta$  in the middle of the inequality inside the probability statement. One approach is to use the Central Limit Theorem (CLT) to approximate normality and follow the normal-mean construction, replacing the unknown s.d. by its estimate.

#### 4.4.4.2 Confidence intervals for $\mu$ using the CLT

The CLT allows us to assume the large sample approximation

$$\sqrt{n} \frac{(\bar{X} - \mu)}{\sigma} \stackrel{\text{approx}}{\sim} N(0, 1) \text{ as } n \rightarrow \infty.$$

Thus an approximate 95% CI for  $\mu$  is  $\bar{x} \pm 1.96 \sigma/\sqrt{n}$ . Since  $\sigma$  is unknown, we replace it with an estimate (the s.e.), giving the familiar form:

$$\text{Estimate} \pm \text{Critical value} \times \text{Standard error}$$

Suppose that we do not assume any distribution for the sampled random variable  $X$  but assume only that  $X_1, \dots, X_n$  are i.i.d, following the distribution of  $X$  where  $E(X) = \mu$  and  $\text{Var}(X) = \sigma^2$ . We know that the standard error of  $\bar{X}$  is  $s/\sqrt{n}$  where  $s$  is the sample standard deviation with divisor  $n - 1$ . Then the following provides an (approximate) 95% CI for  $\mu$ :

$$\bar{x} \pm 1.96 \frac{s}{\sqrt{n}}.$$

**Example 4.5** (Computer failures). For the computer failure data,  $\bar{x} = 3.75$ ,  $s = 3.381$  and  $n = 104$ . Under the model that the data are observations of i.i.d. random variables with population mean  $\mu$  (but no other assumptions about the underlying distribution), we compute a 95% confidence interval for  $\mu$  to be

$$\left( 3.75 - 1.96 \frac{3.381}{\sqrt{104}}, 3.75 + 1.96 \frac{3.381}{\sqrt{104}} \right) = (3.10, 4.40).$$

If we can assume a distribution for  $X$ , i.e. a parametric model for  $X$ , then we can do slightly better in estimating the standard error of  $\bar{X}$  and as a result we can improve upon the previously obtained 95% CI. Two examples follow.

**Example 4.6** (Poisson). If  $X_1, \dots, X_n$  are modelled as i.i.d. Poisson ( $\lambda$ ) random variables, then  $\mu = \lambda$  and  $\sigma^2 = \lambda$ . We know  $\text{Var}(\bar{X}) = \sigma^2/n = \lambda/n$ . Hence a standard error is  $\sqrt{\hat{\lambda}/n} = \sqrt{\bar{x}/n}$  since  $\hat{\lambda} = \bar{X}$  is an unbiased estimator of  $\lambda$ . Thus a 95% CI for  $\mu = \lambda$  is given by

$$\bar{x} \pm 1.96\sqrt{\frac{\bar{x}}{n}}.$$

For the computer failure data,  $\bar{x} = 3.75$ ,  $s = 3.381$  and  $n = 104$ . Under the model that the data are observations of i.i.d. random variables following a Poisson distribution with population mean  $\lambda$ , we compute a 95% confidence interval for  $\lambda$  as

$$\bar{x} \pm 1.96\sqrt{\frac{\bar{x}}{n}} = 3.75 \pm 1.96\sqrt{3.75/104} = (3.38, 4.12).$$

We see that this interval is narrower ( $0.74 = 4.12 - 3.38$ ) than the earlier interval  $(3.10, 4.40)$ , which has a length of 1.3. We prefer narrower confidence intervals as they facilitate more accurate inference regarding the unknown parameter.

**Example 4.7** (Bernoulli). If  $X_1, \dots, X_n$  are modelled as i.i.d. Bernoulli ( $p$ ) random variables, then  $\mu = p$  and  $\sigma^2 = p(1 - p)$ . We know  $\text{Var}(\bar{X}) = \sigma^2/n = p(1 - p)/n$ . Hence a standard error is  $\sqrt{\hat{p}(1 - \hat{p})/n} = \sqrt{\bar{x}(1 - \bar{x})/n}$ , since  $\hat{p} = \bar{X}$  is an unbiased estimator of  $p$ . Thus a 95% CI for  $\mu = p$  is given by

$$\bar{x} \pm 1.96\sqrt{\frac{\bar{x}(1 - \bar{x})}{n}}.$$

For the example, suppose  $\bar{x} = 0.2$  and  $n = 10$ . Then we obtain the 95% CI as

$$0.2 \pm 1.96\sqrt{(0.2 \times 0.8)/10} = (-0.048, 0.448)$$

Here  $n$  is too small for the large-sample approximation to be accurate.

#### 4.4.5 Exact confidence interval for the normal mean

For normal models we do not have to rely on large sample approximations, because it turns out that the distribution of

$$T = \frac{\sqrt{n}(\bar{X} - \mu)}{S},$$

where  $S^2$  is the sample variance with divisor  $n - 1$ , is standard (easily calculated) and thus the statistic  $T = T(\mathbf{X}, \mu)$  can be an exact pivot for any sample size  $n > 1$ .

For any chosen  $1 - \alpha$  (e.g., 0.95) we can find  $h$  such that  $P(-h < T < h) = 1 - \alpha$ . Note that  $T$  does not involve the unknown variance  $\sigma^2$ . Proceed as follows to obtain the exact CI for  $\mu$ :

$$\begin{aligned} P(-h \leq T \leq h) &= 1 - \alpha \\ \text{i.e. } P\left(-h \leq \sqrt{n} \frac{(\bar{X} - \mu)}{S} \leq h\right) &= 0.95 \\ \Rightarrow P\left(\bar{X} - h \frac{S}{\sqrt{n}} \leq \mu \leq \bar{X} + h \frac{S}{\sqrt{n}}\right) &= 0.95 \end{aligned}$$

The observed value of this interval,  $(\bar{x} \pm h \frac{s}{\sqrt{n}})$ , is the 95% confidence interval for  $\mu$ . Remarkably, this also of the general form,

$$\text{Estimate} \pm \text{Critical value} \times \text{Standard error},$$

where the critical value is  $h$  and the standard error of the sample mean is  $\frac{s}{\sqrt{n}}$ . Now, how do we find the critical value  $h$  for a given  $1 - \alpha$ ? We need to introduce the  $t$ -distribution.

Let  $X_1, \dots, X_n$  be i.i.d  $N(\mu, \sigma^2)$  random variables. Define  $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$  and

$$S^2 = \frac{1}{n-1} \left( \sum_{i=1}^n X_i^2 - n\bar{X}^2 \right).$$

Then, it can be shown (and will be in MATH2011) that

$$\sqrt{n} \frac{(\bar{X} - \mu)}{S} \sim t_{n-1},$$

where  $t_{n-1}$  denotes the standard  $t$  distribution with  $n - 1$  degrees of freedom. The standard  $t$  distribution is a family of distributions which depend on one parameter called the degrees-of-freedom (df) which is  $n - 1$  here. The concept of degrees of freedom is that it is usually the number of independent random samples,  $n$  here, minus the number of linear parameters estimated, 1 here for  $\mu$ . Hence the df is  $n - 1$ .

