

# Symmetry 9

<https://github.com/heptagons/lenses>

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## Abstract

Let  $\mathcal{L}$  be a set of connected lines, all of them of length 1 in  $\mathbb{R}^2$ . Let  $S = \{3, 5, 7, 9, \dots\}$  be a set of odd symmetries  $\geq 3$ . Then let  $\mathcal{L}(s)$ ,  $s \in S$  be the set of connected lines forming exclusively angles equal to  $2\pi u/s$  where  $u = 1, 2, \dots, s-1$ . We show  $\mathcal{L}(s)$  lines can form simple or complex polygons of sizes  $n = \{6, 8, 10, \dots\}$ . With software we look for simple polygons and dissect bigger ones into smaller ones.

## 1 Symmetries

### 1.1 Odd symmetries

Let  $S = \{3, 5, 7, 9, \dots\}$  be a set of odd symmetries. For every symmetry  $s \in S$  we define  $\omega = 2\pi/s$  as the symmetry minimal angle and we have  $s-1$  different angles with values  $u\omega$  for  $u = 1, 2, 3, \dots, s-1$ . Table 1 show the symmetries angles up to  $s = 21$ .

$s$	Name	$\omega$	Different angles
3	$\alpha$	$2\pi/3$	$\alpha, 2\alpha$
5	$\beta$	$2\pi/5$	$\beta, 2\beta, 3\beta, 4\beta$
7	$\gamma$	$2\pi/7$	$\gamma, 2\gamma, \dots, 6\gamma$
9	$\delta$	$2\pi/9$	$\delta, 2\delta, \dots, 8\delta$
11	$\epsilon$	$2\pi/11$	$\epsilon, 2\epsilon, \dots, 10\epsilon$
13	$\zeta$	$2\pi/13$	$\zeta, 2\zeta, \dots, 11\zeta$
15	$\eta$	$2\pi/15$	$\eta, 2\eta, \dots, 14\eta$
17	$\theta$	$2\pi/17$	$\theta, 2\theta, \dots, 16\theta$
19	$\iota$	$2\pi/19$	$\iota, 2\iota, \dots, 18\iota$
21	$\kappa$	$2\pi/21$	$\kappa, 2\kappa, \dots, 20\kappa$

Table 1: Angles of symmetries  $\{3, 5, 7, 9, 11, 13, 15, 17, 19, 21\}$ .

For symmetry  $s \in S$  we define  $t = \frac{s+1}{2}$ . Then we define  $t$  independent abscissas and ordinates  $x_i, y_i$  as follows:

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} \cos((i-1)\omega) \\ \sin((i-1)\omega) \end{bmatrix} \quad i = 1, 2, \dots, t \quad (1)$$

### 1.2 Points

Let define a point  $p_0$  in  $\mathbb{R}^2$  located at coordinates  $(0, 0)$ . Then for every symmetry  $s \in S$  there are exactly  $s$  different points  $p_1, p_2, \dots, p_s$  equidistant to point  $p_0$  located at coordinates  $\langle x_i, y_i \rangle$  of equation 1 as follows:

$$p_i \equiv \begin{cases} \langle +x_j, +y_j \rangle & \text{for } i \leq t \quad j = i \\ \langle +x_j, -y_j \rangle & \text{for } i > t \quad j = s + 2 - i \end{cases} \quad (2)$$

### 1.3 Vectors

We map every point  $p_i$  of equation 2 to a vector  $v_i$  which holds only the pairs  $(x, y)$  indices with the signs:

$$v_i \equiv \begin{cases} (+j, +j) & \text{for } i \leq t \quad j = i \\ (+j, -j) & \text{for } i > t \quad j = s + 2 - i \end{cases} \quad (3)$$

Any vector  $v_i \equiv (k_1, k_2)$  can be rotated by  $180^\circ$  around  $p_0$ . The vector after the rotation is denoted as  $\overline{v}_i$  and has the original vector indices  $k_1, k_2$  but with the signs changed:

$$\overline{v}_i \equiv (-k_1, -k_2) \quad (4)$$

### 1.4 Polyline

We define  $\mathcal{L}(s)$  the set of lines connected at their vertices and forming angles of the symmetry  $s$ . Consecutive lines connected at their vertices form a **polyline**. First we start at the vertex  $o$  located at coordinates  $(0, 0)$  and then add vectors alternating the type  $v_i$  and the type  $\overline{v}_i$  in order the angles at the vertices are multiples of  $\omega$ . For example, the polyline with vectors  $\overrightarrow{P_1P_2} = v_a$ ,  $\overrightarrow{P_2P_3} = \overline{v}_b$ ,  $\overrightarrow{P_3P_4} = v_c$ , ... is denoted as:

$$\overrightarrow{P_1P_2P_3P_4...} = \{[], [v_a], [v_a, \overline{v}_b], [v_a, \overline{v}_b, v_c], \dots\} \quad 1 \leq a, b, c \leq t \quad (5)$$

