Lenses

https://github.com/heptagons/lenses

2024/1/8

Abstract

Lenses are equilateral hexagons resembling concave and convex optical lenses. Lenses consecutive six internal angles are $(\theta_1, \theta_2, \theta_3, \theta_1, \theta_2, \theta_3)$ where $\theta_1 = x\theta_0$, $\theta_2 = y\theta_0$, and $\theta_3 = z\theta_0$ where $\theta_0 = 2\pi/m$ is the base angle of symmetry $m = x + y + z$. Lenses can be formed adding and substracting rhombi or by intersecting equi-stars with others.

1 Equi-stars

Equi-stars are equilateral polygons with an even number of sides and vertices of at most two different angles. These stars can be defined with only two numbers: A symmetry integer m and a minimum angle integer a so the star is defined as $S(m, a)$. Here we are interested only in symmetries of the form $m = 2n+1$ for $n = 1, 2, 3....$ Every symmetry $m = 2n+1$ has exactly n different stars: $S(m, n), S(m, n-1), ..., S(m, 1)$. Stars of the form $S(m, n)$ correspond to the regular polygons of $2m$ sides.

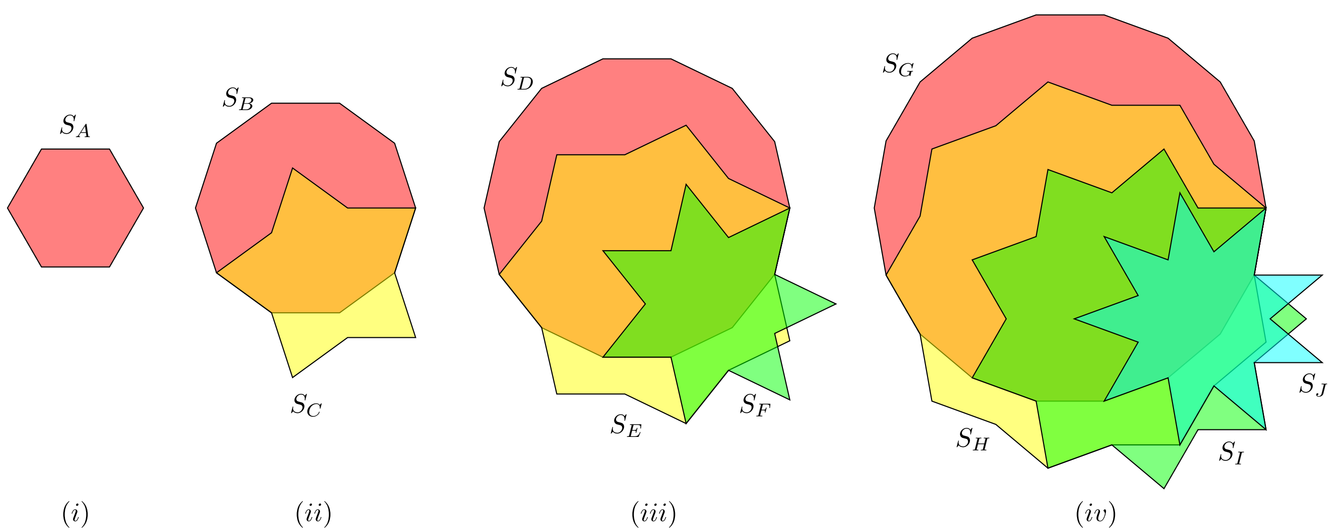


Figure 1: Equi-stars of symmetries $m = \{3, 5, 7, 9\}$.

Figure 1 show the stars for the smaller symmetries in translucent colors and intersecting with others of the same symmetry. At (i) we have for the symmetry 3 the only star in red $S_A \equiv S(3, 1)$ which is the regular hexagon. At (ii) we have for the symmetry 5 the regular decagon in red $S_B \equiv S(5, 2)$ and the star $S_C \equiv S(5, 1)$ in yellow; the region in orange is the intersection of the two stars. At (iii) for symmetry 7 we have three stars: The regular 14-gon $S_D \equiv S(7, 3)$ in red, the star $S_E \equiv S(7, 2)$ in yellow and the star $S_F \equiv S(7, 1)$ in green. At (iv) we have for the symmetry 9 four stars: The regular 18-gon $S_G \equiv S(9, 4)$ in red, the star $S_H \equiv S(9, 3)$ in yellow, the star $S_I \equiv S(9, 2)$ in green and the star $S_J \equiv S(9, 1)$ in blue.

