Symmetry 9

https://github.com/heptagons/lenses

2024/1/22

Abstract

We build equilateral polygons that can tesselate the plane using the symmetry 9. We analyze lower odd symmetries $3, 5, 7$ also to find patterns. We analize vectors which describe symmetry vertices. Then we define polygons as rhombus, hexagons, octagons and stars build with the vectors.

1 Symmetries

We are interested in odd symmetries starting with $s = 3$. Table 1 show the symmetries up to $s = 9$. The value of the angle of the symmetry s is $\theta = \frac{2\pi}{s}$.

s	Angle	θ
3	α	$2\pi/3$
5	β	$2\pi/5$
7	γ	$2\pi/7$
9	δ	$2\pi/9$

Table 1: Symmetries $s = \{3, 5, 7, 9\}$.

For every symmetry s we identify $t = \frac{s+1}{2}$ independent abscissas and ordinates pairs $\langle x_i, y_i \rangle$:

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} \cos((i-1)\theta) \\ \sin((i-1)\theta) \end{bmatrix} \quad i = 1, 2, ..., t \tag{1}$$

We define a relative origin point v_0. There are s different vectors $v_1, v_2, ...v_s$ of magnitud 1 and equidistant to point v_0 defined with coordinates x_i, y_i as follows.

$$v_i \equiv \begin{cases} \langle x_i, y_i \rangle & \text{for} \quad i \leq t \\ \langle x_j, -y_j \rangle & \text{for} \quad i > t \quad j = s + 2 - i \end{cases} \tag{2}$$

Any vector $v_m \equiv \langle x_m, y_m \rangle$ where $1 \leq m \leq s$ can be rotated by 180° around v_0 and is denoted as $\overline{v_m}$ and the pair signs are changed:

$$\overline{v_m} \equiv \langle -x_m, -y_m \rangle \tag{3}$$

We define an absolute origin point o located at $(0, 0)$. Starting at o we can form polylines adding vectors alternating the type v_m and the type $\overline{v_m}$ in order the consecutive points angles are always multiples of θ. The polyline with vectors v_a, $\overline{v_b}$, v_c, ... is denoted as:

$$\{P_1, P_2, P_3, P_4, ...\} = \{[], [v_a], [v_a, \overline{v_b}], [v_a, \overline{v_b}, v_c], ...\} \quad 1 \leq a, b, c \leq t \tag{4}$$

where point P_1 is at origin and the rest of points are array of vectors. If point $P_k = [..., v_m]$ and $P_{k+1} = [..., v_m, \overline{v_n}]$ for $k > 0$ then the angle at point P_k is $u\theta$, where u is:

$$u \equiv s + m - n \pmod{s} \qquad 1 \leq m, n \leq t \tag{5}$$

To record and draw polylines absolutely we assign to each point $\mathbf{P}$ `accumulators` consistent of two vectors $\mathbf{X}$ and $\mathbf{Y}$ holding t integers each:

$$\mathbf{X} = \begin{bmatrix} X_1 & X_2 & X_3 & ... & X_t \end{bmatrix} \quad X_i \in \mathbb{Z} \tag{6}$$

$$\mathbf{Y} = \begin{bmatrix} Y_1 & Y_2 & Y_3 & ... & Y_t \end{bmatrix} \quad Y_i \in \mathbb{Z} \tag{7}$$

$$\mathbf{P}^A \equiv \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} = \begin{bmatrix} X_1 & X_2 & X_3 & ... & X_t \\ Y_1 & Y_2 & Y_3 & ... & Y_t \end{bmatrix} \tag{8}$$

The absolute position of point P can be calculated as follows:

$$(P_x, P_y) = \left(\sum_{i=1}^{t} X_i x_i, \sum_{i=1}^{t} Y_i y_i \right) \tag{9}$$

1.1 Symmetry $m = 9$

Figure 1 (i) show the nine different vectors v_1, v_2, ... v_9 in symmetry $m = 9$. Using the vectors equation 2 we get the nine vectors: $v_1 = \langle x_1, y_1 \rangle$, $v_2 = \langle x_2, y_2 \rangle$, $v_3 = \langle x_3, y_3 \rangle$, $v_4 = \langle x_4, y_4 \rangle$, $v_5 = \langle x_5, y_5 \rangle$, $v_6 = \langle x_5, -y_5 \rangle$, $v_7 = \langle x_4, -y_4 \rangle$, $v_8 = \langle x_3, -y_3 \rangle$ and $v_9 = \langle x_2, -y_2 \rangle$.