$$= P_s(0, a, b, c, \dots) \quad \text{simplified form} \quad (6)$$

For two consecutive points  $P_k = [..., v_m]$  and  $P_{k+1} = [..., v_m, \overline{v}_n]$  for  $k > 0$  we have that the angle at point  $P_k$  is  $u\omega$ , where  $u$  is:

$$u \equiv s + m - n \pmod{s} \quad 1 \leq m, n \leq t \quad (7)$$

### 1.5 Accumulators

Every polyline vertex can be located independently by pre-calculating **accumulators**  $\mathbf{P}^A$  which are integer matrices of size  $t \times 2$ :

$$\mathbf{X} = \begin{bmatrix} X_1 & X_2 & X_3 & \dots & X_t \end{bmatrix} \quad X_i \in \mathbb{Z} \quad (8)$$

$$\mathbf{Y} = \begin{bmatrix} Y_1 & Y_2 & Y_3 & \dots & Y_t \end{bmatrix} \quad Y_i \in \mathbb{Z} \quad (9)$$

$$\mathbf{P}^A \equiv \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} = \begin{bmatrix} X_1 & X_2 & X_3 & \dots & X_t \\ Y_1 & Y_2 & Y_3 & \dots & Y_t \end{bmatrix} \quad (10)$$

By walking through the polyline the accumulators are copied from previous points and incremented by +1 or decremented by -1 depending in the indices of the vector  $v = (k_x, k_y)$ :

$$X_i = \begin{cases} X_i + 1 & \text{for } i = k_x \\ X_i - 1 & \text{for } i = -k_x \end{cases} \quad (11)$$

$$Y_i = \begin{cases} Y_i + 1 & \text{for } i = k_y \\ Y_i - 1 & \text{for } i = -k_y \end{cases} \quad (12)$$

### 1.6 Coordinates

The coordinates of every polyline's point  $P$  in  $\mathbb{R}^2$  is calculated adding the products of the accumulators  $\mathbf{P}^A$  of equation 10 and the coordinates of points  $p_i$  of equation 2:

$$\langle P_x, P_y \rangle = \left\langle \sum_{i=1}^t X_i x_i, \sum_{i=1}^t Y_i y_i \right\rangle \quad (13)$$

## 2 Symmetry $s = 9$

Let  $\mathcal{L}(9)$  be the set of connected lines of length 1 forming angles exclusively of symmetry 9. First we find the vectors to be used to build valid polylines. Using equation 3 we get the nine vectors:

$$\begin{aligned} v_1 &= (1, 1) \\ v_2 &= (2, 2) \\ v_3 &= (3, 3) \\ v_4 &= (4, 4) \\ v_5 &= (5, 5) \\ v_6 &= (11 - 6, -(11 - 6)) = (5, -5) \\ v_7 &= (11 - 7, -(11 - 7)) = (4, -4) \\ v_8 &= (11 - 8, -(11 - 8)) = (3, -3) \\ v_9 &= (11 - 9, -(11 - 9)) = (2, -2) \end{aligned}$$

Figure 1 (i) show the nine different vectors  $v_1, v_2, \dots, v_9$  in symmetry  $s = 9$ . With the vectors we can update the lines points accumulators. Some examples for the **X** row:

$$X[a, b, c, d, e] + v_i = \begin{cases} X[a + 1, b, c, d, e] & \text{when } v_1 \equiv (1, 1), \quad \text{because } x = +1 \\ X[a, b + 1, c, d, e] & \text{when } v_9 \equiv (2, -2), \quad \text{because } x = +2 \\ \dots \\ X[a, b, c, d, e + 1] & \text{when } v_6 \equiv (5, -5), \quad \text{because } x = +5 \\ X[a - 1, b, c, d, e] & \text{when } \bar{v}_1 \equiv (-1, -1), \quad \text{because } x = -1 \\ \dots \end{cases}$$

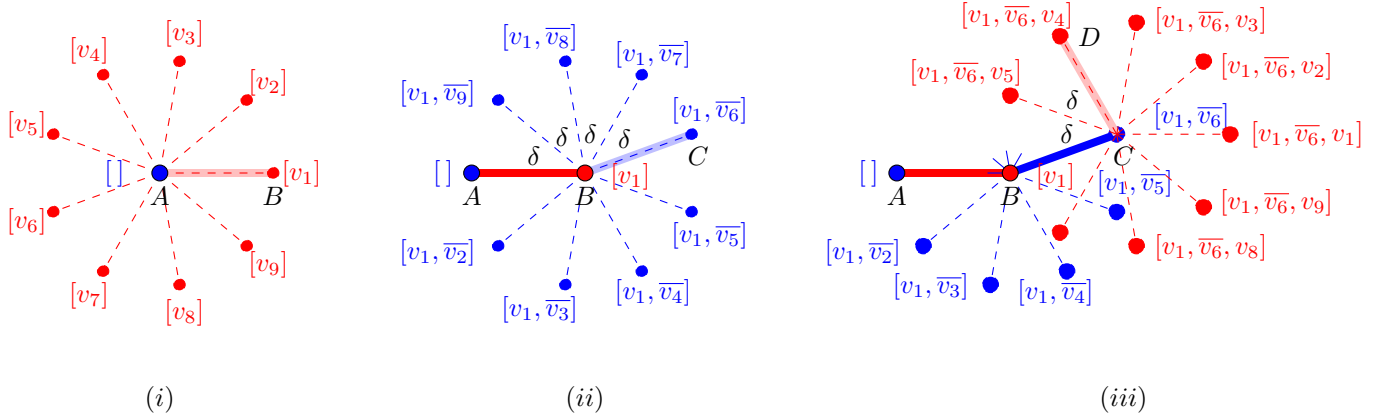


Figure 1: The symmetry 9 vectors, points, angles and polylines.

