Lenses

Abstract

Lenses are equilateral hexagons resembling concave and convex optical lenses. Lenses consecutive six internal angles are $(\theta_1, \theta_2, \theta_3, \theta_1, \theta_2, \theta_3)$ where $\theta_1 = x\theta_0$, $\theta_2 = y\theta_0$, and $\theta_3 = z\theta_0$ where $\theta_0 = 2\pi/m$ is the base angle of symmetry $m = x + y + z$. Lenses can be formed adding and substracting rhombi or by intersecting equi-stars with others.

1 Equi-stars

Equi-stars are equilateral polygons with an even number of sides and vertices of at most two different angles. These stars can be defined with only two numbers: A symmetry integer m and a minimum angle integer a so the star is defined as $S(m, a)$. Here we are interested only in symmetries of the form $m = 2n+1$ for $n = 1, 2, 3....$ Every symmetry $m = 2n+1$ has exactly n different stars: $S(m, n), S(m, n-1), ..., S(m, 1)$. Stars of the form $S(m, n)$ correspond to the regular polygons of $2m$ sides.

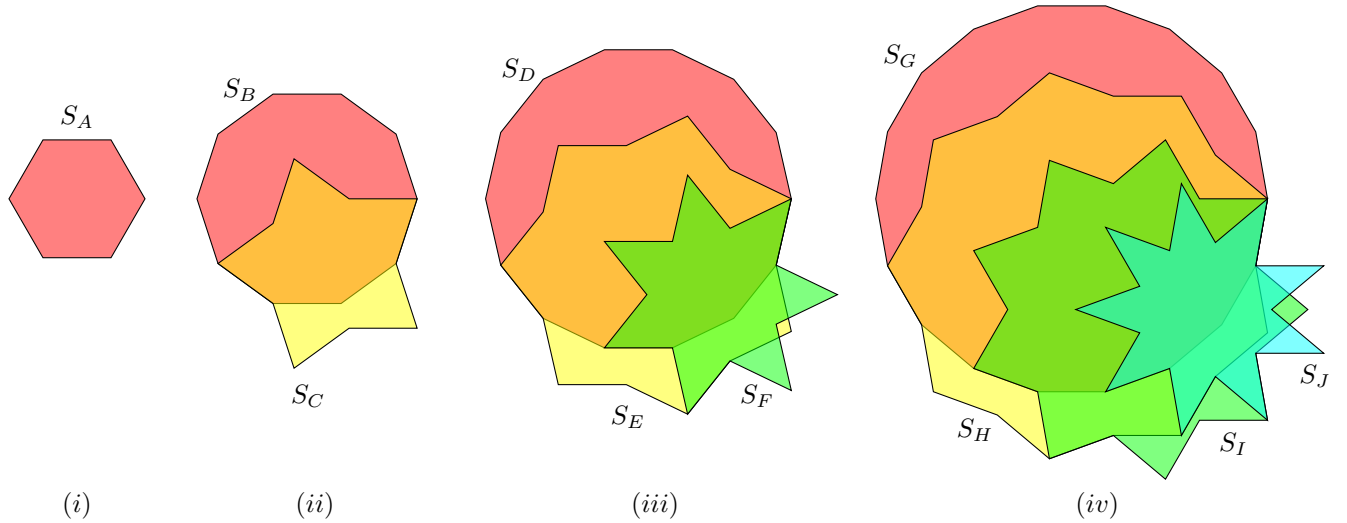Figure 1: Equi-stars of symmetries $m = \{3, 5, 7, 9\}$.

