

# Lenses

<https://github.com/heptagons/lenses>

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## Abstract

Lenses are equilateral hexagons resembling concave and convex optical lenses. Lenses consecutive six internal angles are  $(\theta_1, \theta_2, \theta_3, \theta_1, \theta_2, \theta_3)$  where  $\theta_1 = X\theta_0$ ,  $\theta_2 = Y\theta_0$ , and  $\theta_3 = Z\theta_0$  where  $\theta_0 = 2\pi/S$  is the base angle of symmetry  $S = X + Y + Z$ .

## 1 Stars

Stars are polygons with a pair number of sides and defined by three numbers: A symmetry number and a pair of angles multiples of the symmetry. The symmetry angle is defined as  $\omega = \frac{2\pi}{s}$  where  $s$  is the symmetry number.

## 2 Symmetry 5

Symmetry 5 is based in angle  $\beta = \frac{2\pi}{5}$  and produces the two rhombi  $(b, c)$  and the two lenses  $(B, C)$ .

### 2.1 Rhombi $(b, c)$

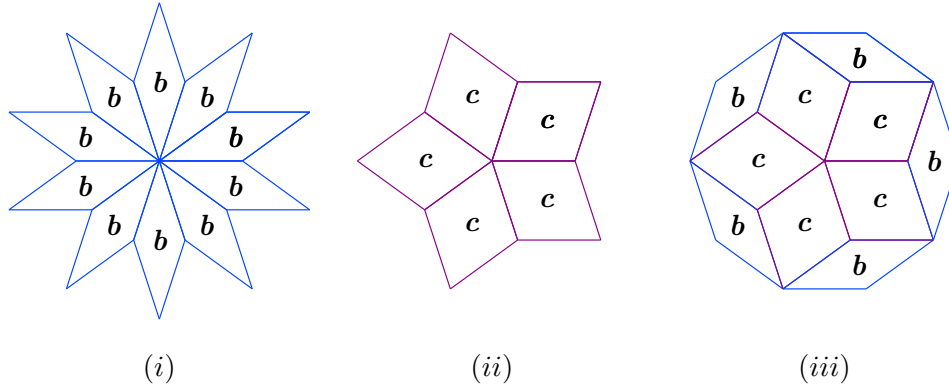


Figure 1: Rhombi  $(b, c)$  from dissecting stars  $S_{10}$ .

Figure 1 show the rhombi  $(b, c)$ . Inspecting the stars we get the areas simply adding their rhombi. At (i) the star  $S_{10}(1, 8)$  with area  $A = 10b$ . At (ii) the star  $S_{10}(2, 6) = S_5(1, 3)$  with area  $A = 5c$ . At (iii) the regular decagon equivalent to stars  $S_{10}(4, 4) = S_5(2, 2)$  with area  $A = 5b + 5c$ . Table 1 show the rhombi  $(b, c)$  internal angles in terms of angle  $\beta = 2\pi/5$  and areas for side equals to 1. Dividing areas we find  $\frac{c}{b} = 2 \cos\left(\frac{\pi}{5}\right) = \frac{\sqrt{5} + 1}{2}$ .

Rhombus	$\theta_1$	$\theta_2$	Area
$\mathbf{b}$	$\beta/2$	$4\beta/2$	$\sin(2\beta) = \frac{\sqrt{10 - 2\sqrt{5}}}{4}$
$\mathbf{c}$	$2\beta/2$	$3\beta/2$	$\sin(\beta) = \frac{\sqrt{10 + 2\sqrt{5}}}{4} = \mathbf{b} \cos\left(\frac{\pi}{5}\right) = \mathbf{b} \left(\frac{\sqrt{5} + 1}{2}\right)$

Table 1: Rhombi ( $\mathbf{b}, \mathbf{c}$ ) internal angles and areas.  $\theta_1 + \theta_2 = \pi$  and  $\beta = 2\pi/5$ .

