Symmetry 9

https://github.com/heptagons/lenses

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Abstract

Symmetry 9

1 Rhombi

Rhombus	Name	θ_1	θ_2	Area
$R_3(\frac{1}{2}, 1)$	$\boldsymbol{a}$	$\alpha/2$	$2\alpha/2$	$\sin(\alpha) \approx 0.866$
$R_5(\frac{1}{2}, 2)$	$\boldsymbol{b}$	$\beta/2$	$4\beta/2$	$\sin(2\beta) \approx 0.587$
$R_5(1, \frac{3}{2})$	$\boldsymbol{c}$	$2\beta/2$	$3\beta/2$	$\sin(\beta) \approx 0.951$
$R_7(\frac{1}{2}, 3)$	$\boldsymbol{d}$	$\gamma/2$	$6\gamma/2$	$\sin(3\gamma) \approx 0.433$
$R_7(1, \frac{5}{2})$	$\boldsymbol{e}$	$2\gamma/2$	$5\gamma/2$	$\sin(\gamma) \approx 0.781$
$R_7(\frac{3}{2}, 2)$	$\boldsymbol{f}$	$3\gamma/2$	$4\gamma/2$	$\sin(2\gamma) \approx 0.974$
$R_9(\frac{1}{2}, 4)$	$\boldsymbol{g}$	$\delta/2$	$8\delta/2$	$\sin(4\delta) \approx 0.342$
$R_9(1, \frac{7}{2})$	$\boldsymbol{h}$	$2\delta/2$	$7\delta/2$	$\sin(\delta) \approx 0.642$
$R_9(\frac{3}{2}, 3)$	$\boldsymbol{a}$	$3\delta/2$	$6\delta/2$	$\sin(3\delta) \approx 0.866$
$R_9(2, \frac{5}{2})$	$\boldsymbol{i}$	$4\delta/2$	$5\delta/2$	$\sin(2\delta) \approx 0.984$

Table 1: Rhombi $R_m(\omega_1, \omega_2)$ for symmetries $m = \{3, 5, 7, 9\}$ internal angles $\theta_1 < \theta_2$ ($\theta_1 + \theta_2 = \pi$) and areas. $\alpha = 2\pi/3$, $\beta = 2\pi/5$, $\gamma = 2\pi/7$ and $\delta = 2\pi/9$ ($3\delta = \alpha$).

Star	Name	Area	Polygon
$S_3(\frac{1}{2},1)$	-	$6\boldsymbol{a}$	$\|6/2\|$ hexagram
$S_3(1,1)$	$\mathcal{A}$	$3\boldsymbol{a}$	Regular hexagon
$S_5(\frac{1}{2},4)$	-	$5\boldsymbol{b}$	$\|10/4\|$ decagram
$S_5(1,3)$	$\mathcal{B}$	$5\boldsymbol{c}$	$\|(5/2)_\alpha\|$ decagram
$S_5(2,2)$	$\mathcal{C}$	$5(\boldsymbol{c}+\boldsymbol{b})$	Regular decagon
$S_7(\frac{1}{2},6)$	-	$7\boldsymbol{d}$	$\|14/6\|$ 14-gram
$S_7(1,5)$	$\mathcal{D}$	$7\boldsymbol{e}$	$\|(7/4)_\alpha\|$ 14-gram
$S_7(2,4)$	$\mathcal{E}$	$7(\boldsymbol{e}+\boldsymbol{f})$	$\|(7/2)_\alpha\|$ 14-gram
$S_7(3,3)$	$\mathcal{F}$	$7(\boldsymbol{e}+\boldsymbol{f}+\boldsymbol{d})$	Regular 14-gon
$S_9(\frac{1}{2},7)$	-	$9\boldsymbol{g}$	$\|18/8\|$ 18-gram
$S_9(1,6)$	$\mathcal{G}$	$9\boldsymbol{h}$	$\|(9/6)_\alpha\|$ 18-gram
$S_9(2,5)$	$\mathcal{H}$	$9(\boldsymbol{h}+\boldsymbol{i})$	$\|(9/4)_\alpha\|$ 18-gram
$S_9(3,4)$	$\mathcal{I}$	$9(\boldsymbol{h}+\boldsymbol{i}+\boldsymbol{a})$	$\|(9/2)_\alpha\|$ 18-gram
$S_9(4,4)$	$\mathcal{J}$	$9(\boldsymbol{h}+\boldsymbol{i}+\boldsymbol{a}+\boldsymbol{g})$	Regular 18-gon

Table 2: Stars $S_m(\omega_1,\omega_2)$ for symmetries $m=\{3,5,7,9\}$. We use the names $\{\mathcal{A},\mathcal{B},...\mathcal{J}\}$ when both ω_1 and ω_2 are integers.

