Symmetry 9

https://github.com/heptagons/lenses

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Abstract

We define the odd symmetry $s = \{3, 5, 7, 9, ...\}$ as the **set of pairs of lines** of length 1 all connected at their vertices and forming always angles equal to $2\pi u/s, u = 1, 2, ..., s-1$. The lines form cyclic polylines as hexagons, octagons and grater equilateral polygons that can tesselate the Euclidean plane.

1 Symmetries

We are interested in odd symmetries starting with $s = 3$. Table 1 show the symmetries up to $s = 9$. The minimum possible angle of the symmetry s is $\theta = \dfrac{2\pi}{s}$.

s	Angle θ	Minimal value
3	α	$2\pi/3$
5	β	$2\pi/5$
7	γ	$2\pi/7$
9	δ	$2\pi/9$

Table 1: Angles of symmetries $s = \{3, 5, 7, 9\}$.

Let $t = \dfrac{s+1}{2}$. We define t independent abscissas and ordinates x_i, y_i as follows:

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} \cos((i-1)\theta) \\ \sin((i-1)\theta) \end{bmatrix} \quad i = 1, 2, ..., t \tag{1}$$

Let define a point p_0 located at coordinates $(0, 0)$. Then there are exactly s different points $p_1, p_2, ...p_s$ equidistant to point p_0 located at coordinates $\langle x_i, y_i\rangle$ as follows:

$$p_i \equiv \begin{cases} \langle x_j, y_j\rangle & \text{for} \quad i \le t \quad j = i \\ \langle x_j, -y_j\rangle & \text{for} \quad i > t \quad j = s+2-i \end{cases} \tag{2}$$

We transform each point p_i into a vector v_i which holds only the pairs (x, y) indices and the signs:

$$v_i \equiv \begin{cases} (j, j) & \text{for} \quad i \le t \quad j = i \\ (j, -j) & \text{for} \quad i > t \quad j = s+2-i \end{cases} \tag{3}$$

Any vector $v_i \equiv (k_1, k_2)$ can be rotated by 180° around p_0. The vector after the rotation is denoted as $\overline{v_i}$ and the original vector indices k_1, k_2 signs are changed:

$$\overline{v_i} \equiv (-k_1, -k_2) \tag{4}$$

Consecutive lines connected at their vertices to form a **polyline**. First we start at the vertice o located at coordinates $(0, 0)$ and then add vectors alternating the type v_i and the type $\overline{v_i}$ in order the angles at the vertices are multiples of θ. For example, the polyline with vectors $\overrightarrow{P_1P_2} = v_a$, $\overrightarrow{P_2P_3} = \overline{v_b}$, $\overrightarrow{P_3P_4} = v_c$, ... is denoted as:

$$\begin{aligned} \overline{P_1P_2P_3P_4...} &= \{[], [v_a], [v_a, \overline{v_b}], [v_a, \overline{v_b}, v_c], ...\} & 1 \leq a, b, c \leq t \quad (5) \\ &= P_s(0, a, b, c, ...) & \text{simplifyed form} \quad (6) \end{aligned}$$

For two consecutive points $P_k = [..., v_m]$ and $P_{k+1} = [..., v_m, \overline{v_n}]$ for $k > 0$ we have that the angle at point P_k is $u\theta$, where u is:

$$u \equiv s + m - n \pmod{s} \qquad 1 \leq m, n \leq t \tag{7}$$

Every point can be located independently by pre-calculating **accumulators** $\mathbf{P}^A$ wich are integer matrices of size $t \times 2$:

$$\mathbf{X} = \left[\begin{array}{ccccc} X_1 & X_2 & X_3 & ... & X_t \end{array} \right] \quad X_i \in \mathbb{Z} \tag{8}$$

$$\mathbf{Y} = \left[\begin{array}{ccccc} Y_1 & Y_2 & Y_3 & ... & Y_t \end{array} \right] \quad Y_i \in \mathbb{Z} \tag{9}$$

$$\mathbf{P}^A \equiv \left[\begin{array}{c} \mathbf{X} \\ \mathbf{Y} \end{array} \right] = \left[\begin{array}{ccccc} X_1 & X_2 & X_3 & ... & X_t \\ Y_1 & Y_2 & Y_3 & ... & Y_t \end{array} \right] \tag{10}$$

By walking through the polyline the accumulators are copied from previous points and incremented by $+1$ or decremented by -1 depending in the indices of the vector $v = (k_x, k_y)$:

$$X_i = \begin{cases} X_i + 1 & \text{for} \quad i = k_x \\ X_i - 1 & \text{for} \quad i = -k_x \end{cases} \tag{11}$$

$$Y_i = \begin{cases} Y_i + 1 & \text{for} \quad i = k_y \\ Y_i - 1 & \text{for} \quad i = -k_y \end{cases} \tag{12}$$