The probability density function of the  $t_k$  distribution is similar to a standard normal, in that it is symmetric around zero and ‘bell-shaped’, but the  $t$ -distribution is more heavy-tailed, giving greater probability to observations further away from zero. Figure 4.2 illustrates the  $t_k$  density function for various values of  $k$ .

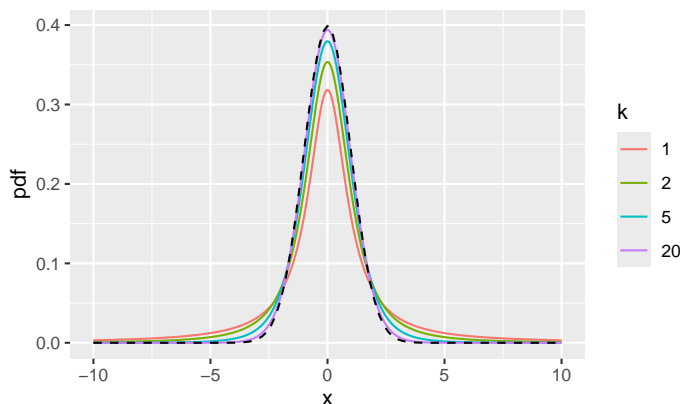


Figure 4.2: The pdf of the  $t_k$  distribution for various values of  $k$ . The dashed line is the standard normal pdf.

The values of  $h$  for a given  $1 - \alpha$  can be obtained using the R command `qt` (abbreviation for quantile of  $t$ ). For example, we can find  $h$  for  $1 - \alpha = 0.95$  and  $n = 20$

```
qt(0.975, df = 19)
```

```
## [1] 2.093024
```

Note that it should be 0.975 so that we are splitting 0.05 probability between the two tails equally and the df should be  $n - 1 = 19$ . Indeed, modifying the above command, we obtain the following critical values for the 95% interval for different values of the sample size  $n$ .

$n$	2	5	10	15	20	30	50	100	1000
$h$	12.71	2.78	2.26	2.14	2.09	2.05	2.01	1.98	1.96

As  $n \rightarrow \infty$ ,  $h$  approaches 1.96 (the normal critical value), since  $t_{n-1}$  converges to the standard normal.

If you can justify that the underlying distribution is normal then you can use the  $t$ -distribution-based confidence interval.

**Example 4.8** (Fast food waiting times). We would like to find a confidence interval for the true mean waiting time. If  $X$  denotes the waiting time in seconds, we have  $n = 20$ ,  $\bar{x} = 67.85$ ,  $s = 18.36$ . Hence, recalling that the critical value  $h = 2.093$ , from the command `qt(0.975, df = 19)`, a 95% confidence interval for  $\mu$  is

$$67.85 - 2.093 \times 18.36/\sqrt{20} \leq \mu \leq 67.85 + 2.093 \times 18.36/\sqrt{20} \\ \Rightarrow 59.26 \leq \mu \leq 76.44.$$

In R, if the vector `fastfood` contains all the service times, we can find a 95% confidence interval with

```
mean(fastfood) + c(-1, 1) * qt(0.975, df = 19) * sd(fastfood) / sqrt(20)
```

```
## [1] 59.25467 76.44533
```

or a 90% confidence interval with

```
mean(fastfood) + c(-1, 1) * qt(0.95, df = 19) * sd(fastfood) / sqrt(20)
```

```
## [1] 60.74905 74.95095
```

or a 99% confidence interval with

```
mean(fastfood) + c(-1, 1) * qt(0.995, df = 19) * sd(fastfood) / sqrt(20)
```

```
## [1] 56.10113 79.59887
```

We see that the interval widens as the confidence level increases.

**Example 4.9** (Weight gain). Returning to the weight gain data from Example 1.3, we would like to find a confidence interval for the true average weight gain (final weight - initial weight). Here  $n = 68$ ,  $\bar{x} = 0.8672$  and  $s = 0.9653$ . Hence, a 95% confidence interval for  $\mu$  is

$$0.8672 - 1.996 \times 0.9653/\sqrt{68} \leq \mu \leq 0.8672 + 1.996 \times 0.9653/\sqrt{68} \\ \Rightarrow 0.6335 \leq \mu \leq 1.1008$$

In R, we obtain the critical value 1.996 by

```
qt(0.975, df = 67)
```

```
## [1] 1.996008
```

Note that the interval here does not include the value 0, so it is very likely that the weight gain is significantly positive, which we will justify using what is called hypothesis testing.

## 4.5 Hypothesis testing

### 4.5.1 Hypothesis testing in general

#### 4.5.1.1 Introduction

Managers and researchers often make claims (e.g., average waiting time  $< 1$  minute; phones never fail in three years; students gain weight in their first year away from home). How can we assess such claims? We use hypothesis testing.

In statistical inference, we use observations  $x_1, \dots, x_n$  of univariate random variables  $X_1, \dots, X_n$  in order to draw inferences about the probability distribution  $f(x)$  of the underlying random variable  $X$ . So far, we have mainly been concerned with estimating features (usually unknown parameters) of  $f(x)$ . It is often of interest to compare alternative specifications for  $f(x)$ . If we have a set of competing probability models which might have generated the observed data, we may want to determine which of the models is most appropriate. A proposed (hypothesised) model for  $X_1, \dots, X_n$  is then referred to as a hypothesis, and pairs of models are compared using hypothesis tests.

For example, let  $H_0$  and  $H_1$  specify competing distributions  $f^{(0)}(x)$  and  $f^{(1)}(x)$  that fully determine the joint distribution of  $X_1, \dots, X_n$ ; these are simple hypotheses. Often both take the same parametric form  $f(x; \theta)$  with different specified values  $\theta^{(0)} \neq \theta^{(1)}$ .

More generally, hypotheses may leave parameters unspecified (composite hypotheses). For instance,  $X_1, \dots, X_n$  i.i.d. from  $f(x; \theta)$  with  $\theta < 0$  does not fully specify probabilities without a particular  $\theta$ .