1.1 Stars from rhombi

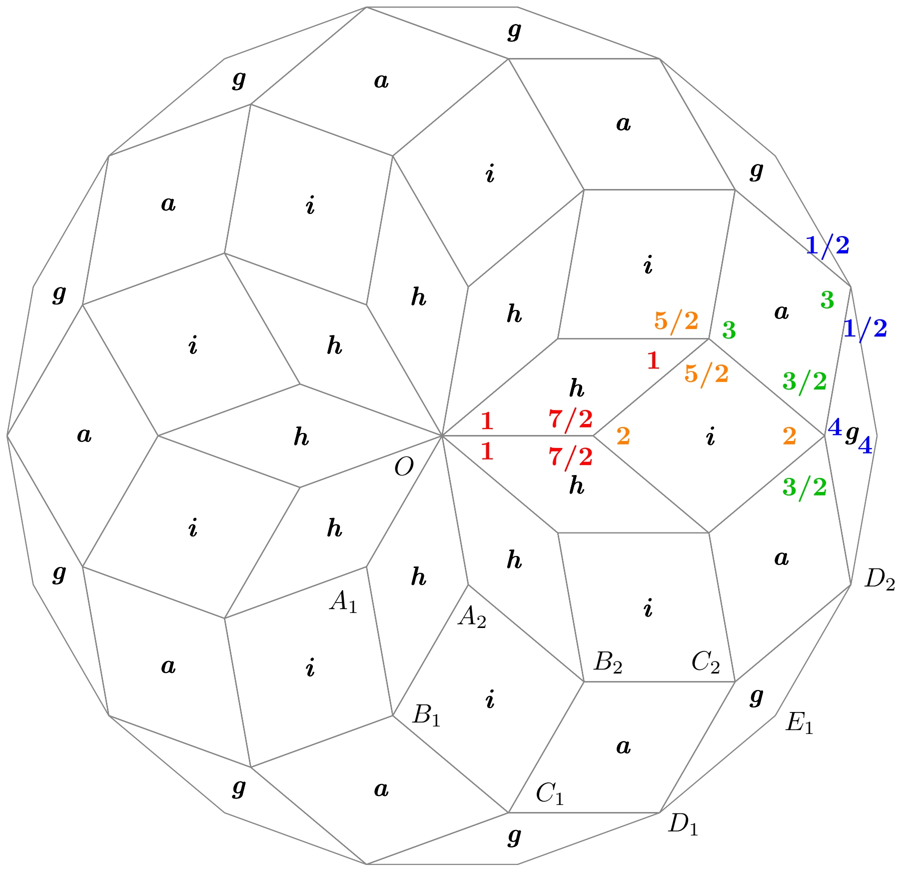

Figure 1: The symmetry 9 four rhombi $\{\boldsymbol{h}, \boldsymbol{i}, \boldsymbol{a}, \boldsymbol{g}\}$ produce the four stars $\{\mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}\}$ with areas $9\boldsymbol{h}$, $9(\boldsymbol{h}+\boldsymbol{i})$, $9(\boldsymbol{h}+\boldsymbol{i}+\boldsymbol{a})$ and $9(\boldsymbol{h}+\boldsymbol{i}+\boldsymbol{a}+\boldsymbol{g})$ respectivelly.

Figure 1 show nine copies of symmetry-9 rhombi $\{\boldsymbol{h}, \boldsymbol{i}, \boldsymbol{a}, \boldsymbol{g}\}$ to form four stars.