The coordinates of point P in $\mathbb{R}^2$ is calculated adding the products of the accumulators $\mathbf{P}^A$ of equation 10 and the coordinates of points p_i of equation 2:

$$\langle P_x, P_y \rangle = \left\langle \sum_{i=1}^{t} X_i x_i, \sum_{i=1}^{t} Y_i y_i \right\rangle \tag{13}$$

2 Symmetry $m = 9$

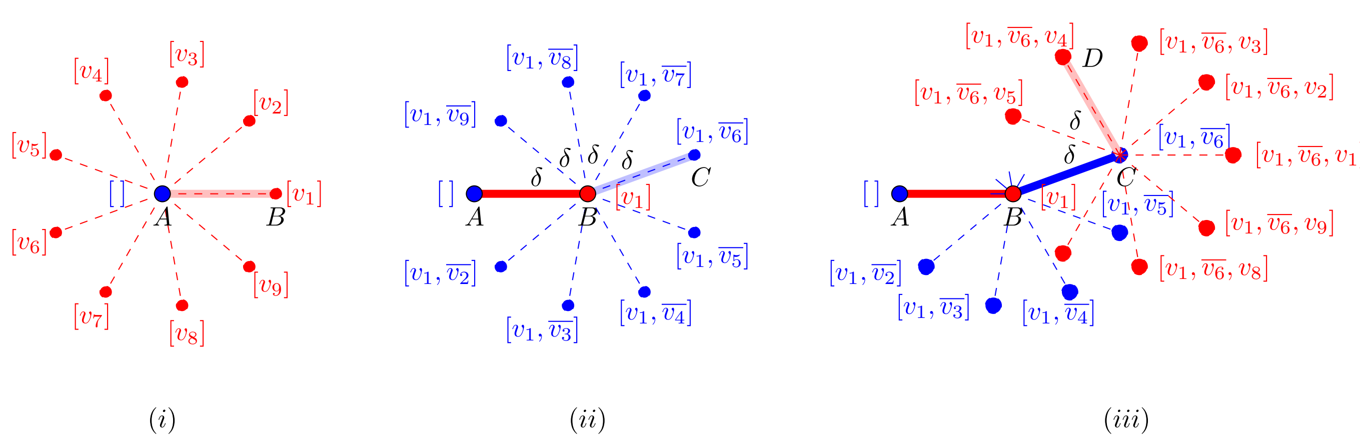

Figure 1: The symmetry 9 vectors, points, angles and polylines.

2.1 Vectors

Figure 1 (i) show the nine different vectors v_1, v_2, ... v_9 in symmetry $m = 9$. Using the vectors equation 3 we get the nine vectors: $v_1 = (1,1)$, $v_2 = (2,2)$, $v_3 = (3,3)$, $v_4 = (4,4)$, $v_5 = (5,5)$, $v_6 = (11-6, -(11-6)) = (5,-5)$, $v_7 = (11-7, -(11-7)) = (4,-4)$, $v_8 = (11-8, -(11-8)) = (3,-3)$ and $v_9 = (11-9, -(11-9)) = (2,-2)$.

2.2 Polylines

Figure 1 (ii) show polyline $\overline{ABC}$ and (iii) show polyline $\overline{ABCD}$. The alternating vectors of the type $[..., v_m]$ are show in red and the vectors of the type $[..., v_m, \overline{v_n}]$ are shown in blue. Angle $\angle ABC = 4\delta$ and angle $\angle BCD = 2\delta$. The polyline $\{A, B, C, D\}$ is denoted by $\{[\,], [v_1], [v_1, \overline{v_6}], [v_1, \overline{v_6}, v_4]\}$ or in compact form $P_9(0,1,6,4)$.

2.3 Accumulators

With the vertices $v_1, ...v_9$ of symmetry 9 already calculated, we know how to calculate the accumulators, for example, for the **X** row we have:

$$X[a,b,c,d,e] + v_i = \begin{cases} X[a+1,b,c,d,e] & \text{for} \quad v_1 \equiv (1,1) \\ X[a,b+1,c,d,e] & \text{for} \quad v_2 \equiv (2,2), v_9 \equiv (2,-2) \\ \quad ... & \\ X[a,b,c,d,e+1] & \text{for} \quad v_5 \equiv (5,5), v_6 \equiv (5,-5) \\ X[a-1,b,c,d,e] & \text{for} \quad \overline{v_1} \equiv (-1,-1) \\ \quad ... & \end{cases} \tag{14}$$

Lets get the accumulators of the points of polyline $\overline{ABCD} = \{[\,], [v_1], [v_1, \overline{v_6}], [v_1, \overline{v_6}, v_4]\} = P_9(0,1,6,4)$ shown in figure (iii). First by definition the accumulators at $o \equiv (0,0)$ which is point $A = [\,]$ are:

$$[\,]^A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

Then we use the accumulators of vector v_1 to get tha accumulators of point $B = [v_1]$:

$$\begin{aligned} v_1 &= (1,1) \\ v_1^A &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \\ [v_1]^A &= [\,]^A + v_1^A \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{16}$$