Figure 1 show the stars for the smaller symmetries in translucent colors and intersecting with others of the same symmetry. At (i) we have for the symmetry 3 the only star in red $S_A \equiv S(3, 1)$ which is the regular hexagon. At (ii) we have for the symmetry 5 the regular decagon in red $S_B \equiv S(5, 2)$ and the star $S_C \equiv S(5, 1)$ in yellow; the region in orange is the intersection of the two stars. At (iii) for symmetry 7 we have three stars: The regular 14-gon $S_D \equiv S(7, 3)$ in red, the star $S_E \equiv S(7, 2)$ in yellow and the star $S_F \equiv S(7, 1)$ in green. At (iv) we have for the symmetry 9 four stars: The regular 18-gon $S_G \equiv S(9, 4)$ in red, the star $S_H \equiv S(9, 3)$ in yellow, the star $S_I \equiv S(9, 2)$ in green and the star $S_J \equiv S(9, 1)$ in blue.
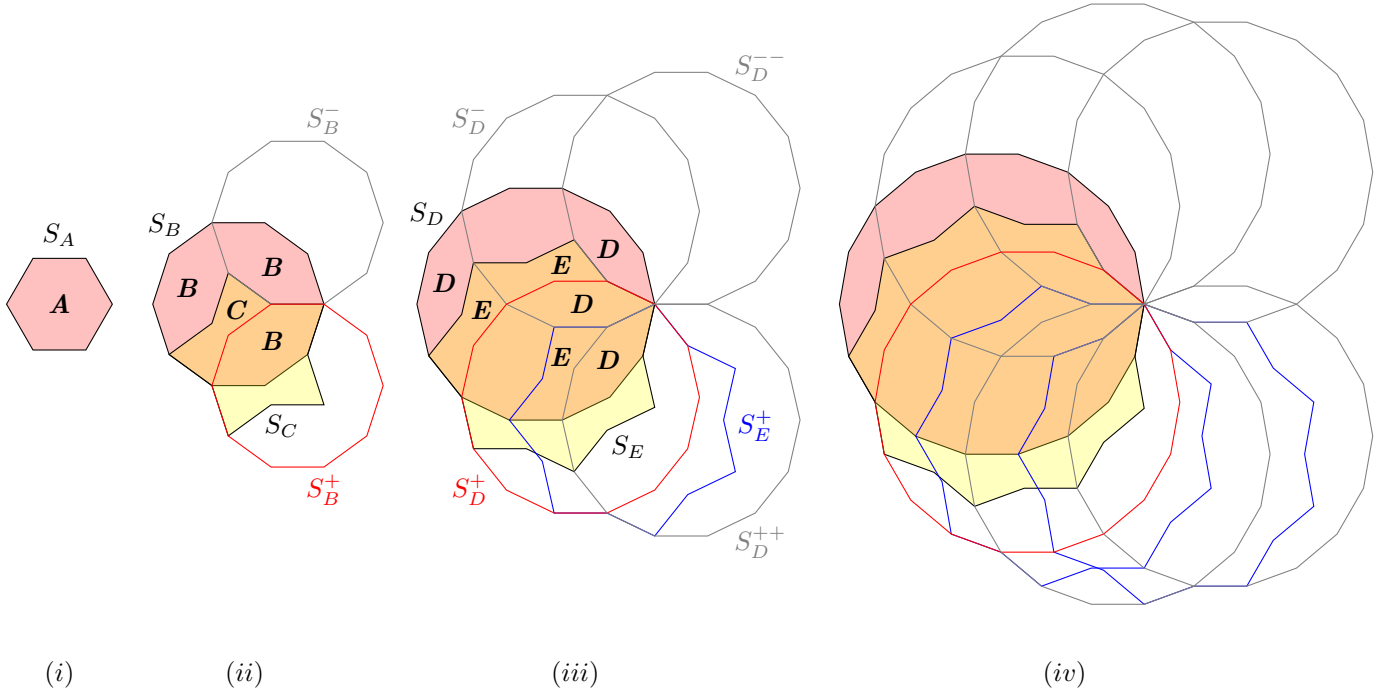
1.1 Lenses

Figure 2: Lenses build from the intersection of three stars: One $S(m,n)$ at the top, one $S(m, n-1)$ at the center and another $S(m,n)$ at the bottom, for symmetries $m = \{3, 5, 7, 9\}$.

Figure 2 show the first lenses produced by the intersections of three stars: One $S(m,n)$ in pink at the top, one $S(m, n-1)$ in orange at the center and another $S(m,n)$ in pink at the bottom. The intersections produce five regions: α congruent with ϵ, β congruent with δ and γ. The lenses emerge after dissecting regions α and β. For every symmetry $m = 2n + 1$ we get n distinct lenses.

At (i) for symmetry $m = 3$ we have the single lense ($\boldsymbol{A}$) equivalent to the regular hexagon. Lense $\boldsymbol{A}$ is congruent with S_A. At (ii) for symmetry $m = 5$ we have the two lenses ($\boldsymbol{B}, \boldsymbol{C}$). The α region of S_B area equals $2\boldsymbol{B}$. The β region area equals $\boldsymbol{C}$. At (iii) for symmetry $m = 7$ we have the three lenses ($\boldsymbol{D}, \boldsymbol{E}, \boldsymbol{F}$). The region α area equals $2\boldsymbol{D} + \boldsymbol{F}$ and the region β area equals $2\boldsymbol{E}$. At (iv) for symmetry $m = 9$ we have the four lenses ($\boldsymbol{G}, \boldsymbol{H}, \boldsymbol{I}, \boldsymbol{J}$). The α region area equals $2(\boldsymbol{G} + \boldsymbol{I})$ and the β region area equals $2\boldsymbol{H} + \boldsymbol{J}$.

