

Lenses

<https://github.com/heptagons/lenses>

2024/1/8

Abstract

Lenses are equilateral hexagons resembling concave and convex optical lenses. Lenses consecutive six internal angles are $(\theta_1, \theta_2, \theta_3, \theta_1, \theta_2, \theta_3)$ where $\theta_1 = x\theta_0$, $\theta_2 = y\theta_0$, and $\theta_3 = z\theta_0$ where $\theta_0 = 2\pi/m$ is the base angle of symmetry $m = x + y + z$. Lenses can be formed adding and subtracting rhombi or by intersecting equi-stars with others.

1 Equi-stars

Equi-stars are equilateral polygons with an even number of sides and vertices of at most two different angles. These stars can be defined with only two numbers: A symmetry integer m and a minimum angle integer a so the star is defined as $S(m, a)$. Here we are interested only in symmetries of the form $m = 2n + 1$ for $n = 1, 2, 3, \dots$. Every symmetry $m = 2n + 1$ has exactly n different stars: $S(m, n), S(m, n-1), \dots, S(m, 1)$. Stars of the form $S(m, n)$ correspond to the regular polygons of $2m$ sides.

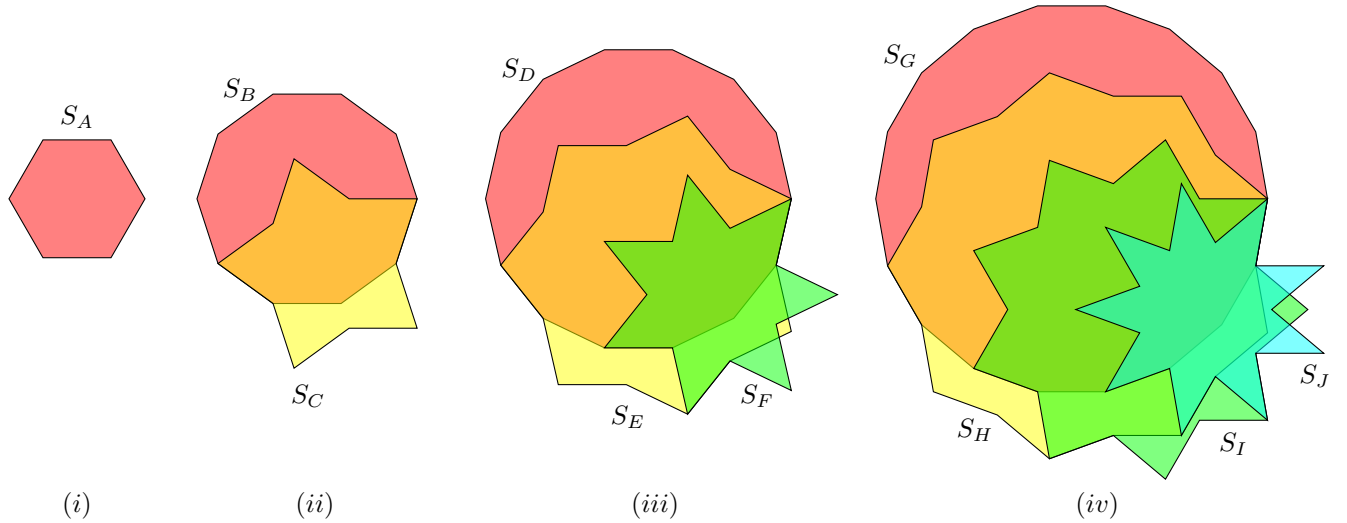


Figure 1: Equi-stars of symmetries $m = \{3, 5, 7, 9\}$.

Figure 1 show the stars for the smaller symmetries in translucent colors and intersecting with others of the same symmetry. At (i) we have for the symmetry 3 the only star in red $S_A \equiv S(3, 1)$ which is the regular hexagon. At (ii) we have for the symmetry 5 the regular decagon in red $S_B \equiv S(5, 2)$ and the star $S_C \equiv S(5, 1)$ in yellow; the region in orange is the intersection of the two stars. At (iii) for symmetry 7 we have three stars: The regular 14-gon $S_D \equiv S(7, 3)$ in red, the star $S_E \equiv S(7, 2)$ in yellow and the star $S_F \equiv S(7, 1)$ in green. At (iv) we have for the symmetry 9 four stars: The regular 18-gon $S_G \equiv S(9, 4)$ in red, the star $S_H \equiv S(9, 3)$ in yellow, the star $S_I \equiv S(9, 2)$ in green and the star $S_J \equiv S(9, 1)$ in blue.