Then we get the accumulators of vector $\overline{v_6}$:

$$\begin{aligned} v_6 &\equiv (5,-5) \\ \overline{v_6} &= (-5,5) \\ \overline{v_6}^A &= \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{17}$$

And then use $\overline{v_6}^A$ to get the accumulators of point $C = [v_1, \overline{v_6}]$:

$$\begin{aligned} [v_1, \overline{v_6}]^A = v_1^A + \overline{v_6}^A &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{18}$$

Finally we get the accumulator of vector v_4 and use it to get the accumulator of point $D = [v_1, \overline{v_6}, v_4]$:

$$\begin{aligned}
v_4 &\equiv (4,4) \\
v_4^A &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\
[v_1, \overline{v_6}, v_4]^A = [v_1, \overline{v_6}]^A + v_4^A &= \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}
\end{aligned} \tag{19}$$

From the last equation of the accumulators of point $D = [v_1, \overline{v_6}, v_4]$ we have as different of zero these values: $X_1 = X_4 = 1$, $X_5 = -1$, $Y_1 = Y_4 = Y_5 = 1$. Using the equation 13 we calculate the absolute coordinates:

$$\begin{aligned}
D_x &= x_1 + x_2 - x_5 \\
&= \cos(0) + \cos(3\delta) - \cos(4\delta) \\
&= 1 + \cos(120°) - \cos(160°) \\
&\approx 1.439 \\
D_y &= y_1 + y_2 + y_2 \\
&= \sin(0) + \sin(3\delta) + \sin(4\delta) \\
&= \sin(120°) + \sin(160°) \\
&\approx 1.208
\end{aligned}$$

Using equation 7 we calculate the angles multiples u at points B and C. From $C = [v_1, \overline{v_6}]$ we have that $m_B = 1, n_B = 6$ and we calculate u at point B:

$$u_B = 9 + 1 - 6 \pmod 9 = 4 \tag{20}$$

From $D = [v_1, \overline{v_6}, v_4]$ we have that $m_C = 6, n_C = 4$ and we calculate u at point C:

$$u_C = 9 + 6 - 4 \pmod 9 = 2 \tag{21}$$

3 Rhombi

Rhombi are equilateral tetragons. For every odd symmetry s we can build exactly $n = \dfrac{s-1}{2}$ different rhombi. The rhombi angles $u_1 < u_2$ are calculated as follows:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{i}{2} \\ n + \frac{i-1}{2} \end{bmatrix} \quad i = 1, ..., n \tag{22}$$

Any rhombus can be identified with only two numbers, the symmetry s and the angle u_1 with the notation $R_s(u_1)$ so for symmetry $s = 9$ we have these four rhombi:

$$R_9 = \left\{ R_9\left(\frac{1}{2}\right), R_9(1), R_9\left(\frac{3}{2}\right), R_9(2) \right\} \tag{23}$$

Table 2 show the rhombi for odd symmetries up to 9. Every rhombus is labeled with a consecutive lowercase letter $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$,... The first rhombi $\boldsymbol{a}$ is found first in symmetry $s = 3$ and then again in symmetry $s = 9$ so we prevent renaming congruent rhombi.

The rhombi angles $\theta_1 = u_1\theta$ and $\theta_2 = u_2\theta$ are expressed in function of the angles $\{\alpha, \beta, \gamma, \delta\}$ of symmetries $s = \{3, 5, 7, 9\}$ where $\theta_1 < \theta_2$ and $\theta_1 + \theta_2 = \pi$.

$R_s(u_1)$	u_2	Label	θ_1	θ_2	Area
$R_3(\frac{1}{2})$	1	$\boldsymbol{a}$	$\alpha/2$	$2\alpha/2$	$\sin(\alpha) \approx 0.866$
$R_5(\frac{1}{2})$	2	$\boldsymbol{b}$	$\beta/2$	$4\beta/2$	$\sin(2\beta) \approx 0.587$
$R_5(1)$	$\frac{3}{2}$	$\boldsymbol{c}$	$2\beta/2$	$3\beta/2$	$\sin(\beta) \approx 0.951$
$R_7(\frac{1}{2})$	3	$\boldsymbol{d}$	$\gamma/2$	$6\gamma/2$	$\sin(3\gamma) \approx 0.433$
$R_7(1)$	$\frac{5}{2}$	$\boldsymbol{e}$	$2\gamma/2$	$5\gamma/2$	$\sin(\gamma) \approx 0.781$
$R_7(\frac{3}{2})$	2	$\boldsymbol{f}$	$3\gamma/2$	$4\gamma/2$	$\sin(2\gamma) \approx 0.974$
$R_9(\frac{1}{2})$	4	$\boldsymbol{g}$	$\delta/2$	$8\delta/2$	$\sin(4\delta) \approx 0.342$
$R_9(1)$	$\frac{7}{2}$	$\boldsymbol{h}$	$2\delta/2$	$7\delta/2$	$\sin(\delta) \approx 0.642$
$R_9(\frac{3}{2})$	3	$\boldsymbol{a}$	$3\delta/2$	$6\delta/2$	$\sin(3\delta) \approx 0.866$
$R_9(2)$	$\frac{5}{2}$	$\boldsymbol{i}$	$4\delta/2$	$5\delta/2$	$\sin(2\delta) \approx 0.984$

Table 2: Rhombi $R_s(u_1)$ for symmetries $s = \{3, 5, 7, 9\}$.

