MA1521 CALCULUS FOR COMPUTING[1]

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January 9, 2023

[1]This notes is exclusively for students taking MA1521 in AY2022/23 Semester 2.

Contents

Chapter 0

Real Numbers and Functions

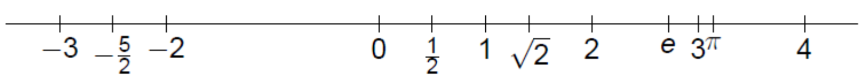
Read Thomas' Calculus, Chapter 1.

0.1 Numbers

The collection of all *real numbers* is denoted by $\mathbb{R}$. Thus $\mathbb{R}$ includes the integers

$$\ldots, -2, -1, 0, 1, 2, 3 \ldots,$$

the *rational numbers*, p/q, where p and q are integers ($q \neq 0$), and the *irrational numbers*, like $\sqrt{2}, \pi, e$, etc.



The number line

$a \in \mathbb{R}$ means a is a member of the set $\mathbb{R}$. In other words, a is a real number. Given two real numbers a and b with $a < b$, the *closed interval* $[a, b]$ consists of all x such that $a \leq x \leq b$, and the *open interval* (a, b) consists of all x such that $a < x < b$. Similarly, we may form the half-open intervals $[a, b)$ and $(a, b]$.

0.2 Absolute Value

The *absolute value* of a number $a \in \mathbb{R}$ is written as $|a|$ and is defined as

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases}$$

For example, $|2| = 2, |-2| = 2$.

Some properties of $|x|$ are summarized as follows:

1. $|-x| = |x|$, for all $x \in \mathbb{R}$.
2. $|xy| = |x||y|$, for all $x, y \in \mathbb{R}$.
3. $-|x| \le x \le |x|$, for all $x \in \mathbb{R}$.
4. For a fixed $r > 0$, $|x| < r$ if and only if $x \in (-r, r)$.
5. $\sqrt{x^2} = |x|$, $x \in \mathbb{R}$.
6. (*Triangle Inequality*) $|x + y| \le |x| + |y|$ for all $x, y \in \mathbb{R}$.

Example 0.1. *Solve the inequality* $\dfrac{2x-1}{2x+1} < 1$.

Solution.

$$
\begin{aligned}
& \frac{2x-1}{2x+1} < 1 \\
\Leftrightarrow\ & 0 < 1 - \frac{2x-1}{2x+1} \\
\Leftrightarrow\ & 0 < \frac{2x+1-2x+1}{2x+1} \\
\Leftrightarrow\ & 0 < \frac{2}{2x+1} \\
\Leftrightarrow\ & 0 < 2x+1 \\
\Leftrightarrow\ & -\frac{1}{2} < x.
\end{aligned}
$$

■

Example 0.2. *Solve the inequality* $|x+1| \le |2x-1|$.

Solution.

$$
\begin{aligned}
& |x+1| \le |2x-1| \\
\Leftrightarrow\ & |x+1|^2 \le |2x-1|^2 \\
\Leftrightarrow\ & x^2 + 2x + 1 \le 4x^2 - 4x + 1 \\
\Leftrightarrow\ & 0 \le 3x^2 - 6x \\
\Leftrightarrow\ & 0 \le 3x(x-2) \\
\Leftrightarrow\ & x \le 0 \text{ or } x \ge 2 \\
\Leftrightarrow\ & x \in (-\infty, 0] \cup [2, \infty).
\end{aligned}
$$

■

Exercise 0.1. *Let $r > 0$. Prove that $|x-a| < r$ if and only if $x \in (-r+a, a+r)$.*

Exercise 0.2. *Prove the triangle inequality $|x+y| \leq |x|+|y|$.*

Exercise 0.3. *Prove that for any $x, y \in \mathbb{R}$, $\big||x|-|y|\big| \leq |x-y|$.*

0.3 Functions

A *function* $f : A \longrightarrow B$ is a rule that assigns to each $a \in A$ one specific member $f(a)$ of B. Symbolically we may denote the function by $a \mapsto f(a)$. We can specify a function f by giving the rule for $f(x)$.

Example 0.3. *$f(x) = x^2/(1-x)$ assigns the number $x^2/(1-x)$ to each $x \neq 1$ in $\mathbb{R}$.*

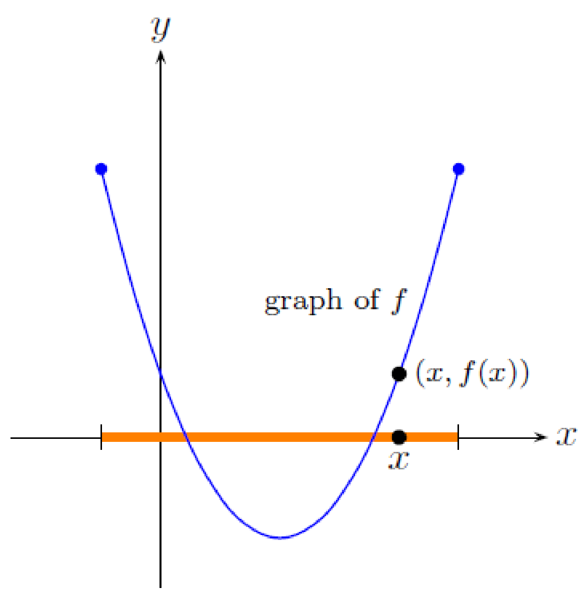
The set A is called the *domain* of f and B is the *codomain* of f.

The *range* of f is the subset of B consisting of all the values of f. That is, the range of $f = \{f(x) \in B \mid x \in A\}$.

Given $f : A \longrightarrow \mathbb{R}$, it means that f assigns a value $f(x)$ in $\mathbb{R}$ to each $x \in A$.

Such a function is called a *real-valued function*.

For a real-valued function $f : A \longrightarrow \mathbb{R}$ defined on a subset A of $\mathbb{R}$, the *graph* of f consists of all the points $(x, f(x))$ in the xy-plane.



The graph of a function f

If $f : A \to B$ and $g : B \to C$, then the composite function of f and g is the function $g \circ f : A \to C$ given by $g \circ f(x) = g(f(x))$.

Example 0.4. *Let $f(x) = \frac{1}{x}$ and $g(x) = x^2 - 1$. Find $g \circ f$ and $f \circ g$.*

Solution. $g \circ f(x) = g(f(x)) = g(\frac{1}{x}) = (\frac{1}{x})^2 - 1 = \frac{1}{x^2} - 1$.
$f \circ g(x) = f(g(x)) = f(x^2 - 1) = \frac{1}{x^2-1}$. ∎

Let $f : A \to B$. If $g : B \to A$ is a function such that $f(g(x)) = x$ for all $x \in B$ and $g(f(x)) = x$ for all $x \in A$, then g is called the inverse of f. Similarly, f is the inverse of g. The inverse function of f is usually denoted by f^{-1}.

Let $f : A \to B$. f is called an *injective* function if for any $x, y \in A$, $f(x) = f(y) \Rightarrow x = y$. f is called a *surjective* function if for any $z \in B$, there is an $x \in A$ such that $f(x) = z$. f is called a *bijective* function if f is injective and surjective.

Exercise 0.4. *Prove that if f^{-1} exists, then f is a bijective function.*

0.4 Polynomials

A function of the form $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, \ldots, a_n$ are constants, is called a polynomial of degree n.

For example, a quadratic function $p(x) = ax^2 + bx + c$ is a polynomial of degree 2.

A polynomial of degree n can be factored as a product of linear and quadratic factors.

For example, $x^4 - 1 = (x^2 + 1)(x + 1)(x - 1)$.

In general a polynomial of degree n has at most n real roots.

For example, $x^4 - 1$ has only two real roots -1 and 1.

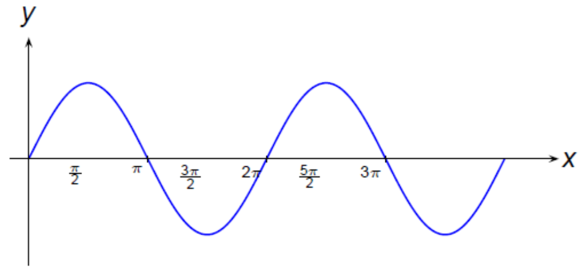
0.5 Rational Functions

A rational function is a function of the form $\dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials. The domain of $\dfrac{p(x)}{q(x)}$ consists of all real numbers except the roots of $q(x)$.

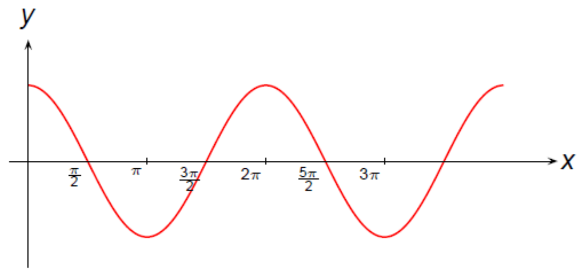
For example, the domain of $\dfrac{x^3 + 3}{x^4 - 1}$ is $\mathbb{R} \setminus \{-1, 1\}$.

0.6 Trigonometric Functions

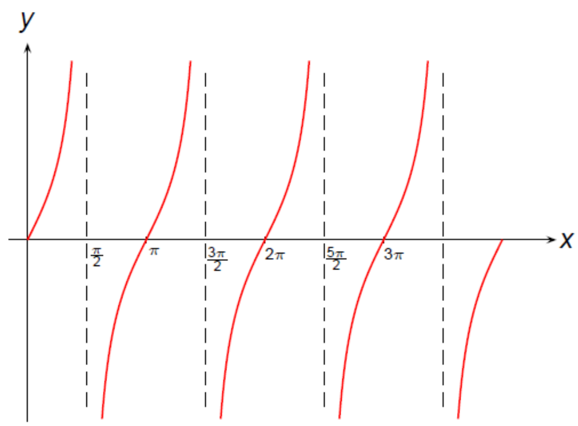
The 6 trigonometric functions are $\sin x, \cos x, \tan x, \csc x, \sec x, \cot x$. They are periodic functions of period 2π.



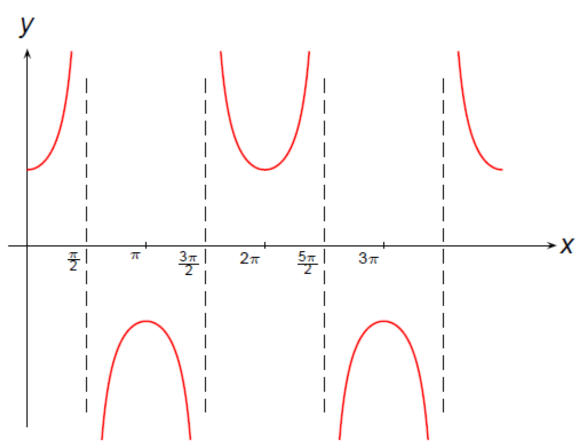
$y = \sin x$



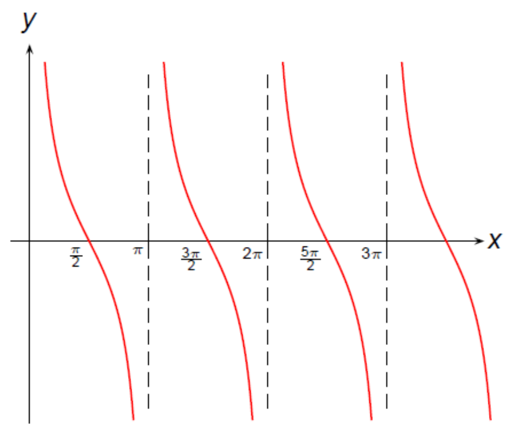
$y = \cos x$



$y = \tan x$



$y = \sec x$



$y = \cot x$

Exercise 0.5. *Sketch the graph of* $\csc(x)$.

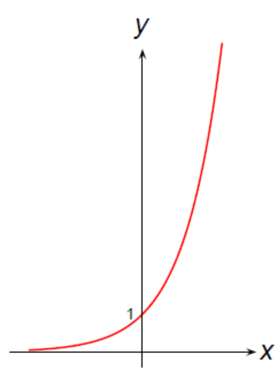
0.7 Exponential and Logarithmic Function

A function of the form $f(x) = a^x$, where $a > 0$ is called an exponential function. It's inverse function, denoted by $\log_a x$ is called the logarithmic function to the base a. ($a > 0$ and $a \neq 1$.)

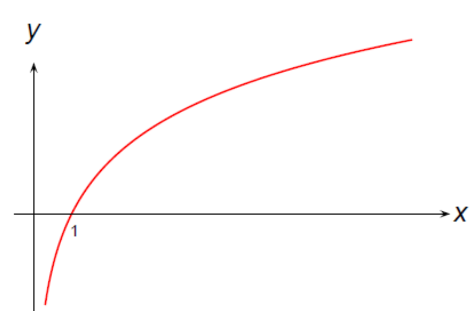
Let $e = 2.71828182845904523536028 7$ be the Euler constant. Then the inverse of the exponential function e^x is the natural logarithm $\ln x$.

We have $e^{\ln x} = x$ for $x > 0$ and $\ln e^x = x$ for all x.

The domain of e^x is $\mathbb{R}$ and the range is the set $\mathbb{R}^+$ of all positive real numbers.



$y = e^x$



$y = \ln x$

Example 0.5. *Sketch the graph of* $y = \frac{4}{x+2} - 3$.

Solution. The domain of the function y is $\mathbb{R} \setminus \{-2\}$.

When $x > -2$ and close to -2, the value of y is large and tends to positive infinity.

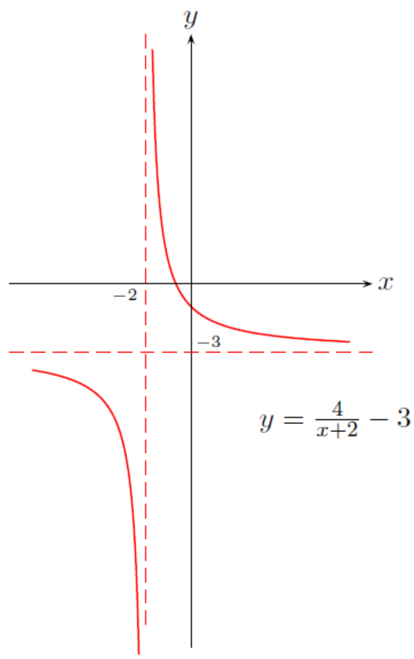
When $x < -2$ and close to -2, the value of y is large and tends to negative infinity.

When x is large and positive, the term $\frac{4}{x+2}$ is positive and small, the value of y is bigger than -3 but close to -3.

When x is large and negative, the term $\frac{4}{x+2}$ is negative and small, the value of y is smaller than -3 but close to -3.

Also if $x_2 > x_1 > -2$, then $\frac{4}{x_2+2} - 3 < \frac{4}{x_1+2} - 3$. That is $y_2 < y_1$, meaning y is a decreasing function for $x > -2$. Similarly, y is a decreasing function for $x < -2$.

With this information, we can sketch the graph of y as follow.



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0.8 The Range of a Function

In general it is not so easy to determine the range of a function. In some simple cases, basic algebraic techniques can be used to find the range of a function.

Example 0.6. *Find the maximal domain and the range of* $f(x) = \frac{1}{x-1}$.

Solution. The maximal domain of f is $\mathbb{R} \setminus \{1\}$.

Recall that the range of $f = \{f(x) \in \mathbb{R} \mid x \neq 1\}$.

To find the range of f, let $y = f(x)$. That is $y = \frac{1}{x-1}$. Solving for x, we get $x = 1 + \frac{1}{y}$. From this we see that if $y \neq 0$ then we may choose $x = 1 + \frac{1}{y}$ to get $f(x) = y$. Thus the range of f is $\mathbb{R} \setminus \{0\}$. ■

Example 0.7. *Find the maximal domain and range of* $f(x) = x^2 - x + 1$.

Solution. The maximal domain of f is $\mathbb{R}$.
To find the range of f, let $y = f(x)$. That is $y = x^2 - x + 1$. Solving for x, we get $x = \frac{1}{2}(1 \pm \sqrt{1 - 4(1 - y)})$. That is $x = \frac{1}{2}(1 \pm \sqrt{4y - 3})$.

From this we see that if $y \geq \frac{3}{4}$ then we may choose $x = \frac{1}{2}(1 \pm \sqrt{4y - 3})$ to get $f(x) = y$. Thus the range of f is $[\frac{3}{4}, \infty)$. ■

Exercise 0.6. *Let* $f(x) = x + 5$ *and* $g(x) = x^2 - 3$. *Find the maximal domain and range of* $g(f(x))$.

Ans: Maximal domain is $\mathbb{R}$, range is $[-3, \infty)$.

Chapter 1

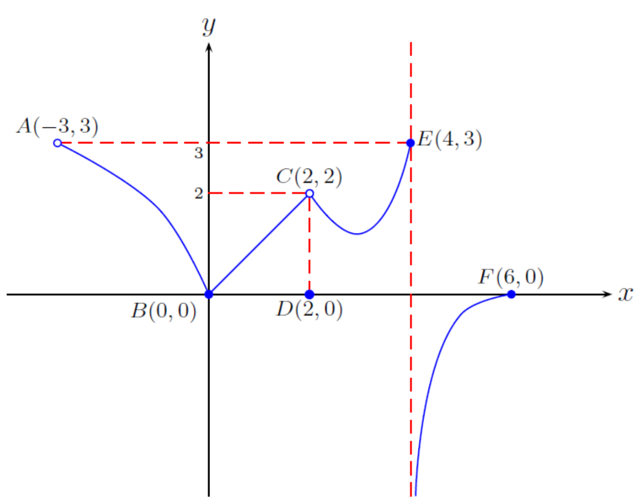
Limits and Continuity

Read Thomas' Calculus, Chapter 2.

1.1 Limits

Let f be a real-valued function defined on some interval I (e.g. (a,b), or $(a,b]$ or (a,∞)). Let c be a point in I.

- $\lim_{x\to c^-} f(x)$ is the value that $f(x)$ approaches when x approaches c from the left.
- $\lim_{x\to c^+} f(x)$ is the value that $f(x)$ approaches when x approaches c from the right.
- Let c be an interior point (i.e. not an end point). If $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L \in \mathbb{R}$, we say that $\lim_{x\to c} f(x)$ exist and has value L.



c	Left Limit $\lim_{x\to c^-} f(x)$	Right Limit $\lim_{x\to c^+} f(x)$	Limit $\lim_{x\to c} f(x)$	$f(x)$
0				
2				
4				
-3				
6				

1.2 Continuity

Let f be a real-valued function defined on some interval I (e.g. (a,b), or $(a,b]$ or (a,∞)). Let c be a point in I.

Continuity at a point

Case 1 c is an interior point

- f is continuous at $x = c$ if

(i) $\lim_{x\to c} f(x)$ exists,

(ii) $\lim_{x\to c} f(x) = f(c)$.

Case 2 c is the left end-point

- f is continuous at $x = c$ if

(i) $\lim_{x\to c^+} f(x)$ exists,

(ii) $\lim_{x\to c^+} f(x) = f(c)$.

Case 3 c is the right end-point

- f is continuous at $x = c$ if

(i) $\lim_{x\to c^-} f(x)$ exists,

(ii) $\lim_{x\to c^-} f(x) = f(c)$.

Continuity on an interval

- f is continuous on an interval if f is continuous at $x = c$ for all points c in I.

Results

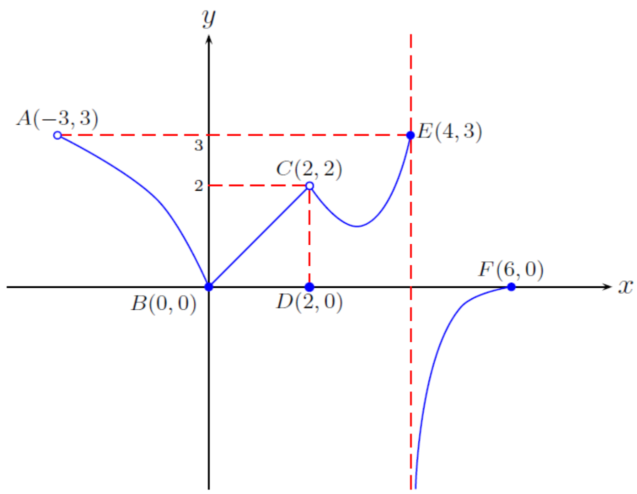
- If f and g are continuous at $x = c$, then for any constant k and any positive constant n, each of the following functions is continuous at $x = c$.

(i) $f+g$, (ii) f^n, (iii) kf, (iv) fg, (v) f/g provided $g(c) \neq 0$.

- If g is continuous at $x=c$ and f is continuous at $x=g(c)$, then the composite function $f \circ g$ is continuous at $x=c$.

  (Note $(f \circ g)(x) = f(g(x))$).

Example 1.1. *Find the points of discontinuity of the function f whose graph on $(-3,6]$ is given below.*



Solution.

Point of discontinuity	Reason
$x=2$	$\lim_{x\to 2} f(x) \neq f(2)$
$x=4$	$\lim_{x\to 4} f(x)$ does not exist

1.3 Evaluation of limits

Results (Law of limits)

The following results are true provided all the limits involved exist. The limit could be one-sided or two-sided. The number k is a constant.

1. $\lim_{x\to c}(f(x) \pm g(x)) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$
2. $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$
3. $\lim_{x\to c}(f(x)g(x)) = (\lim_{x\to c} f(x))(\lim_{x\to c} g(x))$

4. $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$

5. If g is continuous at the point b and $\lim_{x\to c} f(x) = b$, then $\lim_{x\to c} g(f(x)) = g(b) = g(\lim_{x\to c} f(x))$.

Result. The following functions are continuous on any interval contained in their maximal domain.

1. Polynomials
2. Trigonometric Functions
3. Exponential Functions
4. Logarithmic Functions
5. A combination of any of the above on the domain it is defined.

For example, a rational function $P(x)/Q(x)$ is continuous at all points x where $Q(x) \neq 0$.

Since $\lim_{x\to c} f(x) = f(c)$ when f is continuous at $x = c$, finding the limit at $x = c$ of any of the above functions is a matter of evaluating f at $x = c$.

Example 1.2. *Evaluate* $\lim_{x\to -2} \dfrac{x + \ln(x+3)}{\sqrt{x+6}}$.

Solution. The function $\frac{x+\ln(x+3)}{\sqrt{x+6}}$ is continuous at $x = -2$. Thus

$$\lim_{x\to -2} \frac{x + \ln(x+3)}{\sqrt{x+6}} = \frac{-2 + \ln(-2+3)}{\sqrt{-2+6}} = -1.$$

■

Exercise 1.1. *Evaluate* $\lim_{x\to 0} \tan^3(\sin x)$.

Ans: 0.

1.4 Limits at infinity

Let f be defined on $\mathbb{R}$.

- $\lim_{x\to\infty} f(x)$ is the value $f(x)$ approaches as x tends to positive infinity.
- $\lim_{x\to-\infty} f(x)$ is the value $f(x)$ approaches as x tends to negative infinity.

Graphically, if $\lim_{x\to\infty} f(x) = c \in \mathbb{R}$ or $\lim_{x\to-\infty} f(x) = c$, then the line $y = c$ is a horizontal asymptote of the graph of $f(x)$.

Example 1.3. *Evaluate* $\lim_{x\to\infty}\left(\frac{3}{2x}+\sqrt{4-e^{-x}}\right)^2$.

Solution. $\lim_{x\to\infty}\left(\frac{3}{2x}+\sqrt{4-e^{-x}}\right)^2 = (0+\sqrt{4})^2 = 4.$ ■

Exercise 1.2. *Evaluate* $\lim_{x\to-\infty}\ln\left(3-2\sin\frac{4}{x}\right)$.

Ans: $\ln 3$.

1.5 More on Limits

Indeterminate forms.

(a) A limit of the form $\lim_{x\to c}\frac{f(x)}{g(x)}$ where $f(x)\to 0$ and $g(x)\to 0$ as $x\to c$ is called an indeterminate form of the type $\frac{0}{0}$.

(b) A limit of the form $\lim_{x\to c}\frac{f(x)}{g(x)}$ where $f(x)\to\infty$ and $g(x)\to\infty$ as $x\to c$ is called an indeterminate form of the type $\frac{\infty}{\infty}$.

Replacement rule. Let I be an open interval containing the point $x=c$. Suppose $f(x)=g(x)$ for all $x\in I$, except possibly at $x=c$. Then $\lim_{x\to c}f(x)=\lim_{x\to c}g(x)$.

Example 1.4. *Evaluate* $\lim_{x\to 6}\frac{x^2-7x+6}{36-x^2}$.

Solution. $\lim_{x\to 6}\frac{x^2-7x+6}{36-x^2} = \lim_{x\to 6}\frac{(x-1)(x-6)}{-(6+x)(x-6)} == \lim_{x\to 6}\frac{x-1}{-(6+x)} = -\frac{5}{12}.$ ■

Exercise 1.3. *Evaluate* $\lim_{x\to-3}\frac{\sqrt{x+12}-\sqrt{6-x}}{18-2x^2}$.

Ans: $\frac{1}{36}$.

Result. Limits of the form $\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x.

$$\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)} = \lim_{x\to\pm\infty}\frac{\overbrace{Ax^{\alpha}}^{\textit{leading term}} \quad +\cdots}{\underbrace{Bx^{\beta}}_{\textit{leading term}} \quad +\cdots} = \begin{cases} 0 & \text{if } \alpha<\beta \\ \frac{A}{B} & \text{if } \alpha=\beta \\ \underbrace{\infty \text{ or } -\infty}_{\textit{depends on the question}} & \text{if } \alpha>\beta \end{cases}$$

Example 1.5. *Evaluate* $\lim_{x\to\infty}\frac{(18x^2+5x-1)(2\sqrt{x}-1)^3}{(3x-1)^4}$.

Solution. $\lim_{x\to\infty}\frac{(18x^2+5x-1)(2\sqrt{x}-1)^3}{(3x-1)^4} = \lim_{x\to\infty}\frac{144x^{\frac{7}{2}}+\cdots}{81x^4+\cdots} = 0.$ ■

Exercise 1.4. *Evaluate* $\lim_{x\to-\infty}\frac{(1+2x)^3}{\sqrt{16x^6+9x-1}}$.

Ans: -2.

Useful results.

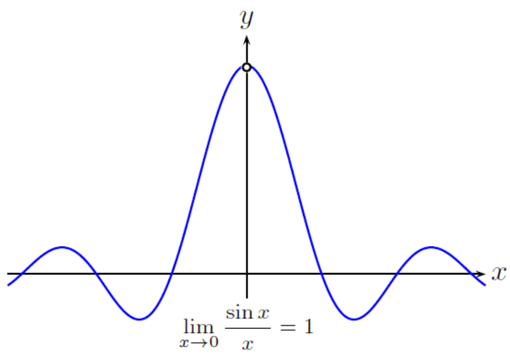
If $\lim_{x\to c} g(x)=0$, then

- $\lim_{x\to c}\frac{\sin(g(x))}{g(x)} = \lim_{x\to c}\frac{g(x)}{\sin(g(x))} = 1,$
- $\lim_{x\to c}\frac{\tan(g(x))}{g(x)} = \lim_{x\to c}\frac{g(x)}{\tan(g(x))} = 1.$

In particular, when $c=0$ and $g(x)=x$,

- $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{x}{\sin x} = 1,$
- $\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{x}{\tan x} = 1.$

For example, $\lim_{x\to 0}\frac{\sin 3x}{3x} = 1$, $\lim_{x\to 1}\frac{\ln x}{\tan(\ln x)} = 1$, $\lim_{x\to\infty}\frac{\sin(e^{-x})}{e^{-x}} = 1$.



Example 1.6. *Evaluate* $\lim\limits_{x\to 0} \dfrac{x\tan(2x)+\sin^2 x}{\sin(3x^2)+x\tan(2x)}$.

Solution.

$$\lim_{x\to 0} \frac{x\tan(2x)+\sin^2 x}{\sin(3x^2)+x\tan(2x)} = \lim_{x\to 0} \frac{2x^2\frac{\tan(2x)}{2x}+x^2\frac{\sin^2 x}{x^2}}{3x^2\frac{\sin(3x^2)}{3x^2}+2x^2\frac{\tan(2x)}{2x}} = \lim_{x\to 0} \frac{2\frac{\tan(2x)}{2x}+\frac{\sin^2 x}{x^2}}{3\frac{\sin(3x^2)}{3x^2}+2\frac{\tan(2x)}{2x}} = \frac{2+1}{3+2} = \frac{3}{5}. \quad \blacksquare$$

Exercise 1.5. *Evaluate* $\lim\limits_{x\to 0^+} \left(x^2\cot(2x)\csc^2(3\sqrt{x})\right)$.

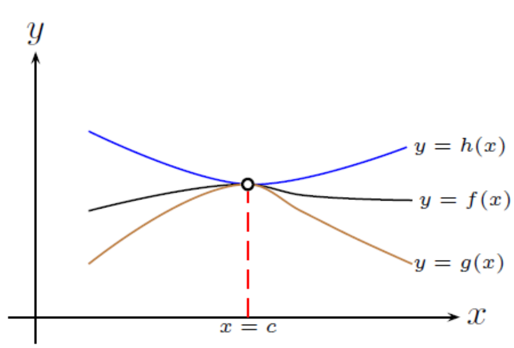
Ans: $\frac{1}{18}$.

Exercise 1.6. *Evaluate* $\lim\limits_{x\to 4} \dfrac{\tan(\sqrt{x}-2)}{\sin(16-x^2)}$.

Ans: $-\frac{1}{32}$.

1.6 Squeeze (Sandwich) Theorem

Squeeze Theorem I. Suppose $g(x) \le f(x) \le h(x)$ for all x in some open interval containing a point c, except possibly at $x = c$. If $\lim\limits_{x\to c} g(x) = \lim\limits_{x\to c} h(x) = L$, then $\lim\limits_{x\to c} f(x) = L$.



Example 1.7. *It is given that* $3 - x^2 \leq f(x) \leq 1 + 2e^x$ *for all* x. *Find* $\lim_{x\to 0} f(x)$.

Solution. As $\lim_{x\to 0} 3 - x^2 = 3$ and $\lim_{x\to 0} 1 + 2e^x = 3$, we have by Squeeze Theorem I that $\lim_{x\to 0} f(x) = 3$. ■

Exercise 1.7. *Use Squeeze Theorem I to show that* $\lim_{x\to c} |f(x)| = 0 \Rightarrow \lim_{x\to c} f(x) = 0$.

Remark The converse of the above result, namely $\lim_{x\to c} f(x) = 0 \Rightarrow \lim_{x\to c} |f(x)| = 0$ is true.

Hence, we have

Result. $\lim_{x\to c} f(x) = 0 \Leftrightarrow \lim_{x\to c} |f(x)| = 0$.

Squeeze Theorem II. If $\lim_{x\to c} g(x) = 0$, then for any function h,

$$\lim_{x\to c} g(x)\sin(h(x)) = 0 \quad \text{and} \quad \lim_{x\to c} g(x)\cos(h(x)) = 0.$$

Example 1.8. *Evaluate* $\lim_{x\to 0}\left(x^3\cos(\frac{2}{\sin x})\right)$.

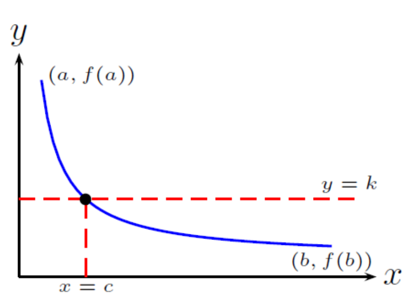
Solution. Since $\lim_{x\to 0} x^3 = 0$, we have by Squeeze Theorem II, $\lim_{x\to 0}\left(x^3\cos(\frac{2}{\sin x})\right) = 0$. ■

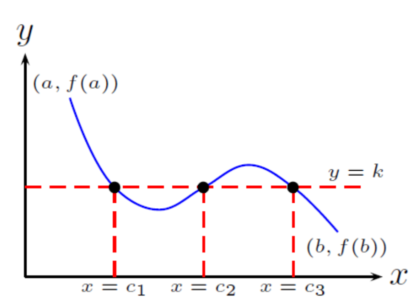
Exercise 1.8. *Evaluate* $\lim_{x\to\infty}\left(\frac{1}{\ln x}\sin(2\ln x) + 2x\sin(\frac{1}{x})\right)$.

Ans: 2.

1.7 Intermediate Value Theorem (IVT)

If f is continuous on $[a,b]$ and k is a number between $f(a)$ and $f(b)$, then $f(c) = k$ for some $c \in [a,b]$.





Example 1.9. *Show that the equation* $x^3 e^x = 10$ *has a root between 1 and 1.5.*

Solution. Let $f(x) = x^3 e^x$. f is continuous on $\mathbb{R}$. We have $f(1) = e = 2.718$ and $f(1.5) = 1.5^3 e^{1.5} = 15.126$. Thus $f(1) < 10 < f(1.5)$. By Intermediate Value Theorem, there is a root to $f(x) = 10$ between 1 and 1.5.

Exercise 1.9. *Show that the equation* $10 = x + 2\tan(2x)$ *has a root between 3 and 4.*

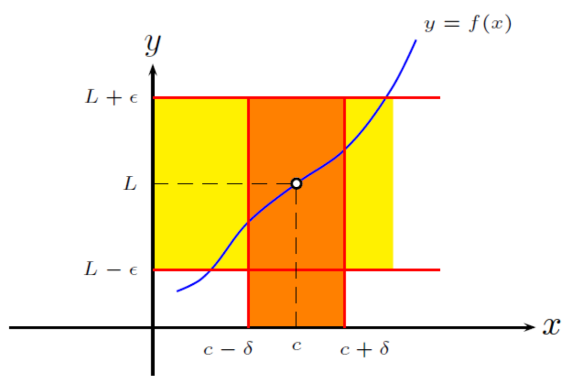
1.8 The Precise Definition of the Limit of a Function

Let $f(x)$ be defined on an open interval containing the point c, except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is the number L, and write

$$\lim_{x \to c} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$



The $\epsilon - \delta$ definition of the limit of a function

Example 1.10. *Prove from definition that* $\lim_{x \to 1} 5x - 3 = 2$.

Solution. Note that $|(5x - 3) - 2| = 5|x - 1|$. Given $\epsilon > 0$, we choose $\delta = \epsilon/5$. Then for all x, $0 < |x - 1| < \delta \Rightarrow 5|x - 1| < 5\delta \Rightarrow |(5x - 3) - 2| < \epsilon$.

■

Exercise 1.10. *Prove from definition that* $\lim_{x \to -6} \dfrac{x}{3} + 3 = 1$.

Chapter 2

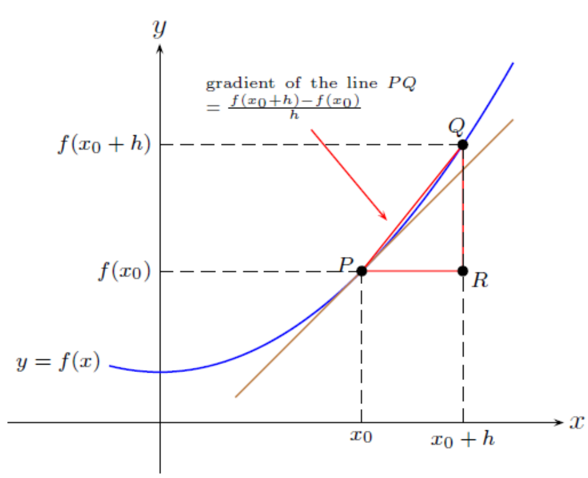
Derivatives

Read Thomas' Calculus, Chapter 3.

2.1 Differentiability

Definition 2.1. *The* **derivative** *of a function f at the point x_0 denoted by $f'(x_0)$ is given by the following limit*

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$



Consider the tangent to the curve $y = f(x)$ at the point $(x_0, f(x_0))$. From the diagram, as $h \to 0$, Q approaches P and hence, the gradient of the chord PQ, namely

$$\frac{f(x_0+h) - f(x_0)}{h},$$

approaches the limiting value

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=f'(x_0).$$

When $f'(x_0)$ exists, we say that f is differentiable at $x=x_0$.

Definition 2.2. *Suppose the derivative $f'(x)$ exists for all x in an open interval I. We can then treat $f'(x)$ as a function defined on I. The process of finding the derivative of a function is called differentiation. If $y=f(x)$, we can also write $\frac{d}{dx}f(x)$, $\frac{dy}{dx}$ or $\frac{df}{dx}$ to denote $f'(x)$. An alternative formula for the derivative $f'(x)$ is*

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}.$$

Differentiability implies continuity.

Theorem 2.1. *If f is differentiable at $x=x_0$, then f is continuous at $x=x_0$.*

Remark. The converse of the above result is not true in general. For example the absolute value function $|x|$ is continuous at $x=0$ but not differentiable at $x=0$.

Example 2.1.

(a) Use the above definition of derivative to differentiate the function $\dfrac{1}{2+\sqrt{x}}$.

(b) Find the equation of the tangent to the curve $y=\frac{1}{2+\sqrt{x}}$ at the point $(1,\frac{1}{3})$.

Ans: (a) $\frac{-1}{2\sqrt{x}(2+\sqrt{x})^2}$, (b) $(y-\frac{1}{3})=-\frac{1}{18}(x-1)$.

Differentiability on Intervals.

Definition 2.3. *A function f is said to be differentiable on an interval I if it is differentiable at every point in I.*

Remark. If the interval has endpoints, then the limit in defining the derivative should be replaced by the appropriate one-sided limit.

Exercise 2.1. *Show that $f(x)=|x^2-2x|$ is* **not** *differentiable at $x=2$.*

2.2 Standard Derivatives & Differentiation Rules

Table 1

Function	**Derivative**
x^n	nx^{n-1}
$\cos(x)$	$-\sin(x)$
$\sin(x)$	$\cos(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\cot(x)$	$-\csc^2(x)$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$
$\cot^{-1}(x)$	$-\frac{1}{1+x^2}$
$\sec^{-1}(x)$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}},\ \lvert x\rvert > 1$
$\csc^{-1}(x)$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}},\ \lvert x\rvert > 1$

Table 2

Function	**Derivative**
$(f(x))^n$	$nf'(x)(f(x))^{n-1}$
$\cos(f(x))$	$-f'(x)\sin(f(x))$
$\sin(f(x))$	$f'(x)\cos(f(x))$
$\tan(f(x))$	$f'(x)\sec^2(f(x))$
$\sec(f(x))$	$f'(x)\sec(f(x))\tan(f(x))$
$\csc(f(x))$	$-f'(x)\csc(f(x))\cot(f(x))$
$\cot(f(x))$	$-f'(x)\csc^2(f(x))$
$e^{f(x)}$	$f'(x)e^{f(x)}$
$\ln(f(x))$	$\frac{f'(x)}{f(x)}$
$\sin^{-1}(f(x))$	$\frac{f'(x)}{\sqrt{1-f(x)^2}}$
$\cos^{-1}(f(x))$	$-\frac{f'(x)}{\sqrt{1-f(x)^2}}$
$\tan^{-1}(f(x))$	$\frac{f'(x)}{1+f(x)^2}$
$\cot^{-1}(f(x))$	$-\frac{f'(x)}{1+f(x)^2}$
$\sec^{-1}(f(x))$	$\frac{f'(x)}{\lvert f(x)\rvert\sqrt{f(x)^2-1}},\ \lvert f(x)\rvert > 1$
$\csc^{-1}(f(x))$	$-\frac{f'(x)}{\lvert f(x)\rvert\sqrt{f(x)^2-1}},\ \lvert f(x)\rvert > 1$

Note that when $f(x) = x$, the formulae in table 2 reduce to those in table 1.

Rules of Differentiation

Let u and v be differentiable functions of x and c be a constant.

Constant Rule	$\frac{d}{dx}(c) = 0$
Constant Multiple Rule	$\frac{d}{dx}(cu) = c\frac{du}{dx}$
Sum Rule	$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$
Product Rule	$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$
Quotient Rule	$\frac{d}{dx}(\frac{u}{v}) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$

Let $f(u)$ be differentiable at $u = g(x)$ and g be a differentiable function of x.

Chain Rule	$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$

Example 2.2. *Differentiate* $\ln\left(\frac{x^2}{(6x-7)^2}\right)$ *with respect to* x.

Ans: $\frac{2}{x} - \frac{12}{6x-7}$.

Exercise 2.2. *Differentiate with respect to* x.
(a) $(x+1)^2\tan^{-1}(\sqrt{x})$
(b) $\frac{\sin^{-1}(2x)}{\sqrt{1-4x^2}}$

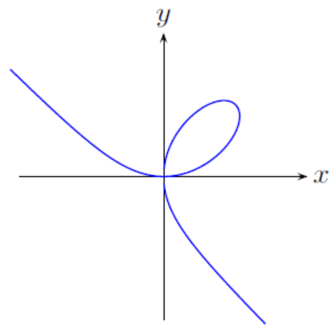
Ans: (a) $2(x+1)\tan^{-1}(\sqrt{x}) + \frac{x+1}{2\sqrt{x}}$, (b) $\frac{2}{1-4x^2} + \frac{4x\sin^{-1}(2x)}{(1-4x^2)^{\frac{3}{2}}}$.

2.3 Implicit Differentiation

The diagram shows the curve given implicitly by the equation

$$x^3 + y^3 - 9xy = 0$$

known as the *folium of Descartes*.



Folium of Descartes $x^3 + y^3 = 9xy$

Suppose we wish to find the gradient of the curve at the point $(2,4)$. For this example, getting an explicit expression for y in terms of x is challenging. It turns out that it is possible to find $\frac{dy}{dx}$ by a method known as *implicit differentiation*. This consists of differentiating both sides of the given equation with respect to x and solving the resulting equation for $\frac{dy}{dx}$. When differentiating a function in y with respect to x, we need the following result.

$$\frac{d}{dx}g(y) = g'(y)\frac{dy}{dx}.$$

Example 2.3. *Consider the curve $x^3 + y^3 - 9xy = 0$. Find $\frac{dy}{dx}$.*

Ans: $\frac{dy}{dx} = -\frac{3x^2-9y}{3y^2-9x}$.

Example 2.4. *Find $\frac{dy}{dx}$ for points on the curve $x^3e^y + \cos(xy) = 2017$.*

Solution. Differentiating both sides of the equation, we have

$$\overbrace{3x^2e^y + x^3e^y\frac{dy}{dx}}^{\text{Apply product rule to } x^3e^y} - \sin(xy)\left(x\frac{dy}{dx} + y\right) = 0.$$

Solving for $\frac{dy}{dx}$, we obtain $\dfrac{dy}{dx} = \dfrac{3x^2e^y - y\sin(xy)}{x\sin(xy) - x^3e^y}$.

∎

In general, for a given equation of the form $f(x,y) = 0$, let

f_x denote the expression obtained by differentiating $f(x,y)$ with respect to x, treating y as a constant,

f_y denote the expression obtained by differentiating $f(x,y)$ with respect to y, treating x as a constant.

Then,

$$\frac{dy}{dx} = -\frac{f_x}{f_y}.$$

2.4 Derivatives of Inverse Functions

Recall that

(i) a bijective function (or one-one function) f has an inverse f^{-1} defined on the range of f.

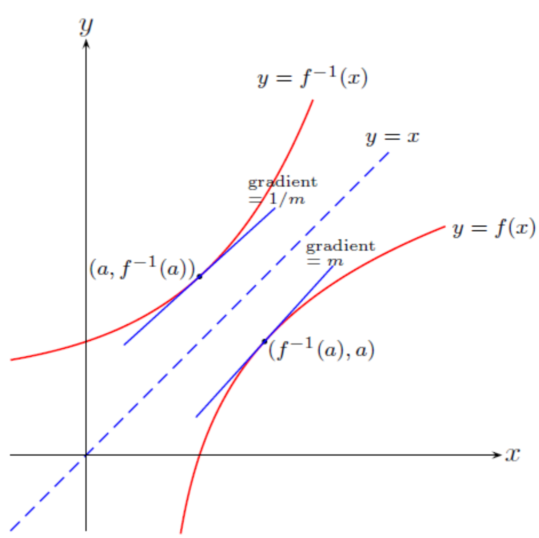
(ii) increasing or decreasing functions are bijective.

Theorem 2.2. *Let f be bijective and differentiable on an open interval I. Then*

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

At the point $(a, f^{-1}(a)) = (a,b)$ on the curve $y = f^{-1}(x)$, the gradient of the tangent is

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))} = \frac{1}{f'(b)}.$$



Example 2.5. *Show that* $\dfrac{d}{dx}\sin^{-1}(x) = \dfrac{1}{\sqrt{1-x^2}}$.

