Meccano triangles

https://github.com/heptagons/meccano/nest

Abstract

We construct meccano triangles. Basic triangles has the three sides as integers and calculate the internal diagonal distances. Such diagonals then are used as the new side of more complicated triangles and then again we calculate new distances formed and so on. Eventually we expect to find certain angles joining the triangles which can be used to construct regular polygons or more figures.

1 Triangles (a, b, c)

Triangles (a, b, c) have three sides a, b and c where $a, b, c \in \mathbb{N}$. To avoid repetitions and get only valid triangles, we consider only the cases:

$$a \geq b \geq c \tag{1}$$

$$a < b + c \tag{2}$$

The cosines of the three angles of triangle(a, b, c) are rationals:

$$\cos \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} \dfrac{b^2 + c^2 - a^2}{2bc} \\ \dfrac{c^2 + a^2 - b^2}{2ca} \\ \dfrac{a^2 + b^2 - c^2}{2ab} \end{pmatrix} \equiv \begin{pmatrix} \dfrac{a_n}{a_d} \\ \dfrac{b_n}{b_d} \\ \dfrac{c_n}{c_d} \end{pmatrix} \in \mathbb{Q} \tag{3}$$

The numerators are integers $a_n, b_n, c_n \in \mathbb{Z}$. The denominators are naturals $a_d, b_d, c_d \in \mathbb{N}$. For $x = \{a, b, c\}$ $x_n <= x_d$.

The sines of the three angles are algebraic:

$$\sin \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} \sqrt{1 - \cos^2 A} \\ \sqrt{1 - \cos^2 B} \\ \sqrt{1 - \cos^2 C} \end{pmatrix} = \begin{pmatrix} \dfrac{\sqrt{a_d^2 - a_n^2}}{a_d} \\ \dfrac{\sqrt{b_d^2 - b_n^2}}{b_d} \\ \dfrac{\sqrt{c_d^2 - c_n^2}}{c_d} \end{pmatrix} \equiv \begin{pmatrix} \dfrac{\sqrt{a_s}}{a_d} \\ \dfrac{\sqrt{b_s}}{b_d} \\ \dfrac{\sqrt{c_s}}{c_d} \end{pmatrix} \in \mathbb{A} \tag{4}$$

Numbers inside square roots are naturals $a_s, b_s, c_s \in \mathbb{N}$. For $x = \{a, b, c\}$ $x_s = x_d^2 - x_n^2 \geq 0$.

Table 1: Small triangles (a, b, c) vertices A, B, C cosines and sines.

	(a,b,c)	$\cos A$	$\cos B$	$\cos C$	$\sin A$	$\sin B$	$\sin C$
1	(1,1,1)	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
2	(2,2,1)	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{7}{8}$	$\frac{\sqrt{15}}{4}$	$\frac{\sqrt{15}}{4}$	$\frac{\sqrt{15}}{8}$
3	(3,2,2)	$-\frac{1}{8}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3\sqrt{7}}{8}$	$\frac{\sqrt{7}}{4}$	$\frac{\sqrt{7}}{4}$
4	(3,3,1)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{17}{18}$	$\frac{\sqrt{35}}{6}$	$\frac{\sqrt{35}}{6}$	$\frac{\sqrt{35}}{18}$
5	(3,3,2)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{7}{9}$	$\frac{2\sqrt{2}}{3}$	$\frac{2\sqrt{2}}{3}$	$\frac{4\sqrt{2}}{9}$
6	(4,3,2)	$-\frac{1}{4}$	$\frac{11}{16}$	$\frac{7}{8}$	$\frac{\sqrt{15}}{4}$	$\frac{3\sqrt{15}}{16}$	$\frac{\sqrt{15}}{8}$
7	(4,3,3)	$\frac{1}{9}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{4\sqrt{5}}{9}$	$\frac{\sqrt{5}}{3}$	$\frac{\sqrt{5}}{3}$
8	(4,4,1)	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{31}{32}$	$\frac{3\sqrt{7}}{8}$	$\frac{3\sqrt{7}}{8}$	$\frac{3\sqrt{7}}{32}$
9	(4,4,3)	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{23}{32}$	$\frac{\sqrt{55}}{8}$	$\frac{\sqrt{55}}{8}$	$\frac{3\sqrt{55}}{32}$
10	(5,3,3)	$-\frac{7}{18}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5\sqrt{11}}{18}$	$\frac{\sqrt{11}}{6}$	$\frac{\sqrt{11}}{6}$
11	(5,4,2)	$-\frac{5}{16}$	$\frac{13}{20}$	$\frac{37}{40}$	$\frac{\sqrt{231}}{16}$	$\frac{\sqrt{231}}{20}$	$\frac{\sqrt{231}}{40}$
12	(5,4,3)	0	$\frac{3}{5}$	$\frac{4}{5}$	1	$\frac{4}{5}$	$\frac{3}{5}$
13	(5,4,4)	$\frac{7}{32}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{5\sqrt{39}}{32}$	$\frac{\sqrt{39}}{8}$	$\frac{\sqrt{39}}{8}$
14	(5,5,1)	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{49}{50}$	$\frac{3\sqrt{11}}{10}$	$\frac{3\sqrt{11}}{10}$	$\frac{3\sqrt{11}}{50}$
15	(5,5,2)	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{23}{25}$	$\frac{2\sqrt{6}}{5}$	$\frac{2\sqrt{6}}{5}$	$\frac{4\sqrt{6}}{25}$
16	(5,5,3)	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{41}{50}$	$\frac{\sqrt{91}}{10}$	$\frac{\sqrt{91}}{10}$	$\frac{3\sqrt{91}}{50}$
17	(5,5,4)	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{17}{25}$	$\frac{\sqrt{21}}{5}$	$\frac{\sqrt{21}}{5}$	$\frac{4\sqrt{21}}{25}$
18	(7,6,5)	$\frac{1}{5}$	$\frac{19}{35}$	$\frac{5}{7}$	$\frac{2\sqrt{6}}{5}$	$\frac{12\sqrt{6}}{35}$	$\frac{2\sqrt{6}}{7}$

Data from `github.com/heptagons/meccano/nest/t_test.go TestTCosSin`

2 Triangles (a, b, c) pairs

We can attach two triangles (a, b, c) sharing a side and vertex to get more angles.

2.1 Two angles sum

We can use the equations 3 and 4 to calculate the sum of two angles of two triangles:

$$\begin{aligned}\cos(X+Y) &= \cos X \cos Y - \sin X \sin Y \\ &= \frac{x_n}{x_d} \times \frac{y_n}{y_d} - \frac{\sqrt{x_s}}{x_d} \times \frac{\sqrt{y_s}}{y_d} \\ &= \frac{x_n y_n - \sqrt{x_s y_s}}{x_d y_d} \qquad (5)\end{aligned}$$

2.2 Two angles sum examples

From the list of triangles including the range $(1, 1, 1)$ to $(3, 3, 2)$ we can calculate the cosines of sums of the angles in pairs. Now some cosines values are algebraic.