3.1 Rhombi as cycled polylines

We cannot use the rhombi alone in our symmetry tessellations since both u_1 and u_2 are not integers at the same time. But we can attach rhombi by their sides forming hexagons, octagons, decagons and more polygons with n sides where $n = 4 + 2i, i = 1, 2, 3...$ always the polygons n angles result all integers.

By attaching we mean adding and substracting rhombi preventing the intersection of any edges. This method permits to calculate the n-gons areas as the additions and substraction of the known rhombi areas. Knowing any polygons area in function of the at most $(s-1)/2$ different rhombi of symmetry s is very helpful as a necessary, but not suffient condition for dissections.

4 Stars

For every odd symmetry s we have **stars** that are equilateral $2s$-gons with at most two different angles u_1 and u_2 at the vertices. We find exactly $n = \dfrac{s-1}{2}$ different stars with angles $u_1 \leq u_2$ as integers.

Table 3 show the stars for odd symmetries up to 9. Every star with u_1 as integer are labeled with a consecutive calligraphy letter $\mathcal{A}, \mathcal{B}, \mathcal{C}$.... Any star is identified as $S_s(u_1)$ using only two numbers the the symmetry s and the smallest angle u_1. The another angle is calculated as $u_2 = s - u_1 - 1$. The stars are easily build with copies of rhombi so the table show the stars area in function of them.

$S_s(u_1)$	u_2	Label	Area	Polygon
$S_3(\frac{1}{2})$	2	-	$6\boldsymbol{a}$	$\|6/2\|$ (12-gon)
$S_3(1)$	1	$\mathcal{A}$	$3\boldsymbol{a}$	Regular hexagon
$S_5(\frac{1}{2})$	4	-	$5\boldsymbol{b}$	$\|10/4\|$ (20-gon)
$S_5(1)$	3	$\mathcal{B}$	$5\boldsymbol{c}$	$\|(5/2)_\alpha\|$ decagram
$S_5(2)$	2	$\mathcal{C}$	$5(\boldsymbol{c}+\boldsymbol{b})$	Regular decagon
$S_7(\frac{1}{2})$	6	-	$7\boldsymbol{d}$	$\|14/6\|$ (28-gon)
$S_7(1)$	5	$\mathcal{D}$	$7\boldsymbol{e}$	$\|(7/4)_\alpha\|$ 14-gram
$S_7(2)$	4	$\mathcal{E}$	$7(\boldsymbol{e}+\boldsymbol{f})$	$\|(7/2)_\alpha\|$ 14-gram
$S_7(3)$	3	$\mathcal{F}$	$7(\boldsymbol{e}+\boldsymbol{f}+\boldsymbol{d})$	Regular 14-gon
$S_9(\frac{1}{2})$	8	-	$9\boldsymbol{g}$	$\|18/8\|$ (36-gon)
$S_9(1)$	7	$\mathcal{G}$	$9\boldsymbol{h}$	$\|(9/6)_\alpha\|$ 18-gram
$S_9(2)$	6	$\mathcal{H}$	$9(\boldsymbol{h}+\boldsymbol{i})$	$\|(9/4)_\alpha\|$ 18-gram
$S_9(3)$	5	$\mathcal{I}$	$9(\boldsymbol{h}+\boldsymbol{i}+\boldsymbol{a})$	$\|(9/2)_\alpha\|$ 18-gram
$S_9(4)$	4	$\mathcal{J}$	$9(\boldsymbol{h}+\boldsymbol{i}+\boldsymbol{a}+\boldsymbol{g})$	Regular 18-gon

Table 3: Stars $S_s(u_1)$ for symmetries $s = \{3, 5, 7, 9\}$.

Figure 2 show nine copies of symmetry-9 rhombi $\{\boldsymbol{h}, \boldsymbol{i}, \boldsymbol{a}, \boldsymbol{g}\}$ forming the four stars $S_9(1)$, $S_9(2)$, $S_9(3)$ and $S_9(4)$ labeled respectivelly $\mathcal{G}$, $\mathcal{H}$, $\mathcal{I}$ and $\mathcal{J}$. Note how the rhombi half angles always are added together to produce only angles as integers for the stars.