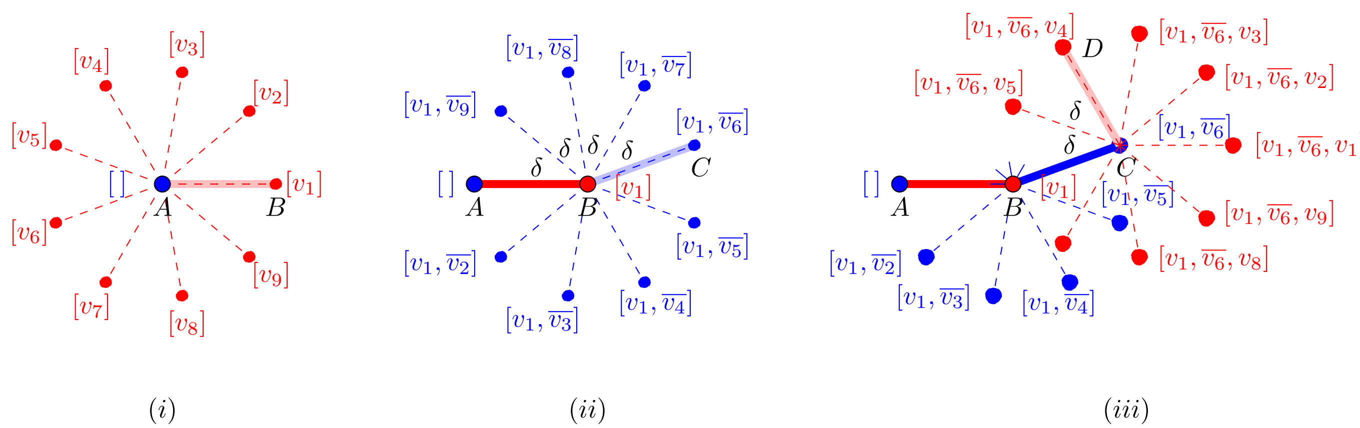

Figure 1: The symmetry 9 vectors and polylines.

Figure 1 (ii) show polyline $\overline{ABC}$ and (iii) show polyline $\overline{ABCD}$ alternating vectors of the type $[..., v_m]$ (show in red) with vectors of the type $[..., v_m, \overline{v_n}]$ (shown in blue) to guarantee the angles between edges are always multiples of δ: angle $\angle ABC = 4\delta$, angle $\angle BCD = 2\delta$. New vertices accumulate the preceding vertices by notation. The four vertices $\{A, B, C, D\}$ are denoted as $\{[\,], [v_1], [v_1, \overline{v_6}], [v_1, \overline{v_6}, v_2]\}$.

The vertice o at the origin is assigned with accumulators $\mathbf{X}, \mathbf{Y}$ initialized to zero. The next point accumulator increments by $+1$ or -1 the position $1 \leq j \leq t$ according the components x_j, y_j the vector v_m has defined. For example we have:

$$X[a,b,c,d,e] + v_m = \begin{cases} X[a+1,b,c,d,e] & \text{for} \quad v_1 \equiv \langle x_1, y_1 \rangle \\ X[a,b+1,c,d,e] & \text{for} \quad v_2 \equiv \langle x_2, y_2 \rangle, v_9 \equiv \langle x_2, -y_2 \rangle \\ \quad ... & \\ X[a,b,c,d,e+1] & \text{for} \quad v_5 \equiv \langle x_5, y_5 \rangle, v_6 \equiv \langle x_5, -y_5 \rangle \\ X[a-1,b,c,d,e] & \text{for} \quad \overline{v_1} \equiv \langle -x_1, -y_1 \rangle \\ \quad ... & \end{cases} \tag{10}$$

Lets get the accumulators of the polyline $\{[\,], [v_1], [v_1, \overline{v_6}], [v_1, \overline{v_6}, v_4]\}$ shown in figure (iii). By definition the accumulators at $o \equiv (0,0)$ which is point $A = [\,]$ are:

$$[\,]^A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

First we get accumulators of vector v_1 and point $B = [v_1]$:

$$v_1 = \langle x_1, y_1 \rangle \tag{12}$$

$$v_1^A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

$$[v_1]^A = []^A + v_1^A \tag{14}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

Then we get accumulators of vector $\overline{v_6}$:

$$v_6 \equiv \langle x_6, y_6 \rangle = \langle x_{11-6}, -y_{11-6} \rangle = \langle x_5, -y_5 \rangle \tag{16}$$

$$\overline{v_6} = -\langle x_5, -y_5 \rangle = \langle -x_5, y_5 \rangle \tag{17}$$

$$\overline{v_6}^A = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

And then we get the accumulators of point $C = [v_1, \overline{v_6}]$:

$$[v_1, \overline{v_6}]^A = v_1^A + \overline{v_6}^A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{19}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{20}$$

Finally we get the accumulator of vector v_4 and use it to get the accumulator of point $D = [v_1, \overline{v_6}, v_4]$:

$$v_4 \equiv \langle x_4, y_4 \rangle$$

$$v_4^A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{21}$$

$$[v_1, \overline{v_6}, v_4]^A = [v_1, \overline{v_6}]^A + v_4^A = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{22}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{23}$$

Then the accumulators of point $D = [v_1, \overline{v_6}, v_4]$ have different of zero these values: $X_1 = X_4 = 1$, $X_5 = -1$, $Y_1 = Y_4 = Y_5 = 1$. Using the equation 9 we calculate the absolute coordinates:

$$\begin{aligned} D_x &= x_1 + x_2 - x_5 \\ &= \cos(0) + \cos(3\delta) - \cos(4\delta) \\ &= 1 + \cos(120°) - \cos(160°) \\ D_y &= y_1 + y_2 + y_2 \\ &= \sin(0) + \sin(3\delta) + \sin(4\delta) \\ &= \sin(120°) + \sin(160°) \end{aligned}$$