Pair	$vertex_X$	$vertex_Y$	$\cos(X+Y)$
1	(1,1,1)[A]	(1,1,1)[A]	$-\frac{1}{2}$
2	(2,2,1)[A]	(1,1,1)[A]	$\frac{1-3\sqrt{5}}{8}$
3	(2,2,1)[C]	(1,1,1)[A]	$\frac{7-3\sqrt{5}}{16}$
4	(2,2,1)[A]	(2,2,1)[A]	$-\frac{7}{8}$
5	(2,2,1)[A]	(2,2,1)[C]	$-\frac{1}{4}$
6	(2,2,1)[C]	(2,2,1)[C]	$\frac{17}{32}$
7	(3,2,2)[A]	(1,1,1)[A]	$-\frac{1+3\sqrt{21}}{16}$
8	(3,2,2)[B]	(1,1,1)[A]	$\frac{3-\sqrt{21}}{8}$
9	(3,2,2)[A]	(2,2,1)[A]	$-\frac{1+3\sqrt{105}}{32}$
10	(3,2,2)[A]	(2,2,1)[C]	$-\frac{7+3\sqrt{105}}{64}$
11	(3,2,2)[B]	(2,2,1)[A]	$\frac{3-\sqrt{105}}{16}$
12	(3,2,2)[B]	(2,2,1)[C]	$\frac{21-\sqrt{105}}{32}$
13	(3,2,2)[A]	(3,2,2)[A]	$-\frac{31}{32}$
14	(3,2,2)[A]	(3,2,2)[B]	$-\frac{3}{4}$
15	(3,2,2)[B]	(3,2,2)[B]	$\frac{1}{8}$
16	(3,3,1)[A]	(1,1,1)[A]	$\frac{1-\sqrt{105}}{12}$
17	(3,3,1)[C]	(1,1,1)[A]	$\frac{17-\sqrt{105}}{36}$
18	(3,3,1)[A]	(2,2,1)[A]	$\frac{1-5\sqrt{21}}{24}$
19	(3,3,1)[A]	(2,2,1)[C]	$\frac{7-5\sqrt{21}}{48}$
20	(3,3,1)[C]	(2,2,1)[A]	$\frac{17-5\sqrt{21}}{72}$
21	(3,3,1)[C]	(2,2,1)[C]	$\frac{119-5\sqrt{21}}{144}$
22	(3,3,1)[A]	(3,2,2)[A]	$-\frac{1+21\sqrt{5}}{48}$
23	(3,3,1)[A]	(3,2,2)[B]	$\frac{3-7\sqrt{5}}{24}$
24	(3,3,1)[C]	(3,2,2)[A]	$-\frac{17+21\sqrt{5}}{144}$
25	(3,3,1)[C]	(3,2,2)[B]	$\frac{51-7\sqrt{5}}{72}$
26	(3,3,1)[A]	(3,3,1)[A]	$-\frac{17}{18}$
27	(3,3,1)[A]	(3,3,1)[C]	$-\frac{1}{6}$
28	(3,3,1)[C]	(3,3,1)[C]	$\frac{127}{162}$
29	(3,3,2)[A]	(1,1,1)[A]	$\frac{1-2\sqrt{6}}{6}$
30	(3,3,2)[C]	(1,1,1)[A]	$\frac{7-4\sqrt{6}}{18}$
31	(3,3,2)[A]	(2,2,1)[A]	$\frac{1-2\sqrt{30}}{12}$
32	(3,3,2)[A]	(2,2,1)[C]	$\frac{7-2\sqrt{30}}{24}$
33	(3,3,2)[C]	(2,2,1)[A]	$\frac{7-4\sqrt{30}}{36}$
34	(3,3,2)[C]	(2,2,1)[C]	$\frac{49-4\sqrt{30}}{72}$
35	(3,3,2)[A]	(3,2,2)[A]	$-\frac{1+6\sqrt{14}}{24}$
36	(3,3,2)[A]	(3,2,2)[B]	$\frac{3-2\sqrt{14}}{12}$
37	(3,3,2)[C]	(3,2,2)[A]	$-\frac{7+12\sqrt{14}}{72}$

Pair	$vertex_X$	$vertex_Y$	$\cos(X+Y)$
38	(3,3,2)[C]	(3,2,2)[B]	$\frac{21-4\sqrt{14}}{36}$
39	(3,3,2)[A]	(3,3,1)[A]	$\frac{1-2\sqrt{70}}{18}$
40	(3,3,2)[A]	(3,3,1)[C]	$\frac{17-2\sqrt{70}}{54}$
41	(3,3,2)[C]	(3,3,1)[A]	$\frac{7-4\sqrt{70}}{54}$
42	(3,3,2)[C]	(3,3,1)[C]	$\frac{119-4\sqrt{70}}{162}$
43	(3,3,2)[A]	(3,3,2)[A]	$-\frac{7}{9}$
44	(3,3,2)[A]	(3,3,2)[C]	$-\frac{1}{3}$
45	(3,3,2)[C]	(3,3,2)[C]	$\frac{17}{81}$

Calculations by `github.com/heptagons/meccano/nest/t_test.go TestTCosXY`
We can also consider bigger triangles and filter, for example, the cosines having the term $\sqrt{5}$:

Pair	$vertex_X$	$vertex_Y$	$\cos(X+Y)$
1	(2,2,1)[A]	(1,1,1)[A]	$\frac{1-3\sqrt{5}}{8}$
2	(2,2,1)[C]	(1,1,1)[A]	$\frac{7-3\sqrt{5}}{16}$
3	(3,3,1)[A]	(3,2,2)[A]	$-\frac{1+21\sqrt{5}}{48}$
4	(3,3,1)[A]	(3,2,2)[B]	$\frac{3-7\sqrt{5}}{24}$
5	(3,3,1)[C]	(3,2,2)[A]	$-\frac{17+21\sqrt{5}}{144}$
6	(3,3,1)[C]	(3,2,2)[B]	$\frac{51-7\sqrt{5}}{72}$
7	(4,3,2)[A]	(1,1,1)[A]	$-\frac{1+3\sqrt{5}}{8}$
8	(4,3,2)[B]	(1,1,1)[A]	$\frac{11-9\sqrt{5}}{32}$
9	(4,4,1)[A]	(3,3,1)[A]	$\frac{1-21\sqrt{5}}{48}$
10	(4,4,1)[A]	(3,3,1)[C]	$\frac{17-21\sqrt{5}}{144}$
11	(4,4,1)[C]	(3,3,1)[A]	$\frac{31-21\sqrt{5}}{192}$
12	(4,4,1)[C]	(3,3,1)[C]	$\frac{527-21\sqrt{5}}{576}$
13	(5,3,3)[A]	(4,4,3)[A]	$-\frac{21+55\sqrt{5}}{144}$
14	(5,3,3)[A]	(4,4,3)[C]	$-\frac{161+165\sqrt{5}}{576}$
15	(5,3,3)[B]	(4,4,3)[A]	$\frac{15-11\sqrt{5}}{48}$
16	(5,3,3)[B]	(4,4,3)[C]	$\frac{115-33\sqrt{5}}{192}$
17	(5,4,3)[A]	(4,3,3)[A]	$\frac{-4\sqrt{5}}{9}$
18	(5,4,3)[A]	(4,3,3)[B]	$\frac{-\sqrt{5}}{3}$
19	(5,4,3)[B]	(4,3,3)[A]	$\frac{3-16\sqrt{5}}{45}$
20	(5,4,3)[B]	(4,3,3)[B]	$\frac{6-4\sqrt{5}}{15}$
21	(5,4,3)[C]	(4,3,3)[A]	$\frac{4-12\sqrt{5}}{45}$
22	(5,4,3)[C]	(4,3,3)[B]	$\frac{8-3\sqrt{5}}{15}$
23	(5,5,1)[A]	(4,4,3)[A]	$\frac{3-33\sqrt{5}}{80}$
24	(5,5,1)[A]	(4,4,3)[C]	$\frac{23-99\sqrt{5}}{320}$
25	(5,5,1)[C]	(4,4,3)[A]	$\frac{147-33\sqrt{5}}{400}$
26	(5,5,1)[C]	(4,4,3)[C]	$\frac{1127-99\sqrt{5}}{1600}$

3 Triangles (a, b, c) triplets

We can attach three triangles to share a common vertex and two sides.