2 Hexagons

2.1 Hexagons angles

Hexagon	Name	$(\omega_1, \omega_2, \omega_3)$	Polygon
$H_3(1,1)$	$\boldsymbol{A}$	$(\mathbf{1,\ 1,\ 1})$	Regular hexagon
$H_5(1,1)$	$\boldsymbol{B}$	$(\mathbf{1,\ 1,\ 3})$	Sormeh Dan Girih tile
$H_5(1,2)$	$\boldsymbol{C}$	$(\mathbf{1,\ 2,\ 2})$	Shesh Band Girih tite
$H_7(1,1)$	-	$(\mathbf{1,\ 1,\ 5})$	self-intersecting
$H_7(1,2)$	$\boldsymbol{D}$	$(\mathbf{1,\ 2,\ 4})$	
$H_7(1,3)$	$\boldsymbol{E}$	$(\mathbf{1,\ 3,\ 3})$	
$H_7(2,2)$	$\boldsymbol{F}$	$(\mathbf{2,\ 2,\ 3})$	
$H_9(1,1)$	-	$(\mathbf{1,\ 1,\ 7})$	self-intersecting
$H_9(1,2)$	$\boldsymbol{G}$	$(\mathbf{1,\ 2,\ 6})$	
$H_9(1,3)$	$\boldsymbol{H}$	$(\mathbf{1,\ 3,\ 5})$	
$H_9(1,4)$	$\boldsymbol{I}$	$(\mathbf{1,\ 4,\ 4})$	
$H_9(2,2)$	$\boldsymbol{J}$	$(\mathbf{2,\ 2,\ 5})$	
$H_9(2,3)$	$\boldsymbol{K}$	$(\mathbf{2,\ 3,\ 4})$	
$H_9(3,3)$	$\boldsymbol{A}$	$(\mathbf{3,\ 3,\ 3})$	equivalent to $H_3(1,1)$

Table 3: Hexagons $H_m(\omega_1, \omega_2)$ for symmetries $m = \{3, 5, 7, 9\}$.

Figure 3 show the hexagons defined as $H_m(\omega_1, \omega_2)$ for symmetries $\{3, 5, 7, 9\}$. Always $\omega_1 \leq \omega_2 \leq \omega_3$ and $\omega_1 + \omega_2 + \omega_3 = m$. The six consecutive angles of the hexagons are $(\omega_1, \omega_2, \omega_3, \omega_1, \omega_2, \omega_3)$.

2.2 Hexagons areas

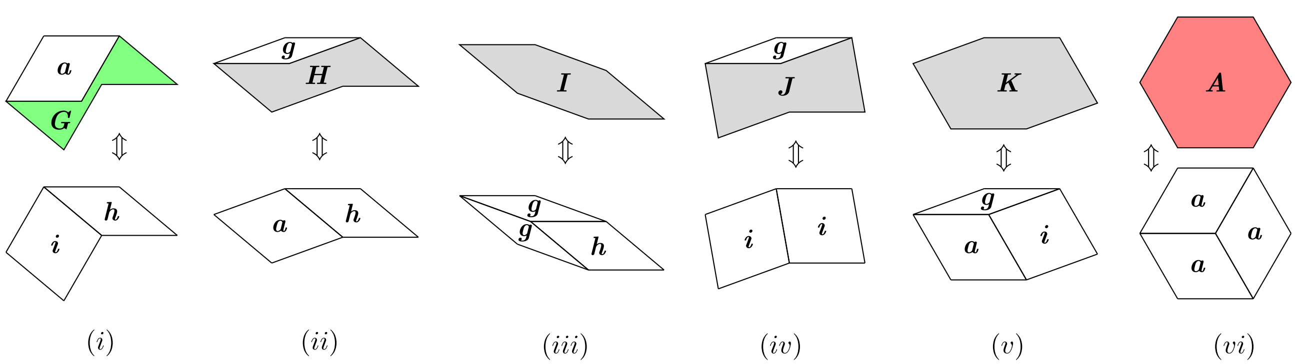

Figure 2: Hexagons formed adding and substracting rhombi.