2 Rhombi

Rhombi are equilateral tetragons. For every odd symmetry m we find $n = \dfrac{m-1}{2}$ different rhombi. The rhombi smallest angles are defined as $\omega_1 \equiv \dfrac{i}{2}$ while the other as $\omega_2 \equiv n + \dfrac{i-1}{2}$ where $i = 1, ..., n$. We identify the rhombi with the symmetry m and angle ω_1 only. So for symmetry 9 we have the four rhombi: $R_9\left(\frac{1}{2}\right)$, $R_9(1)$, $R_9\left(\frac{3}{2}\right)$ and $R_9(2)$.

Figure 2 show the rhombi for symmetries up to 9. Every rhombus is labeled with a consecutive lowercase letter $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$,... The first rhombi $\boldsymbol{a}$ is found again in symmetry 9 so we prevent renaming identical rhombi.

The rhombi angles θ_1 and θ_2 are expressed in function of the angles $\{\alpha, \beta, \gamma, \delta\}$ of symmetries $m = \{3, 5, 7, 9\}$ where $\theta_1 < \theta_2$ and $\theta_1 + \theta_2 = \pi$. We cannot use the rhombi isolated in our tessellations since not both ω_1 and ω_2 are integers. But we'll add and substract rhombi together to build hexagons, octagons and beyond where all polygon's angles ω_n are integers.

$R_m(\omega_1)$	ω_2	Name	θ_1	θ_2	Area
$R_3(\frac{1}{2})$	1	$\boldsymbol{a}$	$\alpha/2$	$2\alpha/2$	$\sin(\alpha) \approx 0.866$
$R_5(\frac{1}{2})$	2	$\boldsymbol{b}$	$\beta/2$	$4\beta/2$	$\sin(2\beta) \approx 0.587$
$R_5(1)$	$\frac{3}{2}$	$\boldsymbol{c}$	$2\beta/2$	$3\beta/2$	$\sin(\beta) \approx 0.951$
$R_7(\frac{1}{2})$	3	$\boldsymbol{d}$	$\gamma/2$	$6\gamma/2$	$\sin(3\gamma) \approx 0.433$
$R_7(1)$	$\frac{5}{2}$	$\boldsymbol{e}$	$2\gamma/2$	$5\gamma/2$	$\sin(\gamma) \approx 0.781$
$R_7(\frac{3}{2})$	2	$\boldsymbol{f}$	$3\gamma/2$	$4\gamma/2$	$\sin(2\gamma) \approx 0.974$
$R_9(\frac{1}{2})$	4	$\boldsymbol{g}$	$\delta/2$	$8\delta/2$	$\sin(4\delta) \approx 0.342$
$R_9(1)$	$\frac{7}{2}$	$\boldsymbol{h}$	$2\delta/2$	$7\delta/2$	$\sin(\delta) \approx 0.642$
$R_9(\frac{3}{2})$	3	$\boldsymbol{a}$	$3\delta/2$	$6\delta/2$	$\sin(3\delta) \approx 0.866$
$R_9(2)$	$\frac{5}{2}$	$\boldsymbol{i}$	$4\delta/2$	$5\delta/2$	$\sin(2\delta) \approx 0.984$

Table 2: Rhombi $R_m(\omega_1, \omega_2)$ for symmetries $m = \{3, 5, 7, 9\}$.

3 Stars

For every odd symmetry m we have several stars that are equilateral $2m$-gons with at most two different angles ω_1 and ω_2 at the vertices. We find exactly $n = \dfrac{m-1}{2}$ different stars with angles $\omega_1 \leq \omega_2$ as integers.

Figure 3 show the stars for symmetries up to 9. Every star with both ω_1 and ω_2 integers are labeled with a consecutive uppercase letter $\mathcal{A}, \mathcal{B}, \mathcal{C}$.... The stars are easily build with rhombi so the table show the stars area in function of them.