3.1 Triple angles $2A + D$ and $A + 2D$

Lets add three angles, two angles A from a pair of triangles (a, b, c) and one angle D from single triangle (d, e, f). As with equations 3 and 4 lets define cosines and sines of triangle (d, e, f):

$$\begin{pmatrix} \cos & \sin \end{pmatrix} \begin{pmatrix} D \\ E \\ F \end{pmatrix} = \begin{pmatrix} \frac{d_n}{d_d} & \frac{\sqrt{d_s}}{d_d} \\ \frac{e_n}{e_d} & \frac{\sqrt{e_s}}{e_d} \\ \frac{f_n}{f_d} & \frac{\sqrt{f_s}}{f_d} \end{pmatrix} \tag{6}$$

For triangle (a, b, c) and using the equations 3 and 4 we have:

$$\cos(2A) = 2\cos^2 A - 1 = 2\frac{a_n^2}{a_d^2} - 1 = \frac{2a_n^2 - a_d^2}{a_d^2} \tag{7}$$

$$\sin(2A) = 2\sin A \cos A = \frac{2a_n\sqrt{a_s}}{a_d^2} \tag{8}$$

Using $\cos(2A)$, $\sin(2A)$, $\cos D$ and $\sin D$ we have:

$$\begin{aligned} \cos(2A + D) &= \cos(2A)\cos D - \sin(2A)\sin D \\ &= \frac{2a_n^2 - a_d^2}{a_d^2} \times \frac{d_n}{d_d} - \frac{2a_n\sqrt{a_s}}{a_d^2} \times \frac{\sqrt{d_s}}{d_d} \\ &= \frac{(2a_n^2 - a_d^2)d_n - 2a_n\sqrt{a_s d_s}}{a_d^2 d_d} \end{aligned} \tag{9}$$

Table 4: Pair of triangles (X_{ABC}) and (Y_{ABC}) adding three angles in two different forms $(2X + Y)$ and $(X + 2Y)$.

Pair	$X = A \vee B \vee C$	$Y = A \vee B \vee C$	$\cos(2X + Y)$	$\cos(X + 2Y)$
1	(1,1,1)[A]	(1,1,1)[A]	-1	-1
2	(2,2,1)[A]	(1,1,1)[A]	$-\frac{7+3\sqrt{5}}{16}$	$-\frac{1+3\sqrt{5}}{8}$
3	(2,2,1)[C]	(1,1,1)[A]	$\frac{17-21\sqrt{5}}{64}$	$-\frac{7+3\sqrt{5}}{16}$
4	(2,2,1)[A]	(2,2,1)[A]	$-\frac{11}{16}$	$-\frac{11}{16}$
5	(2,2,1)[A]	(2,2,1)[C]	-1	$-\frac{11}{16}$
6	(2,2,1)[C]	(2,2,1)[A]	$-\frac{11}{16}$	-1
7	(2,2,1)[C]	(2,2,1)[C]	$\frac{7}{128}$	$\frac{7}{128}$
8	(3,2,2)[A]	(1,1,1)[A]	$-\frac{31-3\sqrt{21}}{64}$	$\frac{1-3\sqrt{21}}{16}$
9	(3,2,2)[B]	(1,1,1)[A]	$\frac{1-3\sqrt{21}}{16}$	$-\frac{3\sqrt{21}}{8}$
10	(3,2,2)[A]	(2,2,1)[A]	$-\frac{31-3\sqrt{105}}{128}$	$\frac{7-3\sqrt{105}}{64}$
11	(3,2,2)[A]	(2,2,1)[C]	$-\frac{217-3\sqrt{105}}{256}$	$-\frac{17+21\sqrt{105}}{256}$

Table 4: Pair of triangles (X_{ABC}) and (Y_{ABC}) adding three angles in two different forms $(2X+Y)$ and $(X+2Y)$.

Pair	$X = A \vee B \vee C$	$Y = A \vee B \vee C$	$\cos(2X+Y)$	$\cos(X+2Y)$
12	(3,2,2)[B]	(2,2,1)[A]	$\frac{1-3\sqrt{105}}{32}$	$-\frac{21\sqrt{105}}{32}$
13	(3,2,2)[B]	(2,2,1)[C]	$\frac{7-3\sqrt{105}}{64}$	$\frac{51-7\sqrt{105}}{128}$
14	(3,2,2)[A]	(3,2,2)[A]	$\frac{47}{128}$	$\frac{47}{128}$
15	(3,2,2)[A]	(3,2,2)[B]	$-\frac{9}{16}$	-1
16	(3,2,2)[B]	(3,2,2)[A]	-1	$-\frac{9}{16}$
17	(3,2,2)[B]	(3,2,2)[B]	$-\frac{9}{16}$	$-\frac{9}{16}$
18	(3,3,1)[A]	(1,1,1)[A]	$-\frac{17\sqrt{105}}{36}$	$-\frac{1\sqrt{105}}{12}$
19	(3,3,1)[C]	(1,1,1)[A]	$\frac{127-17\sqrt{105}}{324}$	$-\frac{17\sqrt{105}}{36}$
20	(3,3,1)[A]	(2,2,1)[A]	$-\frac{17+5\sqrt{21}}{72}$	$-\frac{7+5\sqrt{21}}{48}$
21	(3,3,1)[A]	(2,2,1)[C]	$-\frac{119+5\sqrt{21}}{144}$	$\frac{17-35\sqrt{21}}{192}$
22	(3,3,1)[C]	(2,2,1)[A]	$\frac{127-85\sqrt{21}}{648}$	$-\frac{119+5\sqrt{21}}{144}$
23	(3,3,1)[C]	(2,2,1)[C]	$\frac{889-85\sqrt{21}}{1296}$	$\frac{289-35\sqrt{21}}{576}$
24	(3,3,1)[A]	(3,2,2)[A]	$\frac{17-21\sqrt{5}}{144}$	$-\frac{31-21\sqrt{5}}{192}$
25	(3,3,1)[A]	(3,2,2)[B]	$-\frac{51+7\sqrt{5}}{72}$	$\frac{1-21\sqrt{5}}{48}$
26	(3,3,1)[C]	(3,2,2)[A]	$-\frac{127+357\sqrt{5}}{1296}$	$-\frac{527-21\sqrt{5}}{576}$
27	(3,3,1)[C]	(3,2,2)[B]	$\frac{381-119\sqrt{5}}{648}$	$\frac{17-21\sqrt{5}}{144}$
28	(3,3,1)[A]	(3,3,1)[A]	$-\frac{13}{27}$	$-\frac{13}{27}$
29	(3,3,1)[A]	(3,3,1)[C]	-1	$-\frac{13}{27}$
30	(3,3,1)[C]	(3,3,1)[A]	$-\frac{13}{27}$	-1
31	(3,3,1)[C]	(3,3,1)[C]	$\frac{391}{729}$	$\frac{391}{729}$
32	(3,3,2)[A]	(1,1,1)[A]	$-\frac{7+4\sqrt{6}}{18}$	$-\frac{1+2\sqrt{6}}{6}$
33	(3,3,2)[C]	(1,1,1)[A]	$\frac{17-56\sqrt{6}}{162}$	$-\frac{7+4\sqrt{6}}{18}$
34	(3,3,2)[A]	(2,2,1)[A]	$-\frac{7+4\sqrt{30}}{36}$	$-\frac{7+2\sqrt{30}}{24}$
35	(3,3,2)[A]	(2,2,1)[C]	$-\frac{49+4\sqrt{30}}{72}$	$\frac{17-14\sqrt{30}}{96}$
36	(3,3,2)[C]	(2,2,1)[A]	$\frac{17-56\sqrt{30}}{324}$	$-\frac{49+4\sqrt{30}}{72}$
37	(3,3,2)[C]	(2,2,1)[C]	$\frac{119-56\sqrt{30}}{648}$	$\frac{119-28\sqrt{30}}{288}$
38	(3,3,2)[A]	(3,2,2)[A]	$\frac{7-12\sqrt{14}}{72}$	$-\frac{31-6\sqrt{14}}{96}$
39	(3,3,2)[A]	(3,2,2)[B]	$-\frac{21+4\sqrt{14}}{36}$	$\frac{1-6\sqrt{14}}{24}$
40	(3,3,2)[C]	(3,2,2)[A]	$-\frac{17+168\sqrt{14}}{648}$	$-\frac{217-12\sqrt{14}}{288}$
41	(3,3,2)[C]	(3,2,2)[B]	$\frac{51-56\sqrt{14}}{324}$	$\frac{7-12\sqrt{14}}{72}$
42	(3,3,2)[A]	(3,3,1)[A]	$-\frac{7+4\sqrt{70}}{54}$	$-\frac{17+2\sqrt{70}}{54}$
43	(3,3,2)[A]	(3,3,1)[C]	$-\frac{119+4\sqrt{70}}{162}$	$\frac{127-34\sqrt{70}}{486}$
44	(3,3,2)[C]	(3,3,1)[A]	$\frac{17-56\sqrt{70}}{486}$	$-\frac{119+4\sqrt{70}}{162}$
45	(3,3,2)[C]	(3,3,1)[C]	$\frac{289-56\sqrt{70}}{1458}$	$\frac{889-68\sqrt{70}}{1458}$
46	(3,3,2)[A]	(3,3,2)[A]	$-\frac{23}{27}$	$-\frac{23}{27}$