Next we are going to show the regions δ for any symmetry $m \geq 5$ can be dissected with the lenses.
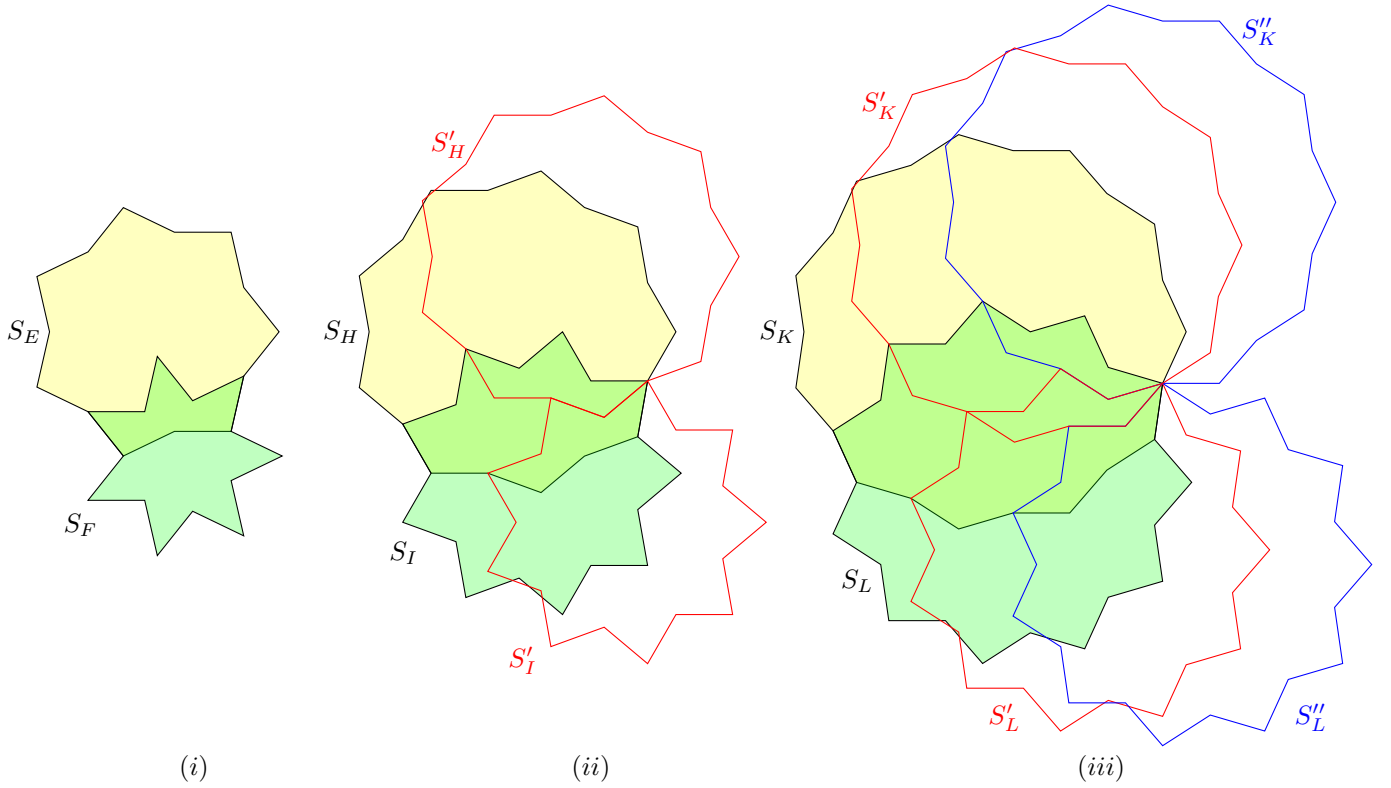
2 Crowns

Figure 3: Crowns.

3 Symmetry 5

Symmetry 5 is based in angle $\beta = \dfrac{2\pi}{5}$ and produces the two rhombi $(\boldsymbol{b}, \boldsymbol{c})$ and the two lenses $(\boldsymbol{B}, \boldsymbol{C})$.