$S_m(\omega_1)$	ω_2	Name	Area	Polygon
$S_3(\frac{1}{2})$	2	-	$6\boldsymbol{a}$	$\|6/2\|$ (12-gon)
$S_3(1)$	1	$\mathcal{A}$	$3\boldsymbol{a}$	Regular hexagon
$S_5(\frac{1}{2})$	4	-	$5\boldsymbol{b}$	$\|10/4\|$ (20-gon)
$S_5(1)$	3	$\mathcal{B}$	$5\boldsymbol{c}$	$\|(5/2)_\alpha\|$ decagram
$S_5(2)$	2	$\mathcal{C}$	$5(\boldsymbol{c}+\boldsymbol{b})$	Regular decagon
$S_7(\frac{1}{2})$	6	-	$7\boldsymbol{d}$	$\|14/6\|$ (28-gon)
$S_7(1)$	5	$\mathcal{D}$	$7\boldsymbol{e}$	$\|(7/4)_\alpha\|$ 14-gram
$S_7(2)$	4	$\mathcal{E}$	$7(\boldsymbol{e}+\boldsymbol{f})$	$\|(7/2)_\alpha\|$ 14-gram
$S_7(3)$	3	$\mathcal{F}$	$7(\boldsymbol{e}+\boldsymbol{f}+\boldsymbol{d})$	Regular 14-gon
$S_9(\frac{1}{2})$	8	-	$9\boldsymbol{g}$	$\|18/8\|$ (36-gon)
$S_9(1)$	7	$\mathcal{G}$	$9\boldsymbol{h}$	$\|(9/6)_\alpha\|$ 18-gram
$S_9(2)$	6	$\mathcal{H}$	$9(\boldsymbol{h}+\boldsymbol{i})$	$\|(9/4)_\alpha\|$ 18-gram
$S_9(3)$	5	$\mathcal{I}$	$9(\boldsymbol{h}+\boldsymbol{i}+\boldsymbol{a})$	$\|(9/2)_\alpha\|$ 18-gram
$S_9(4)$	4	$\mathcal{J}$	$9(\boldsymbol{h}+\boldsymbol{i}+\boldsymbol{a}+\boldsymbol{g})$	Regular 18-gon

Table 3: Stars $S_m(\omega_1, \omega_2)$ for symmetries $m = \{3, 5, 7, 9\}$.

Figure 2 show nine copies of symmetry-9 rhombi $\{\boldsymbol{h}, \boldsymbol{i}, \boldsymbol{a}, \boldsymbol{g}\}$ forming the four stars $S_9(1)$, $S_9(2)$, $S_9(3)$ and $S_9(4)$ named respectivelly $\mathcal{G}$, $\mathcal{H}$, $\mathcal{I}$ and $\mathcal{J}$.

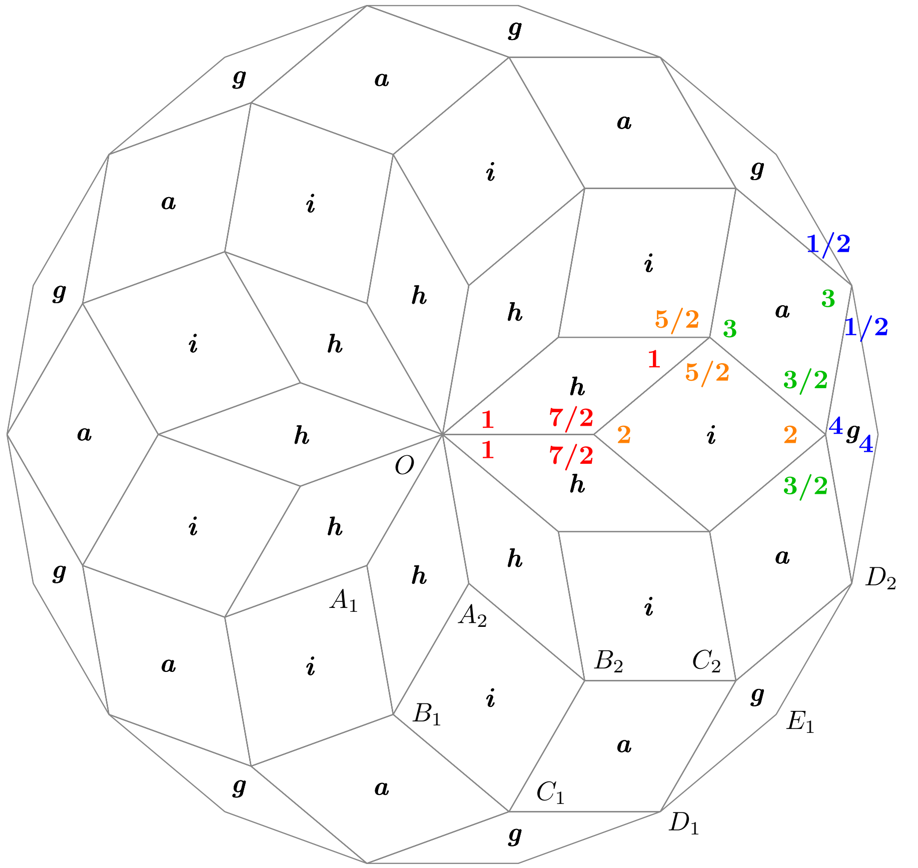

Figure 2: The symmetry 9 four rhombi $\{\boldsymbol{h}, \boldsymbol{i}, \boldsymbol{a}, \boldsymbol{g}\}$ produce the four stars $\{\mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}\}$ with areas $9\boldsymbol{h}$, $9(\boldsymbol{h}+\boldsymbol{i})$, $9(\boldsymbol{h}+\boldsymbol{i}+\boldsymbol{a})$ and $9(\boldsymbol{h}+\boldsymbol{i}+\boldsymbol{a}+\boldsymbol{g})$ respectivelly.