Examples of hypotheses:

- $X_1, \dots, X_n \sim N(\mu, \sigma^2)$  with  $\mu = 0, \sigma^2 = 2$ .
- $X_1, \dots, X_n \sim N(\mu, \sigma^2)$  with  $\mu = 0, \sigma^2 \in \mathcal{R}_+$ .
- $X_1, \dots, X_n \sim N(\mu, \sigma^2)$  with  $\mu \neq 0, \sigma^2 \in \mathcal{R}_+$ .
- $X_1, \dots, X_n \sim \text{Bernoulli}(p)$  with  $p = \frac{1}{2}$ .
- $X_1, \dots, X_n \sim \text{Bernoulli}(p)$  with  $p \neq \frac{1}{2}$ .
- $X_1, \dots, X_n \sim \text{Bernoulli}(p)$  with  $p > \frac{1}{2}$ .
- $X_1, \dots, X_n \sim \text{Poisson}(\lambda)$  with  $\lambda = 1$ .
- $X_1, \dots, X_n \sim \text{Poisson}(\theta)$  with  $\theta > 1$ .

#### 4.5.1.2 Hypothesis testing procedure

A hypothesis test provides a mechanism for comparing two competing statistical models,  $H_0$  and  $H_1$ . A hypothesis test does not treat the two hypotheses (models) symmetrically. One hypothesis,  $H_0$ , is given special status, and referred to as the null hypothesis. The null hypothesis is the reference model, and is assumed to be appropriate unless the observed data strongly indicate that  $H_0$  is inappropriate, and that  $H_1$  (the alternative hypothesis) should be preferred. Thus, failure to reject  $H_0$  is not evidence that  $H_0$  is true or better supported than  $H_1$ —only that the data do not provide sufficient evidence against  $H_0$ .

A hypothesis test is defined by its critical region or rejection region, which we shall denote by  $C$ .  $C$  is a subset of  $\mathcal{R}^n$  and is the set of possible observed values of  $\mathbf{X}$  which, if observed, would lead to rejection of  $H_0$  in favour of  $H_1$ , i.e.

$$\begin{aligned} \text{If } \mathbf{x} \in C & \quad H_0 \text{ is rejected in favour of } H_1 \\ \text{If } \mathbf{x} \notin C & \quad H_0 \text{ is not rejected} \end{aligned}$$

Because  $\mathbf{X}$  is random, tests can err. We define:

Type I error: $H_0$ is rejected when it is true
Type II error: $H_0$ is not rejected when it is false

The following table helps to understand further:

	$H_0$ true	$H_0$ false
Reject $H_0$	Type I error	Correct decision
Do not reject $H_0$	Correct decision	Type II error



When  $H_0$  and  $H_1$  are simple hypotheses, we can define

$$\begin{aligned}\alpha &= P(\text{Type I error}) = P(\mathbf{X} \in C) \quad \text{if } H_0 \text{ is true} \\ \beta &= P(\text{Type II error}) = P(\mathbf{X} \notin C) \quad \text{if } H_1 \text{ is true}\end{aligned}$$

**Example 4.10** (Uniform). Suppose that we have one observation from the uniform distribution on the range  $(0, \theta)$ . In this case,  $f(x) = 1/\theta$  if  $0 < x < \theta$  and  $P(X \leq x) = \frac{x}{\theta}$  for  $0 < x < \theta$ . We want to test  $H_0 : \theta = 1$  against the alternative  $H_1 : \theta = 2$ . Suppose we decide arbitrarily that we will reject  $H_0$  if  $X > 0.75$ . Then

$$\begin{aligned}\alpha &= P(\text{Type I error}) = P(X > 0.75) \text{ if } H_0 \text{ is true} \\ \beta &= P(\text{Type II error}) = P(X < 0.75) \text{ if } H_1 \text{ is true}\end{aligned}$$

which will imply:

$$\begin{aligned}\alpha &= P(X > 0.75 \mid \theta = 1) = 1 - 0.75 = 0.25, \\ \beta &= P(X < 0.75 \mid \theta = 2) = 0.75/2 = 0.375.\end{aligned}$$

**Example 4.11** (Poisson). The daily demand for a product has a Poisson distribution with mean  $\lambda$ , the demands on different days being statistically independent. It is desired to test the hypotheses  $H_0 : \lambda = 0.7$  against the alternative  $H_1 : \lambda = 0.3$ . The daily demand will be measured for 20 days, and the null hypothesis will be rejected if there is no demand on at least 15 of these 20 days. Calculate the Type I and Type II error probabilities.

Let  $p$  denote the probability that the demand on a given day is zero. Then

$$p = e^{-\lambda} = \begin{cases} e^{-0.7} & \text{under } H_0 \\ e^{-0.3} & \text{under } H_1 \end{cases}$$

If  $X$  denotes the number of days out of 20 with zero demand, it follows that

$$\begin{aligned}X &\sim \text{Binomial}(20, e^{-0.7}) \quad \text{under } H_0, \\ X &\sim \text{Binomial}(20, e^{-0.3}) \quad \text{under } H_1.\end{aligned}$$

Thus

$$\begin{aligned}\alpha &= P(\text{Reject } H_0 \mid H_0 \text{ true}) \\ &= P(X \geq 15 \mid X \sim \text{Binomial}(20, e^{-0.7})) \\ &= 1 - P(X \leq 14 \mid X \sim \text{Binomial}(20, 0.4966)) \\ &= 1 - 0.98028 \\ &= 0.01923(1 - \text{pbinom}(14, \text{size} = 20, \text{prob} = 0.4966))\end{aligned}$$

Furthermore

$$\begin{aligned}\beta &= P(\text{Do not reject } H_0 \mid H_1 \text{ true}) \\ &= P(X \leq 14 \mid X \sim \text{Binomial}(20, e^{-0.3})) \\ &= P(X \leq 14 \mid X \sim \text{Binomial}(20, 0.7408)) \\ &= 0.42023(1 - \text{pbinom}(14, \text{size} = 20, \text{prob} = 0.7408))\end{aligned}$$

The size (significance level) is  $\alpha$ ; the power is  $1 - \beta$ . We want small  $\alpha$  and large power, but they trade off: e.g.,  $C = \emptyset$  gives  $\alpha = 0$  and  $\beta = 1$ ;  $C = \mathbb{R}^n$  gives  $\alpha = 1$  and  $\beta = 0$ .

In this section we have seen how to compute the type I and II error rates given a hypothesis testing procedure. But how should we construct a hypothesis testing procedure in the first place? We typically proceed via a test statistic: a data-derived quantity with a known distribution under  $H_0$ . We then assess whether the observed value is plausible under this null distribution. Next we illustrate.

## 4.5.2 Testing a normal mean (t-test)

### 4.5.2.1 Introduction

Suppose that we observe data  $x_1, \dots, x_n$  which are modelled as observations of i.i.d.  $N(\mu, \sigma^2)$  random variables  $X_1, \dots, X_n$ , and we want to test the null hypothesis

$$H_0 : \mu = \mu_0$$

against the alternative hypothesis

$$H_1 : \mu \neq \mu_0.$$

We recall that

$$\sqrt{n} \frac{(\bar{X} - \mu)}{S} \sim t_{n-1}$$

and therefore, when  $H_0$  is true, often written as under  $H_0$ ,

$$\sqrt{n} \frac{(\bar{X} - \mu_0)}{S} \sim t_{n-1}.$$

Here  $t = \sqrt{n}(\bar{X} - \mu_0)/S$  is a test statistic with known null distribution:  $t_{n-1}$ .