### 2.1 Example polylines

Figure 1 show polylines in three stages. At (i) we have polyline  $\overline{AB}$ , at (ii) we have the polyline  $\overline{ABC}$  and at (iii) we have the polyline  $\overline{ABCD}$ .

The alternating vectors of the type  $[..., v_m]$  are show in red and the vectors of the type  $[..., v_m, \bar{v}_n]$  are shown in blue.

At (ii) we notice the angle  $\angle ABC = 4\delta$  and at (iii) we notice the angle  $\angle BCD = 2\delta$ .

At (iii) we have the polyline  $\{A, B, C, D\}$  is denoted by  $\{[], [v_1], [v_1, \bar{v}_6], [v_1, \bar{v}_6, v_4]\}$  or in compact form  $P_9(0, 1, 6, 4)$ .

Lets get the accumulators of the points of polyline  $\overline{ABCD} = \{[], [v_1], [v_1, \overline{v_6}], [v_1, \overline{v_6}, v_4]\} = P_9(0, 1, 6, 4)$  shown in figure (iii). First by definition the accumulators at  $o \equiv (0, 0)$  which is point  $A = []$  are:

$$[]^A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (14)$$

Then we use the accumulators of vector  $v_1$  to get the accumulators of point  $B = [v_1]$ :

$$\begin{aligned} v_1 &= (1, 1) \\ v_1^A &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \\ [v_1]^A &= []^A + v_1^A \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (15)$$

Then we get the accumulators of vector  $\overline{v_6}$ :

$$\begin{aligned} v_6 &\equiv (5, -5) \\ \overline{v_6} &= (-5, 5) \\ \overline{v_6}^A &= \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \quad (16)$$

And then use  $\overline{v_6}^A$  to get the accumulators of point  $C = [v_1, \overline{v_6}]$ :

$$\begin{aligned} [v_1, \overline{v_6}]^A &= v_1^A + \overline{v_6}^A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \quad (17)$$

Finally we get the accumulator of vector  $v_4$  and use it to get the accumulator of point  $D = [v_1, \overline{v_6}, v_4]$ :

$$\begin{aligned} v_4 &\equiv (4, 4) \\ v_4^A &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\ [v_1, \overline{v_6}, v_4]^A &= [v_1, \overline{v_6}]^A + v_4^A = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} \end{aligned} \quad (18)$$

From the last equation of the accumulators of point  $D = [v_1, \overline{v_6}, v_4]$  we have as different of zero these values:  $X_1 = X_4 = 1$ ,  $X_5 = -1$ ,  $Y_1 = Y_4 = Y_5 = 1$ . Using the equation 13 we calculate the absolute coordinates of point at  $D$ :

$$\begin{aligned} D_x &= x_1 + x_2 - x_5 \\ &= \cos(0) + \cos(3\delta) - \cos(4\delta) \\ &= 1 + \cos(120^\circ) - \cos(160^\circ) \\ &\approx 1.439 \\ D_y &= y_1 + y_2 + y_3 \\ &= \sin(0) + \sin(3\delta) + \sin(4\delta) \\ &= \sin(120^\circ) + \sin(160^\circ) \\ &\approx 1.208 \end{aligned}$$

## 2.2 Example angles

Using equation 7 we can calculate the angles  $u\delta$  at vertices  $B$  and  $C$ . From  $C = [v_1, \overline{v_6}]$  we have that  $m_B = 1, n_B = 6$  and we calculate  $u$  at point  $B$ :

$$\begin{aligned} u_B &\equiv s + m_B - n_B \pmod{s} \\ &= 9 + 1 - 6 \pmod{9} \\ &= 4 \end{aligned}$$

From  $D = [v_1, \overline{v_6}, v_4]$  we have that  $m_C = 6, n_C = 4$  and we calculate  $u$  at point  $C$ :

$$\begin{aligned} u_C &\equiv s + m_C - n_C \pmod{s} \\ &= 9 + 6 - 4 \pmod{9} \\ &= 2 \end{aligned}$$

From figure 1 we notice that the angles values at  $B$  and  $C$  which are  $4\delta$  and  $2\delta$  match our calculations above.