4 Hexagons

For any odd symmetry m, we can obtain equilateral hexagons by connecting six equal edges at integral values of ω or by adding and substracting rhombi or by dissecting the areas left after stars intersections. In any case we find the hexagons has at most three different internal angles in this order $(\omega_1, \omega_2, \omega_3, \omega_1, \omega_2, \omega_3)$ where $\omega_1 + \omega_2 + \omega_3 = m$.

4.1 Hexagons angles

Figure 4 show the hexagons defined as $H_m(\omega_1, \omega_2)$ for symmetries $m = \{3, 5, 7, 9\}$ where $\omega_1 \leq \omega_2 \leq \omega_3$. We labeled the non self-intersecting hexagons with uppercase letters $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, ... We prevent to name differently any hexagon which is equivalent to a previous one having the same angles as what happens with hexagon $\boldsymbol{A}$ of symmetries 3 and 9.

$H_m(\omega_1, \omega_2)$	Name	$(\omega_1, \omega_2, \omega_3)$	Polygon
$H_3(1,1)$	$\boldsymbol{A}$	$(\mathbf{1,\ 1,\ 1})$	Regular hexagon
$H_5(1,1)$	$\boldsymbol{B}$	$(\mathbf{1,\ 1,\ 3})$	Sormeh Dan Girih tile
$H_5(1,2)$	$\boldsymbol{C}$	$(\mathbf{1,\ 2,\ 2})$	Shesh Band Girih tite
$H_7(1,1)$	-	$(\mathbf{1,\ 1,\ 5})$	self-intersecting
$H_7(1,2)$	$\boldsymbol{D}$	$(\mathbf{1,\ 2,\ 4})$	
$H_7(1,3)$	$\boldsymbol{E}$	$(\mathbf{1,\ 3,\ 3})$	
$H_7(2,2)$	$\boldsymbol{F}$	$(\mathbf{2,\ 2,\ 3})$	
$H_9(1,1)$	-	$(\mathbf{1,\ 1,\ 7})$	self-intersecting
$H_9(1,2)$	$\boldsymbol{G}$	$(\mathbf{1,\ 2,\ 6})$	
$H_9(1,3)$	$\boldsymbol{H}$	$(\mathbf{1,\ 3,\ 5})$	
$H_9(1,4)$	$\boldsymbol{I}$	$(\mathbf{1,\ 4,\ 4})$	
$H_9(2,2)$	$\boldsymbol{J}$	$(\mathbf{2,\ 2,\ 5})$	
$H_9(2,3)$	$\boldsymbol{K}$	$(\mathbf{2,\ 3,\ 4})$	
$H_9(3,3)$	$\boldsymbol{A}$	$(\mathbf{3,\ 3,\ 3})$	equivalent to $H_3(1,1)$

Table 4: Hexagons $H_m(\omega_1, \omega_2)$ for symmetries $m = \{3, 5, 7, 9\}$.

4.2 Hexagons areas

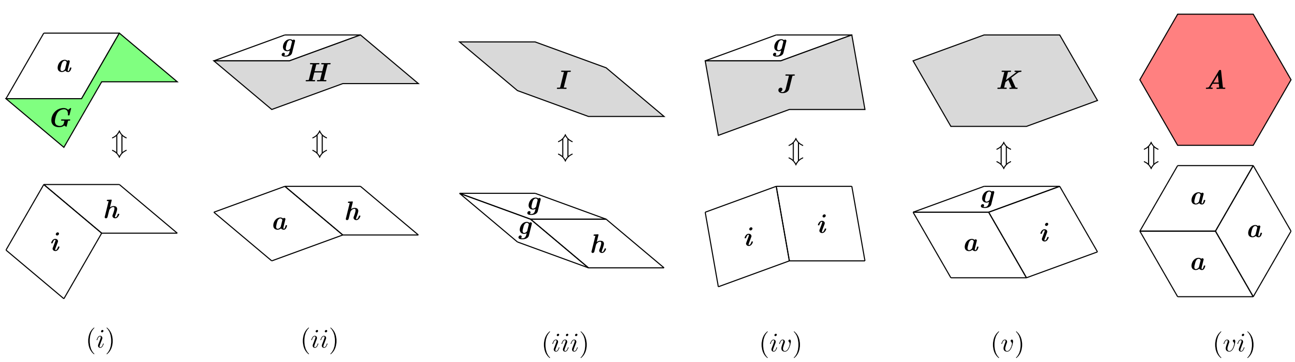

Figure 3: Symmetry 9 hexagons formed adding and substracting rhombi.