1.1 Lenses

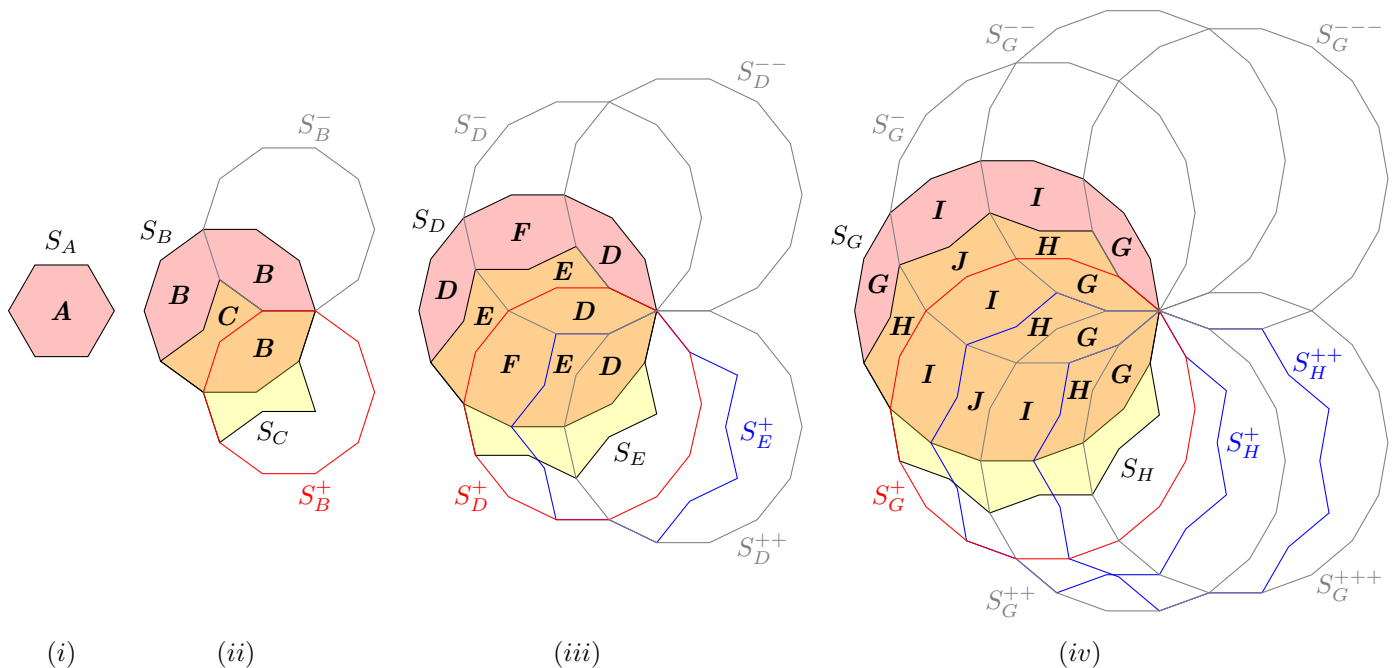


Figure 2: Lenses build from the intersection of two types of stars: $S(m, n)$ and $S(m, n - 1)$ for symmetries $m = \{3, 5, 7, 9\}$.

Figure 2 show how to dispose several stars to build all the lenses for symmetries $m = \{3, 5, 7, 9\}$ but the process works for greater symmetries. For every symmetry $m = 2n + 1$ we obtain n different lenses, using $2n - 1$ stars $S(m, n)$ and using $n - 1$ stars $S(m, n - 1)$. Both stars can be dissected with the lenses.

