Meccano triangles

https://github.com/heptagons/meccano/nest

Abstract

We construct meccano triangles. Basic triangles has the three sides as integers and calculate the internal diagonal distances. Such diagonals then are used as the new side of more complicated triangles and then again we calculate new distances formed and so on. Eventually we expect to find certain angles joining the triangles which can be used to construct regular polygons or more figures.

1 Triangles (a, b, c)

Triangles (a, b, c) have three sides a, b and c where $a, b, c \in \mathbb{N}$. To avoid repetitions and get only valid triangles, we consider only the cases:

$$a \geq b \geq c \tag{1}$$

$$a < b + c \tag{2}$$

We calculate the three angles cosines. The three cosines are rationals:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \in \mathbb{Q} \tag{3}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} \qquad \in \mathbb{Q} \tag{4}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \qquad \in \mathbb{Q} \tag{5}$$

1.1 Triangle (a, b, c) diagonals

To calculate the diagonals we use the law of cosines. With the $\cos A$ we can calculate every diagonal $\overline{b_m c_n}$ with:

$$\overline{b_m c_n} = \sqrt{m^2 + n^2 - 2mn \cos A} \tag{6}$$

$$= \sqrt{m^2 + n^2 - 2mn\frac{b^2 + c^2 - a^2}{2bc}} \tag{7}$$

$$= \frac{\sqrt{b^2c^2(m^2 + n^2) - bcmn(b^2 + c^2 - a^2)}}{bc} \qquad \in \mathbb{A} \tag{8}$$

where $1 \leq m \leq b$, $1 \leq n \leq c$ and $m - n \geq 0$. Similarly:

$$\overline{c_m a_n} = \frac{\sqrt{c^2a^2(m^2 + n^2) - camn(c^2 + a^2 - b^2)}}{ac} \qquad \in \mathbb{A} \tag{9}$$

$$\overline{a_m b_n} = \frac{\sqrt{a^2b^2(m^2 + n^2) - abmn(a^2 + b^2 - c^2)}}{ab} \qquad \in \mathbb{A} \tag{10}$$

The diagonals are algebraic of the form $\frac{y\sqrt{z}}{x}$.

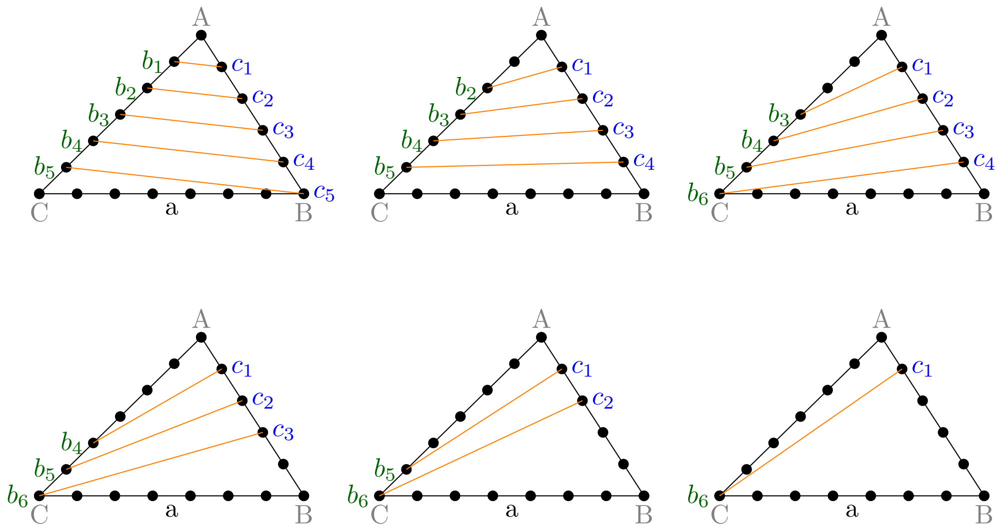

Figure 1: Triangle $(7, 6, 5)$ $b_m c_n$ diagonals $(m \geq n)$. For top to bottom and left to right we have six groups of diagonals. Each group is defined by m and n indices difference: $m - n = 0$, $m - n = 1$, ... $m - n = 5$.