Figure 3 show how to calculate the area of the symmetry 9 hexagons in function of the symmetry 9 rhombi. From (i) to (vi) we equate the area of sum of the polygons in the top with the area of the sum of the polygons of the bottom. We have for the six cases these six equations:

$$\boldsymbol{a} + \boldsymbol{G} = \boldsymbol{i} + \boldsymbol{h} \tag{24}$$

$$\boldsymbol{g} + \boldsymbol{H} = \boldsymbol{a} + \boldsymbol{h} \tag{25}$$

$$\boldsymbol{I} = 2\boldsymbol{g} + \boldsymbol{h} \tag{26}$$

$$\boldsymbol{g} + \boldsymbol{J} = 2\boldsymbol{i} \tag{27}$$

$$\boldsymbol{K} = \boldsymbol{g} + \boldsymbol{a} + \boldsymbol{i} \tag{28}$$

$$\boldsymbol{A} = 3\boldsymbol{a} \tag{29}$$

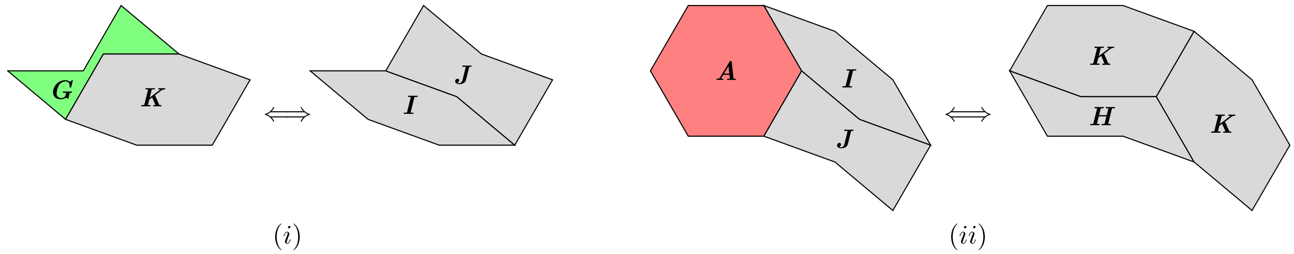

Figure 4: Hexagons $\{\boldsymbol{G}, \boldsymbol{A}\}$ formed adding and substracting hexagons $\{\boldsymbol{H}, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}\}$.

Figure 4 show how to express the area of the six hexagons in function of only four. For (i) and (ii) we equate the area of the sum of hexagons of the left with the area of the hexagons of the right. We have for the two cases these two equations:

$$\boldsymbol{G} + \boldsymbol{K} = \boldsymbol{I} + \boldsymbol{J} \tag{30}$$

$$\boldsymbol{A} + \boldsymbol{I} + \boldsymbol{J} = \boldsymbol{H} + 2\boldsymbol{K} \tag{31}$$

Using the last eight equations we form the table 5 which show the areas of the six hexagons in function of four rhombi $\boldsymbol{g,h,a,i}$ and in function of only four hexagons $\boldsymbol{H,I,J,K}$.

Hexagon	$\boldsymbol{g,h,a,i}$ area	$\boldsymbol{H,I,J,K}$ area
$\boldsymbol{H}$	$\boldsymbol{a}+\boldsymbol{h}-\boldsymbol{g}$	$\boldsymbol{H}$
$\boldsymbol{I}$	$2\boldsymbol{g}+\boldsymbol{h}$	$\boldsymbol{I}$
$\boldsymbol{J}$	$2\boldsymbol{i}-\boldsymbol{g}$	$\boldsymbol{J}$
$\boldsymbol{K}$	$\boldsymbol{g}+\boldsymbol{a}+\boldsymbol{i}$	$\boldsymbol{K}$
$\boldsymbol{A}$	$3\boldsymbol{a}$	$2\boldsymbol{K}+\boldsymbol{H}-\boldsymbol{I}-\boldsymbol{J}$
$\boldsymbol{G}$	$\boldsymbol{i}+\boldsymbol{h}-\boldsymbol{a}$	$\boldsymbol{I}+\boldsymbol{J}-\boldsymbol{K}$

Table 5: Symmetry 9 hexagon areas.

4.3 Hexagons from stars

Figure 5 show the disposition of the symmetry 9 four stars. We label the 18 vertices of stars $\{\mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}\}$ as $\{G_0, G_1, ..., G_{17}\}$, $\{H_0, H_1, ..., H_{17}\}$, $\{I_0, I_1, ..., I_{17}\}$ and $\{I_0, G_1, ..., I_{17}\}$ respectivelly. For simplicity, only some vertices are show. First we make coincident at vertice O all the vertices G_0, H_0, I_0, J_0. With the center at O we rotate all stars to make coincidents G_{17}, H_{17}, I_{17} and J_{17}. The rotations also joined another different vertices.

First we add three new edges (in red) joining the stars $\mathcal{J}$ and $\mathcal{I}$ vertices: $\overline{J_3I_2}$, $\overline{J_5I_4}$ and $\overline{J_7I_6}$ dissecting the red region into four hexagons, two of them essentially different. The three consective angles of the two hexagons are shown: **I (1,4,4)** and **K (3,4,2)**.

Then we add three new edges (in orange) joining the stars $\mathcal{I}$ and $\mathcal{H}$ vertices: $\overline{I_3H_2}$, $\overline{I_5H_4}$ and $\overline{I_7H_6}$ dissecting the orange region into four hexagons, two of them new. The three consective angles of the the two hexagons are show: **H (1,5,3)** and **A (3,3,3)**.

