

Lenses

<https://github.com/heptagons/lenses>

2024/1/1

Abstract

Lenses are equilateral hexagons resembling concave and convex optical lenses. Lenses consecutive six internal angles are $(\theta_1, \theta_2, \theta_3, \theta_1, \theta_2, \theta_3)$ where $\theta_1 = X\theta_0$, $\theta_2 = Y\theta_0$, and $\theta_3 = Z\theta_0$ where $\theta_0 = 2\pi/S$ is the base angle of symmetry $S = X + Y + Z$.

1 Lenses

2 Stars

3 Symmetry 5

Symmetry 5 is based in angle $\beta = \frac{2\pi}{5}$ and produces the two rhombi (\mathbf{b}, \mathbf{c}) and the two lenses (\mathbf{B}, \mathbf{C}) .

3.1 Rhombi (\mathbf{b}, \mathbf{c})

Rhombus	θ_1	θ_2
\mathbf{b}	$\beta/2$	$4\beta/2$
\mathbf{c}	$2\beta/2$	$3\beta/2$

Table 1: Rhombi (\mathbf{b}, \mathbf{c}) internal angles. $\theta_1 + \theta_2 = \pi$ and $\beta = 2\pi/5$.

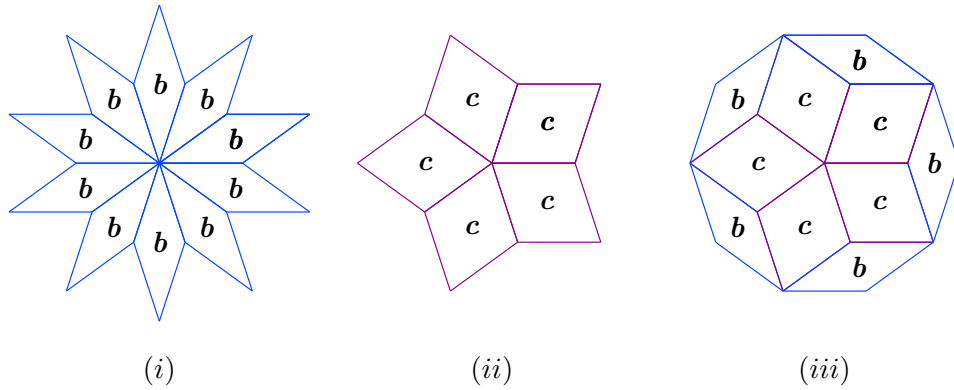


Figure 1: Rhombi (\mathbf{b}, \mathbf{c}) from dissecting stars S_{10} .

Table 1 show the rhombi (\mathbf{b}, \mathbf{c}) internal angles in terms of angle $\beta = 2\pi/5$. Figure 1 show the rhombi (\mathbf{b}, \mathbf{c}) . Inspecting the stars we get the areas simply adding their rhombi. At (i) the star $S_{10}(1,8)$ with area

$A = 10b$. At (ii) the star $S_{10}(2, 6) = S_5(1, 3)$ with area $A = 5c$. At (iii) the regular decagon equivalent to stars $S_{10}(4, 4) = S_5(2, 2)$ with area $A = 5b + 5c$.

3.2 Regular pentagon and star $|5/2|$

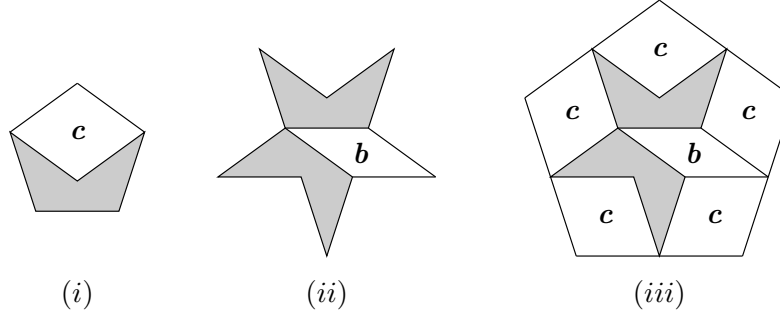


Figure 2: Regular pentagon $|5/1|$ at (i). Star $|5/2|$ at (ii). Double pentagon at (iii).