Figure 2 show how to calculate the area of the symmetry 9 hexagons in function of the symmetry 9 rhombi. From (i) to (vi) we equate the area of sum of the polygons in the top with the area of the sum of the polygons of the bottom:

$$\boldsymbol{a} + \boldsymbol{G} = \boldsymbol{i} + \boldsymbol{h} \tag{1}$$
$$\boldsymbol{g} + \boldsymbol{H} = \boldsymbol{a} + \boldsymbol{h} \tag{2}$$
$$\boldsymbol{I} = 2\boldsymbol{g} + \boldsymbol{h} \tag{3}$$
$$\boldsymbol{g} + \boldsymbol{J} = 2\boldsymbol{i} \tag{4}$$
$$\boldsymbol{K} = \boldsymbol{g} + \boldsymbol{a} + \boldsymbol{i} \tag{5}$$
$$\boldsymbol{A} = 3\boldsymbol{a} \tag{6}$$

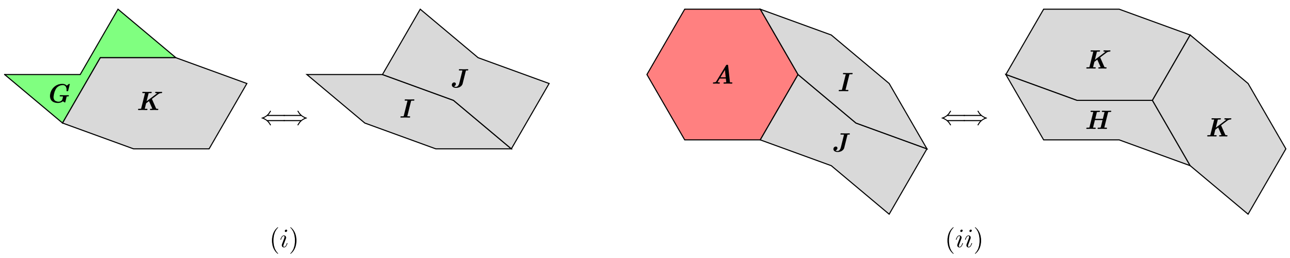

Figure 3: Hexagons $\{\boldsymbol{G}, \boldsymbol{A}\}$ formed adding and substracting hexagons $\{\boldsymbol{H}, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}\}$.

Figure 3 show how to express the area of the six hexagons in function of only four. For (i) and (ii) we equate the area of the sum of hexagons of the left with the area of the hexagons of the right:

$$\boldsymbol{G} + \boldsymbol{K} = \boldsymbol{I} + \boldsymbol{J} \tag{7}$$
$$\boldsymbol{A} + \boldsymbol{I} + \boldsymbol{J} = \boldsymbol{H} + 2\boldsymbol{K} \tag{8}$$

Using the last equations we form the table 4.

Hexagon	$\boldsymbol{g}$,$\boldsymbol{h}$,$\boldsymbol{a}$,$\boldsymbol{i}$ area	$\boldsymbol{H}$,$\boldsymbol{I}$,$\boldsymbol{J}$,$\boldsymbol{K}$ area
$\boldsymbol{H}$	$\boldsymbol{a} + \boldsymbol{h} - \boldsymbol{g}$	$\boldsymbol{H}$
$\boldsymbol{I}$	$2\boldsymbol{g} + \boldsymbol{h}$	$\boldsymbol{I}$
$\boldsymbol{J}$	$2\boldsymbol{i} - \boldsymbol{g}$	$\boldsymbol{J}$
$\boldsymbol{K}$	$\boldsymbol{g} + \boldsymbol{a} + \boldsymbol{i}$	$\boldsymbol{K}$
$\boldsymbol{A}$	$3\boldsymbol{a}$	$2\boldsymbol{K} + \boldsymbol{H} - \boldsymbol{I} - \boldsymbol{J}$
$\boldsymbol{G}$	$\boldsymbol{i} + \boldsymbol{h} - \boldsymbol{a}$	$\boldsymbol{I} + \boldsymbol{J} - \boldsymbol{K}$

Table 4: Areas of the six hexagons in function of four rhombi $\boldsymbol{g}$,$\boldsymbol{h}$,$\boldsymbol{a}$,$\boldsymbol{i}$ and four hexagons $\boldsymbol{H}$,$\boldsymbol{I}$,$\boldsymbol{J}$,$\boldsymbol{K}$.