## 2.2 Regular pentagon and star $|5/2|$

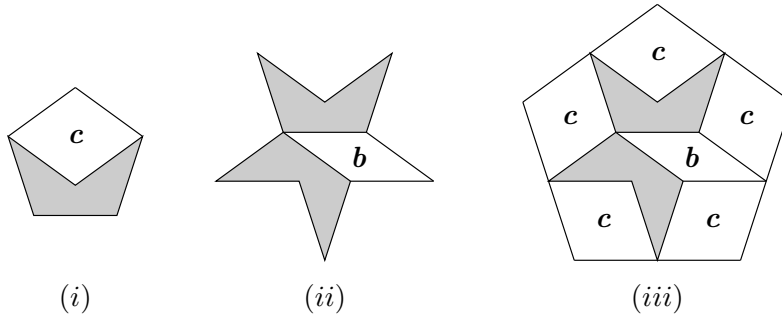


Figure 2: Regular pentagon  $|5/1|$  at (i). Star  $|5/2|$  at (ii). Double pentagon at (iii).

Figure 2 show regular pentagon and isotoxal star  $|5/2|$  dissected with rhombi ( $\mathbf{b}, \mathbf{c}$ ) plus concave pentagons (in gray). Let  $\mathbf{x}$  be the area of such gray piece. By inspection the area of regular pentagon at (i) is  $A_1 = \mathbf{c} + \mathbf{x}$  and the area of regular pentagon at (iii) is  $P_2 = \mathbf{b} + 5\mathbf{c} + 2\mathbf{x}$ . Since the side of  $A_2$  is the double of  $A_1$  its area is four times so we can get the value of  $\mathbf{x}$  in terms of ( $\mathbf{b}, \mathbf{c}$ )

$$\begin{aligned}
4P_1 &= P_4 \\
4(\mathbf{c} + \mathbf{x}) &= \mathbf{b} + 5\mathbf{c} + 2\mathbf{x} \\
\mathbf{x} &= \frac{\mathbf{b} + \mathbf{c}}{2}
\end{aligned} \tag{1}$$

We use the value of  $\mathbf{x}$  to get the areas of pentagon (i) and star (ii):

$$\begin{aligned}
A|5/1| &= \mathbf{c} + \mathbf{x} \\
&= \frac{\mathbf{b} + 3\mathbf{c}}{2}
\end{aligned} \tag{2}$$

$$\begin{aligned}
A|5/2| &= \mathbf{b} + 2\mathbf{x} \\
&= 2\mathbf{b} + \mathbf{c}
\end{aligned} \tag{3}$$

### 2.3 Lenses ( $B, C$ )

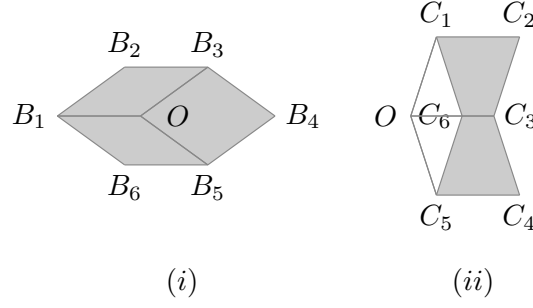


Figure 3: Lenses ( $B, C$ ) build with rhombi ( $b, c$ ).

Figure 3 show lenses ( $B, C$ ) construction and two stars formed with them. At (i) we form the lense  $B$  with perimeter  $\overline{B_1...B_6}$  adding two rhombi  $b$  ( $\overline{B_1B_2B_3O}$  and  $\overline{B_1OB_5B_6}$ ) and adding one rhombus  $c$  ( $\overline{OB_3B_4B_5}$ ) so its area is  $2b + c$ . Lense  $B$  is equivalent to the hexagon  $H_5(1, 2, 2)$ . At (ii) we form the lense  $C$  with perimeter  $\overline{C_1...C_6}$  adding two rhombi  $c$  ( $\overline{OC_1C_2C_3}$  and  $\overline{OC_3C_4C_5}$ ) and subtracting one rhombus  $b$  ( $\overline{OC_1C_6C_5}$ ) so its area is  $2c - b$ . Lense  $C$  is equivalent to the hexagon  $H_5(1, 1, 3)$ . Table 2 show the lenses ( $B, C$ ) internal angles and areas.

Lense	$\theta_1$	$\theta_2$	$\theta_3$	Area
$B$	$\beta$	$2\beta$	$2\beta$	$2b + c$
$C$	$\beta$	$\beta$	$3\beta$	$-b + 2c$

Table 2: Lenses ( $B, C$ ) internal angles and areas in terms of rhombi ( $b, c$ ).  $\theta_1 + \theta_2 + \theta_3 = 2\pi$  where  $\beta = 2\pi/5$ .

