Symmetry 9

https://github.com/heptagons/lenses

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Abstract

We build equilateral polygons that can tesselate the plane using the symmetry 9. We analyze lower odd symmetries $3, 5, 7$ also to find patterns. We analize vectors which describe symmetry vertices. Then we define polygons as rhombus, hexagons, octagons and stars build with the vectors.

1 Symmetries

We are interested in odd symmetries starting with $s = 3$. Table 1 show the symmetries up to $s = 9$. The value of the angle of the symmetry s is $\theta = \dfrac{2\pi}{s}$.

s	Angle	θ
3	α	$2\pi/3$
5	β	$2\pi/5$
7	γ	$2\pi/7$
9	δ	$2\pi/9$

Table 1: Symmetries $s = \{3, 5, 7, 9\}$.

For every symmetry s we identify $t = \dfrac{s+1}{2}$ independent abscissas x_i and t independent ordinates y_i:

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} \cos((i-1)\theta) \\ \sin((i-1)\theta) \end{bmatrix} \quad i = 1, 2, ..., t \tag{1}$$

We define a relative origin point p_0. There are s different points $p_1, p_2, ...p_s$ equidistant to point p_0 located at coordinates $\langle x_i, y_i \rangle$ as follows:

$$p_i \equiv \begin{cases} \langle x_j, y_j \rangle & \text{for} \quad i \leq t \quad j = i \\ \langle x_j, -y_j \rangle & \text{for} \quad i > t \quad j = s + 2 - i \end{cases} \tag{2}$$

The points are transformed into vectors which hold only the (x, y) pair of indices:

$$v_i \equiv \begin{cases} (j, j) & \text{for} \quad i \leq t \quad j = i \\ (j, -j) & \text{for} \quad i > t \quad j = s + 2 - i \end{cases} \tag{3}$$

Any vector $v_i \equiv (k_1, k_2)$ can be rotated by $180°$ around p_0. The vector after the rotation is denoted as $\overline{v_i}$ and the original indices k_1, k_2 signs are changed:

$$\overline{v_i} \equiv (-k_1, -k_2) \tag{4}$$

We define an absolute origin point o located at $(0,0)$. Starting at o we can form polylines adding vectors alternating the type v_i and the type $\overline{v_i}$ in order the points angles are always multiples of θ. The polyline with vectors $\overrightarrow{P_1P_2} = v_a$, $\overrightarrow{P_2P_3} = \overline{v_b}$, $\overrightarrow{P_3P_4} = v_c$, ... is denoted as:

$$\overline{P_1P_2P_3P_4...} = \{[], [v_a], [v_a, \overline{v_b}], [v_a, \overline{v_b}, v_c], ...\} \quad 1 \leq a, b, c \leq t \tag{5}$$

where the point P_1 is at origin and the rest of points are non-empty arrays of vectors. A simpler notation for the last polyline for symmetry s is:

$$\overline{P_1P_2P_3P_4...} = P_s(0, a, b, c, ...) \tag{6}$$

For two consecutive points $P_k = [..., v_m]$ and $P_{k+1} = [..., v_m, \overline{v_n}]$ for $k > 0$ we have that the angle at point P_k is $u\theta$, where u is:

$$u \equiv s + m - n \pmod{s} \qquad 1 \leq m, n \leq t \tag{7}$$

Every point $\mathbf{P}$ has `accumulators` composed of two vectors $\mathbf{X}$ and $\mathbf{Y}$ holding t integers each:

$$\mathbf{X} = \left[\begin{array}{ccccc} X_1 & X_2 & X_3 & ... & X_t \end{array}\right] \quad X_i \in \mathbb{Z} \tag{8}$$

$$\mathbf{Y} = \left[\begin{array}{ccccc} Y_1 & Y_2 & Y_3 & ... & Y_t \end{array}\right] \quad Y_i \in \mathbb{Z} \tag{9}$$

$$\mathbf{P}^A \equiv \left[\begin{array}{c} \mathbf{X} \\ \mathbf{Y} \end{array}\right] = \left[\begin{array}{ccccc} X_1 & X_2 & X_3 & ... & X_t \\ Y_1 & Y_2 & Y_3 & ... & Y_t \end{array}\right] \tag{10}$$

Accumulators are copied from previous points to the new ones and incremented in $+1$ or decremented in -1 depending in the indices of the vector $v = (k_x, k_y)$ which created the new point:

$$X_i = \begin{cases} X_i + 1 & \text{for} \quad i = k_x \\ X_i - 1 & \text{for} \quad i = -k_x \end{cases} \tag{11}$$

$$Y_i = \begin{cases} Y_i + 1 & \text{for} \quad i = k_y \\ Y_i - 1 & \text{for} \quad i = -k_y \end{cases} \tag{12}$$