2.3 Hexagons from stars

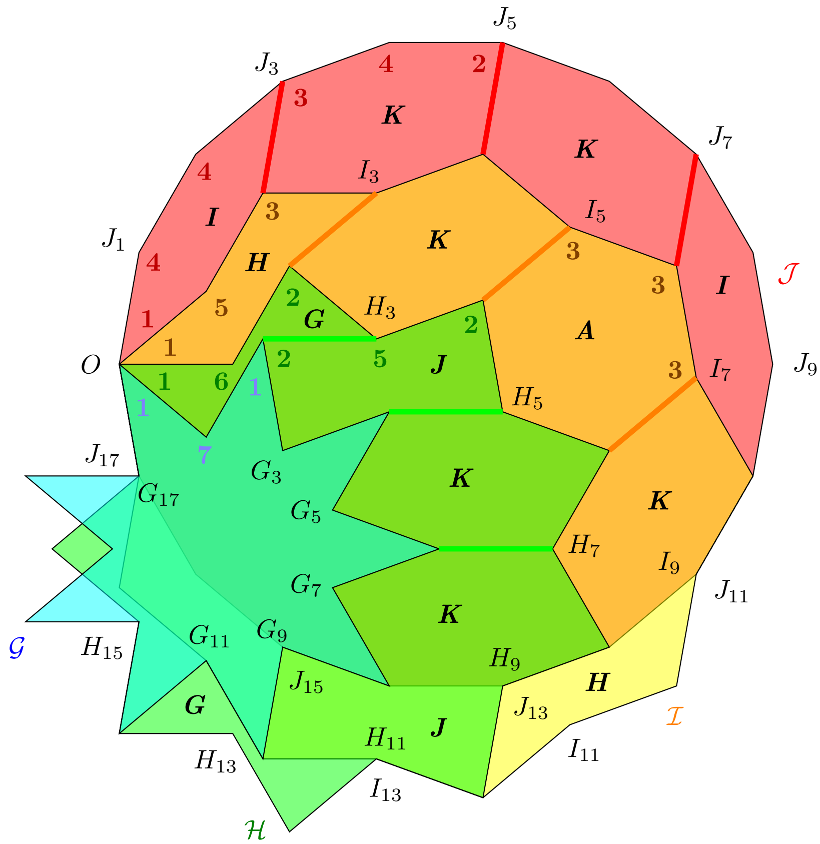

Figure 4: Symmetry 9 stars $\{\mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}\}$ dissected to get the six hexagons $\{\boldsymbol{G}, \boldsymbol{H}, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}, \boldsymbol{A}\}$.

Figure 4 show the disposition of the symmetry 9 four stars. We denote the 18 vertices of stars $\{\mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}\}$ as $\{G_0, G_1, ..., G_{17}\}$, $\{H_0, H_1, ..., H_{17}\}$, $\{I_0, I_1, ..., I_{17}\}$ and $\{I_0, G_1, ..., I_{17}\}$ respectivelly. For simplification only some vertices are labeled in the figure. First we make coincident at vertice O all the vertices G_0, H_0, I_0, J_0. With the center at O we rotate all stars to make coincidents G_{17}, H_{17}, I_{17} and J_{17}. The rotations also joined another different vertices.

First we add three new edges (in red) joining the stars $\mathcal{J}$ and $\mathcal{I}$ vertices: $\overline{J_3I_2}$, $\overline{J_5I_4}$ and $\overline{J_7I_6}$ dissecting the red region into four hexagons, two of them essentially different. The three consective angles of the two hexagons are shown: **I (1,4,4)** and **K (3,4,2)**.

Then we add three new edges (in orange) joining the stars $\mathcal{I}$ and $\mathcal{H}$ vertices: $\overline{I_3H_2}$, $\overline{I_5H_4}$ and $\overline{I_7H_6}$ dissecting the orange region into four hexagons, two of them new. The three consective angles of the the two hexagons are show: **H (1,5,3)** and **A (3,3,3)**.

Finally we add three more edges (in green) joining the stars $\mathcal{H}$ and $\mathcal{G}$ vertices: $\overline{H_3G_2}$, $\overline{H_5G_4}$ and $\overline{H_7G_6}$ dissecting the green region into four hexagons, two of them new. The three consective angles of the the two hexagons are show: **G (1,6,2)** and **J (2,5,2)**.