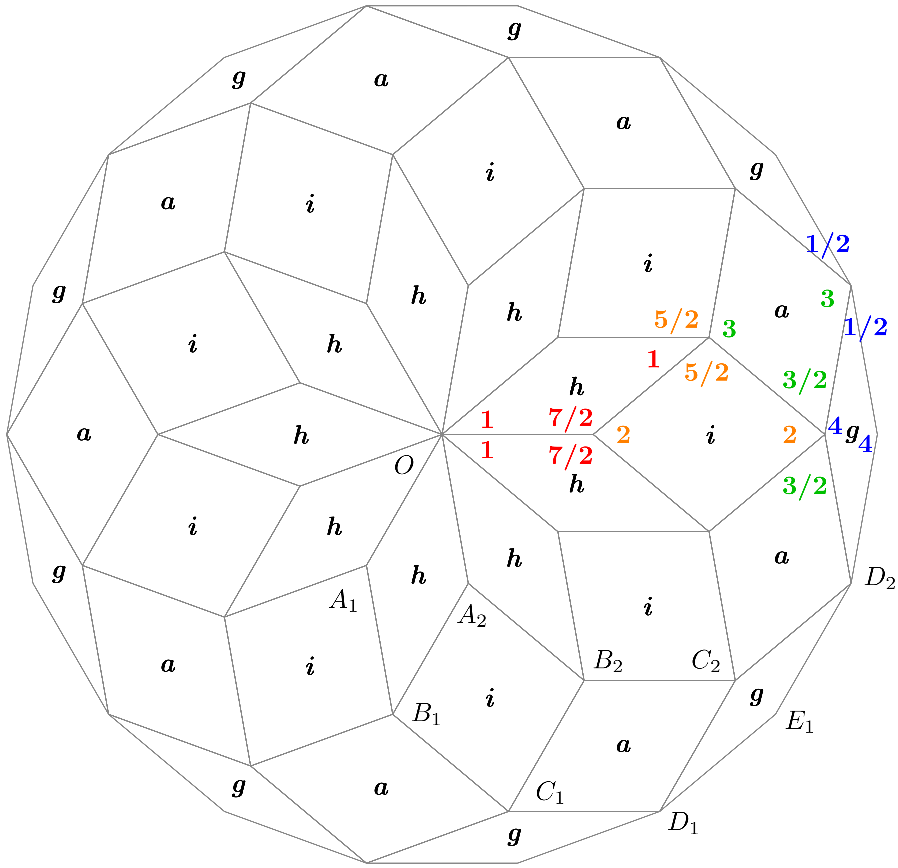

Figure 2: The symmetry 9 four rhombi $\{\boldsymbol{h}, \boldsymbol{i}, \boldsymbol{a}, \boldsymbol{g}\}$ produce the four stars $\{\mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}\}$ with areas $9\boldsymbol{h}$, $9(\boldsymbol{h}+\boldsymbol{i})$, $9(\boldsymbol{h}+\boldsymbol{i}+\boldsymbol{a})$ and $9(\boldsymbol{h}+\boldsymbol{i}+\boldsymbol{a}+\boldsymbol{g})$ respectivelly.

5 Hexagons

For any odd symmetry s, we can obtain equilateral hexagons by connecting six equal edges at integral values of u or by adding and substracting rhombi or by dissecting the areas left after stars intersections. In any case we find the hexagons has at most three different internal angles in this order $(u_1, u_2, u_3, u_1, u_2, u_3)$ where $u_1 + u_2 + u_3 = s$.

5.1 Hexagons angles

Figure 4 show the hexagons defined as $H_s(u_1, u_2)$ for symmetries $s = \{3, 5, 7, 9\}$ where $u_1 \leq u_2 \leq u_3$. We labeled the non self-intersecting hexagons with uppercase letters $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, ... We prevent to name differently any hexagon which is equivalent to a previous one having the same angles as what happens with hexagon $\boldsymbol{A}$ of symmetries 3 and 9.

$H_s(u_1,u_2)$	Label	$(u_1,u_2,u_3,u_4,u_5,u_6)$	Polygon
$H_3(1)$	$\boldsymbol{A}$	$(\mathbf{1,1,1,1,1,1})$	Regular hexagon
$H_5(1,1,3)$	$\boldsymbol{B}$	$(\mathbf{1,1,3,1,1,3})$	Sormeh Dan Girih tile
$H_5(1,2,2)$	$\boldsymbol{C}$	$(\mathbf{1,2,2,1,2,2})$	Shesh Band Girih tite
$H_7(1,1,5)$	-	$(\mathbf{1,1,5,1,1,5})$	self-intersecting
$H_7(1,2,4)$	$\boldsymbol{D}$	$(\mathbf{1,2,4,1,2,4})$	
$H_7(1,3,3)$	$\boldsymbol{E}$	$(\mathbf{1,3,3,1,3,3})$	
$H_7(2,2,3)$	$\boldsymbol{F}$	$(\mathbf{2,2,3,2,2,3})$	
$H_9(3)$	$\boldsymbol{A}$	$(\mathbf{3,3,3,3,3,3})$	congruent with $H_3(1)$
$H_9(1,5)$	$\boldsymbol{A}_1$	$(\mathbf{1,5,1,5,1,5})$	
$H_9(2,4)$	$\boldsymbol{A}_2$	$(\mathbf{2,4,2,4,2,4})$	
$H_9(1,1,7)$	-	$(\mathbf{1,1,7,1,1,7})$	self-intersecting
$H_9(1,2,6)$	$\boldsymbol{G}$	$(\mathbf{1,2,6,1,2,6})$	
$H_9(1,3,5)$	$\boldsymbol{H}$	$(\mathbf{1,3,5,1,3,5})$	
$H_9(1,4,4)$	$\boldsymbol{I}$	$(\mathbf{1,4,4,1,4,4})$	
$H_9(2,2,5)$	$\boldsymbol{J}$	$(\mathbf{2,2,5,2,2,5})$	
$H_9(2,3,4)$	$\boldsymbol{K}$	$(\mathbf{2,3,4,2,3,4})$	