Solution. Let $y = \sin^{-1}(x)$. Then $x = \sin(y)$.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\frac{d}{dy}(\sin y)} = \frac{1}{\cos y}.$$

Note that $x = \sin(y) \Rightarrow \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$, as $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Thus $\dfrac{d}{dx}\sin^{-1}(x) = \dfrac{1}{\sqrt{1 - x^2}}$.

■

Exercise 2.3. *Let* $f(x) = x^5 - 2e^{-2x} + 3$.

(a) Show that f *is bijective by showing that* f *is increasing on* $\mathbb{R}$.

(b) Find the gradient of the tangent to the curve $y = f^{-1}(x)$ *at the point* $(1, 0)$.

(c) Let $g(x) = \dfrac{1}{2x^{-1} + 3f^{-1}(x)}$. *Find the value of* $g'(1)$.

Ans: (b) $\frac{1}{4}$, (c) $\frac{5}{16}$.

2.5 Higher-order Derivatives

Given a differentiable function f, we can find the derivative of its derivative function f' with respect to x to get a new function, denoted by f'' or $f^{(2)}$, called the *second derivative* of f provided f' is differentiable. That is,

$$f''(x) = \frac{d}{dx}f'(x).$$

Notations. Let $y = f(x)$.

$$f^{(2)}(x) = f''(x) = \frac{d^2y}{dx^2} = y'' = D^2 f(x).$$

In general, we can define the nth order derivative of f for any positive integer n provided the derivative exists. For example, the third derivative of $y = f(x)$ is defined by

$$f^{(3)}(x) = f'''(x) = \frac{d}{dx}f^{(2)}(x) = \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}(f(x))\right)\right).$$

We shall denote the nth order derivative of f by

$$f^{(n)}(x) = \frac{d^n y}{dx^n} = D^n y = D^n f(x).$$

2.6 Parametric Equations

A curve defined by the parametric equations

$$x = f(t) \text{ and } y = g(t), \ (t \text{ is the parameter})$$

is differentiable at point where $t = t_0$ if both f and g are differentiable at $t = t_0$. Usually we also assume $f'(t_0) \neq 0$ or $g'(t_0) \neq 0$.
By chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{g'(t)}{f'(t)},$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \div \frac{dx}{dt} = \frac{\frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)}{f'(t)} = \frac{g''(t)f'(t) - g'(t)f''(t)}{f'(t)^3}.$$

Examples of parametric curves

- **Ellipses**

  $$x = a\cos t + x_0 \text{ and } y = b\sin t + y_0,$$

  where $a > 0, b > 0, x_0$ and y_0 are fixed constants and $0 \leq t < 2\pi$.

- **Circles**

  $$x = r\cos t + x_0 \text{ and } y = r\sin t + y_0,$$

  where $r > 0, x_0$ and y_0 are fixed constants and $0 \leq t < 2\pi$.

- **Hyperbolas**

  $$x = a\sec t + x_0 \text{ and } y = b\tan t + y_0,$$

  or

  $$x = a\tan t + x_0 \text{ and } y = b\sec t + y_0,$$

  where $a > 0, b > 0, x_0$ and y_0 are fixed constants and $-\pi \leq t \leq \pi$, $t \neq -\frac{\pi}{2}, \frac{\pi}{2}$.

Example 2.6. *For the parametric curve given by*

$$x = 2t - t^2, \ y = t - t^3,$$

find the point(s) on the curve at which the tangent is parallel to the line $2y = x + 2017$.

Ans: $(0,0)$ and $(\frac{5}{9}, \frac{8}{27})$.

2.7 Miscellaneous examples

Functions of the form $f(x)^{g(x)}$.

$$\frac{d}{dx}f(x)^{g(x)} = f(x)^{g(x)}\left(g'(x)\ln f(x) + \frac{f'(x)}{f(x)}g(x)\right).$$

In particular, when $f(x)$ is a constant function, say $f(x) = a$, $a > 0$, we have

$$\frac{d}{dx}a^{g(x)} = a^{g(x)}g'(x)\ln a.$$

Example 2.7. *Differentiate with respect to x.*

(a) $(x^2 - e^{3x})^{4\tan x}$

(b) $5^{x\ln x}$

Ans: (a) $(x^2 - e^{3x})^{4\tan x}\left(4\sec^2 x\ln(x^2 - e^{3x}) + \frac{2x-3e^{3x}}{x^2-e^{3x}}\cdot 4\tan x\right)$, (b) $5^{x\ln x}\ln 5(1 + \ln x)$.

Change of base formulae

$$\log_a x = \frac{\ln x}{\ln a}, a > 0 \text{ and } a \neq 1.$$

Example 2.8. *Differentiate* $\log_{(2+x^2)}\sqrt{1+2x}$ *with respect to x.*

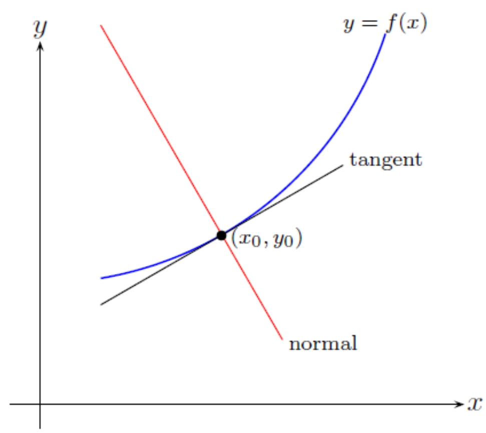
Ans: $\frac{1}{(\ln(2+x^2))^2}\left(\frac{\ln(2+x^2)}{1+2x} - \frac{x\ln(1+2x)}{2+x^2}\right)$.

Chapter 3

Applications of Differentiation

Read Thomas' Calculus, Chapter 4.

3.1 Tangents and Normals



- The tangent at the point $(x_0, f(x_0))$ on the graph of a differentiable function f has equation
$$y - f(x_0) = m(x - x_0).$$
- The normal at the point $(x_0, f(x_0))$ on the graph of a differentiable function f has equation
$$y - f(x_0) = -\frac{1}{m}(x - x_0),$$
where $m = f'(x_0)$.

Example 3.1. *The curve C has equation $x^2 + y^2 + 3xy = 5$.*

(a) Find the equations of the tangent and normal at the point $(1, 1)$.

(b) Find (if any) the equations of the tangents that are parallel to the axes.

Ans: (a) tangent: $y = -x + 2$*, normal:* $y = x$*.*

If a curve is defined parametrically by

$$x = x(t) \text{ and } y = y(t)$$

then,

- the equation of the tangent to the point where $t = t_0$ is
$$y - y(t_0) = m(x - x(t_0)),$$
- the equation of the normal to the point where $t = t_0$ is
$$y - y(t_0) = -\frac{1}{m}(x - x(t_0)),$$
where m is the value of $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt}$ at $t = t_0$.

Example 3.2. *A curve is defined by*

$$x = t^2 - t \text{ and } y = (t+1)^2.$$

The tangent at the point A on the curve passes through $(-1, 0)$ *and* $(1, 8)$*. Find the equation of the normal at the point A.*

Ans: $y = -\frac{x}{4} + 4$.

3.2 Increasing and Decreasing Functions

Definition 3.1. *(a) The function* f *is* increasing *on an interval I if* $f(x_2) > f(x_1)$ *for* $x_1, x_2 \in I$ *with* $x_2 > x_1$*.*

(b) The function f *is* decreasing *on an interval I if* $f(x_2) < f(x_1)$ *for* $x_1, x_2 \in I$ *with* $x_2 > x_1$*.*

Theorem 3.1. *Let* f *be differentiable on* (a, b) *and continuous on* $[a, b]$*.*

(a) f *is increasing on* $[a, b]$ *if* $f'(x) > 0$ *for all x in* (a, b)*.*

(b) f *is decreasing on* $[a, b]$ *if* $f'(x) < 0$ *for all x in* (a, b)*.*

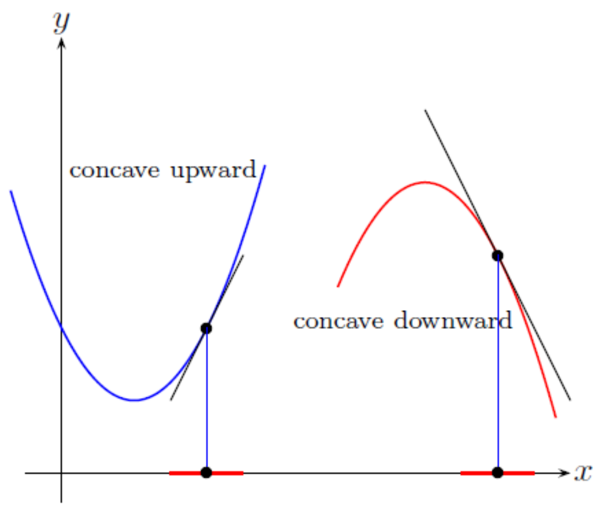
Thus if $f'(x) > 0 (< 0)$ on (a, b) except possibly at a finite number of points at which $f'(x) = 0$, then f is increasing (decreasing) on $[a, b]$.

Example 3.3. *Show that the function* $f(x) = 3x^3 - 3e^{-x} - \frac{4}{x}$ *is bijective (one-one) on the interval* $(0, \infty)$ *to its range* $\mathbb{R}$*.*

3.3 Concave Upward and Concave Downward Functions

Definition 3.2. *(a) The graph of f is concave upward (downward) at $(c, f(c))$ if $f'(c)$ exists and there is an open interval I containing c such that for all $x \neq c$ in I, the point $(x, f(x))$ on the graph of f is above (below) the tangent line to the graph of f at $x = c$.*

(b) The graph of f is concave upward (downward) on (a, b) if it is concave upward (downward) at every point in (a, b).

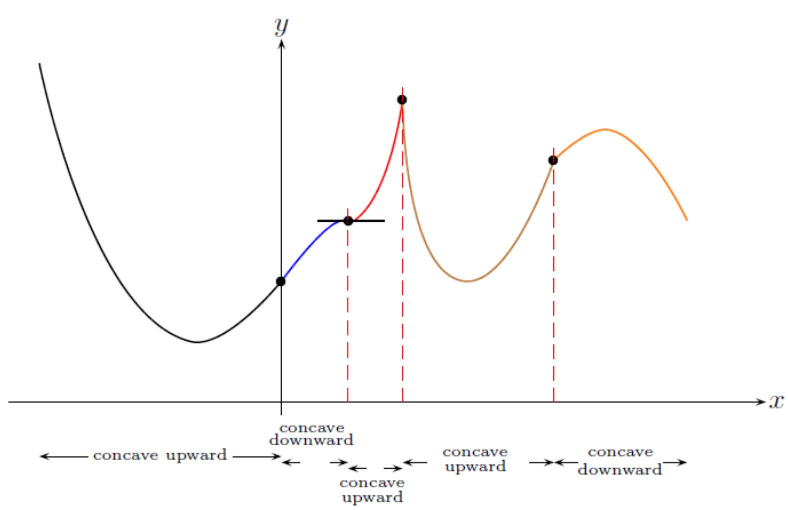


The following theorem gives a test for concavity.

Theorem 3.2. *Let f be differentiable on (a, b). Let $c \in (a, b)$.*

(a) If $f''(c) > 0$, then the graph of f is concave upward at $(c, f(c))$.

(b) If $f''(c) < 0$, then the graph of f is concave downward at $(c, f(c))$.



Note that the converse of the theorem is not true. For example, if $f(x) = x^4$, then the graph of f is concave upward at $(0,0)$, but $f''(0) = 0$.

Definition 3.3. *A point $(c, f(c))$ where the graph of a function f has a tangent line and where the concavity changes is called a point of inflection.*

Point of inflection: change of concavity

Theorem 3.3. *Let f be differentiable on (a,b). Let $c \in (a,b)$. If $(c, f(c))$ is a point of inflection of the graph of f and $f''(c)$ exists, then $f''(c) = 0$.*

Example 3.4. *Determine the intervals on which the function*

$$f(x) = -2x^3 + 15x^2 - 24x + 7$$

is

(i) increasing, (ii) decreasing, and its graph is (iii) concave upward (iv) concave downward. (v) Find the point(s) of inflection of the graph of f.

Ans: (i) $[1,4]$, (ii) $(-\infty,1]\cup[4,\infty)$, (iii) $(-\infty,\frac{5}{2})$, (iv) $(\frac{5}{2},\infty)$, (v) $(\frac{5}{2},\frac{19}{2})$.

Solution. $f'(x) = -6x^2 + 30x - 24 = -6(x-1)(x-4)$.

x	$x<1$	$x=1$	$1<x<4$	$x=4$	$4<x$
$f'(x)$	$-$	0	$+$	0	$-$

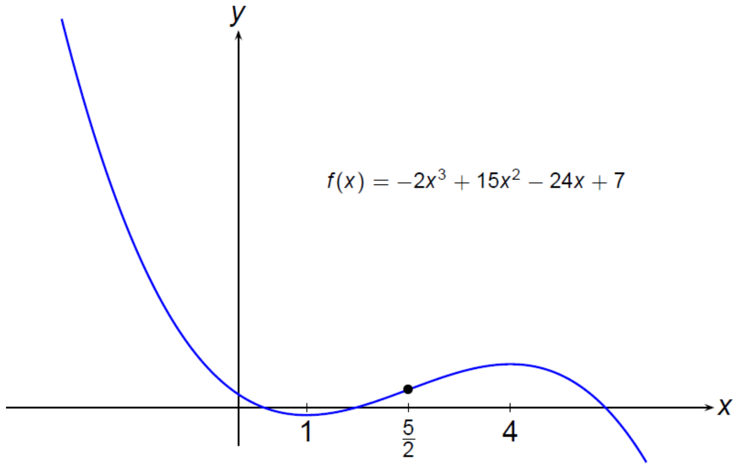
Thus f is increasing on $[1,4]$, and decreasing on $(-\infty,1]\cup[4,\infty)$,

We have $f''(x) = -12x + 30 = -12(x - \frac{5}{2})$.

x	$x<\frac{5}{2}$	$x=\frac{5}{2}$	$\frac{5}{2}<x$
$f''(x)$	$+$	0	$-$

Thus the graph of f is concave upward on $(-\infty,\frac{5}{2})$ and concave downward on $(\frac{5}{2},\infty)$.

There is a change of concavity of the graph of f at $(\frac{5}{2},\frac{19}{2})$. Thus $(\frac{5}{2},\frac{19}{2})$ is a point of inflection of the graph of f.



Summarizing, (i) increasing on $[1,4]$, (ii) decreasing on $(-\infty, 1]$, $[4, \infty)$, (iii) concave upward on $(-\infty, \frac{5}{2})$, (iv) concave downward $(\frac{5}{2}, \infty)$, (v) inflection point at $x = \frac{5}{2}$.

3.4 Related Rates

Let $y = f(x)$ and let x and y be functions of a third variable t that represents for example time. By the Chain Rule

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

Example 3.5. *Water is flowing at a rate of* 100 *cm*^{3} *per second into an inverted conical flask of height* 16*cm and base radius* 4*cm. At the instant when the height of water level is* 12*cm, the water level is rising at the rate of* 3*cm per sec. Calculate the rate at which water is leaking from the flask.*

Ans: 15.18 cm^3 per second.

Exercise 3.1. *A particle is moving horizontally in the* x–y *plane along the line* $y = 5$ *in such a way that its distance from the origin* $(0,0)$ *is increasing at a rate of* 1 *unit per sec. Calculate the rate at which the particle is moving horizontally at the instant when it is* 13 *units from* $(0,0)$.

Ans: $\frac{13}{12}$ unit per sec.

3.5 Maximum and Minimum Values

Definition 3.4. *(Absolute Extrema) A function* f *has an*

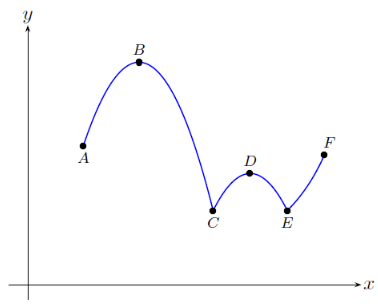
(a) absolute/global maximum at $x = c$ *if* $f(x) \leq f(c)$ *for all* x *in the domain of* f.

(b) *absolute/global minimum at $x = c$ if $f(x) \geq f(c)$ for all x in the domain of f.*

Definition 3.5. *(Local Extrema) A function f defined on some interval I has a*

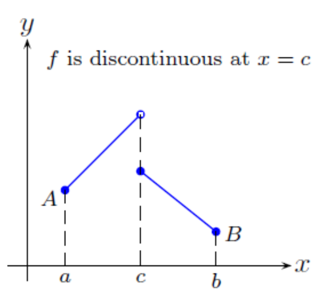
(a) *relative/local maximum at $x = c$ if $f(x) \leq f(c)$ for x in some open interval containing $x = c$.*

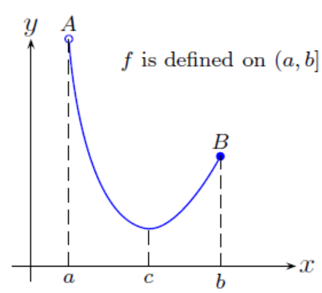
(b) *relative/local minimum at $x = c$ if $f(x) \geq f(c)$ for x in some open interval containing $x = c$.*



Theorem 3.4. *(Extreme Value Theorem) If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value and an absolute minimum value at some points in $[a, b]$.*

Question. What if f is not continuous or if the domain is not a closed interval of the form $[a, b]$?





Theorem 3.5. *If f is differentiable on an open interval containing $x = c$ and f has a local extremum at $x = c$, then $f'(c) = 0$.*

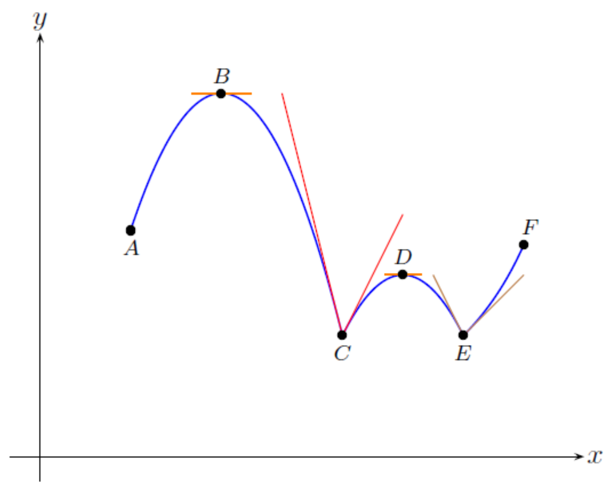
Definition 3.6. *(Critical Point) A number c in the domain of a function f is a* **critical point** *of f if the following 2 conditions hold:*

(i) it is not an end-point,

(ii) either $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem 3.6. *If f has a local minimum/maximum at $x = c$, then c is a critical point of f.*

However, the converse of this result is not true in general.



From the above results, we conclude that an absolute extremum occurs either at the end point or at a critical point.

Hence, to find absolute extrema of a continuous function f defined on $[a,b]$, we

(1) find the values of f at all critical points of f on (a,b),

(2) find the values of $f(a)$ and $f(b)$.

The largest and smallest values from steps (1) and (2) are the absolute maximum and absolute minimum respectively.

Remark. If the function f is defined on $(a,b]$ and the largest (smallest) value of f obtained from steps 1 and 2 occurs at $x = a$, then f has no absolute maximum (minimum).

Similarly, if the function f is defined on $[a,b)$ and the largest (smallest) value of f obtained from steps 1 and 2 occurs at $x = b$, then f has no absolute maximum (minimum).

Example 3.6. *Find the absolute maximum and minimum values of*

$$g(x) = \frac{x}{x^2+1}$$

on (a) $[-2,1]$, *(b)* $(-2,1)$, *(c)* $(-1,1]$.

Ans: (a) absolute maximum value = $\frac{1}{2}$, absolute minimum value = $-\frac{1}{2}$, (b) no absolute maximum value, absolute minimum value = $-\frac{1}{2}$, (c) absolute maximum value = $\frac{1}{2}$, no absolute minimum value.

Exercise 3.2. *Find the absolute maximum and minimum values of*

$$h(x) = x^{5/3} - x^{2/3}$$

on (a) $[-1,8]$, *(b)* $(-1,1)$.

Ans: (a) absolute maximum value = 28, absolute minimum value = -2.

Exercise 3.3. *Let* $f(x) = \begin{cases} |x| & -5 \le x < 2 \\ x^2 - 6x + 10 & 2 \le x \le 4 \end{cases}$. *Find*

(a) the critical points of f,

(b) the absolute maximum and minimum values of f.

Ans: (a) critical points at $x = 0, 2, 3$, (b) absolute maximum value = 5, absolute minimum = 0.

Theorem 3.7. *(First Derivative Test for Local Extrema) Let* f *be differentiable on an open interval containing a critical point* c *except possibly at* c *and* f *is continuous at* c.

(1) If f' *changes from positive to negative at* $x = c$, *then* f *has a local maximum at* c.

(2) If f' *changes from negative to positive at* $x = c$, *then* f *has a local minimum at* c.

(3) If f' *does not change sign at* $x = c$, *then* f *has no local extremum at* c.

Theorem 3.8. *(First Derivative Test for Absolute Extrema) Let* f *be differentiable on an open interval containing a critical point* c *except possibly at* c *and* f *is continuous at* c.

(1) If $f'(x) > 0$ *for all* $x < c$ *and* $f'(x) < 0$ *for all* $x > c$, *then* f *has an absolute maximum at* c.

(2) If $f'(x) < 0$ *for all* $x < c$ *and* $f'(x) > 0$ *for all* $x > c$, *then* f *has an absolute minimum at* c.

Theorem 3.9. *(Second Derivative Test) Let f be a twice differentiable function defined in an open interval containing c.*

(1) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

(2) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(3) No conclusion can be drawn if $f''(c) = 0$.

a local minimum at $x = c$ a local maximum at $x = c$

Remark. The functions x^4, $-x^4$, x^3, has a local min, max, neither a max nor a min at $x = 0$, respectively, but have 0 second derivative at $x = 0$.

Example 3.7. *Let* $f(x) = \dfrac{\ln x}{x}$, $x > 0$.

(a) Find the critical points of f.

(b) Determine whether a local maximum or local minimum or neither occurs at each of these points.

Ans: (a) critical point at $x = e$, (b) f has a local maximum at $x = e$.

Exercise 3.4. *Let* $f(x) = \begin{cases} 2x - x^2, & 0 \le x < 2 \\ (x-2)^2, & x \ge 2 \end{cases}$.

(a) Find the critical points of f.

(b) Determine whether a local maximum or local minimum or neither occurs at each of these points.

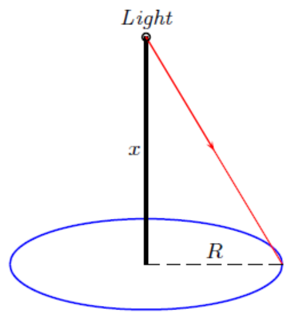
Ans: (a) critical points at $x = 1$ and $x = 2$, (b) f has a local maximum at $x = 1$ and a local minimum at $x = 2$.

3.6 Applied Maximum and Minimum Problems

Example 3.8. *A light bulb is placed at the top of a vertical pole to illuminate a circular field of fixed radius R metres. The intensity of illumination, I, is proportional to*

$$\frac{x}{(x^2+R^2)^{\frac{3}{2}}},$$

where x is the height of the pole. Find the value of x that maximises I. Justify that your answer indeed corresponds to the absolute maximum value of I.



Ans: $x = \frac{R}{\sqrt{2}}$.

Exercise 3.5. *Determine the coordinates of the points on the curve $y = \frac{16}{x}$ that are closest to the origin. With the aid of a graph, justify that your answer indeed gives the shortest distance.*

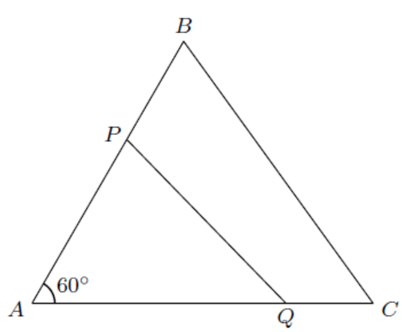
Ans: $(4,4)$, $(-4,-4)$.

Exercise 3.6. *A triangular plot ABC in which $\angle A = 60°$ and AB is 80 metres long is to be divided into two plots of equal areas by a fence built along the line PQ. The fence costs \$10 per metre. Let the lengths of AC and AP be $10b$ and $10x$ respectively.*

(a) Show that the length of PQ is $10z$, where $z^2 = x^2 + \frac{16b^2}{x^2} - 4b$.

(b) Show that $4 \le x \le 8$.

(c) Determine, to the nearest dollar, the minimum cost of fencing, and the corresponding value of x, when (i) $b = 9$, (ii) $b = 25$.

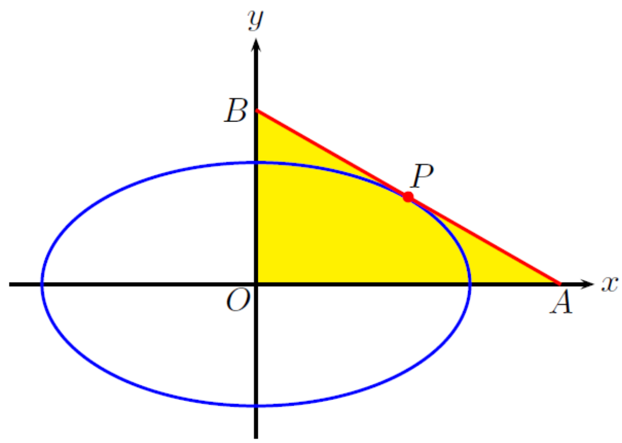


Ans: (a) \$600, *(b)* \$1096.59.

Exercise 3.7. *A bus carries* 60 *passengers each day from a train station to a shopping mall. It costs* \$1.50 *per passenger to ride the bus. Research reveals that* 4 *more (fewer) people would ride the bus for each* 5 *cents decrease (increase) in bus fare. Determine the bus fare (to the nearest cent) per passenger that will maximise revenue of the bus operator.*

Ans: \$1.12.

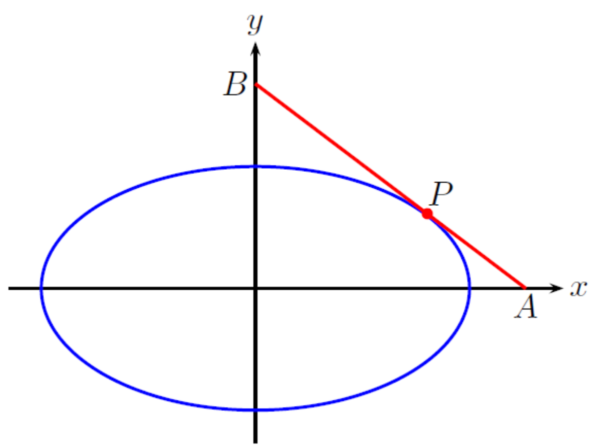
Exercise 3.8. *Find the point P on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *in the first quadrant such that the triangle bounded by the axes of the ellipse and the tangent to the ellipse at the point P has the least area.*



$\triangle OAB$ has the least area.

Ans: $(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})$.

Exercise 3.9. *Find the point P on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *in the first quadrant such that the line segment tangent to the ellipse at the point P with endpoints on the axes of the ellipse has the least length.*



AB has the shortest length.

Ans: $\left(\frac{a^{\frac{3}{2}}}{(a+b)^{\frac{1}{2}}}, \frac{b^{\frac{3}{2}}}{(a+b)^{\frac{1}{2}}}\right)$.

3.7 L'Hôpital's Rule

Theorem 3.10. *Let f and g be differentiable at all points in some open interval containing $x=c$ (except possibly at c). If $\lim_{x\to c} f(x) = 0 = \lim_{x\to c} g(x)$ or $\lim_{x\to c} f(x) = \infty = \lim_{x\to c} g(x)$, then*

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)},$$

provided the limit on the right exists or equals ∞ or $-\infty$.

The result also holds

(1) for limits at infinity, i.e. $c = \pm\infty$.

(2) for one-sided limits.

Example 3.9. *Evaluate*

(a) $\lim_{x\to\infty}\dfrac{3 + x\ln x}{x^2 + 2\ln x}$

(b) $\lim_{x\to 0}\left(2\csc 2x - \dfrac{1}{x}\right)$

(c) $\lim_{x\to 0^+} x\ln x$

Ans: (a) 0, (b) 0, (c) 0.

There are 3 indeterminate forms of the following types:

(1) 0^0, (2) ∞^0, (3) 1^∞.

Example 3.10.

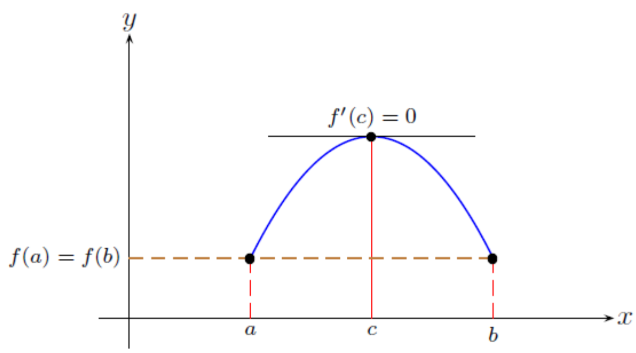
0^0: $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^{\lim_{x\to 0^+} x\ln x} = e^{\lim_{x\to 0^+}\frac{\ln x}{x^{-1}}} = e^{\lim_{x\to 0^+}\frac{x^{-1}}{-x^{-2}}} = e^{\lim_{x\to 0^+} -x} = e^0 = 1.$

∞^0: $\lim_{x\to 0^+}(\frac{1}{x})^x = \lim_{x\to 0^+}\frac{1}{x^x} = \frac{1}{1} = 1.$

1^∞: $\lim_{x\to 0}(\cos x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{\ln\cos x}{x}} = e^{\lim_{x\to 0}\frac{\ln\cos x}{x}} = e^{\lim_{x\to 0}\frac{\frac{1}{\cos x}(-\sin x)}{1}} = e^{\lim_{x\to 0} -\tan x} = e^0 = 1.$

3.8 Rolle's Theorem and Mean Value Theorem

Theorem 3.11. *(Rolle's Theorem) Let f be continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$, then there is at least one number c in (a,b) such that $f'(c) = 0$.*



Exercise 3.10. *Use the Rolle's Theorem to prove that the equation*

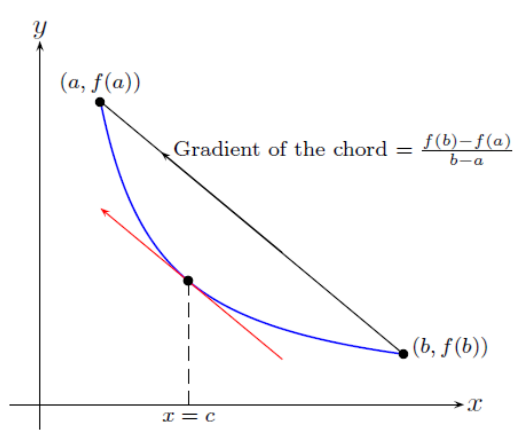
$$2e^x + x^2 + 3x = 0$$

has at most two real roots.

The Rolle's Theorem can be used to prove the following result.

Theorem 3.12. *(Mean Value Theorem) Let f be continuous on $[a,b]$ and differentiable on (a,b). Then, there is at least one number c in (a,b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Theorem 3.13. *Let f be continuous on $[a,b]$ and differentiable on (a,b). If $f'(x) > 0 (< 0)$ for all $x \in (a,b)$, the f is increasing (decreasing) on $[a,b]$.*

Proof. Let's suppose $f'(x) > 0$ for all $x \in (a,b)$. Let $x_1, x_2 \in [a,b]$ and $x_1 < x_2$. Thus f is continuous on $[x_1,x_2]$ and differentiable on (x_1,x_2) as f is continuous on $[a,b]$ and differentiable on (a,b). By mean value theorem applied to f on $[x_1,x_2]$, there is a number c in (x_1,x_2) such that

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c).$$

Since $f'(c) > 0$ and $x_2 > x_1$, we have $f(x_2) > f(x_1)$. That is f is increasing on $[a,b]$.

■

Exercise 3.11. *Use the Mean Value Theorem to prove that for all real numbers x and y,*

$$|\cos x - \cos y| \le |x-y|.$$

Exercise 3.12. *Use the Mean Value Theorem to show that if $f'(x) = 0$ for all x in (a,b), then f is a constant function on (a,b).*

Exercise 3.13. *Use the Mean Value Theorem to prove that there exists a real number $\theta \in (\frac{\pi}{5}, \frac{\pi}{4})$ such that*

$$\sin\frac{\pi}{5} = \sin\frac{\pi}{4} - \frac{\pi}{20}\cos\theta.$$

Deduce that $\sin\frac{\pi}{5} < \frac{\sqrt{2}}{40}(20-\pi)$.

Chapter 4

Integrals

Read Thomas' Calculus, Chapter 5.

4.1 Antiderivatives

Definition 4.1. *F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.*

Theorem 4.1. *(1) If F is an antiderivative of f on an interval I, then so is $F + C$ for any constant C. This can be expressed as*

$$\int f(x)\,dx = F(x) + C.$$

$\int f(x)\,dx$ is called an indefinite integral.

(2) Let α and β be any constants. Then

$$\int \alpha f(x) + \beta g(x)\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx.$$

Example 4.1. *An anti-derivative of x^n is $\frac{1}{n+1}x^{n+1}$, where $n \neq -1$ as $\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1}\right) = x^n$. Thus*

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C.$$

Example 4.2. *Find all the anti-derivatives of $2x - \cos 3x$.*

Solution. $\int 2x - \cos 3x\,dx = x^2 - \frac{1}{3}\sin 3x + C$.

4.2 Standard Integrals

1. $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a} + C \quad (n \neq -1)$

2. $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$

3. $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$

4. $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$

5. $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$

6. $\int \tan(ax+b)\,dx = \frac{1}{a}\ln|\sec(ax+b)| + C$

7. $\int \sec(ax+b)\,dx = \frac{1}{a}\ln|\sec(ax+b)+\tan(ax+b)| + C$

8. $\int \csc(ax+b)\,dx = -\frac{1}{a}\ln|\csc(ax+b)+\cot(ax+b)| + C$

9. $\int \cot(ax+b)\,dx = -\frac{1}{a}\ln|\csc(ax+b)| + C$

10. $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$

11. $\int \csc^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + C$

12. $\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + C$

13. $\int \csc(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\csc(ax+b) + C$

14. $\int \frac{1}{a^2+(x+b)^2}\,dx = \frac{1}{a}\tan^{-1}(\frac{x+b}{a}) + C$

15. $\int \frac{1}{\sqrt{a^2-(x+b)^2}}\,dx = \sin^{-1}(\frac{x+b}{a}) + C$

16. $\int \frac{-1}{\sqrt{a^2-(x+b)^2}}\,dx = \cos^{-1}(\frac{x+b}{a}) + C$

17. $\int \frac{1}{a^2-(x+b)^2}\,dx = \frac{1}{2a}\ln\left|\frac{x+b+a}{x+b-a}\right| + C$

18. $\int \frac{1}{(x+b)^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x+b-a}{x+b+a}\right| + C$

19. $\int \frac{1}{\sqrt{(x+b)^2+a^2}}\,dx = \ln\left|(x+b)+\sqrt{(x+b)^2+a^2}\right| + C$

20. $\int \frac{1}{\sqrt{(x+b)^2-a^2}}\,dx = \ln\left|(x+b)+\sqrt{(x+b)^2-a^2}\right| + C$

21. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

22. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}| + C$

Example 4.3. *Let $a \neq 0$. Show that* $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$, *for* $x \neq -\frac{b}{a}$.

Solution. We have to show $\frac{d}{dx}(\frac{1}{a}\ln|ax+b|) = \frac{1}{ax+b}$.

For $x > -\frac{b}{a}$, we have $|ax+b| = ax+b$. Thus

$$\frac{d}{dx}(\frac{1}{a}\ln|ax+b|) = \frac{d}{dx}(\frac{1}{a}\ln(ax+b)) = \frac{1}{ax+b}.$$

For $x < -\frac{b}{a}$, we have $|ax+b| = -(ax+b)$. Thus

$$\frac{d}{dx}(\frac{1}{a}\ln|ax+b|) = \frac{d}{dx}(\frac{1}{a}\ln(-(ax+b))) = \frac{1}{a}\frac{-a}{-(ax+b)} = \frac{1}{ax+b}.$$

■

Example 4.4. *Find*

(a) $\int \frac{1}{\sqrt{x^2-4x+29}}\,dx$

(b) $\int \frac{1}{\sqrt{3+6x-9x^2}}\,dx$

Solution. (a)
$$\begin{aligned}\int \frac{1}{\sqrt{x^2-4x+29}}\,dx &= \int \frac{1}{\sqrt{(x-2)^2+5^2}}\,dx\\ &= \ln\left|(x-2)+\sqrt{(x-2)^2+5^2}\right| + C\\ &= \ln\left|(x-2)+\sqrt{x^2-4x+29}\right| + C.\end{aligned}$$

(b)
$$\begin{aligned}\int \frac{1}{\sqrt{3+6x-9x^2}}\,dx &= \int \frac{1}{\sqrt{2^2-(3x-1)^2}}\,dx\\ &= \frac{1}{3}\int \frac{1}{\sqrt{(\frac{2}{3})^2-(x-\frac{1}{3})^2}}\,dx\\ &= \frac{1}{3}\sin^{-1}\left(\frac{x-\frac{1}{3}}{\frac{2}{3}}\right) + C\\ &= \frac{1}{3}\sin^{-1}(\frac{3x-1}{2}) + C.\end{aligned}$$

■

Exercise 4.1. *Find*

(a) $\int \left(\frac{3x-1}{2x+1}\right)^2 dx$

(b) $\int \frac{(2e^{2x-1}-e^{-x})^2}{e^{x+1}}\,dx$

Ans: (a) $\frac{9x}{4} - \frac{15}{4}\ln|2x+1| - \frac{25}{8(2x+1)} + C$, (b) $\frac{4}{3}e^{3x-3} - 4e^{-2}x - \frac{1}{3}e^{-3x-1} + C$.

Trigonometric Identities Useful for Integration

1. $\sec^2 x - 1 = \tan^2 x$
2. $\csc^2 x - 1 = \cot^2 x$
3. $\sin A \cos A = \frac{1}{2}\sin 2A$
4. $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$
5. $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$
6. $\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$
7. $\cos A \sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$
8. $\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$
9. $\sin A \sin B = -\frac{1}{2}(\cos(A+B) - \cos(A-B))$

Example 4.5. *Find* $\int \cos\frac{x}{6}\sin\frac{x}{3}\,dx$

Solution.

$$\begin{aligned}
\int \cos\tfrac{x}{6}\sin\tfrac{x}{3}\,dx &= \int \tfrac{1}{2}(\sin\tfrac{x}{2} - \sin(-\tfrac{x}{6}))\,dx \\
&= \tfrac{1}{2}\int \sin\tfrac{x}{2} + \sin\tfrac{x}{6}\,dx \\
&= \tfrac{1}{2}(-2\cos\tfrac{x}{2} - 6\cos\tfrac{x}{6}) + C \\
&= -\cos\tfrac{x}{2} - 3\cos\tfrac{x}{6} + C.
\end{aligned}$$

■

Example 4.6. *Show that* $\int \cos^4 x\,dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$

Solution. First we have

$$\begin{aligned}
\cos^4 x &= (\tfrac{1}{2}(1+\cos 2x))^2 = \tfrac{1}{4}(1 + 2\cos 2x + \cos^2 2x) \\
&= \tfrac{1}{4}(1 + 2\cos 2x + \tfrac{1}{2}(1+\cos 4x)) \\
&= \tfrac{3}{8} + \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x.
\end{aligned}$$

Thus $\int \cos^4 x\,dx = \int \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\,dx$
$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$

■

Exercise 4.2. *Find* $\int \left(\frac{\sin 4x}{1+\cos 4x}\right)^2 dx$

Ans: $\frac{1}{2}\tan 2x - x + C.$

4.3 Partial Fractions

Let $P(x)$ and $Q(x)$ be two polynomials. Suppose $Q(x)$ is a product of linear or quadratic factors with real coefficients. Then, the rational function $\frac{P(x)}{Q(x)}$ can be expressed as a sum of simple fractions whose denominators are factors of $Q(x)$.

Factors of $Q(x)$	Partial fractions
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^2$	$\dfrac{A}{ax+b} + \dfrac{B}{(ax+b)^2}$
$ax^2+bx+c,\ b^2-4ac<0$	$\dfrac{Ax+B}{ax^2+bx+c}$

The rational function $\frac{P(x)}{Q(x)}$ is said to be a proper fraction if the degree of $P(x)$ is smaller that the degree of $Q(x)$. Otherwise, it is called an improper fraction. If $\frac{P(x)}{Q(x)}$ is an improper fraction, one can perform long division to write it as $A(x) + \frac{B(x)}{Q(x)}$, where $\frac{B(x)}{Q(x)}$ is a proper fraction.

Examples

(1) $\dfrac{2x+4}{x^2-9} = \dfrac{\frac{5}{3}}{x-3} + \dfrac{\frac{1}{3}}{x+3}.$

(2) $\dfrac{3x^2+x+4}{x^2+x-2} = 3 - \dfrac{\frac{14}{3}}{x+2} + \dfrac{\frac{8}{3}}{x-1}.$ (2) is an example of an improper fraction.

Example 4.7. *Find* $\displaystyle\int \frac{3x^2+x+4}{x^2+x-2}\,dx.$

Solution.

$$\int \frac{3x^2+x+4}{x^2+x-2}\,dx = \int 3 - \frac{\frac{14}{3}}{x+2} + \frac{\frac{8}{3}}{x-1}\,dx = 3x - \frac{14}{3}\ln|x+2| + \frac{8}{3}\ln|x-1| + C.$$

■

4.4 Integration by Substitution

Theorem 4.2. *Let $u = g(x)$ be a differentiable function whose range is some interval I and let f be continuous on I. Then,*

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Example 4.8. *Find* $\displaystyle\int \frac{e^{3x}}{\sqrt{2e^{3x}+4}}\,dx.$

Solution. Let $u = 2e^{3x}+4$. Then $du = 6e^{3x}dx$.

$$\int \frac{e^{3x}}{\sqrt{2e^{3x}+4}}\,dx = \int \frac{1}{6\sqrt{u}}\,du = \frac{1}{3}\sqrt{u} + C = \frac{1}{3}\sqrt{2e^{3x}+4} + C.$$

■

Exercise 4.3. *Find* $\displaystyle\int \frac{(3-\tan 4x)^5}{\cos^2 4x}\,dx.$

Ans: $-\frac{1}{24}(3-\tan 4x)^6 + C$.

Exercise 4.4. *Find* $\displaystyle\int \frac{8}{x\sqrt{\ln x}}\,dx.$

Ans: $16\sqrt{\ln x} + C$.