This test is called a  $t$ -test. We reject the null hypothesis  $H_0$  in favour of the alternative  $H_1$  if the observed test statistic seems unlikely to have been generated by the null distribution.

**Example 4.12** (Weight gain). For the weight gain data, if  $x$  denotes the differences in weight gain, we have  $\bar{x} = 0.8672$ ,  $s = 0.9653$  and  $n = 68$ . Hence our test statistic for the null hypothesis  $H_0 : \mu = \mu_0 = 0$  is

$$\sqrt{n} \frac{(\bar{x} - \mu_0)}{s} = 7.41.$$

The observed value of 7.41 does not seem reasonable from Figure 4.3, which shows the density of the  $t$ -distribution with 67 degrees of freedom, and a vertical line at the observed value of 7.41. So there may be evidence here to reject  $H_0 : \mu = 0$ .

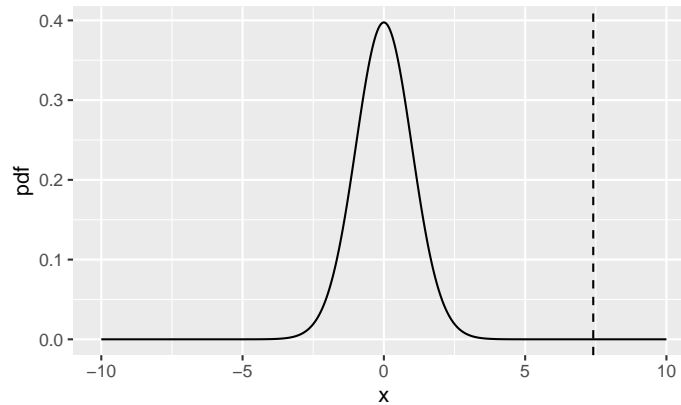


Figure 4.3: The density of the  $t_{67}$  distribution. The dashed line shows the observed value of the test statistic for the weight gain data.

**Example 4.13** (Fast food waiting times). Suppose the manager of the fast food outlet claims that the average waiting time is only 60 seconds. So, we want to test  $H_0 : \mu = 60$ . We have  $n = 20$ ,  $\bar{x} = 67.85$ ,  $s = 18.36$ . Hence our test statistic for the null hypothesis  $H_0 : \mu = \mu_0 = 60$  is

$$\sqrt{n} \frac{(\bar{x} - \mu_0)}{s} = \sqrt{20} \frac{(67.85 - 60)}{18.36} = 1.91.$$

The observed value of 1.91 may or may not be reasonable from Figure 4.4, which shows the density of the  $t$ -distribution with 19 degrees of freedom, and a vertical line at the observed value of 1.91. This value is a bit out in the tail but we are not sure, unlike in the previous weight gain example. So how can we decide whether to reject the null hypothesis?

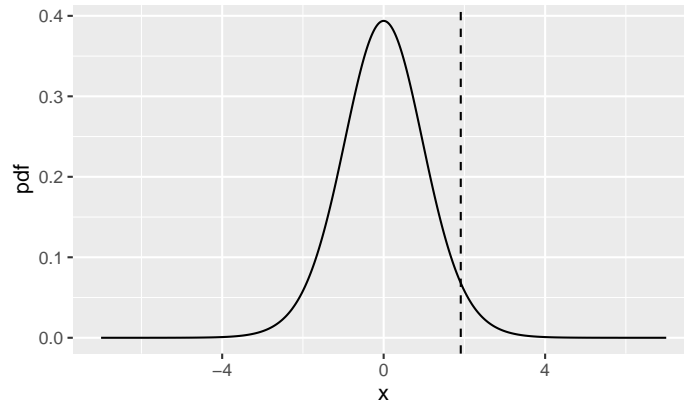


Figure 4.4: The density of the  $t_{19}$  distribution. The dashed line shows the observed value of the test statistic for the fast food data.

#### 4.5.2.2 The significance level

The significance level  $\alpha$  is the Type I error rate. Smaller  $\alpha$  reduces false rejections but increases the chance of failing to reject poor models (Type II error). A common choice is  $\alpha = 0.05$ ; we reject when the test statistic falls in a region of total null probability  $\alpha$ .

#### 4.5.2.3 Rejection region

For the  $t$ -test, the null distribution is  $t_{n-1}$  where  $n$  is the sample size, so the rejection region for the test corresponds to a region of total probability  $\alpha = 0.05$  comprising the ‘most extreme’ values in the direction of the alternative hypothesis. If the alternative hypothesis is two-sided, e.g.  $H_1 : \mu \neq \mu_0$ , then this is obtained as below, where the two shaded regions both have area (probability)  $\alpha/2 = 0.025$ .

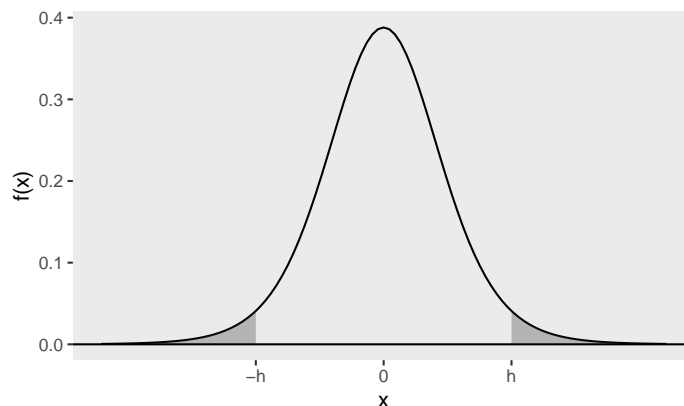


Figure 4.5: The shaded area shows the rejection region for a  $t$ -test.  $h$  is chosen to make the shaded area equal to the significance level  $1 - \alpha$

The value of  $h$  depends on the sample size  $n$  and can be found in R with the `qt` command. Note that

we need to put  $n - 1$  in the `df` argument of `qt`. So for  $n = 100$ , we can find  $h$  with

```
qt(0.975, df = 99)
```

```
## [1] 1.984217
```

For some other values of  $n$ , we have

$n$	2	5	10	15	20	30	50	100	$\infty$
$h$	12.71	2.78	2.26	2.14	2.09	2.05	2.01	1.98	1.96

where the last value for  $n = \infty$  is obtained from the normal distribution.