Table 4: Pair of triangles (X_{ABC}) and (Y_{ABC}) adding three angles in two different forms ($2X + Y$) and ($X + 2Y$).

Pair	$X = A \vee B \vee C$	$Y = A \vee B \vee C$	$\cos(2X + Y)$	$\cos(X + 2Y)$
47	(3,3,2)[A]	(3,3,2)[C]	-1	$-\frac{23}{27}$
48	(3,3,2)[C]	(3,3,2)[A]	$-\frac{23}{27}$	-1
49	(3,3,2)[C]	(3,3,2)[C]	$-\frac{329}{729}$	$-\frac{329}{729}$

Table 5: Pair of triangles (X_{ABC}) and (Y_{ABC}) adding three angles in two different forms ($2X + Y$) and ($X + 2Y$).

Pair	$X = A \vee B \vee C$	$Y = A \vee B \vee C$	$Z = A \vee B \vee C$	$\cos(X + Y + Z)$
	pending...			

Data from: `github.com/heptagons/meccano/nest/t_test.go TestTCosXYZ`

4 Triangle (a, b, c) diagonals

Within the triangle (a, b, c) we can form diagonals as lines joining integer points of a given side a, b, c with others points of another side. To calculate the diagonals we use the law of cosines. Using equation 3 we can calculate diagonals $\overline{b_i c_j}$:

$$\begin{aligned} \overline{b_i c_j} &= \sqrt{i^2 + j^2 - 2ij\cos A} = \sqrt{i^2 + j^2 - 2ij\frac{a_n}{a_d}} \\ &= \frac{\sqrt{a_d^2(i^2 + j^2) - 2a_n ij}}{a_d} \end{aligned} \tag{10}$$

where $i, j \in \mathbb{N}$ are sides points positions starting with 1 (don't confuse i with $\sqrt{-1}$) and$1 \le i \le b$, $1 \le j \le c$ and $i \ge j$. For the whole triangle we have:

$$\begin{pmatrix} \overline{b_i c_j} \\ \overline{a_i c_j} \\ \overline{a_i b_j} \end{pmatrix} = \begin{pmatrix} \dfrac{\sqrt{a_d^2(i^2 + j^2) - 2a_n ij}}{a_d} \\ \dfrac{\sqrt{b_d^2(i^2 + j^2) - 2b_n ij}}{b_d} \\ \dfrac{\sqrt{c_d^2(i^2 + j^2) - 2c_n ij}}{c_d} \end{pmatrix} \in \mathbb{A} \tag{11}$$

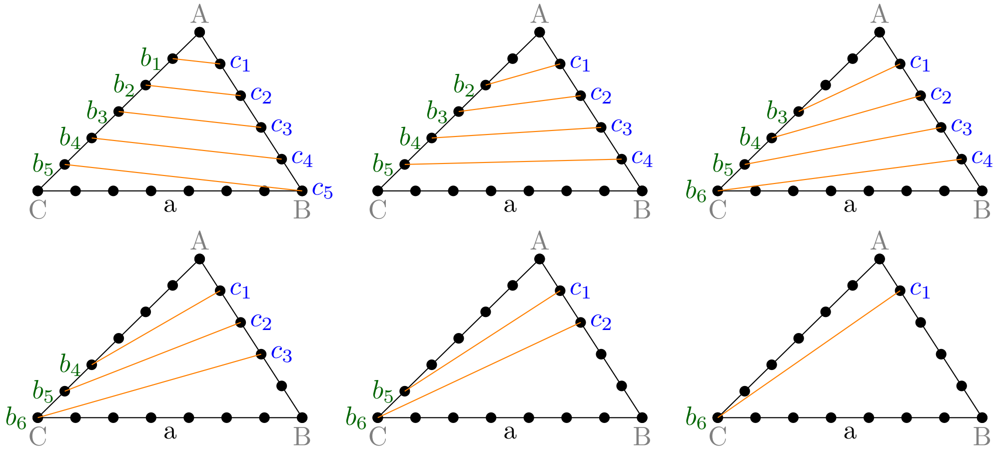

Figure 1: Triangle $(7,6,5)$ b_ic_j diagonals $(i \geq j)$.

