Meccano nonagon

https://github.com/heptagons/meccano/nona

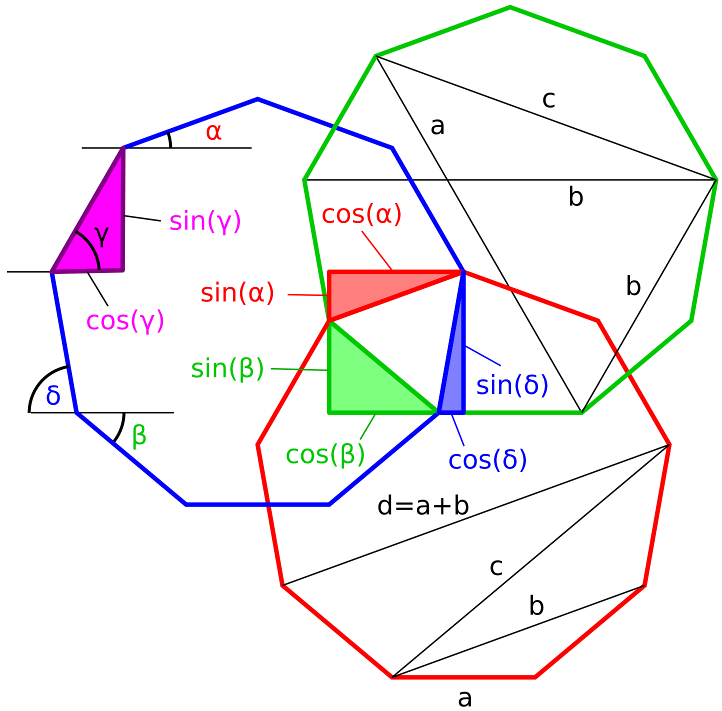

Figure 1: Three regular nonagons connected by an equilateral triangle to prove two relations: $\cos\alpha = \cos\beta + \cos\delta$ and $\sin\alpha + \sin\beta = \sin\delta$.

1 Algebra

Figure 1 shows three regular nonagons connected by an equilateral triangle. Four angles appear orthogonally in any regular nonagon:

$$\alpha = \frac{\pi}{9}, \quad \beta = \frac{2\pi}{9}, \quad \gamma = \frac{3\pi}{9}, \quad \delta = \frac{4\pi}{9} \tag{1}$$

$$\alpha + \beta = \delta - \alpha = \gamma \tag{2}$$

The relations of angle γ are those of equilateral triangle:

$$\cos\gamma = -\frac{1}{2}, \quad \sin\gamma = \frac{\sqrt{3}}{2} \tag{3}$$

From the figure 1, cosines of angles α, β, δ are related as:

$$\begin{aligned}
\cos\alpha &= \cos\beta + \cos\delta && (4)\\
&= \cos(2\alpha) + \cos(4\alpha)\\
&= (2\cos^2\alpha - 1) + (1 - 8\cos^2\alpha + 8\cos^4\alpha)\\
&= 8\cos^4\alpha - 6\cos^2\alpha\\
1 &= 8\cos^3\alpha - 6\cos\alpha && (5)
\end{aligned}$$

The last equation is a depressed cubic equation with a negative discriminant:

$$t^3 + pt + q = 0 \quad \texttt{where} \; p = -\frac{3}{4}, q = -\frac{1}{8} \tag{6}$$
$$\Delta = \frac{q^2}{4} + \frac{p^3}{27} = -\frac{3}{64}$$

The negative discriminant means we have three real roots which can be found by trigonometry:

$$\begin{aligned}
t_k &= 2\sqrt{-\frac{p}{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right) - k\frac{2\pi}{3}\right) && \texttt{for}\; k = 0, 1, 2.\\
&= \cos\left(\frac{1}{3}\arccos\left(\frac{1}{2}\right) - k\frac{2\pi}{3}\right) && \texttt{for}\; k = 0, 1, 2.\\
&= \cos\left(\frac{1}{3}\left(\frac{\pi}{3}\right) - k\frac{2\pi}{3}\right) && \texttt{for}\; k = 0, 1, 2.\\
t_0 &= \cos\left(\frac{\pi}{9}\right) && = \cos\alpha \approx +0.939692620785908 \quad (7)\\
t_1 &= \cos\left(-\frac{2\pi}{9}\right) && = -\cos\beta \approx -0.766044443118978 \quad (8)\\
t_2 &= \cos\left(-\frac{4\pi}{9}\right) && = -\cos\delta \approx -0.173648177666930 \quad (9)
\end{aligned}$$

From equation 6 we know the sum of roots squared is $-2p = \frac{3}{2}$:

$$\begin{aligned}
\cos^2\alpha + \cos^2\beta + \cos^2\delta &= \frac{3}{2} && (10)\\
1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\delta &= \frac{3}{2}\\
\sin^2\alpha + \sin^2\beta + \sin^2\delta &= \frac{3}{2} && (11)
\end{aligned}$$