1.1 Lenses

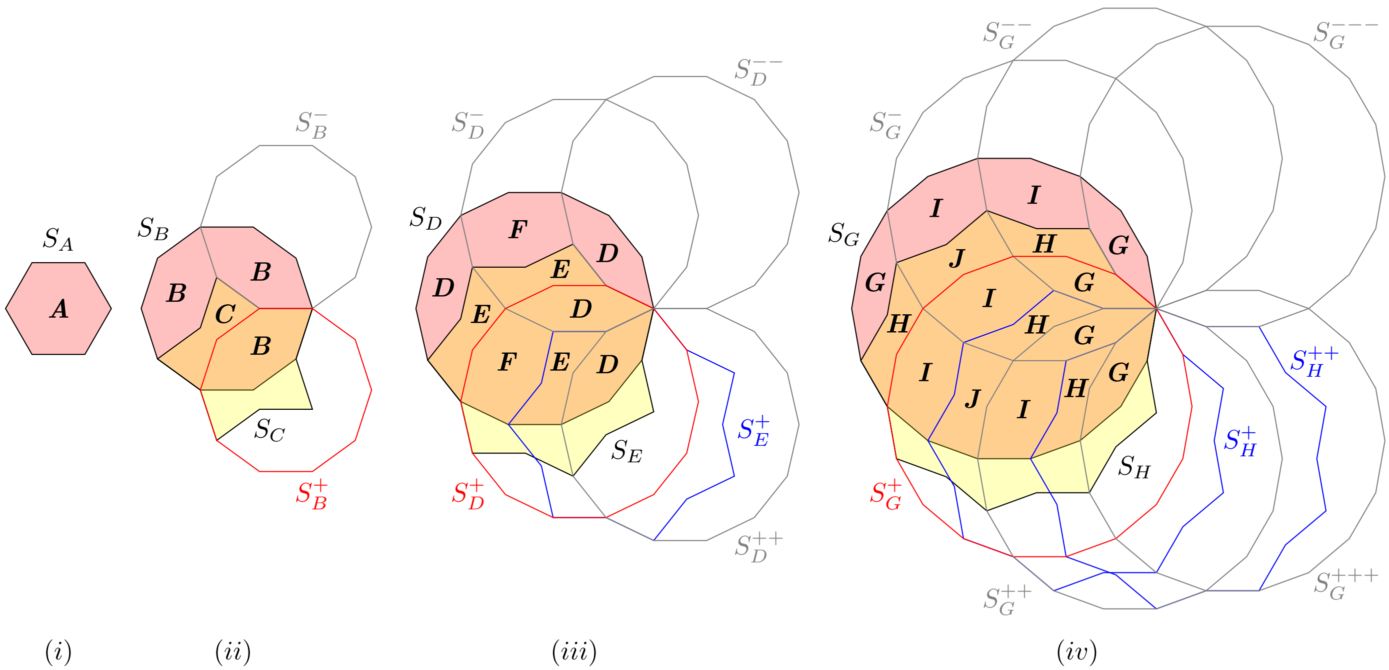


Figure 2: Lenses build from the intersection of two types of stars: $S(m,n)$ and $S(m,n-1)$ for symmetries $m = \{3,5,7,9\}$.

Figure 2 show how to dispose several stars to build all the lenses for symmetries $m = \{3,5,7,9\}$ but the process works for greater symmetries. For every symmetry $m = 2n+1$ we obtain n different lenses, using $2n-1$ stars $S(m,n)$ and using $n-1$ stars $S(m,n-1)$. Both stars can be dissected with the lenses.

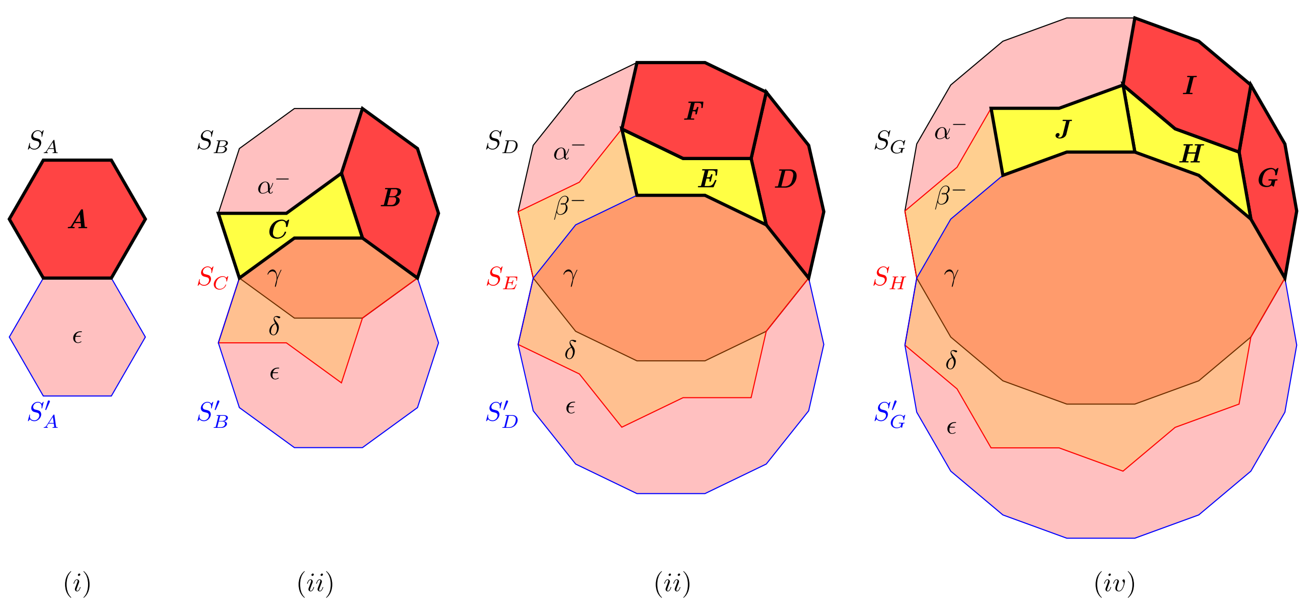


Figure 3: First lenses from the intersection of red stars $S(m,n)$ and yellow stars $S(m,n-1)$ for ymmetries $m = \{3,5,7,9\}$. Here the lenses appear once and after dissecting the areas remaining red and yellow.