For a one-sided alternative (e.g.,  $H_1 : \mu > \mu_0$ ), the rejection region lies in the right tail: reject if the statistic exceeds  $h$ , chosen so that the right-tail area under  $t_{n-1}$  is  $\alpha$ . So for  $n = 100$  and  $\alpha = 0.05$ , we can find this  $h$  with

```
qt(0.95, df = 99)
```

```
## [1] 1.660391
```

For some other values of  $n$ , we have

$n$	2	5	10	15	20	30	50	100	$\infty$
$h$	6.31	2.13	1.83	1.76	1.73	1.70	1.68	1.66	1.64

#### 4.5.2.4 Summary of the t-test procedure

Suppose that we observe data  $x_1, \dots, x_n$  which are modelled as observations of i.i.d.  $N(\mu, \sigma^2)$  random variables  $X_1, \dots, X_n$  and we want to test the null hypothesis  $H_0 : \mu = \mu_0$  against the alternative hypothesis  $H_1 : \mu \neq \mu_0$ :

1. Compute the test statistic

$$t = \sqrt{n} \frac{(\bar{x} - \mu_0)}{s}.$$

2. For a chosen  $\alpha$  (usually 0.05), the two-sided rejection region is  $|t| > h$ , where  $-h$  is the  $\alpha/2$  percentile of  $t_{n-1}$ .
3. If your computed  $t$  lies in the rejection region, i.e.  $|t| > h$ , you report that  $H_0$  is rejected in favour of  $H_1$  at the chosen level of significance. If  $t$  does not lie in the rejection region, you report that  $H_0$  is not rejected. (Never refer to ‘accepting’ a hypothesis.)

#### 4.5.2.5 Examples

**Example 4.14** (Fast food waiting times). We would like to test  $H_0 : \mu = 60$  against the alternative  $H_1 : \mu > 60$ , as this alternative will refute the claim of the store manager that customers only wait for a maximum of one minute. We calculated the observed value to be 1.91. This is a one-sided test and for a 5% level of significance, the critical value  $h$  will come from `qt(0.95, df = 19) = 1.73`. Thus the observed value is higher than the critical value so we will reject the null hypothesis, disputing the manager’s claim regarding a minute wait.

**Example 4.15** (Weight gain). For the weight gain example  $\bar{x} = 0.8671, s = 0.9653, n = 68$ . Then, we would be interested in testing  $H_0 : \mu = 0$  against the alternative hypothesis  $H_1 : \mu \neq 0$  in the model that the data are observations of i.i.d.  $N(\mu, \sigma^2)$  random variables.

- We obtain the test statistic

$$t = \sqrt{n} \frac{(\bar{x} - \mu_0)}{s} = \sqrt{68} \frac{(0.8671 - 0)}{0.9653} = 7.41.$$

- Under  $H_0$  this is an observation from a  $t_{67}$  distribution. For significance level  $\alpha = 0.05$  the rejection region is  $|t| > 1.996$ .
- Our computed test statistic lies in the rejection region, i.e.  $|t| > 1.996$ , so  $H_0$  is rejected in favour of  $H_1$  at the 5% level of significance.

In R we can perform the test as follows:

```
wtgain$diff <- wtgain$final - wtgain$initial
t.test(wtgain$diff)

##
##  One Sample t-test
##
## data:  wtgain$diff
## t = 7.4074, df = 67, p-value = 2.813e-10
## alternative hypothesis: true mean is not equal to 0
## 95 percent confidence interval:
##  0.6334959 1.1008265
## sample estimates:
## mean of x
## 0.8671612
```

This yields  $t = 7.4074$  with  $df = 67$ .

#### 4.5.2.6 p-values

The result of a test is most commonly summarised by rejection or non-rejection of  $H_0$  at the stated level of significance. An alternative, which you may see in practice, is the computation of a p-value. This is the probability, under  $H_0$ , of observing the realised test statistic or something more extreme. Small p-values indicate evidence against  $H_0$ . A common threshold is 0.05.

For the t-test with a two-sided alternative, the p-value is given by:

$$p = P(|T| > |t_{\text{obs}}|) = 2P(T > |t_{\text{obs}}|),$$

where  $T$  has a  $t_{n-1}$  distribution and  $t_{\text{obs}}$  is the observed sample value.

However, if the alternative is one-sided and to the right then the p-value is given by:

$$p = P(T > t_{\text{obs}}),$$

where  $T$  has a  $t_{n-1}$  distribution and  $t_{\text{obs}}$  is the observed sample value.

A small p-value corresponds to an observation of  $T$  that is improbable (since it is far out in the low probability tail area) under  $H_0$  and hence provides evidence against  $H_0$ . The p-value should not be misinterpreted as the probability that  $H_0$  is true.  $H_0$  is not a random event (under our models) and so cannot be assigned a probability. The null hypothesis is rejected at significance level  $\alpha$  if the p-value for the test is less than  $\alpha$ .

Reject  $H_0$  if p-value  $< \alpha$ .

**Example 4.16** (Fast food waiting times). In the fast food example, a test of  $H_0 : \mu = 60$  resulted in a test statistic  $t = 1.91$ . Then the p-value is given by

$$p = P(T > 1.91) = 0.036, \text{ when } T \sim t_{19}.$$

This is the area of the shaded region in Figure 4.6. In R it is

```
1 - pt(1.91, df = 19)
```

```
## [1] 0.03567359
```

The p-value 0.036 indicates some evidence against the manager's claim at the 5% level of significance but not the 1% level of significance.

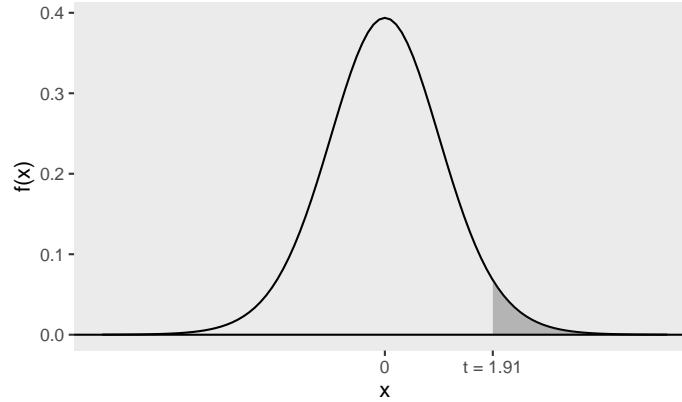


Figure 4.6: The area of the shaded region (under the  $t_{19}$  pdf) is the p-value for the one-sided hypothesis test in the fast food example

When the alternative hypothesis is two-sided the p-value has to be calculated from  $P(|T| > t_{\text{obs}})$ , where  $t_{\text{obs}}$  is the observed value and  $T$  follows the  $t$ -distribution with  $n - 1$  df.