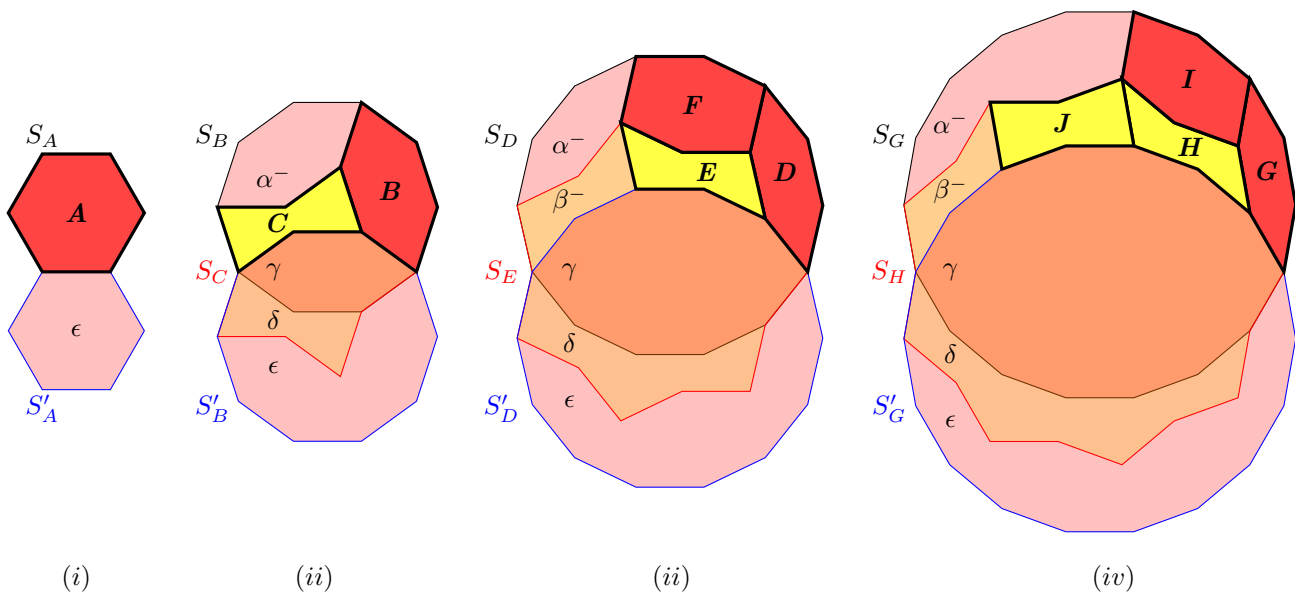


Figure 3: First lenses from the intersection of red stars $S(m, n)$ and yellow stars $S(m, n - 1)$ for ymmetries $m = \{3, 5, 7, 9\}$. Here the lenses appear once and after dissecting the areas remaining red and yellow.

Figure ?? show the first lenses produced by the intersections of three stars: One $S(m, n)$ in pink at the top, one $S(m, n - 1)$ in orange at the center and another $S(m, n)$ in pink at the bottom. The intersections

produce five regions: α congruent with ϵ , β congruent with δ and γ . The lenses emerge after dissecting regions α and β . For every symmetry $m = 2n + 1$ we get n distinct lenses.

At (i) for symmetry $m = 3$ we have the single lense (**A**) equivalent to the regular hexagon. Lense **A** is congruent with S_A . At (ii) for symmetry $m = 5$ we have the two lenses (**B**, **C**). The α region of S_B area equals $2\mathbf{B}$. The β region area equals \mathbf{C} . At (iii) for symmetry $m = 7$ we have the three lenses (**D**, **E**, **F**). The region α area equals $2\mathbf{D} + \mathbf{F}$ and the region β area equals $2\mathbf{E}$. At (iv) for symmetry $m = 9$ we have the four lenses (**G**, **H**, **I**, **J**). The α region area equals $2(\mathbf{G} + \mathbf{I})$ and the β region area equals $2\mathbf{H} + \mathbf{J}$.

Next we are going to show the regions δ for any symmetry $m \geq 5$ can be dissected with the lenses.