Figure **??** show the first lenses produced by the intersections of three stars: One $S(m,n)$ in pink at the top, one $S(m,n-1)$ in orange at the center and another $S(m,n)$ in pink at the bottom. The intersections produce five regions: α congruent with ϵ, β congruent with δ and γ. The lenses emerge after dissecting regions α and β. For every symmetry $m = 2n + 1$ we get n distinct lenses.

At (i) for symmetry $m = 3$ we have the single lense ($\boldsymbol{A}$) equivalent to the regular hexagon. Lense $\boldsymbol{A}$ is congruent with S_A. At (ii) for symmetry $m = 5$ we have the two lenses ($\boldsymbol{B}$, $\boldsymbol{C}$). The α region of S_B area equals $2\boldsymbol{B}$. The β region area equals $\boldsymbol{C}$. At (iii) for symmetry $m = 7$ we have the three lenses ($\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{F}$). The region α area equals $2\boldsymbol{D} + \boldsymbol{F}$ and the region β area equals $2\boldsymbol{E}$. At (iv) for symmetry $m = 9$ we have the four lenses ($\boldsymbol{G}$, $\boldsymbol{H}$, $\boldsymbol{I}$, $\boldsymbol{J}$). The α region area equals $2(\boldsymbol{G} + \boldsymbol{I})$ and the β region area equals $2\boldsymbol{H} + \boldsymbol{J}$.

Next we are going to show the regions δ for any symmetry $m \geq 5$ can be dissected with the lenses.

2 Crowns

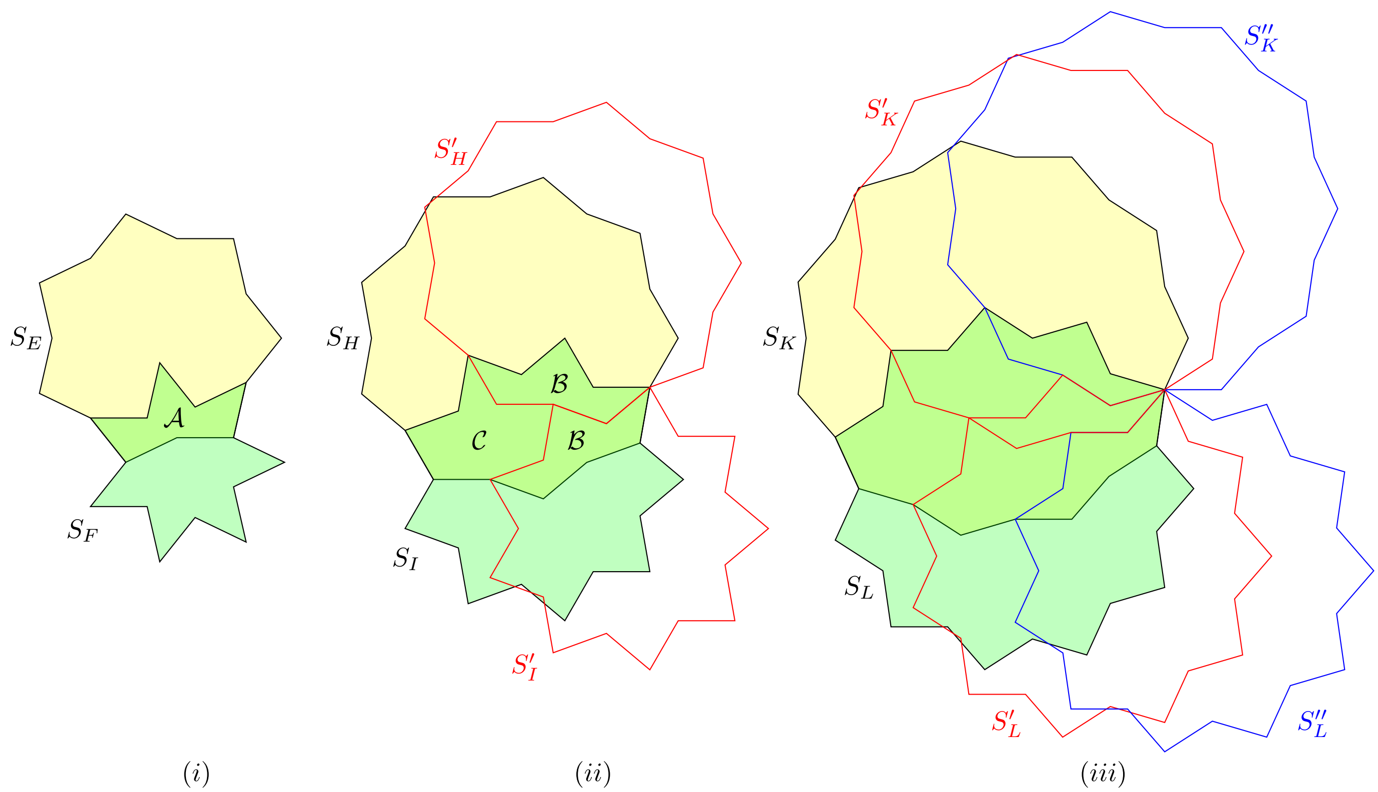


Figure 4: Crowns.