Figure 2 show regular pentagon and isotoxal star $|5/2|$ dissected with rhombi (b, c) plus concave pentagons (in gray). Let x be the area of such gray piece. By inspection the area of regular pentagon at (i) is $A_1 = c + x$ and the area of regular pentagon at (iii) is $P_2 = b + 5c + 2x$. Since the side of A_2 is the double of A_1 its area is four times so we can get the value of x in terms of (b, c)

$$\begin{aligned} 4P_1 &= P_4 \\ 4(c + x) &= b + 5c + 2x \\ x &= \frac{b + c}{2} \end{aligned} \tag{1}$$

We use the value of x to get the areas of pentagon (i) and star (ii):

$$\begin{aligned} A|5/1| &= c + x \\ &= \frac{b + 3c}{2} \end{aligned} \tag{2}$$

$$\begin{aligned} A|5/2| &= b + 2x \\ &= 2b + c \end{aligned} \tag{3}$$

3.3 Lenses (B, C)

Lense	θ_1	θ_2	θ_3
B	β	2β	2β
C	β	β	3β

Table 2: Lenses (B, C) internal angles $\theta_1 + \theta_2 + \theta_3 = 2\pi$ where $\beta = 2\pi/5$.

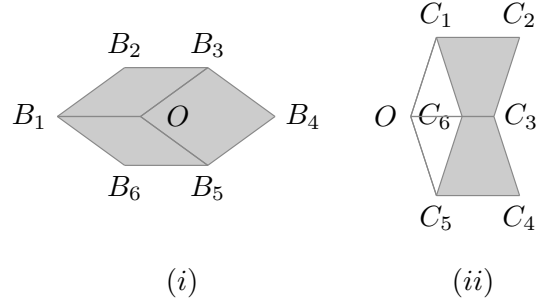


Figure 3: Lenses (\mathbf{B} , \mathbf{C}) build with rhombi (\mathbf{b} , \mathbf{c}).

Table 2 show the lenses (\mathbf{B} , \mathbf{C}) internal angles. Figure 3 show lenses (\mathbf{B} , \mathbf{C}) construction and two stars formed with them. At (i) we form the lense \mathbf{B} with perimeter $\overline{B_1...B_6}$ adding two rhombi \mathbf{b} ($\overline{B_1B_2B_3O}$ and $\overline{B_1OB_5B_6}$) and adding one rhombus \mathbf{c} ($\overline{OB_3B_4B_5}$) so its area is $2\mathbf{b} + \mathbf{c}$. Lense \mathbf{B} is equivalent to the hexagon $H_5(1, 2, 2)$. At (ii) we form the lense \mathbf{C} with perimeter $\overline{C_1...C_6}$ adding two rhombi \mathbf{c} ($\overline{OC_1C_2C_3}$ and $\overline{OC_3C_4C_5}$) and subtracting one rhombus \mathbf{b} ($\overline{OC_1C_6C_5}$) so its area is $2\mathbf{c} - \mathbf{b}$. Lense \mathbf{C} is equivalent to the hexagon $H_5(1, 1, 3)$.

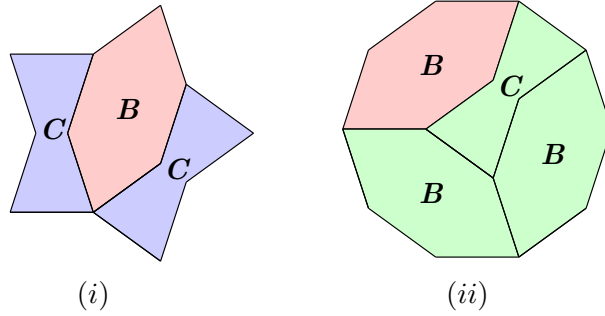


Figure 4: Two stars dissected with lenses (\mathbf{B} , \mathbf{C}).