**Example 4.17** (Weight gain). Because the alternative is two-sided, the p-value is given by:

$$p = P(|T| > 7.41) = 2.78 \times 10^{-10} \approx 0.0, \text{ when } T \sim t_{67}.$$

This very small p-value indicates very strong evidence against the null hypothesis of no weight gain in the first year of college.

#### 4.5.2.7 Equivalence of testing and interval estimation

Note that the 95% confidence interval for  $\mu$  in the weight gain example has previously been calculated to be (0.6335, 1.1008) in Example 4.9. This interval does not include the hypothesised value 0 of  $\mu$ . Hence we can conclude that the hypothesis test at the 5% level of significance will reject the null hypothesis  $H_0 : \mu = 0$ .

This is because

$$\left| T_{\text{obs}} = \frac{\sqrt{n}(\bar{x} - \mu_0)}{s} \right| > h$$

implies and is implied by  $\mu_0$  being outside the interval

$$(\bar{x} - hs/\sqrt{n}, \bar{x} + hs/\sqrt{n}).$$

Notice that  $h$  is the same in both. For this reason we often just calculate the confidence interval and take the reject/do not reject decision merely by inspection.

### 4.5.3 Two sample t-tests

Suppose we observe two samples  $x_1, \dots, x_n$  and  $y_1, \dots, y_m$ , modelled as

$$X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} N(\mu_X, \sigma_X^2)$$

and

$$Y_1, \dots, Y_m \stackrel{\text{i.i.d.}}{\sim} N(\mu_Y, \sigma_Y^2)$$

respectively, where it is also assumed that the  $X$  and  $Y$  variables are independent of each other. Suppose that we want to test the hypothesis that the distributions of  $X$  and  $Y$  are identical, that is

$$H_0 : \mu_X = \mu_Y, \quad \sigma_X = \sigma_Y = \sigma$$

against the alternative hypothesis

$$H_1 : \mu_X \neq \mu_Y.$$

In Chapter 3 we proved that

$$\bar{X} \sim N(\mu_X, \sigma_X^2/n) \text{ and } \bar{Y} \sim N(\mu_Y, \sigma_Y^2/m)$$

and therefore

$$\bar{X} - \bar{Y} \sim N\left(\mu_X - \mu_Y, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right).$$

Hence, under  $H_0$ ,

$$\bar{X} - \bar{Y} \sim N\left(0, \sigma^2 \left[\frac{1}{n} + \frac{1}{m}\right]\right) \Rightarrow \sqrt{\frac{nm}{n+m}} \frac{(\bar{X} - \bar{Y})}{\sigma} \sim N(0, 1).$$

The involvement of the (unknown)  $\sigma$  above means that this is not a pivotal test statistic. It will be proved in MATH2011 that if  $\sigma^2$  is replaced by its unbiased estimator  $S^2$ , which here is the two-sample estimator of the common variance, given by

$$S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2 + \sum_{i=1}^m (Y_i - \bar{Y})^2}{n + m - 2},$$

then

$$\sqrt{\frac{nm}{n+m}} \frac{(\bar{X} - \bar{Y})}{S} \sim t_{n+m-2}.$$

Hence

$$t = \sqrt{\frac{nm}{n+m}} \frac{(\bar{x} - \bar{y})}{s}$$

is a test statistic. For a two-sided test, reject if  $|t| > h$ , where  $-h$  is the  $\alpha/2$  percentile of  $t_{n+m-2}$ .

From the hypothesis testing, a  $100(1 - \alpha)\%$  confidence interval for  $\mu_X - \mu_Y$  is given by

$$\bar{x} - \bar{y} \pm h \sqrt{\frac{n+m}{nm}} s,$$

where  $-h$  is the  $\alpha/2$  (usually 0.025) percentile of  $t_{n+m-2}$ .

**Example 4.18** (Fast food waiting times). In this example, we would like to know if there are significant differences between the AM and PM waiting times. Here  $n = m = 10$ ,  $\bar{x} = 68.9$ ,  $\bar{y} = 66.8$ ,  $s_x^2 = 538.22$  and  $s_y^2 = 171.29$ . From this we calculate,

$$s^2 = \frac{(n-1)s_x^2 + (m-1)s_y^2}{n+m-2} = 354.8,$$

and

$$t_{\text{obs}} = \sqrt{\frac{nm}{n+m}} \frac{(\bar{x} - \bar{y})}{s} = 0.25.$$

This is not significant as the critical value  $h = \text{qt}(0.975, \text{df} = 18) = 2.10$  is larger in absolute value than 0.25. We can conduct the test easily in R:

```

fastfood_am <- c(38, 100, 64, 43, 63, 59, 107, 52, 86, 77)
fastfood_pm <- c(45, 62, 52, 72, 81, 88, 64, 75, 59, 70)
t.test(fastfood_am, fastfood_pm)

##
## Welch Two Sample t-test
##
## data: fastfood_am and fastfood_pm
## t = 0.24929, df = 14.201, p-value = 0.8067
## alternative hypothesis: true difference in means is not equal to 0
## 95 percent confidence interval:
## -15.9434 20.1434
## sample estimates:
## mean of x mean of y
##      68.9      66.8

```

R reports the test statistic (0.249), p-value (0.8067), and the 95% CI (−15.94, 20.14).

#### 4.5.4 Paired t-test

Sometimes  $X$  and  $Y$  are not independent by design, e.g., paired measurements on the same subjects before ( $X$ ) and after ( $Y$ ) treatment (as in the weight gain example). In such cases, analyse the differences

$$z_i = x_i - y_i, \quad i = 1, \dots, n$$

and modelling these differences as observations of i.i.d.  $N(\mu_z, \sigma_z^2)$  variables  $Z_1, \dots, Z_n$ . Then, a test of the hypothesis  $\mu_X = \mu_Y$  is achieved by testing  $\mu_Z = 0$ , which is just a standard (one sample) t-test, as described previously.

**Example 4.19.** Water-quality researchers wish to measure the biomass to chlorophyll ratio for phytoplankton (in milligrams per litre of water). There are two possible tests, one less expensive than the other. To see whether the two tests give the same results, ten water samples were taken and each was measured both ways. The results are as follows:

Test 1 ( $x$ )	45.9	57.6	54.9	38.7	35.7	39.2	45.9	43.2	45.4	54.8
Test 2 ( $y$ )	48.2	64.2	56.8	47.2	43.7	45.7	53.0	52.0	45.1	57.5

To test the null-hypothesis

$$H_0 : \mu_Z = 0 \text{ against } H_1 : \mu_Z \neq 0$$

we use the test statistic  $t = \sqrt{n} \frac{\bar{z}}{s_z}$ , where  $s_z^2 = \frac{1}{n-1} \sum_{i=1}^n (z_i - \bar{z})^2$ .