## 3 Rhombi

Rhombi are equilateral tetragons. For every odd symmetry  $s$  we can build exactly  $n = \frac{s-1}{2}$  different rhombi. The rhombi angles  $u_1 < u_2$  are calculated as follows:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{i}{2} \\ n + \frac{i-1}{2} \end{bmatrix} \quad i = 1, \dots, n \quad (19)$$

Any rhombus can be identified with only two numbers, the symmetry  $s$  and the angle  $u_1$  with the notation  $R_s(u_1)$  so for symmetry  $s = 9$  we have these four rhombi:

$$R_9 = \left\{ R_9\left(\frac{1}{2}\right), R_9(1), R_9\left(\frac{3}{2}\right), R_9(2) \right\} \quad (20)$$

Table 2 show the rhombi for odd symmetries up to 9. Every rhombus is labeled with a consecutive lowercase letter **a**, **b**, **c**,... The first rhombi **a** is found first in symmetry  $s = 3$  and then again in symmetry  $s = 9$  so we prevent renaming congruent rhombi.

The rhombi angles  $\omega_1 = u_1\omega$  and  $\omega_2 = u_2\omega$  are expressed in function of the angles  $\{\alpha, \beta, \gamma, \delta\}$  of symmetries  $s = \{3, 5, 7, 9\}$  where  $\omega_1 < \omega_2$  and  $\omega_1 + \omega_2 = \pi$ .

### 3.1 Rhombi as cycled polylines

We cannot use the rhombi alone in our symmetry tessellations since both  $u_1$  and  $u_2$  are not integers at the same time. But we can attach rhombi by their sides forming hexagons, octagons, decagons and more polygons with  $n$  sides where  $n = 4 + 2i, i = 1, 2, 3, \dots$  always the polygons  $n$  angles result all integers.

By attaching we mean adding and subtracting rhombi preventing the intersection of any edges. This method permits to calculate the  $n$ -gons areas as the additions and subtraction of the known rhombi areas. Knowing any polygons area in function of the at most  $(s-1)/2$  different rhombi of symmetry  $s$  is very helpful as a necessary, but not sufficient condition for dissections.

$R_s(u_1)$	$u_2$	Label	$\omega_1$	$\omega_2$	Area
$R_3(\frac{1}{2})$	1	<b>a</b>	$\alpha/2$	$2\alpha/2$	$\sin(\alpha) \approx 0.866$
$R_5(\frac{1}{2})$	2	<b>b</b>	$\beta/2$	$4\beta/2$	$\sin(2\beta) \approx 0.587$
$R_5(1)$	$\frac{3}{2}$	<b>c</b>	$2\beta/2$	$3\beta/2$	$\sin(\beta) \approx 0.951$
$R_7(\frac{1}{2})$	3	<b>d</b>	$\gamma/2$	$6\gamma/2$	$\sin(3\gamma) \approx 0.433$
$R_7(1)$	$\frac{5}{2}$	<b>e</b>	$2\gamma/2$	$5\gamma/2$	$\sin(\gamma) \approx 0.781$
$R_7(\frac{3}{2})$	2	<b>f</b>	$3\gamma/2$	$4\gamma/2$	$\sin(2\gamma) \approx 0.974$
$R_9(\frac{1}{2})$	4	<b>g</b>	$\delta/2$	$8\delta/2$	$\sin(4\delta) \approx 0.342$
$R_9(1)$	$\frac{7}{2}$	<b>h</b>	$2\delta/2$	$7\delta/2$	$\sin(\delta) \approx 0.642$
$R_9(\frac{3}{2})$	3	<b>a</b>	$3\delta/2$	$6\delta/2$	$\sin(3\delta) \approx 0.866$
$R_9(2)$	$\frac{5}{2}$	<b>i</b>	$4\delta/2$	$5\delta/2$	$\sin(2\delta) \approx 0.984$

Table 2: Rhombi  $R_s(u_1)$  for symmetries  $s = \{3, 5, 7, 9\}$ .

## 4 Stars

For every odd symmetry  $s$  we have **stars** that are equilateral  $2s$ -gons with at most two different angles  $u_1$  and  $u_2$  at the vertices. We find exactly  $n = \frac{s-1}{2}$  different stars with angles  $u_1 \leq u_2$  as integers.

Table 3 show the stars for odd symmetries up to 9. Every star with  $u_1$  as integer are labeled with a consecutive calligraphy letter  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$ . Any star is identified as  $S_s(u_1)$  using only two numbers the the symmetry  $s$  and the smallest angle  $u_1$ . The another angle is calculated as  $u_2 = s - u_1 - 1$ . The stars are easily build with copies of rhombi so the table show the stars area in function of them.