The absolute position coordinates of point P are calculated adding the products of the accumulators $\mathbf{P}^A$ of equation 10 and the coordinates of points p_i of equation 2:

$$\langle P_x, P_y \rangle = \left\langle \sum_{i=1}^{t} X_i x_i, \sum_{i=1}^{t} Y_i y_i \right\rangle \tag{13}$$

2 Symmetry $m = 9$

Figure 1 (i) show the nine different vectors v_1, v_2, ... v_9 in symmetry $m = 9$. Using the vectors equation 3 we get the nine vectors: $v_1 = (1,1)$, $v_2 = (2,2)$, $v_3 = (3,3)$, $v_4 = (4,4)$, $v_5 = (5,5)$, $v_6 = (11 - 6, -(11 - 6)) = (5, -5)$, $v_7 = (11 - 7, -(11 - 7)) = (4, -4)$, $v_8 = (11 - 8, -(11 - 8)) = (3, -3)$ and $v_9 = (11 - 9, -(11 - 9)) = (2, -2)$.

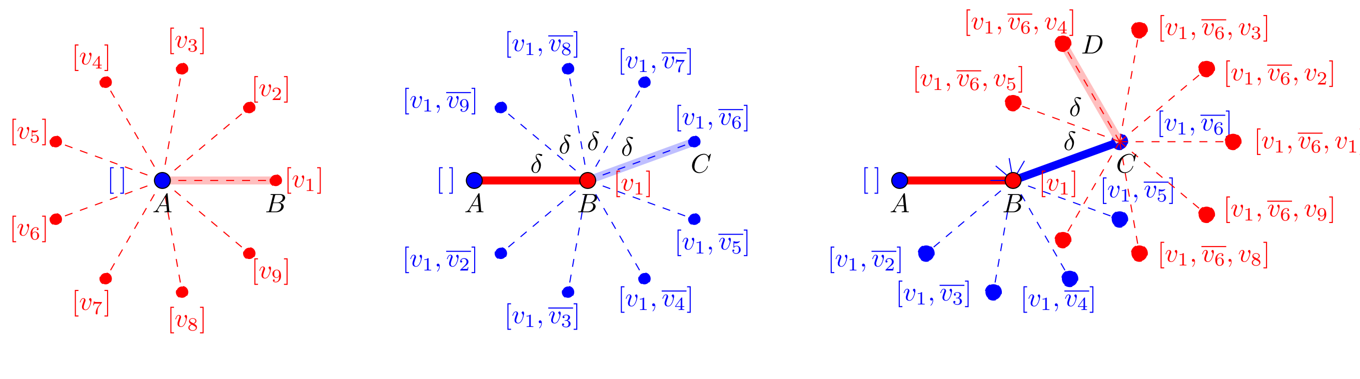

Figure 1: The symmetry 9 vectors, points, angles and polylines.

Figure 1 (ii) show polyline $\overline{ABC}$ and (iii) show polyline $\overline{ABCD}$ alternating vectors of the type $[..., v_m]$ (show in red) with vectors of the type $[..., v_m, \overline{v_n}]$ (shown in blue) to guarantee the angles between edges are always multiples of δ: angle $\angle ABC = 4\delta$, angle $\angle BCD = 2\delta$. The polyline $\{A, B, C, D\}$ is denoted by $\{[\,], [v_1], [v_1, \overline{v_6}], [v_1, \overline{v_6}, v_2]\}$ or in compact form $V_9(0, 1, 6, 2)$.

The vertice o at the origin is assigned with accumulators $\mathbf{X}, \mathbf{Y}$ initialized to zero. For the next point the accumulators are updated at positions $1 \leq k \leq t$ according the components of the vector $v_m \equiv (k_x, k_y)$ of the new point:

$$X[a,b,c,d,e] + v_i = \begin{cases} X[a+1,b,c,d,e] & \text{for} \quad v_1 \equiv (1,1) \\ X[a,b+1,c,d,e] & \text{for} \quad v_2 \equiv (2,2), v_9 \equiv (2,-2) \\ \quad ... & \\ X[a,b,c,d,e+1] & \text{for} \quad v_5 \equiv (5,5), v_6 \equiv (5,-5) \\ X[a-1,b,c,d,e] & \text{for} \quad \overline{v_1} \equiv (-1,-1) \\ \quad ... & \end{cases} \tag{14}$$

Lets get the accumulators of the polyline $\{[\,], [v_1], [v_1, \overline{v_6}], [v_1, \overline{v_6}, v_4]\}$ shown in figure (iii). By definition the accumulators at $o \equiv (0,0)$ which is point $A = [\,]$ are:

$$[\,]^A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

First we get accumulators of vector v_1 and point $B = [v_1]$:

$$v_1 = (1,1)$$

$$v_1^A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

$$[v_1]^A = [\,]^A + v_1^A \tag{17}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{18}$$