From the hypothesis testing, a  $100(1 - \alpha)\%$  confidence interval for  $\mu_Z$  is given by  $\bar{z} \pm h \frac{s_z}{\sqrt{n}}$ , where  $h$  is the critical value of the  $t$  distribution with  $n - 1$  degrees of freedom. In R we perform the test as follows:

```

x <- c(45.9, 57.6, 54.9, 38.7, 35.7, 39.2, 45.9, 43.2, 45.4, 54.8)
y <- c(48.2, 64.2, 56.8, 47.2, 43.7, 45.7, 53.0, 52.0, 45.1, 57.5)

t.test(x, y, paired = TRUE)

##
## Paired t-test
##
## data: x and y
## t = -5.0778, df = 9, p-value = 0.0006649
## alternative hypothesis: true mean difference is not equal to 0

```



```
## 95 percent confidence interval:  
## -7.531073 -2.888927  
## sample estimates:  
## mean difference  
## -5.21
```

This gives the test statistic  $t_{\text{obs}} = -5.0778$  with a df of 9 and a p-value = 0.0006649. Thus we reject the null hypothesis. The associated 95% CI is  $(-7.53, -2.89)$ .

The second test yields significantly higher values than the first, so it cannot replace the first.



## Chapter 5

# Simple Linear Regression

### 5.1 What is regression?

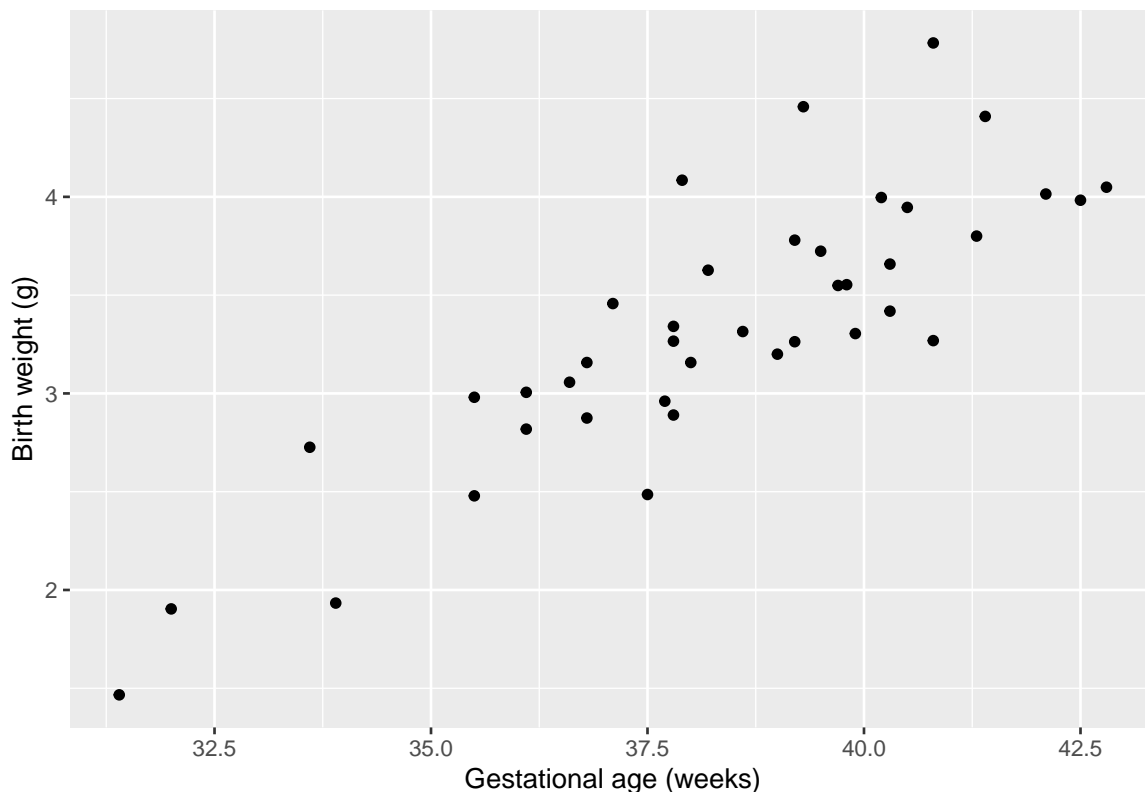
So far, we have looked at simple statistical models, with a single variable of interest. We have estimated the distribution of that variable based on random samples from that one variable. In many real-world problems, we instead want to understand the relationship between multiple variables.

Regression seeks to model how one “output” variable, called the **response** variable,  $Y$ , depends on one or more “input” variables  $x$ , which we will call **covariates**.

The response  $Y$  is sometimes called the dependent variable. The covariates  $x$  are sometimes called independent or explanatory variables.

In MATH1063, we will study only **simple** regression models, which have a single covariate  $x$ .

For instance, suppose we are interested in how the birth weight of babies depends on gestational age (in weeks). The response,  $Y$ , is birth weight in  $g$ . The covariate,  $x$  is gestational age (in weeks). Suppose we have the following data (this data is fictional, but based on real data for male singleton births in Canada).



## 5.2 Simple linear regression

In simple linear regression, we model the dependence of the response on the covariate as a straight line plus an error term. That is

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad i = 1, \dots, n,$$

where  $\beta_0$  is an **intercept** parameter,  $\beta_1$  is a **slope** parameter and  $\epsilon_i$  is a random error term. We usually model  $\epsilon_i \sim N(0, \sigma^2)$ , with variance parameter  $\sigma^2$ . There are three unknown parameters,  $\beta_0$ ,  $\beta_1$  and  $\sigma^2$ , which we want to estimate from the data.

We have

$$E(Y_i) = \beta_0 + \beta_1 x_i.$$

Any choice of  $\beta_0$  and  $\beta_1$  give a different straight line. We call  $\beta_0$  and  $\beta_1$  the **regression parameters**.

Out of all possible choices of straight line (choices of the regression parameters  $\beta_0$  and  $\beta_1$ ), which one should we choose?

## 5.3 Estimating the regression parameters

To estimate the unknown regression parameters, we attempt to make the associated straight line as close as possible to the data. We measure the distance between the line and the data with the sum of squares criterion:

$$SS(\beta_0, \beta_1) = \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2.$$

To estimate  $\beta_0$  and  $\beta_1$ , we find the values which minimise this sum of squares criterion. These estimates are called the **least squares estimates**.

To find the least squares estimates, we take partial derivatives of  $SS(\beta_0, \beta_1)$  with respect to  $\beta_0$  and  $\beta_1$ , set them to zero, and solve for the parameters.