Trigonometric Substitution

Expression	Substitution	Identity involved
$\sqrt{a^2-(x+b)^2}$	$x+b = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+(x+b)^2}$	$x+b = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{(x+b)^2-a^2}$	$x+b = a\sec\theta,\ 0 < \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Example 4.9. *Find* $\int \frac{\sqrt{25-4x^2}}{x^2}\,dx.$

Solution. Let $x = \frac{5}{2}\sin\theta$. That is $\theta = \sin^{-1}\frac{2x}{5}$. Also $2x = 5\sin\theta$ so that $\sqrt{25-4x^2} = 5\cos\theta$ and $dx = \frac{5}{2}\cos\theta d\theta$.

$$\int \frac{\sqrt{25-4x^2}}{x^2}\,dx = \int \frac{5\cos\theta}{(\frac{5}{2}\sin\theta)^2}\cdot\frac{5}{2}\cos\theta\,d\theta$$

$$= \int 2\cot^2\theta\,d\theta = 2\int \csc^2\theta - 1\,d\theta$$

$$= -2\cot\theta - 2\theta + C$$

$$= -\frac{1}{x}\sqrt{25-4x^2} - 2\sin^{-1}(\frac{2x}{5}) + C.$$

■

Example 4.10. *Find* $\int \frac{1}{x\sqrt{9x^2+1}}\,dx.$

Solution. Let $x = \frac{1}{3}\tan\theta$. That is $\theta = \tan^{-1}(3x)$. Also $\sqrt{1+9x^2} = \sec\theta$ and $dx = \frac{1}{3}\sec^2\theta d\theta$.

$$\int \frac{1}{x\sqrt{9x^2+1}}\,dx = \int \frac{\frac{1}{3}\sec^2\theta}{\frac{1}{3}\tan\theta\sec\theta}\,d\theta$$

$$= \int \csc\theta\,d\theta = -\ln|\csc\theta + \cot\theta| + C$$

$$= -\ln\left|\frac{\sqrt{9x^2+1}}{3x} + \frac{1}{3x}\right| + C$$

$$= -\ln\left|\frac{\sqrt{9x^2+1}+1}{3x}\right| + C.$$

■

Exercise 4.5. *Find* $\int \sqrt{6x-x^2}\,dx.$

Ans: $\frac{9}{2}\sin^{-1}(\frac{x-3}{3}) + \frac{1}{2}(x-3)\sqrt{6x-x^2} + C$

4.5 Integration by Parts

Recall the product rule of differentiation:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides of the above equation with respect to x gives

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx,$$

or

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx.$$

This suggests the following way of performing integration by parts:

$$\int f'(x)g(x)\,dx = \overbrace{f(x)}^{\substack{\text{integrate}\\ f'}} \cdot \overbrace{g(x)}^{\text{keep } g} - \int \overbrace{f(x)}^{\text{keep } f}\ \overbrace{g'(x)}^{\substack{\text{differentiate}\\ g}}\ dx.$$

Example 4.11. *Find* $\int x \sin 3x\,dx$.

Solution. $$\int x\sin 3x\,dx = \overbrace{\frac{-\cos 3x}{3}}^{\text{integrate } \sin 3x}\ \overbrace{x}^{\text{keep } x} - \int \overbrace{\frac{-\cos 3x}{3}}^{\text{keep integral of } \sin 3x}\ \overbrace{1}^{\text{differentiate } x}\ dx$$
$$= \frac{-x\cos 3x}{3} + \frac{\sin 3x}{9} + C.$$

■

Example 4.12. *Find* $\int x \ln x\,dx$.

Solution. $$\int x\ln x\,dx = \overbrace{\frac{x^2}{2}}^{\text{integrate } x}\ \overbrace{\ln x}^{\text{keep } \ln x} - \int \overbrace{\frac{x^2}{2}}^{\substack{\text{keep integral of}\\ x}}\ \overbrace{\frac{1}{x}}^{\text{differentiate } \ln x}\ dx$$
$$= \frac{x^2\ln x}{2} - \frac{x^2}{4} + C.$$

■

Basic rules to determine which function to integrate and which function to differentiate.

The choice of integration follows the reverse order of the following:

Types of functions	**Examples**	**Remark**
Logarithmic Function	$\ln(ax+b)$ or its higher powers	differentiate it
Inverse Trigonometric Functions	$\sin^{-1}(ax+b)$, $\cos^{-1}(ax+b)$, $\tan^{-1}(ax+b)$	differentiate it
Algebraic Functions	Power functions x^a, polynomials	differentiate it
Trigonometric Functions	$\sin(ax+b)$, $\cos(ax+b)$, $\tan(ax+b)$, $\csc(ax+b)$, $\sec(ax+b)$, $\cot(ax+b)$, or a combinations of these	differentiate it integrate it
Exponential Functions	e^{ax+b}	integrate it

Exercise 4.6. *Find* $\displaystyle\int (2x+1)\ln(2x-3)\,dx.$

Ans: $(x^2+x)\ln(2x-3)-\frac{x^2}{2}-\frac{5x}{2}-\frac{15}{4}\ln(2x-3)+C.$

Exercise 4.7. *Find* $\displaystyle\int x\tan^{-1}(2x)\,dx.$

Ans: $(\frac{1}{8}+\frac{x^2}{2})\tan^{-1}2x-\frac{x}{4}+C.$

Exercise 4.8. *Find* $\displaystyle\int \frac{\sin 2x}{e^{2x}}\,dx.$

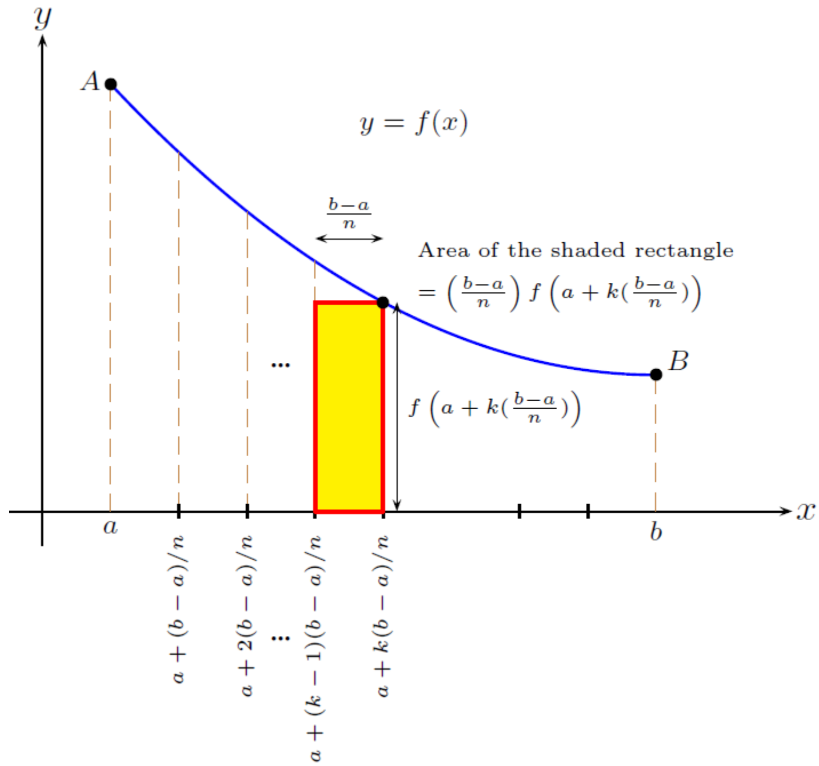
Ans: $-\frac{1}{4}e^{-2x}(\sin 2x+\cos 2x)+C.$

Exercise 4.9. *Find* $\displaystyle\int \sec^3 x\,dx.$

Ans: $\frac{1}{2}(\sec x\tan x+\ln|\sec x+\tan x|)+C.$

4.6 Riemann Sums and Definite Integrals

Let f be continuous on $[a,b]$.



The following limit exists and is known as the *definite integral* of f from $x = a$ to $x = b$.

$$\lim_{n\to\infty}\left\{\sum_{k=1}^{n}\left(\frac{b-a}{n}\right)f\left(a+k(\frac{b-a}{n})\right)\right\}.$$

It is denoted by

$$\int_a^b f(x)\,dx.$$

This is basically the algebraic area of the region under the graph of f from $x = a$ to $x = b$. The numbers a and b are respectively the lower and upper limits of the integral.

The finite series $\displaystyle\sum_{k=1}^{n}\left(\frac{b-a}{n}\right)f\left(a+k(\frac{b-a}{n})\right)$ is known as a *Riemann sum* of f.

Summing up, we have the following definition of $\displaystyle\int_a^b f(x)\,dx$.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left\{\sum_{k=1}^{n}\left(\frac{b-a}{n}\right)f\left(a+k(\frac{b-a}{n})\right)\right\}.$$

Approximation. For sufficiently large n,

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} \left(\frac{b-a}{n}\right) f\left(a + k(\frac{b-a}{n})\right).$$

Example 4.13. *Use Riemann sum to compute* $\displaystyle\int_0^3 x^2\,dx$.

The summation formula $\displaystyle\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$ *is needed to compute the sum.*

Ans: 9.

4.7 Fundamental Theorem of Calculus (FTC)

The method of using Riemann sums to evaluate definite integrals is tedious. The following result provides us with a simpler way of calculating definite integrals when the anti-derivatives of the integrand can be found.

Theorem 4.3 (FTC 1)**.** *Let f be continuous on $[a,b]$ and let F be an anti-derivative of f. Then,*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

That is,

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Example 4.14. *Evaluate* $\displaystyle\int_1^e \frac{(\ln x)^{\frac{1}{3}}}{x}\,dx$.

Ans: $\frac{3}{4}$.

Example 4.15. *Evaluate* $\displaystyle\int_0^{\frac{\pi}{2}} x\cos x\,dx$.

Ans: $\frac{\pi}{2} - 1$.

Example 4.16. *Use a Riemann sum to show that* $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{3k+n} = \frac{1}{3}\ln 4$.

Solution. Recall that $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{b-a}{n}\right) f\left(a + k(\frac{b-a}{n})\right) = \int_a^b f(x)\,dx$.

To express $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{3k+n}$ as a Riemann sum, we need to identify the function $f(x)$ and the interval $[a,b]$.

First we have $\sum_{k=1}^{n}\frac{1}{3k+n}=\sum_{k=1}^{n}\frac{1}{k(\frac{3}{n})+1}\frac{1}{n}=\frac{1}{3}\sum_{k=1}^{n}\frac{1}{1+k(\frac{4-1}{n})}\frac{4-1}{n}$. From this we see that $a=1$, $b=4$ and the function is $f(x)=\frac{1}{x}$.

Therefore, $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{3k+n}=\frac{1}{3}\sum_{k=1}^{n}\frac{1}{1+k(\frac{4-1}{n})}\frac{4-1}{n}=\frac{1}{3}\int_1^4\frac{1}{x}\,dx=\frac{1}{3}[\ln x]_1^4=\frac{1}{3}\ln 4$.

Theorem 4.4 (FTC 2)**.** *Let f be continuous on $[a,b]$. The function g defined by*

$$\int_a^x f(t)\,dt, \quad a\le x\le b$$

is continuous and differentiable on (a,b), and $g'(x)=f(x)$. That is

$$\frac{d}{dx}\int_a^x f(t)\,dt=f(x).$$

Remark. The second fundamental theorem of Calculus (FTC 2) essentially says that the area function $\int_a^x f(t)\,dt$ is an anti-derivative of f.

By the Chain Rule, it can be shown that if $u(x)$ is differentiable, then

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt=f(u(x))u'(x).$$

Example 4.17. *Find* $\dfrac{d}{dx}\displaystyle\int_{-2}^{\sin x}\sqrt{1+t^6}\,dt$.

Ans: $\cos x\sqrt{1+\sin^6 x}$.

Properties of Definite Integrals

Let $c\in[a,b]$ and $\alpha,\beta\in\mathbb{R}$.

1. $\int_a^b \alpha\,dx=\alpha(b-a)$
2. $\int_c^c f(x)\,dx=0$
3. $\int_a^b(\alpha f(x)+\beta g(x))\,dx=\int_a^b\alpha f(x)\,dx+\int_a^b\beta g(x)\,dx$
4. $\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$
5. $\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx$

6. $\int_a^b f(x)\,dx \geq 0$ if $f(x) \geq 0$ for $a \leq x \leq b$

7. $\int_a^b f(x)\,dx \leq 0$ if $f(x) \leq 0$ for $a \leq x \leq b$

8. $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$ if $f(x) \geq g(x)$ for $a \leq x \leq b$

9. $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$ if $f(x) \leq g(x)$ for $a \leq x \leq b$

10. $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$ if $m \leq f(x) \leq M$ for $a \leq x \leq b$

11. $\int_{-a}^a f(x)\,dx = 0$ if f is an odd function defined on $[-a,a]$

12. $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ if f is an even function defined on $[-a,a]$

4.8 Miscellaneous Examples

Example 4.18. *(Integrals of the type* $\displaystyle\int \frac{px+q}{ax^2+bx+c}\,dx$*)*

Evaluate $\displaystyle\int_{-2}^{-1} \frac{3x+7}{x^2+4x+5}\,dx$.

Solution.

$$\int_{-2}^{-1} \frac{3x+7}{x^2+4x+5}\,dx$$

$$=\int_{-2}^{-1} \frac{3}{2}\frac{2x+4}{x^2+4x+5} + \frac{1}{x^2+4x+5}\,dx$$

$$=\int_{-2}^{-1} \frac{3}{2}\frac{2x+4}{x^2+4x+5} + \frac{1}{1+(x+2)^2}\,dx$$

$$=\left[\frac{3}{2}\ln|x^2+4x+5|\right]_{-2}^{-1} + \left[\tan^{-1}(x+2)\right]_{-2}^{-1}$$

$$=\frac{3}{2}(\ln 2 - \ln 1) + \tan^{-1}(1) - \tan^{-1}(0)$$

$$=\frac{3}{2}\ln 2 + \frac{\pi}{4}.$$

■

Example 4.19. *(Integrals of the type* $\displaystyle\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$*)*

Show that $\displaystyle\int_1^2 \frac{2x+3}{\sqrt{4x-x^2}}\,dx = 2\sqrt{3}+\frac{7\pi}{6}-4.$

Exercise 4.10. *Using the identity* $1+\cos x+\sin x = 2\cos^2\frac{x}{2}(1+\tan\frac{x}{2})$*, evaluate*

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x+\sin x}\,dx.$$

Ans: $\ln 2$.

Exercise 4.11. *Using the substitution* $x=\frac{\pi}{2}-y$*, show that*

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\cos x+\sin x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{1+\cos x+\sin x}\,dx.$$

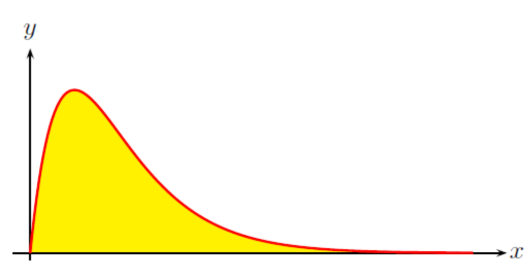
Exercise 4.12. *Using the results in exercise 4.10 and 4.11 and the identity* $\sin^2 x+\cos^2 x = 1$*, show that*

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\cos x+\sin x}\,dx = \frac{\ln 2}{2}.$$

4.9 Improper Integrals

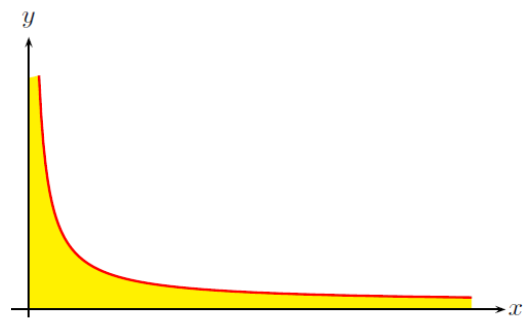
The definite integrals that we have studied so far have the following characteristics: (i) the domain of integration is a finite closed interval $[a,b]$, (ii) the function or the integrand has finite values on the domain of integration. It is possible that we would encounter problems that do not meet these conditions.

The integral for the area under the curve $y = xe^{-x}$ from $x=0$ to $x=\infty$ has domain of integration which is infinite.



The graph of $y = xe^{-x}$.

The integral for the area under the curve $y = \frac{1}{\sqrt{x}}$ from $x = 0$ to $x = 10$ requires us to integrate the function $\frac{1}{\sqrt{x}}$ whose value at $x = 0$ is infinite.



The graph of $y = \frac{1}{\sqrt{x}}$.

In either case, the integrals are called *improper integrals* and are calculated as limits.

Definition 4.2. *Integrals with infinite limits of integration are improper integrals of* **Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite, we say that the improper integral converges and that the limit is the value of the improper integral. If the limit fails to exist, the improper integral diverges.

Definition 4.3. *Integrals of functions that become infinite at a point within the interval of integration are improper integrals of* **Type II**.

1. *If $f(x)$ is continuous on $(a,b]$ and is discontinuous at a, then*

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx.$$

2. *If $f(x)$ is continuous on $[a,b)$ and is discontinuous at b, then*

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx.$$

3. *If $f(x)$ is discontinuous at c with $a<c<b$, then*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite, we say that the improper integral converges and that the limit is the value of the improper integral. If the limit fails to exist, the improper integral diverges.

Example 4.20. *Evaluate the Type I improper integral*

$$\int_1^\infty \frac{\ln x}{x^2}\,dx.$$

Solution. We first use integration by parts to compute the indefinite integral.

$$\int \ln x\cdot\frac{1}{x^2}\,dx = \ln x\cdot\frac{-1}{x} - \int\frac{1}{x}\cdot\frac{-1}{x}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C = -\frac{(1+\ln x)}{x} + C.$$

Thus

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{(1+\ln x)}{x}\right]_1^b = \lim_{b\to\infty} 1 - \frac{1+\ln b}{b} = 1.$$

Here $\lim_{b\to\infty}\frac{\ln b}{b} = 0$ by L'Hôpital's rule. ■

Example 4.21. *Evaluate the Type I improper integral*

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx.$$

Solution. We need to evaluate $\int_{-\infty}^0 \frac{1}{1+x^2}\,dx$ and $\int_0^\infty \frac{1}{1+x^2}\,dx$.

$$\int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \lim_{b\to-\infty}\int_b^0 \frac{1}{1+x^2}\,dx = \lim_{b\to-\infty}(0-\tan^{-1} b) = -(-\frac{\pi}{2}) = \frac{\pi}{2}.$$

Similarly,

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}.$$

Therefore,

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \pi.$$

■

Example 4.22. *Evaluate the Type II improper integral*

$$\int_0^1 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx.$$

Solution.

$$\int_0^1 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx = \lim_{c\to 1^-}\int_0^c \frac{1}{(x-1)^{\frac{2}{3}}}\,dx = \lim_{c\to 1^-}\left[3(x-1)^{\frac{1}{3}}\right]_0^c = \lim_{c\to 1^-} 3(c-1)^{\frac{1}{3}} + 3 = 3.$$

■

Exercise 4.13. *Evaluate the Type I improper integral*

$$\int_0^\infty \frac{\tan^{-1} x}{1+x^2}\,dx.$$

Ans: $\frac{\pi^2}{8}$.

Exercise 4.14. *Evaluate the Type II improper integral*

$$\int_0^1 \frac{1}{2\sqrt{x}(1+x)}\,dx.$$

Ans: $\frac{\pi}{4}$.

Exercise 4.15. *Evaluate the Type I improper integral*

$$\int_{-\infty}^\infty \frac{1}{e^{-x}+e^x}\,dx.$$

Ans: $\frac{\pi}{2}$.

Chapter 5

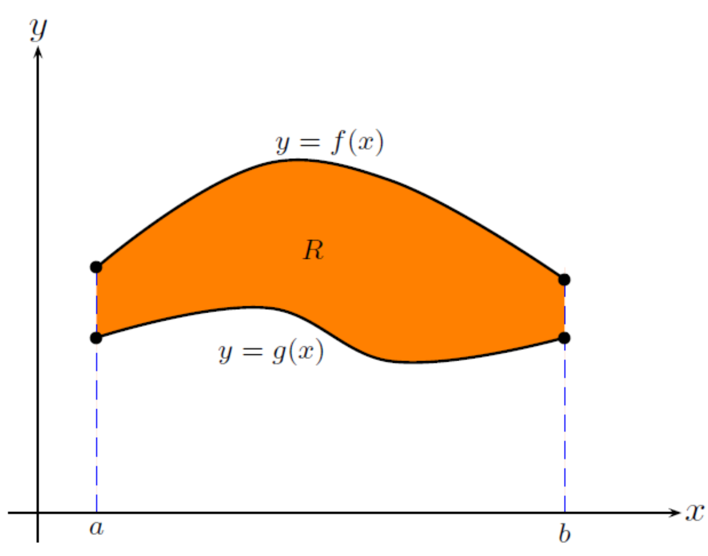
Applications of Integration

Read Thomas' Calculus, Chapter 6.

5.1 Area Between Curves

Theorem 5.1. *Let f and g be continuous on $[a,b]$ with $f(x) \geq g(x)$ for all $a \leq x \leq b$. The area of the region bounded by the curves $y = f(x), y = g(x)$, and the lines $x = a$ and $x = b$ is given by*

$$A = \int_a^b (f(x) - g(x))\,dx.$$



In particular, if $g(x) = 0$, we obtain the following result.

Theorem 5.2. *Let f be continuous on $[a,b]$ with $f(x) \geq 0$ for all $a \leq x \leq b$. The area of the region bounded by the curve $y = f(x)$, and the lines $x = a$ and $x = b$ is given by*

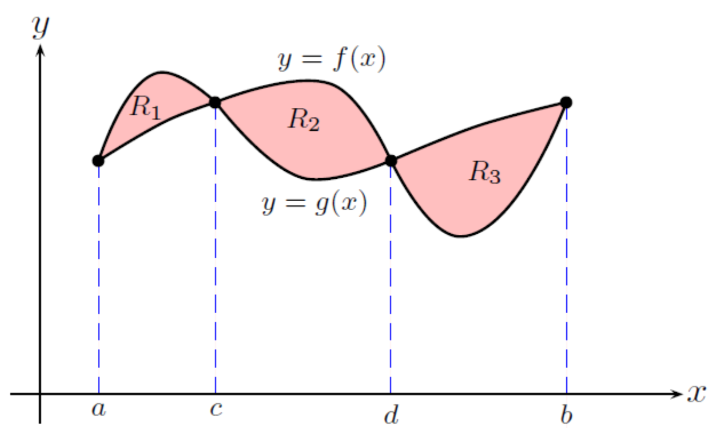
$$A = \int_a^b f(x)\,dx.$$

In general we have the following result.

Theorem 5.3. *Let f and g be continuous on $[a,b]$ (not necessarily with $f(x) \geq g(x)$ for all $a \leq x \leq b$). The area of the region bounded by the curves $y = f(x), y = g(x)$, and the lines $x = a$ and $x = b$ is given by*

$$A = \int_a^b |f(x) - g(x)|\,dx.$$

To evaluate the above integral, we split it into two or more integrals, each corresponding to the region where either $f(x) - g(x) \geq 0$ or $f(x) - g(x) \leq 0$.



In particular, if $g(x) = 0$, then we obtain the following result.

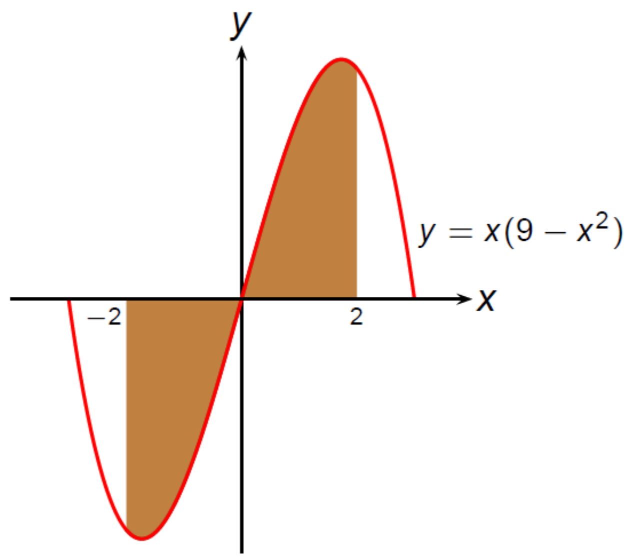
Theorem 5.4. *Let f be continuous on $[a,b]$. The area of the region bounded by the curve $y = f(x)$, and the lines $x = a$ and $x = b$ is given by*

$$A = \int_a^b |f(x)|\,dx.$$

To evaluate the above integral, we split it into two or more integrals, each corresponding to the region where either $f(x) \geq 0$ or $f(x) \leq 0$.

Example 5.1. *Find the area of the region bounded the curve $y = x(9-x^2)$, $(-2 \leq x \leq 2)$, the x-axis, the line $x = -2$ and the line $x = 2$.*

Solution. Note that $f(x) = x(9 - x^2)$ is an odd function.



$$\begin{aligned} \text{Area} &= 2\int_0^2 x(9-x^2)\,dx \\ &= 2\int_0^2 9x - x^3\,dx \\ &= 2\left[\frac{9x^2}{2} - \frac{x^4}{4}\right]_0^2 \\ &= 2(18-4) = 28. \end{aligned}$$

■

Example 5.2. *Find the area of the region bounded by the curves* $y = e^{2x} - 2$, $(x \geq 0)$, *and* $y = 10 - e^x$, $(x \geq 0)$, *and*

(a) the x-axis,

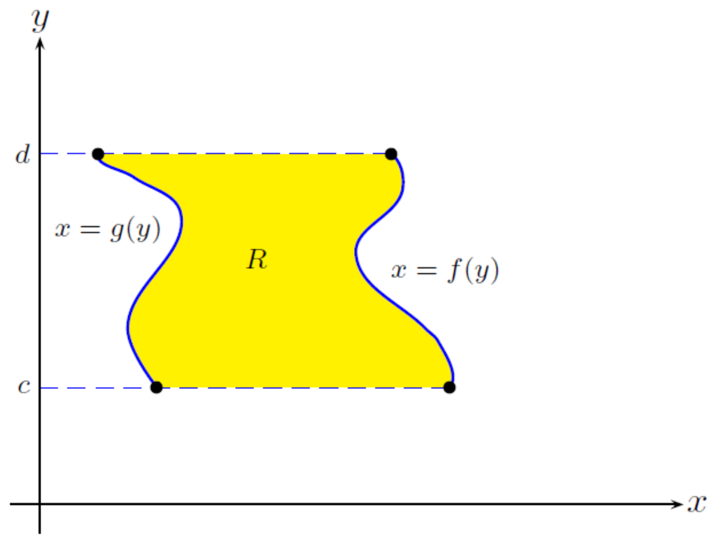
(b) the y-axis.

Ans: (a) $-\frac{7}{2} - 12\ln 3 + 10\ln 10 + \ln 2$,

(b) $12\ln 3 - 6$.

Theorem 5.5. *Let f and g be continuous on $[c,d]$ with $f(y) \geq g(y)$ for all $c \leq y \leq d$. The area of the region bounded by the curves $x = f(y)$, $x = g(y)$, and the lines $y = c$ and $y = d$ is given by*

$$A = \int_c^d (f(y) - g(y))\,dy.$$



In general, we have the following result.

Theorem 5.6. *Let f and g be continuous on $[c,d]$ (not necessarily with $f(y) \geq g(y)$ for all $c \leq y \leq d$). The area of the region bounded by the curves $x = f(y), x = g(y)$, and the lines $y = c$ and $y = d$ is given by*

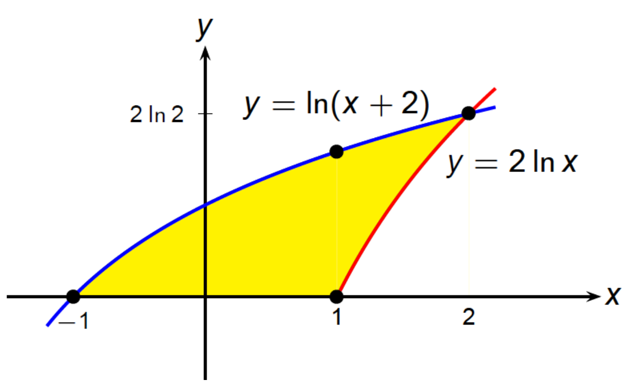
$$A = \int_c^d |f(y) - g(y)|\,dy.$$

To evaluate the above integral, we split it into two or more integrals, each corresponding to the region where either $f(y) - g(y) \geq 0$ or $f(y) - g(y) \leq 0$.

Example 5.3. *Find the area of the region bounded the curve $y = \ln(x+2)$, $y = 2\ln x$, the x-axis.*

Ans: $4\ln 2 - 1$.

Solution.

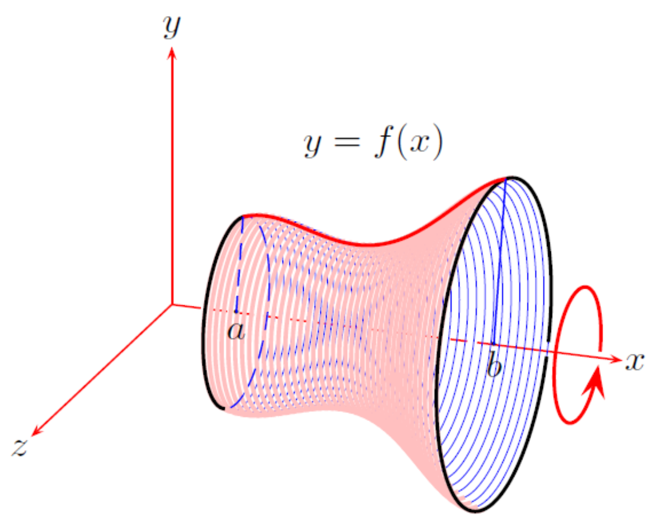


Solving $y = \ln(x+2)$ and $y = 2\ln x$ gives $(x, y) = (2, \ln 2)$. Also $y = \ln(x+2) \Leftrightarrow x = e^y - 2$ and $y = 2\ln x \Leftrightarrow x = e^{\frac{y}{2}}$.

Thus the area is $A = \int_0^{2\ln 2} e^{\frac{y}{2}} - e^y + 2\,dy = \left[2e^{\frac{y}{2}} - e^y + 2y\right]_0^{2\ln 2} = 4\ln 2 - 1$.

■

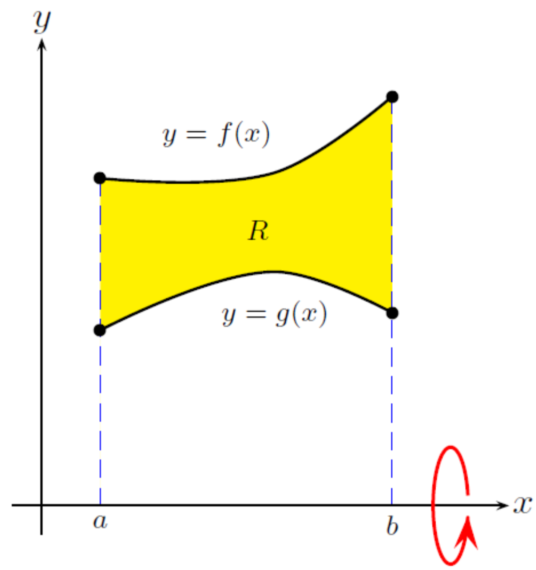
5.2 Volume of Solid of Revolution by Disk Method



Theorem 5.7. *When the plane region bounded by the curve $y = f(x)$ and the lines $x = a$ and $x = b$ is revolved completely about the x-axis, the volume of the solid formed is*

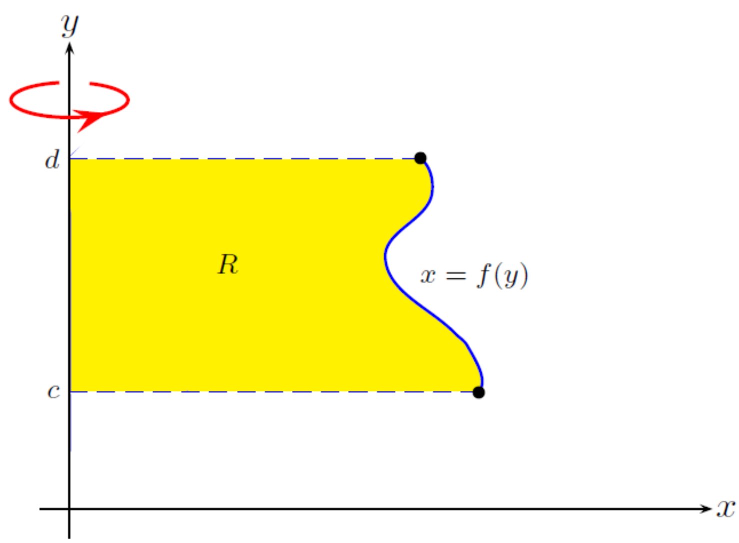
$$V = \pi \int_a^b f(x)^2\,dx.$$

The above formula is known as the disk method.



Theorem 5.8. *Let f and g be continuous on $[a,b]$ with $f(x) \geq g(x)$ for all $a \leq x \leq b$. When the region bounded by the curves $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$ is revolved completely about the x-axis, the volume of the solid formed is*

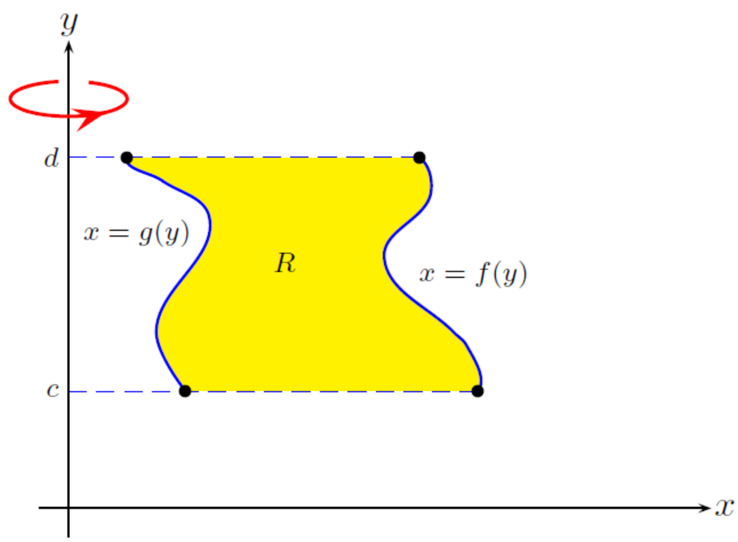
$$V = \pi \int_a^b f(x)^2\,dx - \pi \int_a^b g(x)^2\,dx.$$



Theorem 5.9. *Let f be continuous on $[c,d]$. When the plane region bounded by the curve $x = f(y)$ and the lines $y = c$ and $y = d$ is revolved completely about the y-axis, the volume of*

the solid formed is

$$V = \pi \int_a^b f(y)^2 \, dy.$$



Theorem 5.10. *Let f and g be continuous on $[c, d]$ with $f(y) \geq g(y)$ for all $c \leq y \leq d$. When the region bounded by the curves $x = f(y)$ and $x = g(y)$ for $c \leq y \leq d$ is revolved completely about the y-axis, the volume of the solid formed is*

$$V = \pi \int_c^d f(y)^2 \, dy - \pi \int_c^d g(y)^2 \, dy.$$

Example 5.4. *Find the volume of the solid generated by rotating completely the region bounded by the curve $y^2 = 4x$, the line $y = 2x - 4$ and the y-axis about the*

(a) x-axis,

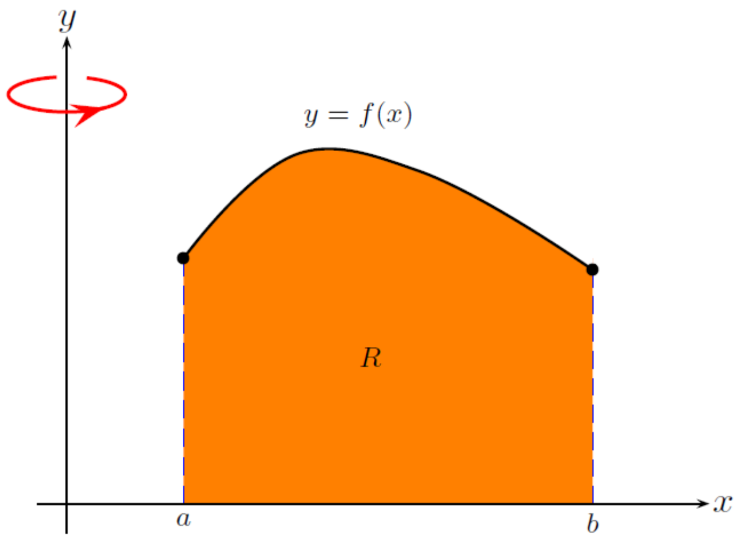
(b) y-axis.

Ans: (a) $\frac{22\pi}{3}$,

(b) $\frac{16\pi}{15}$.

5.3 Cylindrical Shell Method

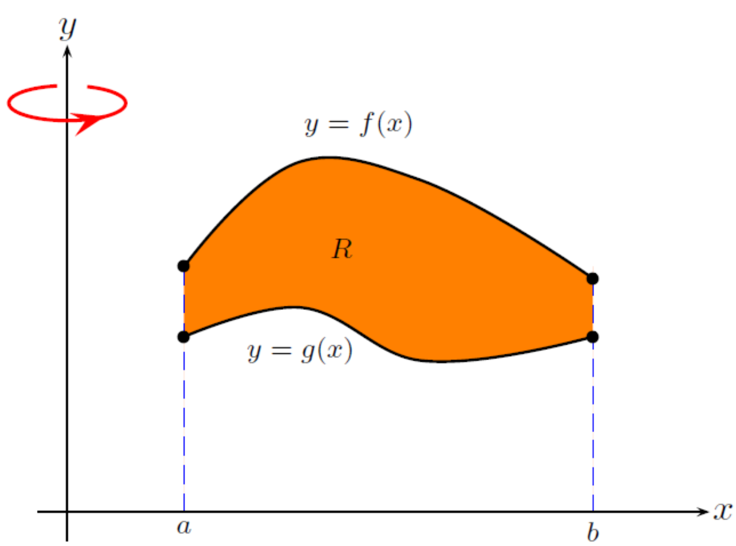
Consider the solid of revolution obtained by revolving the region R about the y-axis. If we apply the disk method, it is necessary to express the equation $y = f(x)$ in the form $x = g(y)$. Quite often, it is difficult or even impossible to do so.



The following result, known as the method of cylindrical shell, provides a solution to this problem.

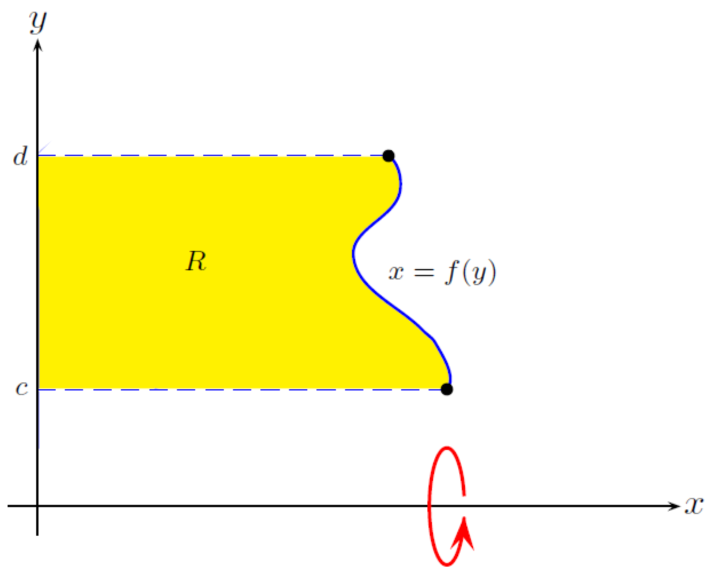
Theorem 5.11. *When the plane region bounded by the curve $y = f(x)$ and the lines $x = a$ and $x = b$, where $0 \leq a < b$, is revolved completely about the y-axis, the volume of the solid formed is*

$$V = 2\pi \int_a^b x|f(x)|\,dx.$$



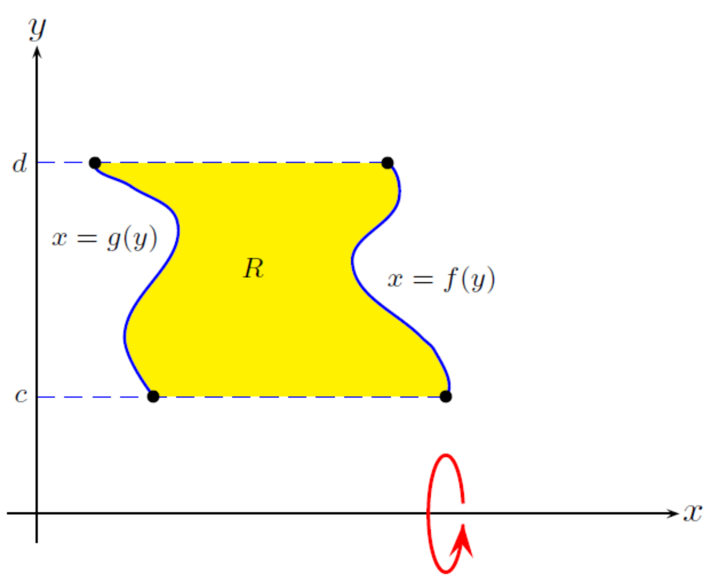
Theorem 5.12. *When the plane region bounded by the curve $y = f(x), y = g(x)$ and the lines $x = a$ and $x = b$, where $0 \leq a < b$, is revolved completely about the y-axis, the volume of the solid formed is*

$$V = 2\pi \int_a^b x|f(x) - g(x)|\,dx.$$



Theorem 5.13. *When the plane region bounded by the curve $x = f(y)$ and the lines $y = c$ and $y = d$, where $0 \leq c < d$, is revolved completely about the x-axis, the volume of the solid formed is*

$$V = 2\pi \int_c^d y|f(y)|\,dy.$$



Theorem 5.14. *When the plane region bounded by the curve $x = f(y), x = g(y)$ and the lines $y = c$ and $y = d$, where $0 \le c < d$, is revolved completely about the x-axis, the volume of the solid formed is*

$$V = 2\pi \int_c^d y|f(y) - g(y)|\, dy.$$

Example 5.5. *The regions bounded by the curve $y = 2x - x^2$ $(1 \le x \le 3)$, the line $x = 1$, the line $x = 3$ and the x-axis is revolved completely about the y-axis. Calculate the volume of the solid generated.*

Ans: 9π.

Example 5.6. *Sketch the curve whose equation is $y = \ln(2x - 1)$ for $x > \frac{1}{2}$.*

The region bounded by this curve, the axes and the line $y = \ln 3$ is rotated completely about the x-axis. Calculate the volume of the solid generated.

Ans: $\pi(3\ln 3 + \frac{1}{2}(\ln 3)^2 - 2)$.

5.4 Arc Length of a curve

Let f be continuous on $[a, b]$. Using Riemann sums, we can prove:

The length of the curve $y = f(x), a \le x \le b$ is

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Example 5.7. *Calculate the length of the curve*

$$y = \frac{x^4 + 3}{6x},\ 1 \le x \le 2.$$

Solution. Given $y = \frac{x^4+3}{6x} = \frac{1}{6}(x^3 + \frac{3}{x})$, we have

$$y' = \frac{1}{6}(3x^2 - \frac{3}{x^2}) = \frac{1}{2}(x^2 - \frac{1}{x^2}).$$

Thus $1 + y'^2 = 1 + \frac{1}{4}(x^2 - \frac{1}{x^2})^2 = \left(\frac{1}{2}(x^2 + \frac{1}{x^2})\right)^2$ so that $\sqrt{1 + y'^2} = \frac{1}{2}(x^2 + \frac{1}{x^2})$.

The arclength is $\int_1^2 \sqrt{1 + y'^2}\, dx = \int_1^2 \frac{1}{2}(x^2 + \frac{1}{x^2})\, dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{1}{x}\right]_1^2 = \frac{17}{12}$.

■

Similarly, if the curve is given as the graph of a function of y, we the the following formula.

The length of the curve $x = g(y), p \leq y \leq q$ is

$$\int_p^q \sqrt{1 + g'(y)^2}\, dy.$$

Exercise 5.1. *Calculate the length of the curve*

$$y = 1 + \left(\frac{3x}{2}\right)^{\frac{2}{3}}, \ 0 \leq x \leq \frac{16}{3}.$$

Ans: $\frac{2}{3}(5^{\frac{3}{2}} - 1)$.

Exercise 5.2. *Sketch the curves $y = 2 - e^{x-1}$ and $y = 4x^2 - 3$ for $x \geq 0$ on a single diagram. It is given that the two curves meet at a point where $x = 1$. Calculate the area of the region bounded by the two curves and the y-axis.*

Ans: $\frac{8}{3} + \frac{1}{e}$.

Exercise 5.3. *Let $I_n = \int_0^1 (2x+1)^n e^{-x}\, dx$, where n is a non-negative integer.*
(a) Show that for $n \geq 1$,

$$I_n = (1 - \frac{3^n}{e}) + 2nI_{n-1}.$$

(b) The region bounded by the curve $y = (2x+1)^2 e^{-x}$, the axes and the line $x = 1$ is rotated completely about the y-axis. Use the result in (a) to find the value of the solid generated.

Ans: (b) $\pi(66 - \frac{172}{e})$.