Then we get accumulators of vector $\overline{v_6}$:

$$v_6 \equiv (5,-5)$$

$$\overline{v_6} = (-5,5)$$

$$\overline{v_6}^A = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{19}$$

And then we get the accumulators of point $C = [v_1, \overline{v_6}]$:

$$[v_1, \overline{v_6}]^A = v_1^A + \overline{v_6}^A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{20}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{21}$$

Finally we get the accumulator of vector v_4 and use it to get the accumulator of point $D = [v_1, \overline{v_6}, v_4]$:

$$v_4 \equiv (4, 4)$$

$$v_4^A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{22}$$

$$[v_1, \overline{v_6}, v_4]^A = [v_1, \overline{v_6}]^A + v_4^A = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{23}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{24}$$

Then the accumulators of point $D = [v_1, \overline{v_6}, v_4]$ have different of zero these values: $X_1 = X_4 = 1$, $X_5 = -1$, $Y_1 = Y_4 = Y_5 = 1$. Using the equation 13 we calculate the absolute coordinates:

$$\begin{aligned} D_x &= x_1 + x_2 - x_5 \\ &= \cos(0) + \cos(3\delta) - \cos(4\delta) \\ &= 1 + \cos(120°) - \cos(160°) \\ D_y &= y_1 + y_2 + y_2 \\ &= \sin(0) + \sin(3\delta) + \sin(4\delta) \\ &= \sin(120°) + \sin(160°) \end{aligned}$$

Using equation 7 we calculate the angles multiples u at points B and C. From $C = [v_1, \overline{v_6}]$ we have that $m_B = 1, n_B = 6$ and we calculate u at point B:

$$u_B = 9 + 1 - 6 \pmod 9 = 4 \tag{25}$$

From $D = [v_1, \overline{v_6}, v_4]$ we have that $m_C = 6, n_C = 4$ and we calculate u at point C:

$$u_C = 9 + 6 - 4 \pmod 9 = 2 \tag{26}$$

3 Rhombi

Rhombi are equilateral tetragons. For every odd symmetry s we have $n = \dfrac{s-1}{2}$ different rhombi. The rhombi angles $u_1 < u_2$ are calculated as follows:

$$(u_1, u_2) = \left(\frac{i}{2}, n + \frac{i-1}{2}\right) \quad i = 1, ..., n \tag{27}$$

We identify any rhombus as $R_s(u_1)$ so for symmetry $s = 9$ we have the four rhombi:

$$R_9 = \left\{ R_9\left(\frac{1}{2}\right), R_9(1), R_9\left(\frac{3}{2}\right), R_9(2) \right\} \tag{28}$$

Table 2 show the rhombi for odd symmetries up to 9. Every rhombus is labeled with a consecutive lowercase letter $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$,... The first rhombi $\boldsymbol{a}$ is found first in symmetry $s = 3$ and then again in symmetry $s = 9$ so we prevent renaming congruent rhombi.

The rhombi angles $\theta_1 = u_1\theta$ and $\theta_2 = u_2\theta$ are expressed in function of the angles $\{\alpha, \beta, \gamma, \delta\}$ of symmetries $s = \{3, 5, 7, 9\}$ where $\theta_1 < \theta_2$ and $\theta_1 + \theta_2 = \pi$. We cannot use the rhombi isolated in our tessellations since not both u_1 and u_2 are integers. But we'll add and substract rhombi together to build hexagons, octagons and beyond where all polygon's angles u_n are integers.

$R_s(u_1)$	u_2	Label	θ_1	θ_2	Area
$R_3(\frac{1}{2})$	1	$\boldsymbol{a}$	$\alpha/2$	$2\alpha/2$	$\sin(\alpha) \approx 0.866$
$R_5(\frac{1}{2})$	2	$\boldsymbol{b}$	$\beta/2$	$4\beta/2$	$\sin(2\beta) \approx 0.587$
$R_5(1)$	$\frac{3}{2}$	$\boldsymbol{c}$	$2\beta/2$	$3\beta/2$	$\sin(\beta) \approx 0.951$
$R_7(\frac{1}{2})$	3	$\boldsymbol{d}$	$\gamma/2$	$6\gamma/2$	$\sin(3\gamma) \approx 0.433$
$R_7(1)$	$\frac{5}{2}$	$\boldsymbol{e}$	$2\gamma/2$	$5\gamma/2$	$\sin(\gamma) \approx 0.781$
$R_7(\frac{3}{2})$	2	$\boldsymbol{f}$	$3\gamma/2$	$4\gamma/2$	$\sin(2\gamma) \approx 0.974$
$R_9(\frac{1}{2})$	4	$\boldsymbol{g}$	$\delta/2$	$8\delta/2$	$\sin(4\delta) \approx 0.342$
$R_9(1)$	$\frac{7}{2}$	$\boldsymbol{h}$	$2\delta/2$	$7\delta/2$	$\sin(\delta) \approx 0.642$
$R_9(\frac{3}{2})$	3	$\boldsymbol{a}$	$3\delta/2$	$6\delta/2$	$\sin(3\delta) \approx 0.866$
$R_9(2)$	$\frac{5}{2}$	$\boldsymbol{i}$	$4\delta/2$	$5\delta/2$	$\sin(2\delta) \approx 0.984$

Table 2: Rhombi $R_s(u_1)$ for symmetries $s = \{3, 5, 7, 9\}$.