2 Crowns

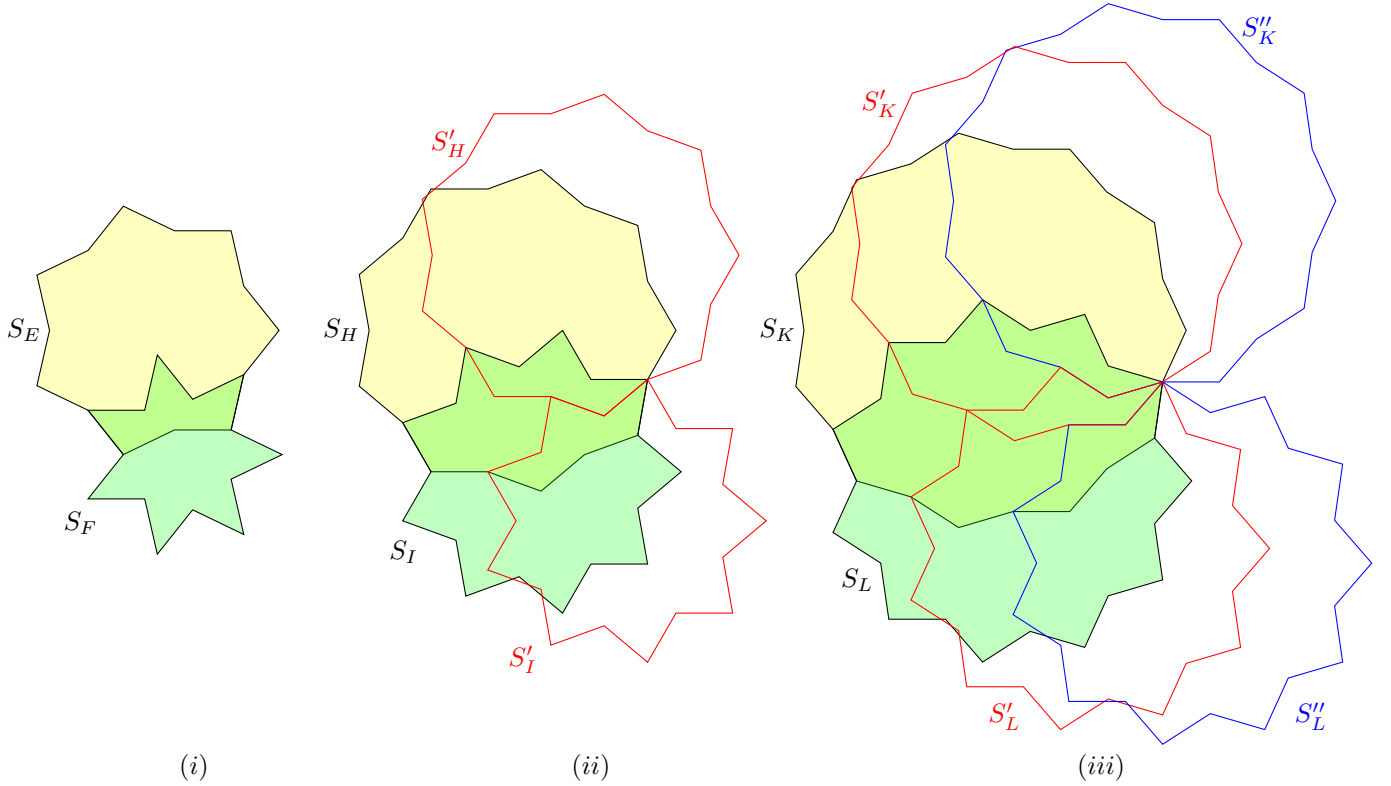


Figure 4: Crowns.

3 Symmetry 5

Symmetry 5 is based in angle $\beta = \frac{2\pi}{5}$ and produces the two rhombi (**b**, **c**) and the two lenses (**B**, **C**).

3.1 Rhombi (b, c)

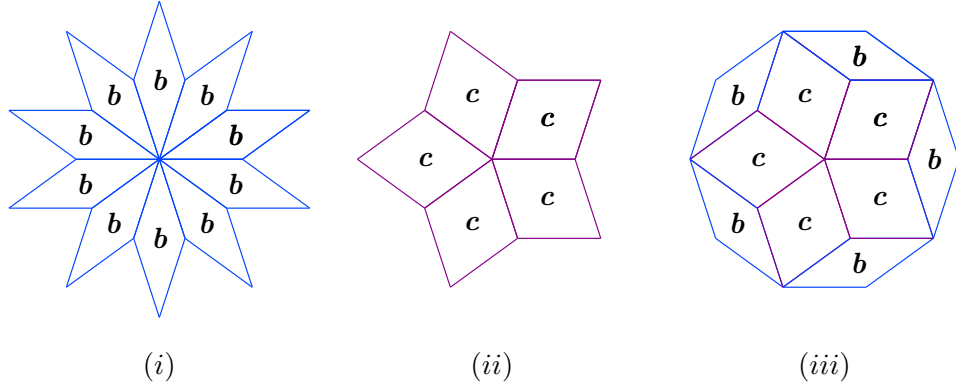


Figure 5: Rhombi (b, c) from dissecting stars S_{10} .

Figure 5 show the rhombi (b, c). Inspecting the stars we get the areas simply adding their rhombi. At (i) the star $S_{10}(1, 8)$ with area $A = 10b$. At (ii) the star $S_{10}(2, 6) = S_5(1, 3)$ with area $A = 5c$. At (iii) the regular decagon equivalent to stars $S_{10}(4, 4) = S_5(2, 2)$ with area $A = 5b + 5c$. Table 1 show the rhombi (b, c) internal angles in terms of angle $\beta = 2\pi/5$ and areas for side equals to 1. Dividing areas we find $\frac{c}{b} = 2 \cos\left(\frac{\pi}{5}\right) = \frac{\sqrt{5} + 1}{2}$.

Rhombus	θ_1	θ_2	Area
b	$\beta/2$	$4\beta/2$	$\sin(2\beta) = \frac{\sqrt{10 - 2\sqrt{5}}}{4}$
c	$2\beta/2$	$3\beta/2$	$\sin(\beta) = \frac{\sqrt{10 + 2\sqrt{5}}}{4} = b \cos\left(\frac{\pi}{5}\right) = b \left(\frac{\sqrt{5} + 1}{2}\right)$

Table 1: Rhombi (b, c) internal angles and areas. $\theta_1 + \theta_2 = \pi$ and $\beta = 2\pi/5$.

3.2 Regular pentagon and star $|5/2|$

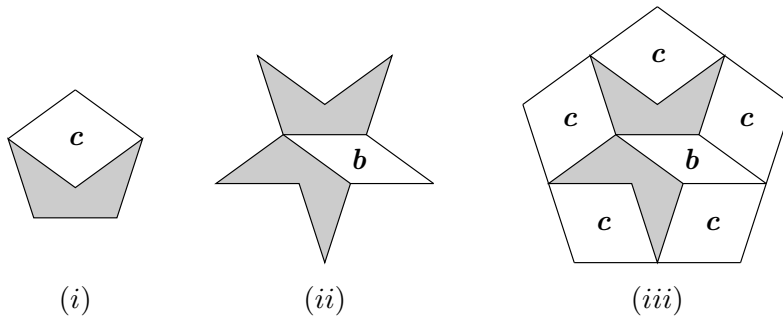


Figure 6: Regular pentagon $|5/1|$ at (i). Star $|5/2|$ at (ii). Double pentagon at (iii).