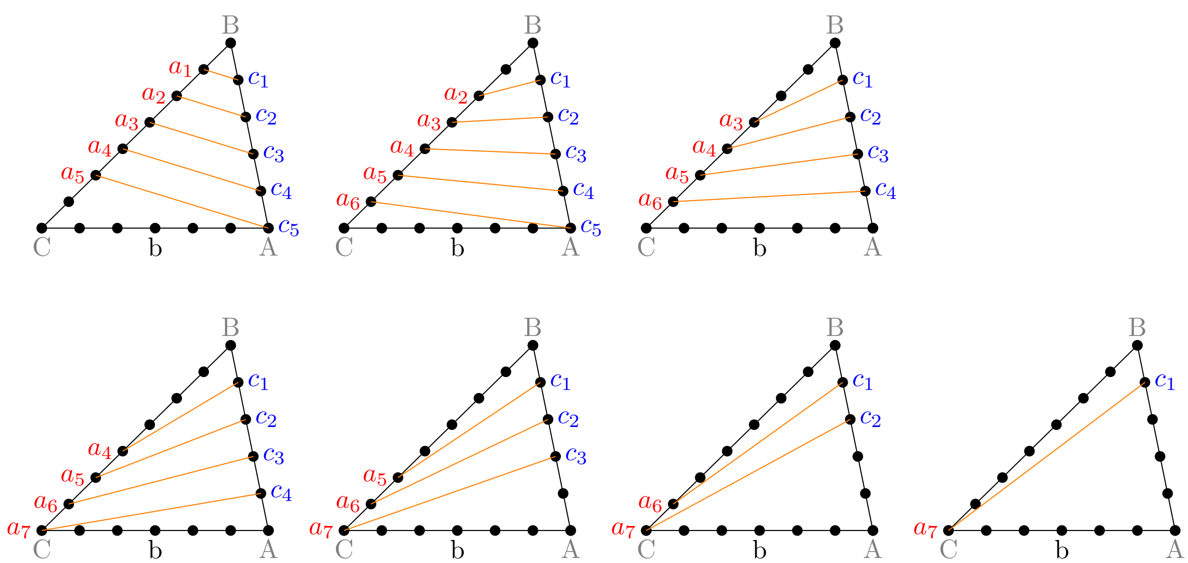

Figure 2: Triangle $(7, 6, 5)$, $a_m c_n$ diagonals $(m \geq n)$. We have also six group. But here we found diagonals repetead already found in previous figure. Diagonals repeated are all including a_7 points.

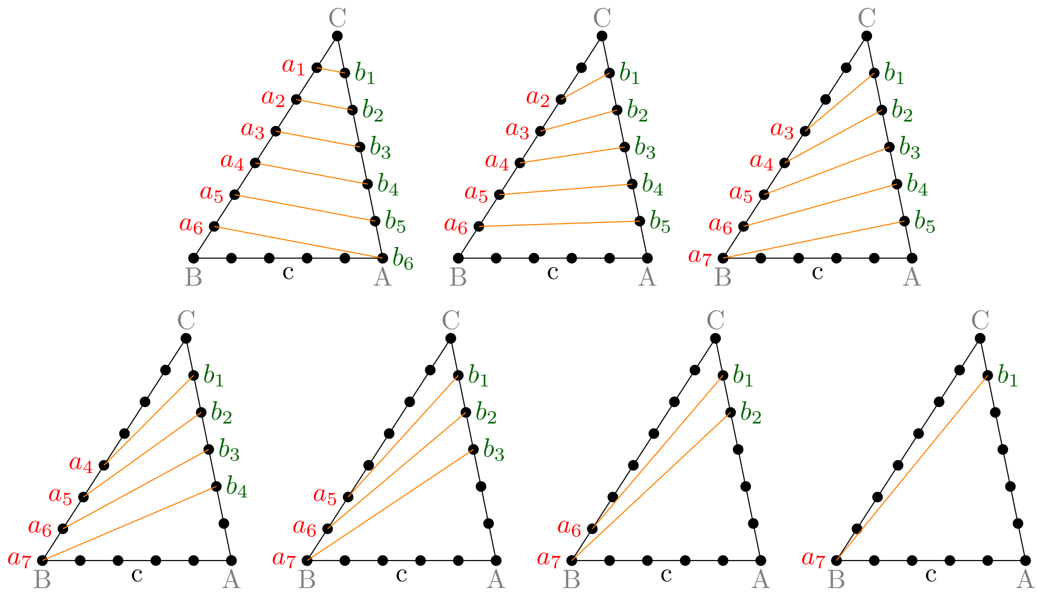

Figure 3: Triangle $(7, 6, 5)$, $a_m b_n$ diagonals $(m \geq n)$. Here we have diagonals repeteated already in previous figure. Using matrices we can reject such diagonals rejecting matrices columns.