4.1 Example triangle (7,6,5)

Figure 1 show triangle $(7,6,5)$ diagonals b_mc_n for vertex A. The diagonals values are set in a first matrix with columns $b_1, ..., b_6$ and rows $c_1, ..., c_5$. Empty cells are repetitions.

$$b_{1:6}c_{1:5} = \begin{pmatrix} \frac{2\sqrt{10}}{5} & \frac{\sqrt{105}}{5} & \frac{2\sqrt{55}}{5} & \frac{\sqrt{385}}{5} & 2\sqrt{6} & \frac{\sqrt{865}}{5} \\ & \frac{4\sqrt{10}}{5} & \frac{\sqrt{265}}{5} & \frac{2\sqrt{105}}{5} & 5 & \frac{4\sqrt{55}}{5} \\ & & \frac{6\sqrt{10}}{5} & \frac{\sqrt{505}}{5} & 2\sqrt{7} & \frac{3\sqrt{105}}{5} \\ & & & \frac{8\sqrt{10}}{5} & \sqrt{33} & \frac{2\sqrt{265}}{5} \\ & & & & 2\sqrt{10} & \end{pmatrix} \tag{12}$$

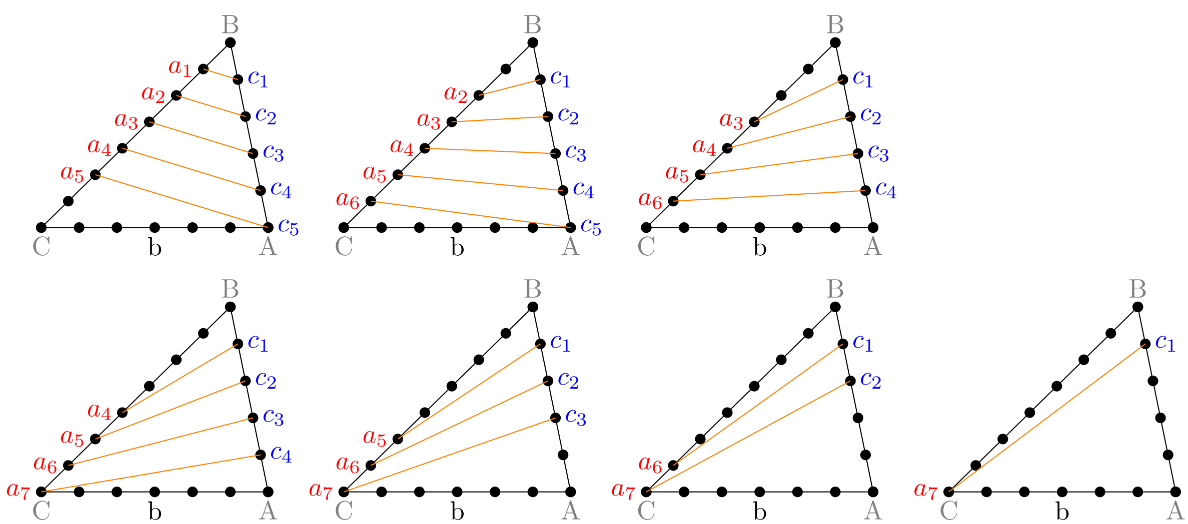

Figure 2: Triangle $(7,6,5)$, a_ic_j diagonals $(i \geq j)$.

Figure 2 show triangle $(7,6,5)$ diagonals a_ic_j for vertex B. The diagonals are set in a second matrix with columns $a_1, ..., a_7$ and rows $c_1, ..., c_5$. Emtpy cells are repetitions. Values at column 7 are repeated and already accounted in previous matrix.

$$a_{1:7}c_{1:5} \begin{pmatrix} \frac{4\sqrt{70}}{35} & \frac{3\sqrt{385}}{35} & \frac{2\sqrt{2065}}{35} & \frac{\sqrt{15505}}{35} & \frac{12\sqrt{7}}{7} & \frac{\sqrt{37345}}{35} & \frac{2\sqrt{265}}{5} \\ & \frac{8\sqrt{70}}{35} & \frac{\sqrt{7945}}{35} & \frac{6\sqrt{385}}{35} & \frac{\sqrt{889}}{7} & \frac{4\sqrt{2065}}{35} & \frac{3\sqrt{105}}{5} \\ & & \frac{12\sqrt{70}}{35} & \frac{\sqrt{14665}}{35} & \frac{2\sqrt{217}}{7} & \frac{9\sqrt{385}}{35} & \frac{4\sqrt{55}}{5} \\ & & & \frac{16\sqrt{70}}{35} & \frac{3\sqrt{105}}{7} & \frac{2\sqrt{7945}}{35} & \frac{\sqrt{865}}{5} \\ & & & & \frac{4\sqrt{70}}{7} & \frac{\sqrt{1393}}{7} & \end{pmatrix} \tag{13}$$

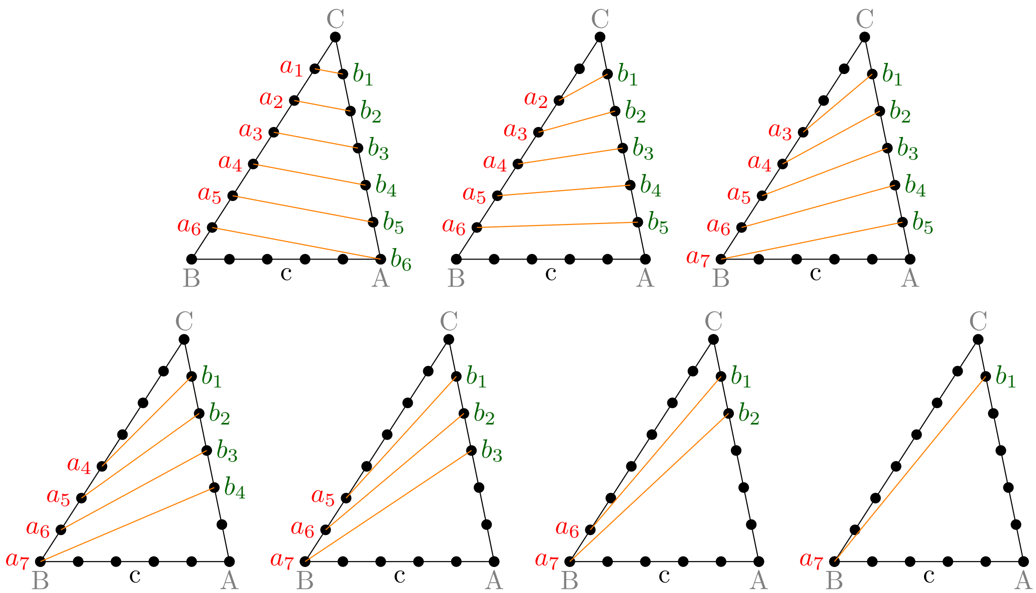

Figure 3: Triangle $(7, 6, 5)$, a_ib_j diagonals $(i \geq j)$

Figure 3 show triangle $(7, 6, 5)$ diagonals a_ib_j for vertex C. The diagonals are set in a third matrix with columns $a_1, ..., a_7$ and rows $b_1, ..., b_6$. Empty cells are repetitions. Values at columns 6 and 7 are repeated and already in previous matrices.

$$a_{1:7}b_{1:6} = \begin{pmatrix} \frac{2\sqrt{7}}{7} & \frac{\sqrt{105}}{7} & \frac{2\sqrt{70}}{7} & \frac{\sqrt{553}}{7} & \frac{2\sqrt{231}}{7} & \frac{\sqrt{1393}}{7} & 2\sqrt{10} \\ & \frac{4\sqrt{7}}{7} & \frac{\sqrt{217}}{7} & \frac{2\sqrt{105}}{7} & \frac{\sqrt{721}}{7} & \frac{4\sqrt{70}}{7} & \sqrt{33} \\ & & \frac{6\sqrt{7}}{7} & \frac{\sqrt{385}}{7} & \frac{2\sqrt{154}}{7} & \frac{3\sqrt{105}}{7} & 2\sqrt{7} \\ & & & \frac{8\sqrt{7}}{7} & \frac{\sqrt{609}}{7} & \frac{2\sqrt{217}}{7} & 5 \\ & & & & \frac{10\sqrt{7}}{7} & \frac{\sqrt{889}}{7} & 2\sqrt{6} \\ & & & & & \frac{12\sqrt{7}}{7} & \end{pmatrix} \tag{14}$$

The values of the matrices are calculated with the code: `github.com/heptagons/meccano/nest/t_test.go TestT765diags`.