Figure 6 show regular pentagon and isotoxal star $|5/2|$ dissected with rhombi (b, c) plus concave pentagons (in gray). Let x be the area of such gray piece. By inspection the area of regular pentagon at (i) is $A_1 = c + x$ and the area of regular pentagon at (iii) is $P_2 = b + 5c + 2x$. Since the side of A_2 is the double

of A_1 its area is four times so we can get the value of x in terms of (b, c)

$$\begin{aligned}
4P_1 &= P_4 \\
4(c + x) &= b + 5c + 2x \\
x &= \frac{b + c}{2}
\end{aligned} \tag{1}$$

We use the value of x to get the areas of pentagon (i) and star (ii):

$$\begin{aligned}
A|5/1| &= c + x \\
&= \frac{b + 3c}{2}
\end{aligned} \tag{2}$$

$$\begin{aligned}
A|5/2| &= b + 2x \\
&= 2b + c
\end{aligned} \tag{3}$$

3.3 Lenses (B, C)

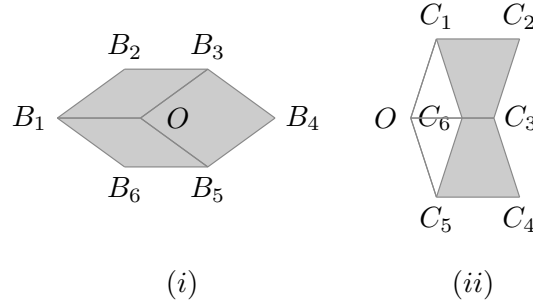


Figure 7: Lenses (B, C) build with rhombi (b, c) .

Figure 7 show lenses (B, C) construction and two stars formed with them. At (i) we form the lense B with perimeter $\overline{B_1...B_6}$ adding two rhombi b ($\overline{B_1B_2B_3O}$ and $\overline{B_1OB_5B_6}$) and adding one rhombus c ($\overline{OB_3B_4B_5}$) so its area is $2b + c$. Lense B is equivalent to the hexagon $H_5(1, 2, 2)$. At (ii) we form the lense C with perimeter $\overline{C_1...C_6}$ adding two rhombi c ($\overline{OC_1C_2C_3}$ and $\overline{OC_3C_4C_5}$) and substracting one rhombus b ($\overline{OC_1C_6C_5}$) so its area is $2c - b$. Lense C is equivalent to the hexagon $H_5(1, 1, 3)$. Table 2 show the lenses (B, C) internal angles and areas.

Lense	θ_1	θ_2	θ_3	Area
B	β	2β	2β	$2b + c$
C	β	β	3β	$-b + 2c$

Table 2: Lenses (B, C) internal angles and areas in terms of rhombi (b, c) . $\theta_1 + \theta_2 + \theta_3 = 2\pi$ where $\beta = 2\pi/5$.

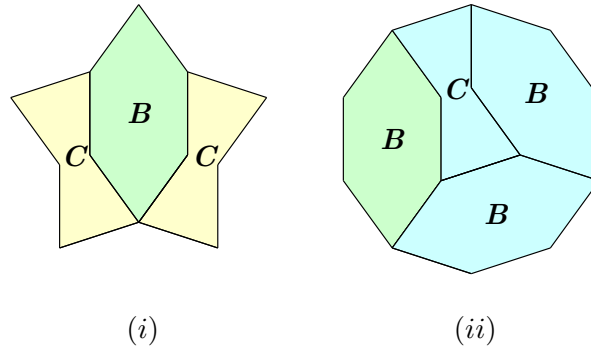


Figure 8: Two stars dissected with lenses (B , C).

Figure 8 show two stars dissected with lenses (B , C). At (i) the star $S_5(1,3)$ dissection implies its area is $A = B + 2C = 5c$. At (ii) the regular decagon or star $S_5(2,2)$ dissection implies its area is $A = 3B + C = 5(b + c)$.

4 Symmetry 7

Symmetry 7 is based in angle $\gamma = \frac{2\pi}{7}$ and produces the three rhombi (d, f, e) and the three lenses (D, E, F).