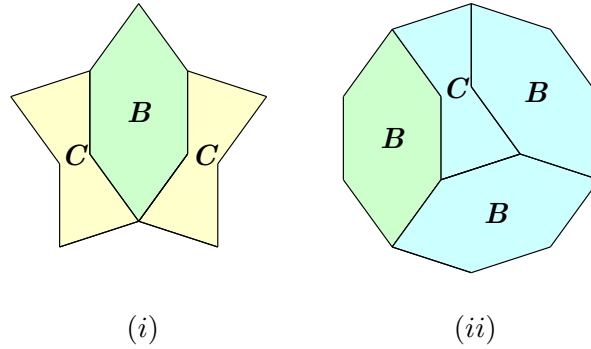


Figure 4: Two stars dissected with lenses ( $B, C$ ).

Figure 4 show two stars dissected with lenses ( $B, C$ ). At (i) the star  $S_5(1, 3)$  dissection implies its area is  $A = B + 2C = 5c$ . At (ii) the regular decagon or star  $S_5(2, 2)$  dissection implies its area is  $A = 3B + C = 5(b + c)$ .

### 3 Symmetry 7

Symmetry 7 is based in angle  $\gamma = \frac{2\pi}{7}$  and produces the three rhombi ( $d, f, e$ ) and the three lenses ( $D, E, F$ ).

### 3.1 Rhombi $(d, e, f)$

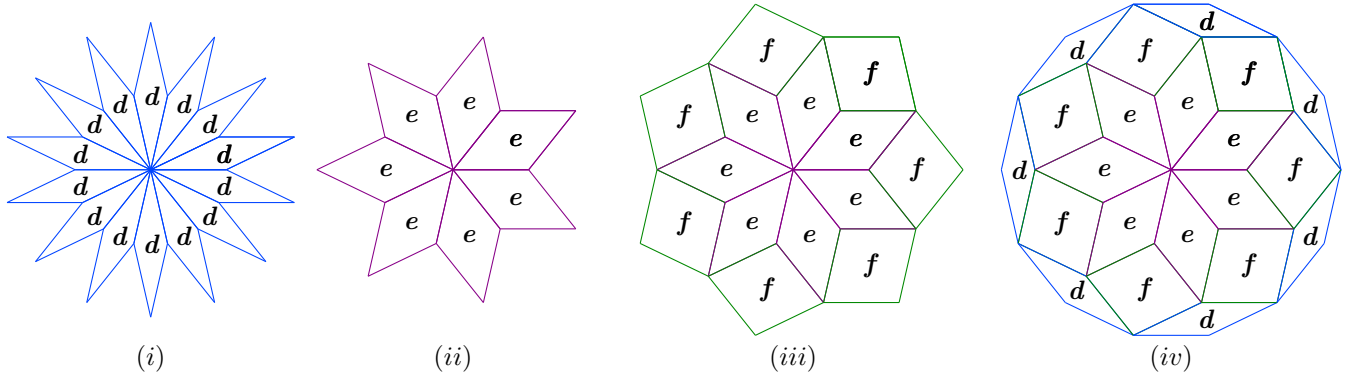


Figure 5: Rhombi  $(d, e, f)$  from dissected stars  $S_{14}$ .

Figure 5 show rhombi  $(d, e, f)$ . Inspecting the stars we get the areas simply adding their rhombi. At (i) the star  $S_{14}(1, 12)$  with area  $A = 14d$ . At (ii) the star  $S_{14}(2, 10) = S_7(1, 5)$  with area  $A = 7e$ . At (iii) the star  $S_{14}(4, 8) = S_7(2, 4)$  with area  $A = 7(e + f)$ . At (iv) the regular 14-gon equivalent to stars  $S_{14}(6, 6) = S_7(3, 3)$  with area  $A = 7(d + e + f)$ . Table 3 show the symmetry 7 lenses internal angles based in angle  $\gamma = 2\pi/7$  and the areas.

Rhombus	$\theta_1$	$\theta_2$	Area
$d$	$\gamma/2$	$6\gamma/2$	$\sin(3\gamma)$
$e$	$2\gamma/2$	$5\gamma/2$	$\sin(\gamma)$
$f$	$3\gamma/2$	$4\gamma/2$	$\sin(2\gamma)$

Table 3: Rhombi  $(d, e, f)$  internal angles.  $\theta_1 + \theta_2 = \pi$  and  $\gamma = 2\pi/7$ .

