Symmetry 9

https://github.com/heptagons/lenses

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Abstract

Symmetry 9

1 Rhombi

Rhombus	Name	θ_1	θ_2	Area
$R_3(\frac{1}{2},1)$	$\boldsymbol{a}$	$\alpha/2$	$2\alpha/2$	$\sin(\alpha) \approx 0.866$
$R_5(\frac{1}{2},2)$	$\boldsymbol{b}$	$\beta/2$	$4\beta/2$	$\sin(2\beta) \approx 0.587$
$R_5(1,\frac{3}{2})$	$\boldsymbol{c}$	$2\beta/2$	$3\beta/2$	$\sin(\beta) \approx 0.951$
$R_7(\frac{1}{2},3)$	$\boldsymbol{d}$	$\gamma/2$	$6\gamma/2$	$\sin(3\gamma) \approx 0.433$
$R_7(1,\frac{5}{2})$	$\boldsymbol{e}$	$2\gamma/2$	$5\gamma/2$	$\sin(\gamma) \approx 0.781$
$R_7(\frac{3}{2},2)$	$\boldsymbol{f}$	$3\gamma/2$	$4\gamma/2$	$\sin(2\gamma) \approx 0.974$
$R_9(\frac{1}{2},4)$	$\boldsymbol{g}$	$\delta/2$	$8\delta/2$	$\sin(4\delta) \approx 0.342$
$R_9(1,\frac{7}{2})$	$\boldsymbol{h}$	$2\delta/2$	$7\delta/2$	$\sin(\delta) \approx 0.642$
$R_9(\frac{3}{2},3)$	$\boldsymbol{a}$	$3\delta/2$	$6\delta/2$	$\sin(3\delta) \approx 0.866$
$R_9(2,\frac{5}{2})$	$\boldsymbol{i}$	$4\delta/2$	$5\delta/2$	$\sin(2\delta) \approx 0.984$

Table 1: Rhombi $R_m(\omega_1,\omega_2)$ for symmetries $m = \{3,5,7,9\}$ internal angles $\theta_1 < \theta_2$ ($\theta_1 + \theta_2 = \pi$) and areas. $\alpha = 2\pi/3$, $\beta = 2\pi/5$, $\gamma = 2\pi/7$ and $\delta = 2\pi/9$ ($3\delta = \alpha$).

Star	Name	Area	Polygon
$S_3(\frac{1}{2},1)$	-	$6\boldsymbol{a}$	$\|6/2\|$ hexagram
$S_3(1,1)$	$\mathcal{A}$	$3\boldsymbol{a}$	Regular hexagon
$S_5(\frac{1}{2},4)$	-	$5\boldsymbol{b}$	$\|10/4\|$ decagram
$S_5(1,3)$	$\mathcal{B}$	$5\boldsymbol{c}$	$\|(5/2)_\alpha\|$ decagram
$S_5(2,2)$	$\mathcal{C}$	$5(\boldsymbol{c}+\boldsymbol{b})$	Regular decagon
$S_7(\frac{1}{2},6)$	-	$7\boldsymbol{d}$	$\|14/6\|$ 14-gram
$S_7(1,5)$	$\mathcal{D}$	$7\boldsymbol{e}$	$\|(7/4)_\alpha\|$ 14-gram
$S_7(2,4)$	$\mathcal{E}$	$7(\boldsymbol{e}+\boldsymbol{f})$	$\|(7/2)_\alpha\|$ 14-gram
$S_7(3,3)$	$\mathcal{F}$	$7(\boldsymbol{e}+\boldsymbol{f}+\boldsymbol{d})$	Regular 14-gon
$S_9(\frac{1}{2},7)$	-	$9\boldsymbol{g}$	$\|18/8\|$ 18-gram
$S_9(1,6)$	$\mathcal{G}$	$9\boldsymbol{h}$	$\|(9/6)_\alpha\|$ 18-gram
$S_9(2,5)$	$\mathcal{H}$	$9(\boldsymbol{h}+\boldsymbol{i})$	$\|(9/4)_\alpha\|$ 18-gram
$S_9(3,4)$	$\mathcal{I}$	$9(\boldsymbol{h}+\boldsymbol{i}+\boldsymbol{a})$	$\|(9/2)_\alpha\|$ 18-gram
$S_9(4,4)$	$\mathcal{J}$	$9(\boldsymbol{h}+\boldsymbol{i}+\boldsymbol{a}+\boldsymbol{g})$	Regular 18-gon

Table 2: Stars $S_m(\omega_1, \omega_2)$ for symmetries $m = \{3, 5, 7, 9\}$. We use the names $\{\mathcal{A}, \mathcal{B}, ...\mathcal{J}\}$ when both ω_1 and ω_2 are integers.