4.1 Rhombi (d, e, f)

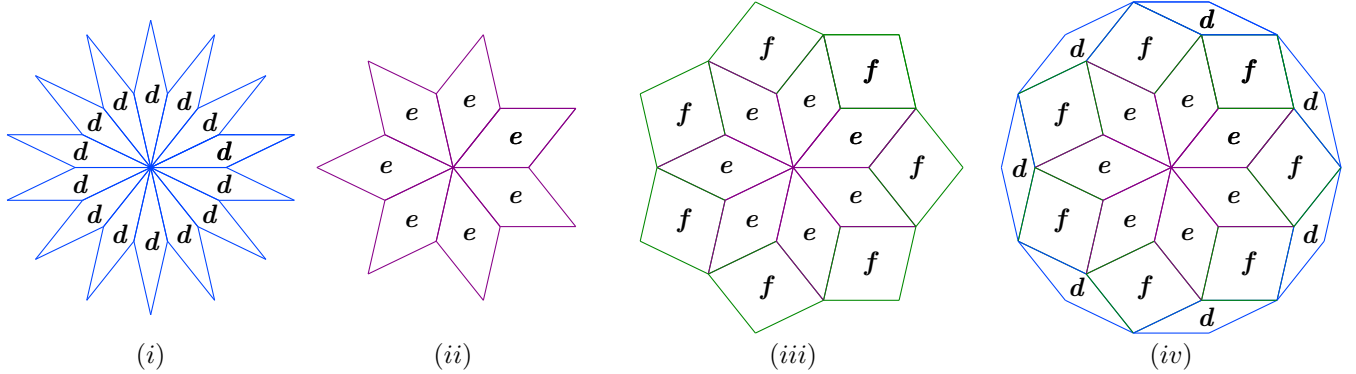


Figure 9: Rhombi (d, e, f) from dissected stars S_{14} .

Figure 9 show rhombi (d, e, f). Inspecting the stars we get the areas simply adding their rhombi. At (i) the star $S_{14}(1,12)$ with area $A = 14d$. At (ii) the star $S_{14}(2,10) = S_7(1,5)$ with area $A = 7e$. At (iii) the star $S_{14}(4,8) = S_7(2,4)$ with area $A = 7(e + f)$. At (iv) the regular 14-gon equivalent to stars $S_{14}(6,6) = S_7(3,3)$ with area $A = 7(d + e + f)$. Table 3 show the symmetry 7 lenses internal angles based in angle $\gamma = 2\pi/7$ and the areas.

Rhombus	θ_1	θ_2	Area
d	$\gamma/2$	$6\gamma/2$	$\sin(3\gamma)$
e	$2\gamma/2$	$5\gamma/2$	$\sin(\gamma)$
f	$3\gamma/2$	$4\gamma/2$	$\sin(2\gamma)$

Table 3: Rhombi (d, e, f) internal angles. $\theta_1 + \theta_2 = \pi$ and $\gamma = 2\pi/7$.

4.2 Regular heptagon and stars $|7/3|$ and $|7/2|$

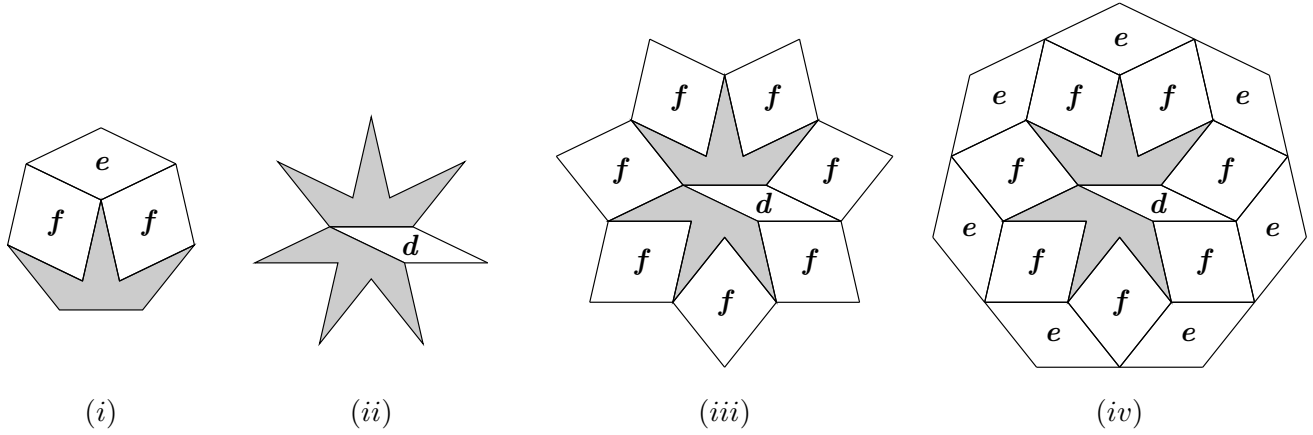


Figure 10: Heptagon $|7/1|$ at (i). Star $|7/3|$ at (ii). Star $|7/2|$ at (iii). Double heptagon at (iv).