### 3.2 Regular heptagon and stars $|7/3|$ and $|7/2|$

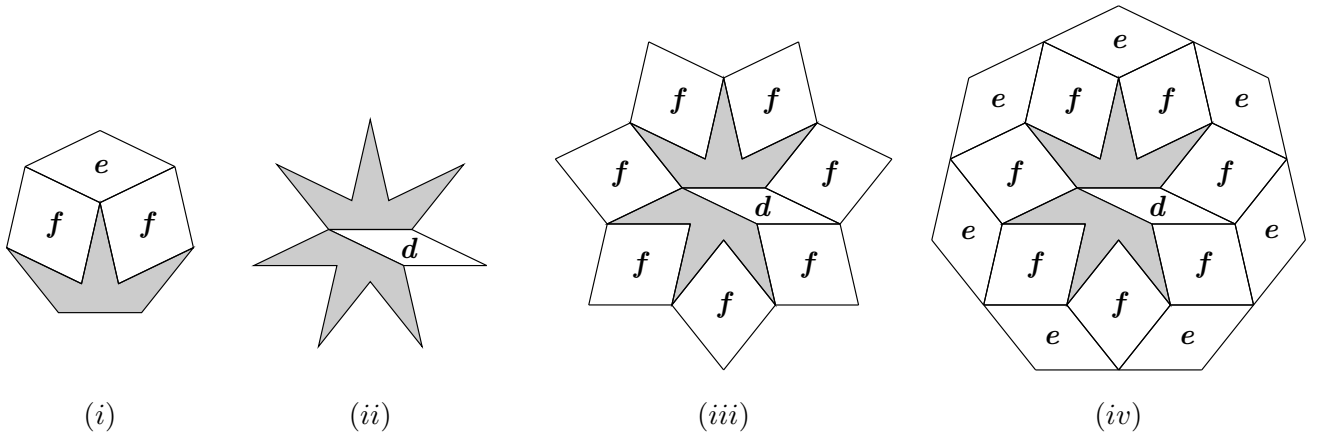


Figure 6: Heptagon  $|7/1|$  at (i). Star  $|7/3|$  at (ii). Star  $|7/2|$  at (iii). Double heptagon at (iv).

Figure 6 show regular heptagon and heptagrams dissected with rhombi  $(c, d, e)$  plus equilateral concave heptagons (in gray). Let  $y$  be the area of such gray piece. By inspection the area of regular heptagon at

(i) is  $A_1 = \mathbf{e} + 2\mathbf{f} + \mathbf{y}$  while the area of regular heptagon at (iv) is  $A_2 = \mathbf{d} + 7(\mathbf{e} + \mathbf{f}) + 2\mathbf{y}$ . Since the side of  $A_2$  is the double of  $A_1$  its area is four times so we can get the value of  $\mathbf{y}$  in terms of  $(\mathbf{d}, \mathbf{e}, \mathbf{f})$ :

$$\begin{aligned} 4A_1 &= A_2 \\ 4(\mathbf{e} + 2\mathbf{f} + \mathbf{y}) &= \mathbf{d} + 7(\mathbf{e} + \mathbf{f}) + 2\mathbf{y} \\ \mathbf{y} &= \frac{\mathbf{d} + 3\mathbf{e} - \mathbf{f}}{2} \end{aligned} \quad (4)$$

We use the value of  $\mathbf{y}$  to calculate the areas of heptagon (i) and stars (ii) and (iii) in terms of  $(\mathbf{d}, \mathbf{e}, \mathbf{f})$ :

$$\begin{aligned} A|7/1| &= \mathbf{e} + 2\mathbf{f} + \mathbf{y} \\ &= \frac{\mathbf{d} + 5\mathbf{e} + 3\mathbf{f}}{2} \end{aligned} \quad (5)$$

$$\begin{aligned} A|7/3| &= \mathbf{d} + 2\mathbf{y} \\ &= 2\mathbf{d} + 3\mathbf{e} - \mathbf{f} \end{aligned} \quad (6)$$

$$\begin{aligned} A|7/2| &= A\{7/3\} + 7\mathbf{f} \\ &= 2\mathbf{d} + 3\mathbf{e} + 6\mathbf{f} \end{aligned} \quad (7)$$