5 Triangles($\sqrt{\alpha}, b, c$)

Triangles($\sqrt{\alpha}, b, c$) have three sides with lengths $a = \sqrt{\alpha}$, b and c where $\alpha, b, c \in \mathbb{N}$ and α is square-free. We have:

$$\sqrt{\alpha} > b \geq c \qquad \Longrightarrow \alpha > b^2 \geq c^2 \tag{15}$$

$$\sqrt{\alpha} < b + c \qquad \Longrightarrow \alpha < (b+c)^2 \tag{16}$$

We calculate the triangle cosines. $\cos A_\alpha$ is rational and $\cos B_\alpha$ and $\cos C_\alpha$ are algebraic:

$$\cos A_\alpha = \frac{b^2 + c^2 - (\sqrt{\alpha})^2}{2bc} = \frac{b^2 + c^2 - \alpha}{2bc} \qquad \equiv \frac{\alpha_n}{\alpha_d} \in \mathbb{Q} \tag{17}$$

$$\cos B_\alpha = \frac{(\sqrt{\alpha})^2 + c^2 - b^2}{2\sqrt{\alpha}c} = \frac{(\alpha + c^2 - b^2)\sqrt{\alpha}}{2\alpha c} \qquad \in \mathbb{A} \tag{18}$$

$$\cos C_\alpha = \frac{(\sqrt{\alpha})^2 + b^2 - c^2}{2\sqrt{\alpha}b} = \frac{(\alpha + b^2 - c^2)\sqrt{\alpha}}{2\alpha b} \qquad \in \mathbb{A} \tag{19}$$

5.1 Triangle ($\sqrt{\alpha}, b, c$) diagonals

The only possible diagonals are for sides with integers points, that is segments $\overline{b_i c_j}$. Using the law of cosines:

$$\begin{aligned}
\overline{b_i c_j} &= \sqrt{i^2 + j^2 - 2ij\cos A_\alpha} \\
&= \sqrt{i^2 + j^2 - 2ij\frac{\alpha_n}{\alpha_d}} \\
&= \frac{\sqrt{\alpha_d^2(i^2 + j^2) - 2\alpha_n ij}}{\alpha_d} \qquad \in \mathbb{A}
\end{aligned} \tag{20}$$

where $1 \leq i \leq b$, $1 \leq j \leq c$ and $i \geq j$.

5.2 Example triangles($2\sqrt{6}, b, c$)

In this case $\sqrt{\alpha} = 2\sqrt{6}$ so $\alpha = 24$. Then sets $i = j = \{1, 2, 3, 4\}$ because sets $b^2 = c^2 = \{1, 4, 9, 16\} < 24$. We form a matrix with the values $(b+c)^2$ and satisfying $b \geq c$:

$$(b_i + c_j)^2 = \begin{array}{c} \\ c=1 \\ c=2 \\ c=3 \\ c=4 \end{array} \begin{array}{c} \begin{array}{cccc} b=1 & b=2 & b=3 & b=4 \end{array} \\ \begin{pmatrix} 2 & 9 & 16 & 25 \\ \times & 16 & 25 & 36 \\ \times & \times & 36 & 49 \\ \times & \times & \times & 64 \end{pmatrix} \end{array} \tag{21}$$

Then we remove the cells that don't satisfy the condition $\alpha < (b+c)^2$:

$$(b_m + c_n)^2 = \begin{array}{c} \\ c=1 \\ c=2 \\ c=3 \\ c=4 \end{array} \begin{array}{c} \begin{array}{cccc} b=1 & b=2 & b=3 & b=4 \end{array} \\ \begin{pmatrix} \times & \times & \times & 25 \\ & \times & 25 & 36 \\ & & 36 & 49 \\ & & & 64 \end{pmatrix} \end{array} \tag{22}$$

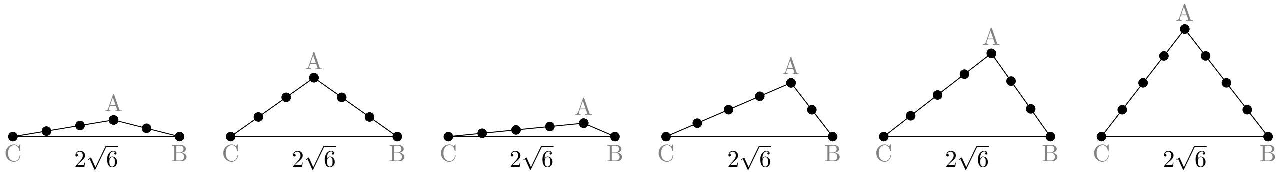

Figure 4: All triangles with sides $a = 2\sqrt{6} > b \geq c$

Each remaining cell in the matrix corresponds to a particular triangle:

$$(2\sqrt{6}, b, c) = \begin{array}{c} \\ (2\sqrt{6},3,2) \\ (2\sqrt{6},3,3) \\ (2\sqrt{6},4,1) \\ (2\sqrt{6},4,2) \\ (2\sqrt{6},4,3) \\ (2\sqrt{6},4,4) \end{array} \begin{array}{c} \begin{array}{ccc} \cos A_\alpha & \cos B_\alpha & \cos C_\alpha \end{array} \\ \begin{pmatrix} -\frac{11}{12} & \frac{19\sqrt{6}}{48} & \frac{29\sqrt{6}}{72} \\ -\frac{1}{3} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{3} \\ -\frac{7}{8} & \frac{3\sqrt{6}}{8} & \frac{13\sqrt{6}}{32} \\ -\frac{1}{4} & \frac{\sqrt{6}}{4} & \frac{3\sqrt{6}}{8} \\ \frac{1}{24} & \frac{17\sqrt{6}}{72} & \frac{31\sqrt{6}}{96} \\ \frac{1}{4} & \frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} \end{pmatrix} \end{array} \tag{23}$$

Figure 4 show the triangles $(2\sqrt{6}, b, c)$. The cosines are calculated by code at:

`github.com/heptagons/meccano/nest/t_test.go TestTslursA`

6 Triangles$(a, \sqrt{\beta}, c)$

Triangles$(a, \sqrt{\beta}, c)$ have the three sides a, $b = \sqrt{\beta}$ and c where $a, \beta, c \in \mathbb{N}$ and β is square-free. We have:

$$a > \sqrt{\beta} > c \qquad \Longrightarrow a^2 > \beta > c^2 \tag{24}$$

$$a < \sqrt{\beta} + c \qquad \Longrightarrow (a-c)^2 < \beta \tag{25}$$

We calculate the triangle cosines:

$$\cos A_\beta = \frac{(\sqrt{\beta})^2 + c^2 - a^2}{2\sqrt{\beta}c} = \frac{(\beta + c^2 - a^2)\sqrt{\beta}}{2\beta c} \qquad \in \mathbb{A} \tag{26}$$

$$\cos B_\beta = \frac{a^2 + c^2 - (\sqrt{\beta})^2}{2ac} = \frac{a^2 + c^2 - \beta}{2ac} \qquad \equiv \frac{\beta_n}{\beta_d} \in \mathbb{Q} \tag{27}$$

$$\cos C_\beta = \frac{a^2 + (\sqrt{\beta})^2 - c^2}{2a\sqrt{\beta}} = \frac{(a^2 + \beta - c^2)\sqrt{\beta}}{2a\beta} \qquad \in \mathbb{A} \tag{28}$$

6.1 Triangle $(a, \sqrt{\beta}, c)$ diagonals

The only possible diagonals are for sides with integers points, that is segments $\overline{a_i c_j}$. Using the law of cosines:

$$\overline{a_i c_j} = \sqrt{i^2 + j^2 - 2ij \cos B_\beta} \tag{29}$$

$$= \sqrt{i^2 + j^2 - 2ij \frac{\beta_n}{\beta_d}} \tag{30}$$

$$= \frac{\sqrt{\beta_d^2 (i^2 + j^2) - 2\beta_n ij}}{\beta_d} \quad \in \mathbb{A} \tag{31}$$

where $1 \leq i \leq a$, $1 \leq j \leq c$ and $i \geq j$.