1.1 Stars from rhombi

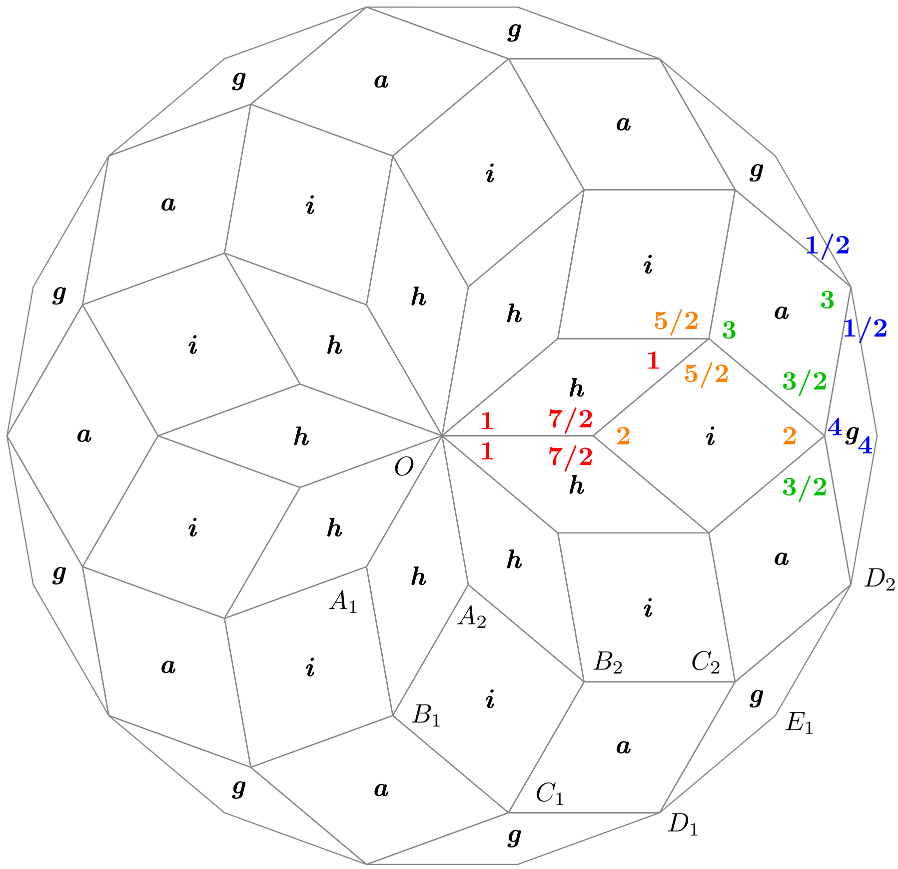

Figure 1: The symmetry 9 four rhombi $\{\boldsymbol{h}, \boldsymbol{i}, \boldsymbol{a}, \boldsymbol{g}\}$ produce the four stars $\{\mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}\}$ with areas $9\boldsymbol{h}$, $9(\boldsymbol{h}+\boldsymbol{i})$, $9(\boldsymbol{h}+\boldsymbol{i}+\boldsymbol{a})$ and $9(\boldsymbol{h}+\boldsymbol{i}+\boldsymbol{a}+\boldsymbol{g})$ respectivelly.

Figure 1 show nine copies of symmetry-9 rhombi $\{\boldsymbol{h}, \boldsymbol{i}, \boldsymbol{a}, \boldsymbol{g}\}$ to form four stars.

2 Hexagons

2.1 Hexagons angles

Hexagon	Name	$(\omega_1, \omega_2, \omega_3)$	Polygon
$H_3(1,1)$	$\boldsymbol{A}$	$(\mathbf{1},\ \mathbf{1},\ \mathbf{1})$	Regular hexagon
$H_5(1,1)$	$\boldsymbol{B}$	$(\mathbf{1},\ \mathbf{1},\ \mathbf{3})$	Sormeh Dan Girih tile
$H_5(1,2)$	$\boldsymbol{C}$	$(\mathbf{1},\ \mathbf{2},\ \mathbf{2})$	Shesh Band Girih tite
$H_7(1,1)$	-	$(\mathbf{1},\ \mathbf{1},\ \mathbf{5})$	self-intersecting
$H_7(1,2)$	$\boldsymbol{D}$	$(\mathbf{1},\ \mathbf{2},\ \mathbf{4})$	
$H_7(1,3)$	$\boldsymbol{E}$	$(\mathbf{1},\ \mathbf{3},\ \mathbf{3})$	
$H_7(2,2)$	$\boldsymbol{F}$	$(\mathbf{2},\ \mathbf{2},\ \mathbf{3})$	
$H_9(1,1)$	-	$(\mathbf{1},\ \mathbf{1},\ \mathbf{7})$	self-intersecting
$H_9(1,2)$	$\boldsymbol{G}$	$(\mathbf{1},\ \mathbf{2},\ \mathbf{6})$	
$H_9(1,3)$	$\boldsymbol{H}$	$(\mathbf{1},\ \mathbf{3},\ \mathbf{5})$	
$H_9(1,4)$	$\boldsymbol{I}$	$(\mathbf{1},\ \mathbf{4},\ \mathbf{4})$	
$H_9(2,2)$	$\boldsymbol{J}$	$(\mathbf{2},\ \mathbf{2},\ \mathbf{5})$	
$H_9(2,3)$	$\boldsymbol{K}$	$(\mathbf{2},\ \mathbf{3},\ \mathbf{4})$	
$H_9(3,3)$	$\boldsymbol{A}$	$(\mathbf{3},\ \mathbf{3},\ \mathbf{3})$	equivalent to $H_3(1,1)$