$S_s(u_1)$	$u_2$	Label	Area	Group	Polygon
$S_3(\frac{1}{2})$	2	-	<b>6a</b>	$D_6$	$ 6/2 $ (12-gon)
$S_3(1)$	1	$\mathcal{A}$	<b>3a</b>	$D_6$	Regular hexagon
$S_5(\frac{1}{2})$	4	-	<b>5b</b>	$D_{10}$	$ 10/4 $ (20-gon)
$S_5(1)$	3	$\mathcal{B}$	<b>5c</b>	$D_5$	$ (5/2)_\alpha $ decagram
$S_5(2)$	2	$\mathcal{C}$	<b>5(c + b)</b>	$D_{10}$	Regular decagon
$S_7(\frac{1}{2})$	6	-	<b>7d</b>	$D_{14}$	$ 14/6 $ (28-gon)
$S_7(1)$	5	$\mathcal{D}$	<b>7e</b>	$D_7$	$ (7/4)_\alpha $ 14-gram
$S_7(2)$	4	$\mathcal{E}$	<b>7(e + f)</b>	$D_7$	$ (7/2)_\alpha $ 14-gram
$S_7(3)$	3	$\mathcal{F}$	<b>7(e + f + d)</b>	$D_{14}$	Regular 14-gon
$S_9(\frac{1}{2})$	8	-	<b>9g</b>	$D_{18}$	$ 18/8 $ (36-gon)
$S_9(1)$	7	$\mathcal{G}$	<b>9h</b>	$D_9$	$ (9/6)_\alpha $ 18-gram
$S_9(2)$	6	$\mathcal{H}$	<b>9(h + i)</b>	$D_9$	$ (9/4)_\alpha $ 18-gram
$S_9(3)$	5	$\mathcal{I}$	<b>9(h + i + a)</b>	$D_9$	$ (9/2)_\alpha $ 18-gram
$S_9(4)$	4	$\mathcal{J}$	<b>9(h + i + a + g)</b>	$D_{18}$	Regular 18-gon

Table 3: Stars  $S_s(u_1)$  for symmetries  $s = \{3, 5, 7, 9\}$ .

Figure 2 show nine copies of symmetry-9 rhombi  $\{\mathbf{h}, \mathbf{i}, \mathbf{a}, \mathbf{g}\}$  forming the four stars  $S_9(1)$ ,  $S_9(2)$ ,  $S_9(3)$  and  $S_9(4)$  labeled respectively  $\mathcal{G}$ ,  $\mathcal{H}$ ,  $\mathcal{I}$  and  $\mathcal{J}$ . Note how the rhombi half angles always are added together to produce only angles as integers for the stars.

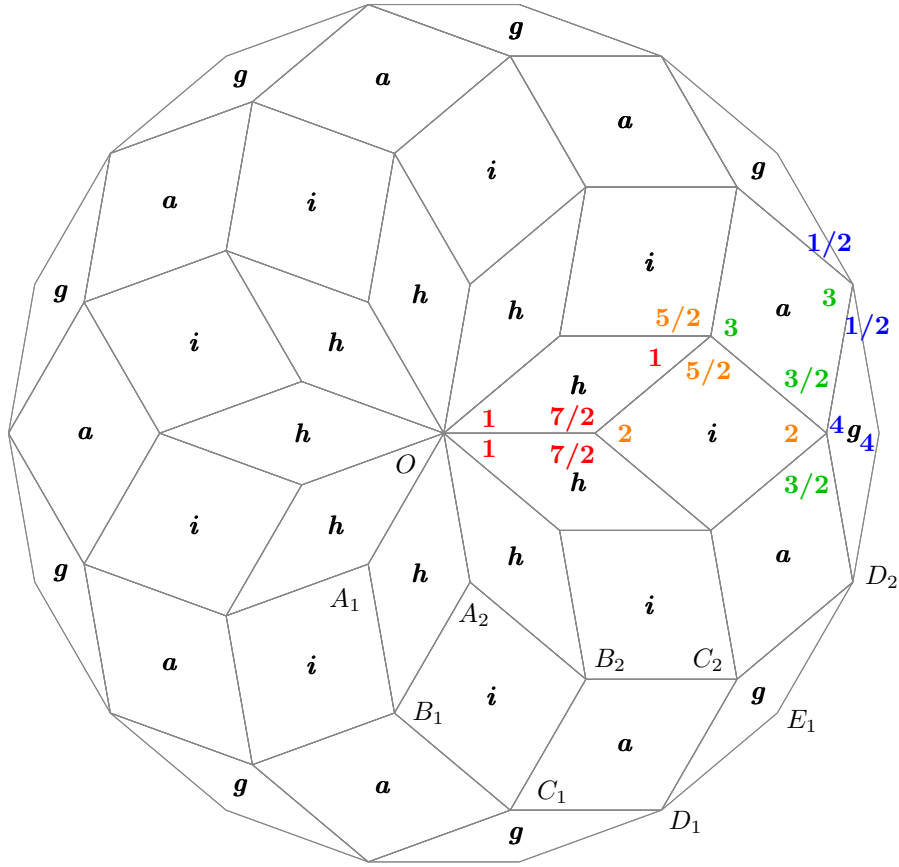


Figure 2: The symmetry 9 four rhombi  $\{h, i, a, g\}$  produce the four stars  $\{\mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}\}$  with areas  $9h$ ,  $9(h + i)$ ,  $9(h + i + a)$  and  $9(h + i + a + g)$  respectively.

## 5 Hexagons

For every odd symmetry  $s \geq 3$ , we can obtain equilateral hexagons by connecting six equal edges at angles  $\frac{2\pi u}{s}$ ,  $u = 1, 2, 3, \dots, s - 1$  or by adding and subtracting rhombi or by dissecting the areas left after stars intersections. In any case we find the hexagons has at most three different internal angles in this order  $(u_1, u_2, u_3, u_1, u_2, u_3)$  where  $u_1 + u_2 + u_3 = s$ .