First, differentiate with respect to  $\beta_0$ , to give

$$\frac{\partial SS}{\partial \beta_0} = -2 \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i).$$

Setting to zero:

$$\sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i) = 0,$$

so

$$n\beta_0 + \beta_1 \sum_{i=1}^n x_i = \sum_{i=1}^n y_i,$$

which gives

$$\beta_0 = \bar{y} - \beta_1 \bar{x}.$$

Next, differentiate with respect to  $\beta_1$ , to give

$$\frac{\partial SS}{\partial \beta_1} = -2 \sum_{i=1}^n x_i (y_i - \beta_0 - \beta_1 x_i).$$

Setting to zero gives

$$\sum_{i=1}^n x_i (y_i - \beta_0 - \beta_1 x_i) = 0,$$

or

$$\sum_{i=1}^n x_i y_i - \beta_0 \sum_{i=1}^n x_i - \beta_1 \sum_{i=1}^n x_i^2 = 0.$$

Substituting  $\beta_0 = \bar{y} - \beta_1 \bar{x}$  gives

$$\sum_{i=1}^n x_i y_i - (\bar{y} - \beta_1 \bar{x}) \sum_{i=1}^n x_i - \beta_1 \sum_{i=1}^n x_i^2 = 0,$$

so

$$\sum_{i=1}^n x_i y_i - n\bar{y}\bar{x} = \beta_1 \left( \sum_{i=1}^n x_i^2 - n\bar{x}^2 \right).$$

So

$$\beta_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}.$$

Thus, the least squares estimates are

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}, \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}.$$

These are unbiased estimators of the regression parameters. We do not derive this result here; the derivation will be covered in the second-year module, Statistical Modelling I).

## 5.4 Estimating the variance parameter

In addition to estimating the regression parameters  $\beta_0$  and  $\beta_1$ , we also need to estimate the variance parameter  $\sigma^2$  of the error terms.

We estimate  $\sigma^2$  with

$$\hat{\sigma}^2 = \frac{1}{n-2} \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2,$$

where  $\hat{\beta}_0$  and  $\hat{\beta}_1$  are the least squares estimates.

This estimator is unbiased for  $\sigma^2$ . We do not derive this result here; the derivation will be covered in the second-year module, Statistical Modelling I.

## 5.5 Model fitting in R

In practice, we do not need to apply these formulas by hand to find the least squares estimates. Instead, we can use the `lm` function in R to find the estimates. For instance, if the birth weight data is stored in a data frame called `bw`, with columns `weight` for birth weight (which is the response,  $y$ ) and `ga` for gestational age (which is the covariate,  $x$ ), then we can fit the simple linear regression model with:

```
mod <- lm(weight ~ ga, data = bw)
```

We can inspect the fitted model:

```
mod

##
## Call:
## lm(formula = weight ~ ga, data = bw)
##
## Coefficients:
## (Intercept)          ga
##    -5.1062      0.2203
```

We see  $\hat{\beta}_0 = -5.1062$  and  $\hat{\beta}_1 = 0.2203$ .

We can get more information about the model fit, including standard errors for the estimates, by using the `summary` function in R:

```
summary(mod)

##
## Call:
## lm(formula = weight ~ ga, data = bw)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.67051 -0.26954 -0.08779  0.17978  0.90572
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -5.10619    0.86147  -5.927 7.16e-07 ***
## ga           0.22033    0.02245   9.814 5.74e-12 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
```

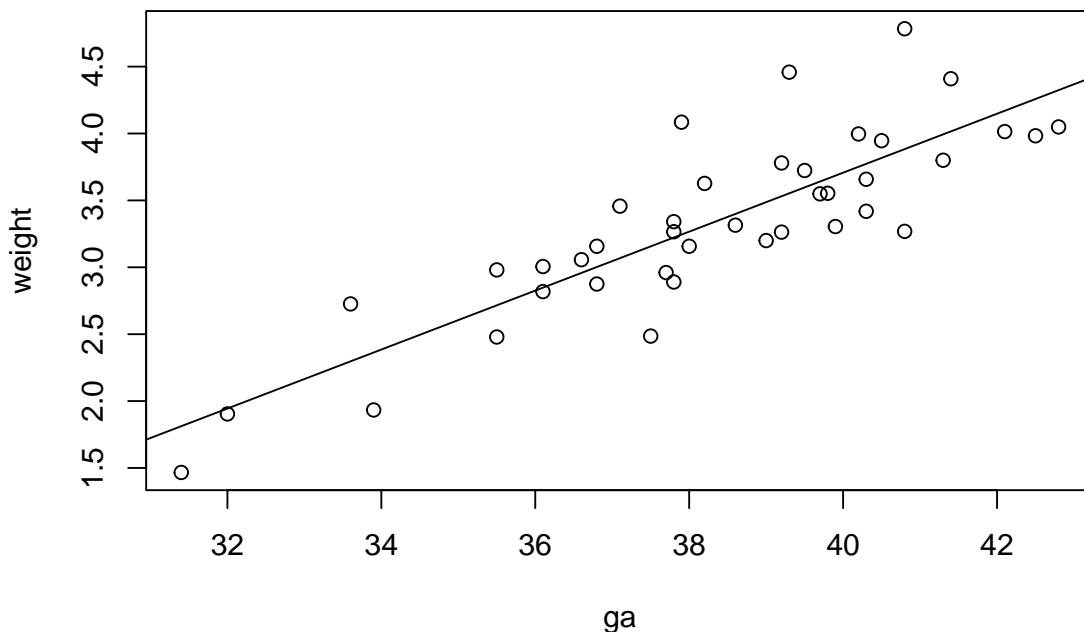
```
## Residual standard error: 0.3731 on 38 degrees of freedom
## Multiple R-squared:  0.7171, Adjusted R-squared:  0.7096
## F-statistic: 96.32 on 1 and 38 DF,  p-value: 5.735e-12
```

You can learn more about expressing uncertainty in simple linear regression the second-year module, Statistical Modelling I.

The “Residual standard error” (0.3731) is an estimate of  $\sigma$ . The variance estimate  $\hat{\sigma}^2$  can be found by squaring this number (in this case,  $\hat{\sigma}^2 = 0.1392$ ).

We can also plot the data with our fitted line overlaid:

```
plot(weight ~ ga, data = bw)
abline(mod)
```



## 5.6 Limitations of simple linear regression

While simple linear regression is a powerful and widely used tool, it has several important limitations:

- **Only one covariate:** Simple linear regression models the relationship between the response and a single covariate. In many real-world situations, multiple variables may influence the response.
- **Linearity assumption:** The model assumes that the relationship between the response and the covariate is linear. If the true relationship is non-linear, the model may not fit well.
- **Constant variance (homoscedasticity):** The errors are assumed to have constant variance across all values of the covariate. If the variance of the errors changes (heteroscedasticity), the model's estimates and inferences may be unreliable.
- **Normality of errors:** The errors are assumed to be normally distributed. If this assumption is violated, inference based on the model may not be valid.

These limitations, and methods to address them, will be covered in more detail in the second-year module, Statistical Modelling I.