Table 3: Hexagons $H_m(\omega_1, \omega_2)$ for symmetries $m = \{3, 5, 7, 9\}$.

Figure 3 show the hexagons defined as $H_m(\omega_1, \omega_2)$ for symmetries $\{3, 5, 7, 9\}$. Always $\omega_1 \le \omega_2 \le \omega_3$ and $\omega_1 + \omega_2 + \omega_3 = m$. The six consecutive angles of the hexagons are $(\omega_1, \omega_2, \omega_3, \omega_1, \omega_2, \omega_3)$.

2.2 Hexagons areas

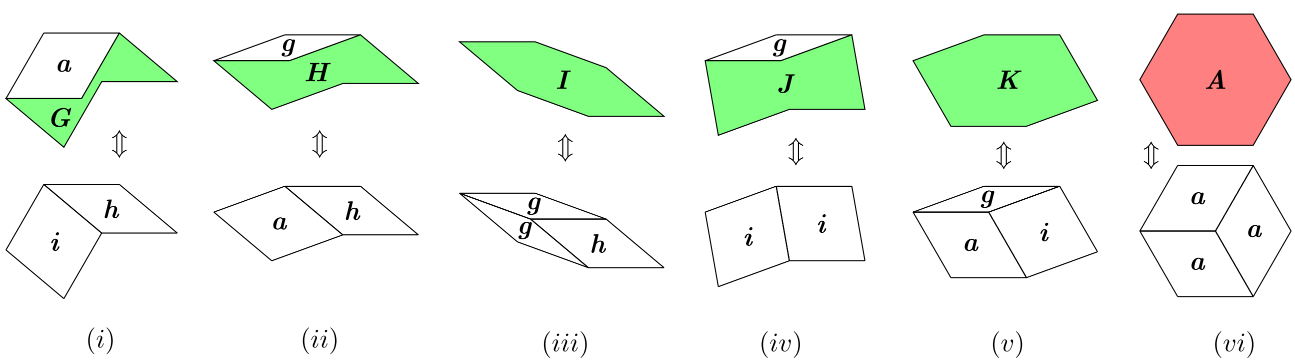

Figure 2: Hexagons formed adding and substracting rhombi.

Figure 2 show how to calculate the area of the symmetry 9 hexagons in function of the symmetry 9 rhombi. From (i) to (vi) we equate the area of sum of the polygons in the top with the area of the sum of the polygons of the bottom:

$$\boldsymbol{a} + \boldsymbol{G} = \boldsymbol{i} + \boldsymbol{h} \tag{1}$$
$$\boldsymbol{g} + \boldsymbol{H} = \boldsymbol{a} + \boldsymbol{h} \tag{2}$$
$$\boldsymbol{I} = 2\boldsymbol{g} + \boldsymbol{h} \tag{3}$$
$$\boldsymbol{g} + \boldsymbol{J} = 2\boldsymbol{i} \tag{4}$$
$$\boldsymbol{K} = \boldsymbol{g} + \boldsymbol{a} + \boldsymbol{i} \tag{5}$$
$$\boldsymbol{A} = 3\boldsymbol{a} \tag{6}$$

Using the last equations we form the table 4

Hexagon	$\boldsymbol{g,h,a,i}$ area	$\boldsymbol{H,I,J,K}$ area
$\boldsymbol{G}$	$\boldsymbol{i} + \boldsymbol{h} - \boldsymbol{a}$	$\boldsymbol{I} + \boldsymbol{J} - \boldsymbol{K}$
$\boldsymbol{H}$	$\boldsymbol{a} + \boldsymbol{h} - \boldsymbol{g}$	$\boldsymbol{H}$
$\boldsymbol{I}$	$2\boldsymbol{g} + \boldsymbol{h}$	$\boldsymbol{I}$
$\boldsymbol{J}$	$2\boldsymbol{i} - \boldsymbol{g}$	$\boldsymbol{J}$
$\boldsymbol{K}$	$\boldsymbol{g} + \boldsymbol{a} + \boldsymbol{i}$	$\boldsymbol{K}$
$\boldsymbol{A}$	$3\boldsymbol{a}$	$2\boldsymbol{K} + \boldsymbol{H} - \boldsymbol{I} - \boldsymbol{J}$

Table 4: Symmetry 9 hexagons areas in function of rhombi $\boldsymbol{g,h,a,i}$ and hexagons $\boldsymbol{H,I,J,K}$.