Exercise 5.4. *Consider the region R bounded by $y = 2x^2$, the line $x = 2$ and the x-axis. For $0 < p < 2$, the vertical line $x = p$ divides R into two parts R_1 and R_2, where R_1 denotes the part on the right of $x = p$ and R_2 denotes the part on the left of $x = p$. Let V_1 be the volume of the solid generated by revolving R_1 about the x-axis, and V_2 be the volume of the solid generated by revolving R_2 about the y-axis. Find R_1 and R_2 in terms of p. Find also the value of p that maximizes the total volume given by $V = V_1 + V_2$.*

Ans: $p = 1$ gives the maximum V.

Chapter 6

Sequences and Series

Read Thomas' Calculus, Chapter 10.

6.1 Sequences

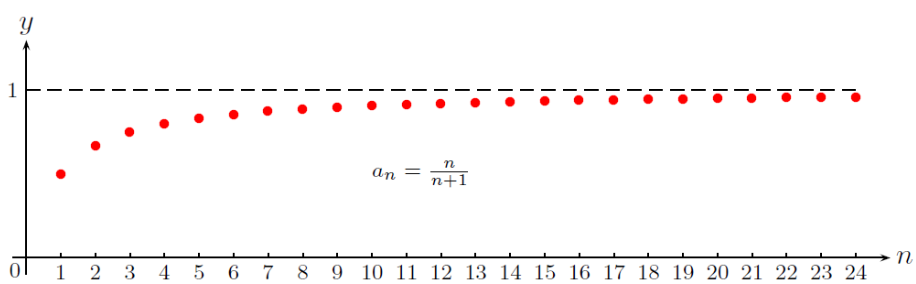
Definition 6.1. *An infinite sequence of numbers is a function whose domain is the set of positive integers.*

We usually denote a sequence by $\{a_n\}_{n=1}^{\infty}$ (or simply $\{a_n\}$ when the reference to n is clear).

Example 6.1. *The sequence of arithmetic progression is given by $\{a+(n-1)d\}_{n=1}^{\infty}$, where a is the first term of the sequence and d is called the common difference.*

The sequence of geometric progression is given by $\{ar^{n-1}\}_{n=1}^{\infty}$, where a is the first term of the sequence and r is known as the common ratio of the geometric progression.

We now consider the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = \frac{n}{n+1}$, and determine what happens to a_n when n is getting large. The following graph shows how the terms approach 1.



We say that the sequence $\{\frac{n}{n+1}\}$ approaches 1 as n increases and write

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

We now state formally the meaning of a limit of a sequence.

Definition 6.2. *The sequence* $\{a_n\}$ *converges to the number* L *if for every positive number* ϵ*, there corresponds an integer* N *such that for all* n*,*

$$n > N \text{ implies that } |a_n - L| < \epsilon.$$

If no such number exists, we say that $\{a_n\}$ *diverges. If* $\{a_n\}$ *converges to* L*, we write*

$$\lim_{n\to\infty} a_n = L$$

or simply $a_n \to L$ *and call* L *the limit of the sequence.*

We write $\lim_{n\to\infty} a_n = \infty$ if a_n is arbitrarily large by taking n sufficiently large. Formally, it means for every $M > 0$, there exists a number N such that

$$n > N \Rightarrow a_n > M.$$

Example 6.2. *Show that* $\lim_{n\to\infty} \dfrac{1}{n} = 0$.

Solution. Given $\epsilon > 0$, choose a positive integer N such that $\frac{1}{N} < \epsilon$. Then for any integer $n > N$, we have $|\frac{1}{n} - 0| = \frac{1}{n} < \frac{1}{N} < \epsilon$. This verifies the definition of limit. Therefore we have shown that $\lim_{n\to\infty} \dfrac{1}{n} = 0$.

The sequence $\{(-1)^n\}$ is an example of a divergent sequence.

■

6.2 Finding the Limit of a Sequence

This following theorem gives a shortcut to evaluate the limit of some sequences using the limit of functions.

Theorem 6.1. *Let* f *be a function, and* $\{a_n\}$ *be a sequence such that* $f(n) = a_n$ *for all* n*. If* $\lim_{x\to\infty} f(x) = L$*, then* $\lim_{n\to\infty} a_n = L$*.*

Example 6.3. *Find* $\lim_{n\to\infty} \dfrac{\ln n}{n}$.

Solution. By L'Hôspital's rule, $\lim_{x\to\infty} \dfrac{\ln x}{x} = \lim_{x\to\infty} \dfrac{1/x}{1} = \lim_{x\to\infty} \dfrac{1}{x} = 0$. By the theorem 6.1, $\lim_{n\to\infty} \dfrac{\ln n}{n} = 0$.

■

6.3 Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then we have

- $\lim\limits_{n\to\infty} ca_n = c \lim\limits_{n\to\infty} a_n$.
- $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$.
- $\lim\limits_{n\to\infty} a_n b_n = \lim\limits_{n\to\infty} a_n \lim\limits_{n\to\infty} b_n$.
- $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, if $\lim\limits_{n\to\infty} b_n \neq 0$.

Theorem 6.2. *(Squeeze Theorem for Sequence) If $a_n \le b_n \le c_n$ for all n and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.*

Example 6.4. *Show that if $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.*

Solution. Since $-|a_n| \le a_n \le |a_n|$ for all n, the result follows from Squeeze Theorem.

■

Example 6.5. *Show that $\lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$.*

Solution. Since $0 \le \frac{n!}{n^n} = \frac{n}{n} \times \frac{n-1}{n} \times \frac{n-2}{n} \times \cdots \times \frac{2}{n} \times \frac{1}{n} \le \frac{1}{n}$ for all n and $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, the result follows from Squeeze Theorem.

■

6.4 Series

Given a sequence $\{a_n\}$, we can construct a new sequence defined by

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

$\{S_n\}$ is called the sequence of *partial sums*. We write its limit as

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n.$$

$\sum_{n=1}^{\infty} a_n$ is called an *infinite series* or simply a *series*.

A series is called *convergent* if its corresponding sequence of partial sums $\{S_n\}$ is convergent, and it is called *divergent* otherwise.

Example 6.6. *(a) An important example of an infinite series is the geometric series*

$$\sum_{n=1}^{\infty} ar^{n-1}, \quad (a \neq 0).$$

It is convergent to $\frac{a}{1-r}$ *when* $|r| < 1$*, and it is divergent when* $|r| \geq 1$*.*

(b) Is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ *convergent or divergent?*

(c) Find the sum of the series $\sum_{n=1}^{\infty} x^n$*, for* $|x| < 1$*.*

(d) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ *is convergent and find its sum.*

Solution. (a) $S_n = \sum_{i=1}^{n} ar^{i-1} = \frac{a(1-r^n)}{1-r}$. We know that $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$ and $\lim_{n\to\infty} r^n$ does not exist if $|r| \geq 1$. Thus $\sum_{n=1}^{\infty} ar^{n-1} = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$ when $|r| < 1$, and it is divergent when $|r| \geq 1$.

(b) $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = 3 \sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$ is a geometric series with $r = \frac{4}{3} > 1$, thus it is divergent.

(c) By (a), $\sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty} xx^{n-1} = \frac{x}{1-x}$.

(d) $S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) = 1 - \frac{1}{1+n}$. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} \left(1 - \frac{1}{1+n}\right) = 1.$$

■

Theorem 6.3. *If* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are convergent series, so are the series* $\sum_{n=1}^{\infty} ca_n$ *(where* c *is a constant) and* $\sum_{n=1}^{\infty}(a_n + b_n)$. *Moreover,*

(a) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$, *and*

(b) $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

Lemma 6.4. *If the series* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof. Suppose $\sum_{n=1}^{\infty} a_n = L$. Let $S_n = a_1 + \cdots + a_n$. Then $\lim_{n\to\infty} S_n = L$. Note that $a_n = S_n - S_{n-1}$ for all $n \geq 2$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - S_{n-1} = L - L = 0.$$

■

Theorem 6.5. *(Test for Divergence) If* $\lim_{n\to\infty} a_n$ *does not exist or if* $\lim_{n\to\infty} a_n \neq 0$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

This is also known as the n^{th} term test for divergence.

Example 6.7. *Is the series* $\sum_{n=1}^{\infty} \frac{n^2}{7n^2+3}$ *convergent or divergent?*

Solution. $\lim_{n\to\infty} \frac{n^2}{7n^2+3} = \lim_{n\to\infty} \frac{1}{7+\frac{3}{n}} = \frac{1}{7} \neq 0$. Thus by Theorem 6.5, the series $\sum_{n=1}^{\infty} \frac{n^2}{7n^2+3}$ is divergent.

Remark. The test for divergence is inconclusive if $\lim_{n\to\infty} a_n = 0$.

■

For series of nonnegative terms, we have the following fundamental result which follows from the Monotone Convergence Theorem for sequences.

Theorem 6.6. *A series* $\sum_{n=1}^{\infty} a_n$ *of nonnegative terms converges if and only if its partial sums are bounded from above (i.e. there exists a constant* K *such that* $S_n < K$ *for all* n*.)*

In general, it is difficult to determine if a given series in convergent. In the next few sections, we will discuss some methods that will enable us to test the convergence of certain series.

6.5 Integral Test

Example 6.8. *The harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$ *is divergent.*

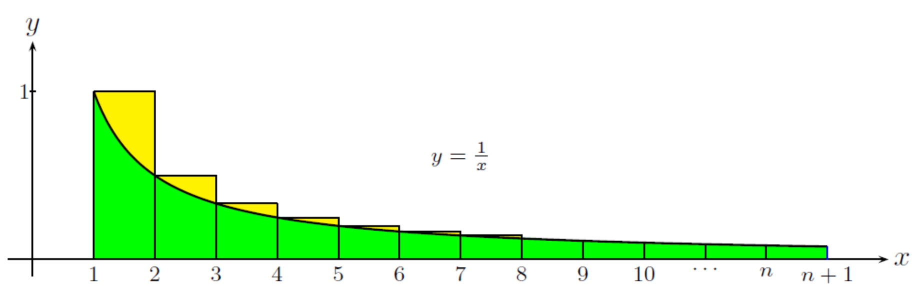
Solution. Observe that

$$1+\frac{1}{2}+\underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{\geq\frac{2}{4}=\frac{1}{2}}+\underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{\geq\frac{4}{8}=\frac{1}{2}}+\underbrace{\left(\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16}\right)}_{\geq\frac{8}{16}=\frac{1}{2}}+\cdots$$

In general, the sum of 2^n terms ending with $\frac{1}{2^{n+1}}$ is greater than $2^n/2^{n+1} = 1/2$. If $n = 2^k$, the partial sum S_n is greater than $k/2$, so the sequence of partial sums is not bounded from above. By Theorem 6.6, the harmonic series diverges.

■

For $n = 1, 2, \ldots$, the rectangle with height $\frac{1}{n}$ erected on the unit interval $[n, n+1]$ has area $\frac{1}{n}$. The harmonic series may be viewed as the sum of the areas of these rectangles. Graphically we see that the union of all these rectangles contains the region R bounded by the graph of $y = \frac{1}{x}$ and the x-axis for $x \geq 1$. However, the region R has an infinite area as the improper integral $\int_1^{\infty} \frac{1}{x}\,dx$ diverges. This implies the harmonic series diverges.



Theorem 6.7. *(Integral Test) Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \ge 1$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:*

(i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 6.9. *Test the convergence of the series* $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Solution. $\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} [\tan^{-1} x]_1^b = \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$. By the integral test, $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges, but $\frac{\pi}{4}$ is *not* the sum of the series.

■

Theorem 6.8. *(The p-series) The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $p > 1$.*

Proof. Let $p > 1$. $\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b = \lim_{b\to\infty} \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right) = \frac{1}{p-1}$. Thus the series converges by the integral test.

Let $p \le 0$. Then $\lim_{n\to\infty} \frac{1}{n^p} \ne 0$. By Theorem 6.5, the series diverges.

Let $0 < p < 1$. Then $1 - p > 0$. $\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \frac{1}{1-p}\left(b^{1-p} - 1\right) = \infty$. Thus the series diverges by the integral test.

If $p = 1$, then the series is the harmonic series which is divergent.

■

6.6 The Comparison Test

Theorem 6.9. *Suppose* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are series with positive terms such that* $a_n \le b_n$ *for all* n*.*

(i) If $\sum_{n=1}^{\infty} b_n$ *is convergent, then* $\sum_{n=1}^{\infty} a_n$ *is convergent.*

(ii) If $\sum_{n=1}^{\infty} a_n$ *is divergent, then* $\sum_{n=1}^{\infty} b_n$ *is divergent.*

Example 6.10. *Determine the convergence of the series* $\sum_{n=1}^{\infty} \frac{7}{2n^2+4n+3}$.

Solution. Note that $\frac{7}{2n^2+4n+3} \le \frac{7}{n^2}$ for all $n \ge 1$. We know $\sum_{n=1}^{\infty} \frac{7}{n^2}$ converges as it is a p-series with $p = 2$. Thus by the comparison test, $\sum_{n=1}^{\infty} \frac{7}{2n^2+4n+3}$ converges.

■

Example 6.11. *Determine the convergence of the series* $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$.

Solution. Note that $\frac{1}{2^n-1} \le \frac{1}{2^{n-1}}$ for all $n \ge 1$. The geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is convergent. Thus by the comparison test, $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ is convergent.

■

6.7 The Ratio Test and Root Test

Theorem 6.10. *(The Ratio Test) Suppose* $\sum_{n=1}^{\infty} a_n$ *is a series such that* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ *(*L *is a finite number or* ∞*).*

(i) If $0 \le L < 1$*, then* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent. That is* $\sum_{n=1}^{\infty} |a_n|$ *is convergent.*

(ii) If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $L = 1$, then the ratio test is inconclusive.

Proof. Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$. Choose r such that $L < r < 1$. Then there exists a positive integer N such that

$$n > N \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| < r.$$

Then $|a_n| \le |a_{N+1}| r^{n-N-1}$ for all $n > N$. Since the geometric series $\sum |a_{N+1}| r^{n-N-1}$ is convergent, by comparison test, $\sum |a_n|$ is convergent.

Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$. Choose r such that $1 < r < L$. Then there exists a positive integer N such that

$$n > N \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| > r.$$

Then $|a_n| \ge |a_{N+1}| r^{n-N-1}$ for all $n > N$. Since $\lim_{n\to\infty} |a_{N+1}| r^{n-N-1} = \infty$ as $r > 1$, we have $\lim_{n\to\infty} |a_n| = \infty$. Thus $\lim_{n\to\infty} a_n \ne 0$. Therefore, $\sum_{n=1}^{\infty} a_n$ is divergent by the test for divergence (Theorem 6.5).

■

Example 6.12. *Test the series*

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$$

for absolute convergence.

Solution.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} \frac{(n+1)^3}{3^{n+1}}}{(-1)^n \frac{n^3}{3^n}} \right| = \lim_{n\to\infty} \frac{1}{3} \left(\frac{n+1}{n}\right)^3 = \lim_{n\to\infty} \frac{1}{3} \left(1 + \frac{1}{n}\right)^3 = \frac{1}{3} < 1.$$

Thus by the ratio test, $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ converges absolutely.

■

Example 6.13. *Test the series*

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

for absolute convergence.

Solution.

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}\right| = \lim_{n\to\infty}\frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n = 1.$$

The ratio test is inconclusive.

However, $0 \le \frac{n!}{n^n} = \frac{n}{n}\times\frac{n-1}{n}\times\frac{n-2}{n}\times\cdots\times\frac{2}{n}\times\frac{1}{n} \le \frac{2}{n^2}$ for all $n \ge 1$. Since $\sum_{n=1}^{\infty}\frac{2}{n^2}$ converges, we have by the comparison test that $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ converges.

■

Exercise 6.1. *Test for convergence of the following series.*

(a) $\sum_{n=1}^{\infty}\frac{2^n+5}{3^n}$ *(b)* $\sum_{n=1}^{\infty}\frac{(2n)!}{n!n!}$ *(c)* $\sum_{n=1}^{\infty}\frac{4^n(n!)^2}{(2n)!}$

Ans. (a) convergent, (b) divergent, (c) divergent.

Theorem 6.11. *(The Root Test) Suppose $\sum_{n=1}^{\infty}a_n$ is a series such that $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L$ (L is a finite number or ∞).*

(i) If $0 \le L < 1$, then $\sum_{n=1}^{\infty}a_n$ is absolutely convergent.

(ii) If $L > 1$, then $\sum_{n=1}^{\infty}a_n$ is divergent.

(iii) If $L = 1$, then the root test is inconclusive.

Example 6.14. *Test the series*

$$\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$$

for convergence.

Solution.

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{2n+3}{3n+2}\right)^n\right|} = \lim_{n\to\infty}\frac{2n+3}{3n+2} = \frac{2}{3} < 1.$$

By the root test, $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$ is convergent.

■

Example 6.15. *Test the series*

$$\sum_{n=1}^{\infty} \frac{n^n}{3^{3n+1}}$$

for convergence.

Solution. $\lim_{n\to\infty} \sqrt[n]{\left|\frac{n^n}{3^{3n+1}}\right|} = \lim_{n\to\infty} \frac{n}{3^{3+\frac{1}{n}}} = \infty$. By the root test, $\sum_{n=1}^{\infty} \frac{n^n}{3^{3n+1}}$ is divergent.

■

Exercise 6.2. *Test for convergence of the following series.*

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ *(b)* $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ *(c)* $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

Ans. (a) convergent, (b) divergent, (c) convergent.

6.8 Alternating Series

An *alternating series* is a series whose terms are alternatively positive and negative.

Theorem 6.12. *(The Alternating Series Test) If b_n is a sequence of positive numbers such that*

(i) b_n *is decreasing, and*

(ii) $\lim_{n\to\infty} b_n = 0$,

then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

is convergent.

Proof. Let $S_n = \sum_{i=1}^{n} (-1)^{i-1} b_i$. Then $S_{2n} = (b_1 - b_2) + \cdots + (b_{2n-1} - b_{2n}) = b_1 - (b_2 - b_3) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}$. The first equality shows that $S_{2n} \geq 0$ and $S_{2(n+1)} \geq S_{2n}$, The second equality shows that $S_{2n} \leq b_1$ for all n. The sequence $\{S_{2n}\}$ is non-decreasing and bounded above, so it have a limit, say

$$\lim_{n\to\infty} S_{2n} = L.$$

As $S_{2n+1} = S_{2n} + b_{2n+1}$, we have

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} b_{2n+1} = L + 0 = L.$$

Lastly, for any term S_n with $n \geq 2$, there exists a positive integer m such that $S_{2m} \leq S_n \leq S_{2m+1}$. [For example, $S_2 \leq S_2 \leq S_3, S_2 \leq S_3 \leq S_3, S_4 \leq S_4 \leq S_5, S_4 \leq S_5 \leq S_5$, etc.] Also, as n tends to infinity, m tends to infinity. Thus

$$\lim_{m\to\infty} S_{2m} \leq \lim_{n\to\infty} S_n \leq \lim_{m\to\infty} S_{2m+1}.$$

Since $\lim_{m\to\infty} S_{2m} = \lim_{m\to\infty} S_{2m+1} = L$, we have by Squeeze Theorem that $\lim_{m\to\infty} S_n = L$. This means the series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ is convergent.

∎

Example 6.16. *Show that the series* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ *is convergent.*

Example 6.17. *Show that the series* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{n^3+1}$ *is convergent.*

6.9 Absolute Convergence

Given a series $\sum_{n=1}^{\infty} a_n$, we can construct a new series $\sum_{n=1}^{\infty} |a_n|$, whose terms are the absolute values of the terms of the original series.

Theorem 6.13. *If* $\sum_{n=1}^{\infty} |a_n|$ *is convergent, then* $\sum_{n=1}^{\infty} a_n$ *is convergent.*

Proof. Note that $0 \leq (a_n + |a_n|) \leq 2|a_n|$ for all n. Since $\sum_{n=1}^{\infty} 2|a_n|$ converges, we have by comparison test, $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges too. Therefore,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$$

converges.

∎

Definition 6.3. *A series* $\sum_{n=1}^{\infty} a_n$ *is called absolutely convergent if* $\sum_{n=1}^{\infty} |a_n|$ *is convergent. It is called conditionally convergent if it is convergent but not absolutely convergent.*

Theorem 6.13 states that every absolutely convergent series is convergent.

Example 6.18. *Show that the series* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ *is absolutely convergent.*

Example 6.19. *Show that the series* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ *is conditionally convergent.*

Example 6.20. *Show that the series* $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ *is absolutely convergent.*

6.10 Power Series

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots,$$

where x is a variable, and the c's are constants called coefficients of the series. For each fixed x, the power series is a series of numbers that we can test for convergence or divergence.

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots,$$

is called a power series centred at a or a power series about a.

Note that the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ always converges at $x = a$.

Example 6.21. *For what values of x is the series* $\sum_{n=0}^{\infty} n! x^n$ *convergent?*

Solution. If $x \neq 0$, then $\lim_{n\to\infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| = \lim_{n\to\infty} (n+1)|x| = \infty$. By ratio test, $\sum_{n=0}^{\infty} n! x^n$ diverges.

Therefore, $\sum_{n=0}^{\infty} n! x^n$ converges if and only if $x = 0$.

■

Example 6.22. *For what values of x is the series* $\displaystyle\sum_{n=0}^{\infty} \frac{(x-7)^n}{n}$ *convergent?*

Solution. $\displaystyle\lim_{n\to\infty} \left| \frac{\frac{(x-7)^{n+1}}{n+1}}{\frac{(x-7)^n}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1}|x-7| = |x-7|$. By ratio test, we have the following conclusions (i) and (ii).

(i) If $|x-7| < 1$, then $\displaystyle\sum_{n=0}^{\infty} \frac{(x-7)^n}{n}$ is absolutely convergent.

(ii) If $|x-7| > 1$, then $\displaystyle\sum_{n=0}^{\infty} \frac{(x-7)^n}{n}$ is divergent.

(iii) If $x = 6$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$, which is an alternating series. By the alternating series test, it is convergent.

(iv) If $x = 8$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n}$, which is the harmonic series which is divergent.

Summarizing, the series $\displaystyle\sum_{n=0}^{\infty} \frac{(x-7)^n}{n}$ is convergent if and only if $x \in [6, 8)$.

■

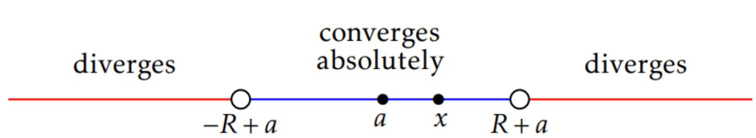
Theorem 6.14. *For a given power series* $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$, *exactly one of the following possibilities holds:*

(i) The series converges at $x = a$ only.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges absolutely if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R in case (iii) is called the radius of convergence of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii). The interval of convergence of a power series is the interval consisting of all values of x for which the series converges.



The interval of convergence can be $(a-R, a+R), [a-R, a+R), (a-R, a+R], [a-R, a+R]$.

In some cases, we can compute R by the following method.

Theorem 6.15. *Consider the power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$, *where* $c_n \neq 0$ *for all* n. *If* $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L$ *or* $\lim_{n\to\infty}\sqrt[n]{|c_n|} = L$, *where* L *is a real number or* ∞, *then* $R = \frac{1}{L}$.

By convention, if $L = 0$, *then* $R = \infty$, *and if* $L = \infty$, *then* $R = 0$.

Proof. Suppose $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L$.

Thus $\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right||x-a| = L|x-a|$.

By ratio test, the series converges absolutely for $L|x-a| < 1$, that is $|x-a| < \frac{1}{L}$; and the series diverges for $L|x-a| > 1$, that is $|x-a| > \frac{1}{L}$. Therefore the radius of convergence is $\frac{1}{L}$.

The second case where $\lim_{n\to\infty}\sqrt[n]{|c_n|} = L$ follows similarly by root test.

■

Example 6.23. *Find the radius of convergence and interval of convergence of the power series* $\sum_{n=0}^{\infty}\frac{(-3)^n x^n}{\sqrt{n+1}}$.

Solution. $\lim_{n\to\infty}\left|\frac{\frac{(-3)^{n+1}}{\sqrt{n+2}}}{\frac{(-3)^n}{\sqrt{n+1}}}\right| = \lim_{n\to\infty} 3\sqrt{\frac{n+1}{n+2}} = 3$. Thus the radius of convergence is $\frac{1}{3}$.

When $x = \frac{1}{3}$, the series becomes $\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$, which is an alternating series. By the alternating series test, it is convergent.

When $x = -\frac{1}{3}$, the series becomes $\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}}$, which is divergent by integral test.

Consequently, the interval of convergent is $(-\frac{1}{3}, \frac{1}{3}]$.

■

Example 6.24. *Find the radius of convergence and interval of convergence of the power series* $\sum_{n=0}^{\infty}\frac{n(x+2)^n}{3^{n+1}}$.

Solution. $\lim_{n\to\infty}\left|\frac{\frac{n+1}{3^{n+2}}}{\frac{n}{3^{n+1}}}\right| = \lim_{n\to\infty}\frac{n+1}{3n} = \frac{1}{3}$. Thus the radius of convergence is 3. The centre of the power series is at $x = -2$. Next we consider the two endpoints $x = -5, 1$.

When $x = -5$, the series becomes $\sum_{n=0}^{\infty}\frac{n(-3)^n}{3^{n+1}}$, that is $\sum_{n=0}^{\infty}\frac{n(-1)^n}{3}$, which is divergent by the test for divergence.

When $x = 1$, the series becomes $\sum_{n=0}^{\infty}\frac{n(3)^n}{3^{n+1}}$, that is $\sum_{n=0}^{\infty}\frac{n}{3}$, which is also divergent by the test for divergence.

Consequently, the interval of convergent is $(-5, 1)$.

■

Example 6.25. *Find the radius of convergence and interval of convergence of the power series* $\sum_{n=1}^{\infty}\frac{(4x-3)^{2n+1}}{4^n n^{4/3}}$.

Solution. Note that we cannot apply Theorem 6.15 directly since all the coefficients of the even powers of x are zero. Let $u_n = \frac{(4x-3)^{2n+1}}{4^n n^{4/3}}$.

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(4x-3)^{2n+3}}{4^{n+1}(n+1)^{4/3}}\cdot\frac{4^n n^{4/3}}{(4x-3)^{2n+1}}\right|$$
$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{\frac{4}{3}}\frac{|4x-3|^2}{4} = \frac{|4x-3|^2}{4}.$$

By ratio test, the power series converges absolutely for $\frac{|4x-3|^2}{4} < 1 \Leftrightarrow \frac{|4x-3|}{2} < 1 \Leftrightarrow |x-\frac{3}{4}| < \frac{1}{2}$, and diverges for $|x-\frac{3}{4}| > \frac{1}{2}$.

Therefore, the radius of convergence is $\frac{1}{2}$.

At $x = \frac{5}{4}$, the series is $\sum_{n=1}^{\infty}\frac{2^{2n+1}}{4^n n^{4/3}} = \sum_{n=1}^{\infty}\frac{2}{n^{4/3}}$, which is convergent since it is a p-series with $p = \frac{4}{3} > 1$.

At $x = \frac{1}{4}$, the series is $\sum_{n=1}^{\infty}\frac{(-2)^{2n+1}}{4^n n^{4/3}} = \sum_{n=1}^{\infty}\frac{-2}{n^{4/3}}$, which is convergent since it is a p-series with $p = \frac{4}{3} > 1$.

Therefore, the interval of convergence is $[\frac{1}{4}, \frac{5}{4}]$.

■

6.11 Power Series Representation

Recall that for $|x| < 1$, the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{1}{1-x}.$$

$\sum_{n=0}^{\infty} x^n$ is called a power series representation of the function $\frac{1}{1-x}$ about $x = 0$.

Example 6.26. *Find a power series representation of $\frac{x^3}{x+2}$ about $x = 0$.*

Solution. We make use of the above geometric series.

$$\frac{x^3}{x+2} = \frac{x^3}{2}\frac{1}{1-(-\frac{x}{2})} = \frac{x^3}{2}(1 - \frac{x}{2} + \frac{x^2}{2^2} - \frac{x^3}{2^3} + \cdots + (-\frac{x}{2})^n + \cdots),$$

which is valid for $|-\frac{x}{2}| < 1$. That is

$$\frac{x^3}{x+2} = \sum_{n=0}^{\infty} \frac{1}{2}(-\frac{1}{2})^n x^{n+3},$$

for $|x| < 2$.

■

Example 6.27. *Find a power series representation of $\frac{1}{x^2+3x+2}$ about $x = 0$.*

Solution.

$$\frac{1}{x^2+3x+2} = \frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{1+x} - \frac{1}{2}\frac{1}{1+\frac{x}{2}} = \sum_{n=0}^{\infty}(-1)^n x^n - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(\frac{x}{2})^n.$$

Thus

$$\frac{1}{x^2+3x+2} = \sum_{n=0}^{\infty}(-1)^n(1 - \frac{1}{2^{n+1}})x^n,$$

for $|x| < 1$.

■

Theorem 6.16. *If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable on the interval $|x-a| < R$ and *(i)* $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$, *for* $|x-a| < R$.

(ii) $\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$, *for* $|x-a| < R$.

Example 6.28. *Find a power series representation of* $\ln(1-x)$ *and its radius of convergence.*

Solution. For $|x| < 1$, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Thus by Theorem 6.16 (ii),

$$-\ln(1-x) = \int \frac{1}{1-x}\,dx + C = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C,$$

for $|x| < 1$. When $x = 0$, we have $0 = 0 + C$ so that $C = 0$. Therefore,

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.$$

The radius of convergence is 1 by ratio test.

■

Example 6.29. *Find a power series representation of* $\tan^{-1} x$.

Solution. For $|x| < 1$, $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$. by Theorem 6.16 (ii),

$$\tan^{-1} x = \int \frac{1}{1+x^2}\,dx + C = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} + C.$$

When $x = 0$, we have $0 = 0 + C$ so that $C = 0$. Therefore,

$$\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}.$$

■

6.12 Taylor and Maclaurin Serise

By repeated use of Theorem 6.16 (i), we deduce the following.

Theorem 6.17. *If f has a power series representation at $x = a$, that is*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad |x-a| < R, \text{ for some } R > 0,$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

If f has a power series representation at $x = a$, then it is unique and has the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This is called the Taylor series of f at $x = a$.

The Maclaurin series of f is the special case of Taylor series when $a = 0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Exercise 6.3. *Assume that each of the functions e^x, $\sin x$ and $\cos x$ has a Maclaurin series representation. Find the Maclaurin series and its radius of convergence for (a) e^x, (b) $\sin x$, (c) $\cos x$.*

Ans: (a) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, (b) $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, (c) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.

All three series converge for all x in $\mathbb{R}$.

Chapter 7

Vectors and Geometry of Space

Read Thomas' Calculus, Chapter 12.

In this chapter,

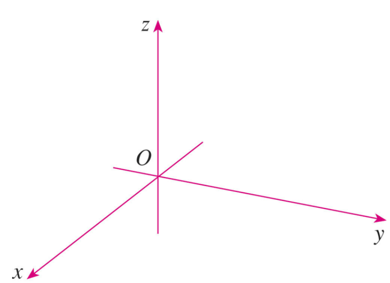
• we introduce the coordinate systems for three-dimensional space $\mathbb{R}^3$. This provides the setting for our study of calculus of functions of two and three variables. The simplest geometric notion is the distance between two points in space.

• we define vectors geometrically followed by the study their algebraic properties. We emphasize the power of algebraic manipulation of vectors. In particular, we look at dot product and cross product of vectors and their applications.

• we define lines and planes in $\mathbb{R}^3$.

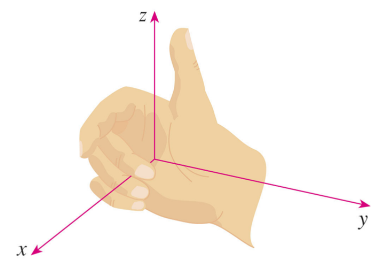
7.1 The 3D-Coordinate System

We set up the 3D coordinate system by fixing a point O in space (called the origin) and take three lines passing through O that are perpendicular to each other. These lines are labeled as x-axis, y-axis and z-axis respectively.

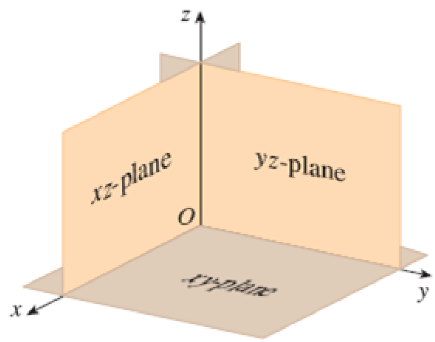


A point P in space can be represented by an ordered triple (a, b, c) where a , b and c are projections of the point P onto the x-, y- and z-axis respectively. The three dimensional space is also called the xyz-space.

The direction of the z-axis is determined by the right-hand rule:

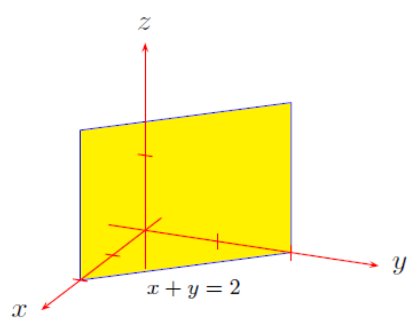


Any two of the axes determine a plane:



Example 7.1. *Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $x + y = 2$.*

Solution. Note that $x+y = 2$ represents a line on the xy-plane. However, in $\mathbb{R}^3$, it represents the plane containing all points whose x- and y-coordinate sum to 2. This is a vertical plane.



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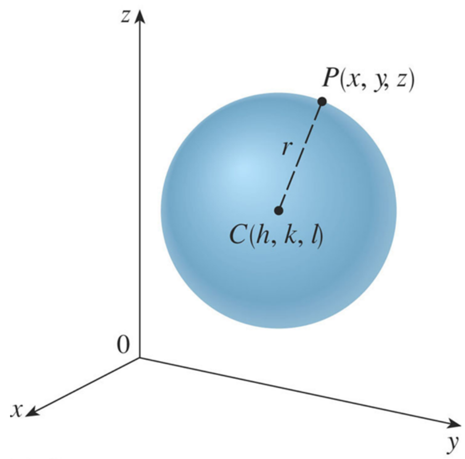
Theorem 7.1 (Distance Formula)**.**
The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Consequently, we have the following equation of a sphere:

Theorem 7.2 (Equation of Sphere)**.**
An equation of a sphere with center $C(h,k,l)$ and radius r is

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2.$$



7.2 Vectors

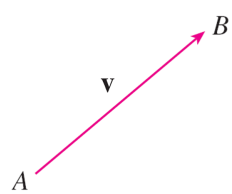
It can be complicated (messy) to describe an object in space directly using the coordinates x, y and z. It turns out to be easier by using **vectors**.

A vector is often represented by an arrow.

• The length of the arrow represents the magnitude of the vector.

• The arrow points in the direction of the vector.

For instance, suppose a particle moves along a line segment from point A to point B.



The vector $\mathbf{v}$ has initial point A (the tail) and terminal point B (the tip). We indicate this by writing $\mathbf{v}=\overrightarrow{AB}$. Call this the **displacement vector** of a particle from A to B.

We denote a vector by either:

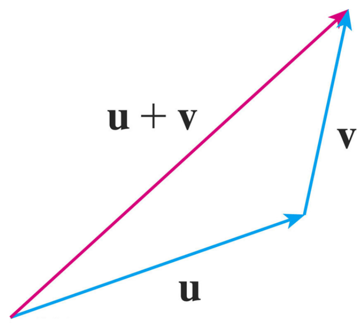
• Printing a letter in boldface $\mathbf{v}$, or

• Putting an arrow above the letter $\vec{v}$.

The zero vector, denoted by $\mathbf{0}$, has length 0. It is the only vector with no specific direction.

Note that a vector does not depend on its initial point.

Definition 7.1 (Adding Vectors – The Triangle Law)**.**
Let $\mathbf{u}$ *and* $\mathbf{v}$ *be two vectors. Then their sum* $\mathbf{u}+\mathbf{v}$ *is the vector from the initial point of* $\mathbf{u}$ *to the terminal point of* $\mathbf{v}$ *when we position the vectors so that the initial point of* $\mathbf{v}$ *coincide with the terminal point of* $\mathbf{u}$.



Definition 7.2 (Scalar Multiplication)**.**
Let $c \in \mathbb{R}$ *and* $\mathbf{u}$ *be a vector. The scalar multiple* $c\mathbf{u}$ *is the vector whose length is* $|c|$ *times the length of* $\mathbf{u}$ *and whose direction is the same as* $\mathbf{u}$ *if* $c > 0$ *and is opposite to* $\mathbf{u}$ *if* $c < 0$. *If* $c = 0$ *or* $\mathbf{u} = \mathbf{0}$, *then* $c\mathbf{u} = \mathbf{0}$.

Notice two nonzero vectors are parallel if they are scalar multiple of each other. By the difference $\mathbf{u}-\mathbf{v}$, we mean

$$\mathbf{u}-\mathbf{v} = \mathbf{u}+(-\mathbf{v}).$$

To treat vectors systematically (algebraically), we place the initial point of a vector $\mathbf{u}$ at the origin O. In doing so, the terminal point has the coordinate (u_1, u_2) or (u_1, u_2, u_3) for some $u_1, u_2, u_3 \in \mathbb{R}$, depending on whether $\mathbf{u}$ is a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Denote $\mathbf{u}$ by

$$\mathbf{u} = \langle u_1, u_2, u_3 \rangle.$$

u_1, u_2, u_3 are called the **components** of $\mathbf{u}$.

$\langle u_1, u_2, u_3 \rangle$ is also called the **position vector** of the point (u_1, u_2, u_3).

In other words, position vectors are vectors whose initial point is the origin.

Any vector can be represented by a position vector.

What is so nice about position vectors? The main advantage of representing vectors using position vectors is that we can simplify calculations by algebraic manipulations!

• To add position vectors, we can just add their corresponding components. If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ then

$$\mathbf{a}+\mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle.$$

• To multiply $\mathbf{a}$ by a scalar c, we multiply each component by that scalar.

$$c\mathbf{a} = \langle ca_1, ca_2, ca_3 \rangle.$$

Theorem 7.3.
Given the points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$, the vector $\mathbf{a}$ representing $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle.$$

Proof.

$$\begin{aligned}
\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\
&= \overrightarrow{OB} - \overrightarrow{OA} \\
&= \langle x_2, y_2, z_2\rangle - \langle x_1, y_1, z_1\rangle \\
&= \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle.
\end{aligned}$$

■

Theorem 7.4 (Properties of Vectors)**.**
Suppose $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors, and $c, d \in \mathbb{R}$ are scalars. Then

(1) $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$	(2) $\mathbf{a}+(\mathbf{b}+\mathbf{c}) = (\mathbf{a}+\mathbf{b})+\mathbf{c}$
(3) $\mathbf{a}+\mathbf{0} = \mathbf{a}$	(4) $\mathbf{a}+(-\mathbf{a}) = \mathbf{0}$
(5) $c(\mathbf{a}+\mathbf{b}) = c\mathbf{a}+c\mathbf{b}$	(6) $(c+d)\mathbf{a} = c\mathbf{a}+d\mathbf{a}$
(7) $(cd)\mathbf{a} = c(d\mathbf{a})$	(8) $1\mathbf{a} = \mathbf{a}$

These properties are readily verified geometrically or algebraically.

Three vectors in $\mathbb{V}_3$ play a special role. They are

$$\mathbf{i} = \langle 1,0,0\rangle, \mathbf{j} = \langle 0,1,0\rangle, \mathbf{k} = \langle 0,0,1\rangle.$$

These vectors are called the **standard basis vectors**. They have length 1 and point in the direction of the positive x-, y- and z-axes respectively. Thus, any vector $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ can be written as

$$\begin{aligned}
\mathbf{a} &= \langle a_1, a_2, a_3\rangle \\
&= a_1\langle 1,0,0\rangle + a_2\langle 0,1,0\rangle + a_3\langle 0,0,1\rangle \\
&= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.
\end{aligned}$$

The **length** of a vector $\mathbf{u}$ is the length of any of its representation, and is denoted by $\|\mathbf{u}\|$ (in the textbook, the notation $|\mathbf{u}|$ is used). Using the distance formula, we have the following

The length of the vector $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ is

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

A **unit vector** is a vector whose length is 1. For example, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors.

Theorem 7.5.
If $\mathbf{a} \neq \mathbf{0}$, then a unit vector in the same direction as $\mathbf{a}$ is given by

$$\mathbf{u} = \mathbf{a}/\|\mathbf{a}\|.$$

Notice $1/\|\mathbf{a}\|$ is a positive scalar, so $\mathbf{u}$ is in the same direction as $\mathbf{a}$. Now,

$$\|\mathbf{u}\| = \left\|\frac{\mathbf{a}}{\|\mathbf{a}\|}\right\| = \frac{1}{\|\mathbf{a}\|}\|\mathbf{a}\| = 1.$$

So $\mathbf{u}$ is the unit vector in the same direction as $\mathbf{a}$. ■

7.3 The Dot Product

The **dot product** of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is defined to be

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

The dot product satisfies the following properties

Theorem 7.6 (Properties of Dot Product)**.**
For vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and any scalar d,

(i) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ *(commutativity)*

(ii) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ *(distributive law)*

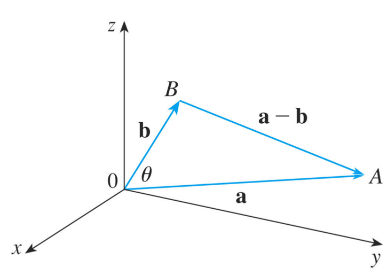
(iii) $(d\mathbf{a}) \cdot \mathbf{b} = d(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (d\mathbf{b})$

(iv) $\mathbf{0} \cdot \mathbf{a} = 0$

(v) $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$.

Notice $\mathbf{a} \cdot \mathbf{b} = 0$ does not imply that $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.

For two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{V}_3$, we define the **angle** θ between them to be the smaller angle between $\mathbf{a}$ and $\mathbf{b}$, formed by placing their initial points at the origin.



- $\mathbf{a}$ and $\mathbf{b}$ have the same direction iff $\theta = 0$.
- $\mathbf{a}$ and $\mathbf{b}$ have opposite direction iff $\theta = \pi$.
- $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (perpendicular) iff $\theta = \frac{\pi}{2}$.

Theorem 7.7.
Let θ be the angle between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$. Then

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta.$$

Proof. Recall that the Law of Cosines says that

$$\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\| \|\mathbf{b}\| \cos\theta.$$

$$\begin{aligned}
\|\mathbf{a} - \mathbf{b}\|^2 &= \|\langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle\|^2 \\
&= (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 \\
&= (a_1^2 - 2a_1 b_1 + b_1^2) + (a_2^2 - 2a_2 b_2 + b_2^2) \\
&\quad +(a_3^2 - 2a_3 b_3 + b_3^2) \\
&= (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) - 2(a_1 b_1 + a_2 b_2 + a_3 b_3) \\
&= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a} \cdot \mathbf{b}.
\end{aligned}$$

Rearranging,

$$\begin{aligned}
2\mathbf{a} \cdot \mathbf{b} &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a} - \mathbf{b}\|^2 \\
&= 2\|\mathbf{a}\| \|\mathbf{b}\| \cos\theta,
\end{aligned}$$

the last equality follows from the Law of Cosines.
So

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta.$$

■

Example 7.2. *Find the angle between the vectors* $\mathbf{a} = \langle 2, 1, -3 \rangle$ *and* $\mathbf{b} = \langle 1, 5, 6 \rangle$.

Solution. We have

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} = \frac{-11}{\sqrt{14}\sqrt{62}}.$$

It follows that

$$\theta = \cos^{-1}\left(\frac{-11}{\sqrt{14}\sqrt{62}}\right) \approx 1.953\text{radian}.$$

■

Theorem 7.8.
Two vectors $\mathbf{a}$ *and* $\mathbf{b}$ *are orthogonal if and only if* $\mathbf{a} \cdot \mathbf{b} = 0$.

Proof. If either $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$ then $\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{a}$ and $\mathbf{b}$ are orthogonal as $\mathbf{0}$ is considered to be orthogonal to every vector.

