Lenses

https://github.com/heptagons/lenses

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Abstract

Lenses are equilateral hexagons resembling concave and convex optical lenses. Lenses consecutive six internal angles are $(\theta_1, \theta_2, \theta_3, \theta_1, \theta_2, \theta_3)$ where $\theta_1 = X\theta_0$, $\theta_2 = Y\theta_0$, and $\theta_3 = Z\theta_0$ where $\theta_0 = 2\pi/S$ is the base angle of symmetry $S = X + Y + Z$.

1 Lenses

2 Stars

3 Symmetry 5

Symmetry 5 is based in angle $\beta = \dfrac{2\pi}{5}$ and produces the two rhombi $(\boldsymbol{b}, \boldsymbol{c})$ and the two lenses $(\boldsymbol{B}, \boldsymbol{C})$.

3.1 Rhombi $(\boldsymbol{b},\boldsymbol{c})$

Rhombus	θ_1	θ_2
$\boldsymbol{b}$	$\beta/2$	$4\beta/2$
$\boldsymbol{c}$	$2\beta/2$	$3\beta/2$

Table 1: Rhombi $(\boldsymbol{b}, \boldsymbol{c})$ internal angles. $\theta_1 + \theta_2 = \pi$ and $\beta = 2\pi/5$.

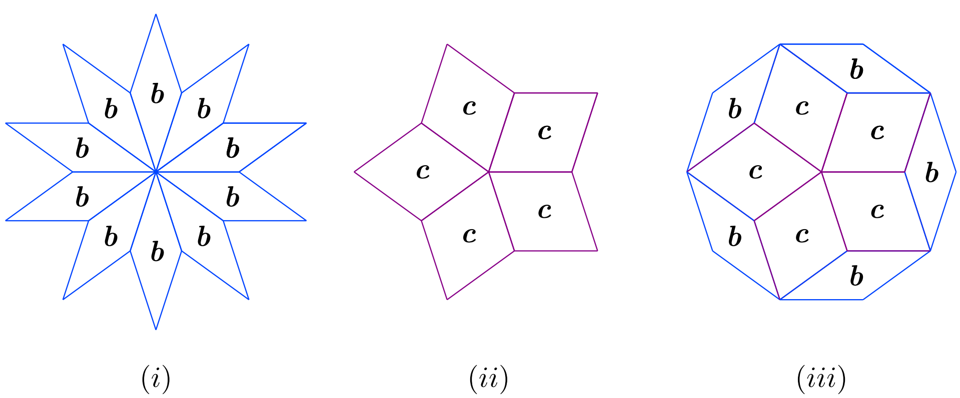


Figure 1: Rhombi $(\boldsymbol{b}, \boldsymbol{c})$ from dissecting stars S_{10}.

Table 1 show the rhombi $(\boldsymbol{b}, \boldsymbol{c})$ internal angles in terms of angle $\beta = 2\pi/5$. Figure 1 show the rhombi $(\boldsymbol{b}, \boldsymbol{c})$. Inspecting the stars we get the areas simply adding their rhombi. At (i) the star $S_{10}(1, 8)$ with area

$A = 10\boldsymbol{b}$. At (ii) the star $S_{10}(2,6) = S_5(1,3)$ with area $A = 5\boldsymbol{c}$. At (iii) the regular decagon equivalvent to stars $S_{10}(4,4) = S_5(2,2)$ with area $A = 5\boldsymbol{b} + 5\boldsymbol{c}$.

3.2 Regular pentagon and star $|5/2|$

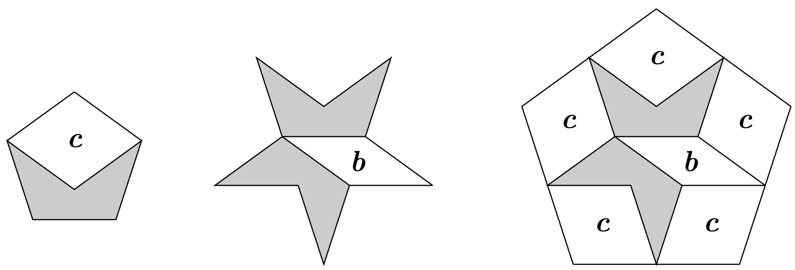


Figure 2: Regular pentagon $|5/1|$ at (i). Star $|5/2|$ at (ii). Double pentagon at (iii).