### 5.1 Hexagons angles

Figure 4 show the hexagons defined as  $H_s(u_1, u_2)$  for symmetries  $s = \{3, 5, 7, 9\}$  where  $u_1 \leq u_2 \leq u_3$ . We labeled the non self-intersecting hexagons with uppercase letters  $A, B, C, \dots$ . We prevent to name differently any hexagon which is equivalent to a previous one having the same angles as what happens with hexagon  $A$  of symmetries 3 and 9.

$H_s(u_1, u_2)$	Label	$(u_1, u_2, u_3, u_4, u_5, u_6)$	Group	Description
$H_3(1)$	<b>A</b>	(1,1,1,1,1,1)	$D_6$	Regular hexagon
$H_5(1, 1, 3)$	<b>B</b>	(1,1,3,1,1,3)	$D_2$	Sormeh Dan Girih tile
$H_5(1, 2, 2)$	<b>C</b>	(1,2,2,1,2,2)	$D_2$	Shesh Band Girih tile
$H_7(1, 1, 5)$	-	(1,1,5,1,1,5)	$D_2$	self-intersecting
$H_7(1, 2, 4)$	<b>D</b>	(1,2,4,1,2,4)	$C_2$	
$H_7(1, 3, 3)$	<b>E</b>	(1,3,3,1,3,3)	$D_2$	
$H_7(2, 2, 3)$	<b>F</b>	(2,2,3,2,2,3)	$D_2$	
$H_9(1, 1, 7)$	-	(1,1,7,1,1,7)	$D_2$	
$H_9(1, 2, 6)$	<b>G</b>	(1,2,6,1,2,6)	$C_2$	self-intersecting
$H_9(1, 3, 5)$	<b>H</b>	(1,3,5,1,3,5)	$C_2$	
$H_9(1, 4, 4)$	<b>I</b>	(1,4,4,1,4,4)	$D_2$	
$H_9(1, 5)$	<b>I'</b>	(1,5,1,5,1,5)	$D_3$	
$H_9(2, 2, 5)$	<b>J</b>	(2,2,5,2,2,5)	$D_2$	
$H_9(2, 3, 4)$	<b>K</b>	(2,3,4,2,3,4)	$C_2$	
$H_9(2, 4)$	<b>M</b>	(2,4,2,4,2,4)	$D_3$	
$H_9(3)$	<b>A</b>	(3,3,3,3,3,3)	$D_6$	

Table 4: Hexagons  $H_s(u_1, u_2, u_3)$  for symmetries  $s = \{3, 5, 7, 9\}$ .

## 5.2 Hexagons areas

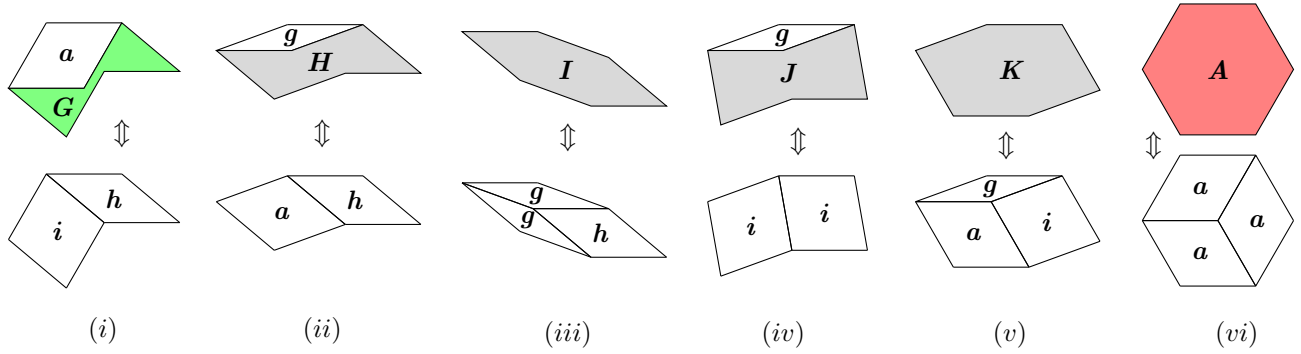


Figure 3: Symmetry 9 hexagons formed adding and subtracting rhombi.

Figure 3 show how to calculate the area of the symmetry 9 hexagons in function of the symmetry 9 rhombi. From (i) to (vi) we equate the area of sum of the polygons in the top with the area of the sum of the polygons of the bottom. We have for the six cases these six equations:

$$a + G = i + h \quad (21)$$

$$g + H = a + h \quad (22)$$

$$I = 2g + h \quad (23)$$

$$g + J = 2i \quad (24)$$

$$K = g + a + i \quad (25)$$

$$A = 3a \quad (26)$$



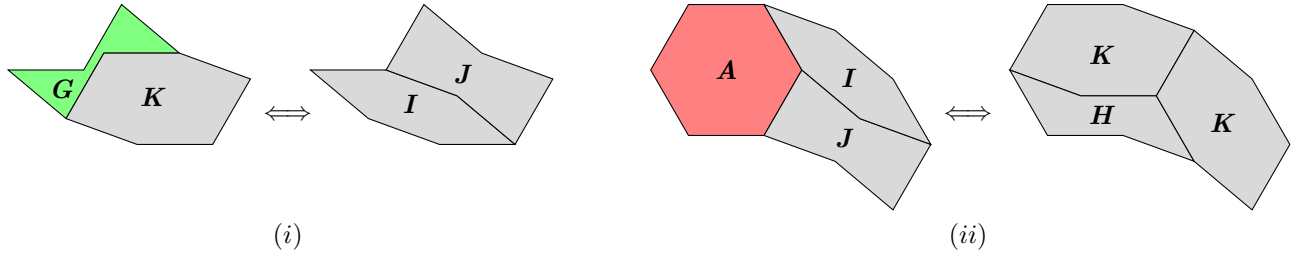


Figure 4: Hexagons  $\{G, A\}$  formed adding and subtracting hexagons  $\{H, I, J, K\}$ .

Figure 4 show how to express the area of the six hexagons in function of only four. For (i) and (ii) we equate the area of the sum of hexagons of the left with the area of the hexagons of the right. We have for the two cases these two equations:

$$G + K = I + J \quad (27)$$

$$A + I + J = H + 2K \quad (28)$$

Using the last eight equations we form the table 5 which show the areas of the six hexagons in function of four rhombi  $g, h, a, i$  and in function of only four hexagons  $H, I, J, K$ .

Hexagon	$g, h, a, i$ area	$H, I, J, K$ area
<b>H</b>	$a + h - g$	<b>H</b>
<b>I</b>	$2g + h$	<b>I</b>
<b>J</b>	$2i - g$	<b>J</b>
<b>K</b>	$g + a + i$	<b>K</b>
<b>A</b>	$3a$	$2K + H - I - J$
<b>G</b>	$i + h - a$	$I + J - K$

Table 5: Symmetry 9 hexagon areas.

### 5.3 Hexagons from stars

Figure 5 show the disposition of the symmetry 9 four stars. We label the 18 vertices of stars  $\{\mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}\}$  as  $\{G_0, G_1, \dots, G_{17}\}$ ,  $\{H_0, H_1, \dots, H_{17}\}$ ,  $\{I_0, I_1, \dots, I_{17}\}$  and  $\{J_0, J_1, \dots, J_{17}\}$  respectively. For simplicity, only some vertices are show. First we make coincident at vertex  $O$  all the vertices  $G_0, H_0, I_0, J_0$ . With the center at  $O$  we rotate all stars to make coincidents  $G_{17}, H_{17}, I_{17}$  and  $J_{17}$ . The rotations also joined another different vertices.

First we add three new edges (in red) joining the stars  $\mathcal{J}$  and  $\mathcal{I}$  vertices:  $\overline{J_3I_2}$ ,  $\overline{J_5I_4}$  and  $\overline{J_7I_6}$  dissecting the red region into four hexagons, two of them essentially different. The three consecutive angles of the two hexagons are shown: **I** (1,4,4) and **K** (3,4,2).

Then we add three new edges (in orange) joining the stars  $\mathcal{I}$  and  $\mathcal{H}$  vertices:  $\overline{I_3H_2}$ ,  $\overline{I_5H_4}$  and  $\overline{I_7H_6}$  dissecting the orange region into four hexagons, two of them new. The three consecutive angles of the the two hexagons are show: **H** (1,5,3) and **A** (3,3,3).

Finally we add three more edges (in green) joining the stars  $\mathcal{H}$  and  $\mathcal{G}$  vertices:  $\overline{H_3G_2}$ ,  $\overline{H_5G_4}$  and  $\overline{H_7G_6}$  dissecting the green region into four hexagons, two of them new. The three consecutive angles of the the two hexagons are show: **G** (1,6,2) and **J** (2,5,2).