1.2 Example triangle (7,6,5)

Figure 1 show triangle $(7, 6, 5)$ diagonals $b_m c_n$ for vertex A. For this figure we calculate the values and form a matrix. Empty cells are reflections:

$$\begin{pmatrix} \frac{2\sqrt{10}}{5} & \frac{\sqrt{105}}{5} & \frac{2\sqrt{55}}{5} & \frac{\sqrt{385}}{5} & 2\sqrt{6} & \frac{\sqrt{865}}{5} \\ & \frac{4\sqrt{10}}{5} & \frac{\sqrt{265}}{5} & \frac{2\sqrt{105}}{5} & \boxed{5} & \frac{4\sqrt{55}}{5} \\ & & \frac{6\sqrt{10}}{5} & \frac{\sqrt{505}}{5} & 2\sqrt{7} & \frac{3\sqrt{105}}{5} \\ & & & \frac{8\sqrt{10}}{5} & \sqrt{33} & \frac{2\sqrt{265}}{5} \\ & & & & 2\sqrt{10} & \boxed{7} \end{pmatrix} \tag{11}$$

Figure 2 show triangle $(7, 6, 5)$ diagonals $a_m c_n$ for vertex B. For this figure we calculate the values for a second matrix. Values at column 7 (at the right of separator |) are repeated and already in previous matrix. Empty cells are reflections.

$$\begin{pmatrix} \frac{4\sqrt{70}}{35} & \frac{3\sqrt{385}}{35} & \frac{2\sqrt{2065}}{35} & \frac{\sqrt{15505}}{35} & \frac{12\sqrt{7}}{7} & \frac{\sqrt{37345}}{35} & | & \frac{2\sqrt{265}}{5} \\ & \frac{8\sqrt{70}}{35} & \frac{\sqrt{7945}}{35} & \frac{6\sqrt{385}}{35} & \frac{\sqrt{889}}{7} & \frac{4\sqrt{2065}}{35} & | & \frac{3\sqrt{105}}{5} \\ & & \frac{12\sqrt{70}}{35} & \frac{\sqrt{14665}}{35} & \frac{2\sqrt{217}}{7} & \frac{9\sqrt{385}}{35} & | & \frac{4\sqrt{55}}{5} \\ & & & \frac{16\sqrt{70}}{35} & \frac{3\sqrt{105}}{7} & \frac{2\sqrt{7945}}{35} & | & \frac{\sqrt{865}}{5} \\ & & & & \frac{4\sqrt{70}}{7} & \frac{\sqrt{1393}}{7} & | & \boxed{6} \end{pmatrix} \tag{12}$$

Figure 3 show triangle $(7, 6, 5)$ diagonals $a_m b_n$ for vertex C. For this figure we calculate the values for a third matrix. Values at columns 6 and 7 (at the right of separator |) are repeated and already in previous matrices.

$$\left(\begin{array}{ccccc|cc}
\frac{2\sqrt{7}}{7} & \frac{\sqrt{105}}{7} & \frac{2\sqrt{70}}{7} & \frac{\sqrt{553}}{7} & \frac{2\sqrt{231}}{7} & \frac{\sqrt{1393}}{7} & 2\sqrt{10} \\
 & \frac{4\sqrt{7}}{7} & \frac{\sqrt{217}}{7} & \frac{2\sqrt{105}}{7} & \frac{\sqrt{721}}{7} & \frac{4\sqrt{70}}{7} & \sqrt{33} \\
 & & \frac{6\sqrt{7}}{7} & \frac{\sqrt{385}}{7} & \frac{2\sqrt{154}}{7} & \frac{3\sqrt{105}}{7} & 2\sqrt{7} \\
 & & & \frac{8\sqrt{7}}{7} & \frac{\sqrt{609}}{7} & \frac{2\sqrt{217}}{7} & \colorbox{green}{5} \\
 & & & & \frac{10\sqrt{7}}{7} & \frac{\sqrt{889}}{7} & 2\sqrt{6} \\
 & & & & & \frac{12\sqrt{7}}{7} & \boxed{5}
\end{array}\right) \tag{13}$$