Figure 2 show regular pentagon and isotoxal star $|5/2|$ dissected with rhombi $(\boldsymbol{b}, \boldsymbol{c})$ plus concave pentagons (in gray). Let $\boldsymbol{x}$ be the area of such gray piece. By inspection the area of regular pentagon at (i) is $A_1 = \boldsymbol{c} + \boldsymbol{x}$ and the area of regular pentagon at (iii) is $P_2 = \boldsymbol{b} + 5\boldsymbol{c} + 2\boldsymbol{x}$. Since the side of A_2 is the double of A_1 its area is four times so we can get the value of $\boldsymbol{x}$ in terms of $(\boldsymbol{b}, \boldsymbol{c})$

$$\begin{aligned} 4P_1 &= P_4 \\ 4(\boldsymbol{c} + x) &= \boldsymbol{b} + 5\boldsymbol{c} + 2x \\ x &= \frac{\boldsymbol{b} + \boldsymbol{c}}{2} \end{aligned} \tag{1}$$

We use the value of $\boldsymbol{x}$ to get the areas of pentagon (i) and star (ii):

$$\begin{aligned} A|5/1| &= \boldsymbol{c} + \boldsymbol{x} \\ &= \frac{\boldsymbol{b} + 3\boldsymbol{c}}{2} \end{aligned} \tag{2}$$

$$\begin{aligned} A|5/2| &= \boldsymbol{b} + 2\boldsymbol{x} \\ &= 2\boldsymbol{b} + \boldsymbol{c} \end{aligned} \tag{3}$$

3.3 Lenses $(\boldsymbol{B},\boldsymbol{C})$

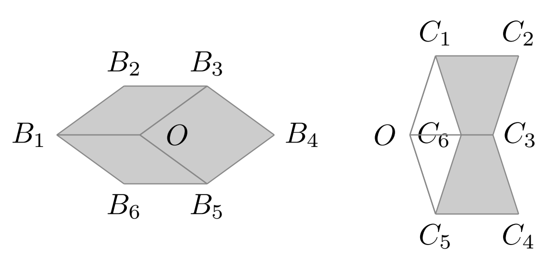


Figure 3: Lenses $(\boldsymbol{B},\ \boldsymbol{C})$ build with rhombi $(\boldsymbol{b},\ \boldsymbol{c})$.

Figure 3 show lenses $(\boldsymbol{B}, \boldsymbol{C})$ construction and two stars formed with them. At (i) we form the lense $\boldsymbol{B}$ with perimeter $\overline{B_1...B_6}$ adding two rhombi $\boldsymbol{b}$ $(\overline{B_1B_2B_3O}$ and $\overline{B_1OB_5B_6})$ and adding one rhombus $\boldsymbol{c}$ $(\overline{OB_3B_4B_5})$

so its area is $2\boldsymbol{b}+\boldsymbol{c}$. Lense $\boldsymbol{B}$ is equivalent to the hexagon $H_5(1,2,2)$. At (ii) we form the lense $\boldsymbol{C}$ with perimeter $\overline{C_1...C_6}$ adding two rhombi $\boldsymbol{c}$ ($\overline{OC_1C_2C3}$ and $\overline{OC_3C_4C_5}$) and substracting one rhombus $\boldsymbol{b}$ ($\overline{OC_1C_6C_5}$) so its area is $2\boldsymbol{c}-\boldsymbol{b}$. Lense $\boldsymbol{C}$ is equivalent to the hexagon $H_5(1,1,3)$. Table 2 show the lenses ($\boldsymbol{B}$, $\boldsymbol{C}$) internal angles and areas.

Lense	θ_1	θ_2	θ_3	Area
$\boldsymbol{B}$	β	2β	2β	$2\boldsymbol{b}+\boldsymbol{c}$
$\boldsymbol{C}$	β	β	3β	$-\boldsymbol{b}+2\boldsymbol{c}$

Table 2: Lenses ($\boldsymbol{B}, \boldsymbol{C}$) internal angles and areas in terms of rhombi ($\boldsymbol{b}, \boldsymbol{c}$). $\theta_1+\theta_2+\theta_3=2\pi$ where $\beta=2\pi/5$.