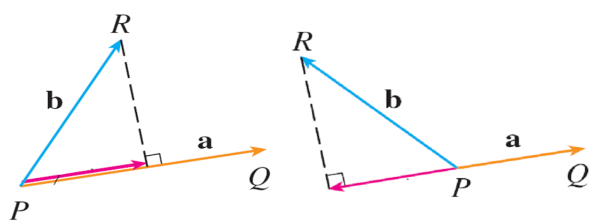
We may assume that $\mathbf{a}$ and $\mathbf{b}$ are nonzero. Then

$$\|\mathbf{a}\| \|\mathbf{b}\| \cos\theta = \mathbf{a} \cdot \mathbf{b} = 0$$

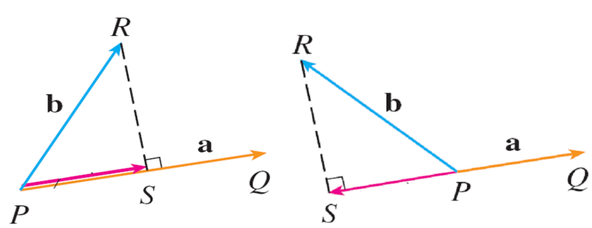
if and only if $\cos\theta = 0$, if and only if $\theta = \frac{\pi}{2}$, which is equivalent to saying that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal. ■

7.4 Projections

The figure shows two vectors $\mathbf{a}$ and $\mathbf{b}$ with the same initial point representing $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.



Let S be the foot of the perpendicular line from R to the line containing $\overrightarrow{PQ}$.



The vector $\overrightarrow{PS}$ is called the **vector projection** of $\mathbf{b}$ **onto** $\mathbf{a}$, denoted by

$$\text{proj}_{\mathbf{a}}\mathbf{b}.$$

The **scalar projection** of $\mathbf{b}$ **onto** $\mathbf{a}$ (also called the **component of** $\mathbf{b}$ **along** $\mathbf{a}$) is defined to be the signed magnitude of the vector projection, and is denoted by

$$\text{comp}_{\mathbf{a}}\mathbf{b}.$$

Notice

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \|\mathbf{b}\|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}.$$

This value is negative if $\frac{\pi}{2} < \theta \le \pi$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

Therefore,

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \text{comp}_{\mathbf{a}}\mathbf{b} \times \frac{\mathbf{a}}{\|\mathbf{a}\|} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\right)\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}.$$

Example 7.3. *Let* $\mathbf{a} = \langle -2, 3, 1\rangle$ *and* $\mathbf{b} = \langle 1, 1, 2\rangle$. *Find the scalar projection and vector projection of* $\mathbf{b}$ *onto* $\mathbf{a}$.

Solution. Notice $\|\mathbf{a}\| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$. So

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}.$$

It follows that

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{3}{\sqrt{14}}\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{3}{14}\mathbf{a} = \langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\rangle.$$

■

Theorem 7.9 (Distance from a point to a plane).
The (shortest) distance from a point $P(x_0, y_0, z_0)$ *to the plane* $ax + by + cz = d$ *is given by*

$$\frac{\left|ax_0 + by_0 + cz_0 - d\right|}{\sqrt{a^2 + b^2 + c^2}}.$$

Proof. A normal vector to the plane is $\mathbf{n} = \langle a, b, c\rangle$. (See Section 7.7.) Pick any point $Q(x_1, y_1, z_1)$ on the plane so that $ax_1 + by_1 + cz_1 = d$. Then the shortest distance from P to the plane is

$$\begin{aligned}
\left\|\text{proj}_{\mathbf{n}}\overrightarrow{QP}\right\| &= \left|\text{comp}_{\mathbf{n}}\overrightarrow{QP}\right| \\
&= \left|\frac{\overrightarrow{QP}\cdot\mathbf{n}}{\|\mathbf{n}\|}\right| \\
&= \frac{\left|\langle x_0 - x_1, y_0 - y_1, z_0 - z_1\rangle\cdot\langle a, b, c\rangle\right|}{\|\mathbf{n}\|} \\
&= \frac{\left|ax_0 + by_0 + cz_0 - d\right|}{\sqrt{a^2 + b^2 + c^2}}.
\end{aligned}$$

■

Example 7.4. *Find the distance from the point* $(2, -3, 4)$ *to the plane* $x + 2y + 3z = 13$.

Solution. We have $(x_0, y_0, z_0) = (2, -3, 4)$ and $a = 1, b = 2, c = 3, d = 13$. Using the formula, the distance is

$$\frac{|2(1) + (-3)(2) + 4(3) - 13|}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{5}{\sqrt{14}}.$$

■

7.5 The Cross Product

We now define a second type of product of vectors, called the cross product or vector product. While the dot product is a scalar, the cross product is a vector.

For two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, define the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ to be

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \\
&= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}.
\end{aligned}$$

To compute $\mathbf{a} \times \mathbf{b}$, we must write the components of $\mathbf{a}$ in the second row of the determinant, and the components of $\mathbf{b}$ in the third row. The order is important!

One of the most important properties of the cross product is the following theorem.

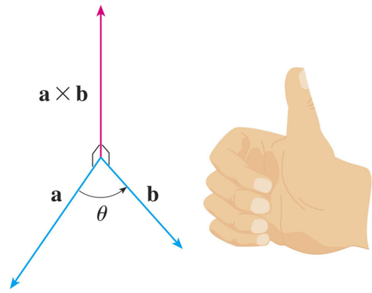
Theorem 7.10.
The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Proof. To show $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$, we compute their dot product as follows:

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\
&= 0.
\end{aligned}$$

A similar computation shows that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$. ■

The vector $\mathbf{a} \times \mathbf{b}$ points in a direction perpendicular to $\mathbf{a}$ and $\mathbf{b}$. This can be given by the right-hand rule as follows:



What is the geometric meaning of the length $\|\mathbf{a}\times\mathbf{b}\|$? This is given by the following theorem.

Theorem 7.11.
If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ then

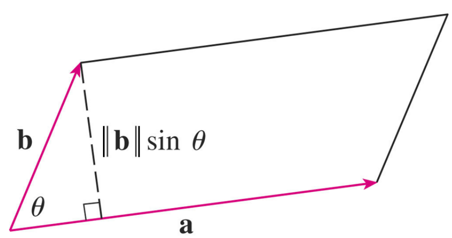
$$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta.$$

We can use cross product

- to find the area of a parallelogram
- to find the distance from a point to a line in $\mathbb{R}^3$.

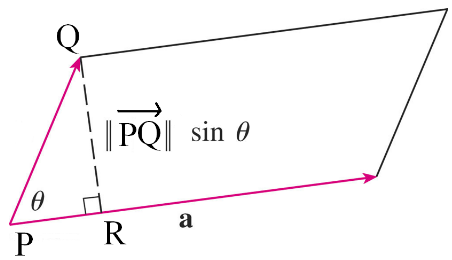
If $\mathbf{a}$ and $\mathbf{b}$ are represented by directed line segments with the same initial point then they determine a parallelogram with base $\|\mathbf{a}\|$, altitude $\|\mathbf{b}\|\sin\theta$. Therefore, the area of the parallelogram is given by

$$\|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta = \|\mathbf{a}\times\mathbf{b}\|.$$



The distance from Q to the line through P and R is

$$\left\|\overrightarrow{PQ}\right\|\sin\theta = \frac{\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\|}{\left\|\overrightarrow{PR}\right\|}.$$



Some of the usual laws of algebra hold for cross products.

Theorem 7.12.
If $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *are vectors and* d *is a scalar, then*

(i) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

(ii) $(d\mathbf{a}) \times \mathbf{b} = d(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (d\mathbf{b})$

(iii) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

(iv) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

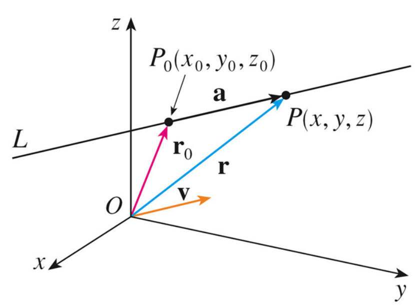
7.6 Lines

How do we write down an equation of a line in space? To do it, we must describe the behavior of a general point on the line. We shall see that vectors can help us to achieve this goal with minimal effort.

So let $P(x, y, z)$ denote an arbitrary point on the line L.

Let $\mathbf{r}$ and $\mathbf{r}_0$ denote the position vectors of P and P_0 respectively, where P_0 is a point on L which we have fixed. Our aim is to describe $\mathbf{r}$, the position vector of an arbitrary point on L.

Let $\mathbf{v}$ be a vector parallel to L, so $\overrightarrow{P_0P} = t\mathbf{v}$ for some scalar t since $\overrightarrow{P_0P}$ and $\mathbf{v}$ are parallel.



Then

$$\mathbf{r} = \mathbf{r}_0 + \overrightarrow{P_0P},$$

so that

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

which is a **vector equation** of L.

Each **parameter** t gives the position vector $\mathbf{r}$ of a point on L. As t varies, the line is traced out by the tip of the vector $\mathbf{r}$.

We can write the vector equation in the component form:

$$\mathbf{v} = \langle a, b, c \rangle, \quad \mathbf{r_0} = \langle x_0, y_0, z_0 \rangle, \quad \mathbf{r} = \langle x, y, z \rangle.$$

Two vectors are equal if and only if the corresponding components are equal. Therefore, we have

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$
$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle.$$

Theorem 7.13 (Parametric Equation of Line)**.**

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

Usually the parameter t (in the parametric equation of line) takes values on the entire $\mathbb{R}$ or an interval I.

The numbers a, b and c are called **direction numbers** of the line L.

The vector equation and parametric equations of a line are not unique.

If we change the point $\mathbf{r}_0$ or the parameter t or choose a different parallel vector $\mathbf{v}$, then the equations change. Therefore, direction numbers are not unique.

Example 7.5. *Find an equation of the line passing through $P(1,2,-1)$ and $Q(5,-3,4)$.*

Solution. A vector parallel to the line is

$$\overrightarrow{PQ} = \langle 5-1, -3-2, 4-(-1) \rangle = \langle 4, -5, 5 \rangle.$$

Pick a point on the line, say $(1,2,-1)$. Then the parametric equations for the line are

$$x = 1 + 4t, y = 2 - 5t, z = -1 + 5t.$$

■

Let L_1 and L_2 be two lines in $\mathbb{R}^3$, with parallel vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, and let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$.

• The lines L_1 and L_2 are parallel whenever $\mathbf{a}$ and $\mathbf{b}$ are parallel.

• If L_1 and L_2 intersect then θ is an angle between L_1 and L_2. Notice $\pi - \theta$ is also an angle between the lines.

In 2-D, two lines are either parallel or intersect. This is not true in 3-D. Nonparallel, nonintersecting lines are called **skew** lines.

Example 7.6. *Show that the lines*

$$L_1 : x - 2 = -t, y - 1 = 2t, z - 5 = 2t,$$

$$L_2 : x - 1 = s, y - 2 = -s, z - 1 = 3s,$$

are skew.

Solution. The lines are not parallel since a vector parallel to L_1 is $\mathbf{a} = \langle -1, 2, 2\rangle$ and a vector parallel to L_2 is $\mathbf{b} = \langle 1, -1, 3\rangle$. Since $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, these vectors are not parallel.

Assume for a contradiction that L_1 and L_2 intersect. Then there must exist a choice of the parameter t and s such that the values of x, y and z are the same. In particular, for the x-coordinate,

$$2 - t = 1 + s,$$

so that $s = 1 - t$.

On the other hand, the y-coordinate must satisfy

$$y = 1 + 2t = 2 - s.$$

Substituting $s = 1 - t$ into the last equation, we have $t = 0$ and so $s = 1$.

Now, the z-coordinate must satisfy

$$z = 5 + 2t = 5,$$

$$z = 1 + 3s = 4,$$

which is absurd!

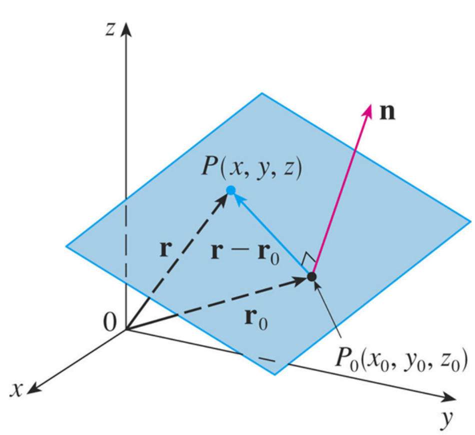
Hence our assumption that L_1 and L_2 intersects was wrong. So the lines are skew, as desired.
■

7.7 Planes

To get an equation for the plane, we need to describe an arbitrary point $P(x, y, z)$ on the plane. Again, we use vectors to help us do that.

Let $\mathbf{r}$ and $\mathbf{r}_0$ denote the position vectors of P and P_0 respectively, where P_0 is a fixed point on the given plane.

Then $\mathbf{r} - \mathbf{r}_0$ is represented by $\overrightarrow{P_0P}$.



The normal vector $\mathbf{n}$ (which is orthogonal to the plane) is always orthogonal to $\mathbf{r} - \mathbf{r}_0$. Therefore we have

Theorem 7.14 (Vector Equation of Plane)**.**

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

which can be written as

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0.$$

To obtain a scalar equation for the plane, write the vectors in component form and equate corresponding components:

$$\mathbf{n} = \langle a, b, c \rangle, \quad \mathbf{r} = \langle x, y, z \rangle, \quad \mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle.$$

Then $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$ becomes

$$\langle a, b, c \rangle \cdot \langle x, y, z \rangle = \langle a, b, c \rangle \cdot \langle x_0, y_0, z_0 \rangle$$

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

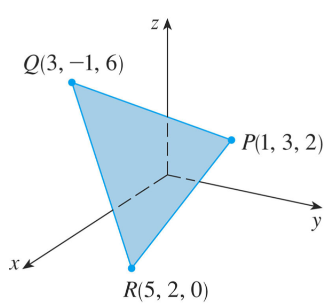
Theorem 7.15 (Linear Equation of Plane)**.**

$$ax + by + cz + d = 0,$$

where

$$d = -(ax_0 + by_0 + cz_0).$$

Example 7.7. *Find an equation of the plane that passes through the points* $P(1, 3, 2)$, $Q(3, -1, 6)$, $R(5, 2, 0)$.



Solution. First, we need a vector $\mathbf{n}$ orthogonal to the plane. This can be given by

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$$

Notice

$$\overrightarrow{PQ} = \langle 2, -4, 4\rangle, \overrightarrow{PR} = \langle 4, -1, -2\rangle.$$

So

$$\begin{aligned}\mathbf{n} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} \\ &= 12\mathbf{i} + 20\mathbf{j} + 14\mathbf{k}.\end{aligned}$$

With the point $P(1,3,2)$ and the normal vector $\mathbf{n}$, an equation of the plane is:

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

or after simplifications,

$$6x + 10y + 7z = 50.$$

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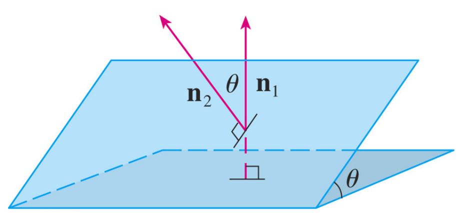
Two planes are parallel if their normal vectors are parallel.

If two planes are not parallel, then

• They intersect in a straight line.

• The angle between the two planes is defined as the **acute** angle between their normal vectors



Example 7.8. *(a) Find the angle between the planes $x + 2y + z = 3$ and $x - 4y + 3z = 5$.*

(b) Find the line of intersection of these two planes.

Solution. (a) The normal vectors of these planes are

$$\mathbf{n}_1 = \langle 1, 2, 1\rangle, \mathbf{n}_2 = \langle 1, -4, 3\rangle.$$

So, if θ is the angle between them, then

$$\begin{aligned}\theta &= \cos^{-1}\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} \\ &= \cos^{-1}\frac{1(1) + 2(-4) + 1(3)}{\sqrt{1+4+1}\sqrt{1+16+9}} \\ &= \cos^{-1}\frac{-4}{\sqrt{156}} \approx 108.7^\circ\end{aligned}$$

Therefore, the angle between the planes is 71.3°.

(b) Solving both equations for x,

$$x = 3 - 2y - z \quad \text{and} \quad x = 5 + 4y - 3z.$$

Setting them to be equal gives

$$3 - 2y - z = 5 + 4y - 3z.$$

Solving for z gives

$$z = 3y + 1.$$

Substituting this into the first equation,

$$x = 3 - 2y - (3y + 1) = -5y + 2.$$

Let $y = t$ be the parameter, we obtain a parametric equation for the line of intersection

$$x = -5t + 2, \quad y = t, \quad z = 3t + 1.$$

■

Exercise 7.1. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be vectors in* $\mathbb{R}^3$. *Prove that* $(\mathbf{a} \cdot \mathbf{b})^2 + |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2$.

Exercise 7.2. *Find the distance from the point* $P(3,3,3)$ *to the line* $\ell: \; x = t, y = \frac{t}{2}, z = t$.

Ans. $\sqrt{2}$.

Exercise 7.3. *Find the point on the surface* $z = x^2 + y^2 + 10$ *that is nearest to the plane* $x + 2y - z = 0$.

Ans. $(\frac{1}{2}, 1, \frac{45}{4})$.

Chapter 8

Functions of Several Variables

Read Thomas' Calculus, Chapter 14.

8.1 Vector Functions of One Variable

Recall the vector equation of line:

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}.$$

We have seen that the tip of the vector $\mathbf{r}(t)$ traces out a line as t varies.

We can rewrite the above as follows:

$$\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle.$$

Notice that each component of $\mathbf{r}(t)$ is a scalar function of t.

In general, a vector-valued function is

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$$

where $f(t)$, $g(t)$ and $h(t)$ are scalar functions of t.

Formally,

A **vector-valued function** $\mathbf{r}(t)$ is a mapping from its domain $D \subseteq \mathbb{R}$ to its range $R \subseteq \mathbb{V}_3$, so that for each $t \in D$, $\mathbf{r}(t) = \mathbf{v}$ for exactly one vector $\mathbf{v} \in \mathbb{V}_3$.

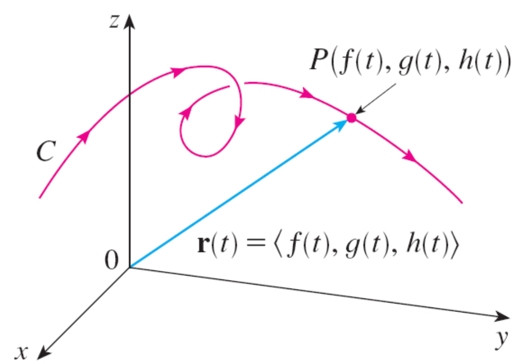
We write a vector-valued function as

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

or

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$$

for some scalar function f, g and h (called the **component functions** of $\mathbf{r}$).



Suppose $\mathbf{r}(t)$ traces out the curve C, we say that $\mathbf{r}(t)$ is a **parametrization** of C.

A curve C can have more than one parametrizations.

For example, both

$$\mathbf{r}(t) = \langle t, t^2 \rangle, \quad t \in \mathbb{R}$$

$$\mathbf{r}(t) = \langle t^3, t^6 \rangle, \quad t \in \mathbb{R}$$

parameterize the parabola $f(x) = x^2$ on the xy-plane.

Example 8.1. *Sketch the curve traced out by the vector-valued function* $\mathbf{r}(t) = \sin t\mathbf{i} - 3\cos t\mathbf{j} + 2t\mathbf{k}$.

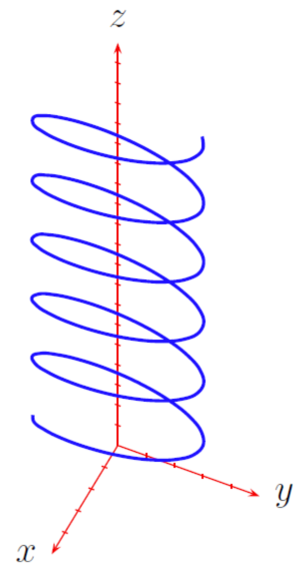
Solution. There is a relationship between x and y here:

$$x^2 + \left(\frac{y}{3}\right)^2 = \sin^2 t + \cos^2 t = 1$$

which is the equation of an ellipse in 2-D. In 3-D, since the equation does not involve z, it becomes the equation of an elliptic cylinder whose axis is the z-axis.

The curve will wind its way up the cylinder anticlockwise as t increases.

We call this curve an **elliptical helix**.



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8.2 Calculus of Vector Functions

To extend differentiation and integration to vector-valued functions, just think

'component-wise'!!!

Definition 8.1. *The* **derivative** $\mathbf{r}'(t)$ *of the vector-valued function* $\mathbf{r}(t)$ *is defined by*

$$\mathbf{r}'(t) = \lim_{\triangle t \to 0} \frac{\mathbf{r}(t+\triangle t) - \mathbf{r}(t)}{\triangle t}$$

for any values of t for which the limit exists.

When the limit exists for $t = a$, we say that $\mathbf{r}$ is **differentiable** at $t = a$.

The derivative of a vector-valued function can be found directly from the derivatives of the components.

Theorem 8.1 (Derivative of Vector-valued Function)**.**
Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ *and suppose that the components* f, g *and* h *are all differentiable at* $t = a$. *Then* $\mathbf{r}$ *is differentiable at* $t = a$ *and its derivative is given by*

$$\mathbf{r}'(a) = \langle f'(a), g'(a), h'(a) \rangle.$$

Theorem 8.2 (Derivative Rules)**.**
Suppose $\mathbf{r}(t)$ *and* $\mathbf{s}(t)$ *are differentiable vector-valued functions,* $f(t)$ *is a differentiable scalar function and* c *is a scalar constant. Then*

(i) $\frac{d}{dt}(\mathbf{r}(t)+\mathbf{s}(t)) = \mathbf{r}'(t)+\mathbf{s}'(t)$

(ii) $\frac{d}{dt}(c\mathbf{r}(t)) = c\mathbf{r}'(t)$

(iii) $\frac{d}{dt}f(t)\mathbf{r}(t) = f'(t)\mathbf{r}(t)+f(t)\mathbf{r}'(t)$

(iv) $\frac{d}{dt}\mathbf{r}(t)\cdot\mathbf{s}(t) = \mathbf{r}'(t)\cdot\mathbf{s}(t)+\mathbf{r}(t)\cdot\mathbf{s}'(t)$

(v) $\frac{d}{dt}(\mathbf{r}(t)\times\mathbf{s}(t)) = \mathbf{r}'(t)\times\mathbf{s}(t)+\mathbf{r}(t)\times\mathbf{s}'(t)$.

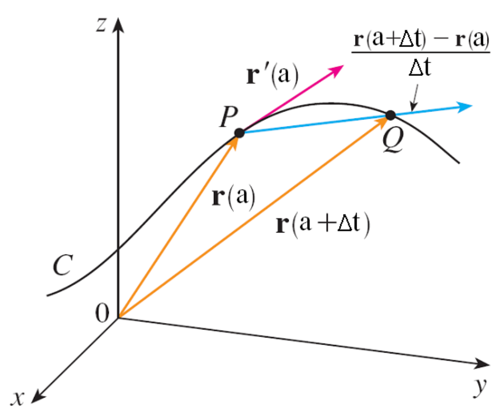
8.3 Tangent Vector and Tangent Line to a Curve

Recall that one interpretation of the derivative of a scalar function is that the value of the derivative at a point gives the slope of the tangent line to the curve at that point.

There is a similar interpretation for the derivative of vector-valued functions.

Recall

$$\mathbf{r}'(a) = \lim_{\triangle t\to 0}\frac{\mathbf{r}(a+\triangle t)-\mathbf{r}(a)}{\triangle t}.$$



The tangent vector

Notice that for $\triangle t>0$, the vector $\frac{\mathbf{r}(a+\triangle t)-\mathbf{r}(a)}{\triangle t}$ points in the same direction as $\mathbf{r}(a+\triangle t)-\mathbf{r}(a)$.

As $\triangle t\to 0$, $\frac{\mathbf{r}(a+\triangle t)-\mathbf{r}(a)}{\triangle t}$ approaches $\mathbf{r}'(a)$.

This is a vector tangent to the curve at $\mathbf{r}(a)$. We call $\mathbf{r}'(a)$ a **tangent vector** to the curve at $t=a$.

Example 8.2. *Find the tangent line L to the curve $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ at $(0, 1, \pi/2)$.*

Solution. At point $(0, 1, \pi/2)$, $t = \pi/2$. Since $\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle$, a direction of the tangent line L is

$$\mathbf{r}'(\pi/2) = \langle -\sin(\pi/2), \cos(\pi/2), 1\rangle = \langle -1, 0, 1\rangle$$

So a parametric equation of L is

$$x = 0 + (-1)t, \quad y = 1 + (0)t, \quad z = \pi/2 + (1)t,$$

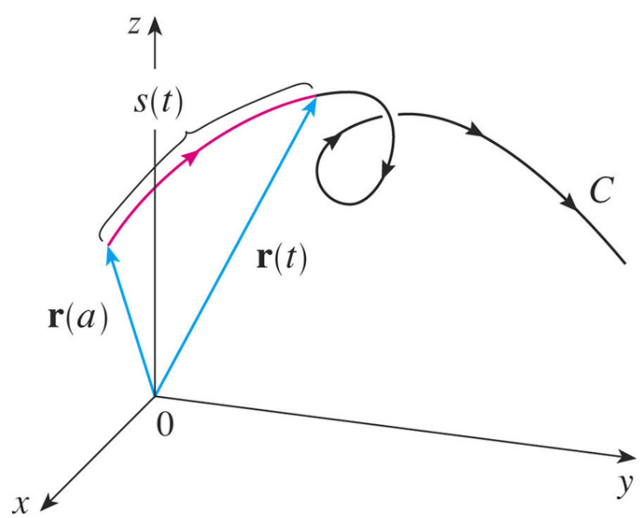
that is

$$x = -t, \quad y = 1, \quad z = \pi/2 + t.$$

■

8.4 Arc Length of a Space Curve

Suppose that a smooth curve C is traced out by the endpoint of $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ where f, f', g, g', h, h' are all continuous for $t \in [a, b]$ where the curve is traversed exactly once as t increases from a to b. Then the arc length of C is given by the following result.



Theorem 8.3 (Arc Length Formula)**.**
Let C be the curve given by

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle, \quad a \le t \le b$$

where f', g' and h' are continuous. If C is traversed exactly once as t increases from a to b, *then its length is*

$$\begin{aligned} s &= \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\,dt \\ &= \int_a^b \left\|\mathbf{r}'(t)\right\| dt \end{aligned}$$

Example 8.3. *Find the arclength of the curve traced out by the endpoint of the vector-valued function* $\mathbf{r}(t) = \langle 2t, \ln t, t^2 \rangle$ *for* $1 \le t \le e$.

Solution.

$$\begin{aligned} s &= \int_1^e \sqrt{2^2 + \left(\frac{1}{t}\right)^2 + (2t)^2}\,dt \\ &= \int_1^e \sqrt{\frac{1 + 4t^2 + 4t^4}{t^2}}\,dt \\ &= \int_1^e \sqrt{\frac{(1 + 2t^2)^2}{t^2}}\,dt \\ &= \int_1^e \frac{1 + 2t^2}{t}\,dt = \int_1^e \left(\frac{1}{t} + 2t\right) dt \\ &= \left(\ln|t| + t^2\right)\Big|_1^e \\ &= (\ln e + e^2) - (\ln 1 + 1) = e^2. \end{aligned}$$

■

So far we have seen functions of one variable, i.e. the domain is a subset of $\mathbb{R}$

function	Domain D	Range R
(scalar) $f(t)$	$D \subseteq \mathbb{R}$	$R \subseteq \mathbb{R}$
(vector) $\mathbf{r}(t)$	$D \subseteq \mathbb{R}$	$R \subseteq \mathbb{V}_2$ or $\mathbb{V}_3$

However, in the real world, physical quantities often depend on two or more variables.

8.5 Functions of Two Variables

Definition 8.2.
A function f of two variables is a rule that assigns to each **ordered pair** *of real numbers (x, y) in a set $D \subseteq \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ a* **unique** *real number denoted by $f(x, y)$.*

If a function f is given by a formula and no domain is specified, then the domain of f is understood to be:

the set of all pairs (x, y) for which the given expression is a well-defined real number.

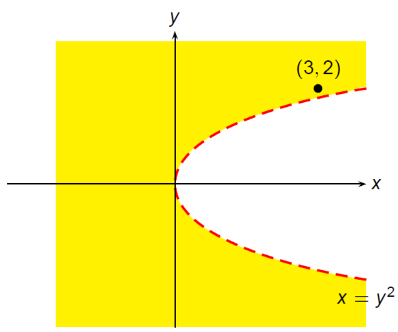
Example 8.4. *Find the domain of*

$$f(x, y) = x \ln(y^2 - x).$$

Solution. $\ln(y^2 - x)$ is defined only when $y^2 - x > 0$, that is, $x < y^2$.

So the domain of f is

$$D = \{(x, y) : x < y^2\}.$$



Domain of the function $f(x, y) = x \ln(y^2 - x)$

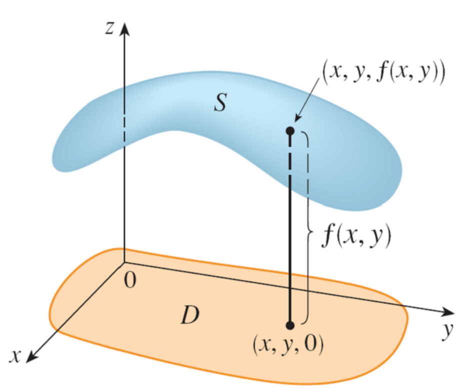
■

One way of visualizing $z = f(x, y)$ is to draw its graph.

If f is a function of two variables with domain D, then the **graph** of f is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ and $(x, y) \in D$.

The graph of a function f of two variables is also called the **surface** S with equation $z = f(x, y)$.

We can visualize the graph S of f as lying directly above or below its domain D in the xy-plane.



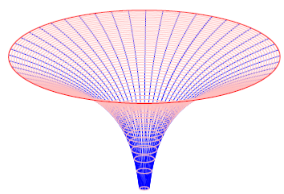
Graphing functions of more than one variable is not simple!

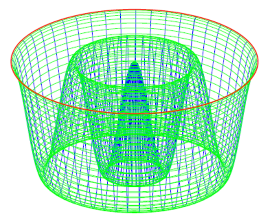
For most functions of two variables, to identify the surface, we must

(1) take hints from the expressions $z = f(x,y)$

(2) think through the traces and piece together the clues

Example 8.5. *Match the functions* $f(x,y) = \ln(x^2 + y^2)$ *and* $g(x,y) = \cos(x^2 + y^2)$ *to the surfaces shown below:*





Solution. Notice both functions contain the expression $x^2 + y^2$. This is significant: given any value r and any point (x,y) on the circle $x^2 + y^2 = r^2$, the height of the surface at the point (x,y) is a constant. That is, the surface has circular cross sections parallel; to the xy-plane.

However, both surfaces shown have this property, so we cannot yet tell which surface is matched by which function.

Notice the cosine of any angle lies between 1 and -1. So the second graph is $g(x,y)$.

An important property of $f(x,y)$ is that the logarithm tends to $-\infty$ as its argument $x^2 + y^2$ approaches 0. This appears in the first graph.

■

Another way to visualize functions of several variables is to use the contour plot which provide the same information condensed into a 2-D picture.

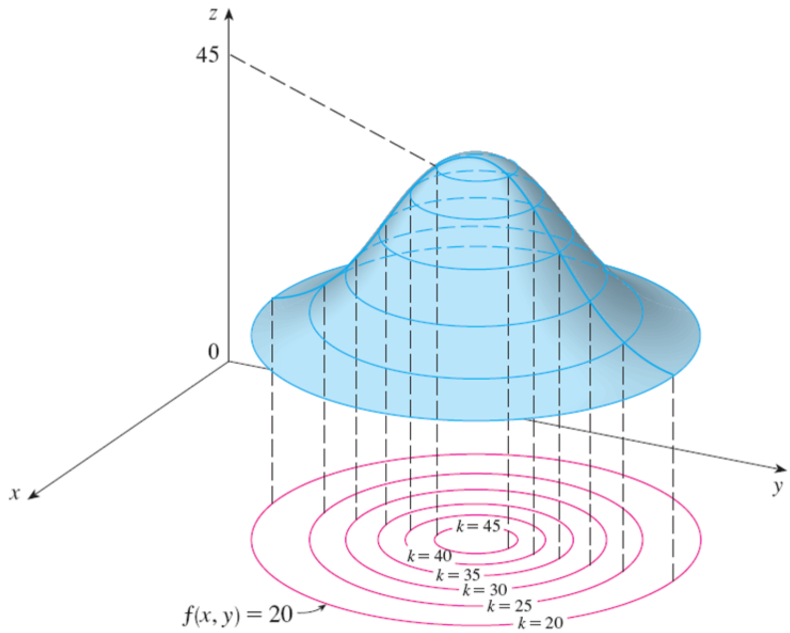
Definition 8.3 (Level Curve)**.**
A **level curve** *of* $f(x,y)$ *is the two-dimensional graph of the equation* $f(x,y) = k$ *for some constant* k.

Definition 8.4 (Contour Plot)**.**
A **contour plot** *of* $f(x,y)$ *is a graph of numerous level curves* $f(x,y) = k$, *for representative values of* k.

To sketch contour plots, we use values of k that are **equally spaced**. The surface is:

• steep where the level curves are close together.

• flatter where the level curves are farther apart.

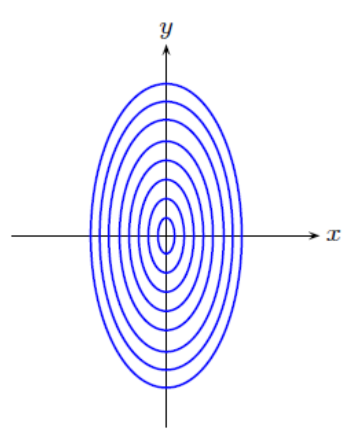


Example 8.6. *Sketch some level curves of* $h(x,y) = 4x^2 + y^2$.

Solution. If $k < 0$, then $4x^2 + y^2 = k$ has no solution, so there is no level curves for $k < 0$. If $k = 0$, then there is only one solution $(0,0)$, so the level curve is a single point $(0,0)$. If $k > 0$, then $4x^2 + y^2 = k$ is an ellipse:

$$\frac{x^2}{(\sqrt{k}/2)^2} + \frac{y^2}{\sqrt{k}^2} = 1.$$

Thus, larger k gives rise to an ellipse with longer major and minor axes.



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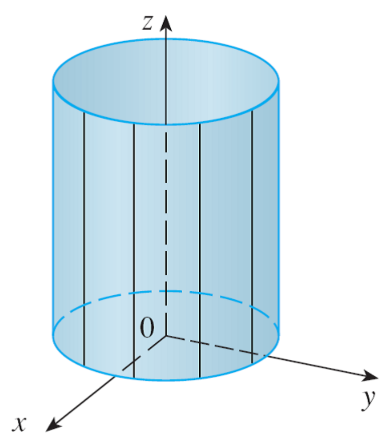
8.6 Cylinders and Quadric Surfaces

We have seen that the graph of functions of two variables are surfaces in space $\mathbb{R}^3$. To appreciate the calculus of functions of two variables, we need more examples of surfaces in space other than planes ($ax + by + cz = d$)and spheres ($(x-a)^2 + (y-b)^2 + (z-c)^2 = d^2$).

It is not easy to draw surfaces in $\mathbb{R}^3$. Our goal here is to identify and sketch some special type surfaces, namely the **cylinders** and some **quadric surfaces**.

8.6.1 Cylinders

When we mention the word cylinder, we probably think of the following object



Actually, the term cylinder is used to refer to a surface more general that the one we saw.

Definition 8.5.
A surface is a cylinder if there is a plane P such that all the planes parallel to P intersect the surface in the same curve (when viewed in 2-dimension).

Example 8.7. *Show that the surface given by*

$$y^2 + z^2 = 1$$

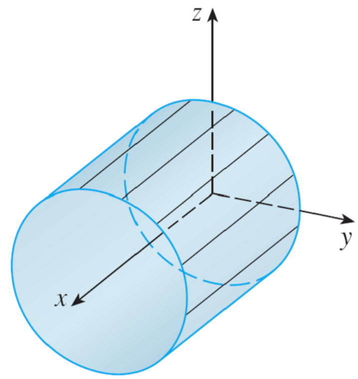
is a cylinder.

Solution. Notice x is missing in the equation. When $x = 0$, $y^2 + z^2 = 1$ is a circle with radius 1 in the yz-plane, which is the intersection of the surface and the yz-plane.

Generally, $x = k$ represent a plane parallel to the yz-plane, and the intersection the surface and this plane is always the circle $y^2 + z^2 = 1$.

Therefore, the surface $y^2 + z^2 = 1$ is a cylinder.

In fact, any equation in x, y and z where one of the variable is missing is a cylinder.



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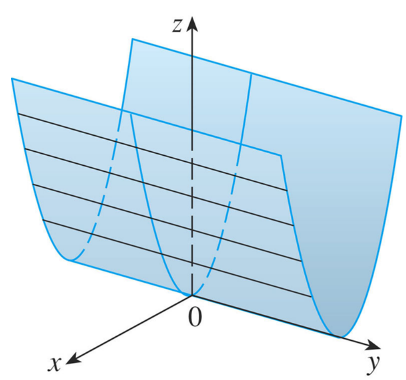
Example 8.8. *Sketch the graph of the surface $z = x^2$.*

Solution. Notice that the equation of the graph, $z = x^2$, does not involve y.

This means that any vertical plane with equation $y = k$ (parallel to the xz-plane) intersects the graph in a curve with equation $z = x^2$. So the surface $z = x^2$ is a cylinder.



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8.6.2 Quadric Surface

Definition 8.6 (Quadric Surface)**.**
A **quadric surface** *is the graph of a second-degree equation in three variables x, y and z:*

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where A, B, …, J are constants.

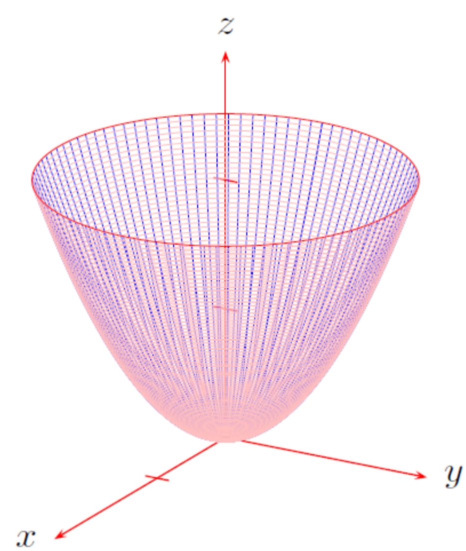
There are six basic quadric surfaces. But we shall focus on two of them:

- Elliptic paraboloid
- Ellipsoid

Definition 8.7 (Elliptic paraboloid – symmetric about the z-axis)**.**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

The graph of the elliptic paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$ when $c > 0$.



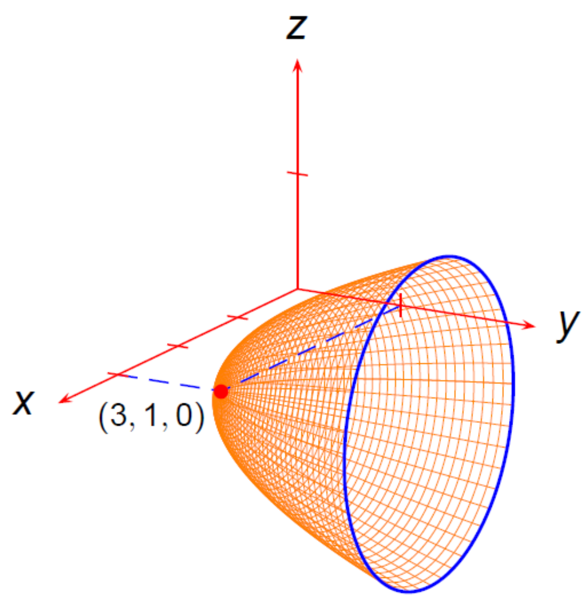
Example 8.9. *Identify and sketch the surface*

$$x^2 + 2z^2 - 6x - y + 10 = 0.$$

Solution. By completing squares, we rewrite the equation as

$$(y-1) = (x-3)^2 + \frac{z^2}{1/2}$$

It represents an elliptic paraboloid. However, it has been shifted so that its vertex is the point $(3,1,0)$, and is symmetric about the line which is parallel to the y-axis and passes through $(3,1,0)$.



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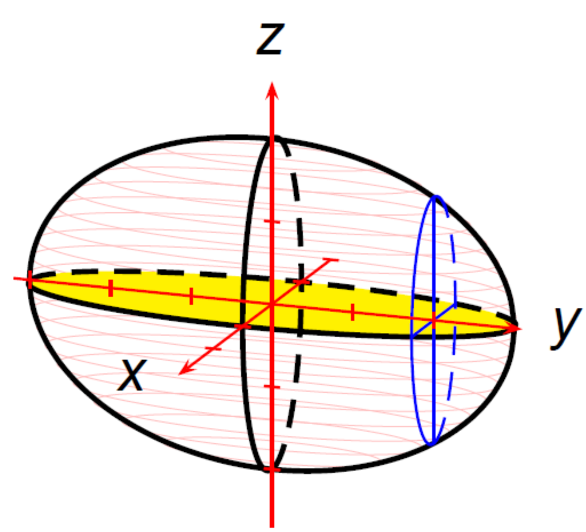
Definition 8.8 (Ellipsoid)**.**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

If $a=b=c$, then the ellipsoid is a sphere.



Example 8.10. *Sketch the quadric surface with equation*

$$x^2+\frac{y^2}{9}+\frac{z^2}{4}=1.$$

Solution. The surface intersects the xy-plane ($z = 0$) in the ellipse

$$x^2 + \frac{y^2}{9} = 1.$$

In general, the surface intersects the plane $z = k$ in the ellipse

$$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4},$$

provided $1 - \frac{k^2}{4} > 0$, that is $-2 < k < 2$.

The surface also intersects every planes $x = k$ which is parallel to the yz-plane ($x = 0$) in ellipse

$$\frac{y^2}{9} + \frac{z^2}{4} = 1 - k^2,$$

provided $1 - k^2 > 0$, that is $-1 < k < 1$.

The surface also intersects every planes $y = k$ which is parallel to the xz-plane ($y = 0$) in ellipse

$$x^2 + \frac{z^2}{4} = 1 - \frac{k^2}{9},$$

provided $1 - \frac{k^2}{9} > 0$, that is $-3 < k < 3$.

■

8.7 Functions of Three Variables

Definition 8.9.
A function f of three variables is a rule that assigns to each **ordered triple** *of real numbers (x, y, z) in a set $D \subseteq \mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ a* **unique** *real number denoted by $f(x, y, z)$.*

Unlike functions of two variables, it is very difficult to visualize a function f of three variables by its graph.

That would lie in a four-dimensional space!!!

However, we do gain some insight into f by examining its level surfaces (counterparts of level curves in two-variable case).

Definition 8.10 (Level Surface)**.**
A **level surface** *of $f(x, y, z)$ is the three-dimensional graph of the equation $f(x, y, z) = k$ for some constant k.*

If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

Example 8.11. *Find the level surfaces of the function*

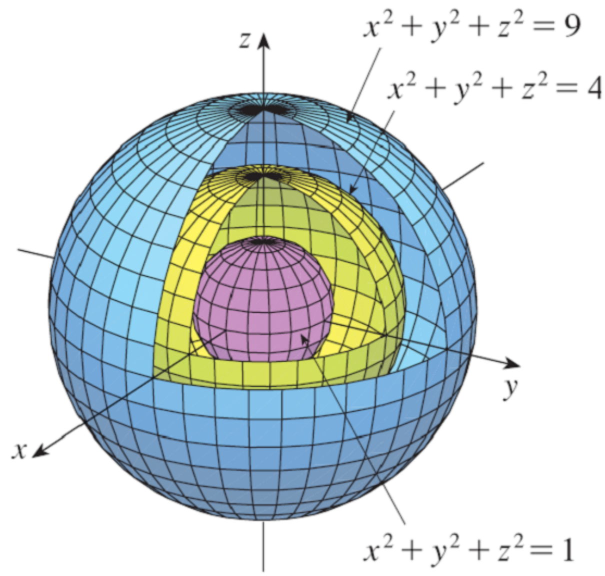
$$f(x,y,z) = x^2 + y^2 + z^2.$$

Solution. The level surfaces are:

$$x^2 + y^2 + z^2 = k$$

where $k \geq 0$.

These form a family of concentric spheres with radius $\sqrt{k}$.



■

8.8 Partial Derivatives

Recall that for a function f of a single variable, we define the derivative function as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for any values of x for which the limit exists.

At a particular point $x = a$, we interpret $f'(a)$ as the instantaneous rate of change of f with respect to x at that point.