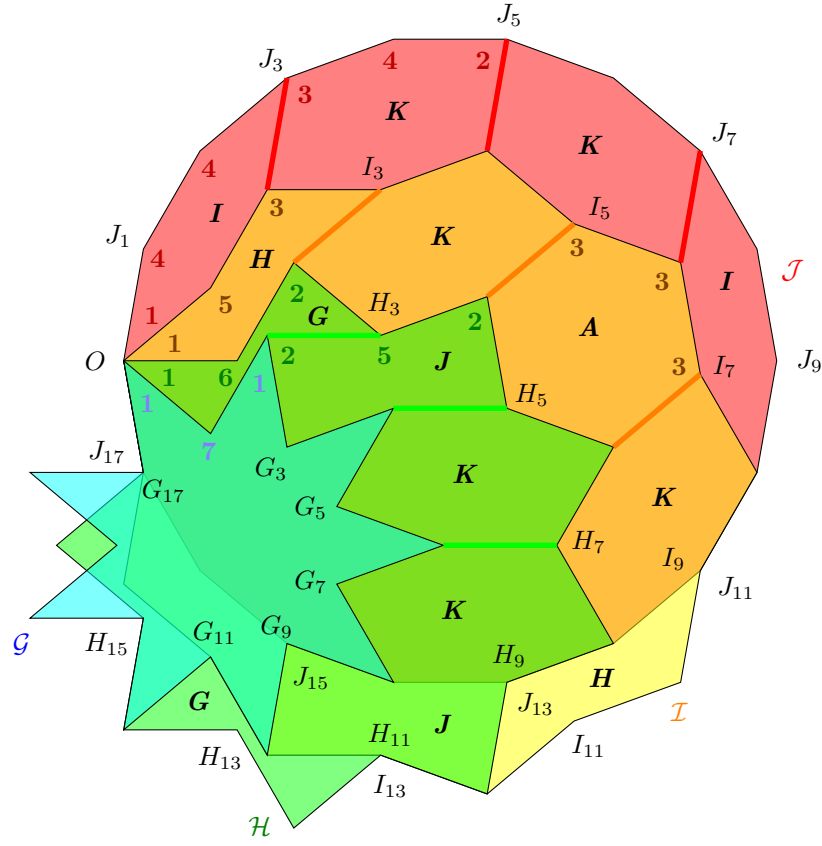


Figure 5: Symmetry 9 stars  $\{\mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}\}$  dissected to get the six hexagons  $\{G, H, I, J, K, A\}$ .

## 6 Octagons

### 6.1 Octagons by stars

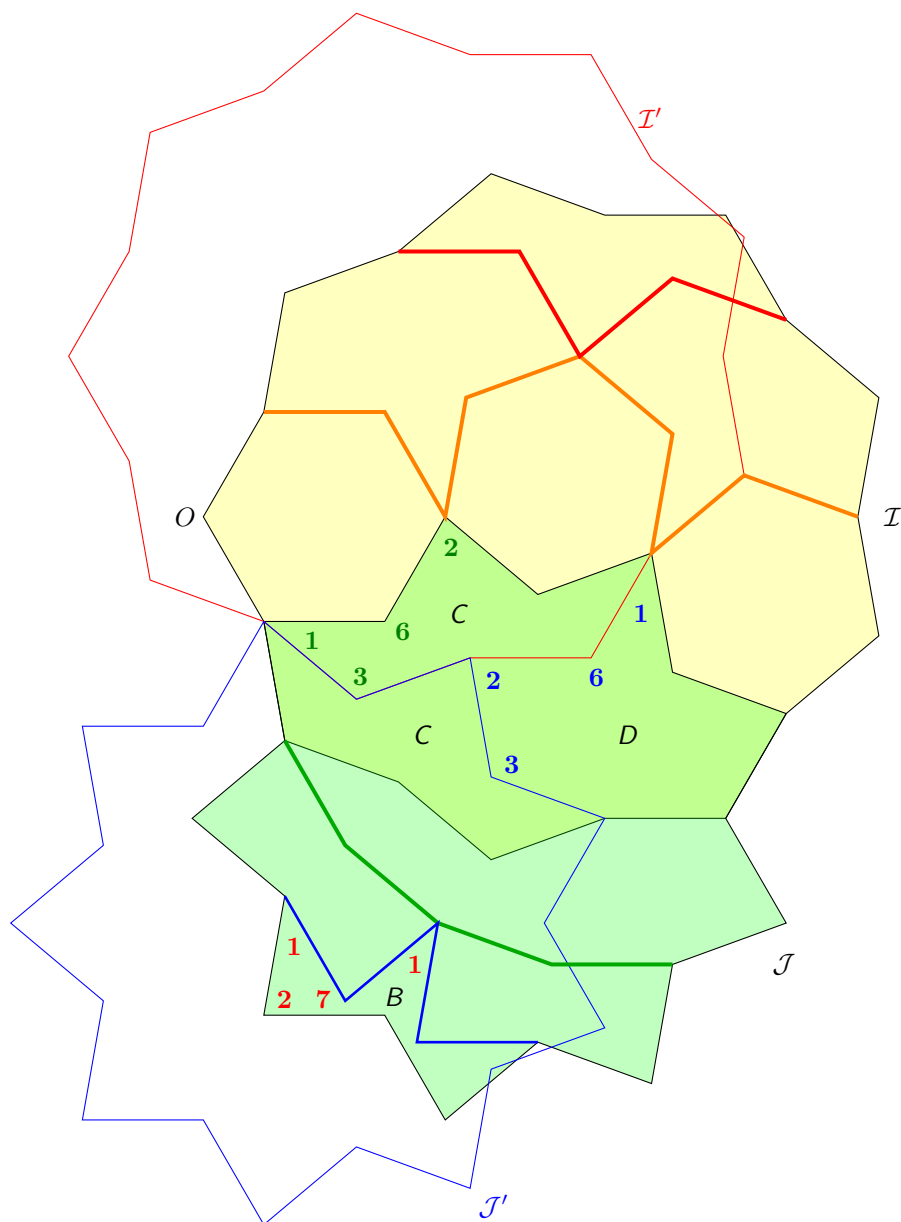


Figure 6: Octagons after intersection of stars  $\mathcal{I}$  and  $\mathcal{J}$ .