3 Symmetry 5

Symmetry 5 is based in angle $\beta = \dfrac{2\pi}{5}$ and produces the two rhombi ($\boldsymbol{b}$, $\boldsymbol{c}$) and the two lenses ($\boldsymbol{B}$, $\boldsymbol{C}$).

3.1 Rhombi $(\boldsymbol{b},\boldsymbol{c})$

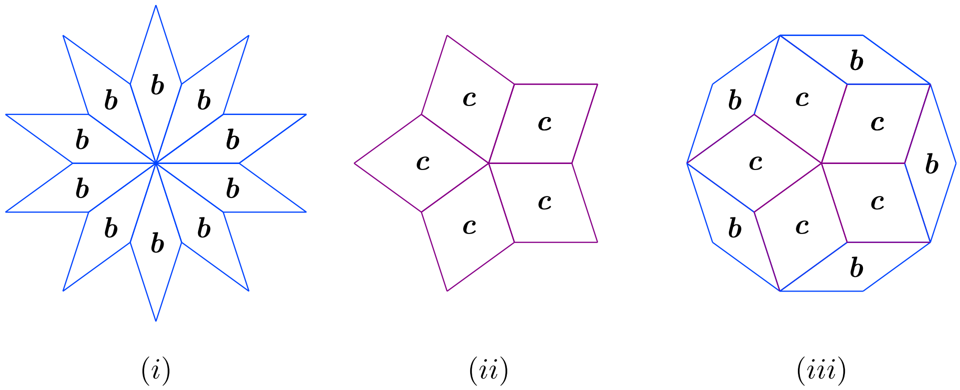


Figure 5: Rhombi $(\boldsymbol{b}, \boldsymbol{c})$ from dissecting stars S_{10}.

Figure 5 show the rhombi $(\boldsymbol{b}, \boldsymbol{c})$. Inspecting the stars we get the areas simply adding their rhombi. At (i) the star $S_{10}(1, 8)$ with area $A = 10\boldsymbol{b}$. At (ii) the star $S_{10}(2, 6) = S_5(1, 3)$ with area $A = 5\boldsymbol{c}$. At (iii) the regular decagon equivalvent to stars $S_{10}(4, 4) = S_5(2, 2)$ with area $A = 5\boldsymbol{b} + 5\boldsymbol{c}$. Table 1 show the rhombi $(\boldsymbol{b}, \boldsymbol{c})$ internal angles in terms of angle $\beta = 2\pi/5$ and areas for side equals to 1. Dividing areas we find $\dfrac{\boldsymbol{c}}{\boldsymbol{b}} = 2\cos\left(\dfrac{\pi}{5}\right) = \dfrac{\sqrt{5}+1}{2}$.

Rhombus	θ_1	θ_2	Area
$\boldsymbol{b}$	$\beta/2$	$4\beta/2$	$\sin(2\beta) = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$
$\boldsymbol{c}$	$2\beta/2$	$3\beta/2$	$\sin(\beta) = \dfrac{\sqrt{10+2\sqrt{5}}}{4} = \boldsymbol{b}\cos\left(\dfrac{\pi}{5}\right) = \boldsymbol{b}\left(\dfrac{\sqrt{5}+1}{2}\right)$

Table 1: Rhombi $(\boldsymbol{b}, \boldsymbol{c})$ internal angles and areas. $\theta_1 + \theta_2 = \pi$ and $\beta = 2\pi/5$.

3.2 Regular pentagon and star $|5/2|$

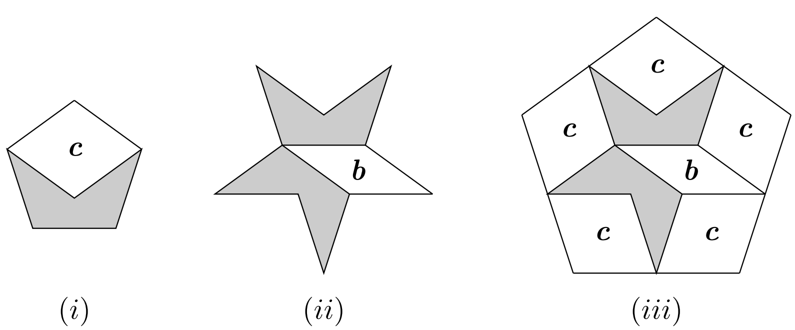


Figure 6: Regular pentagon $|5/1|$ at (i). Star $|5/2|$ at (ii). Double pentagon at (iii).

Figure 6 show regular pentagon and isotoxal star $|5/2|$ dissected with rhombi $(\boldsymbol{b}, \boldsymbol{c})$ plus concave pentagons (in gray). Let $\boldsymbol{x}$ be the area of such gray piece. By inspection the area of regular pentagon at (i) is $A_1 = \boldsymbol{c} + \boldsymbol{x}$ and the area of regular pentagon at (iii) is $P_2 = \boldsymbol{b} + 5\boldsymbol{c} + 2\boldsymbol{x}$. Since the side of A_2 is the double of A_1 its area is four times so we can get the value of $\boldsymbol{x}$ in terms of $(\boldsymbol{b}, \boldsymbol{c})$

$$
\begin{aligned}
4P_1 &= P_4 \\
4(\boldsymbol{c} + x) &= \boldsymbol{b} + 5\boldsymbol{c} + 2x \\
x &= \frac{\boldsymbol{b} + \boldsymbol{c}}{2} \qquad (1)
\end{aligned}
$$