Table 4: Hexagons $H_s(u_1,u_2,u_3)$ for symmetries $s=\{3,5,7,9\}$.

5.2 Hexagons areas

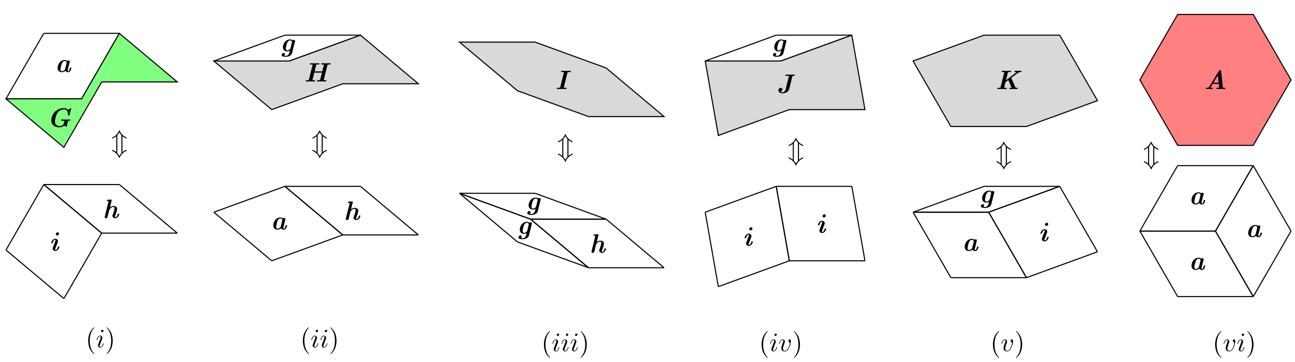

Figure 3: Symmetry 9 hexagons formed adding and substracting rhombi.

Figure 3 show how to calculate the area of the symmetry 9 hexagons in function of the symmetry 9 rhombi. From (i) to (vi) we equate the area of sum of the polygons in the top with the area of the sum of the polygons of the bottom. We have for the six cases these six equations:

$$\boldsymbol{a}+\boldsymbol{G}=\boldsymbol{i}+\boldsymbol{h} \tag{24}$$

$$\boldsymbol{g}+\boldsymbol{H}=\boldsymbol{a}+\boldsymbol{h} \tag{25}$$

$$\boldsymbol{I}=2\boldsymbol{g}+\boldsymbol{h} \tag{26}$$

$$\boldsymbol{g}+\boldsymbol{J}=2\boldsymbol{i} \tag{27}$$

$$\boldsymbol{K}=\boldsymbol{g}+\boldsymbol{a}+\boldsymbol{i} \tag{28}$$

$$\boldsymbol{A}=3\boldsymbol{a} \tag{29}$$

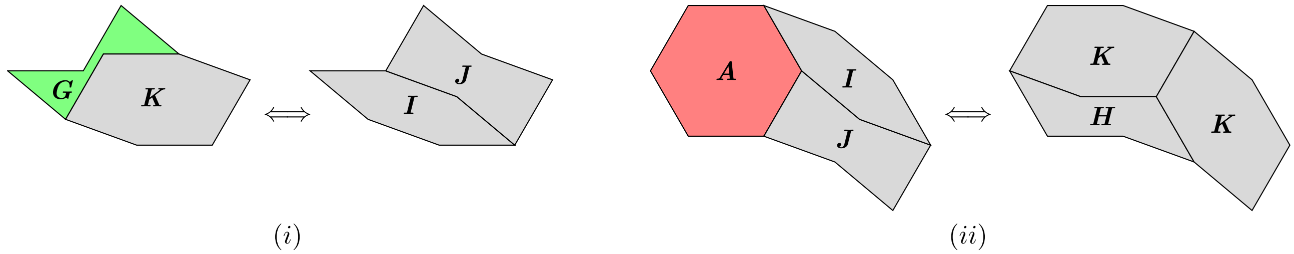

Figure 4: Hexagons $\{\boldsymbol{G}, \boldsymbol{A}\}$ formed adding and substracting hexagons $\{\boldsymbol{H}, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}\}$.