We want to generalize the notion of derivative to functions of more than one variable.

The idea is to 'vary' one variable and keep other variable(s) fixed.

Definition 8.11 (Partial Derivative)**.**
If f is a function of two variables, its partial derivatives are the functions f_x and f_y defined by:

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h},$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}.$$

There are many alternative notations for partial derivatives:

Instead of f_x, we can write f_1 or $D_1 f$ (to indicate differentiation with respect to the first variable x) or

$$\frac{\partial f}{\partial x}.$$

Example 8.12. *For $f(x,y) = e^{xy} + \frac{x}{y}$, compute f_x and f_y.*

Solution. Treating y as a constant, we have

$$f_x(x,y) = ye^{xy} + \frac{1}{y}.$$

Treating x as a constant, we have

$$f_y(x,y) = xe^{xy} - \frac{x}{y^2}.$$

■

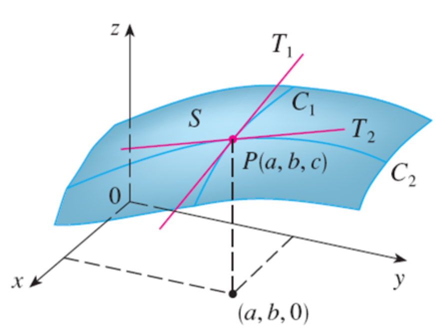
To give a geometric interpretation of partial derivatives, we recall that the equation $z = f(x,y)$ represents a surface S (the graph of f).

If $f(a,b) = c$, then the point $P(a,b,c)$ lies on S.

By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S. That is, C_1 is the trace of S in the plane $y = b$.

Likewise, the vertical plane $x = a$ intersects S in a curve C_2.

Both the curves C_1 and C_2 pass through P.



• The curve C_1 is the graph of the function $g(x) = f(x,b)$. So, the slope of its tangent T_1 at P is: $g'(a) = f_x(a,b)$.

• The curve C_2 is the graph of the function $h(x) = f(a,y)$. So, the slope of its tangent T_2 at P is: $h'(b) = f_y(a,b)$.

Thus, the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ can be interpreted geometrically as:

The slopes of the tangent lines at $P(a,b,c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.

For functions of more than two variables, such as $w = f(x,y,z)$, we can similarly define

$$f_x, f_y, f_z, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \text{ or } \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}.$$

Example 8.13. *Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation $x^3 + y^3 + z^3 + 6xyz = 1$.*

Solution. Take partial derivative with respect to x on both sides:

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6yx \frac{\partial z}{\partial x} = 0.$$

Solving for $\frac{\partial z}{\partial x}$, we have

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}.$$

Similarly,

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}.$$

■

8.9 Higher Order Partial Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables.

So, we can consider their partial derivatives

$$(f_x)_x, (f_x)_y, (f_y)_x, (f_y)_y.$$

These are called the second partial derivatives of f.

If $z = f(x, y)$, we use the following notation:

$$\begin{array}{ccccccccccc}
(f_x)_x & = & f_{xx} & = & f_{11} & = & \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) & = & \frac{\partial^2 f}{\partial x^2} & = & \frac{\partial^2 z}{\partial x^2} \\
(f_x)_y & = & f_{xy} & = & f_{12} & = & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) & = & \frac{\partial^2 f}{\partial y \partial x} & = & \frac{\partial^2 z}{\partial y \partial x} \\
(f_y)_x & = & f_{yx} & = & f_{21} & = & \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) & = & \frac{\partial^2 f}{\partial x \partial y} & = & \frac{\partial^2 z}{\partial x \partial y} \\
(f_y)_y & = & f_{yy} & = & f_{22} & = & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) & = & \frac{\partial^2 f}{\partial y^2} & = & \frac{\partial^2 z}{\partial y^2}
\end{array}$$

Thus, the notation f_{xy} means that we first differentiate with respect to x and then with respect to y.

In computing f_{yx} , the order is reversed.

Example 8.14. *Find all second-order partial derivatives of $f(x, y) = x^2y - y^3 + \ln x$.*

Solution. First, we compute the first-order derivatives:

$$f_x = 2xy + \frac{1}{x},$$
$$f_y = x^2 - 3y^2.$$

Then we have

$$f_{xx} = \frac{\partial}{\partial x}\left(2xy + \tfrac{1}{x}\right) = 2y - \tfrac{1}{x^2},$$

$$\begin{array}{ccccc}
f_{xy} & = & \frac{\partial}{\partial y}\left(2xy + \frac{1}{x}\right) & = & 2x, \\
f_{yx} & = & \frac{\partial}{\partial x}\left(x^2 - 3y^2\right) & = & 2x, \\
f_{yy} & = & \frac{\partial}{\partial y}\left(x^2 - 3y^2\right) & = & -6y.
\end{array}$$

■

Notice $f_{xy} = f_{yx}$ in the preceding example. This is not a coincidence.

It turns out that the mixed partial derivatives f_{xy} and f_{yx} are equal for most (not all) functions that one meets in practice.

The following theorem, discovered by the French mathematician Alexis Clairaut (1713–1765), gives conditions under which we can assert that $f_{xy} = f_{yx}$.

Theorem 8.4 (Clairaut's Theorem)**.**
Suppose f is defined on a disk D that contains (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Partial derivatives of order 3 and higher can also be defined. For example, $f_{xyy} = (f_{xy})_y$.

Using Clairaut's Theorem, it can be shown that

$$f_{xyy} = f_{yxy} = f_{yyx}$$

if these functions are continuous.

8.10 Tangent Planes

Recall that we use derivative $f'(a)$ to get the tangent line to the curve $y = f(x)$ at $x = a$:

$$y = f(a) + f'(a)(x - a).$$

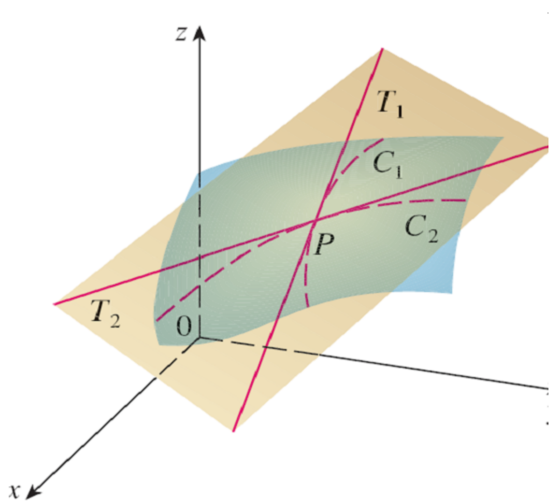
In the same spirit, we shall use partial derivatives to obtain the **tangent plane** to the surface $z = f(x,y)$ at a given point.

Consider the surface S which is the graph of $z = f(x,y)$. Suppose f has continuous first partial derivatives.

Let $P(a,b,c)$ be a point on S. Notice $c = f(a,b)$.

Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = b$ and $x = a$ with the surface S. Notice P lies on both C_1 and C_2.

Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P.



Then, the tangent plane to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2.

How to find an equation for the tangent plane?

Recall that any plane passing through $P(a,b,c)$ has an equation of the form

$$\mathbf{n} \cdot \langle x-a, y-b, z-c \rangle = 0$$

where $\mathbf{n}$ is a vector normal to the plane.

Notice the tangent line T_1 lies on the plane $y = b$. Along T_1 at $x = a$, a change of 1 unit in x corresponds to a change of $f_x(a,b)$ in z (here we require f_x to be continuous). The value of y does not change along the line. A vector with the same direction as T_1 is

$$\langle 1, 0, f_x(a,b) \rangle.$$

Similarly, a vector with the same direction as T_2 is

$$\langle 0, 1, f_y(a,b) \rangle.$$

We have found two vectors parallel to the tangent plane:

$$\langle 1, 0, f_x(a,b) \rangle, \langle 0, 1, f_y(a,b) \rangle.$$

A vector normal to the plane is given by the cross product:

$$\langle 0, 1, f_y(a,b) \rangle \times \langle 1, 0, f_x(a,b) \rangle = \langle f_x(a,b), f_y(a,b), -1 \rangle.$$

Theorem 8.5 (Equation of Tangent Plane)**.**
Suppose $f(x,y)$ has continuous first partial derivatives at (a,b). A normal vector to the tangent plane is

$$\langle f_x(a,b), f_y(a,b), -1 \rangle.$$

Further, an equation of the tangent plane is given by

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z - f(a,b)) = 0$$

or

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Example 8.15. *Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1,1,3)$.*

Solution. Notice

$$f_x(x,y) = 4x, f_x(1,1) = 4,$$

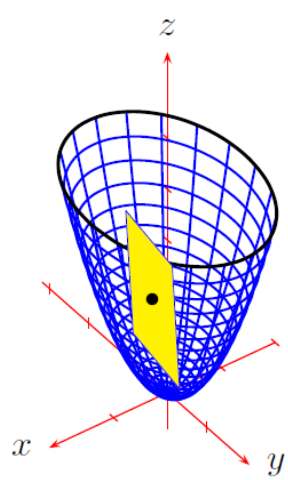
$$f_y(x,y) = 2y, f_y(1,1) = 2.$$

The equation of the plane is

$$z = f(1,1) + 4(x-1) + 2(y-1),$$

$$z = 4x + 2y - 3.$$

The figure shows the elliptic paraboloid and its tangent plane at $(1,1,3)$ that we found in the preceding example



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8.11 Differentiability and Chain Rule

For single-variable function $f(x)$, we say that f is differentiable at a if and only if $f'(a)$ exists. For two-variable function $f(x,y)$, it is tempting to say that f is differentiable at (a,b) if $f_x(a,b)$ and $f_y(a,b)$ exist. However, such definition would fail to capture the true nature of 'differentiability'!

Definition 8.12.
Informally, we say that f is **differentiable** *at (a,b) if the tangent plane at (a,b) is a good approximation to f at points close to (a,b).*

The above definition is not precise since we do not define what do we mean by 'a good approximation'. Do not worry, all the functions we encounter in this course will be differentiable at points in its domain. Recall that the Chain Rule for functions of a single variable gives the following rule for differentiating a composite function.

If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t, and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

We shall extend the chain rule to functions of several variables. This takes several slightly different forms, depending on the number of independent variables.

The first version deals with a function $z = f(x,y)$ where $x = g(t)$ and $y = h(t)$ are both functions of a single variable t:

$$z = f(g(t),h(t)).$$

Theorem 8.6 (The Chain Rule - Case 1)**.**
Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then, z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Example 8.16. *For $z = f(x,y) = x^2e^y$, $x = g(t) = t^2-1$ and $y = h(t) = \sin t$, find the derivative $\frac{dz}{dt}$.*

Solution. First, compute the partial derivatives:

$$\frac{\partial z}{\partial x} = 2xe^y, \quad \frac{\partial z}{\partial y} = x^2e^y.$$

Next, compute the derivatives:

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = \cos t.$$

Therefore, using the chain rule

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= 2xe^y(2t) + x^2e^y\cos t \\
&= 2(t^2-1)e^{\sin t}(2t) + (t^2-1)^2e^{\sin t}\cos t.
\end{aligned}$$

$\blacksquare$

Notice, in the preceding example, you could have first substituted for x and y and then compute the derivative of

$$f(g(t),h(t)) = (t^2-1)^2 e^{\sin t}$$

using the usual rules of differentiation for functions of a single variable.

We can easily extend The Chain Rule to the case of a function $f(x,y)$ where x and y now are both functions of two independent variables s and t, $x = g(s,t)$ and $y = h(s,t)$.

Then, z is indirectly a function of s and t:

$$z = f(g(s,t),h(s,t)).$$

We wish to find

$$\frac{\partial z}{\partial s}, \ \frac{\partial z}{\partial t}.$$

Recall that, in computing $\frac{\partial z}{\partial t}$, we hold s fixed and compute the ordinary derivative of z with respect to t. (This is the situation in The Chain Rule - Case 1)

Similarly, in computing $\frac{\partial z}{\partial s}$, we hold t fixed and compute the ordinary derivative of z with respect to s. (This is the situation in The Chain Rule - Case 1)

We have the following (for free!)

Theorem 8.7 (The Chain Rule - Case 2)**.**
Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(s,t)$ and $y = h(s,t)$ are both differentiable functions of s and t. Then,

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Case 2 of the Chain Rule contains three types of variables:

- s and t are independent variables.
- x and y are called intermediate variables.
- z is the dependent variable.

Example 8.17. *If $z = e^x \sin y$, where $x = st^2$ and $y = s^2 t$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.*

Solution. Applying Case 2 of Chain Rule,

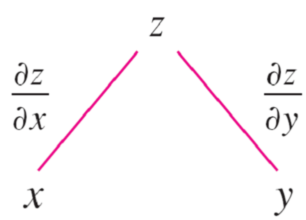
$$\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\
&= (e^x \sin y)(t^2) + (e^x \cos y)(2st) \\
&= t^2 e^{st^2} \sin(s^2 t) + 2ste^{st^2} \cos(s^2 t).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\
&= (e^x \sin y)(2st) + (e^x \cos y)(s^2) \\
&= 2ste^{st^2} \sin(s^2 t) + s^2 e^{st^2} \cos(s^2 t).
\end{aligned}$$

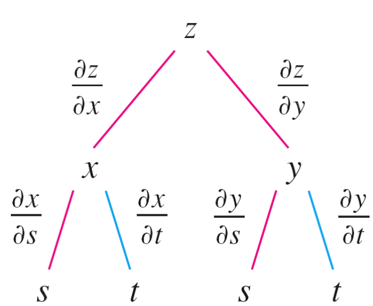
■

To remember the Chain Rule, it's helpful to draw a tree diagram, as follows.

We draw branches from the dependent variable z to the intermediate variables x and y to indicate that z is a function of x and y.



Then, we draw branches from x and y to the independent variables s and t. On each branch, we write the corresponding partial derivative.



To find $\frac{\partial z}{\partial s}$, we find the product of the partial derivatives along each path from z to s and then add these products:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}.$$

Similarly, we find $\frac{\partial z}{\partial t}$ by using the paths from z to t.

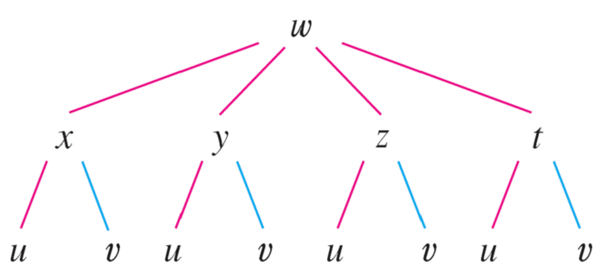
Theorem 8.8 (The Chain Rule - General Version)**.**
Suppose that u is a differentiable function of n variables $x_1, \ldots, x_n$, and each x_j is a differentiable function of m variables $t_1, \ldots, t_m$. Then u is a function of $t_1, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, \ldots, m$.

Example 8.18. *Write out the Chain Rule for the case where $w = f(x,y,z,t)$ and $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$, $t = t(u,v)$.*

Solution. The figure shows the tree diagram.



With the aid of the tree diagram, we can now write the required expressions:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}.$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}.$$

■

Example 8.19. *If $w = f(x^2 - y^2, y^2 - x^2)$ and f is differentiable, show that*

$$y\frac{\partial w}{\partial x} + x\frac{\partial w}{\partial y} = 0.$$

Solution. Introduce intermediate variables:

$$u = x^2 - y^2, \quad v = y^2 - x^2.$$

Using Chain Rule,

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial w}{\partial u}(2x) + \frac{\partial w}{\partial v}(-2x)$$

and

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial w}{\partial u}(-2y) + \frac{\partial w}{\partial v}(2y)$$

Therefore

$$y\frac{\partial w}{\partial x} + x\frac{\partial w}{\partial y}$$

$$= \left(\frac{\partial w}{\partial u}(2xy) + \frac{\partial w}{\partial v}(-2xy)\right) + \left(\frac{\partial w}{\partial u}(-2xy) + \frac{\partial w}{\partial v}(2xy)\right) = 0.$$

■

8.12 Implicit Differentiation

Consider a surface defined by an equation

$$F(x,y,z) = 0$$

where $F(x,y,z)$ is differentiable.

Suppose z is implicitly defined as a function of independent variables x and y, for every choice of x and y, there is a unique z such that $F(x,y,z) = 0$.

Suppose we are interested in $\frac{\partial z}{\partial x}$.

If we can solve the above equation for z, say $z = f(x,y)$, then we can compute $\frac{\partial z}{\partial x}$ directly. But life is complicated enough that we may not be able to solve for z. Using Chain Rule to differentiate the equation $F(x,y,z) = 0$ with respect to x:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0.$$

But

$$\frac{\partial x}{\partial x} = 1, \quad \frac{\partial y}{\partial x} = 0$$

since x and y are independent variables. Therefore, if $\frac{\partial F}{\partial z} \neq 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z}.$$

Theorem 8.9 (Implicit Differentiation: Two Independent Variables)**.**
Suppose the equation $F(x,y,z)=0$, where F is differentiable, defines z implicitly as a differentiable function of x and y. Then,

$$\frac{\partial z}{\partial x}=-\frac{F_x(x,y,z)}{F_z(x,y,z)},\quad \frac{\partial z}{\partial y}=-\frac{F_y(x,y,z)}{F_z(x,y,z)}$$

provided $F_z(x,y,z)\neq 0$.

Example 8.20. *Find $\frac{\partial z}{\partial x}$ if*

$$x^3+y^3+z^3+6xyz=1.$$

Solution. Let $F(x,y,z)=x^3+y^3+z^3+6xyz-1$. Then

$$F_x=3x^2+6yz,\quad F_z=3z^2+6xy.$$

Therefore, by the Implicit Differentiation Theorem,

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{3x^2+6yz}{3z^2+6xy}.$$

■

8.13 Increments and Differentials

Definition 8.13.
Let $z=f(x,y)$. Suppose $\triangle x$ and $\triangle y$ are increments in the independent variable x and y respectively.

Then the **increment** *in z is defined by*

$$\triangle z=f(x+\triangle x,y+\triangle y)-f(x,y).$$

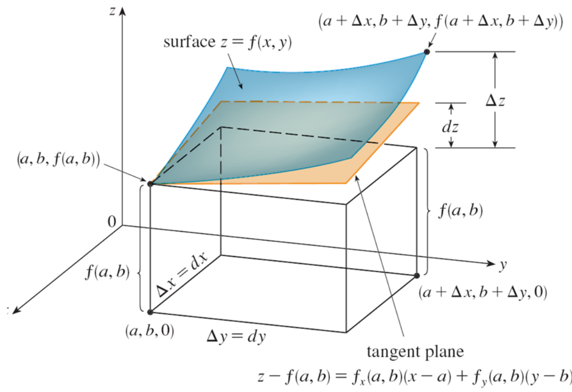
Definition 8.14.
Let $z=f(x,y)$. Suppose $\triangle x$ and $\triangle y$ are increments in the independent variable x and y respectively.

Then the **differentials** *of the independent variables x and y are*

$$dx=\triangle x,\quad dy=\triangle y.$$

The **differential** *(or* **total differential***) of the dependent variable z is*

$$dz = f_x(x,y)dx + f_y(x,y)dy.$$



Notice that

- the increment $\triangle z$ is the change in z as (x,y) changes from (a,b) to $(a+\triangle x, b+\triangle b)$.
- the differential dz is the change in the tangent plane as (x,y) changes from (a,b) to $(a+\triangle x, b+\triangle b)$.

Example 8.21. *Let $z = 2x^2 - xy$. Find $\triangle z$. Use this result to find the change of z if (x,y) changes from $(1,1)$ to $(0.98, 1.03)$.*

Solution.

$$\begin{aligned}
\triangle z &= f(x+\triangle x, y+\triangle y) - f(x,y) \\
&= \Big(2(x+\triangle x)^2 - (x+\triangle x)(y+\triangle y)\Big) - (2x^2 - xy) \\
&= (4x-y)\triangle x - x\triangle y + 2(\triangle x)^2 - \triangle x \triangle y.
\end{aligned}$$

As (x,y) changes from $(1,1)$ to $(0.98,1.03)$, we have $\triangle x = 0.98-1 = -0.02$ and $\triangle y = 1.03-1 = 0.03$. Substituting these values into the expression of $\triangle z$ above, we obtain

$$\triangle z = -0.0886.$$

■

From the previous example, it seems quite complicated to calculate $\triangle z$. Is there a way to approximate $\triangle z$?

It turns out that dz gives a good approximation of $\triangle z$ **provided** $\triangle x$ and $\triangle y$ are small and $f(x,y)$ is differentiable.

Theorem 8.10.
Suppose f is differentiable at (a,b). Let $\triangle x$ and $\triangle y$ be small increments in x and y respectively

from (a,b). *Then*

$$\triangle z \approx dz = f_x(a,b)\,dx + f_y(a,b)\,dy = f_x(a,b)\triangle x + f_y(a,b)\triangle y.$$

Example 8.22. *The base radius and height of a circular cone are measured as* $10cm$ *and* $25cm$ *respectively, with a possible error in measurement of as much as* $0.1cm$ *in each. Use differential to estimate the maximum error in the calculated volume of the cone.*

Solution. The volume of the cone is $V = \pi r^2 h/3$. So

$$dV = V_r dr + V_h dh = \frac{2\pi rh}{3}dr + \frac{\pi r^2}{3}dh.$$

Since each error is at most $0.1cm$, we can take $dr = 0.1$ and $dh = 0.1$ along with $r = 10$, $h = 25$ to give

$$dV = \frac{500\pi}{3}(0.1) + \frac{100\pi}{3}(0.1) = 20\pi.$$

The maximum error required is $20\pi cm^3$. ■

8.14 Directional Derivatives and the Gradient Vector

Imagine you are hiking in the Grand Canyon. Lets think of your altitude at the point given by longitude x and latitude y as a function $f(x,y)$.

Facing east (in the direction of positive x-axis), the slope is given by the partial derivative $\frac{\partial f}{\partial x}$.

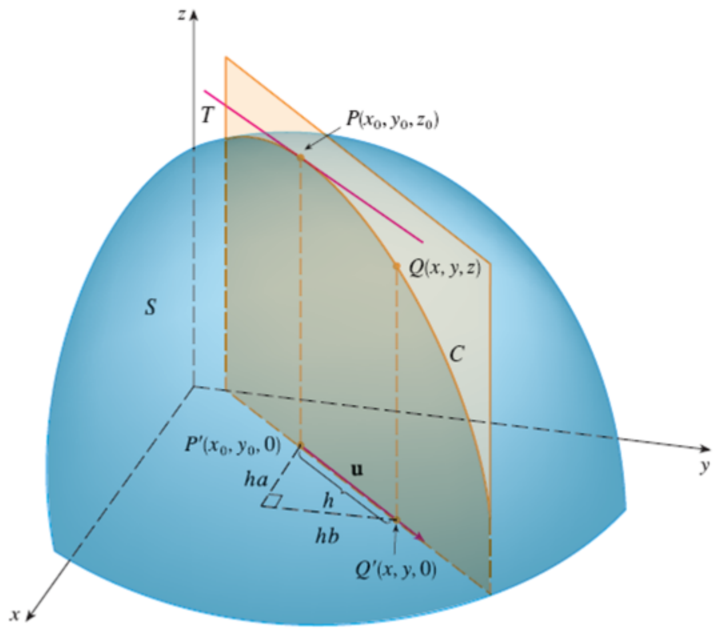
Facing north (in the direction of positive y-axis), the slope is given by the partial derivative $\frac{\partial f}{\partial y}$.

How to compute the slope when you are facing any given direction, say north-east?

Definition 8.15 (Directional Derivative)**.**
The **directional derivative** *of* $f(x,y)$ *at* (x_0,y_0) *in the direction of unit vector* $\mathbf{u} = \langle a,b\rangle$ *is*

$$D_{\mathbf{u}}f(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

provided this limit exists.



By looking at the figure above, we can think of the directional derivative $D_{\mathbf{u}}f(x_0, y_0)$ as the slope to the point $P(x_0, y_0, z_0)$ on the surface in the direction given by $\mathbf{u}$.

Notice

• if $\mathbf{u} = \mathbf{i} = \langle 1, 0\rangle$ then

$$D_{\mathbf{i}}f = f_x.$$

• if $\mathbf{u} = \mathbf{j} = \langle 0, 1\rangle$ then

$$D_{\mathbf{j}}f = f_y.$$

In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivative.

In practice, we do not usually compute the directional derivative using the definition. Instead, we compute it using the dot product of the vector consisting of partial derivatives and the unit direction vector $\mathbf{u}$.

Theorem 8.11 (Computing Directional Derivative)**.**
If $f(x, y)$ is a differentiable function, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

We can rewrite it in terms of vectors:

$$D_{\mathbf{u}}f(x, y) = \langle f_x, f_y\rangle \cdot \langle a, b\rangle = \langle f_x, f_y\rangle \cdot \mathbf{u}.$$

Consider the vector $\langle f_x, f_y\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$.

It turns out that this vector has much significance. So we give it a special name.

Definition 8.16 (Gradient)**.**
The **gradient** *of $f(x,y)$ is the vector-valued function*

$$\nabla f(x,y) = \langle f_x, f_y\rangle = f_x\mathbf{i} + f_y\mathbf{j} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

provided both partial derivatives exist.

∇f is read 'del f'.

With this notation, we have

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y)\cdot \mathbf{u}$$

Example 8.23. *Find the directional derivative of the function $f(x,y) = x^2y^3 - 4y$ at the point $(2,-1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.*

Solution. First compute the gradient vector at $(2,-1)$:

$$\nabla f(x,y) = 2xy^3\mathbf{i} + (3x^2y^2 - 4)\mathbf{j}$$

$$\nabla f(2,-1) = -4\mathbf{i} + 8\mathbf{j}.$$

Notice $\mathbf{v}$ is NOT a unit vector, since

$$\|\mathbf{v}\| = \sqrt{2^2+5^2} = \sqrt{29}.$$

The unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j}.$$

Therefore

$$\begin{aligned} D_{\mathbf{u}} f(2,-1) &= \nabla f(2,-1)\cdot \mathbf{u} \\ &= \langle -4, 8\rangle \cdot \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\rangle \\ &= \frac{32}{\sqrt{29}}. \end{aligned}$$

■

For functions of three variables, we can define directional derivative in a similar manner.

Definition 8.17 (3-D Directional Derivative)**.**
The **directional derivative** *of $f(x,y,z)$ at (x_0,y_0,z_0) in the direction of unit vector $\mathbf{u} = \langle a,b,c\rangle$ is*

$$D_{\mathbf{u}} f(x_0,y_0,z_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb, z_0+hc) - f(x_0,y_0,z_0)}{h}$$

provided this limit exists.

Just as with functions of two variables, we have

Theorem 8.12 (Computing 3-D Directional Derivative)**.**

$$D_{\mathbf{u}} f(x_0,y_0,z_0) = \nabla f(x_0,y_0,z_0)\cdot \mathbf{u}$$

where

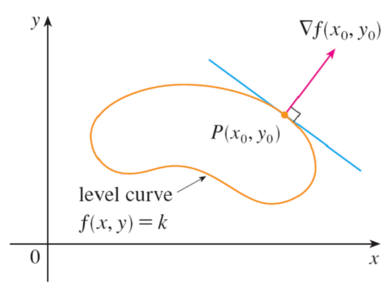
$$\nabla f = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

is the gradient vector.

What is so significant about ∇f?

Theorem 8.13 (Level Curve vs ∇f)**.**
Suppose $f(x,y)$ is differentiable function of x and y at (x_0,y_0).

Suppose $\nabla f(x_0,y_0) \neq \mathbf{0}$. Then $\nabla f(x_0,y_0)$ is perpendicular/normal to the level curve $f(x,y) = k$ at the point (x_0,y_0) where $f(x_0,y_0) = k$.



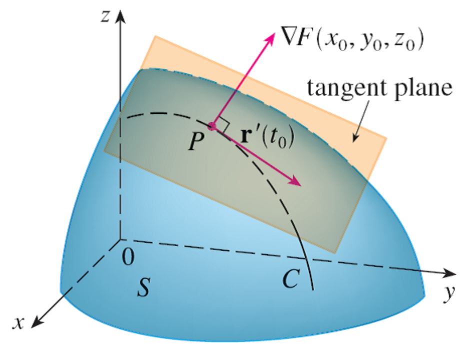
Using a similar argument, we can prove that this phenomenon also holds for level surfaces $F(x,y,z) = k$.

Theorem 8.14 (Level Surface vs ∇f)**.**
Suppose $F(x,y,z)$ is differentiable function of x, y and z at (x_0,y_0,z_0). Suppose S is the level surface $F(x,y,z) = k$ containing (x_0,y_0,z_0). Let C be any curve that lies on S and passes through (x_0,y_0,z_0). Let $\mathbf{r}(t)$ be a parametric equation of C such that $\mathbf{r}(t_0) = \langle x_0,y_0,z_0\rangle$.

Suppose $\nabla F(x_0,y_0,z_0) \neq \mathbf{0}$. Then

$$\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0,$$

That is, the $\nabla F(x_0, y_0, z_0)$ is perpendicular/normal to tangent vector $\mathbf{r}'(t_0)$ to **any** *curve C on the surface S that passes through (x_0, y_0, z_0).*



$$F(x_0, y_0, z_0) = k, \ \nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0.$$

Consequently, the **tangent plane to the level surface** $F(x, y, z) = k$ at (x_0, y_0, z_0) is given by the equation

Theorem 8.15 (Tangent Plane to Level Surface)**.**

$$\nabla F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or equivalently,

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Example 8.24. *Find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to the ellipsoid*

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3.$$

Solution. The ellipsoid is the level surface (with $k = 3$) of the function

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}.$$

Therefore,

$$F_x(x, y, z) = \frac{x}{2}, \ \ F_y(x, y, z) = 2y, \ \ F_z = \frac{2z}{9}$$

$$F_x(-2,1,-3) = -1, \quad F_y(-2,1,-3) = 2, \quad F_z(-2,1,-3) = -\frac{2}{3}.$$

The equation of the tangent plane at $(-2,1,-3)$ is

$$\nabla F(-2,1,-3) \cdot \langle x-(-2), y-1, z-(-3)\rangle$$

$$-1(x+2) + 2(y-1) - \frac{2}{3}(z+3) = 0,$$

which simplifies to

$$3x - 6y + 2z + 18 = 0.$$

The normal vector to the plane is $\langle 3, -6, 2\rangle$. So the parametric equations of the normal line are

$$x = -2+3t, \quad y = 1-6t, \quad z = -3+2t, \quad t \in \mathbb{R}.$$

■

Lets return to the directional derivative $D_{\mathbf{u}} f(x,y,z)$ and ask some questions.

We know that $D_{\mathbf{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \mathbf{u}$ is a scalar function of x, y and z (because it is a dot product of two vectors). Geometrically, we think of $D_{\mathbf{u}} f(x_0,y_0,z_0)$ as the rate of change of f at (x_0,y_0,z_0) in the direction of $\mathbf{u}$.

Question: at a given point (x_0,y_0,z_0), in which direction does f change the fastest? In other words, what is the maximum rate of change of f at (x_0,y_0,z_0)?

The answer lies in ∇f!

Let θ be the angle between ∇f and $\mathbf{u}$. Then

$$\begin{aligned} D_{\mathbf{u}} f &= \nabla f \cdot \mathbf{u} \\ &= \|\nabla f\| \|\mathbf{u}\| \cos\theta \\ &= \|\nabla f\| \cos\theta \quad \text{since } \mathbf{u} \text{ is a unit vector} \end{aligned}$$

$$D_{\mathbf{u}} f = \|\nabla f\| \cos\theta.$$

The maximum value of $\cos\theta$ is 1 and this happens when $\theta = 0$.

So the maximum value of $D_{\mathbf{u}} f$ is $\|\nabla f\|$ and it occurs when $\theta = 0$, i.e. $\mathbf{u}$ points in the direction of ∇f.

The minimum value of $\cos\theta$ is -1 and this happens when $\theta = \pi$.

So the minimum value of $D_{\mathbf{u}} f$ is $-\|\nabla f\|$ and it occurs when $\theta = \pi$, i.e. $\mathbf{u}$ points in the direction of $-\nabla f$.

Theorem 8.16 (Maximizing Rate of Increase/Decrease of f)**.**
Suppose f is a differentiable function of two or three variables. Let P denote a given point. Assume $\nabla f(P) \neq \mathbf{0}$. Let $\mathbf{u}$ be a unit vector making an angle θ with ∇f. Then

$$D_{\mathbf{u}} f(P) = \|\nabla f(P)\| \cos\theta.$$

Moreover,

- *$\nabla f(P)$ points in the direction of maximum rate of increase of f at P (maximum value of $D_{\mathbf{u}} f(P)$ is $\|\nabla f(P)\|$)*
- *$-\nabla f(P)$ points in the direction of maximum rate of decrease of f at P (minimum value of $D_{\mathbf{u}} f(P)$ is $-\|\nabla f(P)\|$)*

Example 8.25. *Let $f(x,y) = xe^y$. In what direction does f have the maximum rate of change at the point $P(2,0)$? What is this maximum rate of change.*

Solution. Note that

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle.$$

f increases fastest in the direction of the gradient vector

$$\nabla f(2,0) = \langle 1, 2 \rangle.$$

The maximum rate of change is

$$\|\nabla f(2,0)\| = \sqrt{1^2 + 2^2} = \sqrt{5}.$$

■

8.15 Extrema of Functions of Two Variables

In the real world, we always seek to optimize our resources.

Given our constraints (our time, ability, finance, health, family background and what not), getting an A in MA1521 itself can be seen as an optimization problem.

Similar to the study of extrema of functions of one variable, two key concepts are the local maximum/minimum and the absolute maximum/minimum.

Definition 8.18 (Local and Absolute Maximum)**.**
Let $f(x,y) : D \to \mathbb{R}$. Then

- *f has a* **local maximum** *at (a,b) if $f(x,y) \leq f(a,b)$ for all points in some disk with center (a,b). The number $f(a,b)$ is called a* **local maximum value**.

• *f has an* **absolute maximum** *at (a,b) if $f(x,y) \leq f(a,b)$ for all points in the domain D. The number $f(a,b)$ is called a* **absolute maximum value**.

Definition 8.19 (Local and Absolute Minimum)**.**
Let $f(x,y) : D \to \mathbb{R}$. Then

• *f has a* **local minimum** *at (a,b) if $f(x,y) \geq f(a,b)$ for all points in some disk with center (a,b). The number $f(a,b)$ is called a* **local minimum value**.

• *f has an* **absolute minimum** *at (a,b) if $f(x,y) \geq f(a,b)$ for all points in the domain D. The number $f(a,b)$ is called a* **absolute minimum value**.

8.15.1 Local Extrema

A key observation that will be used repeatedly when finding local extrema of functions is the following.

Theorem 8.17.
If f has a local maximum or minimum at (a,b) and the first-order derivatives of f exist there, then

$$f_x(a,b) = f_y(a,b) = 0.$$

Proof. Let $g(x) = f(x,b)$. Then g is a function of a single variable x. If f has a local maximum/minimum at $(x,y) = (a,b)$ then g has a local maximim/minimim at $x = a$. So $g'(a) = 0$.

But $g'(a) = f_x(a,b)$. So $f_x(a,b) = 0$.

Similarly, $f_y(a,b) = 0$.

■

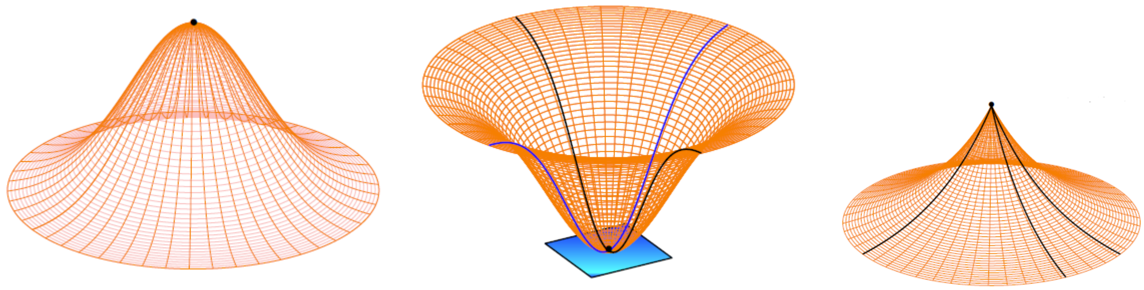
There is a geometric interpretation of the preceding theorem:

If f has a tangent plane at a local maximum/minimum (a,b), then the tangent plane has equation

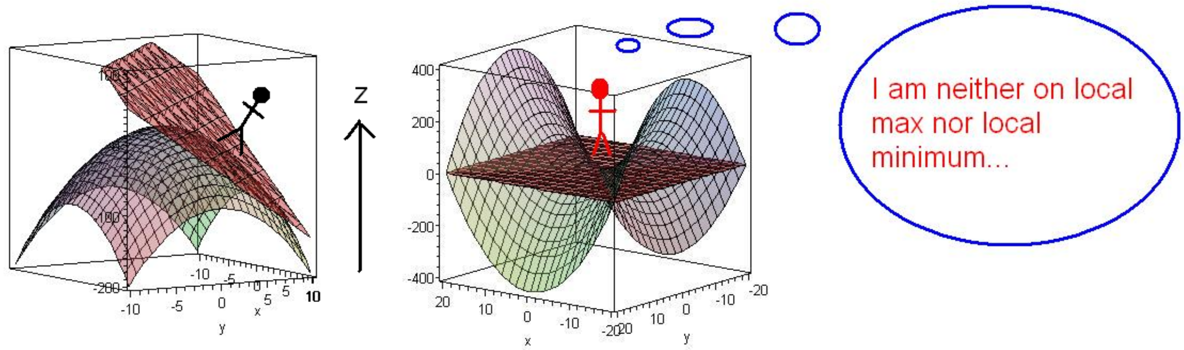
$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) = f(a,b),$$

that is the tangent plane is a horizontal plane parallel to the xy-plane.

If I stand on a local maximum/minimum then ......



In the following, I am definitely NOT standing on local maximum/minimum ......



Definition 8.20 (Critical or Stationary Point)**.**
Let $f(x,y) : D \to \mathbb{R}$. Then a point (a,b) is called a **critical point** *of f if*

- $f_x(a,b) = 0$ *and* $f_y(a,b) = 0$*, OR*
- *one of the partial derivatives does not exist.*

Clearly,

$$(a,b) \text{ local maximum/minimum point} \quad \Longrightarrow \quad (a,b) \text{ critical point.}$$

However, the converse IS NOT TRUE!

Example 8.26. *Find the extreme (maximum/minimum) values of $f(x,y) = y^2 - x^2$.*

Solution. Extreme values can only occur at critical points. Since $f_x = -2x$ and $f_y = 2y$, the only critical point is $(0,0)$.

We still have to check whether $f(0,0)$ is a maximum/minimum value.

Note that $f(0,0) = 0$.

If $y = 0$ and $x \neq 0$, then

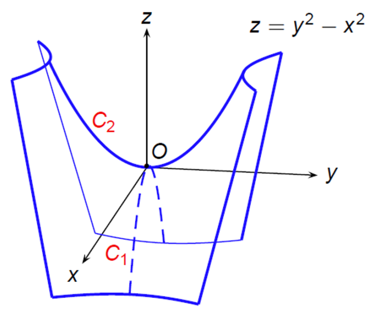
$$f(x,y) = -x^2 < 0 = f(0,0).$$

If $x = 0$ and $y \neq 0$, then

$$f(x,y) = y^2 > 0 = f(0,0).$$

Therefore, $f(0,0)$ cannot be an extreme value for f. So f has no extreme values.

■



In the preceding example, we see that a critical point needs not be an extreme point. But the behavior of the critical point in the preceding example is interesting.

If you look at the graph of $f(x,y) = y^2 - x^2$, you will see that $f(0,0) = 0$ is a maximum in the direction of the x-axis but a minimum in the direction of the y-axis.

This motivates the following definition.

Definition 8.21 (Saddle Point)**.**
Let $f(x,y) : D \to \mathbb{R}$. Then

A point (a,b) is called a **saddle point** *of f if*

• *it is a critical point of f, AND*

• *every open disk centered at (a,b) contains points $(x,y) \in D$ for which $f(x,y) < f(a,b)$ and points $(x,y) \in D$ for which $f(x,y) > f(a,b)$.*

Suppose you are standing on a surface and you are standing upright (parallel to the z-axis). Moreover, when you begin walking, some directions take you uphill while other directions take you downhill.

Then you are standing at a saddle point!

We cannot rely on our visualization of 3D-graphs to locate extreme points.

Luckily, we have the **second derivative test** to determine whether a given critical point is local maximum/minimum, saddle point or neither.

Theorem 8.18 (Second Derivative Test)**.**
Suppose $f(x,y)$ has continuous second-order partial derivatives on some open disk centered at (a,b). Suppose $f_x(a,b) = f_y(a,b) = 0$ (that is (a,b) is a critical point). Define the **discriminant** *D for the point (a,b) by*

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2.$$

(a) *If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.*

(b) *If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.*

(c) *If $D < 0$, then (a,b) is a saddle point of f.*

(d) *If $D = 0$, then no conclusion can be drawn.*

Example 8.27. *Locate and classify all critical points for $f(x,y) = x^3 - 2y^2 - 2y^4 + 3x^2y$.*

Solution. We have

$$f_x = 3x^2 + 6xy, \quad f_y = -4y - 8y^3 + 3x^2.$$

Step 1. Locate critical points.

Let's solve the system

$$\begin{aligned} 3x^2 + 6xy &= 0 && (1) \\ -4y - 8y^3 + 3x^2 &= 0 && (2) \end{aligned}$$

If $x = 0$, then by (2) we have $-4y - 8y^3 = 0 \Leftrightarrow -4y(1+2y^2) = 0 \Leftrightarrow y = 0$. Thus we obtain one solution $(0,0)$.

If $x \neq 0$, then by (1) we have $y = -\frac{x}{2}$. Substituting this into (2), We have

$-4(-\frac{x}{2}) - 8(-\frac{x}{2})^3 + 3x^2 = 0 \Leftrightarrow 2x + x^3 + 3x^2 = 0 \Leftrightarrow x(x+2)(x+1) = 0 \Leftrightarrow x = -1, -2$. Note that $x \neq 0$.

Using $y = -\frac{x}{2}$, we obtain the two solutions $(-1, \frac{1}{2}), (-2, 1)$.

Therefore, the critical points are $(0,0), (-1, \frac{1}{2}), (-2, 1)$.

Step 2. Classify (if possible) these critical points using Second Derivative Test.

We need

$$f_{xx} = 6x + 6y, \quad f_{xy} = 6x, \quad f_{yy} = -4 - 24y^2.$$

Compute and obtain

critical point	D	f_{xx}	2nd Derivative Test's Conclusion
$(0,0)$	0		inconclusive
$(-1, \frac{1}{2})$	$-6 < 0$		saddle point
$(-2, 1)$	$24 > 0$	$-6 < 0$	local maximum

We need a different analysis to deal with the critical point $(0,0)$.

Notice in the plane $y = 0$, $f(x,y) = f(x,0) = x^3$. We know from Calculus that this curve has an inflection point at $x = 0$. So there is no local extremum at this point.

Moreover, when we start walking from $(0,0)$ in positive direction of x, we will be walking uphill; in the negative direction of x, we will be walking downhill.

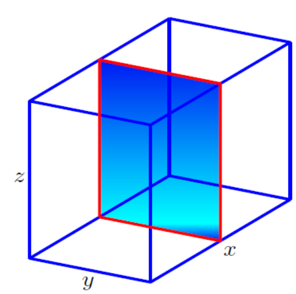
So $(0,0)$ is a saddle point.

■

Exercise 8.1. *Find the critical points of* $f(x,y) = -3xe^{-x^2-y^2}$ *and classify them.*

Ans. Local minimum at $\frac{1}{\sqrt{2}}$, Local maximum at $-\frac{1}{\sqrt{2}}$.

Exercise 8.2. *A delivery company only accepts rectangular boxes the sum of whose length and girth (perimeter of a cross-section) does not exceed 270 cm. Find the dimensions of an acceptable box of largest volume.*



The girth is $2y + 2z$.

Ans. $90\text{cm} \times 45\text{cm} \times 45\text{cm}$.

Chapter 9

Double Integrals

Read Thomas' Calculus, Chapter 15.

9.1 Riemann Sum

Having studied derivatives for functions of several variables and their applications, we now turn to introducing the idea of integral for functions of several variables. It turns out that these ideas are useful in many practical problems.