We use the value of $\boldsymbol{x}$ to get the areas of pentagon (i) and star (ii):

$$
\begin{aligned}
A|5/1| &= \boldsymbol{c} + \boldsymbol{x} \\
&= \frac{\boldsymbol{b} + 3\boldsymbol{c}}{2} \qquad (2) \\
A|5/2| &= \boldsymbol{b} + 2\boldsymbol{x} \\
&= 2\boldsymbol{b} + \boldsymbol{c} \qquad (3)
\end{aligned}
$$

3.3 Lenses $(\boldsymbol{B}, \boldsymbol{C})$

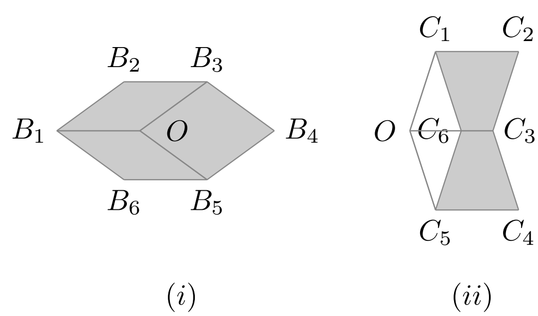


Figure 7: Lenses $(\boldsymbol{B},\ \boldsymbol{C})$ build with rhombi $(\boldsymbol{b},\ \boldsymbol{c})$.

Figure 7 show lenses $(\boldsymbol{B}, \boldsymbol{C})$ construction and two stars formed with them. At (i) we form the lense $\boldsymbol{B}$ with perimeter $\overline{B_1...B_6}$ adding two rhombi $\boldsymbol{b}$ $(\overline{B_1B_2B_3O}$ and $\overline{B_1OB_5B_6})$ and adding one rhombus $\boldsymbol{c}$ $(\overline{OB_3B_4B_5})$ so its area is $2\boldsymbol{b} + \boldsymbol{c}$. Lense $\boldsymbol{B}$ is equivalent to the hexagon $H_5(1, 2, 2)$. At (ii) we form the lense $\boldsymbol{C}$ with perimeter $\overline{C_1...C_6}$ adding two rhombi $\boldsymbol{c}$ $(\overline{OC_1C_2C3}$ and $\overline{OC_3C_4C_5})$ and substracting one rhombus $\boldsymbol{b}$ $(\overline{OC_1C_6C_5})$ so its area is $2\boldsymbol{c} - \boldsymbol{b}$. Lense $\boldsymbol{C}$ is equivalent to the hexagon $H_5(1, 1, 3)$. Table 2 show the lenses $(\boldsymbol{B}, \boldsymbol{C})$ internal angles and areas.

Lense	θ_1	θ_2	θ_3	Area
$\boldsymbol{B}$	β	2β	2β	$2\boldsymbol{b} + \boldsymbol{c}$
$\boldsymbol{C}$	β	β	3β	$-\boldsymbol{b} + 2\boldsymbol{c}$

Table 2: Lenses $(\boldsymbol{B}, \boldsymbol{C})$ internal angles and areas in terms of rhombi $(\boldsymbol{b}, \boldsymbol{c})$. $\theta_1 + \theta_2 + \theta_3 = 2\pi$ where $\beta = 2\pi/5$.

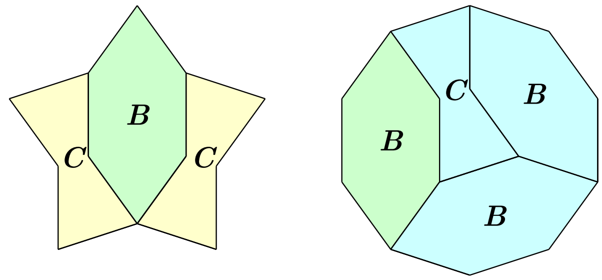


Figure 8: Two stars dissected with lenses ($\boldsymbol{B}$, $\boldsymbol{C}$).

Figure 8 show two stars dissected with lenses ($\boldsymbol{B}$, $\boldsymbol{C}$). At (i) the star $S_5(1,3)$ dissection implies its area is $A = \boldsymbol{B} + 2\boldsymbol{C} = 5\boldsymbol{c}$. At (ii) the regular decagon or star $S_5(2,2)$ dissection implies its area is $A = 3\boldsymbol{B} + \boldsymbol{C} = 5(\boldsymbol{b} + \boldsymbol{c})$.

4 Symmetry 7

Symmetry 7 is based in angle $\gamma = \dfrac{2\pi}{7}$ and produces the three rhombi ($\boldsymbol{d}, \boldsymbol{f}, \boldsymbol{e}$) and the three lenses ($\boldsymbol{D}, \boldsymbol{E}, \boldsymbol{F}$).