3.1 Rhombi $(\boldsymbol{b}, \boldsymbol{c})$
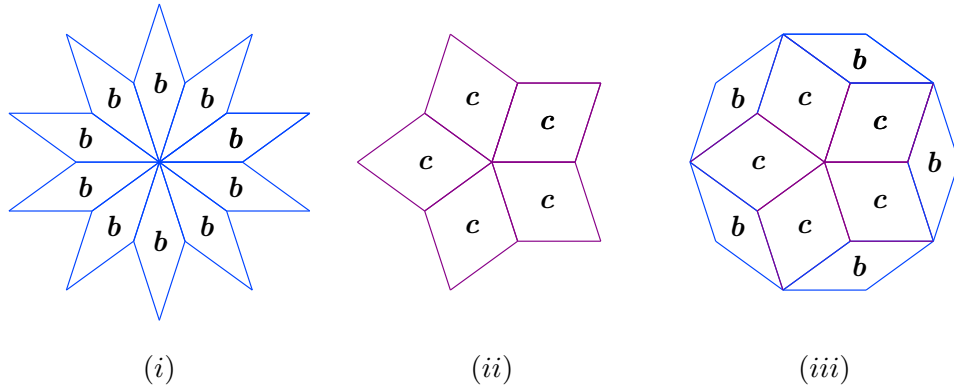
Figure 4: Rhombi $(\boldsymbol{b}, \boldsymbol{c})$ from dissecting stars S_{10}.

Figure 4 show the rhombi $(\boldsymbol{b}, \boldsymbol{c})$. Inspecting the stars we get the areas simply adding their rhombi. At (i) the star $S_{10}(1, 8)$ with area $A = 10\boldsymbol{b}$. At (ii) the star $S_{10}(2, 6) = S_5(1, 3)$ with area $A = 5\boldsymbol{c}$. At (iii) the

regular decagon equivalent to stars $S_{10}(4,4) = S_5(2,2)$ with area $A = 5\boldsymbol{b} + 5\boldsymbol{c}$. Table 1 show the rhombi $(\boldsymbol{b}, \boldsymbol{c})$ internal angles in terms of angle $\beta = 2\pi/5$ and areas for side equals to 1. Dividing areas we find $\dfrac{\boldsymbol{c}}{\boldsymbol{b}} = 2\cos\left(\dfrac{\pi}{5}\right) = \dfrac{\sqrt{5}+1}{2}$.

Rhombus	θ_1	θ_2	Area
$\boldsymbol{b}$	$\beta/2$	$4\beta/2$	$\sin(2\beta) = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$
$\boldsymbol{c}$	$2\beta/2$	$3\beta/2$	$\sin(\beta) = \dfrac{\sqrt{10+2\sqrt{5}}}{4} = \boldsymbol{b}\cos\left(\dfrac{\pi}{5}\right) = \boldsymbol{b}\left(\dfrac{\sqrt{5}+1}{2}\right)$

Table 1: Rhombi $(\boldsymbol{b}, \boldsymbol{c})$ internal angles and areas. $\theta_1 + \theta_2 = \pi$ and $\beta = 2\pi/5$.

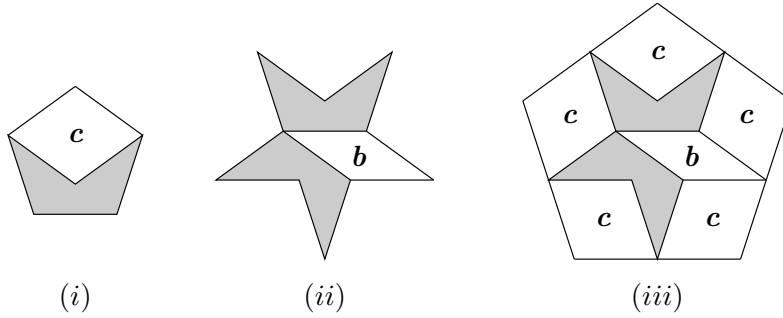## 3.2 Regular pentagon and star $|5/2|$

Figure 5: Regular pentagon $|5/1|$ at (i). Star $|5/2|$ at (ii). Double pentagon at (iii).