6.2 Example triangles$(a, 2\sqrt{6}, c)$

In this case $\sqrt{\beta} = 2\sqrt{6}$ so $\beta = 24$. Then $i = \{5, 6, 7, ...\}$ because $a^2 = \{25, 36, 49, ...\} > 24$ and $j = \{1, 2, 3, 4\}$ because $c^2 = \{1, 4, 9, 16\} < 24$. We form a matrix with the values $(a - c)^2$:

$$(a_i - c_j)^2 = \begin{array}{c} \\ c=1 \\ c=2 \\ c=3 \\ c=4 \end{array} \begin{array}{c} \begin{array}{cccccc} a=5 & a=6 & a=7 & a=8 & a=9 & ... \end{array} \\ \begin{pmatrix} 16 & 25 & 36 & 49 & 64 & ... \\ 9 & 16 & 25 & 36 & 49 & ... \\ 4 & 9 & 16 & 25 & 36 & ... \\ 1 & 4 & 9 & 16 & 25 & ... \end{pmatrix} \end{array} \tag{32}$$

We remove cells which don't satisfy the condition $(a - c)^2 < \beta = 24$:

$$(a_i - c_j)^2 = \begin{array}{c} \\ c=1 \\ c=2 \\ c=3 \\ c=4 \end{array} \begin{array}{c} \begin{array}{cccccc} a=5 & a=6 & a=7 & a=8 & a=9 & ... \end{array} \\ \begin{pmatrix} 16 & \times & \times & \times & \times & ... \\ 9 & 16 & \times & \times & \times & ... \\ 4 & 9 & 16 & \times & \times & ... \\ 1 & 4 & 9 & 16 & \times & ... \end{pmatrix} \end{array} \tag{33}$$

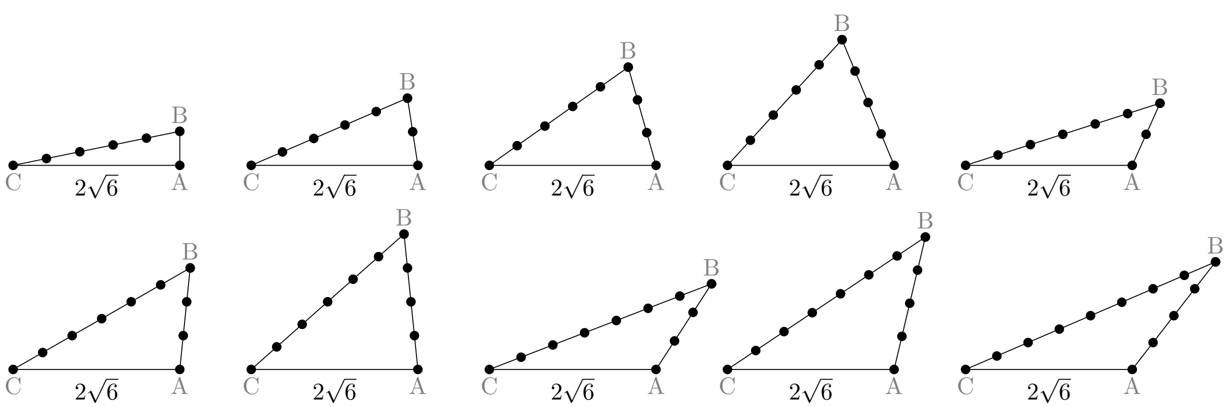

Figure 5: All triangles with sides $a > b = 2\sqrt{6} > c$

So we have ten triangles are valid:

$$
(a, 2\sqrt{6}, c) = \begin{array}{c} \\ (5,2\sqrt{6},1) \\ (5,2\sqrt{6},2) \\ (5,2\sqrt{6},3) \\ (5,2\sqrt{6},4) \\ (6,2\sqrt{6},2) \\ (6,2\sqrt{6},3) \\ (6,2\sqrt{6},4) \\ (7,2\sqrt{6},3) \\ (7,2\sqrt{6},4) \\ (8,2\sqrt{6},4) \end{array} \begin{array}{c} \begin{array}{ccc} \cos A_\beta & \cos B_\beta & \cos C_\beta \end{array} \\ \begin{pmatrix} 0 & \frac{1}{5} & \frac{2\sqrt{6}}{5} \\ \frac{\sqrt{6}}{16} & \frac{1}{4} & \frac{3\sqrt{6}}{8} \\ \frac{\sqrt{6}}{9} & \frac{1}{3} & \frac{\sqrt{6}}{3} \\ \frac{5\sqrt{6}}{32} & \frac{17}{40} & \frac{11\sqrt{6}}{40} \\ \frac{\sqrt{6}}{6} & \frac{2}{3} & \frac{7\sqrt{6}}{18} \\ \frac{\sqrt{6}}{24} & \frac{7}{12} & \frac{17\sqrt{6}}{48} \\ \frac{\sqrt{6}}{24} & \frac{7}{12} & \frac{11\sqrt{6}}{36} \\ -\frac{2\sqrt{6}}{9} & \frac{17}{21} & \frac{8\sqrt{6}}{21} \\ -\frac{3\sqrt{6}}{32} & \frac{41}{56} & \frac{19\sqrt{6}}{56} \\ \frac{\sqrt{6}}{4} & \frac{7}{8} & \frac{3\sqrt{6}}{8} \end{pmatrix} \end{array} \tag{34}
$$

Figure 5 show the triangles $(a, 2\sqrt{6}, c)$. The cosines are calculated by code at:

`github.com/heptagons/meccano/nest/t_test.go TestTslursB`

7 Triangles$(a, b, \sqrt{\gamma})$

Triangles$(a, b, \sqrt{\gamma})$ have three sides a, b, $\sqrt{\gamma}$ where $a, b, \gamma \in \mathbb{N}$ and γ is square-free. We have:

$$
a \geq b > \sqrt{\gamma} \implies a^2 \geq b^2 > \gamma \tag{35}
$$

$$
a < b + \sqrt{\gamma} \implies (a-b)^2 < \gamma \tag{36}
$$

We calculate the triangle cosines:

$$
\cos A_\gamma = \frac{b^2 + \gamma - a^2}{2b\sqrt{\gamma}} = \frac{(b^2 + \gamma - a^2)\sqrt{\gamma}}{2b\gamma} \in \mathbb{A} \tag{37}
$$

$$
\cos B_\gamma = \frac{a^2 + \gamma - b^2}{2a\sqrt{\gamma}} = \frac{(a^2 + \gamma - b^2)\sqrt{\gamma}}{2a\gamma} \in \mathbb{A} \tag{38}
$$

$$
\cos C_\gamma = \frac{a^2 + b^2 - (\sqrt{\gamma})^2}{2ab} = \frac{a^2 + b^2 - \gamma}{2ab} \equiv \frac{\gamma_n}{\gamma_d} \in \mathbb{Q} \tag{39}
$$