4.1 Rhombi ($\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$)

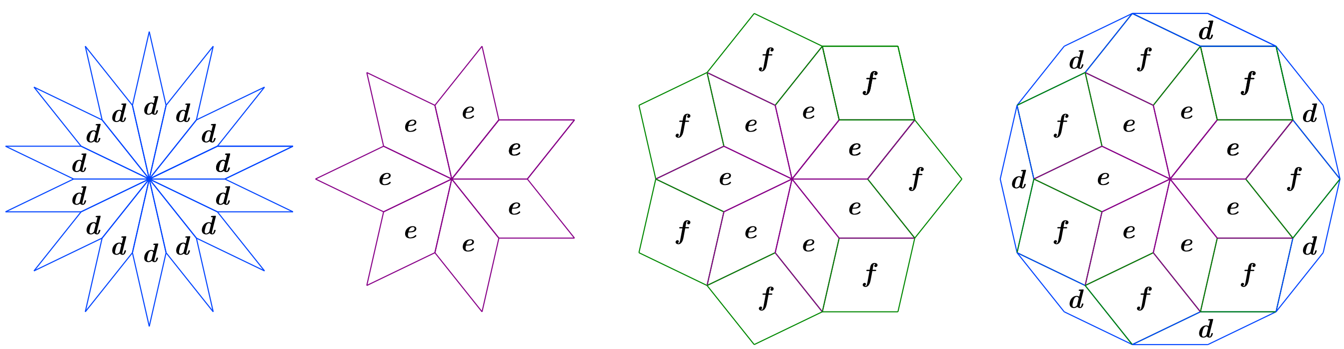


Figure 9: Rhombi ($\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$) from dissected stars S_{14}.

Figure 9 show rhombi ($\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$). Inspecting the stars we get the areas simply adding their rhombi. At (i) the star $S_{14}(1,12)$ with area $A = 14\boldsymbol{d}$. At (ii) the star $S_{14}(2,10) = S_7(1,5)$ with area $A = 7\boldsymbol{e}$. At (iii) the star $S_{14}(4,8) = S_7(2,4)$ with area $A = 7(\boldsymbol{e} + \boldsymbol{f})$. At (iv) the regular 14-gon equivalent to stars $S_{14}(6,6) = S_7(3,3)$ with area $A = 7(\boldsymbol{d} + \boldsymbol{e} + \boldsymbol{f})$. Table 3 show the symmetry 7 lenses internal angles based in angle $\gamma = 2\pi/7$ and the areas.

Rhombus	θ_1	θ_2	Area
$\boldsymbol{d}$	$\gamma/2$	$6\gamma/2$	$\sin(3\gamma)$
$\boldsymbol{e}$	$2\gamma/2$	$5\gamma/2$	$\sin(\gamma)$
$\boldsymbol{f}$	$3\gamma/2$	$4\gamma/2$	$\sin(2\gamma)$

Table 3: Rhombi ($\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$) internal angles. $\theta_1 + \theta_2 = \pi$ and $\gamma = 2\pi/7$.

4.2 Regular heptagon and stars $|7/3|$ and $|7/2|$

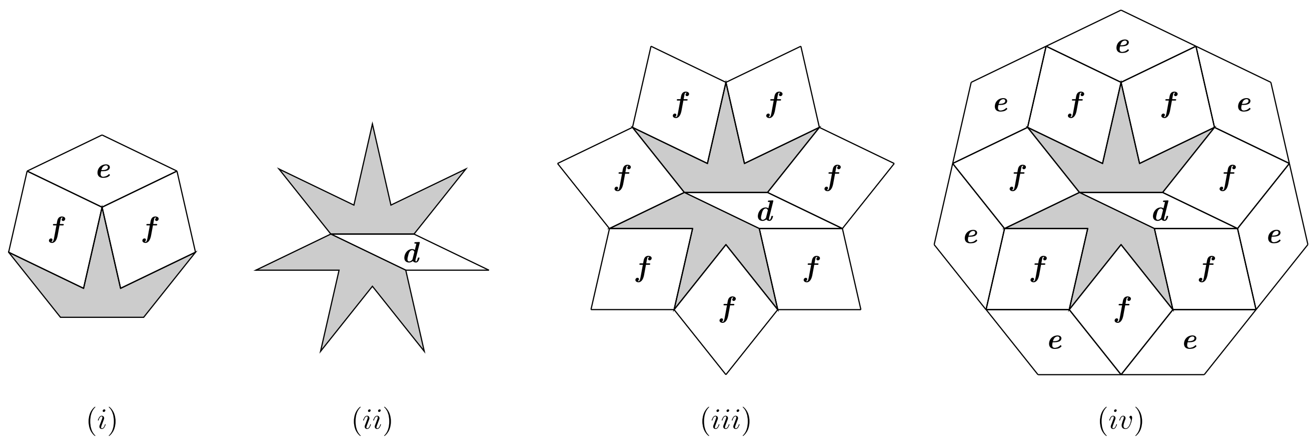


Figure 10: Heptagon $|7/1|$ at (i). Star $|7/3|$ at (ii). Star $|7/2|$ at (iii). Double heptagon at (iv).