Figure 5 show regular pentagon and isotoxal star $|5/2|$ dissected with rhombi $(\boldsymbol{b}, \boldsymbol{c})$ plus concave pentagons (in gray). Let $\boldsymbol{x}$ be the area of such gray piece. By inspection the area of regular pentagon at (i) is $A_1 = \boldsymbol{c} + \boldsymbol{x}$ and the area of regular pentagon at (iii) is $P_2 = \boldsymbol{b} + 5\boldsymbol{c} + 2\boldsymbol{x}$. Since the side of A_2 is the double of A_1 its area is four times so we can get the value of $\boldsymbol{x}$ in terms of $(\boldsymbol{b}, \boldsymbol{c})$

$$4P_1 = P_4$$
$$4(\boldsymbol{c} + x) = \boldsymbol{b} + 5\boldsymbol{c} + 2x$$
$$x = \dfrac{\boldsymbol{b} + \boldsymbol{c}}{2} \tag{1}$$

We use the value of $\boldsymbol{x}$ to get the areas of pentagon (i) and star (ii):

$$A|5/1| = \boldsymbol{c} + \boldsymbol{x}$$
$$= \dfrac{\boldsymbol{b} + 3\boldsymbol{c}}{2} \tag{2}$$
$$A|5/2| = \boldsymbol{b} + 2\boldsymbol{x}$$
$$= 2\boldsymbol{b} + \boldsymbol{c} \tag{3}$$

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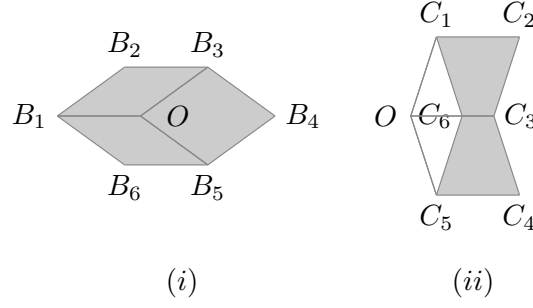
3.3 Lenses (*B*,*C*)

Figure 6: Lenses (**B**, **C**) build with rhombi (**b**, **c**).

Figure 6 show lenses (**B**, **C**) construction and two stars formed with them. At (i) we form the lense **B** with perimeter $\overline{B_1...B_6}$ adding two rhombi **b** ($\overline{B_1B_2B_3O}$ and $\overline{B_1OB_5B_6}$) and adding one rhombus **c** ($\overline{OB_3B_4B_5}$) so its area is $2\boldsymbol{b} + \boldsymbol{c}$. Lense **B** is equivalent to the hexagon $H_5(1,2,2)$. At (ii) we form the lense **C** with perimeter $\overline{C_1...C_6}$ adding two rhombi **c** ($\overline{OC_1C_2C3}$ and $\overline{OC_3C_4C_5}$) and substracting one rhombus **b** ($\overline{OC_1C_6C_5}$) so its area is $2\boldsymbol{c} - \boldsymbol{b}$. Lense **C** is equivalent to the hexagon $H_5(1,1,3)$. Table 2 show the lenses (**B**, **C**) internal angles and areas.

Lense	θ_1	θ_2	θ_3	Area
B	β	2β	2β	$2\boldsymbol{b} + \boldsymbol{c}$
C	β	β	3β	$-\boldsymbol{b} + 2\boldsymbol{c}$

Table 2: Lenses (**B**, **C**) internal angles and areas in terms of rhombi (**b**, **c**). $\theta_1 + \theta_2 + \theta_3 = 2\pi$ where $\beta = 2\pi/5$.
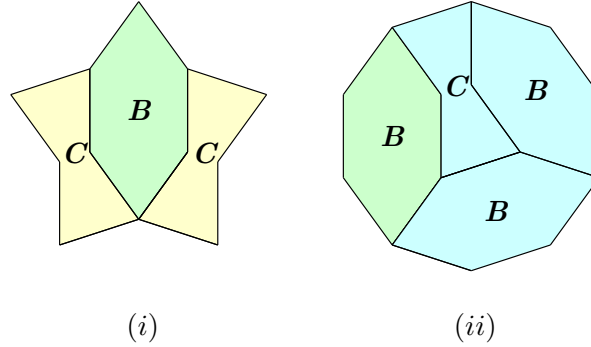

Figure 7: Two stars dissected with lenses (**B**, **C**).

Figure 7 show two stars dissected with lenses (**B**, **C**). At (i) the star $S_5(1,3)$ dissection implies its area is $A = \boldsymbol{B} + 2\boldsymbol{C} = 5\boldsymbol{c}$. At ($ii$) the regular decagon or star $S_5(2,2)$ dissection implies its area is $A = 3\boldsymbol{B} + \boldsymbol{C} = 5(\boldsymbol{b} + \boldsymbol{c})$.