4 Stars

For every odd symmetry s we have stars that are equilateral $2s$-gons with at most two different angles u_1 and u_2 at the vertices. We find exactly $n = \dfrac{s-1}{2}$ different stars with angles $u_1 \leq u_2$ as integers.

Table 3 show the stars for odd symmetries up to 9. Every star with u_1 as integer are labeled with a consecutive calligraphy letter $\mathcal{A}, \mathcal{B}, \mathcal{C}$.... We indentify any stars as $S_s(u_1)$ for the symmetry s. We calculate the other angle as $u_2 = s - u_1 - 1$. The stars are easily build with rhombi so the table show the stars area in function of them.

$S_s(u_1)$	u_2	Label	Area	Polygon
$S_3(\frac{1}{2})$	2	-	$6\boldsymbol{a}$	$\|6/2\|$ (12-gon)
$S_3(1)$	1	$\mathcal{A}$	$3\boldsymbol{a}$	Regular hexagon
$S_5(\frac{1}{2})$	4	-	$5\boldsymbol{b}$	$\|10/4\|$ (20-gon)
$S_5(1)$	3	$\mathcal{B}$	$5\boldsymbol{c}$	$\|(5/2)_\alpha\|$ decagram
$S_5(2)$	2	$\mathcal{C}$	$5(\boldsymbol{c}+\boldsymbol{b})$	Regular decagon
$S_7(\frac{1}{2})$	6	-	$7\boldsymbol{d}$	$\|14/6\|$ (28-gon)
$S_7(1)$	5	$\mathcal{D}$	$7\boldsymbol{e}$	$\|(7/4)_\alpha\|$ 14-gram
$S_7(2)$	4	$\mathcal{E}$	$7(\boldsymbol{e}+\boldsymbol{f})$	$\|(7/2)_\alpha\|$ 14-gram
$S_7(3)$	3	$\mathcal{F}$	$7(\boldsymbol{e}+\boldsymbol{f}+\boldsymbol{d})$	Regular 14-gon
$S_9(\frac{1}{2})$	8	-	$9\boldsymbol{g}$	$\|18/8\|$ (36-gon)
$S_9(1)$	7	$\mathcal{G}$	$9\boldsymbol{h}$	$\|(9/6)_\alpha\|$ 18-gram
$S_9(2)$	6	$\mathcal{H}$	$9(\boldsymbol{h}+\boldsymbol{i})$	$\|(9/4)_\alpha\|$ 18-gram
$S_9(3)$	5	$\mathcal{I}$	$9(\boldsymbol{h}+\boldsymbol{i}+\boldsymbol{a})$	$\|(9/2)_\alpha\|$ 18-gram
$S_9(4)$	4	$\mathcal{J}$	$9(\boldsymbol{h}+\boldsymbol{i}+\boldsymbol{a}+\boldsymbol{g})$	Regular 18-gon

Table 3: Stars $S_s(u_1)$ for symmetries $s = \{3, 5, 7, 9\}$.

Figure 2 show nine copies of symmetry-9 rhombi $\{\boldsymbol{h}, \boldsymbol{i}, \boldsymbol{a}, \boldsymbol{g}\}$ forming the four stars $S_9(1)$, $S_9(2)$, $S_9(3)$ and $S_9(4)$ labeled respectivelly $\mathcal{G}$, $\mathcal{H}$, $\mathcal{I}$ and $\mathcal{J}$. Note how the rhombi half angles always are added together to produce only angles as integers for the stars.

