## Part 1: Warm Up

$y'(x) = y(x)$, with initial condition $y(0) = 1$.

**1.** Solve the above initial value problem of ODE analytically (using a method you learned in ME203), and write down the general solution and then the particular solution by applying the initial condition. This will be used as ground truth to check if your solutions by other methods are right or not.

The equation is a first order separable equation.

$$\begin{aligned}
\frac{dy}{dx} &= y \\
\frac{dy}{y} &= dx \\
\int \frac{dy}{y} &= \int dx \\
\ln|y| &= x + C \\
y &= e^{x+C} \\
y &= e^C e^x
\end{aligned}$$

Substituting the initial condition $y(0) = 1$:

$$\begin{aligned}
1 &= e^C e^0 \\
e^C &= 1
\end{aligned}$$

Therefore, the solution to the ODE is $y = e^x$

**2.** Solve the above ODE by the method of power series.

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n \\
y' &= \sum_{n=1}^{\infty} n c_n x^{n-1} \\
&= \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n
\end{aligned}$$

Subbing these into the original ODE:

$$\begin{aligned}
y' &= y \\
y' - y &= 0 \\
\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=0}^{\infty} c_n x^n &= 0
\end{aligned}$$

This means that for all n, $(n+1)c_{n+1} = c_n$. For $k \in \mathbb{R}$ :

$$\begin{aligned}
n = 0 : &\quad c_1 = c_0 \\
n = 1 : &\quad 2c_2 = c_1 \quad \Rightarrow \quad c_2 = \frac{1}{2}c_0 \\
n = 2 : &\quad 3c_3 = c_2 \quad \Rightarrow \quad c_3 = \frac{1}{6}c_0 \\
n = 3 : &\quad 4c_4 = c_3 \quad \Rightarrow \quad c_4 = \frac{1}{24}c_0 \\
n = 4 : &\quad 5c_5 = c_4 \quad \Rightarrow \quad c_5 = \frac{1}{120}c_0 \\
&\vdots \\
n = k : &\quad (k+1)c_{k+1} = c_k \quad \Rightarrow \quad c_{k+1} = \frac{1}{k!}c_0
\end{aligned}$$

The function $y$ may be written as follows:

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + ... + c_nx^n$$

$$y = c_0 + c_1x + \frac{1}{2!}c_0x^2 + \frac{1}{3!}c_0x^3 + \frac{1}{4!}c_0x^4 + ... + \frac{1}{n!}c_0x^n$$

Subbing in the initial condition:

$$y(0) = 1 = c_0 + 0$$

$$c_0 = 1$$

Therefore, y may be written as follows:

$$y = c_0 + c_1x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + ... + \frac{1}{n!}x^n$$

$$y = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

This is the Maclaurin series representation of $e^x$. Therefore, by power series, we find the same answer as was found in question 1:

$$y = e^x$$

Questions 3-5 will be answered via python code.