Recall that in Calculus of Single Variable, our attempt to find the area under a curve led to the definition of a definite integral.

We now seek to find volume under a surface and in the process we arrive at the definition of a double integral.

We start by reviewing how we arrive at the definite integral of functions of a single variable:

Step 1. Suppose $f(x)$ is defined for $a \le x \le b$. We divide the interval $[a, b]$ into n subintervals of equal size $\triangle x = \frac{b-a}{n}$.

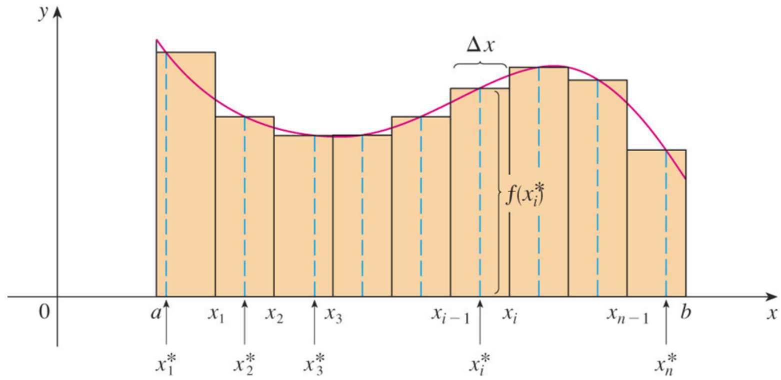
Step 2. We choose sample points x_i^* from these subintervals and form the **Riemann Sum**

$$\sum_{i=1}^{n} f(x_i^*)\triangle x.$$

Step 3. Take the limit of such sum as $n \to \infty$ to obtain the definite integral of f from a to b:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\triangle x.$$

In the special case where $f(x) \ge 0$, the integral $\int_a^b f(x)dx$ represents the area under the curve $f(x)$ from a to b.

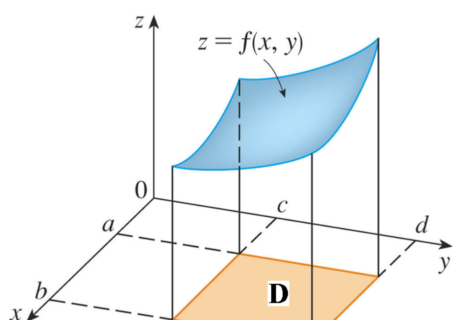


9.2 Volume and Double Integral

Suppose $f(x,y)$ is a function of two variables defined on a closed rectangle

$$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 : a \le x \le b, c \le y \le d\}.$$

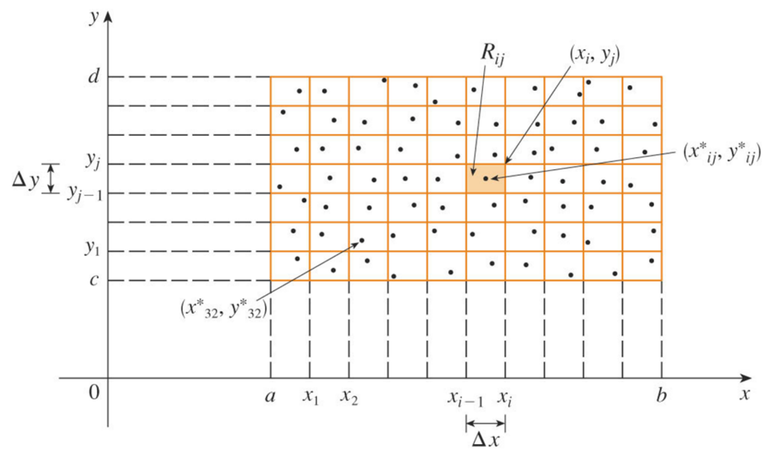
Suppose $f(x,y) \ge 0$. The graph of f is a surface with $z = f(x,y)$ above the region R.
Let S be the solid that lies above R and under the graph of f.
How can we find the volume of S?



We can estimate the volume of S as follows:
Step 1. divide the rectangle R into subrectangles. We do this by
• dividing the interval $[a,b]$ into m subintervals $[x_{i-1}, x_i]$ of equal length $\triangle x = \frac{b-a}{m}$, and
• dividing the interval $[c,d]$ into n subintervals $[y_{j-1}, y_j]$ of equal length $\triangle y = \frac{d-c}{n}$.
Form subrectangles

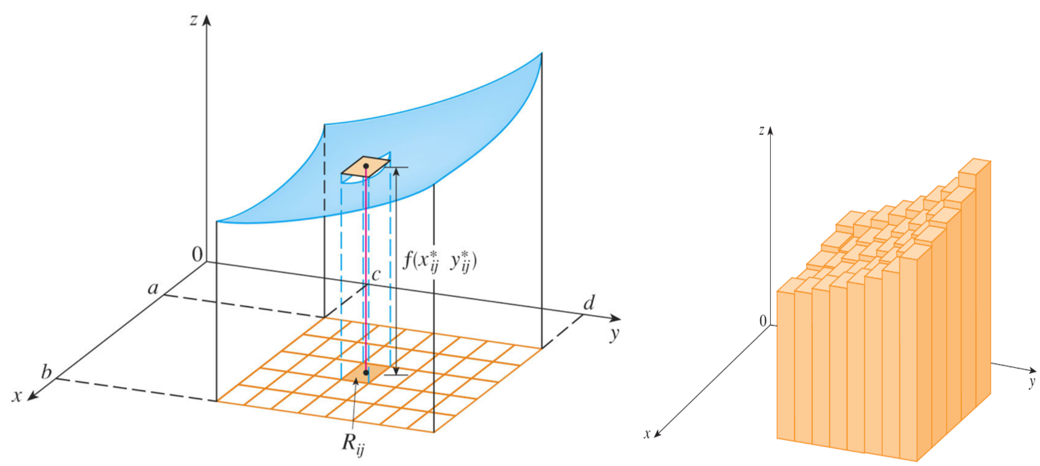
$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j], \quad \text{for all } \ 1 \le i \le m, 1 \le j \le n.$$

Each of these subrectangles has area $\triangle A = \triangle x \triangle y$.



Step 2. Choose a sample point (x^*_{ij}, y^*_{ij}) in each R_{ij}. Then approximate the part of S lies above R_{ij} by a thin rectangle box with base R_{ij} and height $f(x^*_{ij}, y^*_{ij})$. The volume of this box is given by

$$f(x^*_{ij}, y^*_{ij})\triangle A.$$

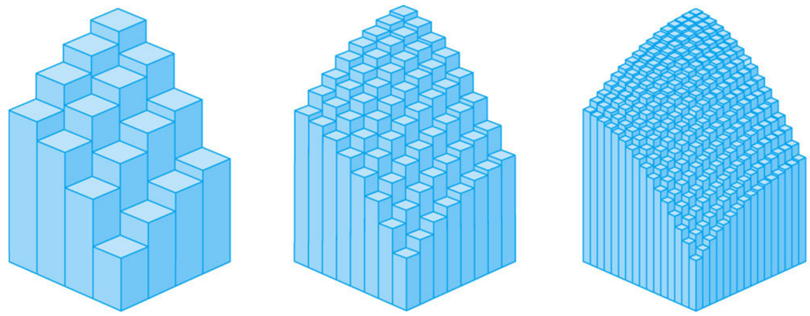


It follows that by adding the volumes of all these thin boxes, we get an approximation of the total volume of S:

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\triangle A.$$

Our intuition tells us that the approximation becomes better as $m, n \to \infty$. So we would expect

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\triangle A.$$



(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

We make the following definition:

Definition 9.1 (Double Integral)**.**
The **double integral** *of f over the rectangle R is*

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\triangle A$$

provided the limit exists and is the same for any choice of the sample points (x_{ij}^*, y_{ij}^*) in R_{ij}, for $1 \le i \le m$, $1 \le j \le n$.

When this happens, we say that f is **integrable** *over R.*

Remark. It can be shown that all continuous functions are integrable.

By comparing our definition of integral and volume, we have

Theorem 9.1 (Volume as a Double Integral)**.**
If $f(x,y) \ge 0$, the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \iint_R f(x,y)\,dA.$$

Some properties of double integral:

Assuming all the integrals exist, we have

1. $\displaystyle\iint_R (f(x,y)+g(x,y))\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA.$

2. $\displaystyle\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA.$

3. If $f(x,y) \geq g(x,y)$ for all $(x,y) \in R$, then

$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA.$$

9.3 Iterated Double Integral

Recall that it is usually difficult to evaluate a single integral directly from definition, but the Fundamental Theorem of Calculus provides a much easier method:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$.

For double integral, it is even more difficult to compute it from first principles. We now see how to express a double integral as an **iterated integral** which can be evaluated by calculating two single integrals.

Suppose $f(x,y)$ is integrable over the rectangle $R = [a,b] \times [c,d]$.

We use the notation $\int_c^d f(x,y)\,dy$ to mean that x is held fixed and $f(x,y)$ is integrated with respect to y from c to d.

This procedure is called **partial integration with respect to** y. Notice the similarity to partial differentiation.

So $\int_c^d f(x,y)\,dy$ is a function of x, as it depends on the value of x: set

$$A(x) = \int_c^d f(x,y)\,dy.$$

We now integrate $A(x)$ from a to b:

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx.$$

The integral on the right-hand side is called an **iterated integral**.

Usually, we omit the brackets:

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx.$$

Definition 9.2 (Iterated Integral).

$$\int_a^b \int_c^d f(x,y)\,dy\,dx$$

means we first integrate with respect to y from c to d (keeping x fixed) and then with respect to x from a to b.

$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$

means we first integrate with respect to x from a to b (keeping y fixed) and then with respect to y from c to d.

Example 9.1. *Evaluate the iterated integral*

$$\int_1^2 \int_0^3 x^2 y\,dx\,dy.$$

Solution. We first integrate with respect to x and then with respect to y:

$$\begin{aligned}
\int_1^2 \int_0^3 x^2 y\,dx\,dy &= \int_1^2 \left[\int_0^3 x^2 y\,dx\right] dy \\
&= \int_1^2 \left[\frac{x^3 y}{3}\right]_0^3 dy \\
&= \int_1^2 9y\,dy \\
&= \left[\frac{9y^2}{2}\right]_1^2 \\
&= \frac{27}{2}.
\end{aligned}$$

■

Example 9.2. *Evaluate the iterated integral*

$$\int_0^3 \int_1^2 x^2 y\,dy\,dx.$$

Solution. We first integrate with respect to y and then with respect to x:

$$\begin{aligned}
\int_0^3 \int_1^2 x^2 y\, dy\, dx &= \int_0^3 \left[\int_1^2 x^2 y\, dy\right] dx \\
&= \int_0^3 \left[\frac{x^2 y^2}{2}\right]_1^2 dx \\
&= \int_0^3 \frac{3}{2} x^2\, dx \\
&= \left[\frac{x^3}{2}\right]_0^3 \\
&= \frac{27}{2}.
\end{aligned}$$

■

Notice in both of the preceding examples, we obtained the same answer.

It seems that the order of integration (with respect to x or y first) does not matter. This is similar to Clairaut's Theorem for mixed partial derivatives.

Indeed, if f is continuous on R, this is always true. Moreover, the (iterated) integral is equal to the corresponding double integral.

The following theorem gives a practical way for evaluating a double integral by expressing it as an iterated integral (in either order):

Theorem 9.2 (Fubini's Theorem)**.**
If f is continuous on the rectangle $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 9.3. *Evaluate*

$$\iint_R y \sin(xy)\, dA$$

where $R = [1,2] \times [0,\pi]$.

Solution 1. Using Fubini's Theorem, let's integrate first with respect to x.

$$
\begin{aligned}
\iint_R y\sin(xy)\,dA &= \int_0^{\pi}\int_1^2 y\sin(xy)\,dx\,dy \\
&= \int_0^{\pi} \left[-\cos(xy)\right]_1^2\,dy \\
&= \int_0^{\pi} (-\cos 2y + \cos y)\,dy \\
&= \left[-\frac{1}{2}\sin 2y + \sin y\right]_0^{\pi} = 0.
\end{aligned}
$$

Solution 2. By Fubini's Theorem, we should get the same answer if we first integrate with respect to y:

$$
\iint_R y\sin(xy)\,dA = \int_1^2\int_0^{\pi} y\sin(xy)\,dy\,dx.
$$

We first need to compute

$$
\int_0^{\pi} y\sin(xy)\,dy.
$$

Using integration by parts,

$$
\begin{aligned}
\int_0^{\pi} y\sin(xy)\,dy &= \left[-\frac{y\cos(xy)}{x}\right]_0^{\pi} - \int_0^{\pi} -\frac{\cos(xy)}{x}\,dy \\
&= -\frac{\pi\cos\pi x}{x} + \frac{1}{x^2}\left[\sin xy\right]_0^{\pi} \\
&= -\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}.
\end{aligned}
$$

Now, integrating the first term by parts, we have

$$
\int\left(-\frac{\pi\cos\pi x}{x}\right)dx = -\frac{\sin\pi x}{x} - \int\frac{\sin\pi x}{x^2}\,dx
$$

So

$$
\int\left(-\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}\right)dx = -\frac{\sin\pi x}{x}.
$$

Hence

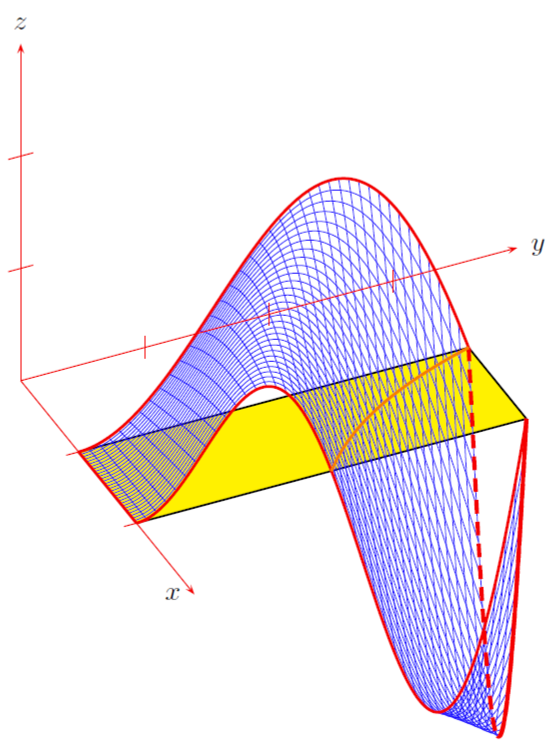
$$
\begin{aligned}
\int_1^2 \int_0^\pi y\sin(xy)\,dy\,dx &= \left[-\frac{\sin \pi x}{x}\right]_1^2 \\
&= -\frac{\sin 2\pi}{2} + \sin\pi = 0.
\end{aligned}
$$

■

Now, we make some observation about the last example.

• Though both solutions give the same answer, the first solution is much easier than the second one. Therefore, when we evaluate double integrals, it is wise to choose the right order of integration that yields simpler calculations.

• Consider the surface $z = y\sin(xy)$.



This function takes both positive and negative values on $R = [1,2]\times[0,\pi]$.

For such a function, $\iint_R f(x,y)\,dA$ is a difference of volumes: $V_1 - V_2$ where V_1 is the volume above R and below the graph of f and V_2 is the volume below R and above the graph.

The fact that the integral is 0 in the preceding example means that these two volumes V_1 and V_2 are equal.

9.4 A Special Case

Sometimes $f(x,y)$ can be factored as the product of a function of x only and a function of y only. That is

$$f(x,y) = g(x)h(y).$$

Then Fubini's Theorem gives

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_c^d \int_a^b g(x)h(y)\,dx\,dy \\ &= \int_c^d \left[\int_a^b g(x)h(y)\,dx\right] dy\end{aligned}$$

In the inner integral, y is a constant, so $h(y)$ is a constant and we can write

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_c^d \left[h(y)\int_a^b g(x)\,dx\right] dy \\ &= \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)\end{aligned}$$

since $\int_a^b g(x)\,dx$ is a constant.

To summarize:

Theorem 9.3 (A Special Case)**.**

$$\iint_R g(x)h(y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)$$

where $R = [a,b]\times[c,d]$.

9.5 Double Integral over General Region

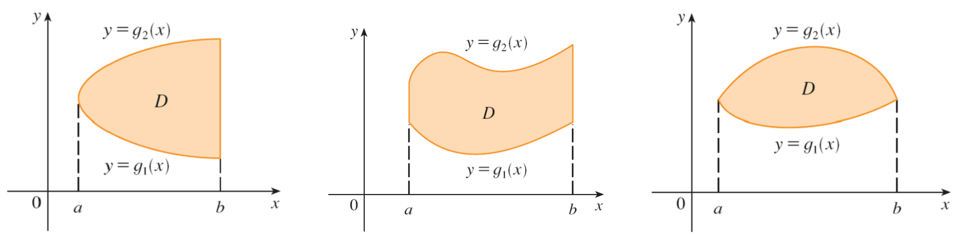
So far we have defined double integrals over domains which are rectangles. In this section, we shall define double integrals over domains which are more general than rectangles. In particular, they are regions which are bounded between two continuous curves. They are called **Type I** and **Type II** regions respectively.

Definition 9.3. *Type I Region*
A plane region D is said to be of **Type I** *if it lies between the graphs of two continuous functions of x, that is,*

$$D = \{(x,y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

where $g_1(x)$ and $g_2(x)$ are continuous on $[a,b]$.

Some examples of Type I region:

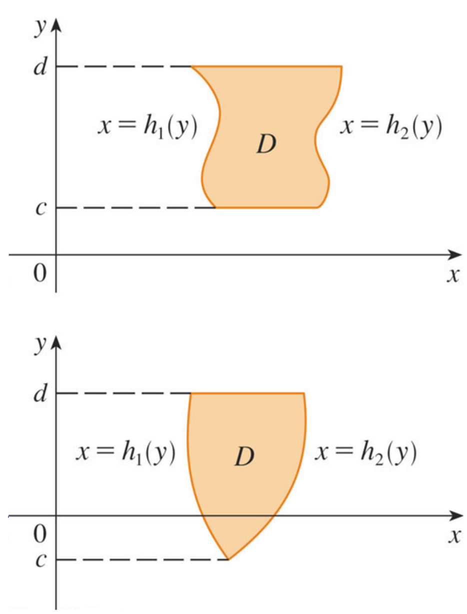


Definition 9.4. *Type II Region*
A plane region D is said to be of **Type II** *if it lies between the graphs of two continuous functions of y, that is,*

$$D = \{(x,y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

where $h_1(y)$ and $h_2(y)$ are continuous on $[c,d]$.

Some examples of Type II region:



How do we compute the integral of $f(x,y)$ over Type I region D?

Theorem 9.4. *Double Integral over Type I Domain*
If f is continuous on a Type I domain D such that

$$D = \{(x, y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

then

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

Observe that the expression on the right-hand side of

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

is an iterated integral similar to the ones we have for rectangle region, except that in the inner integral we regard x as being constant not only in $f(x, y)$ but also in the limits of the integration, $g_1(x)$ and $g_2(x)$.

Similarly, we have

Theorem 9.5. *Double Integral over Type II Domain*
If f is continuous on a Type II domain D such that

$$D = \{(x, y) : c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

then

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Example 9.4. *Evaluate $\iint_D (x + 2y)\, dA$ where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.*

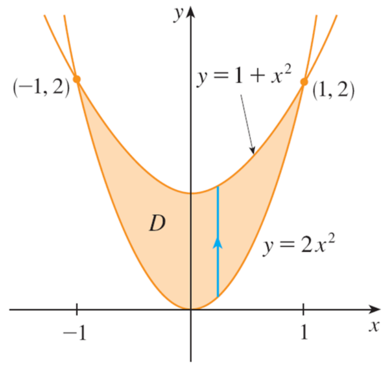
Solution.

Step 1. Identify the region.

Notice the parabolas intersect when $2x^2 = 1 + x^2$, that is, $x = \pm 1$.

We note that D is a Type I region:

$$D = \{(x, y) : -1 \le x \le 1, 2x^2 \le y \le 1 + x^2\}.$$



Step 2. Set up the iterated integral.

Therefore,

$$\iint_D (x+2y)\,dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx.$$

Step 3. Evaluate the inner integral.

$$\begin{aligned}\int_{2x^2}^{1+x^2} (x+2y)\,dy &= \Big[xy+y^2\Big]_{y=2x^2}^{y=1+x^2}\\ &= x(1+x^2)+(1+x^2)^2-x(2x^2)-(2x^2)^2\\ &= -3x^4-x^3+2x^2+x+1.\end{aligned}$$

Step 4. Complete the computation.

$$\begin{aligned}\iint_D (x+2y)\,dA &= \int_{-1}^{1}(-3x^4-x^3+2x^2+x+1)\,dx\\ &= \left[-3\frac{x^5}{5}-\frac{x^4}{4}+2\frac{x^3}{3}+\frac{x^2}{2}+x\right]_{-1}^{1}\\ &= \frac{32}{15}.\end{aligned}$$

■

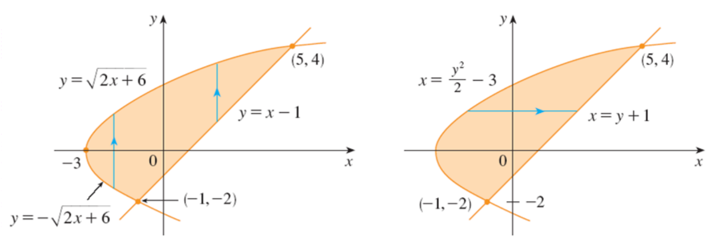
When we set up a double integral, it is helpful to draw a diagram.

For Type I region, it is helpful to draw a **vertical** arrow which starts at the lower boundary $y = g_1(x)$ and ends at the upper boundary $y = g_2(x)$. This corresponds to the inner integral.

For Type II region, it is helpful to draw a **horizontal** arrow which starts at the left boundary $x = h_1(x)$ and ends at the right boundary $x = h_2(x)$. This corresponds to the inner integral.

Example 9.5. *Evaluate* $\iint_D xy\, dA$ *where* D *is the region bounded by the line* $y = x - 1$ *and the parabola* $y^2 = 2x + 6$.

Solution. The region D can be of Type I or II:

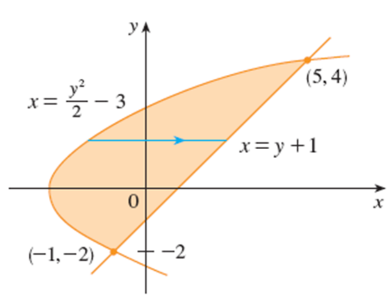


(a) D as a type I region (b) D as a type II region

But we prefer D as Type II because as a Type I region, the lower boundary of D is more complicated, in particular, it consists of two parts: one for $-3 \le x \le -1$ and another for $-1 \le x \le 5$.

Therefore, we let

$$D = \{(x, y) : -2 \le y \le 4, \frac{1}{2}y^2 - 3 \le x \le y + 1\}.$$



D as a type II region

So

$$\iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy.$$

Lets first compute the inner integral:

$$\begin{aligned}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx &= \left[y\frac{x^2}{2}\right]_{x=\frac{1}{2}y^2-3}^{x=y+1}\\ &= \frac{1}{2}\left(-\frac{y^5}{4}+4y^3+2y^2-8y\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\iint_D xy\,dA &= \frac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4}+4y^3+2y^2-8y\right)dy\\ &= \frac{1}{2}\left[-\frac{y^6}{24}+y^4+2\frac{y^3}{3}-4y^2\right]_{-2}^{4}\\ &= 36.\end{aligned}$$

■

Example 9.6. *Find the volume of the tetrahedron T bounded by the planes $x+2y+z=2$, $x=2y$, $x=0$ and $z=0$.*

Solution. For question like this, it is wise to draw two diagrams:

• one for the solid (tetrahedron) T,

• another for the domain D.

How do we start drawing?

There is no general rule, it depends on the problem in the question.

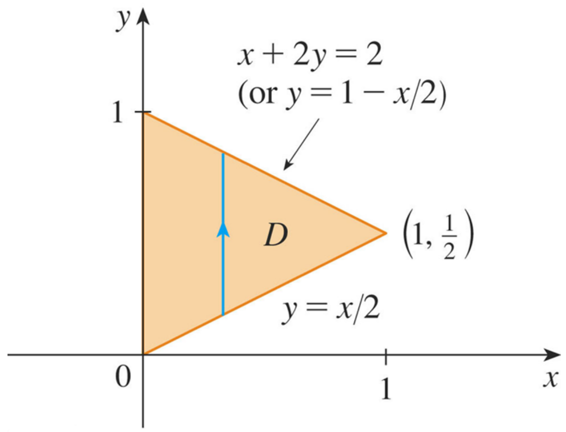
Notice the plane $x+2y+z=2$ intersects the xy-plane in the line $x+2y=2$. (Set $z=0$ in the equation of the plane).

Together with the restrictions that the solid is bounded by $z=0$ (above the xy-plane), $x=0$ (the yz-plane) and the plane $x=2y$, we see that T lies above the region D in the xy-plane bounded by the lines:

• $x=2y$,

• $x+2y=2$ (the intersection of the plane $x+2y+z=2$ and the plane $z=0$),

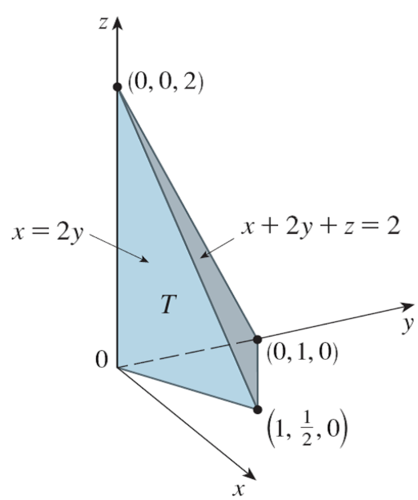
• $x=0$.



Notice that $(1, \frac{1}{2}, 0)$ and $(0, 1, 0)$ are two points on the plane $x + 2y + z = 2$.

There is another point on this plane: $(0, 0, 2)$.

We can now draw the tetrahedron T as follows:



So the required volume V lies under the graph $z = 2 - x - 2y$ and above

$$D = \{(x, y) : 0 \leq x \leq 1, \frac{x}{2} \leq y \leq 1 - \frac{x}{2}\}.$$

Therefore

$$\begin{aligned} V &= \iint_D (2 - x - 2y)\,dA \\ &= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\,dy\,dx \\ &= \int_0^1 \left[2y - xy - y^2\right]_{y=x/2}^{y=1-x/2} dx \\ &= \int_0^1 (x^2 - 2x + 1)\,dx \\ &= \left[\frac{x^3}{3} - x^2 + x\right]_0^1 \\ &= \frac{1}{3}. \end{aligned}$$

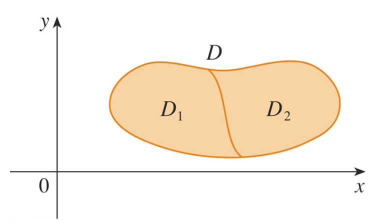
■

9.6 Decomposing Domain into Smaller Domains

Double integrals are additive with respect to the domain: if D is the union of domains $D_1, \ldots, D_n$ that do not overlap except possibly on boundary curves, then

Theorem 9.6. *Additivity With Respect to Domain*

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \cdots + \iint_{D_n} f(x,y)\,dA.$$



Additivity may be used to evaluate double integrals over domain D which is neither of Type I nor II but can be decomposed into finitely many domains of Type I or II.

9.7 Properties of Double Integral

The following properties for double integral over D follow from the corresponding properties for double integrals over a rectangle region R:

Theorem 9.7.

$$\iint_D [f(x,y)+g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$$

Theorem 9.8.

$$\iint_D cf(x,y)\,dA = c\iint_D f(x,y)\,dA$$

Theorem 9.9.
If $f(x,y) \ge g(x,y)$ for **all** $(x,y) \in D$ *then*

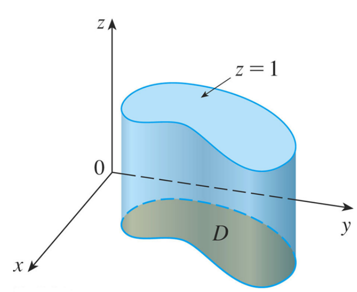
$$\iint_D f(x,y)\,dA \ge \iint_D g(x,y)\,dA.$$

9.8 An Application – Finding Area

We can use double integral to compute area of a region D on the plane:

Theorem 9.10. *Area of plane region*
Let $f(x,y) = 1$ over a given region D. Then the area of D is

$$A(D) = \iint_D 1\,dA.$$



$\iint_D 1\,dA$ is the volume of the solid which is a cylinder whose base is $A(D)$ and height 1. Another way of computing the volume of a cylinder is

$$\text{area of base} \times \text{height}$$

which is

$$A(D) \cdot 1$$

in this case.

So

$$A(D) = \iint_D 1\, dA,$$

as required.

$\blacksquare$

9.9 Double Integrals in Polar Coordinates

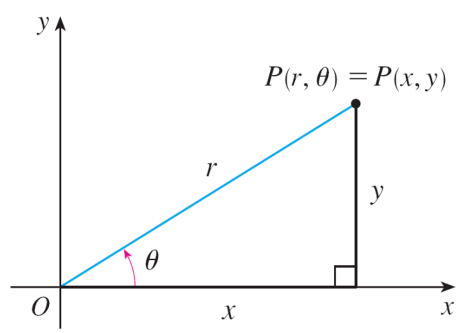
We have learned how to evaluate double integral over D where D is of the following type:

• rectangle;

• region of Type I or Type II.

Sometimes, the region D is not so easily described in terms of x and y coordinates.

Sometimes, such regions can be conveniently described using polar coordinates (r, θ). The following figure which shows the relationship between polar coordinates and the rectangle coordinates:

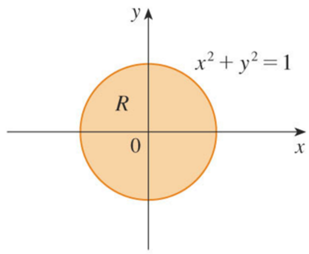


Instead of using (x, y), we note that any point on the xy-plane can be represented by an ordered pair (r, θ) where

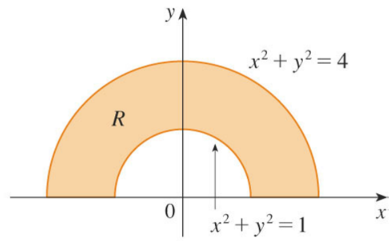
• r is the distance from the origin to the point

• θ is the angle from the positive x-axis to the straight line joining the origin and the point.

For example consider the following region given in terms of its polar coordinates:



(a) $R = \{(r, \theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$



(b) $R = \{(r, \theta) \mid 1 \le r \le 2, 0 \le \theta \le \pi\}$

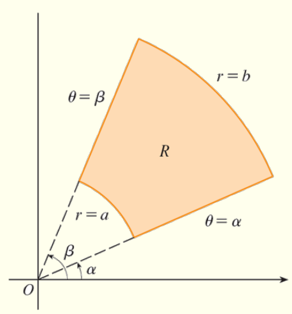
Polar coordinates (r, θ) of a point are related to the rectangle coordinate (x, y) by the equations

Theorem 9.11. *Polar Coordinates Versus Rectangle Coordinates*

$$r^2 = x^2 + y^2, \quad x = r\cos\theta, \quad y = r\sin\theta.$$

Definition 9.5 (Polar Rectangle)**.**
A **polar rectangle** *is a region*

$$R = \{(r, \theta) : a \le r \le b, \ \alpha \le \theta \le \beta\}.$$



How do we compute

$$\iint_R f(x, y)\, dA$$

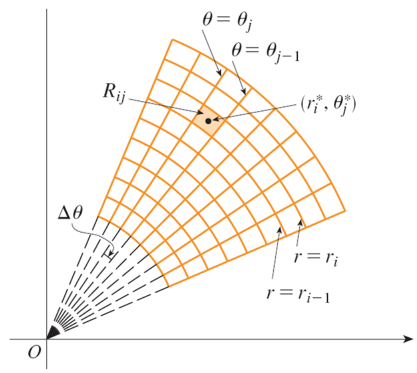
where R is a polar rectangle?

Recall that for $\iint_R f(x, y)\, dA$ over the usual rectangle R, we can think of $dA = dxdy$ as the area of the 'little rectangle' $\triangle A = \triangle x \triangle y$:

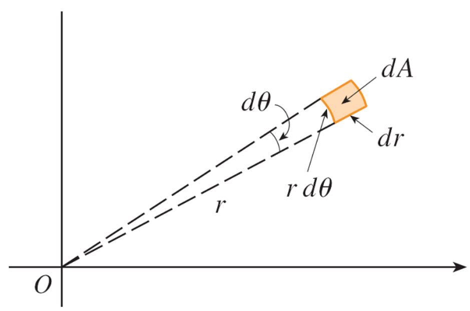
To compute $\iint_R f(x, y)\, dA$ over the **polar rectangle** R given by

$$R = \{(r, \theta) : a \le r \le b, \ \alpha \le \theta \le \beta\}.$$

we partition R as follows:



Then we can think of $dA = r\,dr\,d\theta$ as the area of the 'little polar rectangle' $\triangle A \approx \triangle r(r\triangle\theta)$:



Note that the arc of the polar rectangle is $r\,\triangle\theta$ which depends on r.

Theorem 9.12. *Change to Polar Coordinates in Double Integral*
If f is continuous on a polar rectangle R given by

$$R = \{(r,\theta) : 0 \le a \le r \le b, \alpha \le \theta \le \beta\}$$

where $0 \le \beta - \alpha \le 2\pi$, then

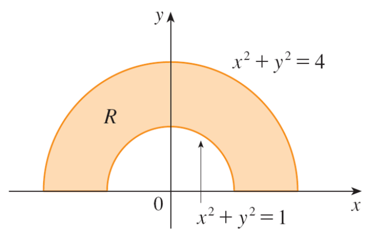
$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$

The formula says that we convert from rectangle to polar coordinates in a double integral by:

- writing $x = r\cos\theta$, $y = r\sin\theta$
- using the appropriate limits of integration for r and θ
- replacing dA by $r\ dr\ d\theta$ (**do not forget the additional** r in $r\ dr\ d\theta$)

Example 9.7. *Evaluate $\iint_R (3x + 4y^2)\,dA$ where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.*

Solution. The region R is shown below:



$R=\{(r,\theta) \mid 1\leqslant r\leqslant 2, 0\leqslant\theta\leqslant\pi\}$

So

$$R=\{(r,\theta): 1\le r\le 2, 0\le\theta\le\pi\}.$$

Changing to polar coordinates for double integral, we have

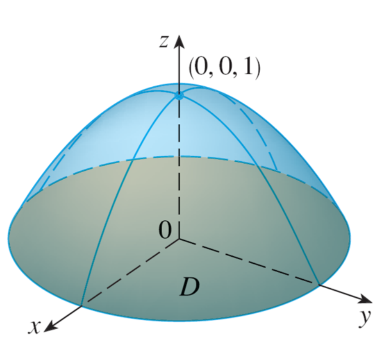
$$\begin{aligned}\iint_R (3x+4y^2)\,dA &= \int_0^{\pi}\int_1^2 (3r\cos\theta+4r^2\sin^2\theta)r\;dr\;d\theta\\ &= \int_0^{\pi}\int_1^2 (3r^2\cos\theta+4r^3\sin^2\theta)\;dr\;d\theta\\ &= \int_0^{\pi}\left[r^3\cos\theta+r^4\sin^2\theta\right]_{r=1}^{r=2} d\theta.\end{aligned}$$

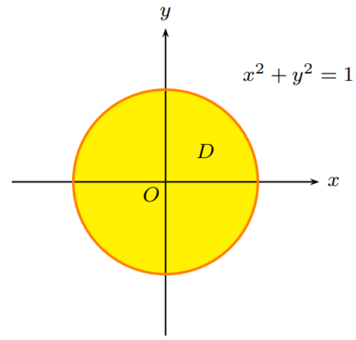
$$\begin{aligned}\iint_R (3x+4y^2)\,dA &= \int_0^{\pi}(7\cos\theta+15\sin^2\theta)\,d\theta\\ &= \int_0^{\pi}\left[7\cos\theta+\frac{15}{2}(1-\cos 2\theta)\right]d\theta\\ &= \left[7\sin\theta+\frac{15\theta}{2}-\frac{15}{4}\sin 2\theta\right]_0^{\pi}\\ &= \frac{15\pi}{2}.\end{aligned}$$

■

Example 9.8. *Find the volume of the solid bounded by the plane $z=0$ and the paraboloid $z=1-x^2-y^2$.*

Solution. Notice the plane and the paraboloid intersect in the circle $x^2+y^2=1$.
So the solid lies under the paraboloid and above the circular disk D given by $x^2+y^2\le 1$.





In polar coordinates, D is given by

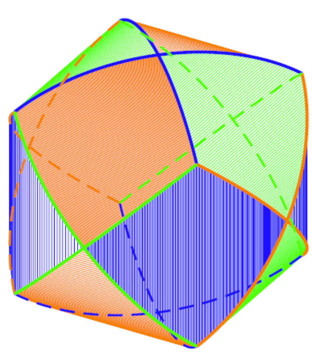
$$D = \{(r, \theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}.$$

Since $1 - x^2 - y^2 = 1 - r^2$, we have

$$\begin{aligned}
\text{Volume} &= \iint_D (1 - x^2 - y^2)\, dA \\
&= \int_0^{2\pi} \int_0^1 (1 - r^2) r\, dr\, d\theta \\
&= \left(\int_0^{2\pi} d\theta \right) \left(\int_0^1 (r - r^3)\, dr \right) \\
&= 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 \\
&= \frac{\pi}{2}.
\end{aligned}$$

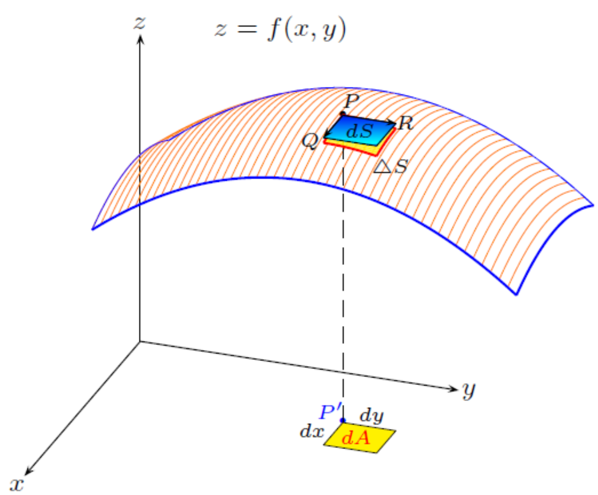
■

Exercise 9.1. *Show that the volume of the solid region bounded by the three cylinders* $x^2 + y^2 = 1$, $y^2 + z^2 = 1$ *and* $x^2 + z^2 = 1$ *is* $16 - 8\sqrt{2}$.



9.10 Surface Area

Let f be a differentiable function of 2 variables defined on a domain D. We wish to find the surface area of the graph of f over D. It is simply equal to $\iint_D dS$. Therefore we need to express the differential of the surface area dS in terms of the differential dA of the domain. To do so, take any point $P'(x,y)$ in D and let P be the corresponding point on the graph of f. Consider an increment dx along the x-direction and an increment dy along the y-direction at the point P'. Thus $dA = |dxdy|$. These increments sweep out an increment of surface area on the surface at P. The differential dS of this area at P is given by the corresponding area on the tangent plane to the surface at P.



Let $\overrightarrow{\boldsymbol{PQ}}$ be the vector on the tangent plane at P with x-component dx, and $\overrightarrow{\boldsymbol{PR}}$ the vector with y-component dy. Thus, $\overrightarrow{\boldsymbol{PQ}} = \langle dx, 0, f_x(x,y)dx\rangle$ and $\overrightarrow{\boldsymbol{PR}} = \langle 0, dy, f_y(x,y)dy\rangle$. The area of the parallelogram spanned by $\overrightarrow{\boldsymbol{PQ}}$ and $\overrightarrow{\boldsymbol{PR}}$ is the magnitude of the cross product $\overrightarrow{\boldsymbol{PQ}} \times \overrightarrow{\boldsymbol{PR}}$.

$$\overrightarrow{\boldsymbol{PQ}} \times \overrightarrow{\boldsymbol{PR}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ dx & 0 & f_x dx \\ 0 & dy & f_y dy \end{vmatrix} = \langle -f_x, -f_y, 1\rangle dxdy.$$

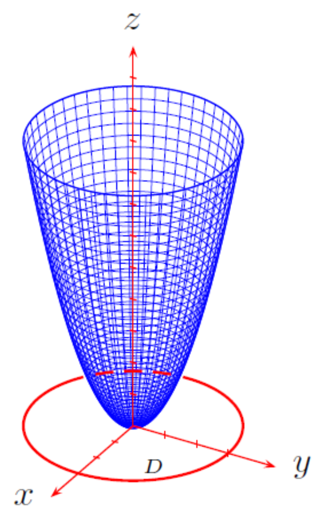
Therefore, $dS = |\langle -f_x, -f_y, 1\rangle dxdy| = \sqrt{f_x^2 + f_y^2 + 1}\, dA$. Consequently,

$$\text{Surface area} = \iint_D dS = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA.$$

Example 9.9. *Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.*

Solution. The paraboloid lies above the circular disk

$$D = \{(r,\theta) \mid 0 \le \theta \le 2\pi, 0 \le r \le 3\}.$$



$$\begin{aligned}
\text{Surface area} &= \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA \\
&= \iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA \\
&= \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\, r\, dr\, d\theta \text{(change to polar coordinates)} \\
&= 2\pi\left[\tfrac{1}{12}(1+4r^2)^{\frac{3}{2}}\right]_0^3 \\
&= \frac{\pi}{6}(37\sqrt{37} - 1).
\end{aligned}$$

■

Exercise 9.2. *Show that the surface area of the solid region bounded by the three cylinders* $x^2 + y^2 = 1$, $y^2 + z^2 = 1$ *and* $x^2 + z^2 = 1$ *is* $48 - 24\sqrt{2}$.

[Hint: Evaluate the surface area of the graph of $f(x,y) = \sqrt{1-x^2}$ over the region $R = \{(r,\theta) \in \mathbb{R}^2 \mid 0 \le r \le 1, 0 \le \theta \le \frac{\pi}{4}\}$.]

Chapter 10

Ordinary Differential Equations

Read Thomas' Calculus, Chapter 9.

10.1 First Order Ordinary Differential Equations

Let y be a function of x. An equation involving x, y and at least one derivative of y is called an ordinary differential equation (ODE). The order of an ODE is the order of the highest derivative that occurs in the equation. We consider only first order ordinary differential equations.

Separable ODE

A separable first order ODE is of the form

$$\frac{dy}{dx} = f(x)g(y).$$

Separating the variables,

$$\frac{1}{g(y)}dy = f(x)dx.$$

Integrating both sides,

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx + C.$$

Example 10.1. *Solve* $y' = (1+y^2)e^x$.

Solution. We separate the variables to obtain $e^x dx = \frac{1}{1+y^2}dy$. Thus $\int e^x dx = \int \frac{1}{1+y^2}dy$. That is $e^x = \tan^{-1} y + C$, or $y = \tan(e^x - C)$.

$\blacksquare$

Example 10.2. *Experiments show that a radioactive substance decomposes at a rate proportional to the amount present. Starting with* 2 *mg at certain time, say* $t = 0$, *what can be said about the amount available at a later time?*

Solution. Let y be the amount of substance in mg at time t in years. Then $\frac{dy}{dt} = -ky$, $y(0) = 2$, where k is a positive constant. Thus $\frac{dy}{y} = -kdt$. Integrating, $\int \frac{dy}{y} = \int -kdt$. That is $\ln|y| = -kt + C$, or equivalently, $|y| = e^{-kt+C} = e^C e^{-kt}$. Therefore, $y = e^C e^{-kt}$ or $y = -e^C e^{-kt}$. In other words, $y = Ae^{-kt}$, where A is a constant. As $y(0) = 2$, we have $2 = Ae^{-k\times 0} = A$. Consequently, $y = 2e^{-kt}$.

■

Remark. How to find k? The value of k depends on the substance. Usually we can calculate k by looking up the half-life of the substance in a chemistry table.

For example, the half-life is T years. From the above solution, we know $y = Ae^{-kt}$. Thus $\frac{A}{2} = Ae^{-kT}$. That is $-\ln 2 = -kT$. From this we obtain $k = \frac{\ln 2}{T}$.