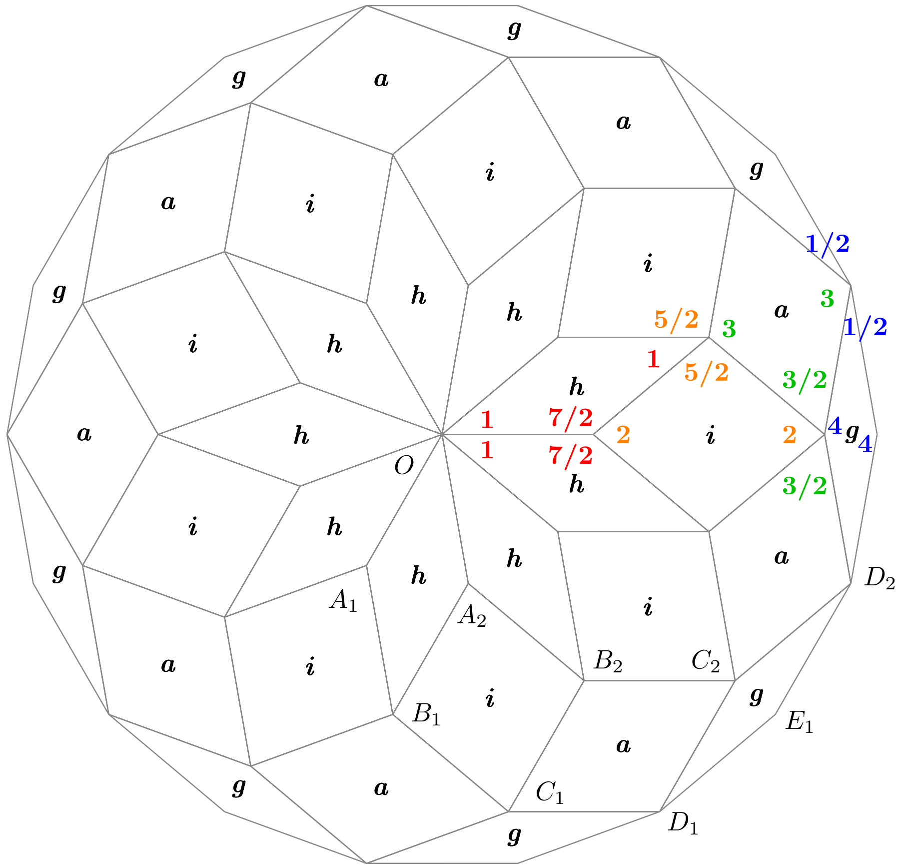

Figure 2: The symmetry 9 four rhombi $\{\boldsymbol{h}, \boldsymbol{i}, \boldsymbol{a}, \boldsymbol{g}\}$ produce the four stars $\{\mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}\}$ with areas $9\boldsymbol{h}$, $9(\boldsymbol{h} + \boldsymbol{i})$, $9(\boldsymbol{h} + \boldsymbol{i} + \boldsymbol{a})$ and $9(\boldsymbol{h} + \boldsymbol{i} + \boldsymbol{a} + \boldsymbol{g})$ respectivelly.

5 Hexagons

For any odd symmetry s, we can obtain equilateral hexagons by connecting six equal edges at integral values of u or by adding and substracting rhombi or by dissecting the areas left after stars intersections. In any case we find the hexagons has at most three different internal angles in this order $(u_1, u_2, u_3, u_1, u_2, u_3)$ where $u_1 + u_2 + u_3 = s$.

5.1 Hexagons angles

Figure 4 show the hexagons defined as $H_s(u_1, u_2)$ for symmetries $s = \{3, 5, 7, 9\}$ where $u_1 \leq u_2 \leq u_3$. We labeled the non self-intersecting hexagons with uppercase letters $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, ... We prevent to name differently any hexagon which is equivalent to a previous one having the same angles as what happens with hexagon $\boldsymbol{A}$ of symmetries 3 and 9.

$H_s(u_1,u_2)$	Label	(u_1,u_2,u_3)	Polygon
$H_3(1,1)$	$\boldsymbol{A}$	$(\mathbf{1},\ \mathbf{1},\ \mathbf{1})$	Regular hexagon
$H_5(1,1)$	$\boldsymbol{B}$	$(\mathbf{1},\ \mathbf{1},\ \mathbf{3})$	Sormeh Dan Girih tile
$H_5(1,2)$	$\boldsymbol{C}$	$(\mathbf{1},\ \mathbf{2},\ \mathbf{2})$	Shesh Band Girih tite
$H_7(1,1)$	-	$(\mathbf{1},\ \mathbf{1},\ \mathbf{5})$	self-intersecting
$H_7(1,2)$	$\boldsymbol{D}$	$(\mathbf{1},\ \mathbf{2},\ \mathbf{4})$	
$H_7(1,3)$	$\boldsymbol{E}$	$(\mathbf{1},\ \mathbf{3},\ \mathbf{3})$	
$H_7(2,2)$	$\boldsymbol{F}$	$(\mathbf{2},\ \mathbf{2},\ \mathbf{3})$	
$H_9(1,1)$	-	$(\mathbf{1},\ \mathbf{1},\ \mathbf{7})$	self-intersecting
$H_9(1,2)$	$\boldsymbol{G}$	$(\mathbf{1},\ \mathbf{2},\ \mathbf{6})$	
$H_9(1,3)$	$\boldsymbol{H}$	$(\mathbf{1},\ \mathbf{3},\ \mathbf{5})$	
$H_9(1,4)$	$\boldsymbol{I}$	$(\mathbf{1},\ \mathbf{4},\ \mathbf{4})$	
$H_9(2,2)$	$\boldsymbol{J}$	$(\mathbf{2},\ \mathbf{2},\ \mathbf{5})$	
$H_9(2,3)$	$\boldsymbol{K}$	$(\mathbf{2},\ \mathbf{3},\ \mathbf{4})$	
$H_9(3,3)$	$\boldsymbol{A}$	$(\mathbf{3},\ \mathbf{3},\ \mathbf{3})$	equivalent to $H_3(1,1)$

Table 4: Hexagons $H_s(u_1,u_2)$ for symmetries $s=\{3,5,7,9\}$.