From equation 6 we know the product of roots is $-q = \frac{1}{8}$ matching the "Morrie's law":

$$\begin{aligned}
\cos\alpha\cos\beta\cos\delta &= \frac{1}{8} && (12)\\
(1 - \sin^2\alpha)(1 - \sin^2\beta)(1 - \sin^2\delta) &= \frac{1}{64}\\
(\sin\alpha\sin\beta)^2 + (\sin\alpha\sin\delta)^2 + (\sin\beta\sin\delta)^2 &= \frac{1}{64} - 1 + \sin^2\alpha + \sin^2\beta + \sin^2\delta + (\sin\alpha\sin\beta\sin\delta)^2\\
&= \frac{1}{64} - 1 + \frac{3}{2} + \left(\frac{\sqrt{3}}{8}\right)^2 = \left(\frac{9}{4}\right)^2 && (13)
\end{aligned}$$

From the figure 1, sines of angles α, β, δ are related as:

$$\begin{aligned}
\sin\alpha + \sin\beta &= \sin\delta \qquad (14)\\
&= \sin(2\alpha+\beta)\\
&= \sin(2\alpha)\cos\beta + \cos(2\alpha)\sin\beta\\
&= (2\sin\alpha\cos\alpha)\cos\beta + (1-2\sin^2\alpha)\sin\beta\\
\sin\alpha &= \sin\alpha(2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta\\
2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta &= 1 \qquad (15)\\
2\cos(\alpha+\beta) = 1 &\implies 2\cos\gamma = 1
\end{aligned}$$

Product of sines of angles α, β, δ is using equations 20 and 21:

$$\begin{aligned}
\sin\alpha\sin\beta\sin\delta &= \frac{1}{2}(2\cos\alpha\cos\beta - 1)(\sin\alpha+\sin\beta)\\
&= \sin(2\alpha)\sin\beta + \sin\alpha\sin(2\beta)\\
&= \frac{\sqrt{3}}{8} \qquad (16)
\end{aligned}$$

Last equation solves this cubic equation:

$$\begin{aligned}
y^3 - \frac{3y}{4} - \frac{3}{8} &= 0\\
y_1 &= -\sin A \approx -0.342020\\
y_2 &= -\sin B \approx -0.642787\\
y_3 &= +\sin C \approx +0.984807
\end{aligned}$$

1.1 Segments a, b, c, d

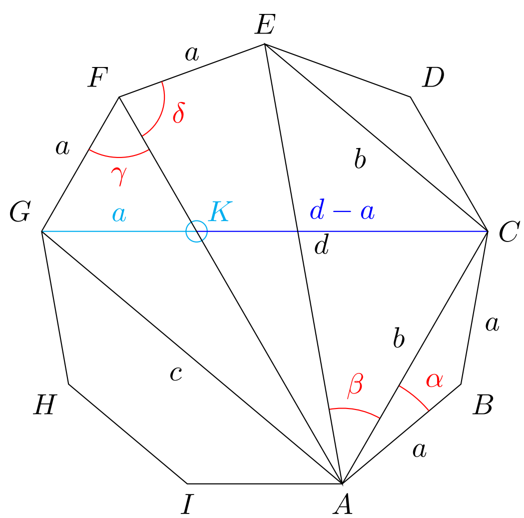

Figure 2: The nonagon perimeter side a and the three internal diagonals b, c, d. Also shown the base angle α and three more $\beta = 2\alpha$, $\gamma = 3\alpha$ and $\delta = 4\alpha$.

From figure 2 we can can calculate $\cos\alpha$ inspecting isosceles $\triangle ABC$ ($\overline{AB} = \overline{AC} = a$):

$$\begin{aligned}
a^2 &= a^2 + b^2 - 2ab\cos\alpha\\
b^2 &= 2ab\cos\alpha \implies \boxed{b = 2a\cos\alpha} \qquad (17)
\end{aligned}$$

We calculate $\cos\beta$ inspecting isosceles $\triangle ACE$ ($\overline{AB} = \overline{AE} = b$):

$$
\begin{aligned}
b^2 &= b^2 + d^2 - 2bd\cos\beta \\
d^2 &= 2bd\cos\beta \implies \boxed{d = 2b\cos\beta} \qquad (18)
\end{aligned}
$$

We calculate $\cos\delta$ inspecting isosceles $\triangle AEF$ ($\overline{AE} = \overline{AF} = $ d):

$$
\begin{aligned}
d^2 &= a^2 + d^2 - 2ad\cos\delta \\
a^2 &= 2ad\cos\delta \implies \boxed{a = 2d\cos\delta} \qquad (19)
\end{aligned}
$$

From equations 17, 18 and 19 we have:

$$
\cos\begin{pmatrix}\alpha\\ \beta\\ \delta\end{pmatrix} = \begin{pmatrix}\dfrac{b}{2a}\\ \dfrac{d}{2b}\\ \dfrac{a}{2d}\end{pmatrix} \qquad (20)
$$