In the report 'Stemming the tide 2020: The reality of the Fukushima radioactive water crisis', Greenpeace claimed that the contaminated water contained "dangerous levels of carbon-14", a radioactive substance that has the "potential to damage human DNA".

Carbon-14 is unstable and has a half-life of 5730 ± 40 years.

Example 10.3. *A copper ball is heated to* $100°$ *C. At time* $t = 0$, *it is placed in water which is maintained at* $30°$ *C. At the end of* 3 *mins, the temperature of the ball is reduced to* $70°$ *C. Find the time at which the temperature of the ball is* $31°$ *C.*

Solution. Physical information: Experiments show that the rate of change $\frac{dT}{dt}$ of the temperature T of the ball with respect to time t is proportional to the difference between T and the temperature of the surrounding medium. Also heat flows so rapidly in copper that at any time the temperature is practically the same at all points of the ball.

Let T be the temperature of the ball at time t. Then $\frac{dT}{dt} = k(T - 30)$, $T(0) = 100, T(3) = 70$. Thus $\int \frac{dT}{T-30} = \int kdt$. That is $\ln|T - 30| = kt + C$, or equivalently, $T - 30 = Ae^{kt}$.

$T(0) = 100 \Rightarrow 100 - 30 = Ae^{k\times 0} \Rightarrow A = 70$. Therefore, $T = 30 + 70e^{kt}$.

$T(3) = 70 \Rightarrow 70 = 30 + 70e^{3k} \Rightarrow 4 = 7e^{3k} \Rightarrow k = \frac{1}{3}\ln\frac{4}{7} = \frac{1}{3}(\ln 4 - \ln 7)$. Therefore, $T = 30 + 70e^{\frac{t}{3}(\ln 4 - \ln 7)}$.

Then $T = 31 \Rightarrow 1 = 70e^{\frac{t}{3}(\ln 4 - \ln 7)} \Rightarrow \frac{t}{3}(\ln 4 - \ln 7) = \ln\frac{1}{70} = -\ln 70 \Rightarrow t = \frac{3\ln 70}{\ln 7 - \ln 4} = 22.78\,\text{min}.$

■

Example 10.4. *A skydiver together with his equipment has a combined weight of m kg. After he jumps and the parachute opens at time* $t = 0$, *he falls freely and is descending with velocity v m/s at the moment when the time is t s. The air resistance against his descending motion is known to be* bv^2 *N, where b is a positive constant, and v is his velocity at that moment. Show that the skydiver eventually approaches a terminal speed of* $k \equiv \sqrt{\frac{mg}{b}}$ *m/s, where* $g = 9.81$ m/s^2 *is the acceleration due to gravity.*

Solution. By Newton's second law, $m\frac{dv}{dt} = mg - bv^2$. Rewrite this as $\frac{dv}{dt} = -\frac{b}{m}(v^2 - \frac{mg}{b}) = -\frac{b}{m}(v^2 - k^2)$. That is $\frac{dv}{v^2-k^2} = -\frac{b}{m}dt \Leftrightarrow \frac{1}{2k}(\frac{1}{v-k} - \frac{1}{v+k})dv = -\frac{b}{m}dt$. Integrating, we have $\ln|v-k| - \ln|v+k| = -\frac{2kb}{m}t + C \Leftrightarrow \ln\left|\frac{v-k}{v+k}\right| = -\frac{2kb}{m}t + C \Leftrightarrow \frac{v-k}{v+k} = Ae^{-\frac{2kb}{m}t}$. Solving for v, we obtain

$$v = \left(\frac{1 + Ae^{-\frac{2kb}{m}t}}{1 - Ae^{-\frac{2kb}{m}t}}\right)k.$$

From this, we see that $\lim_{t\to\infty} v = k$.

■

Exercise 10.1. *Solve the differential equation*

$$\frac{dy}{dx} = xe^{3x-2y}.$$

Ans: $\frac{1}{2}e^{2y} = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$.

Exercise 10.2. *A curve C that passes through the point $(2,1)$ is such that at any point (x,y) on the curve,*

$$x^2\frac{dy}{dx} = y(x^3 + 4).$$

Find the equation of the curve.

Ans: $y = e^{\frac{x^2}{2} - \frac{4}{x}}$.

10.2 Reduction to Separable Form

Certain first order differential equations are not separable but can be made separable by a simple change of variables.

This holds for equations of the form

$$y' = g(\frac{y}{x}),$$

where g is any function of $\frac{y}{x}$. Let $v = \frac{y}{x}$, then $y = vx$ and $y' = v + xv'$. Then the equation $y' = g(\frac{y}{x})$ can be written as $v + xv' = g(v)$, which is separable. That is $\frac{dv}{g(v)-v} = \frac{dx}{x}$. We can now solve for v, hence obtain y.

Example 10.5. *Solve* $2xyy' - y^2 + x^2 = 0$.

Solution. We may rewrite the equation as $y' = \frac{-1+(\frac{y}{x})^2}{2(\frac{y}{x})}$. Let $v = \frac{y}{x}$. Then the equation can be written as $v+xv' = \frac{-1+v^2}{2v}$. That is $xv' = \frac{-1-v^2}{2v} \Leftrightarrow \frac{2v\,dv}{1+v^2} = -\frac{dx}{x}$. Integrating, $\ln|1+v^2| = -\ln|x|+C$. That is $1 + v^2 = Ke^{-\ln|x|} = \frac{K}{|x|}$. Therefore, $1 + \frac{y^2}{x^2} = \frac{A}{x}$, or equivalently, $x^2 + y^2 = Ax$.

■

A differential equation of the form $y' = f(ax + by + c)$ where f is continuous and $b \neq 0$ (if $b = 0$, the equation is separable) can be solved by setting $u = ax + by + c$.

Example 10.6. *Solve* $(2x - 4y + 5)y' + x - 2y + 3 = 0$.

Solution. Let $u = x-2y$. Then $u' = 1-2y'$. Thus the equation becomes $(2u+5)\frac{1}{2}(1-u')+u+3 = 0 \Leftrightarrow (2u + 5)u' = 4u + 11$.

Separating variables, and integrating, $(1 - \frac{1}{4u+11})du = 2dx$. Thus $u - \frac{1}{4}\ln|4u + 11| = 2x + C_1$, or $4x + 8y + \ln|4x - 8y + 11| = C$.

$4x - 8y + 11 = 0$ is also a solution.

■

Exercise 10.3. *Solve the initial value problem* $y' = \frac{y}{x} + \frac{2x^3\cos(x^2)}{y}$, $y(\sqrt{\pi}) = 0$.

Ans: $y = \pm x\sqrt{2\sin(x^2)}$.

Exercise 10.4. *Solve* $(x + 2y - 1) + 3(x + 2y)y' = 0$.

Ans: $x + 3y + C = 3\ln|x + 2y + 2|$, $x + 2y + 2 = 0$.

10.3 Linear First Order ODE

A linear first order ODE is of the form

$$\frac{dy}{dx} + P(x)y = Q(x),$$

where $Q(x)$ is not identically zero. Note that the above ODE is separable if $P(x)$ is identically equal to $Q(x)$. This is the *standard form* of a linear first order ODE.

Let $I(x) = e^{\int P(x)\,dx}$. We call $I(x)$ an integrating factor. Multiplying both sides of the above ODE by $I(x)$, we get

$$\frac{dy}{dx}e^{\int P(x)\,dx} + P(x)e^{\int P(x)\,dx}y = Q(x)e^{\int P(x)\,dx}.$$

But

$$\frac{dy}{dx}e^{\int P(x)\,dx} + P(x)e^{\int P(x)\,dx}y = \frac{d}{dx}\left(ye^{\int P(x)\,dx}\right),$$

which can be shown by applying the product rule and the Fundamental Theorem of Calculus. Hence,

$$\frac{d}{dx}\left(ye^{\int P(x)\,dx}\right) = Q(x)e^{\int P(x)\,dx}.$$

We have thus shown that

$$\frac{d}{dx}(y \cdot I(x)) = Q(x) \cdot I(x).$$

Integrating both sides gives

$$y \cdot I(x) = \int Q(x) \cdot I(x)\,dx$$

from which the solution for y can be obtained.

Example 10.7. *Solve* $xy' - 3y = x^2, x > 0$.

Solution. Rewrite the DE in the standard form $y' - \frac{3}{x}y = x$. An integrating factor is $e^{\int -\frac{3}{x}\,dx} = e^{-3\ln x} = \frac{1}{x^3}$. Multiplying the DE (the standard form) by this integrating factor, we have $(\frac{y}{x^3})' = \frac{1}{x^2}$. Integrating, we have $\frac{y}{x^3} = -\frac{1}{x} + C$. That is $y = -x^2 + Cx^3$.

■

Example 10.8. *Solve* $y' - y = e^{2x}$.

Solution. An integrating factor is $e^{\int -1\,dx} = e^{-x}$. Multiplying the DE by this integrating factor, we have $(ye^{-x})' = e^x$. Integrating, we obtain $ye^{-x} = e^x + C$. That is $y = e^{2x} + Ce^x$.

■

Exercise 10.5. *Solve the differential equation*

$$(x+1)^2\frac{dy}{dx} - (x+1)y = 2, \; x > -1.$$

Ans: $y = -\frac{1}{x+1} + C(x+1)$.

Exercise 10.6. *Solve the differential equation*

$$\frac{dy}{dx} = \frac{4 + y\sin x}{\cos x}, \; -\frac{\pi}{2} < x < \frac{\pi}{2},$$

given that $y = 6$ when $x = 0$.

Ans: $y = \frac{4x+6}{\cos x}$.

Exercise 10.7. *An object of mass m dropped from rest in a medium that offers a resistance proportional to the magnitude of the instantaneous velocity of the object. Let $x(t)$ be the displacement of the object measured vertically downward at time t so that $x(0) = 0$. Show that*

$$x(t) = \frac{mg}{k}t + \frac{m^2 g}{k^2}(e^{-\frac{k}{m}t} - 1),$$

where k is the proportional (positive) constant of the force of resistance of the medium.

[Set up the DE for the velocity first: $m\frac{dv}{dt} = mg - kv$.]

10.4 The Bernoulli Equation.

An ODE in the form

$$y' + p(x)y = q(x)y^n,$$

where $n \neq 0, 1$, is called the *Bernoulli equation*. The functions $p(x)$ and $q(x)$ are continuous functions on an interval J.

Let $u = y^{1-n}$. Substituting into the Bernoulli equation we get

$$u' + (1-n)p(x)u = (1-n)q(x).$$

This is a first order linear ODE.

Example 10.9. *Solve $y' + y = x^2y^2$.*

Solution. Let $z = y^{1-2} = y^{-1}$. Then $z' = -y^{-2}y'$. Thus the given Bernoulli equation can be written as $-y^2z'+y = x^2y^2 \Leftrightarrow z'-y^{-1} = -x^2 \Leftrightarrow z'-z = -x^2$. This is a first order linear equation. Multiplying by the integrating factor $e^{\int -dx} = e^{-x}$, we have $(ze^{-x})' = -x^2e^{-x}$. Integrating, $ze^{-x} = \int -x^2e^{-x}\,dx$. Using integration by parts, we have $\int -x^2e^{-x}\,dx = x^2e^{-x} + 2xe^{-x} + 2e^{-x} + C$. Thus $z = e^x(x^2e^{-x} + 2xe^{-x} + 2e^{-x} + C)$. Therefore, $\frac{1}{y} = x^2 + 2x + 2 + Ce^x$. Consequently, $y = \frac{1}{x^2+2x+2+Ce^x}$.

$y = 0$ is a also a solution.

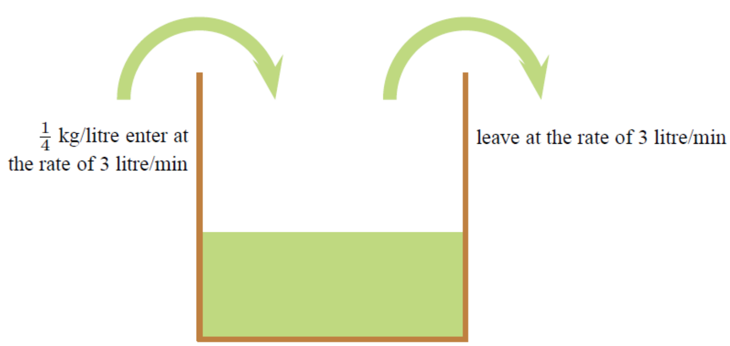
■

Exercise 10.8. Solve $xy' + y = x^4y^3$.

Ans: $\frac{1}{y^2} = -x^4 + cx^2$, or $y = 0$.

10.5 Applications of ODE

Example 10.10. *At time $t = 0$, a tank contains 20 kg of salt dissolved in 100 litres of water. Assume that water containing $\frac{1}{4}$ kg of salt per litre is entering the tank at the rate of 3 litre per min, and the well-stirred solution is leaving the tank at the same rate. Find the amount of salt at any time t.*



At $t = 0$, there is 100 litre of water with 20 kg salt dissolved in it.
At time t, there remains 100 litre of water with Q kg salt dissolved in it.

Solution. First note that the volume of the solution remains constant which is 100 litres. Let Q be the amount of salt in kg at time t. The concentration of salt in the solution is $Q/100$ kg per litre. Suppose at time $t + dt$, the amount of salt is $Q + dQ$. Then

$$dQ = \text{salt input} - \text{salt output} = 3 \times \frac{1}{4} \times dt - 3 \times \frac{Q}{100} \times dt.$$

Thus
$$\frac{dQ}{dt} = \frac{3}{4} - \frac{3Q}{100}.$$
That is
$$\frac{dQ}{dt} = -\frac{3}{100}(Q-25).$$
The general solution to this first order linear DE is $Q = 25 + Ce^{-\frac{3t}{100}}$. Since $Q(0) = 20$, we have $20 = 25 + C$ so that $C = -5$. Consequently, $Q = 25 - 5e^{-\frac{3t}{100}}$.

Note that $\lim_{t\to\infty} Q(t) = 25$. Thus after sufficiently long time, the salt concentration remains constant at 25 kg per 100 litres.

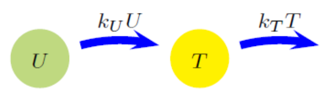
■

Example 10.11. *A body was found at a crime scene. You are a member of the CSI team and you arrived at the crime scene at 8AM. Immediately upon arrival, you took the temperature of the victim and found that it was 26°C. At 9AM, you took the temperature of the victim again and found that it was 24°C. You estimate that the victim's temperature was 37°C just before death and that the temperature at the crime scene stayed approximately constant at 21°C. What is your estimate on the time of death?*

Solution. Set time $t = 0$ at 8AM, where t is measured in hours. Let T be the temperature of the body at time t. We have $\frac{dT}{dt} = k(T-21)$. The general solution is $T = 21 + Ae^{kt}$. As $T(0) = 26$, we have $26 = 21 + A$ so that $A = 5$. Therefore, $T = 21 + 5e^{kt}$. At 9AM, that is 1 hour later, $T(1) = 24$. Thus $24 = 21 + 5e^k$ so that $k = \ln(\frac{3}{5})$. Hence
$$T = 21 + 5e^{\ln(\frac{3}{5})t} = 21 + 5\left(\frac{3}{5}\right)^t.$$
If τ is the time of death, then $T(\tau) = 37$. Therefore,
$$37 = 21 + 5\left(\frac{3}{5}\right)^\tau.$$
That is $\frac{16}{5} = \left(\frac{3}{5}\right)^\tau \Leftrightarrow \ln(\frac{16}{5}) = \tau\ln(\frac{3}{5}) \Leftrightarrow \tau = \ln(\frac{16}{5})/\ln(\frac{3}{5}) = -2.277$ hours, (or equivalently negative 2 hour 17 mins). Thus time of death is about $5:43$AM.

■

Example 10.12. *In example 10.2, we saw that radioactive substance typically decays at a rate proportional to the amount present. Sometimes, the product of a radioactive decay is itself a radioactive substance which in turn decays at a different rate. An interesting example of this is provided by Uranium-Thorium dating, which is a method used by palaeontologists to determine how old certain fossils, especially ancient corals, are. Corals filter the sea water in which they live. Sea water contains a tiny amount Uranium 234 and corals absorb this into their bodies. Uranium 234 decays, with a half-life of* 245000 *years, into Thorium 230, which itself decays with a half-life of* 75000 *years. While a coral is growing, it incorporates a lot of uranium, but no thorium. This* *means that as it ages its thorium/uranium ratio increases at a known rate. It is possible to measure the ratio of the amounts of Uranium and Thorium in any given sample. From this ratio we can work out the age of the sample - the time when it died. Suppose the ratio of the amount of Thorium and Uranium in a sample of coral is* 0.0274. *Determine the age of the coral.*



$k_U \neq k_T, k_U > 0, k_T > 0.$

Solution. Let $U(t)$ be the amount of Uranium in a particular sample of ancient coral and let $T(t)$ be the amount of Thorium. Because each decay of one Uranium atom produces one Thorium atom. Thorium atoms are being born at exactly the same rate at which Uranium atoms die. Thus we have

$$\begin{cases} \frac{dU}{dt} = -k_U U & (1) \\ \frac{dT}{dt} = k_U U - k_T T & (2) \\ U(0) = U_0 & \\ T(0) = 0. & \end{cases}$$

(1) implies that $U = U_0 e^{-k_U t}$. (3)

Since the half-life of Uranium is 245000 years, we get $k_U = \frac{\ln 2}{245000}$. Similarly, $k_T = \frac{\ln 2}{75000}$.

From (2) and (3), we have $\frac{dT}{dt} + k_T T = k_U U_0 e^{-k_U t}$. An integrating factor is $e^{\int k_T\,dt} = e^{k_T t}$. It follows that

$$T = e^{-k_T t}\left(\frac{k_U U_0}{k_T - k_U} e^{(k_T - k_U)t} + C\right).$$

$T(0) = 0 \Rightarrow 0 = \frac{k_U U_0}{k_T - k_U} + C$ so that $C = -\frac{k_U U_0}{k_T - k_U}$. Consequently,

$$T = \frac{k_U U_0}{k_T - k_U}(e^{-k_U t} - e^{-k_T t}). \qquad (4)$$

Dividing (4) by (3), we have

$$\frac{T}{U} = \frac{k_U}{k_T - k_U}(1 - e^{-(k_T - k_U)t}).$$

By measuring $\frac{T}{U}$ at the present time, we can calculate t which gives the age of the sample.

Consider $k_U = \frac{\ln 2}{245000}$ and $k_T = \frac{\ln 2}{75000}$ in our example. Suppose $\frac{T}{U} = 2.74\%$. Then direct computation gives $t \approx 10000$ years.

■

Exercise 10.9. *A fossilized bone is found to contain* 40% *of the original amount of Carbon-14. The half-life of Carbon-14 is* 5600 *years. Estimate the age of the fossil to the nearest 100 years.*

Ans: 7400 years.

Exercise 10.10. *The Jurong lake has a volume of* $700000\,\mathrm{m}^3$. *At time* $t = 0$, *the government starts a water cleaning process so that only fresh clean water flows into the lake. After* 5 *years, it is found that the pollution in the lake is reduced by* 50%. *If fresh water flows into the lake at a rate of* r *cubic metres per year and lake water flows out to the sea at the same rate, what is the value of* r *correct to the nearest thousands?*

Ans: 97000.

Exercise 10.11. *Newton's law of cooling states that the rate of cooling of an object is proportional to the difference in temperature between the object and its surroundings. If an object is kept in an environment whose temperature is kept constant at* 15°C *and the object takes* 20 *minutes to cool from* 95°C *to* 55°C, *determine how much longer it will take for the object to cool down to* 25°C.

Ans: 40 mins more.

Exercise 10.12. *In a chemical reaction, the rate at which the mass,* m *(in grams) of a chemical compound at time* t *(in seconds) is proportional to* $m^2 - 9m + 18$, $(0 < m < 3)$. *Initially* $(t = 0)$, *we assume* $m = 0$. *After* 1 *second, the mass of the chemical compound has increased to* 2g. *Write down a differential equation in* m *and* t, *and show that*

$$\frac{m-6}{m-3} = 2^{t+1}.$$

Find the exact mass of the chemical compound after 2 *seconds.*

Ans: 18/7 grams.

10.6 Malthus Model of Population

The total population $N(t)$ of a country or a colony is clearly a function of time. $N(t)$ *though should be integer valued and is great than* 0, *is considered as a continuous and in fact differentiable function of time, especially its value is usually very huge.*

Given the population now, can one predict the future population?

Suppose B is a function giving the '*per capita birth rate*" in a given society, i.e. B is the number of babies born per second, divided by the total population of the country at that moment. Note that B could be small in a big country and large in a small country - it depends on whether there is a strong social pressure on people to get married and have kids. Now B

could depend on time (people might gradually come to realise that large families are no fun, etc..) and it could depend on N. But suppose you don't believe these things. Instead suppose people will always have as many kids as they can, no matter what. Then B is a constant. Thus

$$\textit{number of babies born in time interval } dt = BN dt.$$

Similarly, let D be the death rate per capita. Again it could be a function of t (in case the society has better medicine, fewer smokers etc) or N (overcrowding leads to famine or disease). But if we assume that it is constant, then

$$\textit{number of deaths in time interval } dt = DN dt.$$

So the change in N, denoted by dN within the time interval dt is

$$dN = \textit{number of births} - \textit{number of deaths},$$

provided there is no emigration or immigration. Thus

$$dN = (B-D)N dt.$$

That is

$$\frac{dN}{dt} = (B-D)N = kN, \tag{1}$$

where $k = B-D$.

This model of society was put forward by Thomas Malthus in 1798. Clearly Malthus was assuming a socially static society in which human reproductive behaviour never changes with time or overcrowding, poverty etc.. What does Malthus' model predict?

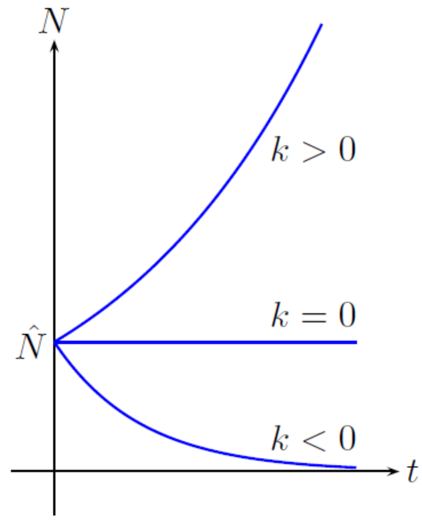
Suppose that the population now is $\hat{N}$ and let $t = 0$ now.

From (1), $\frac{dN}{dt} = kN \Rightarrow \int \frac{dN}{N} = k \int dt = kt + C \Rightarrow \ln(N) = kt + C \Rightarrow N(t) = Ae^{kt}$.

Since $\hat{N} = N(0) = A$, we get

$$N(t) = \hat{N}e^{kt}. \tag{2}$$

(1) is the logistic equation and (2) is the solution of the standard Malthus's model.



Graphs of $N(t)$ for different values of k.

The population collapses if $k < 0$ (more deaths than births per capita), remain stable if (and only if) $k = 0$, and it explodes if $k > 0$ (more births than deaths). Malthus observed that the population of Europe was increasing, so he predicted a catastrophic population explosion; since the food supply could not be expanded so fast, this would be disastrous.

In fact, this didn't happen in Europe. So Malthus' model is not quite correct; as many millions went to the US, and many millions died in wars.

Malthus' model can be improved. Note that Malthus' model is interesting because it shows that static behaviour patterns can lead to disaster. But precisely because the term e^{kt} grows so quickly, Malthus' assumptions must eventually go wrong - obviously there is a limit to the possible population. Eventually, if we don't control B, then D will have to increase. So we have to assume that D is a function of N.

Clearly, D must be an increasing function of N, but which function? The simplest possible choice is

$$D = sN, \tag{3}$$

where s is a constant.

Now we want to solve

$$\frac{dN}{dt} = BN - sN^2, \quad N(0) = \hat{N}.$$

Rewrite the equation as

$$\frac{dN}{dt} - BN = -sN^2.$$

This is a Bernoulli equation.

Let $z = N^{1-2} = \frac{1}{N}$. Then $\frac{dz}{dt} = -\frac{1}{N^2}\frac{dN}{dt}$. Thus $-\frac{N^2 dz}{dt} - BN = -sN^2$. That is $\frac{dz}{dt} + Bz = s$ which is a linear equation in z. An integrating factor is e^{Bt}. Thus

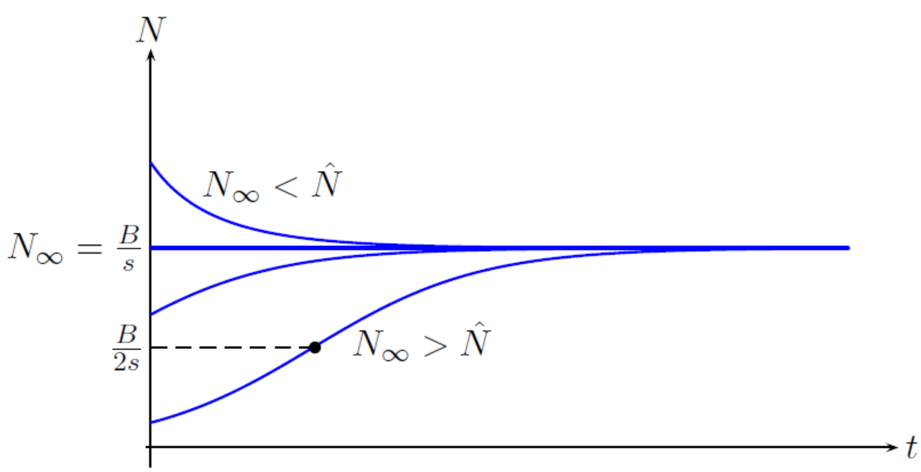
$\frac{dz}{dt} + Bz = s \Leftrightarrow \frac{d(ze^{Bt})}{dt} = se^{Bt} \Leftrightarrow ze^{Bt} = \frac{s}{B}e^{Bt} + C \Leftrightarrow z = \frac{s}{B} + Ce^{-Bt}$. That is $\frac{1}{N} = \frac{s}{B} + Ce^{-Bt}$.
Let $N_\infty = \frac{B}{s}$ which is the *carrying capacity*. Then $\frac{1}{N} = \frac{1}{N_\infty} + Ce^{-Bt}$.
Now $N(0) = \hat{N} \Rightarrow \frac{1}{\hat{N}} = \frac{1}{N_\infty} + C \Rightarrow C = \frac{1}{\hat{N}} - \frac{1}{N_\infty}$.
Thus

$$\frac{1}{N} = \frac{1}{N_\infty} + \left(\frac{1}{\hat{N}} - \frac{1}{N_\infty}\right)e^{-Bt}.$$

Rearranging,

$$N = \frac{N_\infty}{1 + \left(\frac{N_\infty}{\hat{N}} - 1\right)e^{-Bt}}. \tag{4}$$

Note that $\lim_{t\to\infty} N = N_\infty$, as $B > 0$. Also $N(t)$ is increasing if $N_\infty > \hat{N}$, and $N(t)$ is decreasing if $N_\infty < \hat{N}$. Thus $\frac{dN}{dt} \neq 0$ if $N_\infty \neq \hat{N}$. (4) is the solution of the improved Malthus' model.



$N_\infty = \frac{B}{s}$ is the carrying capacity. Point of inflection at $N = \frac{B}{2s}$.

Example 10.13. *The growth of rabbits in your rabbit farm followed a logistic population model with a birth rate per capita of 10 rabbits per rabbit per year. You observed that their number had approached to a logistic equilibrium population of 2500 rabbits. One day your friend Dr. Good visited your farm and suggested that you try to mix some of his latest invention of Vitamin X into your rabbit feed to boost the reproduction rate. You followed his suggestion and after a long period of time, observed that the rabbit population had reached a new logistic equilibrium of 3000 rabbits. If the new rabbit birth rate per capita after Vitamin X was introduced was B rabbits per rabbit per year, what is the value of B?*

Solution. We have $\frac{10}{s} = 2500$ so that $s = \frac{1}{250}$. Thus $\frac{B}{s} = 3000 \Rightarrow B = \frac{3000}{250} = 12$.

■

Exercise 10.13. *Suppose $N_\infty > 2\hat{N}$. Show that there is a point of inflection on the graph of N at $t > 0$.*

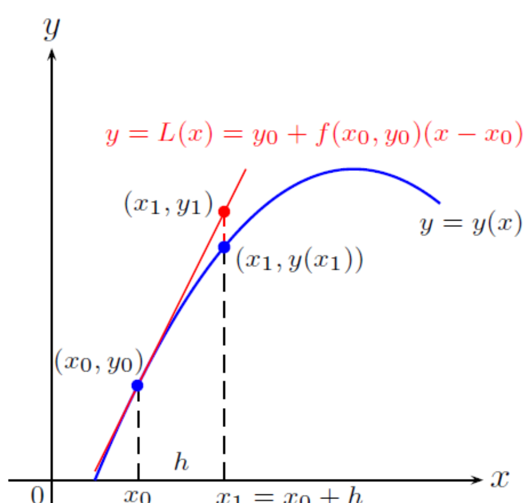
10.7 Euler's Method

Not all the first order ODE's can be solved explicitly in closed form. In that case, we have to rely on numerical solutions. In this section, we introduce Euler's method which is a numerical method in approximating a first order ODE. Given a differential equation $\frac{dy}{dx} = f(x,y)$ and an initial condition $y(x_0) = y_0$, we can approximate the solution $y = y(x)$ by its linearization

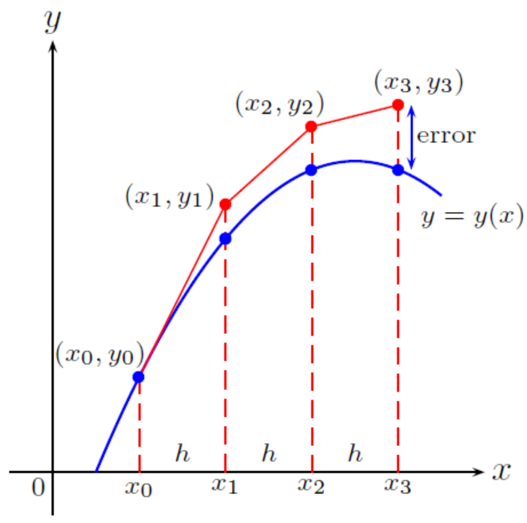
$$L(x) = y(x_0) + y'(x_0)(x - x_0) \text{ or } L(x) = y_0 + f(x_0, y_0)(x - x_0).$$

The linear function $L(x)$ gives a good approximation to the solution in a short interval about x_0. The idea of Euler's method is to put together a sequence of such linearizations in a successive manner to approximate the solution curve over a longer interval.

First we know that the point (x_0, y_0) lies on the solution curve. Consider a small increment from x_0 to $x_1 \equiv x_0 + h$. The graph of $L(x)$ is the tangent line with slope $f(x_0, y_0)$ to the solution curve $y = y(x)$ at the point (x_0, y_0). So if h is small, $y_1 \equiv L(x_1)$ is a good approximation to $y(x_1)$. In other word, the point (x_1, y_1) is close to the solution curve $y = y(x)$.



The linearization $L(x)$ at $x = x_0$.
The first step approximates $y(x_1)$ with $y_1 = L(x_1)$.



The red polygonal curve is Euler's approximation.

Using the point (x_1, y_1) and the slope $f(x_1, y_1)$ of the solution curve through (x_1, y_1), we take a second step. Setting $x_2 = x_1 + h$, we use the linearization of the solution curve through (x_1, y_1) to calculate $y_2 = y_1 + f(x_1, y_1)h$.

This gives the next approximation (x_2, y_2) to the value along the solution curve $y = y(x)$. Continuing in this way, we take a third step from the point (x_2, y_2) with slope $f(x_2, y_2)$ to obtain the third approximation $y_3 = y_2 + f(x_2, y_2)h$, and so on.

In other words, we are building an approximation to one of the solution by following the direction of the slope field of the differential equation.

The following steps summarize Euler's method. Suppose we wish to approximate the solution over the interval $[a, b]$. Choose an integer n as the number of steps. Let $h = \frac{b-a}{n}$. Let

$$\begin{array}{rcl} x_0 & = & a \\ x_1 & = & x_0 + h \\ x_2 & = & x_1 + h \\ & \vdots & \\ b = x_n & = & x_{n-1} + h. \end{array}$$

Then calculate the approximations to the solution as follows.

$$\begin{array}{rcl} y_1 & = & y_0 + f(x_0, y_0)h \\ y_2 & = & y_1 + f(x_1, y_1)h \\ & \vdots & \\ y_n & = & y_{n-1} + f(x_{n-1}, y_{n-1})h. \end{array}$$

The polygonal curve joining the points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ successively is an approximation to the solution curve of the DE $y' = f(x, y)$ through the point (x_0, y_0).

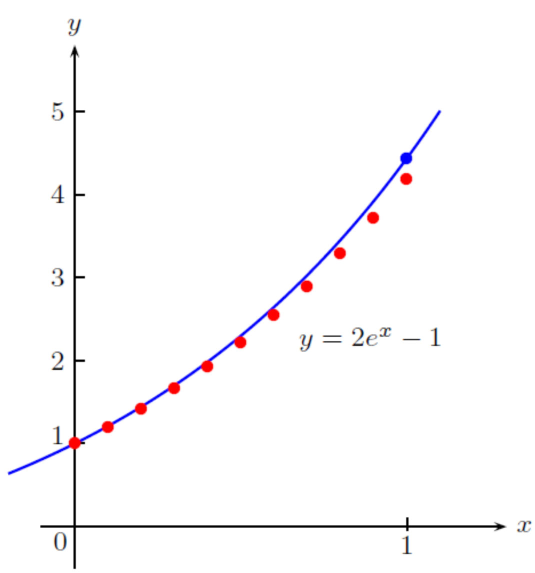
Example 10.14. *Use Euler's method to solve*

$$y' = 1 + y, y(0) = 1,$$

on the interval $[0,1]$ *starting at* $x_0 = 0$ *by taking* $h = 0.1$. *Find the approximate value of* $y(1)$ *and compare it with the exact value.*

Solution. Taking $n = 10$ and $h = 0.1$, the result is tabulated in the following table. The exact solution to the DE is $y = 2e^x - 1$. Thus the exact value at $x = 1$ is $y(1) = 2e - 1 = 4.4366$. The approximate value is 4.1875.

Euler solution of $y' = 1 + y, y(0) = 1$, $h = 0.1$			
x	y(Euler)	y(Exact)	Error
0	1	1	0
0.1	1.2	1.2103	0.0103
0.2	1.42	1.4428	0.0228
0.3	1.662	1.6997	0.0377
0.4	1.9282	1.9836	0.0554
0.5	2.221	2.2974	0.0764
0.6	2.5431	2.6442	0.1011
0.7	2.8974	3.0275	0.1301
0.8	3.2872	3.4511	0.1639
0.9	3.7159	3.9192	0.2033
1	4.1875	4.4366	0.2491



The graph of $y = 2e^x - 1$ and its Euler's approximation

Exercise 10.14. *Use Euler's method to calculate the first three approximations to the initial value problem:*

$$y' = 2xy + 2y,\ y(0) = 3,$$

by taking $h = 0.2$.

Ans: $y_1 = 4.2, y_2 = 6.216, y_3 = 9.697$.

Remark. In the film Hidden Figures, Katherine Goble resorts to Euler's method in calculating the re-entry of astronaut John Glenn from Earth orbit.

10.8 2nd Order Linear Equations with Constant Coefficients

Let us begin with second order homogenous linear equation with constant coefficients

$$y'' + ay' + by = 0, \tag{10.1}$$

where a and b are real constants. We look for a solution of the form $y = e^{\lambda x}$. Plugging into (10.1) we find that, $e^{\lambda x}$ is a solution of (10.1) if and only if

$$\lambda^2 + a\lambda + b = 0. \tag{10.2}$$

(10.2) is called the *auxiliary equation* or *characteristic equation* of (10.1). The roots of (10.2) are called *characteristic values* (or eigenvalues):

$$\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b}),$$
$$\lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b}).$$

1. If $a^2 - 4b > 0$, (10.2) has two distinct real roots λ_1, λ_2, and the general solutions of (10.1) is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

2. If $a^2 - 4b = 0$, (10.2) has one real root λ (we may say that (10.2) has two equal roots $\lambda_1 = \lambda_2$). The general solution of (10.1) is

$$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}.$$

3. If $a^2 - 4b < 0$, (10.2) has a pair of complex conjugate roots

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta.$$

The general solution of (10.1) is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x).$$

Example 10.15. Solve $y'' + y' - 2y = 0$, $y(0) = 4$, $y'(0) = -5$.

Solution. The characteristic values are $1, -2$. Thus the solution is

$$y = e^x + 3e^{-2x}.$$

■

Example 10.16. Solve $y'' - 4y' + 4y = 0$, $y(0) = 3$, $y'(0) = 1$.

Solution. The characteristic values are 2 with multiplicity 2. Thus the solution is

$$y = (3 - 5x)e^{2x}.$$

■

Example 10.17. Solve $y'' - 2y' + 10y = 0$.

Solution. The characteristic values are $1 + 3i$, $1 - 3i$. Thus the solution is

$$y = e^x(c_1 \cos 3x + c_2 \sin 3x).$$

■

10.9 Method of Undetermined Coefficients

Consider the equation $y'' + ay' + by = f(x)$, where a and b are real constants. To solve this non-homogeneous linear DE, we look for a particular solution y_p of $y'' + ay' + by = f(x)$.

Then the general solution is the sum of the general solution y_c of the associated homogeneous linear DE: $y'' + ay' + by = 0$ and this particular solution y_p. That is

$$y = y_c + y_p.$$

Case 1. $f(x) = P_n(x)e^{\alpha x}$, where $P_n(x)$ is a polynomial of degree $n \geq 0$.
We look for a particular solution in the form

$$y = Q(x)e^{\alpha x},$$

where $Q(x)$ is a polynomial. Plugging it into $y'' + ay' + by = f(x)$ we find

$$Q'' + (2\alpha + a)Q' + (\alpha^2 + a\alpha + b)Q = P_n(x). \tag{10.3}$$

Subcase 1.1. If $\alpha^2 + a\alpha + b \neq 0$, namely, α is not a root of the characteristic equation, we choose $Q = R_n$, a polynomial of degree n, and

$$y = R_n(x)e^{\alpha x}.$$

The coefficients of R_n can be determined by comparing the terms of same power in the two sides of (10.3). Note that in this case both sides of (10.3) are polynomials of degree n.

Subcase 1.2. If $\alpha^2 + a\alpha + b = 0$ but $2\alpha + a \neq 0$, namely, α is a simple root of the characteristic equation, then (10.3) is reduced to

$$Q'' + (2\alpha + a)Q' = P_n. \tag{10.4}$$

We choose Q to be a polynomial of degree $n+1$. Since the constant term of Q does not appear in (10.4), we may choose $Q(x) = xR_n(x)$, where $R_n(x)$ is a polynomial of degree n.

$$y = xR_n(x)e^{\alpha x}.$$

Subcase 1.3 If $\alpha^2 + a\alpha + b = 0$ and $2\alpha + a = 0$, namely, α is a root of the characteristic equation with multiplicity 2, then (10.3) is reduced to

$$Q'' = P_n. \tag{10.5}$$

We choose $Q(x) = x^2 R_n(x)$, where $R_n(x)$ is a polynomial of degree n.

$$y = x^2 R_n(x)e^{\alpha x}.$$

Example 10.18. Find the general solution of $y'' - y' - 2y = 4x^2$.

Solution. The homogeneous equation has $\lambda^2 - \lambda - 2 = 0$ as its characteristic equation with roots $\lambda = 2, -1$.

Therefore the general solution of the associated homogeneous equation is $y = c_1 e^{2x} + c_2 e^{-x}$.

Note that $4x^2 = 4x^2 e^{0x}$ and 0 is not a root of the characteristic equation. We can try a particular solution of the form

$$y_p = A + Bx + Cx^2.$$

Substituting this into the equation, we have

$$2C - (B + 2Cx) - 2(A + Bx + Cx^2) = 4x^2.$$

Equating coefficients, we have

$$\begin{aligned} 2C - B - 2A &= 0 \\ -2C - 2B &= 0 \\ -2C &= 4 \end{aligned}$$

Thus $A = -3, B = 2, C = -2$, and $y = -3 + 2x - 2x^2$.

The general solution is

$$y = c_1 e^{2x} + c_2 e^{-x} - 3 + 2x - 2x^2.$$

■

Example 10.19. Solve $y'' - 2y' + y = xe^x$.

Solution. The general solution of the associated homogeneous DE is $C_1e^x + C_2xe^x$.

Here $\alpha = 1$ is a double root of the characteristic equation $\lambda^2 - 2\lambda + 1 = 0$. Therefore, we try a particular solution of the form $y = x^2(A + Bx)e^x$.

We have $y' = (Bx^3 + (A+3B)x^2 + 2Ax)e^x$ and $y'' = (Bx^3 + (A+6B)x^2 + (4A+6B)x + 2A)e^x$.

Substituting these into the DE, we have $(2A + 6Bx)e^x = xe^x$. Thus $A = 0$ and $B = \frac{1}{6}$.

Consequently, the general solution is $y = C_1e^x + C_2xe^x + \frac{1}{6}x^3e^x$.

■

Case 2. $f(x) = P_n(x)e^{\alpha x}\cos(\beta x)$ or $f(x) = P_n(x)e^{\alpha x}\sin(\beta x)$, where $P_n(x)$ is a polynomial of degree $n \geq 0$.

We first look for a solution of

$$y'' + ay' + by = P_n(x)e^{(\alpha+i\beta)x}. \tag{10.6}$$

Using the method in Case 1 we obtain a complex-valued solution

$$z(x) = u(x) + iv(x),$$

where $u(x) = \mathfrak{Re}(z(x))$, $v(x) = \mathfrak{Im}(z(x))$. Substituting $z(x) = u(x) + iv(x)$ into (10.6) and taking the real and imaginary parts, we can show that $u(x) = \mathfrak{Re}(z(x))$ is a solution of

$$y'' + ay' + by = P_n(x)e^{\alpha x}\cos(\beta x), \tag{10.7}$$

and $v(x) = \mathfrak{Im}(z(x))$ is a solution of

$$y'' + ay' + by = P_n(x)e^{\alpha x}\sin(\beta x). \tag{10.8}$$

Example 10.20. Solve $y'' - 2y' + 2y = e^x\cos x$.

Solution. The characteristic equation is $\lambda^2 - 2\lambda + 2 = 0$ with roots $1 + i$ and $1 - i$.

The general solution of the associated homogeneous DE is

$$y = c_1e^x\cos x + c_2e^x\sin x.$$

Now consider the DE $y'' - 2y' + 2y = e^{(1+i)x}$. Let's find a particular solution.

Since $(1+i)$ is a root of the characteristic equation, we should try a particular solution of the form $y = Axe^{(1+i)x}$.

Thus $y' = (A + A(1+i)x)e^{(1+i)x}, y'' = (2A(1+i) + A(1+i)^2x)e^{(1+i)x}$.

Therefore,

$$
\begin{aligned}
&y'' - 2y' + 2 \\
&= \Big[(2A(1+i) + A(1+i)^2 x) - 2(A + A(1+i)x) + 2Ax\Big] e^{(1+i)x} \\
&= 2Aie^{(1+i)x}.
\end{aligned}
$$

From this, $1 = 2Ai$ or $A = -\frac{i}{2}$.
Thus a particular solution is given by $y = -\frac{i}{2}xe^{(1+i)x}$, or equivalently, $y = \frac{1}{2}xe^x \sin x - \frac{i}{2}xe^x \cos x$.
Taking the real part, $y_p = \frac{1}{2}xe^x \sin x$ is a particular solution of the given DE.
Consequently, the general solution is

$$y = c_1 e^x \cos x + c_2 e^x \sin x + \frac{1}{2} x e^x \sin x.$$

■

Remark. Alternatively to solve (10.7) or (10.8), one can try a solution of the form

$$Q_n(x)e^{\alpha x}\cos(\beta x) + R_n(x)e^{\alpha x}\sin(\beta x)$$

if $\alpha + i\beta$ is not a root of $\lambda^2 + a\lambda + b = 0$, and

$$xQ_n(x)e^{\alpha x}\cos(\beta x) + xR_n(x)e^{\alpha x}\sin(\beta x)$$

if $\alpha + i\beta$ is a root of $\lambda^2 + a\lambda + b = 0$, where Q_n and R_n are polynomials of degree n.