4 Symmetry 7

Symmetry 7 is based in angle $\gamma = \dfrac{2\pi}{7}$ and produces the three rhombi (**d**, **f**, **e**) and the three lenses (**D**, **E**, **F**).
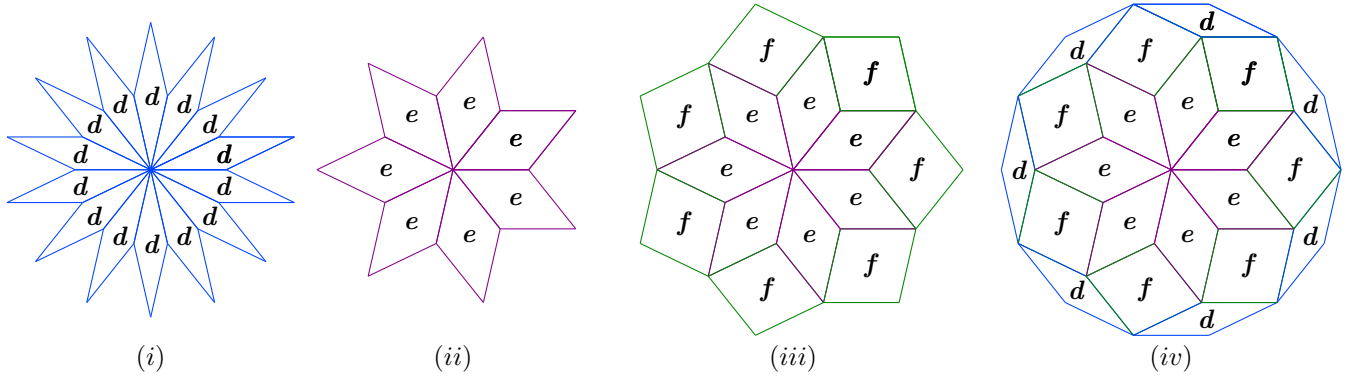
4.1 Rhombi (d, e, f)

Figure 8: Rhombi (d, e, f) from dissected stars S_{14}.

Figure 8 show rhombi (d, e, f). Inspecting the stars we get the areas simply adding their rhombi. At (i) the star $S_{14}(1, 12)$ with area $A = 14d$. At (ii) the star $S_{14}(2, 10) = S_7(1, 5)$ with area $A = 7e$. At (iii) the star $S_{14}(4, 8) = S_7(2, 4)$ with area $A = 7(e + f)$. At (iv) the regular 14-gon equivalent to stars $S_{14}(6, 6) = S_7(3, 3)$ with area $A = 7(d + e + f)$. Table 3 show the symmetry 7 lenses internal angles based in angle $\gamma = 2\pi/7$ and the areas.

Rhombus	θ_1	θ_2	Area
d	$\gamma/2$	$6\gamma/2$	$\sin(3\gamma)$
e	$2\gamma/2$	$5\gamma/2$	$\sin(\gamma)$
f	$3\gamma/2$	$4\gamma/2$	$\sin(2\gamma)$

Table 3: Rhombi (d, e, f) internal angles. $\theta_1 + \theta_2 = \pi$ and $\gamma = 2\pi/7$.
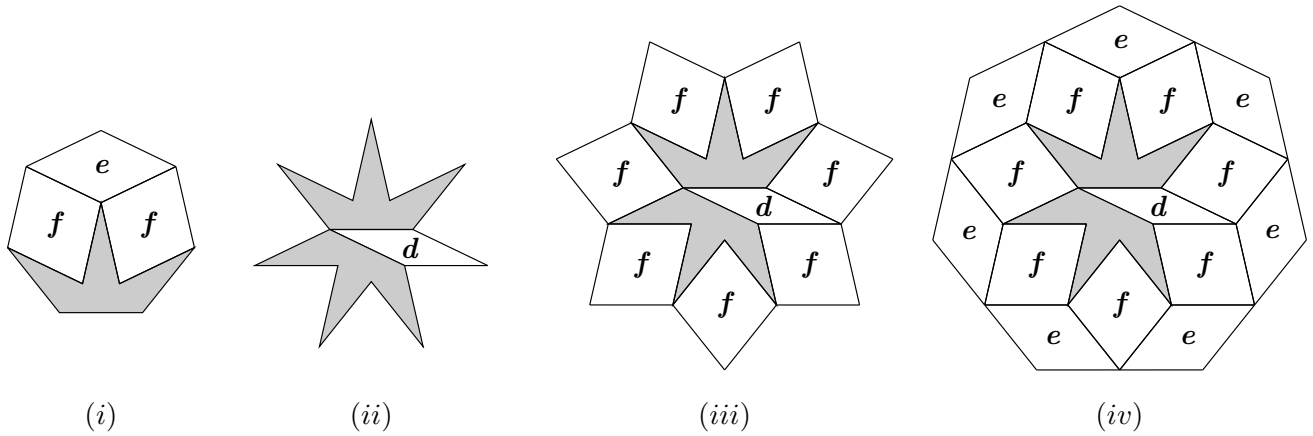
4.2 Regular heptagon and stars $|7/3|$ and $|7/2|$

Figure 9: Heptagon $|7/1|$ at (i). Star $|7/3|$ at (ii). Star $|7/2|$ at (iii). Double heptagon at (iv).