Figure 10 show regular heptagon and heptagrams dissected with rhombi ($\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$) plus equilateral concave heptagons (in gray). Let $\boldsymbol{y}$ be the area of such gray piece. By inspection the area of regular heptagon at (i) is $A_1 = \boldsymbol{e} + 2\boldsymbol{f} + \boldsymbol{y}$ while the area of regular heptagon at (iv) is $A_2 = \boldsymbol{d} + 7(\boldsymbol{e} + \boldsymbol{f}) + 2\boldsymbol{y}$. Since the side of A_2 is the double of A_1 its area is four times so we can get the value of $\boldsymbol{y}$ in terms of ($\boldsymbol{d}$, $\boldsymbol{e}$, $\boldsymbol{f}$):

$$\begin{aligned} 4A_1 &= A_2 \\ 4(\boldsymbol{e} + 2\boldsymbol{f} + \boldsymbol{y}) &= \boldsymbol{d} + 7(\boldsymbol{e} + \boldsymbol{f}) + 2\boldsymbol{y} \\ \boldsymbol{y} &= \frac{\boldsymbol{d} + 3\boldsymbol{e} - \boldsymbol{f}}{2} \end{aligned} \tag{4}$$

We use the value of $\boldsymbol{y}$ to calculate the areas of heptagon (i) and stars (ii) and (iii) in terms of ($\boldsymbol{d}$, $\boldsymbol{e}$, $\boldsymbol{f}$):

$$\begin{aligned} A|7/1| &= \boldsymbol{e} + 2\boldsymbol{f} + \boldsymbol{y} \\ &= \frac{\boldsymbol{d} + 5\boldsymbol{e} + 3\boldsymbol{f}}{2} \qquad (5) \\ A|7/3| &= \boldsymbol{d} + 2\boldsymbol{y} \\ &= 2\boldsymbol{d} + 3\boldsymbol{e} - \boldsymbol{f} \qquad (6) \\ A|7/2| &= A\{7/3\} + 7\boldsymbol{f} \\ &= 2\boldsymbol{d} + 3\boldsymbol{e} + 6\boldsymbol{f} \qquad (7) \end{aligned}$$

4.3 Lenses ($\boldsymbol{D}$,$\boldsymbol{E}$,$\boldsymbol{F}$)

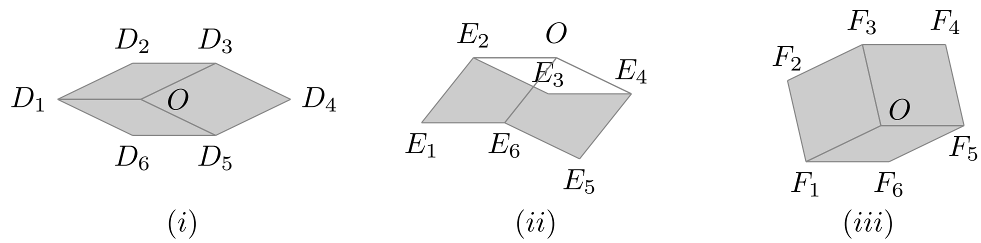


Figure 11: Lenses ($\boldsymbol{D}$,$\boldsymbol{E}$,$\boldsymbol{F}$) build from rhombi ($\boldsymbol{d}$,$\boldsymbol{e}$,$\boldsymbol{f}$).

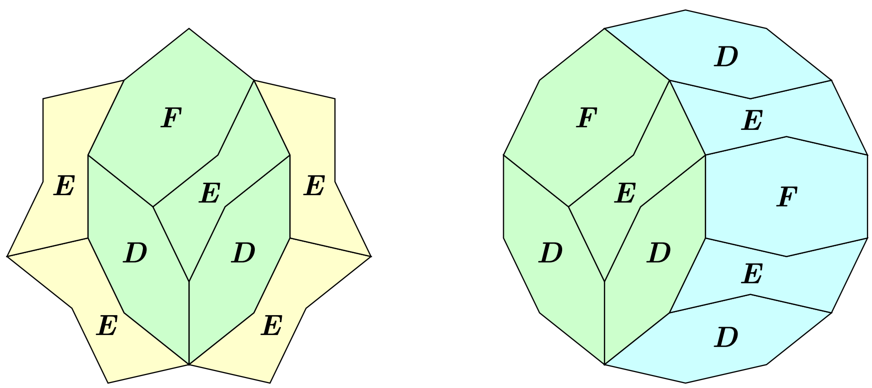
Figure 11 show lenses ($\boldsymbol{B}$, $\boldsymbol{C}$) construction. At (i) we form the lense $\boldsymbol{D}$ with perimeter $\overline{D_1...D_6}$ adding two rhombi $\boldsymbol{d}$ ($\overline{D_1D_2D_3O}$ and $\overline{D_1OD_5D_6}$) and adding one rhombus $\boldsymbol{e}$ ($\overline{OD_3D_4D_5}$) so its area is $2\boldsymbol{d}+\boldsymbol{e}$. Lense $\boldsymbol{D}$ is equivalent to the hexagon $H_7(1,3,3)$.

At (ii) we form the lense $\boldsymbol{E}$ with perimeter $\overline{E_1...E_6}$ adding one rhombus $\boldsymbol{e}$ ($\overline{E_1E_2OE_6}$) adding one rhombus $\boldsymbol{f}$ ($\overline{OE_4E_5E_6}$) and substracting one rhombus $\boldsymbol{d}$ ($\overline{E_2OE_4E_3}$) so its area is $-\boldsymbol{d}+\boldsymbol{e}+\boldsymbol{f}$. Lense $\boldsymbol{E}$ is equivalent to the hexagon $H_7(1,2,4)$.

At (iii) we form the lense $\boldsymbol{F}$ with perimeter $\overline{F_1...F_6}$ adding two rhombi $\boldsymbol{f}$ ($\overline{F_1F_2F_3O}$ and $\overline{F_3F_4F_5O}$) and adding one rhombus $\boldsymbol{d}$ ($\overline{F_1OF_5F_6}$) so its area is $\boldsymbol{d}+2\boldsymbol{f}$. Lense $\boldsymbol{F}$ is equivalent to the hexagon $H_7(2,2,3)$. Table 4 show the lenses ($\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{F}$) internal angles and areas.