From equation 4 $\cos\alpha = \cos\beta + \cos\delta$:

$$
\frac{b}{2a} = \frac{d}{2b} + \frac{a}{2d} \implies \boxed{\frac{1}{a/b} = \frac{1}{b/d} + \frac{1}{d/a}} \qquad (21)
$$

From figure 2 we inpect equilateral triangles $\triangle ACK$ and $\triangle FGK$ and found that:

$$
\begin{aligned}
\overline{FG} = \overline{GK} = \overline{KF} &= a \qquad (22)\\
\overline{AC} = \overline{CK} = \overline{KA} &= b \qquad (23)\\
\overline{CK} &= \overline{CG} - \overline{KG} \\
b = d - a &\implies \boxed{d = a + b} \qquad (24)
\end{aligned}
$$

Replacing d in optic equation 21 we get:

$$
\frac{b}{a} = \frac{a+b}{b} + \frac{a}{a+b} \qquad (25)
$$

Making $e = b/a$ we update last equation as:

$$
\begin{aligned}
e &\equiv \frac{b}{a} \\
e &= \frac{1+e}{e} + \frac{1}{1+e} \implies \boxed{e^3 - 3e - 1 = 0} \qquad (26)\\
e_0 &\approx 1.87938524157182 && = +2\cos\alpha = b/a \\
e_1 &\approx -1.53208888623796 && = -2\cos\beta = (a+b)/b \\
e_2 &\approx -0.347296355333861 && = -2\cos\delta = a/(a+b) \\
e_0 + e_1 + e_2 &= 0 \qquad (27)\\
e_0 e_1 e_2 &= 1 \qquad (28)
\end{aligned}
$$

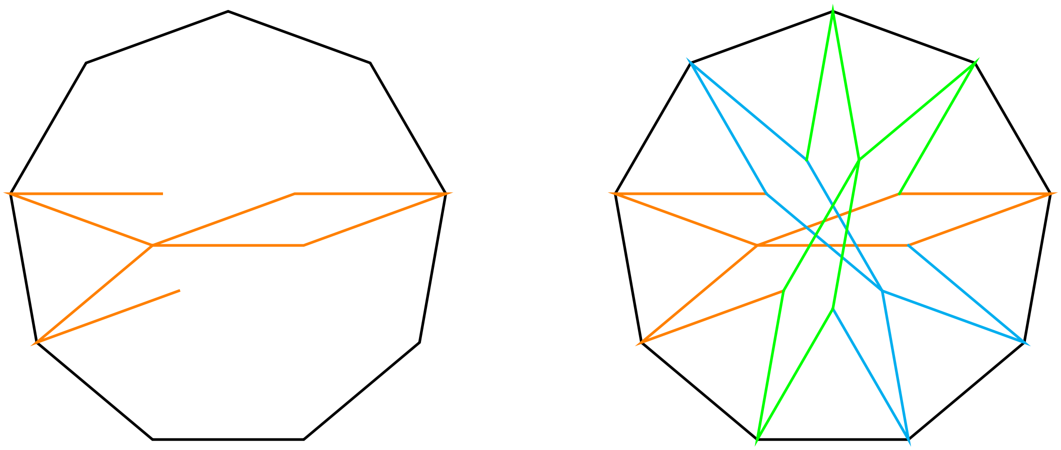

Figure 3: Rigid nonagon. 33 equal strips: Perimeter=9, diagonals=3×8.

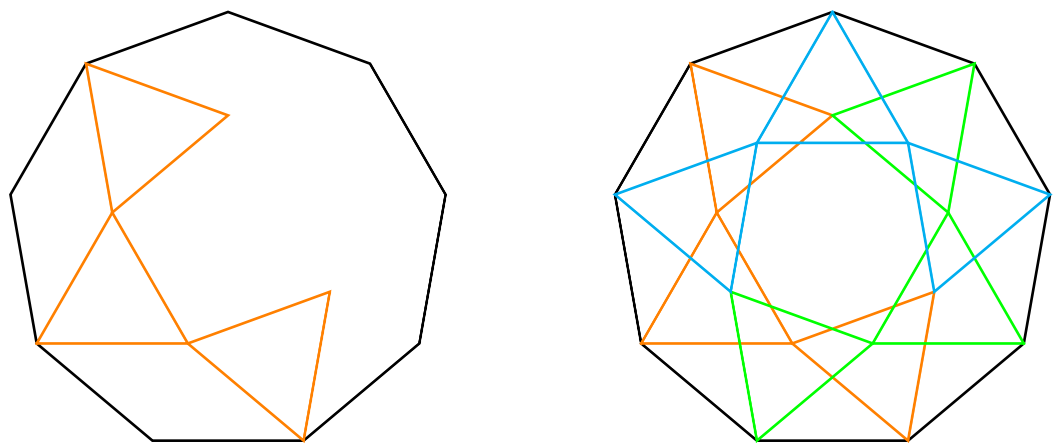

Figure 4: Rigid nonagon. 36 equal strips: Perimeter=9, diagonals=3×9.