Figure 9 show regular heptagon and heptagrams dissected with rhombi (c, d, e) plus equilateral concave heptagons (in gray). Let y be the area of such gray piece. By inspection the area of regular heptagon at

(i) is $A_1 = e + 2f + y$ while the area of regular heptagon at (iv) is $A_2 = d + 7(e + f) + 2y$. Since the side of A_2 is the double of A_1 its area is four times so we can get the value of y in terms of (d, e, f):

$$4A_1 = A_2$$
$$4(e + 2f + y) = d + 7(e + f) + 2y$$
$$y = \frac{d + 3e - f}{2} \tag{4}$$

We use the value of y to calculate the areas of heptagon (i) and stars (ii) and (iii) in terms of (d, e, f):

$$A|7/1| = e + 2f + y$$
$$= \frac{d + 5e + 3f}{2} \tag{5}$$
$$A|7/3| = d + 2y$$
$$= 2d + 3e - f \tag{6}$$
$$A|7/2| = A\{7/3\} + 7f$$
$$= 2d + 3e + 6f \tag{7}$$
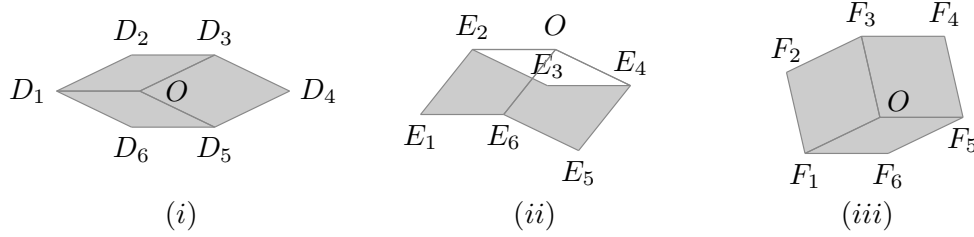
4.3   Lenses (D,E,F)

Figure 10: Lenses (D,E,F) build from rhombi (d,e,f).

Figure 10 show lenses (B, C) construction. At (i) we form the lense D with perimeter $\overline{D_1...D_6}$ adding two rhombi d $(\overline{D_1 D_2 D_3 O}$ and $\overline{D_1 O D_5 D_6})$ and adding one rhombus e $(\overline{O D_3 D_4 D_5})$ so its area is $2d + e$. Lense D is equivalent to the hexagon $H_7(1, 3, 3)$.

At (ii) we form the lense E with perimeter $\overline{E_1...E_6}$ adding one rhombus e $(\overline{E_1 E_2 O E_6})$ adding one rhombus f $(\overline{O E_4 E_5 E_6})$ and substracting one rhombus d $(\overline{E_2 O E_4 E_3})$ so its area is $-d + e + f$. Lense E is equivalent to the hexagon $H_7(1, 2, 4)$.

At (iii) we form the lense F with perimeter $\overline{F_1...F_6}$ adding two rhombi f $(\overline{F_1 F_2 F_3 O}$ and $\overline{F_3 F_4 F_5 O})$ and adding one rhombus d $(\overline{F_1 O F_5 F_6})$ so its area is $d + 2f$. Lense F is equivalent to the hexagon $H_7(2, 2, 3)$. Table 4 show the lenses (D, E, F) internal angles and areas.

