Meccano pentagons diagonals

https://github.com/heptagons/meccano/penta

Abstract

We construct meccano [1] regular pentagons internal diagonals.

1 Regular polygon diagonals

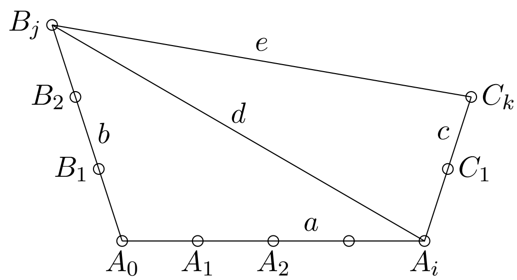

Figure 1: Regular pentagon basic diagonals d and e from sides segments $a \geq\ b \geq c$.

From figure 1 we have the regular internal polygon angle θ:

$$\theta = \angle B_j A_0 A_i = \angle A_0 A_i C_k \tag{1}$$
$$\alpha = \angle A_0 A_i B_j \tag{2}$$
$$\beta = \angle B_j A_i C_k \tag{3}$$
$$\alpha + \beta = \theta \tag{4}$$

We use the cosines sum identity to express $\cos\beta$ in function of the rest of variables. We define $u = \cos\theta$:

$$u \equiv \cos\theta \tag{5}$$
$$\cos(\alpha + \beta) = u \tag{6}$$
$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{7}$$
$$\sin\beta = \frac{\cos\alpha\cos\beta - u}{\sin\alpha} \tag{8}$$
$$\sin^2\beta = \frac{(\cos\alpha\cos\beta - u)^2}{\sin^2\alpha} \tag{9}$$
$$1 - \cos^2\beta = \frac{\cos^2\alpha\cos^2\beta - 2u\cos\alpha\cos\beta + u^2}{sin^2\alpha} \tag{10}$$

We set $X = \cos\beta$ and rearrange the last equation to get:

$$X^2 - 2u\cos\alpha X + u^2 - \sin^2\alpha = 0 \tag{11}$$

[1] Meccano mathematics by ‘t Hooft

And solve the quadratic equation $AX^2 + BX + C = 0$ to get $\cos\beta$:

$$\begin{aligned}
\cos\beta &= \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \\
&= \frac{2u\cos\alpha \pm \sqrt{4u^2\cos^2\alpha - 4(u^2 - \sin^2\alpha)}}{2} \\
&= u\cos\alpha \pm \sqrt{u^2\cos^2\alpha - u^2 + \sin^2\alpha} \qquad (12)
\end{aligned}$$

From the law of cosines we calculate the diagonal d from integers a, b which equal respectively to iterators i, j:

$$\begin{aligned}
d &= \sqrt{a^2 + b^2 - 2ab\cos\theta} \\
&= \sqrt{a^2 + b^2 - 2abu}
\end{aligned} \qquad (13)$$

Using the law of cosines we calculate the angles $\alpha = \angle A_0 A_i B_j$ and $\beta = \angle B_j A_i C_k$:

$$\begin{aligned}
\cos\alpha &= \frac{a^2 + d^2 - b^2}{2ad} \\
&= \frac{a^2 + (a^2 + b^2 - 2abu) - b^2}{2ad} \\
&= \frac{a - bu}{d} \qquad (14) \\
\cos\beta &= \frac{c^2 + d^2 - e^2}{2cd} \\
&= \frac{c^2 + (a^2 + b^2 - 2abu) - e^2}{2cd} \\
&= \frac{a^2 + b^2 + c^2 - e^2 - 2abu}{2cd} \qquad (15)
\end{aligned}$$

We define new variable f to simplify $\cos\beta$ to obtain:

$$f \equiv \frac{a^2 + b^2 + c^2 - e^2}{2} \qquad (16)$$

$$\cos\beta = \frac{f - abu}{cd} \qquad (17)$$

We calculate $\sin^2\alpha = 1 - \cos^2\alpha$:

$$\begin{aligned}
\sin^2\alpha &= 1 - \frac{(a - bu)^2}{d^2} \\
&= \frac{d^2 - a^2 + 2abu - b^2u^2}{d^2} \\
&= \frac{(a^2 + b^2 - 2abu) - a^2 + 2abu - b^2u^2}{d^2} \\
&= \frac{b^2(1 - u^2)}{d^2} \qquad (18)
\end{aligned}$$

We plug the values of $\cos\alpha, \cos\beta, \sin^2\alpha$ in equation 12 to get:

$$\begin{aligned}
\frac{f-abu}{cd} &= \left(\frac{a-bu}{d}\right)u \pm \sqrt{\left(\frac{a-bu}{d}\right)^2 u^2 - u^2 - \frac{b^2(1-u^2)}{d^2}} \\
\frac{f-abu}{c} &= (a-bu)u \pm \sqrt{(a-bu)^2u^2 - d^2u^2 - b^2(1-u^2)} \\
f &= (ab+ac-bcu)u \pm c\sqrt{(a-bu)^2u^2 - d^2u^2 - b^2 + b^2u^2)} \\
&= abu + acu - bcu^2 \pm c\sqrt{a^2u^2 - 2abu^3 + b^2u^4 - d^2u^2 - b^2 + b^2u^2} \qquad (19)
\end{aligned}$$

We define variables m, n to simplify f, so we have:

$$m = abu + acu - bcu^2 \qquad (20)$$

$$n = a^2u^2 - 2abu^3 + b^2u^4 - d^2u^2 - b^2 + b^2u^2 \qquad (21)$$

$$f = m + c\sqrt{n} \qquad (22)$$

1.1 Regular pentagon diagonals

For the regular pentagon we have $u = \cos\theta = \cos(3\pi/5)$:

$$u = \frac{1-\sqrt{5}}{4} \qquad (23)$$

$$u^2 = \frac{3-\sqrt{5}}{8} \qquad (24)$$

$$u^3 = \frac{2-\sqrt{5}}{8} \qquad (25)$$

$$u^4 = \frac{7-3\sqrt{5}}{32} \qquad (26)$$

We plug values u, u^2, u^3, u^4 in equation 22 to get:

$$\begin{aligned}
m &= ab\left(\frac{1-\sqrt{5}}{4}\right) + ac\left(\frac{1-\sqrt{5}}{4}\right) - bc\left(\frac{3-\sqrt{5}}{8}\right) \\
&= \frac{2ab - 2ab\sqrt{5} + 2ac - 2ac\sqrt{5} - 3bc + bc\sqrt{5}}{8} \\
&= \frac{2ab + 2ac - 3bc + (bc - 2ab - 2ac)\sqrt{5}}{8} \qquad (27) \\
n &= a^2\left(\frac{3-\sqrt{5}}{8}\right) - 2ab\left(\frac{2-\sqrt{5}}{8}\right) + b^2\left(\frac{7-3\sqrt{5}}{32}\right) - d^2\left(\frac{3-\sqrt{5}}{8}\right) - b^2 + b^2\left(\frac{3-\sqrt{5}}{8}\right) \\
&= \frac{12a^2 - 4a^2\sqrt{5} - 16ab + 8\sqrt{5} + 7b^2 - 3b^2\sqrt{5} - 12d^2 + 4d^2\sqrt{5} - 32b^2 + 12b^2 - 4b^2\sqrt{5}}{32} \\
&= \frac{12a^2 - 16ab - 13b^2 - 12d^2 + (-4a^2 + 8 - 3b^2 + 4d^2 - 4b^2)\sqrt{5}}{32} \qquad (28)
\end{aligned}$$