2 Triangles$(\sqrt{\alpha}, b, c)$

Triangles$(\sqrt{\alpha}, b, c)$ have three sides with lengths $a = \sqrt{\alpha}$, b and c where $\alpha, b, c \in \mathbb{N}$ and α is squarefree. We have:

$$\sqrt{\alpha} > b \geq c \qquad \Longrightarrow \quad \alpha > b^2 \geq c^2 \tag{14}$$

$$\sqrt{\alpha} < b + c \qquad \Longrightarrow \quad \alpha < (b+c)^2 \tag{15}$$

We calculate the triangle cosines. $\cos A$ is rational and $\cos B$ and $\cos C$ are algebraic:

$$\cos A = \frac{b^2 + c^2 - (\sqrt{\alpha})^2}{2bc} = \frac{b^2 + c^2 - \alpha}{2bc} \qquad \in \mathbb{Q} \tag{16}$$

$$\cos B = \frac{(\sqrt{\alpha})^2 + c^2 - b^2}{2\sqrt{\alpha}c} = \frac{(\alpha + c^2 - b^2)\sqrt{\alpha}}{2\alpha c} \qquad \in \mathbb{A} \tag{17}$$

$$\cos C = \frac{(\sqrt{\alpha})^2 + b^2 - c^2}{2\sqrt{\alpha}b} = \frac{(\alpha + b^2 - c^2)\sqrt{\alpha}}{2\alpha b} \qquad \in \mathbb{A} \tag{18}$$

2.1 Triangle $(\sqrt{\alpha}, b, c)$ diagonals

The only possible diagonals are for sides with integers, that is $\overline{b_m c_n}$. Using the law of cosines:

$$\overline{b_m c_n} = \sqrt{m^2 + n^2 - 2mn\cos A} \tag{19}$$

$$= \sqrt{m^2 + n^2 - 2mn\frac{b^2 + c^2 - \alpha}{2bc}} \tag{20}$$

$$= \frac{\sqrt{b^2c^2(m^2+n^2) - bcmn(b^2 + c^2 - \alpha)}}{bc} \qquad \in \mathbb{A} \tag{21}$$

where $1 \leq m \leq b$, $1 \leq n \leq c$ and $m \geq n$.

2.2 Example triangles$(2\sqrt{6}, b, c)$

In this case $\sqrt{\alpha} = 2\sqrt{6}$ so $\alpha = 24$. Then $m = n = \{1, 2, 3, 4\}$ because $b^2 = c^2 = \{1, 4, 9, 16\} < 24$. We form a matrix with the values $(b+c)^2$ and satisfying $b \geq c$:

$$(b_m + c_n)^2 = \begin{array}{c} \\ c=1 \\ c=2 \\ c=3 \\ c=4 \end{array} \begin{array}{c} \begin{array}{cccc} b=1 & b=2 & b=3 & b=4 \end{array} \\ \begin{pmatrix} 2 & 9 & 16 & 25 \\ \times & 16 & 25 & 36 \\ \times & \times & 36 & 49 \\ \times & \times & \times & 64 \end{pmatrix} \end{array} \tag{22}$$

Then we remove the cells that don't satisfy the condition $\alpha < (b+c)^2$:

$$
(b_m + c_n)^2 = \begin{array}{c} \\ c=1 \\ c=2 \\ c=3 \\ c=4 \end{array} \begin{array}{c} \begin{array}{cccc} b=1 & b=2 & b=3 & b=4 \end{array} \\ \begin{pmatrix} \times & \times & \times & 25 \\ & \times & 25 & 36 \\ & & 36 & 49 \\ & & & 64 \end{pmatrix} \end{array} \tag{23}
$$