### 3.3 Lenses $(\mathbf{D}, \mathbf{E}, \mathbf{F})$

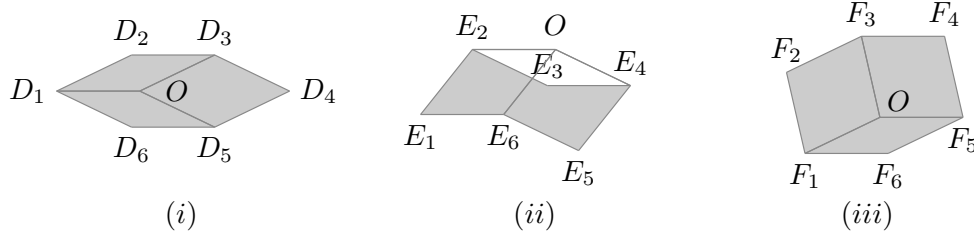


Figure 7: Lenses  $(\mathbf{D}, \mathbf{E}, \mathbf{F})$  build from rhombi  $(\mathbf{d}, \mathbf{e}, \mathbf{f})$ .

Figure 7 show lenses  $(\mathbf{B}, \mathbf{C})$  construction. At (i) we form the lense  $\mathbf{D}$  with perimeter  $\overline{D_1 \dots D_6}$  adding two rhombi  $\mathbf{d}$  ( $\overline{D_1 D_2 D_3 O}$  and  $\overline{D_1 O D_5 D_6}$ ) and adding one rhombus  $\mathbf{e}$  ( $\overline{O D_3 D_4 D_5}$ ) so its area is  $2\mathbf{d} + \mathbf{e}$ . Lense  $\mathbf{D}$  is equivalent to the hexagon  $H_7(1, 3, 3)$ .

At (ii) we form the lense  $\mathbf{E}$  with perimeter  $\overline{E_1 \dots E_6}$  adding one rhombus  $\mathbf{e}$  ( $\overline{E_1 E_2 O E_6}$ ) adding one rhombus  $\mathbf{f}$  ( $\overline{O E_4 E_5 E_6}$ ) and subtracting one rhombus  $\mathbf{d}$  ( $\overline{E_2 O E_4 E_3}$ ) so its area is  $-\mathbf{d} + \mathbf{e} + \mathbf{f}$ . Lense  $\mathbf{E}$  is equivalent to the hexagon  $H_7(1, 2, 4)$ .

At (iii) we form the lense  $\mathbf{F}$  with perimeter  $\overline{F_1 \dots F_6}$  adding two rhombi  $\mathbf{f}$  ( $\overline{F_1 F_2 F_3 O}$  and  $\overline{F_3 F_4 F_5 O}$ ) and adding one rhombus  $\mathbf{d}$  ( $\overline{F_1 O F_5 F_6}$ ) so its area is  $\mathbf{d} + 2\mathbf{f}$ . Lense  $\mathbf{F}$  is equivalent to the hexagon  $H_7(2, 2, 3)$ . Table 4 show the lenses  $(\mathbf{D}, \mathbf{E}, \mathbf{F})$  internal angles and areas.

Lense	$\theta_1$	$\theta_2$	$\theta_3$	Area
$\mathbf{D}$	$\gamma$	$3\gamma$	$3\gamma$	$2\mathbf{d} + \mathbf{e}$
$\mathbf{E}$	$\gamma$	$2\gamma$	$4\gamma$	$-\mathbf{d} + \mathbf{e} + \mathbf{f}$
$\mathbf{F}$	$2\gamma$	$2\gamma$	$3\gamma$	$-\mathbf{d} + 2\mathbf{f}$

Table 4: Lenses  $(\mathbf{D}, \mathbf{E}, \mathbf{F})$  internal angles and areas in terms of rhombi  $(\mathbf{d}, \mathbf{e}, \mathbf{f})$ .  $\theta_1 + \theta_2 + \theta_3 = 2\pi$  and  $\gamma = 2\pi/7$ .