5.2 Hexagons areas

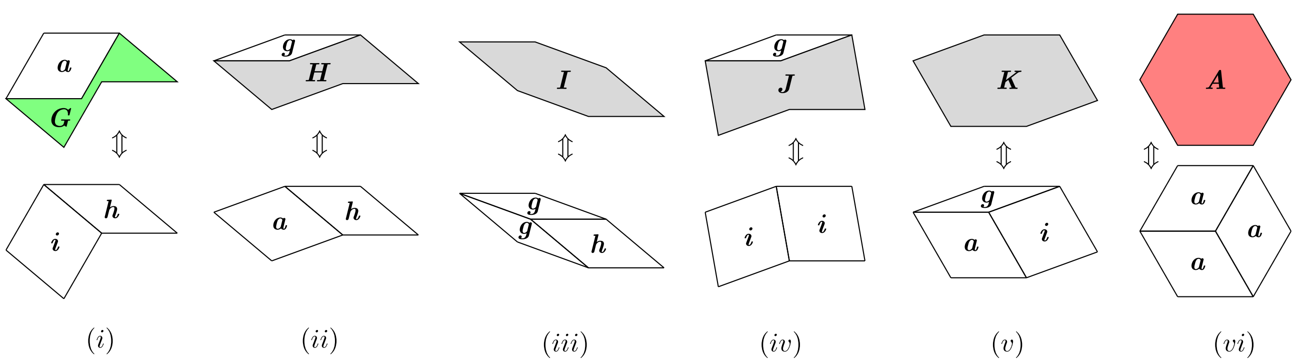

Figure 3: Symmetry 9 hexagons formed adding and substracting rhombi.

Figure 3 show how to calculate the area of the symmetry 9 hexagons in function of the symmetry 9 rhombi. From (i) to (vi) we equate the area of sum of the polygons in the top with the area of the sum of the polygons of the bottom. We have for the six cases these six equations:

$$\boldsymbol{a}+\boldsymbol{G}=\boldsymbol{i}+\boldsymbol{h} \tag{29}$$

$$\boldsymbol{g}+\boldsymbol{H}=\boldsymbol{a}+\boldsymbol{h} \tag{30}$$

$$\boldsymbol{I}=2\boldsymbol{g}+\boldsymbol{h} \tag{31}$$

$$\boldsymbol{g}+\boldsymbol{J}=2\boldsymbol{i} \tag{32}$$

$$\boldsymbol{K}=\boldsymbol{g}+\boldsymbol{a}+\boldsymbol{i} \tag{33}$$

$$\boldsymbol{A}=3\boldsymbol{a} \tag{34}$$

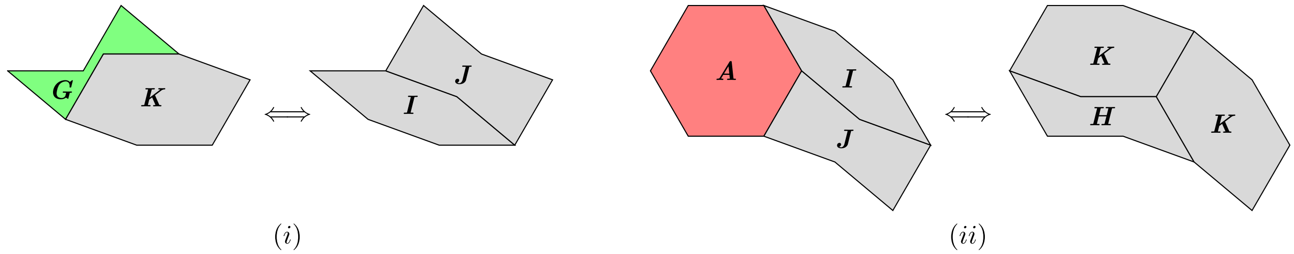

Figure 4: Hexagons $\{\boldsymbol{G}, \boldsymbol{A}\}$ formed adding and substracting hexagons $\{\boldsymbol{H}, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}\}$.

Figure 4 show how to express the area of the six hexagons in function of only four. For (i) and (ii) we equate the area of the sum of hexagons of the left with the area of the hexagons of the right. We have for the two cases these two equations:

$$\boldsymbol{G} + \boldsymbol{K} = \boldsymbol{I} + \boldsymbol{J} \tag{35}$$

$$\boldsymbol{A} + \boldsymbol{I} + \boldsymbol{J} = \boldsymbol{H} + 2\boldsymbol{K} \tag{36}$$

Using the last eight equations we form the table 5 which show the areas of the six hexagons in function of four rhombi $\boldsymbol{g},\boldsymbol{h},\boldsymbol{a},\boldsymbol{i}$ and in function of only four hexagons $\boldsymbol{H},\boldsymbol{I},\boldsymbol{J},\boldsymbol{K}$.