Lense	θ_1	θ_2	θ_3	Area
$\boldsymbol{D}$	γ	3γ	3γ	$2\boldsymbol{d}+\boldsymbol{e}$
$\boldsymbol{E}$	γ	2γ	4γ	$-\boldsymbol{d}+\boldsymbol{e}+\boldsymbol{f}$
$\boldsymbol{F}$	2γ	2γ	3γ	$-\boldsymbol{d}+2\boldsymbol{f}$

Table 4: Lenses ($\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{F}$) internal angles and areas in terms of rhombi ($\boldsymbol{d}$, $\boldsymbol{e}$, $\boldsymbol{f}$). $\theta_1+\theta_2+\theta_3=2\pi$ and $\gamma=2\pi/7$.



(i) (ii)

Figure 12: Stars dissected with only lenses ($\boldsymbol{D}$,$\boldsymbol{E}$,$\boldsymbol{F}$).

Figure 12 show stars $S_7(2,4)$ and $S_7(3,3)$ dissected with lenses ($\boldsymbol{D}$,$\boldsymbol{E}$,$\boldsymbol{F}$). At (i) we have star $S_7(2,4)$ and by inspection we deduce its area is $A=2\boldsymbol{D}+5\boldsymbol{E}+\boldsymbol{F}$. At (ii) we have regular 14-gon (or star $S_7(3,3)$) and by inspection we deduce its area is $4\boldsymbol{D}+3\boldsymbol{E}+2\boldsymbol{F}$. Both stars have in common an area in green resembling a tree leaf. The star at (i) also contains two regions in yellow resembling crowns while the star at (ii) contains a region in cyan resembling a moon phase.

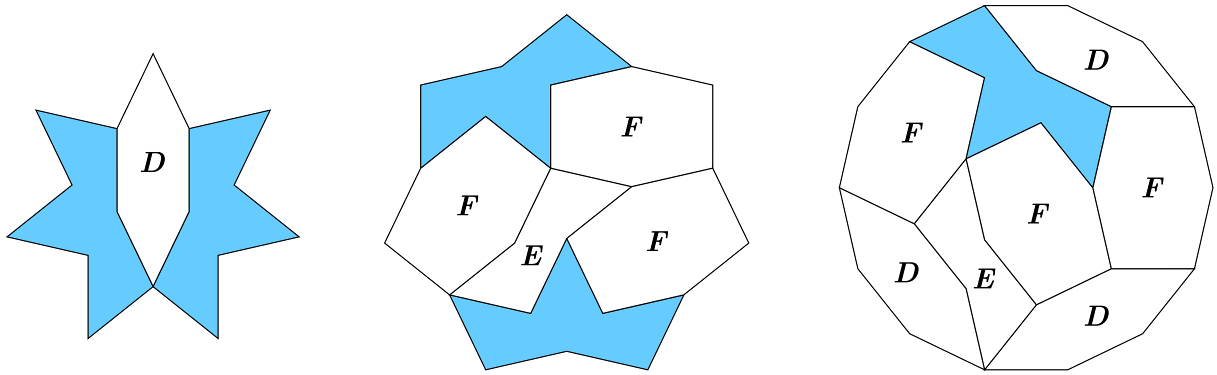


Figure 13: Stars dissected with octagons O_7 (in blue) and lenses ($\boldsymbol{D}$,$\boldsymbol{E}$,$\boldsymbol{F}$).

Figure 13 show Stars $S_7(1,5)$, $S_7(2,4)$ and $S_7(3,3)$ dissected with octagons O_7 (in blue) and lenses ($\boldsymbol{D}$,$\boldsymbol{E}$,$\boldsymbol{F}$). At (i) we have the star $S_7(1,5)$ and by inspection we deduce its area is $A = \boldsymbol{D} + 2O_7$. At (ii) we have the star $S_7(2,4)$ and we can conclude its area is $\boldsymbol{E} + 3\boldsymbol{F} + 2O_7$. Similarly the area of the 14-gon at (iii) is $3\boldsymbol{D} + \boldsymbol{E} + 3\boldsymbol{F} + O_7$. Comparing the areas of the two 14-gons of figures 12 and 13 we can find the area of O_7 in terms of ($\boldsymbol{E}$, $\boldsymbol{F}$, $\boldsymbol{G}$):

$$\begin{aligned} 4\boldsymbol{D} + 3\boldsymbol{E} + 2\boldsymbol{F} &= 3\boldsymbol{D} + \boldsymbol{E} + 3\boldsymbol{F} + O_7 \\ O_7 &= \boldsymbol{D} + 2\boldsymbol{E} - \boldsymbol{F} \end{aligned} \tag{8}$$

So we can calculate the area of star $S_7(1,5)$ in terms of ($\boldsymbol{E}$, $\boldsymbol{F}$, $\boldsymbol{G}$):

$$\begin{aligned} S_7(1,5) &= \boldsymbol{D} + 2(\boldsymbol{D} + 2\boldsymbol{E} - \boldsymbol{F}) \\ &= 3\boldsymbol{D} + 4\boldsymbol{E} - 2\boldsymbol{F} \end{aligned} \tag{9}$$