Figure 4 show two stars dissected with lenses (\mathbf{B} , \mathbf{C}). At (i) the star $S_5(1, 3)$ dissection implies its area is $A = \mathbf{B} + 2\mathbf{C} = 5\mathbf{c}$. At (ii) the regular decagon or star $S_5(2, 2)$ dissection implies its area is $A = 3\mathbf{B} + \mathbf{C} = 5(\mathbf{b} + \mathbf{c})$.

4 Symmetry 7

Symmetry 7 is based in angle $\gamma = \frac{2\pi}{7}$ and produces the three rhombi (\mathbf{d} , \mathbf{f} , \mathbf{e}) and the three lenses (\mathbf{D} , \mathbf{E} , \mathbf{F}).

4.1 Rhombi (\mathbf{d} , \mathbf{e} , \mathbf{f})

Rhombus	θ_1	θ_2
\mathbf{d}	$\gamma/2$	$6\gamma/2$
\mathbf{e}	$2\gamma/2$	$5\gamma/2$
\mathbf{f}	$3\gamma/2$	$4\gamma/2$

Table 3: Rhombi (\mathbf{d} , \mathbf{e} , \mathbf{f}) internal angles. $\theta_1 + \theta_2 = \pi$ and $\gamma = 2\pi/7$.

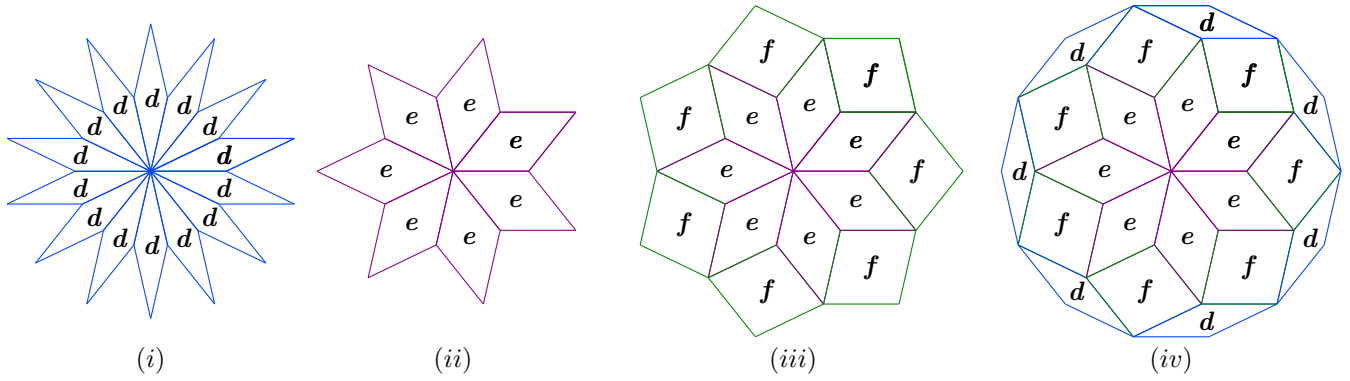


Figure 5: Rhombi (d, e, f) from dissected stars S_{14} .

Table 3 show the symmetry 7 lenses internal angles based in angle $\gamma = 2\pi/7$. Figure 5 show rhombi (d, e, f) . Inspecting the stars we get the areas simply adding their rhombi. At (i) the star $S_{14}(1, 12)$ with area $A = 14d$. At (ii) the star $S_{14}(2, 10) = S_7(1, 5)$ with area $A = 7e$. At (iii) the star $S_{14}(4, 8) = S_7(2, 4)$ with area $A = 7(e + f)$. At (iv) the regular 14-gon equivalent to stars $S_{14}(6, 6) = S_7(3, 3)$ with area $A = 7(d + e + f)$.

