Finite Difference method

Approximate spatial derivatives of function f(x).

Expand f(x) in a Taylor series

$$f(x+\delta x) = f(x) + \delta x\, f'(x) + \frac{\delta x^2}{2} f''(x) + \ldots$$

$$= f(x) + \delta x\, f'(x) + \mathcal{O}(\delta x^2)$$

hence,
$$f'(x) = \frac{f(x+\delta x) - f(x)}{\delta x} + \mathcal{O}(\delta x)$$

order of accuracy = exponent in leading order error

$\frac{f(x+\delta x) - f(x)}{\delta x}$ is a first order approximation

Note: if we halve δx, we expect the error to reduce by a factor of 2.

Note that $\frac{f(x+\delta x) - f(x-\delta x)}{2\delta x}$ is second order accurate; the error is $\mathcal{O}(\delta x^2)$

Higher order derivatives

2nd order central difference:

$$f''(x) = \frac{f(x+\delta x) - 2f(x) + f(x-\delta x)}{\delta x^2} + \mathcal{O}(\delta x^2)$$

(can show error is $\mathcal{O}(\delta x^2)$ using Taylor series)

Vector form

Discretize x to points $[x_0, x_1, \dots, x_N]$

Let $f_i = f(x_i)$

$$f''(x_i) \simeq \frac{f_{i+1} - 2f_i + f_{i-1}}{\delta x^2}$$

$$= \underline{\underline{M}}\,\underline{f} \quad \text{where}$$

$$\underline{f} = [f_0, f_1, \dots, f_N]^T \quad \left(\begin{matrix}\text{column}\\ \text{vector}\end{matrix}\right)$$

$$\underline{\underline{M}} = \frac{1}{\delta x^2}\begin{bmatrix} \ddots & \ddots & & & 0 \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ 0 & & 1 & -2 & 1 \\ & & & \ddots & \ddots \end{bmatrix}$$

Boundary conditions needed for first and last row.

Timestepping

Consider $\frac{dy}{dt} = f(y,t)$

discretize time: $t = t_0, t_1, \dots, t_n, t_{n+1}, \dots t_N$

denote $y_n = y(t_n)$, etc.

Suppose we know y at $t_0, t_1, \dots t_n$ and want to find $y(t_{n+1})$

Explicit methods evaluate f based on y at previous times $y(t_0), y(t_1), \dots y(t_n)$

Implicit methods evaluate f using $y(t_{n+1})$

Examples

Explicit Euler: $\frac{y_{n+1} - y_n}{\Delta t} = f(y_n, t_n)$

$(\Delta t = t_{n+1} - t_n)$

Implicit Euler: $\frac{y_{n+1} - y_n}{\Delta t} = f(y_{n+1}, t_{n+1})$

*note: difficult if f is nonlinear

Crank-Nicolson (semi-implicit)

$$\frac{y_{n+1}-y_n}{\Delta t} = \frac{1}{2}\left(f(y_n, t_n) + f(y_{n+1}, t_{n+1})\right)$$

Stability

Consider model problem: $\frac{dy}{dt} = \lambda y$

w/ $y(0) = y_0$

exact soln: $y = y_0 e^{\lambda t}$

explicit Euler: $y_{n+1} = y_n + \Delta t\, f(y_n, t_n)$

$= y_n + \Delta t \lambda y_n$

$= (1 + \Delta t \lambda) y_n$

$y_1 = (1 + \Delta t \lambda) y_0$

$y_2 = (1 + \Delta t \lambda) y_1 = (1 + \Delta t \lambda)^2 y_0$

$\vdots$

$y_n = (1 + \Delta t \lambda)^n y_0$

let $\sigma = 1 + \Delta t \lambda$

if $|\sigma| < 1$ then $\lim_{n \to \infty} y_n = 0$

and EE scheme is <u>stable</u>

for $\lambda < 0$, need $\Delta t < \frac{-2}{\lambda}$

<u>Implicit Euler</u>

$$y_{n+1} = y_n + \lambda \Delta t \, y_{n+1}$$

$$\Rightarrow \quad y_{n+1} = \frac{y_n}{1 - \lambda \Delta t}$$

$$y_n = \left(\frac{1}{1 - \lambda \Delta t} \right)^n y_0$$

let $\sigma = \frac{1}{1 - \lambda \Delta t}$

stable if $|\sigma| < 1$

$$\Rightarrow \quad |1 - \lambda \Delta t| > 1$$

for $\lambda < 0$ stable for all Δt

for $\lambda > 0$ stable if $\Delta t > \frac{2}{\lambda}$

(even though model problem has unstable exact soln)

<u>Crank - Nicolson</u>

$$y_{n+1} = y_n + \frac{\lambda \Delta t}{2}(y_n + y_{n+1})$$

$$y_{n+1} = \frac{1 + \frac{\lambda \Delta t}{2}}{1 - \frac{\lambda \Delta t}{2}} y_n$$

$$y_n = \left(\frac{1 + \frac{\lambda \Delta t}{2}}{1 - \frac{\lambda \Delta t}{2}} \right)^n y_0$$

let $\sigma = \dfrac{1 + \frac{\lambda \Delta t}{2}}{1 - \frac{\lambda \Delta t}{2}}$

stable if $|\sigma| < 1$

if $\lambda < 0$ stable for all Δt

if $\lambda > 0$ unstable for all Δt

Accuracy

<u>local error</u>: difference between numerical and exact solutions after one timestep.

<u>global error</u>: difference between numerical and exact solutions at the end of a fixed time interval.

Explicit Euler is 2nd order accurate locally and 1st order accurate globally

Crank–Nicolson is 3rd order accurate locally and 2nd order accurate globally