Finally we add three more edges (in green) joining the stars $\mathcal{H}$ and $\mathcal{G}$ vertices: $\overline{H_3G_2}$, $\overline{H_5G_4}$ and $\overline{H_7G_6}$ dissecting the green region into four hexagons, two of them new. The three consective angles of the the two hexagons are show: **G (1,6,2)** and **J (2,5,2)**.

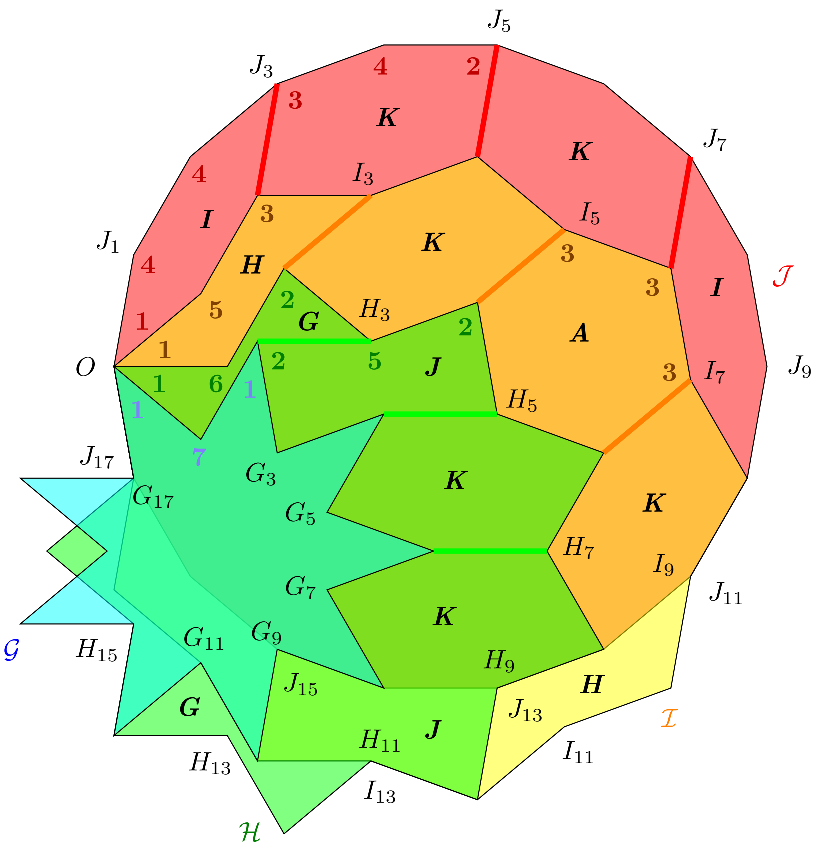

Figure 5: Symmetry 9 stars $\{\mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}\}$ dissected to get the six hexagons $\{\boldsymbol{G}, \boldsymbol{H}, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}, \boldsymbol{A}\}$.