Lense	θ_1	θ_2	θ_3	Area
D	γ	3γ	3γ	$2d + e$
E	γ	2γ	4γ	$-d + e + f$
F	2γ	2γ	3γ	$-d + 2f$

Table 4: Lenses (D, E, F) internal angles and areas in terms of rhombi (d, e, f). $\theta_1 + \theta_2 + \theta_3 = 2\pi$ and $\gamma = 2\pi/7$.
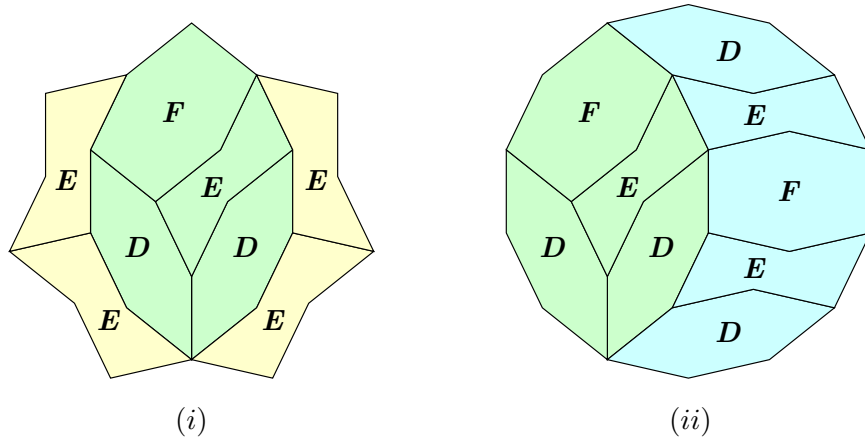
Figure 11: Stars dissected with only lenses (D,E,F).

Figure 11 show stars $S_7(2,4)$ and $S_7(3,3)$ dissected with lenses (D,E,F). At (i) we have star $S_7(2,4)$ and by inspection we deduce its area is $A = 2D + 5E + F$. At (ii) we have regular 14-gon (or star $S_7(3,3)$) and by inspection we deduce its area is $4D + 3E + 2F$. Both stars have in common an area in green resembling a tree leaf. The star at (i) also contains two regions in yellow resembling crowns while the star at (ii) contains a region in cyan resembling a moon phase.
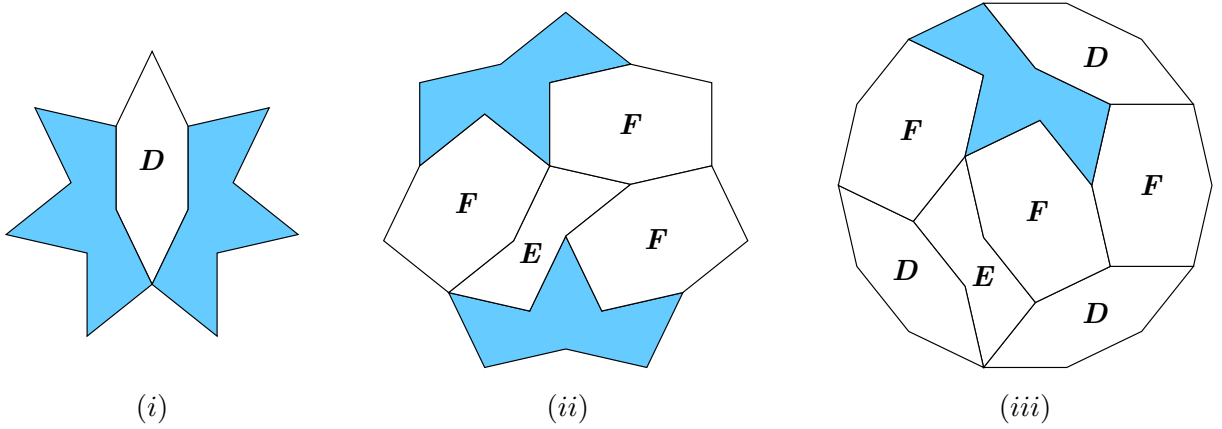

Figure 12: Stars dissected with octagons O_7 (in blue) and lenses (D,E,F).

Figure 12 show Stars $S_7(1,5)$, $S_7(2,4)$ and $S_7(3,3)$ dissected with octagons O_7 (in blue) and lenses (D,E,F). At (i) we have the star $S_7(1,5)$ and by inspection we deduce its area is $A = D + 2O_7$. At (ii) we have the star $S_7(2,4)$ and we can conclude its area is $E + 3F + 2O_7$. Similarly the area of the 14-gon at (iii) is $3D + E + 3F + O_7$. Comparing the areas of the two 14-gons of figures 11 and 12 we can find the area of O_7 in terms of (E, F, G):

$$4D + 3E + 2F = 3D + E + 3F + O_7$$
$$O_7 = D + 2E - F \tag{8}$$

So we can calculate the area of star $S_7(1,5)$ in terms of (E, F, G):

$$S_7(1,5) = D + 2(D + 2E - F)$$
$$= 3D + 4E - 2F \tag{9}$$

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