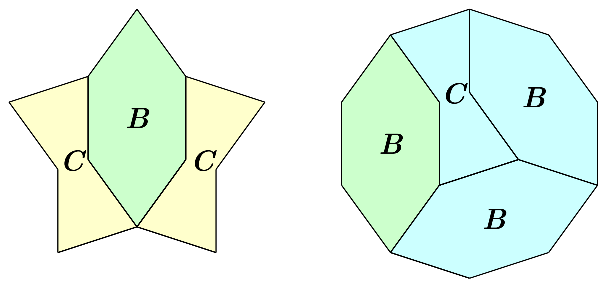


Figure 4: Two stars dissected with lenses ($\boldsymbol{B}$, $\boldsymbol{C}$).

Figure 4 show two stars dissected with lenses ($\boldsymbol{B}$, $\boldsymbol{C}$). At (i) the star $S_5(1,3)$ dissection implies its area is $A=\boldsymbol{B}+2\boldsymbol{C}=5\boldsymbol{c}$. At (ii) the regular decagon or star $S_5(2,2)$ dissection implies its area is $A=3\boldsymbol{B}+\boldsymbol{C}=5(\boldsymbol{b}+\boldsymbol{c})$.

4 Symmetry 7

Symmetry 7 is based in angle $\gamma=\dfrac{2\pi}{7}$ and produces the three rhombi ($\boldsymbol{d}, \boldsymbol{f}, \boldsymbol{e}$) and the three lenses ($\boldsymbol{D}, \boldsymbol{E}, \boldsymbol{F}$).

4.1 Rhombi ($\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$)

Rhombus	θ_1	θ_2
$\boldsymbol{d}$	$\gamma/2$	$6\gamma/2$
$\boldsymbol{e}$	$2\gamma/2$	$5\gamma/2$
$\boldsymbol{f}$	$3\gamma/2$	$4\gamma/2$

Table 3: Rhombi ($\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$) internal angles. $\theta_1+\theta_2=\pi$ and $\gamma=2\pi/7$.

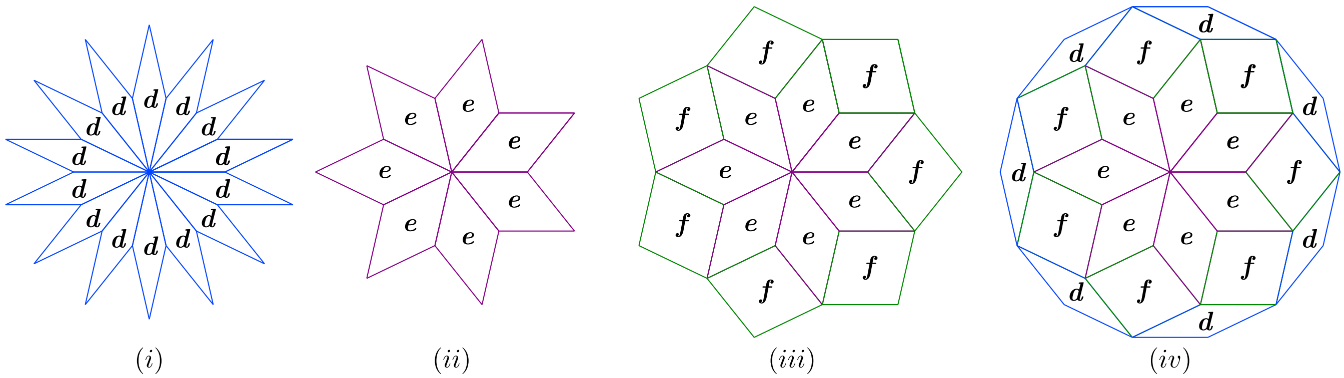


Figure 5: Rhombi $(\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f})$ from dissected stars S_{14}.

Table 3 show the symmetry 7 lenses internal angles based in angle $\gamma = 2\pi/7$. Figure 5 show rhombi $(\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f})$. Inspecting the stars we get the areas simply adding their rhombi. At (i) the star $S_{14}(1, 12)$ with area $A = 14\boldsymbol{d}$. At (ii) the star $S_{14}(2, 10) = S_7(1, 5)$ with area $A = 7\boldsymbol{e}$. At (iii) the star $S_{14}(4, 8) = S_7(2, 4)$ with area $A = 7(\boldsymbol{e} + \boldsymbol{f})$. At (iv) the regular 14-gon equivalent to stars $S_{14}(6, 6) = S_7(3, 3)$ with area $A = 7(\boldsymbol{d} + \boldsymbol{e} + \boldsymbol{f})$.

