

Triple unit

<https://github.com/heptagons/meccano/units/triple>

Abstract

Triple unit is a group of five meccano ¹ strips a, b, c, d, e intended to build regular polygons three consecutive perimeter sides. This unit has three angles equal to the polygon internal angle θ . Triple unit has been using to build the pentagon type 2 mentioned in pentagons paper².

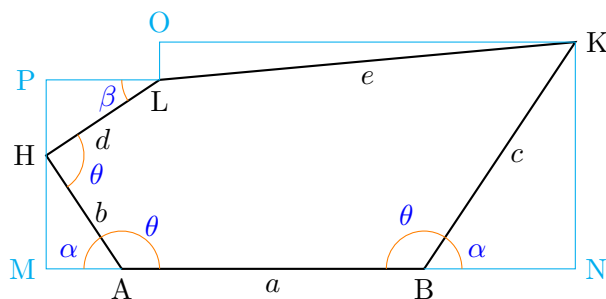


Figure 1: Triple unit has five strips a, b, c, d, e

1 Algebra

From nodes A and B of fig 1 we get α from θ ($\pi = 180^\circ$):

$$\begin{aligned}\theta &= \pi - \alpha \\ \alpha &= \pi - \theta\end{aligned}\tag{1}$$

And from node H we get β from θ :

$$\begin{aligned}\theta &= \alpha + \beta \\ \beta &= \theta - \alpha = \theta - (\pi - \theta) = 2\theta - \pi\end{aligned}\tag{2}$$

We calculate horizontal segment \overline{OK} :

$$\begin{aligned} \overline{OK} &= \overline{MA} + a + \overline{BN} - \overline{PL} \\ &= b \cos \alpha + a + c \cos \alpha - d \cos \beta \\ &= a + (b + c) \cos \alpha - d \cos \beta \\ &= a + (b + c) \cos (\pi - \theta) - d \cos (2\theta - \pi) \\ &= a - (b + c) \cos \theta + d \cos (2\theta) \end{aligned} \quad (3)$$

¹ Meccano mathematics by ‘t Hooft

² Meccano pentagons

And vertical segment \overline{OL} :

$$\begin{aligned}
\overline{OL} &= \overline{KN} - \overline{PH} - \overline{HM} \\
&= c \sin \alpha - d \sin \beta - b \sin \alpha \\
&= (c - b) \sin \alpha - d \sin \beta \\
&= (c - b) \sin (\pi - \theta) - d \sin (2\theta - \pi) \\
&= (c - b) \sin \theta + d \sin (2\theta)
\end{aligned} \tag{4}$$

So we can express e in function of a, b, c, d and angle θ :

$$\begin{aligned}
e^2 &= (\overline{OK})^2 + (\overline{OL})^2 \\
&= (a - (b + c) \cos \theta + d \cos(2\theta))^2 + ((c - b) \sin \theta + d \sin(2\theta))^2 \\
&= a^2 + (b^2 + 2bc + c^2) \cos^2 \theta + d^2 \cos^2(2\theta) + (c^2 - 2cb + b^2) \sin^2 \theta + d^2 \sin^2(2\theta) \\
&\quad - 2a(b + c) \cos \theta + 2ad \cos(2\theta) - 2(b + c)d \cos \theta \cos(2\theta) \\
&\quad + 2(c - b)d \sin \theta \sin(2\theta) \\
&= a^2 + b^2 + c^2 + d^2 + 2bc \cos^2 \theta - 2bc \sin^2 \theta \\
&\quad - 2a(b + c) \cos \theta + 2ad \cos(2\theta) \\
&\quad - 2d((b + c) \cos \theta \cos(2\theta) + (b - c) \sin \theta \sin(2\theta))
\end{aligned} \tag{5}$$

$$\begin{aligned}
&= a^2 + b^2 + c^2 + d^2 + 2bc(\cos^2 \theta - \sin^2 \theta) - 2a(b + c) \cos \theta + 2ad \cos(2\theta) \\
&\quad - 2d(b(\cos \theta \cos(2\theta) + \sin \theta \sin(2\theta)) + c(\cos \theta \cos(2\theta) - \sin \theta \sin(2\theta))) \\
&= a^2 + b^2 + c^2 + d^2 + 2bc \cos(2\theta) - 2a(b + c) \cos \theta + 2ad \cos(2\theta) \\
&\quad - 2d(b \cos(\theta - 2\theta) + c \cos(\theta + 2\theta)) \\
&= a^2 + b^2 + c^2 + d^2 + 2(bc + ad) \cos(2\theta) - 2a(b + c) \cos \theta - 2d(b \cos \theta + c \cos(3\theta)) \\
&= a^2 + b^2 + c^2 + d^2 + 2(bc + ad) \cos(2\theta) - 2(ab + ac) \cos \theta - 2(bd \cos \theta + cd \cos(3\theta)) \\
&= \boxed{a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd) \cos \theta + 2(bc + ad) \cos(2\theta) - 2cd \cos(3\theta)}
\end{aligned} \tag{6}$$

2 Regular polygons

Polygon	θ	$\cos \theta$	$\cos(2\theta)$	$\cos(3\theta)$
Pentagon	$\frac{3\pi}{5}$	$\frac{1 - \sqrt{5}}{4}$	$\frac{-1 - \sqrt{5}}{4}$	$\frac{1 + \sqrt{5}}{4}$
Hexagon	$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\frac{1}{2}$	1
Heptagon	$\frac{5\pi}{7}$			
Octagon	$\frac{3\pi}{4}$			
Decagon	$\frac{4\pi}{5}$			
Dodecagon	$\frac{5\pi}{6}$			

Table 1: Regular polygons internal angles and cosines.

2.1 Equilateral pentagon

We replace the cosines for pentagon in table 1 in e^2 equation:

$$\begin{aligned}
e^2 &= a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd) \cos \theta + 2(bc + ad) \cos(2\theta) - 2cd \cos(3\theta) \\
&= a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd) \left(\frac{1 - \sqrt{5}}{4} \right) + 2(bc + ad) \left(\frac{-1 - \sqrt{5}}{4} \right) - 2cd \left(\frac{1 + \sqrt{5}}{4} \right) \\
&= a^2 + b^2 + c^2 + d^2 - \frac{ab + ac + bd + bc + ad + cd}{2} + \frac{ab + ac + bd - bc - ad - cd}{2} \sqrt{5}
\end{aligned} \tag{7}$$

e cannot to be and integer if the factor of $\sqrt{5}$ is not zero so we force this factor to be zero:

$$\begin{aligned}
ab + ac + bd - bc - ad - cd &= 0 \\
ab + ac + bd &= bc + ad + cd
\end{aligned} \tag{8}$$

We replace $ab + ac + bd$ by $bc + ad + cd$ in the e^2 equation to get:

$$\begin{aligned}
e^2 &= a^2 + b^2 + c^2 + d^2 - \frac{(bc + ad + cd) + bc + ad + cd}{2} + \frac{0}{2} \sqrt{5} \\
&= a^2 + b^2 + c^2 + d^2 - bc - ad - cd \\
e &= \sqrt{a^2 + b^2 + c^2 + d^2 - bc - ad - cd}
\end{aligned} \tag{9}$$

The last formula matches the formula used in the paper Meccano pentagons which finds several pentagons of type 2. Only when we get e integer we have a solution.

2.2 Equilateral hexagon

We replace the cosines for hexagon in table 1 in e^2 equation:

$$e^2 = a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd) \cos \theta + 2(bc + ad) \cos(2\theta) - 2cd \cos(3\theta) \tag{10}$$