Hexagon	$\boldsymbol{g},\boldsymbol{h},\boldsymbol{a},\boldsymbol{i}$ area	$\boldsymbol{H},\boldsymbol{I},\boldsymbol{J},\boldsymbol{K}$ area
$\boldsymbol{H}$	$\boldsymbol{a} + \boldsymbol{h} - \boldsymbol{g}$	$\boldsymbol{H}$
$\boldsymbol{I}$	$2\boldsymbol{g} + \boldsymbol{h}$	$\boldsymbol{I}$
$\boldsymbol{J}$	$2\boldsymbol{i} - \boldsymbol{g}$	$\boldsymbol{J}$
$\boldsymbol{K}$	$\boldsymbol{g} + \boldsymbol{a} + \boldsymbol{i}$	$\boldsymbol{K}$
$\boldsymbol{A}$	$3\boldsymbol{a}$	$2\boldsymbol{K} + \boldsymbol{H} - \boldsymbol{I} - \boldsymbol{J}$
$\boldsymbol{G}$	$\boldsymbol{i} + \boldsymbol{h} - \boldsymbol{a}$	$\boldsymbol{I} + \boldsymbol{J} - \boldsymbol{K}$

Table 5: Symmetry 9 hexagon areas.

5.3 Hexagons from stars

Figure 5 show the disposition of the symmetry 9 four stars. We label the 18 vertices of stars $\{\mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}\}$ as $\{G_0, G_1, ..., G_{17}\}$, $\{H_0, H_1, ..., H_{17}\}$, $\{I_0, I_1, ..., I_{17}\}$ and $\{I_0, G_1, ..., I_{17}\}$ respectivelly. For simplicity, only some vertices are show. First we make coincident at vertice O all the vertices G_0, H_0, I_0, J_0. With the center at O we rotate all stars to make coincidents G_{17}, H_{17}, I_{17} and J_{17}. The rotations also joined another different vertices.

First we add three new edges (in red) joining the stars $\mathcal{J}$ and $\mathcal{I}$ vertices: $\overline{J_3I_2}$, $\overline{J_5I_4}$ and $\overline{J_7I_6}$ dissecting the red region into four hexagons, two of them essentially different. The three consective angles of the two hexagons are shown: **I (1,4,4)** and **K (3,4,2)**.

Then we add three new edges (in orange) joining the stars $\mathcal{I}$ and $\mathcal{H}$ vertices: $\overline{I_3H_2}$, $\overline{I_5H_4}$ and $\overline{I_7H_6}$ dissecting the orange region into four hexagons, two of them new. The three consective angles of the the two hexagons are show: **H (1,5,3)** and **A (3,3,3)**.

Finally we add three more edges (in green) joining the stars $\mathcal{H}$ and $\mathcal{G}$ vertices: $\overline{H_3G_2}$, $\overline{H_5G_4}$ and $\overline{H_7G_6}$ dissecting the green region into four hexagons, two of them new. The three consective angles of the the two hexagons are show: **G (1,6,2)** and **J (2,5,2)**.