Figure 10 show regular heptagon and heptagrams dissected with rhombi ($\mathbf{c}, \mathbf{d}, \mathbf{e}$) plus equilateral concave heptagons (in gray). Let \mathbf{y} be the area of such gray piece. By inspection the area of regular heptagon at (i) is $A_1 = \mathbf{e} + 2\mathbf{f} + \mathbf{y}$ while the area of regular heptagon at (iv) is $A_2 = \mathbf{d} + 7(\mathbf{e} + \mathbf{f}) + 2\mathbf{y}$. Since the side of A_2 is the double of A_1 its area is four times so we can get the value of \mathbf{y} in terms of ($\mathbf{d}, \mathbf{e}, \mathbf{f}$):

$$\begin{aligned} 4A_1 &= A_2 \\ 4(\mathbf{e} + 2\mathbf{f} + \mathbf{y}) &= \mathbf{d} + 7(\mathbf{e} + \mathbf{f}) + 2\mathbf{y} \\ \mathbf{y} &= \frac{\mathbf{d} + 3\mathbf{e} - \mathbf{f}}{2} \end{aligned} \quad (4)$$

We use the value of \mathbf{y} to calculate the areas of heptagon (i) and stars (ii) and (iii) in terms of ($\mathbf{d}, \mathbf{e}, \mathbf{f}$):

$$\begin{aligned} A|7/1| &= \mathbf{e} + 2\mathbf{f} + \mathbf{y} \\ &= \frac{\mathbf{d} + 5\mathbf{e} + 3\mathbf{f}}{2} \end{aligned} \quad (5)$$

$$\begin{aligned} A|7/3| &= \mathbf{d} + 2\mathbf{y} \\ &= 2\mathbf{d} + 3\mathbf{e} - \mathbf{f} \end{aligned} \quad (6)$$

$$\begin{aligned} A|7/2| &= A\{7/3\} + 7\mathbf{f} \\ &= 2\mathbf{d} + 3\mathbf{e} + 6\mathbf{f} \end{aligned} \quad (7)$$

4.3 Lenses ($\mathbf{D}, \mathbf{E}, \mathbf{F}$)

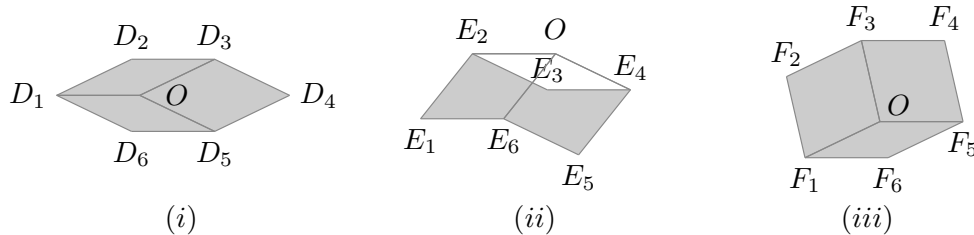


Figure 11: Lenses ($\mathbf{D}, \mathbf{E}, \mathbf{F}$) build from rhombi ($\mathbf{d}, \mathbf{e}, \mathbf{f}$).

Figure 11 show lenses (\mathbf{B}, \mathbf{C}) construction. At (i) we form the lense \mathbf{D} with perimeter $\overline{D_1...D_6}$ adding two rhombi \mathbf{d} ($\overline{D_1D_2D_3O}$ and $\overline{D_1OD_5D_6}$) and adding one rhombus \mathbf{e} ($\overline{OD_3D_4D_5}$) so its area is $2\mathbf{d} + \mathbf{e}$. Lense \mathbf{D} is equivalent to the hexagon $H_7(1, 3, 3)$.

At (ii) we form the lense \mathbf{E} with perimeter $\overline{E_1...E_6}$ adding one rhombus \mathbf{e} ($\overline{E_1E_2OE_6}$) adding one rhombus \mathbf{f} ($\overline{OE_4E_5E_6}$) and subtracting one rhombus \mathbf{d} ($\overline{E_2OE_4E_3}$) so its area is $-\mathbf{d} + \mathbf{e} + \mathbf{f}$. Lense \mathbf{E} is equivalent to the hexagon $H_7(1, 2, 4)$.

At (iii) we form the lense \mathbf{F} with perimeter $\overline{F_1...F_6}$ adding two rhombi \mathbf{f} ($\overline{F_1F_2F_3O}$ and $\overline{F_3F_4F_5O}$) and adding one rhombus \mathbf{d} ($\overline{F_1OF_5F_6}$) so its area is $\mathbf{d} + 2\mathbf{f}$. Lense \mathbf{F} is equivalent to the hexagon $H_7(2, 2, 3)$. Table 4 show the lenses $(\mathbf{D}, \mathbf{E}, \mathbf{F})$ internal angles and areas.

Lense	θ_1	θ_2	θ_3	Area
\mathbf{D}	γ	3γ	3γ	$2\mathbf{d} + \mathbf{e}$
\mathbf{E}	γ	2γ	4γ	$-\mathbf{d} + \mathbf{e} + \mathbf{f}$
\mathbf{F}	2γ	2γ	3γ	$-\mathbf{d} + 2\mathbf{f}$

Table 4: Lenses $(\mathbf{D}, \mathbf{E}, \mathbf{F})$ internal angles and areas in terms of rhombi $(\mathbf{d}, \mathbf{e}, \mathbf{f})$. $\theta_1 + \theta_2 + \theta_3 = 2\pi$ and $\gamma = 2\pi/7$.