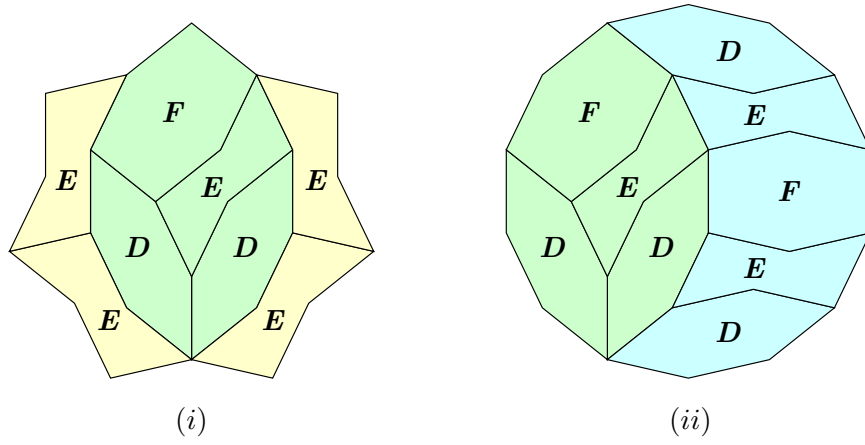


Figure 8: Stars dissected with only lenses ( $D, E, F$ ).

Figure 8 show stars  $S_7(2, 4)$  and  $S_7(3, 3)$  dissected with lenses ( $D, E, F$ ). At (i) we have star  $S_7(2, 4)$  and by inspection we deduce its area is  $A = 2D + 5E + F$ . At (ii) we have regular 14-gon (or star  $S_7(3, 3)$ ) and by inspection we deduce its area is  $4D + 3E + 2F$ . Both stars have in common an area in green resembling a tree leaf. The star at (i) also contains two regions in yellow resembling crowns while the star at (ii) contains a region in cyan resembling a moon phase.

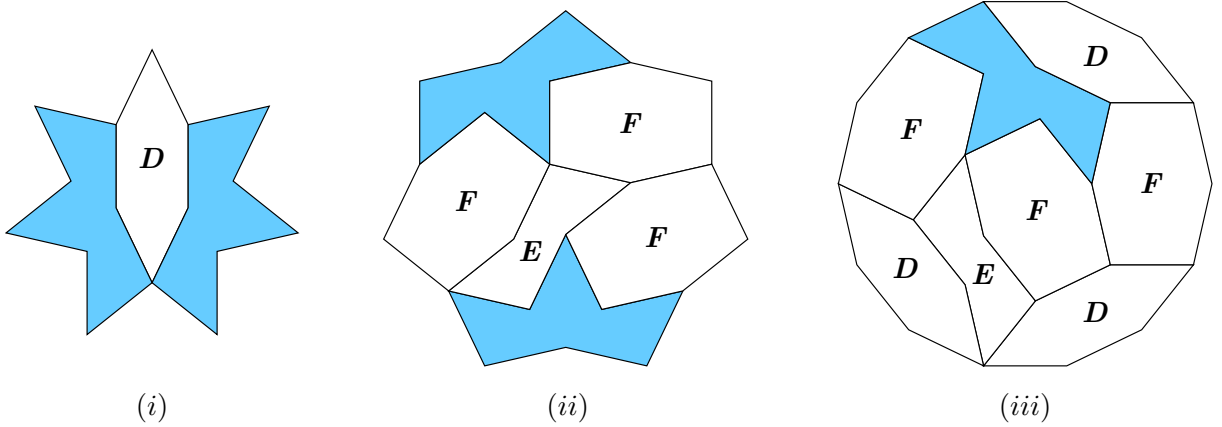


Figure 9: Stars dissected with octagons  $O_7$  (in blue) and lenses ( $D, E, F$ ).

Figure 9 show Stars  $S_7(1, 5)$ ,  $S_7(2, 4)$  and  $S_7(3, 3)$  dissected with octagons  $O_7$  (in blue) and lenses ( $D, E, F$ ). At (i) we have the star  $S_7(1, 5)$  and by inspection we deduce its area is  $A = D + 2O_7$ . At (ii) we have the star  $S_7(2, 4)$  and we can conclude its area is  $E + 3F + 2O_7$ . Similarly the area of the 14-gon at (iii) is  $3D + E + 3F + O_7$ . Comparing the areas of the two 14-gons of figures 8 and 9 we can find the area of  $O_7$  in terms of ( $E, F, G$ ):

$$\begin{aligned} 4D + 3E + 2F &= 3D + E + 3F + O_7 \\ O_7 &= D + 2E - F \end{aligned} \tag{8}$$

So we can calculate the area of star  $S_7(1, 5)$  in terms of ( $E, F, G$ ):

$$\begin{aligned} S_7(1, 5) &= D + 2(D + 2E - F) \\ &= 3D + 4E - 2F \end{aligned} \tag{9}$$