4.2 Regular heptagon and stars $|7/3|$ and $|7/2|$

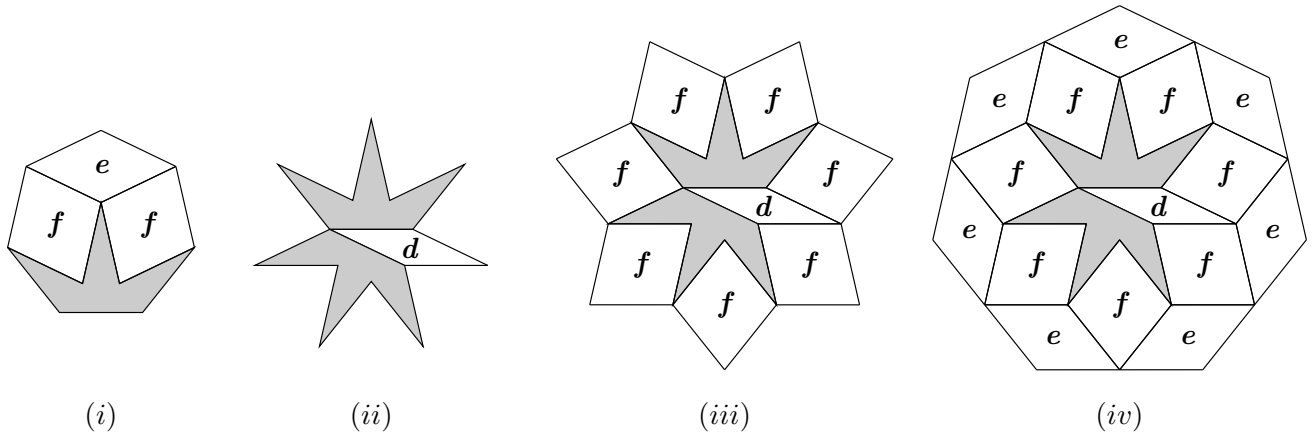


Figure 6: Heptagon $|7/1|$ at (i). Star $|7/3|$ at (ii). Star $|7/2|$ at (iii). Double heptagon at (iv).

Figure 6 show regular heptagon and heptagrams dissected with rhombi (c, d, e) plus equilateral concave heptagons (in gray). Let y be the area of such gray piece. By inspection the area of regular heptagon at (i) is $A_1 = e + 2f + y$ while the area of regular heptagon at (iv) is $A_2 = d + 7(e + f) + 2y$. Since the side of A_2 is the double of A_1 its area is four times so we can get the value of y in terms of (d, e, f) :

$$\begin{aligned}
 4A_1 &= A_2 \\
 4(e + 2f + y) &= d + 7(e + f) + 2y \\
 y &= \frac{d + 3e - f}{2}
 \end{aligned} \tag{4}$$

We use the value of \mathbf{y} to calculate the areas of heptagon (i) and stars (ii) and (iii) in terms of $(\mathbf{d}, \mathbf{e}, \mathbf{f})$:

$$\begin{aligned} A|7/1| &= \mathbf{e} + 2\mathbf{f} + \mathbf{y} \\ &= \frac{\mathbf{d} + 5\mathbf{e} + 3\mathbf{f}}{2} \end{aligned} \quad (5)$$

$$\begin{aligned} A|7/3| &= \mathbf{d} + 2\mathbf{y} \\ &= 2\mathbf{d} + 3\mathbf{e} - \mathbf{f} \end{aligned} \quad (6)$$

$$\begin{aligned} A|7/2| &= A\{7/3\} + 7\mathbf{f} \\ &= 2\mathbf{d} + 3\mathbf{e} + 6\mathbf{f} \end{aligned} \quad (7)$$

4.3 Lenses $(\mathbf{D}, \mathbf{E}, \mathbf{F})$

Lense	θ_1	θ_2	θ_3
\mathbf{D}	γ	3γ	3γ
\mathbf{E}	γ	2γ	4γ
\mathbf{F}	2γ	2γ	3γ

Table 4: Lenses $(\mathbf{D}, \mathbf{E}, \mathbf{F})$ internal angles. $\theta_1 + \theta_2 + \theta_3 = 2\pi$ and $\gamma = 2\pi/7$.