4.2 Regular heptagon and stars $|7/3|$ and $|7/2|$

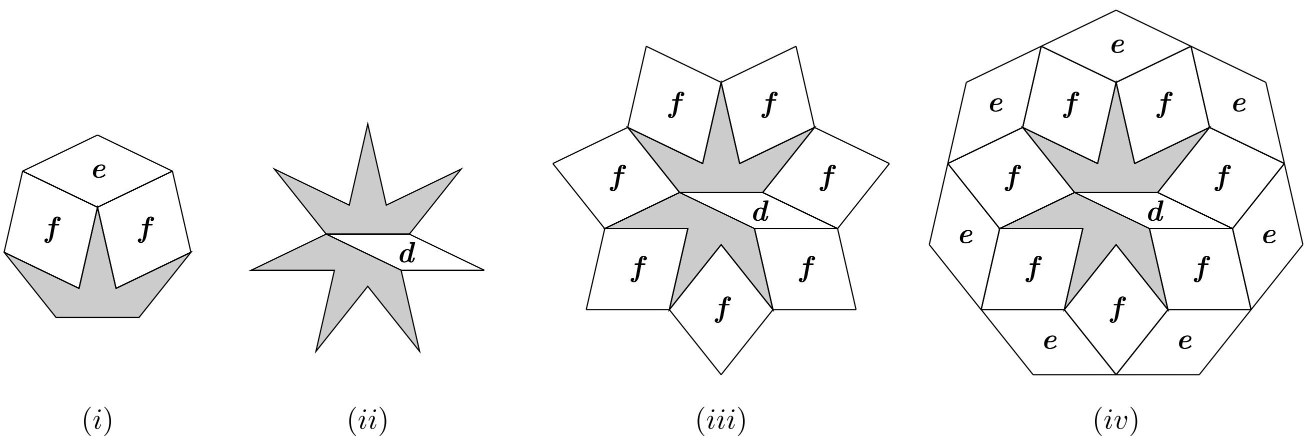


Figure 6: Heptagon $|7/1|$ at (i). Star $|7/3|$ at (ii). Star $|7/2|$ at (iii). Double heptagon at (iv).

Figure 6 show regular heptagon and heptagrams dissected with rhombi $(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e})$ plus equilateral concave heptagons (in gray). Let $\boldsymbol{y}$ be the area of such gray piece. By inspection the area of regular heptagon at (i) is $A_1 = \boldsymbol{e} + 2\boldsymbol{f} + \boldsymbol{y}$ while the area of regular heptagon at (iv) is $A_2 = \boldsymbol{d} + 7(\boldsymbol{e} + \boldsymbol{f}) + 2\boldsymbol{y}$. Since the side of A_2 is the double of A_1 its area is four times so we can get the value of $\boldsymbol{y}$ in terms of $(\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f})$:

$$
\begin{aligned}
4A_1 &= A_2 \\
4(\boldsymbol{e} + 2\boldsymbol{f} + \boldsymbol{y}) &= \boldsymbol{d} + 7(\boldsymbol{e} + \boldsymbol{f}) + 2\boldsymbol{y} \\
\boldsymbol{y} &= \frac{\boldsymbol{d} + 3\boldsymbol{e} - \boldsymbol{f}}{2} \qquad (4)
\end{aligned}
$$

We use the value of $\boldsymbol{y}$ to calculate the areas of heptagon (i) and stars (ii) and (iii) in terms of $(\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f})$:

$$\begin{aligned} A|7/1| &= \boldsymbol{e} + 2\boldsymbol{f} + \boldsymbol{y} \\ &= \frac{\boldsymbol{d} + 5\boldsymbol{e} + 3\boldsymbol{f}}{2} \qquad (5) \\ A|7/3| &= \boldsymbol{d} + 2\boldsymbol{y} \\ &= 2\boldsymbol{d} + 3\boldsymbol{e} - \boldsymbol{f} \qquad (6) \\ A|7/2| &= A\{7/3\} + 7\boldsymbol{f} \\ &= 2\boldsymbol{d} + 3\boldsymbol{e} + 6\boldsymbol{f} \qquad (7) \end{aligned}$$

4.3 Lenses $(\boldsymbol{D},\boldsymbol{E},\boldsymbol{F})$

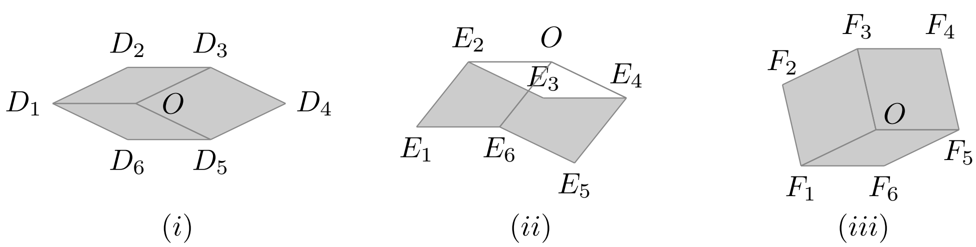


Figure 7: Lenses $(\boldsymbol{D},\boldsymbol{E},\boldsymbol{F})$ build from rhombi $(\boldsymbol{d},\boldsymbol{e},\boldsymbol{f})$.