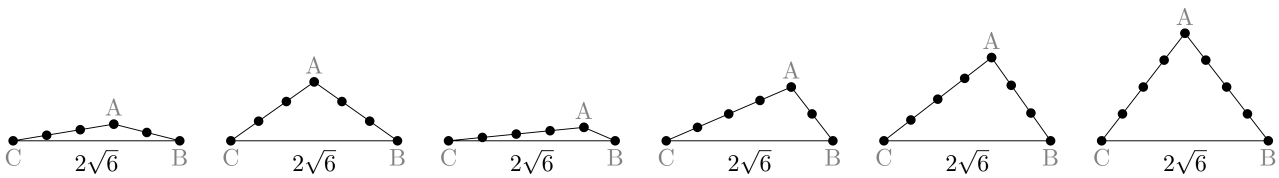

Figure 4: All triangles with sides $a = 2\sqrt{6} > b \geq c$

Each remaining cell in the matrix corresponds to a particular triangle:

$$
(2\sqrt{6}, b, c) = \begin{array}{c} \\ (2\sqrt{6},3,2) \\ (2\sqrt{6},3,3) \\ (2\sqrt{6},4,1) \\ (2\sqrt{6},4,2) \\ (2\sqrt{6},4,3) \\ (2\sqrt{6},4,4) \end{array} \begin{array}{c} \begin{array}{ccc} \cos A & \cos B & \cos C \end{array} \\ \begin{pmatrix} -\frac{11}{12} & \frac{19\sqrt{6}}{48} & \frac{29\sqrt{6}}{72} \\ -\frac{1}{3} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{3} \\ -\frac{7}{8} & \frac{3\sqrt{6}}{8} & \frac{13\sqrt{6}}{32} \\ -\frac{1}{4} & \frac{\sqrt{6}}{4} & \frac{3\sqrt{6}}{8} \\ \frac{1}{24} & \frac{17\sqrt{6}}{72} & \frac{31\sqrt{6}}{96} \\ \frac{1}{4} & \frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} \end{pmatrix} \end{array} \tag{24}
$$

Figure 4 show the triangles $(2\sqrt{6}, b, c)$. The cosines are calculated by code at:

`github.com/heptagons/meccano/nest/t_test.go TestTslursA`

3 Triangles$(a, \sqrt{\beta}, c)$

Triangles$(a, \sqrt{\beta}, c)$ have the three sides a, $b = \sqrt{\beta}$ and c where $a, \beta, c \in \mathbb{N}$ and β is squarefree. We have:

$$
a > \sqrt{\beta} > c \qquad \Longrightarrow \; a^2 > \beta > c^2 \tag{25}
$$

$$
a < \sqrt{\beta} + c \qquad \Longrightarrow \; (a-c)^2 < \beta \tag{26}
$$

We calculate the triangle cosines. $\cos A$ is algebraic, $\cos B$ rational and $\cos C$ algebraic:

$$
\cos A = \frac{(\sqrt{\beta})^2 + c^2 - a^2}{2\sqrt{\beta}c} = \frac{(\beta + c^2 - a^2)\sqrt{\beta}}{2\beta c} \qquad \in \mathbb{A} \tag{27}
$$

$$
\cos B = \frac{a^2 + c^2 - (\sqrt{\beta})^2}{2ac} = \frac{a^2 + c^2 - \beta}{2ac} \qquad \in \mathbb{Q} \tag{28}
$$

$$
\cos C = \frac{a^2 + (\sqrt{\beta})^2 - c^2}{2a\sqrt{\beta}} = \frac{(a^2 + \beta - c^2)\sqrt{\beta}}{2a\beta} \qquad \in \mathbb{A} \tag{29}
$$

3.1 Triangle $(a, \sqrt{\beta}, c)$ diagonals

The only possible diagonals are for sides with integers, that is $\overline{a_m c_n}$. Using the law of cosines:

$$\begin{aligned} \overline{a_m c_n} &= \sqrt{m^2 + n^2 - 2mn\cos B} && (30) \\ &= \sqrt{m^2 + n^2 - 2mn\frac{a^2 + c^2 - \beta}{2ac}} && (31) \\ &= \frac{\sqrt{a^2c^2(m^2 + n^2) - acmn(a^2 + c^2 - \beta)}}{ac} \qquad \in \mathbb{A} && (32) \end{aligned}$$

where $1 \leq m \leq a$, $1 \leq n \leq c$ and $m \geq n$.