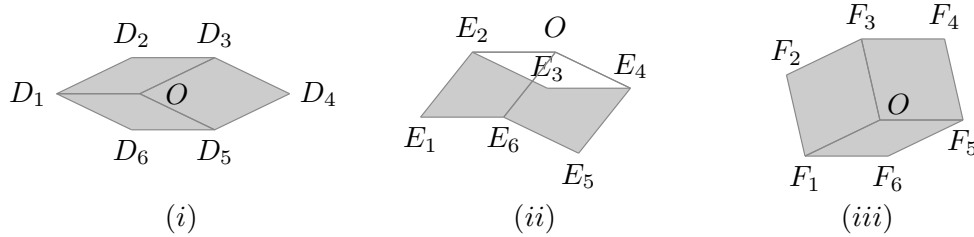


Figure 7: Lenses $(\mathbf{D}, \mathbf{E}, \mathbf{F})$ build from rhombi $(\mathbf{d}, \mathbf{e}, \mathbf{f})$.

Table 4 show the lenses $(\mathbf{D}, \mathbf{E}, \mathbf{F})$ internal angles. Figure 7 show lenses (\mathbf{B}, \mathbf{C}) construction. At (i) we form the lense \mathbf{D} with perimeter $\overline{D_1 \dots D_6}$ adding two rhombi \mathbf{d} ($\overline{D_1 D_2 D_3 O}$ and $\overline{D_1 O D_5 D_6}$) and adding one rhombus \mathbf{e} ($\overline{O D_3 D_4 D_5}$) so its area is $2\mathbf{d} + \mathbf{e}$. Lense \mathbf{D} is equivalent to the hexagon $H_7(1, 3, 3)$.

At (ii) we form the lense \mathbf{E} with perimeter $\overline{E_1 \dots E_6}$ adding one rhombus \mathbf{e} ($\overline{E_1 E_2 O E_6}$) adding one rhombus \mathbf{f} ($\overline{O E_4 E_5 E_6}$) and subtracting one rhombus \mathbf{d} ($\overline{E_2 O E_4 E_3}$) so its area is $-\mathbf{d} + \mathbf{e} + \mathbf{f}$. Lense \mathbf{E} is equivalent to the hexagon $H_7(1, 2, 4)$.

At (iii) we form the lense \mathbf{F} with perimeter $\overline{F_1 \dots F_6}$ adding two rhombi \mathbf{f} ($\overline{F_1 F_2 F_3 O}$ and $\overline{F_3 F_4 F_5 O}$) and adding one rhombus \mathbf{d} ($\overline{F_1 O F_5 F_6}$) so its area is $\mathbf{d} + 2\mathbf{f}$. Lense \mathbf{F} is equivalent to the hexagon $H_7(2, 2, 3)$.