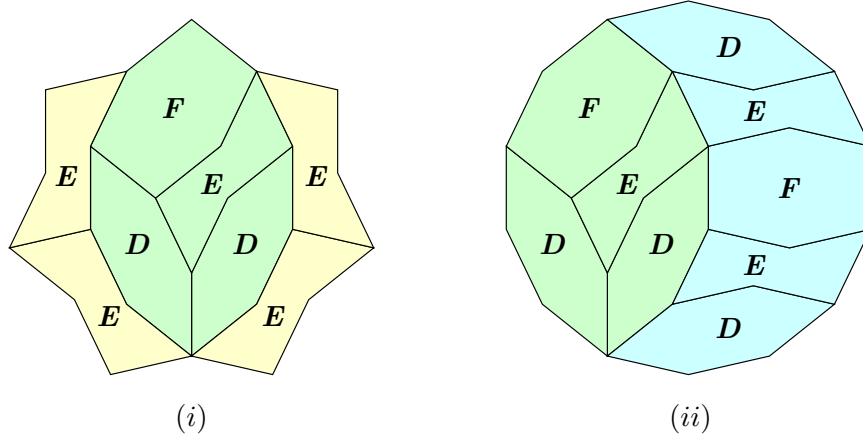


Figure 12: Stars dissected with only lenses $(\mathbf{D}, \mathbf{E}, \mathbf{F})$.

Figure 12 show stars $S_7(2, 4)$ and $S_7(3, 3)$ dissected with lenses $(\mathbf{D}, \mathbf{E}, \mathbf{F})$. At (i) we have star $S_7(2, 4)$ and by inspection we deduce its area is $A = 2\mathbf{D} + 5\mathbf{E} + \mathbf{F}$. At (ii) we have regular 14-gon (or star $S_7(3, 3)$) and by inspection we deduce its area is $4\mathbf{D} + 3\mathbf{E} + 2\mathbf{F}$. Both stars have in common an area in green resembling a tree leaf. The star at (i) also contains two regions in yellow resembling crowns while the star at (ii) contains a region in cyan resembling a moon phase.

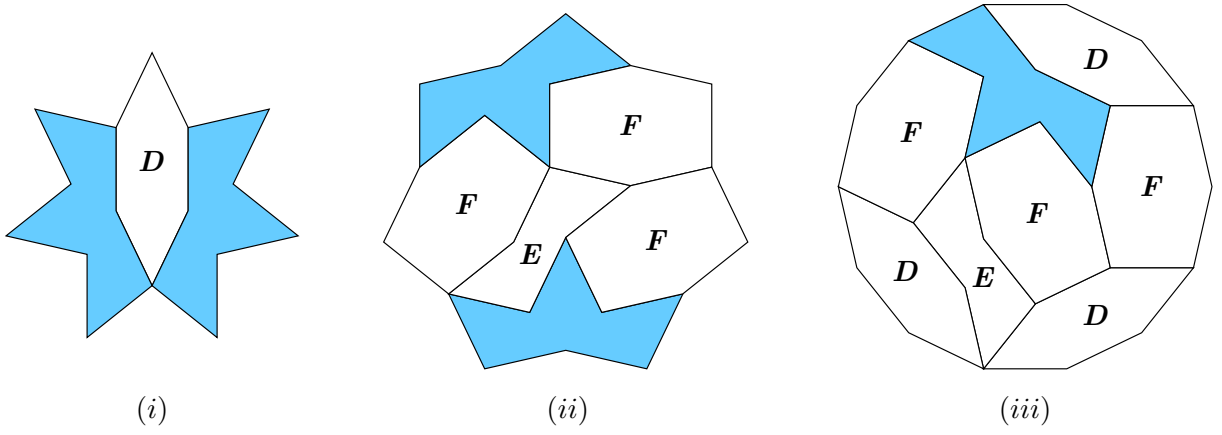


Figure 13: Stars dissected with octagons O_7 (in blue) and lenses (D, E, F) .

Figure 13 show Stars $S_7(1, 5)$, $S_7(2, 4)$ and $S_7(3, 3)$ dissected with octagons O_7 (in blue) and lenses (D, E, F) . At (i) we have the star $S_7(1, 5)$ and by inspection we deduce its area is $A = D + 2O_7$. At (ii) we have the star $S_7(2, 4)$ and we can conclude its area is $E + 3F + 2O_7$. Similarly the area of the 14-gon at (iii) is $3D + E + 3F + O_7$. Comparing the areas of the two 14-gons of figures 12 and 13 we can find the area of O_7 in terms of (E, F, G) :

$$\begin{aligned} 4D + 3E + 2F &= 3D + E + 3F + O_7 \\ O_7 &= D + 2E - F \end{aligned} \tag{8}$$

So we can calculate the area of star $S_7(1, 5)$ in terms of (E, F, G) :

$$\begin{aligned} S_7(1, 5) &= D + 2(D + 2E - F) \\ &= 3D + 4E - 2F \end{aligned} \tag{9}$$