2.3 Hexagons from stars

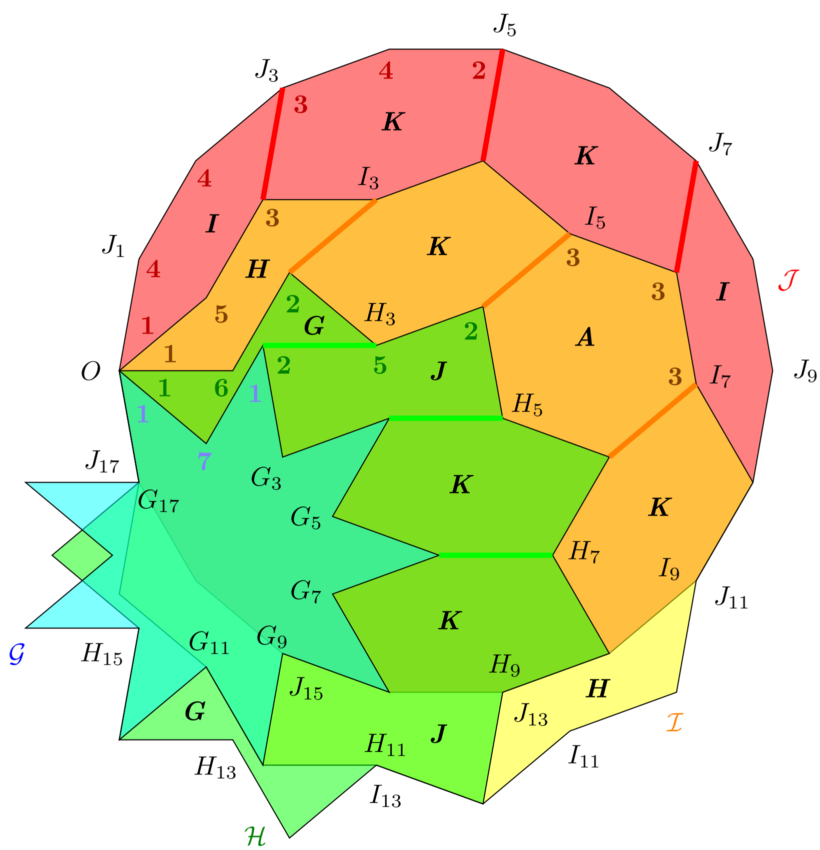

Figure 3: Symmetry 9 stars $\{\mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}\}$ dissected to get the six hexagons $\{\boldsymbol{G}, \boldsymbol{H}, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}, \boldsymbol{A}\}$.

Figure 3 show the disposition of the symmetry 9 four stars. We denote the 18 vertices of stars $\{\mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}\}$ as $\{G_0, G_1, ..., G_{17}\}$, $\{H_0, H_1, ..., H_{17}\}$, $\{I_0, I_1, ..., I_{17}\}$ and $\{I_0, G_1, ..., I_{17}\}$ respectivelly. For simplification only some vertices are labeled in the figure. First we make coincident at vertice O all the vertices G_0, H_0, I_0, J_0. With the center at O we rotate all stars to make coincidents G_{17}, H_{17}, I_{17} and J_{17}. The rotations also joined another different vertices.

First we add three new edges (in red) joining the stars $\mathcal{J}$ and $\mathcal{I}$ vertices: $\overline{J_3 I_2}$, $\overline{J_5 I_4}$ and $\overline{J_7 I_6}$ dissecting the red region into four hexagons, two of them essentially different. The three consective angles of the two hexagons are shown: **I (1,4,4)** and **K (3,4,2)**.

Then we add three new edges (in orange) joining the stars $\mathcal{I}$ and $\mathcal{H}$ vertices: $\overline{I_3 H_2}$, $\overline{I_5 H_4}$ and $\overline{I_7 H_6}$ dissecting the orange region into four hexagons, two of them new. The three consective angles of the the two hexagons are show: **H (1,5,3)** and **A (3,3,3)**.

Finally we add three more edges (in green) joining the stars $\mathcal{H}$ and $\mathcal{G}$ vertices: $\overline{H_3 G_2}$, $\overline{H_5 G_4}$ and $\overline{H_7 G_6}$ dissecting the green region into four hexagons, two of them new. The three consective angles of the the two hexagons are show: **G (1,6,2)** and **J (2,5,2)**.