Figure 4 show how to express the area of the six hexagons in function of only four. For (i) and (ii) we equate the area of the sum of hexagons of the left with the area of the hexagons of the right. We have for the two cases these two equations:

$$\boldsymbol{G} + \boldsymbol{K} = \boldsymbol{I} + \boldsymbol{J} \tag{30}$$

$$\boldsymbol{A} + \boldsymbol{I} + \boldsymbol{J} = \boldsymbol{H} + 2\boldsymbol{K} \tag{31}$$

Using the last eight equations we form the table 5 which show the areas of the six hexagons in function of four rhombi $\boldsymbol{g},\boldsymbol{h},\boldsymbol{a},\boldsymbol{i}$ and in function of only four hexagons $\boldsymbol{H},\boldsymbol{I},\boldsymbol{J},\boldsymbol{K}$.

Hexagon	$\boldsymbol{g},\boldsymbol{h},\boldsymbol{a},\boldsymbol{i}$ area	$\boldsymbol{H},\boldsymbol{I},\boldsymbol{J},\boldsymbol{K}$ area
$\boldsymbol{H}$	$\boldsymbol{a} + \boldsymbol{h} - \boldsymbol{g}$	$\boldsymbol{H}$
$\boldsymbol{I}$	$2\boldsymbol{g} + \boldsymbol{h}$	$\boldsymbol{I}$
$\boldsymbol{J}$	$2\boldsymbol{i} - \boldsymbol{g}$	$\boldsymbol{J}$
$\boldsymbol{K}$	$\boldsymbol{g} + \boldsymbol{a} + \boldsymbol{i}$	$\boldsymbol{K}$
$\boldsymbol{A}$	$3\boldsymbol{a}$	$2\boldsymbol{K} + \boldsymbol{H} - \boldsymbol{I} - \boldsymbol{J}$
$\boldsymbol{G}$	$\boldsymbol{i} + \boldsymbol{h} - \boldsymbol{a}$	$\boldsymbol{I} + \boldsymbol{J} - \boldsymbol{K}$

Table 5: Symmetry 9 hexagon areas.

5.3 Hexagons from stars

Figure 5 show the disposition of the symmetry 9 four stars. We label the 18 vertices of stars $\{\mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}\}$ as $\{G_0, G_1, ..., G_{17}\}$, $\{H_0, H_1, ..., H_{17}\}$, $\{I_0, I_1, ..., I_{17}\}$ and $\{I_0, G_1, ..., I_{17}\}$ respectivelly. For simplicity, only some vertices are show. First we make coincident at vertice O all the vertices G_0, H_0, I_0, J_0. With the center at O we rotate all stars to make coincidents G_{17}, H_{17}, I_{17} and J_{17}. The rotations also joined another different vertices.

First we add three new edges (in red) joining the stars $\mathcal{J}$ and $\mathcal{I}$ vertices: $\overline{J_3I_2}$, $\overline{J_5I_4}$ and $\overline{J_7I_6}$ dissecting the red region into four hexagons, two of them essentially different. The three consective angles of the two hexagons are shown: **I (1,4,4)** and **K (3,4,2)**.

Then we add three new edges (in orange) joining the stars $\mathcal{I}$ and $\mathcal{H}$ vertices: $\overline{I_3H_2}$, $\overline{I_5H_4}$ and $\overline{I_7H_6}$ dissecting the orange region into four hexagons, two of them new. The three consective angles of the the two hexagons are show: **H (1,5,3)** and **A (3,3,3)**.

Finally we add three more edges (in green) joining the stars $\mathcal{H}$ and $\mathcal{G}$ vertices: $\overline{H_3G_2}$, $\overline{H_5G_4}$ and $\overline{H_7G_6}$ dissecting the green region into four hexagons, two of them new. The three consective angles of the the two hexagons are show: **G (1,6,2)** and **J (2,5,2)**.