5 Octagons

5.1 Octagons by stars

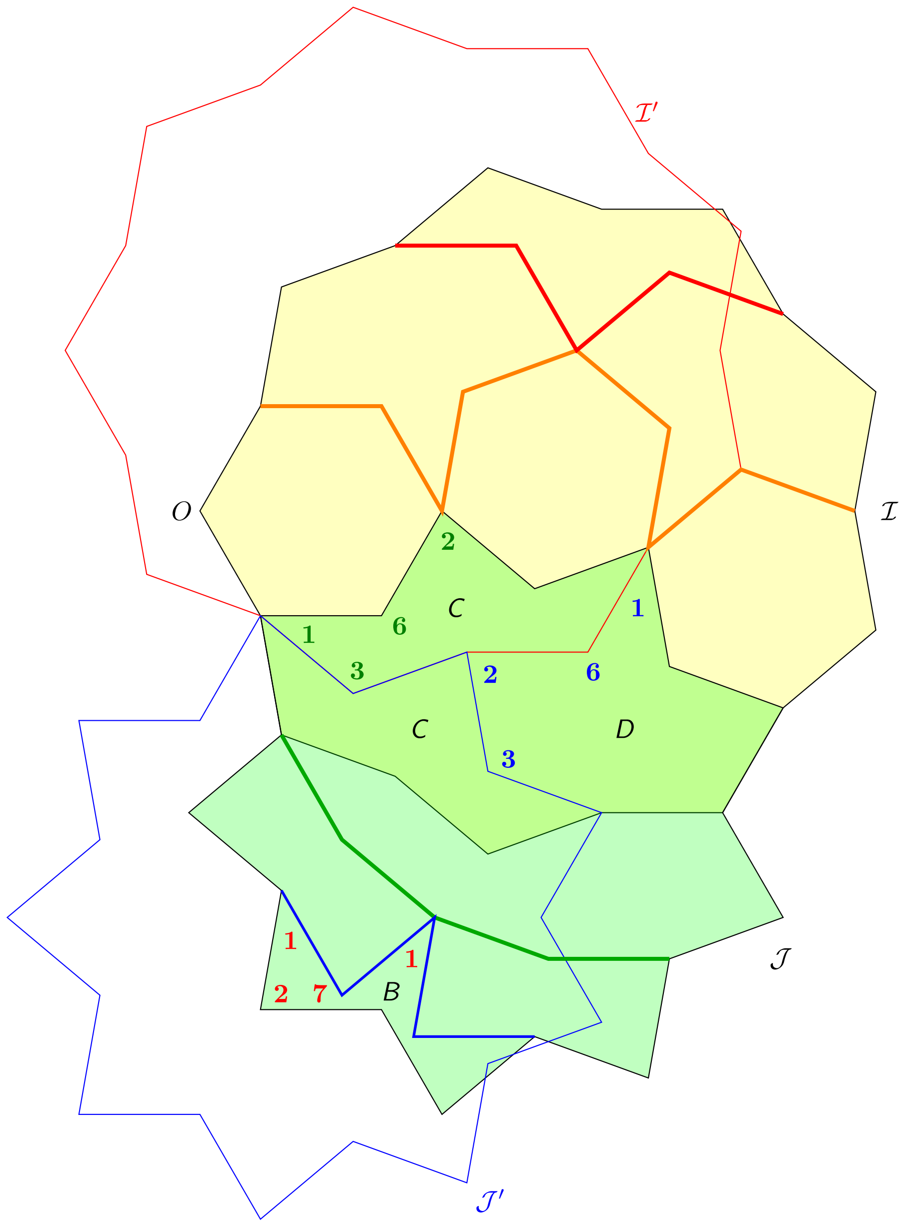

Figure 6: Octagons after intersection of stars $\mathcal{I}$ and $\mathcal{J}$.

6 Vectors

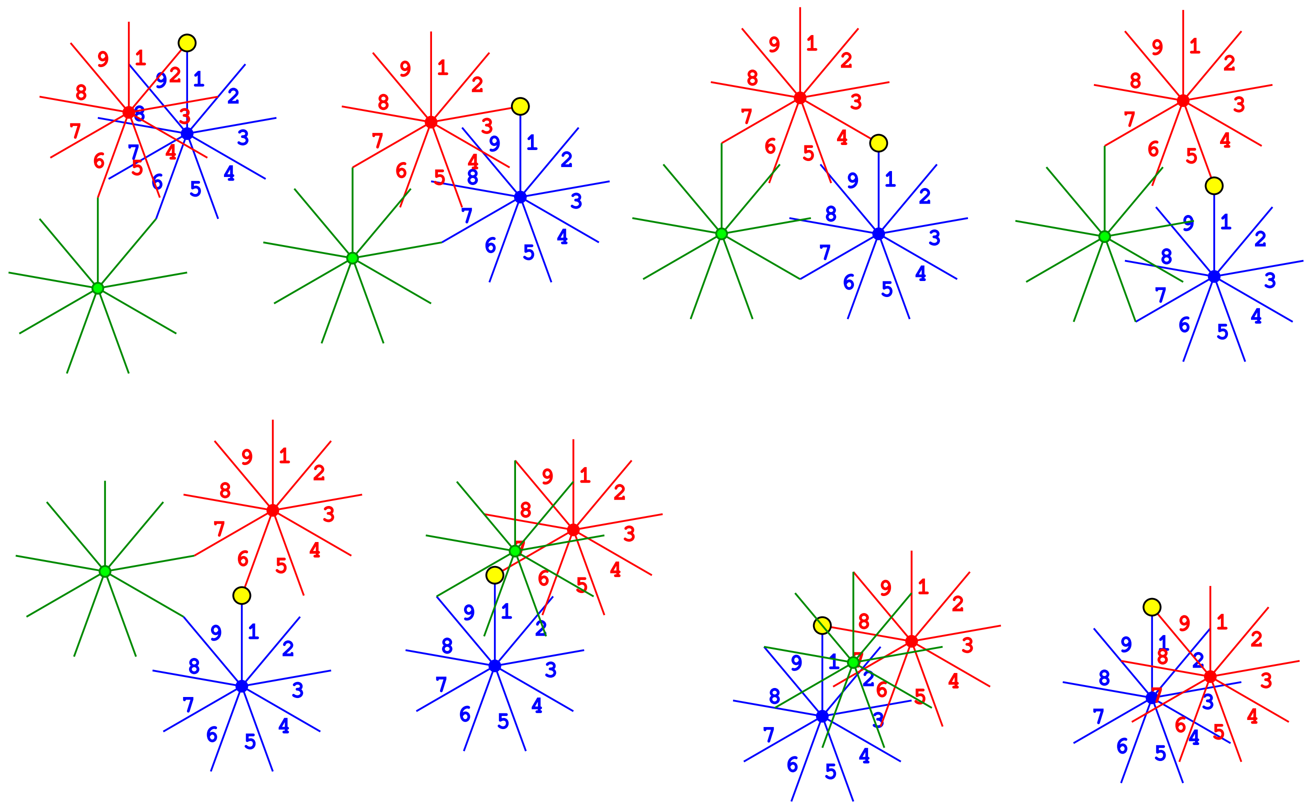

Figure 7: Vectors of symmetry 9.