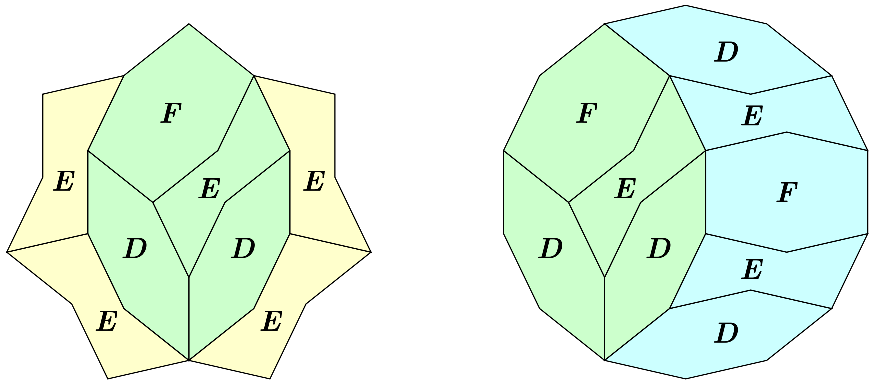
Figure 7 show lenses $(\boldsymbol{B}, \boldsymbol{C})$ construction. At (i) we form the lense $\boldsymbol{D}$ with perimeter $\overline{D_1...D_6}$ adding two rhombi $\boldsymbol{d}$ ($\overline{D_1D_2D_3O}$ and $\overline{D_1OD_5D_6}$) and adding one rhombus $\boldsymbol{e}$ ($\overline{OD_3D_4D_5}$) so its area is $2\boldsymbol{d} + \boldsymbol{e}$. Lense $\boldsymbol{D}$ is equivalent to the hexagon $H_7(1, 3, 3)$.

At (ii) we form the lense $\boldsymbol{E}$ with perimeter $\overline{E_1...E_6}$ adding one rhombus $\boldsymbol{e}$ ($\overline{E_1E_2OE_6}$) adding one rhombus $\boldsymbol{f}$ ($\overline{OE_4E_5E_6}$) and substracting one rhombus $\boldsymbol{d}$ ($\overline{E_2OE_4E_3}$) so its area is $-\boldsymbol{d} + \boldsymbol{e} + \boldsymbol{f}$. Lense $\boldsymbol{E}$ is equivalent to the hexagon $H_7(1, 2, 4)$.

At (iii) we form the lense $\boldsymbol{F}$ with perimeter $\overline{F_1...F_6}$ adding two rhombi $\boldsymbol{f}$ ($\overline{F_1F_2F_3O}$ and $\overline{F_3F_4F_5O}$) and adding one rhombus $\boldsymbol{d}$ ($\overline{F_1OF_5F_6}$) so its area is $\boldsymbol{d} + 2\boldsymbol{f}$. Lense $\boldsymbol{F}$ is equivalent to the hexagon $H_7(2, 2, 3)$. Table 4 show the lenses $(\boldsymbol{D}, \boldsymbol{E}, \boldsymbol{F})$ internal angles and areas.

Lense	θ_1	θ_2	θ_3	Area
$\boldsymbol{D}$	γ	3γ	3γ	$2\boldsymbol{d} + \boldsymbol{e}$
$\boldsymbol{E}$	γ	2γ	4γ	$-\boldsymbol{d} + \boldsymbol{e} + \boldsymbol{f}$
$\boldsymbol{F}$	2γ	2γ	3γ	$-\boldsymbol{d} + 2\boldsymbol{f}$