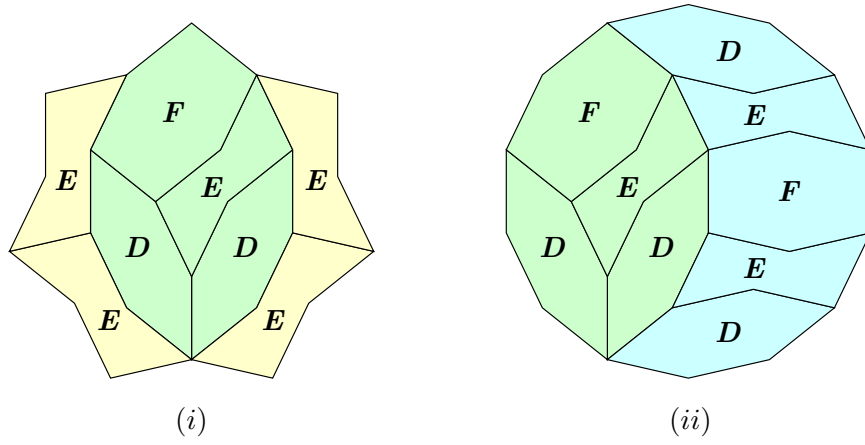


Figure 8: Stars dissected with only lenses (D, E, F).

Figure 8 show stars $S_7(2, 4)$ and $S_7(3, 3)$ dissected with lenses (D, E, F). At (i) we have star $S_7(2, 4)$ and by inspection we deduce its area is $A = 2D + 5E + F$. At (ii) we have regular 14-gon (or star $S_7(3, 3)$) and by inspection we deduce its area is $4D + 3E + 2F$. Both stars have in common an area in green resembling a tree leaf. The star at (i) also contains two regions in yellow resembling crowns while the star at (ii) contains a region in cyan resembling a moon phase.

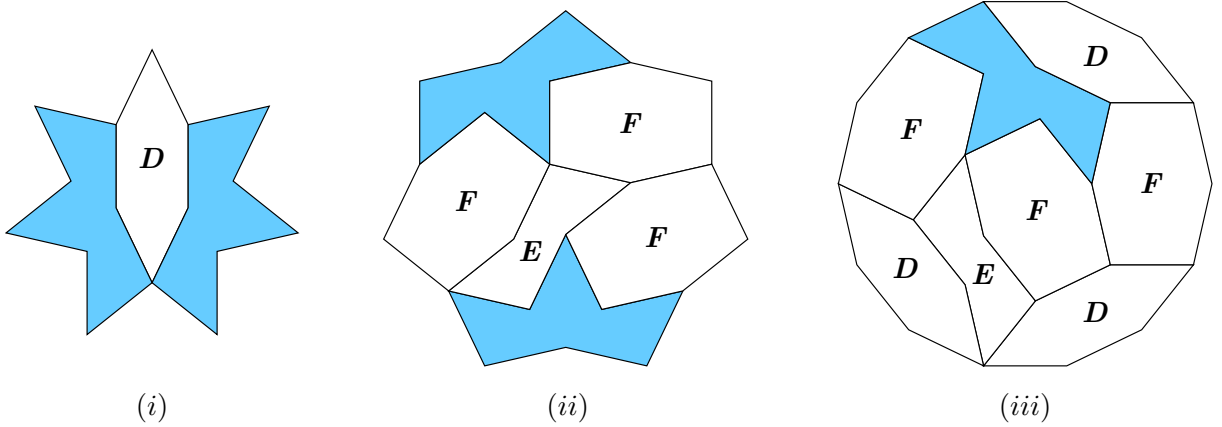


Figure 9: Stars dissected with octagons O_7 (in blue) and lenses (D, E, F).

Figure 9 show Stars $S_7(1, 5)$, $S_7(2, 4)$ and $S_7(3, 3)$ dissected with octagons O_7 (in blue) and lenses (D, E, F). At (i) we have the star $S_7(1, 5)$ and by inspection we deduce its area is $A = D + 2O_7$. At (ii) we have the star $S_7(2, 4)$ and we can conclude its area is $E + 3F + 2O_7$. Similarly the area of the 14-gon at (iii) is $3D + E + 3F + O_7$. Comparing the areas of the two 14-gons of figures 8 and 9 we can find the area of O_7 in terms of (E, F, G):

$$\begin{aligned}
 4D + 3E + 2F &= 3D + E + 3F + O_7 \\
 O_7 &= D + 2E - F
 \end{aligned} \tag{8}$$