7.1 Triangle $(a, b, \sqrt{\gamma})$ diagonals

The only possible diagonals are for sides with integers, that is $\overline{a_i b_j}$. Using the law of cosines:

$$
\overline{a_i b_j} = \sqrt{i^2 + j^2 - 2ij \cos C_\gamma} \tag{40}
$$

$$
= \sqrt{i^2 + j^2 - 2ij\frac{\gamma_n}{\gamma_d}} \tag{41}
$$

$$
= \frac{\sqrt{\gamma_d^2(i^2 + j^2) - 2\gamma_n ij}}{\gamma_d} \in \mathbb{A} \tag{42}
$$

where $1 \leq i < a$, $1 \leq j < b$ and $i \geq j$.

7.2 Example triangles$(a, b, 2\sqrt{6})$

In this case $\sqrt{\gamma} = 2\sqrt{6}$ so $\gamma = 24$. We form a matrix with with the values $(a-b)^2$ satisfying the condition $a^2 \geq b^2 > \gamma$:

$$(a_i - b_j)^2 = \begin{array}{cc} & \begin{array}{ccccccccc} a=5 & a=6 & a=7 & a=8 & a=9 & a=10 & a=11 & a=12 & \dots \end{array} \\ \begin{array}{c} b=5 \\ b=6 \\ b=7 \\ b=8 \\ b=9 \\ b=10 \\ \end{array} & \begin{pmatrix} 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & \dots \\ & 0 & 1 & 4 & 9 & 16 & 25 & 36 & \dots \\ & & 0 & 1 & 4 & 9 & 16 & 25 & \dots \\ & & & 0 & 1 & 4 & 9 & 16 & \dots \\ & & & & 0 & 1 & 4 & 9 & \dots \\ & & & & & 0 & 1 & 4 & \dots \\ & & & & & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array} \tag{43}$$

We remove cells except those satisfying the condition $(a-b)^2 < \gamma$:

$$(a_i - b_j)^2 = \begin{array}{cc} & \begin{array}{ccccccccc} a=5 & a=6 & a=7 & a=8 & a=9 & a=10 & a=11 & a=12 & \dots \end{array} \\ \begin{array}{c} b=5 \\ b=6 \\ b=7 \\ b=8 \\ b=9 \\ b=10 \\ \end{array} & \begin{pmatrix} 0 & 1 & 4 & 9 & 16 & \times & \times & \times & \dots \\ & 0 & 1 & 4 & 9 & 16 & \times & \times & \dots \\ & & 0 & 1 & 4 & 9 & 16 & \times & \dots \\ & & & 0 & 1 & 4 & 9 & 16 & \dots \\ & & & & 0 & 1 & 4 & 9 & \dots \\ & & & & & 0 & 1 & 4 & \dots \\ & & & & & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array} \tag{44}$$

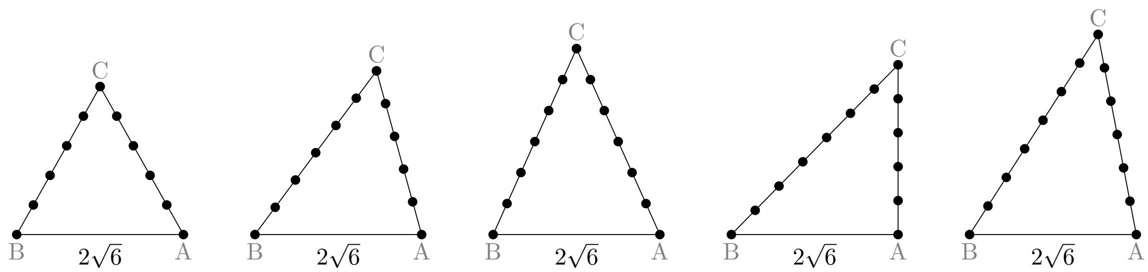

Figure 6: Some triangles with sides $a \geq b > c = 2\sqrt{6}$

So we found that infinite triangles are valid, the smaller ones are:

$$(a, b, 2\sqrt{6}) = \begin{array}{cc} & \begin{array}{ccc} \cos A_\gamma & \cos B_\gamma & \cos C_\gamma \end{array} \\ \begin{array}{c} (5,5,2\sqrt{6}) \\ (6,5,2\sqrt{6}) \\ (6,6,2\sqrt{6}) \\ (7,5,2\sqrt{6}) \\ (7,6,2\sqrt{6}) \\ (7,7,2\sqrt{6}) \\ \dots \end{array} & \begin{pmatrix} \frac{\sqrt{6}}{5} & \frac{\sqrt{6}}{5} & \frac{13}{25} \\ \frac{13\sqrt{6}}{120} & \frac{35\sqrt{6}}{144} & \frac{37}{60} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{2}{3} \\ 0 & \frac{2\sqrt{6}}{7} & \frac{5}{7} \\ \frac{11\sqrt{6}}{144} & \frac{37\sqrt{6}}{168} & \frac{61}{84} \\ \frac{\sqrt{6}}{7} & \frac{\sqrt{6}}{7} & \frac{37}{49} \\ \dots & \dots & \dots \end{pmatrix} \end{array} \tag{45}$$

Figure 6 show some triangles $(a, b, 2\sqrt{6})$. The cosines are calculated by code at:

`github.com/heptagons/meccano/nest/t_test.go TestTslursC`

7.3 Triangles pairs diagonals

So we can calculate new diagonals from one triangle side to another triangle side:

$$\delta = \sqrt{m^2 + n^2 - 2mn\cos Z} \tag{46}$$

$$= \sqrt{m^2 + n^2 - 2mn\frac{b_1 + c_1\sqrt{d_1}}{a_1}} \tag{47}$$

$$= \frac{\sqrt{a_1^2(m^2 + n^2) - 2mn(b_1 + c_1\sqrt{d_1})}}{a_1} \tag{48}$$

$$= \frac{\sqrt{a_1^2(m^2 + n^2) - 2b_1mn - 2c_1mn\sqrt{d_1}}}{a_1} \qquad \equiv \frac{b_2 + c_2\sqrt{d_2 + e_2\sqrt{f_2}}}{a_2} \tag{49}$$

7.4 Triangles pairs surds angles

When we sum two algebraic angles $W = U + V$ when $\cos U = \sqrt{u_n}/u_d$ and $\cos V = \sqrt{v_n}/v_d$ we have:

$$\cos W = \cos U \cos V - \sin U \sin V \tag{50}$$

$$= \cos U \cos V - \sqrt{1 - \cos^2 U}\sqrt{1 - \cos^2 V} \tag{51}$$

$$= \frac{\sqrt{u_n v_n}}{u_d v_d} - \sqrt{\frac{u_d^2 - u_n}{u_d^2}}\sqrt{\frac{v_d^2 - v_n}{v_d^2}} \tag{52}$$

$$= \frac{\sqrt{u_n v_n} - \sqrt{(u_d^2 - u_n)(v_d^2 - v_n)}}{u_d v_d} \tag{53}$$