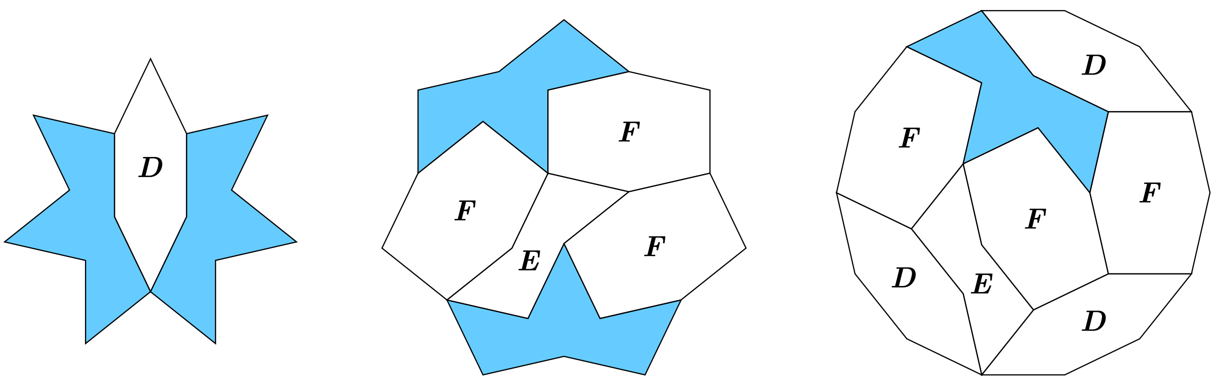
Table 4: Lenses $(\boldsymbol{D}, \boldsymbol{E}, \boldsymbol{F})$ internal angles and areas in terms of rhombi $(\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f})$. $\theta_1 + \theta_2 + \theta_3 = 2\pi$ and $\gamma = 2\pi/7$.



(i) (ii)

Figure 8: Stars dissected with only lenses ($\boldsymbol{D}$,$\boldsymbol{E}$,$\boldsymbol{F}$).

Figure 8 show stars $S_7(2,4)$ and $S_7(3,3)$ dissected with lenses ($\boldsymbol{D}$,$\boldsymbol{E}$,$\boldsymbol{F}$). At (i) we have star $S_7(2,4)$ and by inspection we deduce its area is $A = 2\boldsymbol{D} + 5\boldsymbol{E} + \boldsymbol{F}$. At (ii) we have regular 14-gon (or star $S_7(3,3)$) and by inspection we deduce its area is $4\boldsymbol{D} + 3\boldsymbol{E} + 2\boldsymbol{F}$. Both stars have in common an area in green resembling a tree leaf. The star at (i) also contains two regions in yellow resembling crowns while the star at (ii) contains a region in cyan resembling a moon phase.



(i) (ii) (iii)

Figure 9: Stars dissected with octagons O_7 (in blue) and lenses ($\boldsymbol{D}$,$\boldsymbol{E}$,$\boldsymbol{F}$).

Figure 9 show Stars $S_7(1,5)$, $S_7(2,4)$ and $S_7(3,3)$ dissected with octagons O_7 (in blue) and lenses ($\boldsymbol{D}$,$\boldsymbol{E}$,$\boldsymbol{F}$). At (i) we have the star $S_7(1,5)$ and by inspection we deduce its area is $A = \boldsymbol{D} + 2O_7$. At (ii) we have the star $S_7(2,4)$ and we can conclude its area is $\boldsymbol{E} + 3\boldsymbol{F} + 2O_7$. Similarly the area of the 14-gon at (iii) is $3\boldsymbol{D} + \boldsymbol{E} + 3\boldsymbol{F} + O_7$. Comparing the areas of the two 14-gons of figures 8 and 9 we can find the area of O_7 in terms of ($\boldsymbol{E}$, $\boldsymbol{F}$, $\boldsymbol{G}$):

$$
\begin{aligned}
4\boldsymbol{D} + 3\boldsymbol{E} + 2\boldsymbol{F} &= 3\boldsymbol{D} + \boldsymbol{E} + 3\boldsymbol{F} + O_7 \\
O_7 &= \boldsymbol{D} + 2\boldsymbol{E} - \boldsymbol{F} \qquad (8)
\end{aligned}
$$

So we can calculate the area of star $S_7(1,5)$ in terms of ($\boldsymbol{E}$, $\boldsymbol{F}$, $\boldsymbol{G}$):

$$
\begin{aligned}
S_7(1,5) &= \boldsymbol{D} + 2(\boldsymbol{D} + 2\boldsymbol{E} - \boldsymbol{F}) \\
&= 3\boldsymbol{D} + 4\boldsymbol{E} - 2\boldsymbol{F} \qquad (9)
\end{aligned}
$$