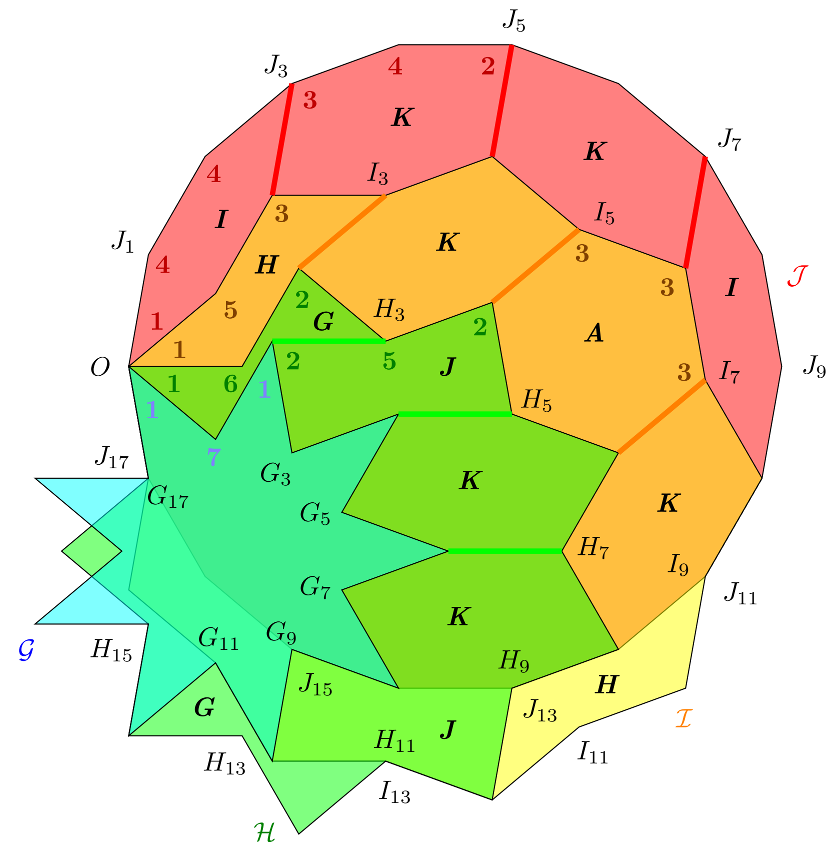

Figure 5: Symmetry 9 stars $\{\mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}\}$ dissected to get the six hexagons $\{\boldsymbol{G}, \boldsymbol{H}, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}, \boldsymbol{A}\}$.

6 Octagons

6.1 Octagons by stars

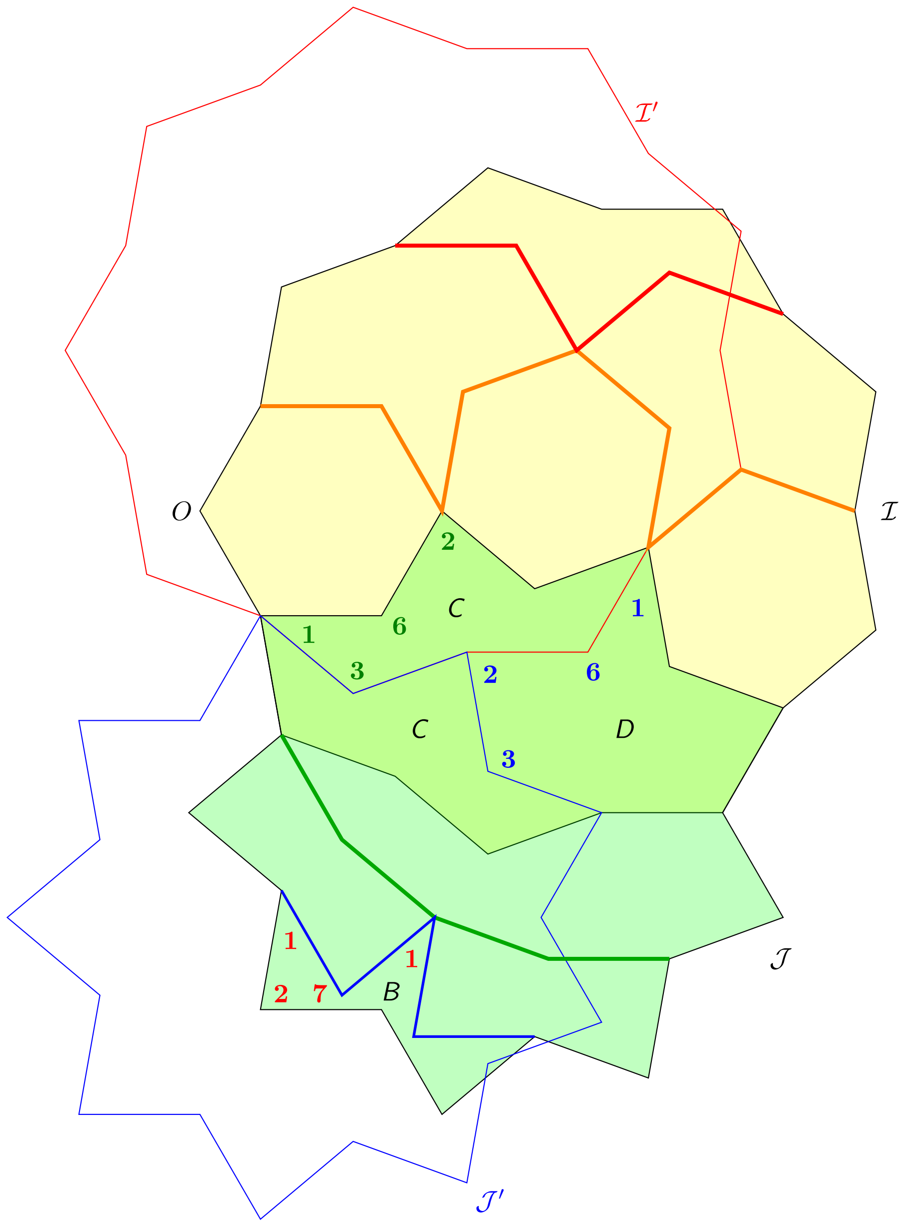

Figure 6: Octagons after intersection of stars $\mathcal{I}$ and $\mathcal{J}$.