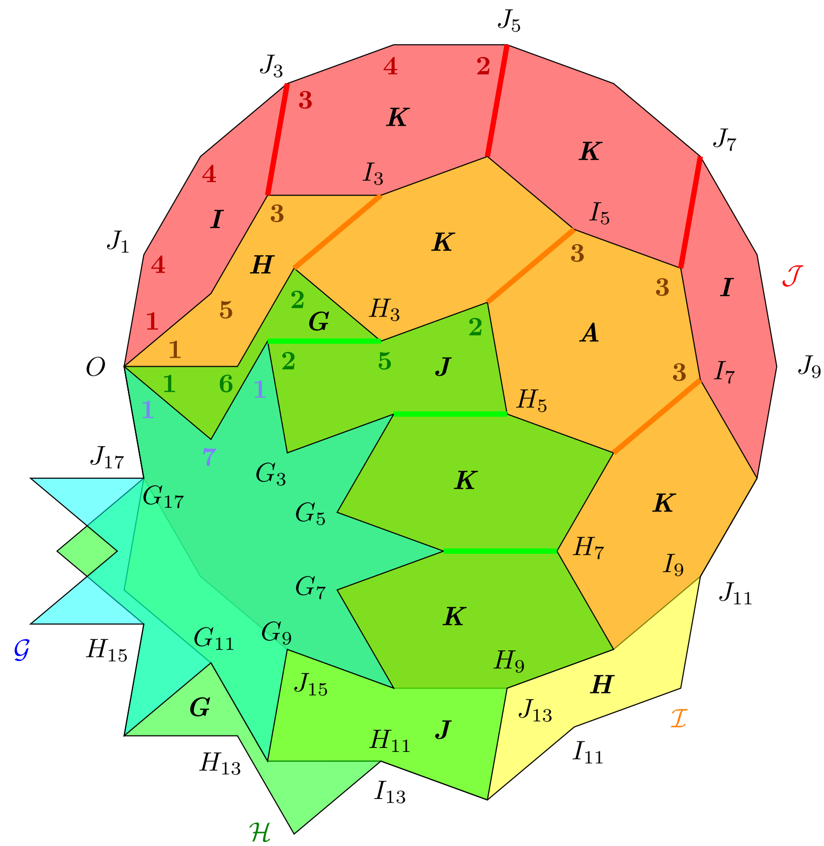

Figure 5: Symmetry 9 stars $\{\mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}\}$ dissected to get the six hexagons $\{\boldsymbol{G}, \boldsymbol{H}, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}, \boldsymbol{A}\}$.

6 Octagons

6.1 Octagons by stars

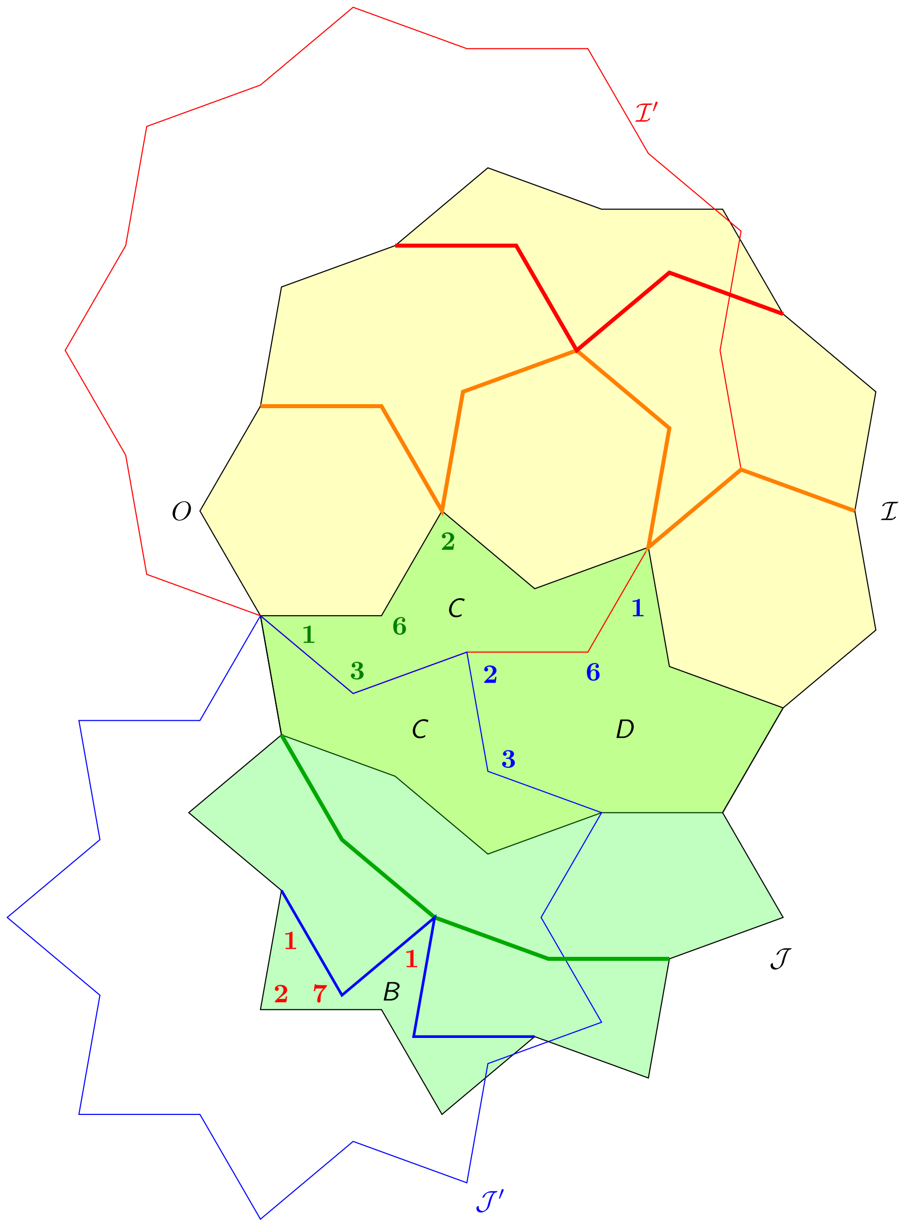

Figure 6: Octagons after intersection of stars $\mathcal{I}$ and $\mathcal{J}$.