3.2 Example triangles$(a, 2\sqrt{6}, c)$

In this case $\sqrt{\beta} = 2\sqrt{6}$ so $\beta = 24$. Then $m = \{5, 6, 7, ...\}$ because $m^2 = \{25, 36, 49, ...\} > 24$ and $n = \{1, 2, 3, 4\}$ because $c^2 = \{1, 4, 9, 16\} < 24$. We form a matrix with the values $(a - c)^2$:

$$(a_m - c_n)^2 = \begin{array}{cc} & \begin{array}{cccccc} a=5 & a=6 & a=7 & a=8 & a=9 & ... \end{array} \\ \begin{array}{c} c=1 \\ c=2 \\ c=3 \\ c=4 \end{array} & \begin{pmatrix} 16 & 25 & 36 & 49 & 64 & ... \\ 9 & 16 & 25 & 36 & 49 & ... \\ 4 & 9 & 16 & 25 & 36 & ... \\ 1 & 4 & 9 & 16 & 25 & ... \end{pmatrix} \end{array} \qquad (33)$$

We remove cells which don't satisfy the condition $(a - c)^2 < \beta$:

$$(a_m - c_n)^2 = \begin{array}{cc} & \begin{array}{cccccc} a=5 & a=6 & a=7 & a=8 & a=9 & ... \end{array} \\ \begin{array}{c} c=1 \\ c=2 \\ c=3 \\ c=4 \end{array} & \begin{pmatrix} 16 & \times & \times & \times & \times & ... \\ 9 & 16 & \times & \times & \times & ... \\ 4 & 9 & 16 & \times & \times & ... \\ 1 & 4 & 9 & 16 & \times & ... \end{pmatrix} \end{array} \qquad (34)$$

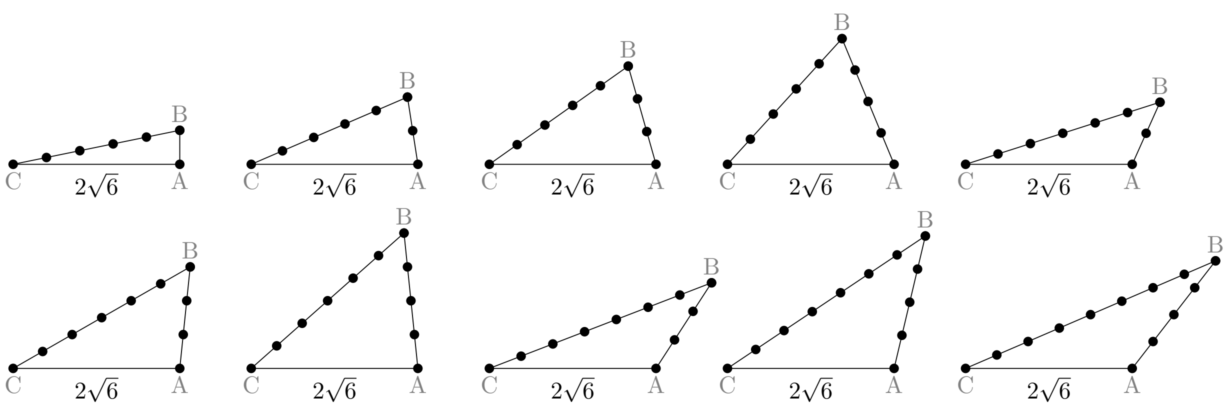

Figure 5: All triangles with sides $a > b = 2\sqrt{6} > c$

So we have ten triangles are valid:

$$
(a, 2\sqrt{6}, c) = \begin{array}{r c}
 & \begin{array}{ccc} \cos A & \cos B & \cos C \end{array} \\
\begin{array}{r}
(5, 2\sqrt{6}, 1) \\
(5, 2\sqrt{6}, 2) \\
(5, 2\sqrt{6}, 3) \\
(5, 2\sqrt{6}, 4) \\
(6, 2\sqrt{6}, 2) \\
(6, 2\sqrt{6}, 3) \\
(6, 2\sqrt{6}, 4) \\
(7, 2\sqrt{6}, 3) \\
(7, 2\sqrt{6}, 4) \\
(8, 2\sqrt{6}, 4)
\end{array} &
\begin{pmatrix}
0 & \frac{1}{5} & \frac{2\sqrt{6}}{5} \\
\frac{\sqrt{6}}{16} & \frac{1}{4} & \frac{3\sqrt{6}}{8} \\
\frac{\sqrt{6}}{9} & \frac{1}{3} & \frac{\sqrt{6}}{3} \\
\frac{5\sqrt{6}}{32} & \frac{17}{40} & \frac{11\sqrt{6}}{40} \\
\frac{\sqrt{6}}{6} & \frac{2}{3} & \frac{7\sqrt{6}}{18} \\
\frac{\sqrt{6}}{24} & \frac{7}{12} & \frac{17\sqrt{6}}{48} \\
\frac{\sqrt{6}}{24} & \frac{7}{12} & \frac{11\sqrt{6}}{36} \\
-\frac{2\sqrt{6}}{9} & \frac{17}{21} & \frac{8/sqrt6}{21} \\
-\frac{3\sqrt{6}}{32} & \frac{41}{56} & \frac{19\sqrt{6}}{56} \\
\frac{\sqrt{6}}{4} & \frac{7}{8} & \frac{3\sqrt{6}}{8}
\end{pmatrix}
\end{array} \tag{35}
$$

Figure 5 show the triangles $(a, 2\sqrt{6}, c)$. The cosines are calculated by code at:

`github.com/heptagons/meccano/nest/t_test.go TestTslursB`

4 Triangles$(a, b, \sqrt{\gamma})$

Triangles$(a, b, \sqrt{\gamma})$ have three sides a, b, $\sqrt{\gamma}$ where $a, b, \gamma \in \mathbb{N}$ and γ is squarefree. We have:

$$a \geq b > \sqrt{\gamma} \qquad \Longrightarrow \; a^2 \geq b^2 > \gamma \tag{36}$$

$$a < b + \sqrt{\gamma} \qquad \Longrightarrow \; (a-b)^2 < \gamma \tag{37}$$

We calculate the triangle cosines. $\cos A$ and $\cos B$ are algebraic and $\cos C$ rational:

$$\cos A = \frac{b^2 + \gamma - a^2}{2b\sqrt{\gamma}} = \frac{(b^2 + \gamma - a^2)\sqrt{\gamma}}{2b\gamma} \qquad \in \mathbb{A} \tag{38}$$

$$\cos B = \frac{a^2 + \gamma - b^2}{2a\sqrt{\gamma}} = \frac{(a^2 + \gamma - b^2)\sqrt{\gamma}}{2a\gamma} \qquad \in \mathbb{A} \tag{39}$$

$$\cos C = \frac{a^2 + b^2 - (\sqrt{\gamma})^2}{2ab} = \frac{a^2 + b^2 - \gamma}{2ab} \qquad \in \mathbb{Q} \tag{40}$$

4.1 Triangle $(a, b, \sqrt{\gamma})$ diagonals

The only possible diagonals are for sides with integers, that is $\overline{a_m b_n}$. Using the law of cosines:

$$\overline{a_m b_n} = \sqrt{m^2 + n^2 - 2mn\cos C} \tag{41}$$

$$= \sqrt{m^2 + n^2 - 2mn\frac{a^2 + b^2 - \gamma}{2ab}} \tag{42}$$

$$= \frac{\sqrt{a^2b^2(m^2 + n^2) - acmn(a^2 + b^2 - \gamma)}}{ab} \qquad \in \mathbb{A} \tag{43}$$

where $1 \leq m < a$, $1 \leq n < b$ and $m \geq n$.

4.2 Example triangles$(a, b, 2\sqrt{6})$

In this case $\sqrt{\gamma} = 2\sqrt{6}$ so $\gamma = 24$. We form a matrix with with the values $(a-b)^2$ satisfying the condition $a^2 \geq b^2 > \gamma$:

$$(a_m - b_n)^2 = \begin{array}{cc} & \begin{array}{ccccccccc} a=5 & a=6 & a=7 & a=8 & a=9 & a=10 & a=11 & a=12 & \cdots \end{array} \\ \begin{array}{c} b=5 \\ b=6 \\ b=7 \\ b=8 \\ b=9 \\ b=10 \\ \end{array} & \begin{pmatrix} 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & \cdots \\ & 0 & 1 & 4 & 9 & 16 & 25 & 36 & \cdots \\ & & 0 & 1 & 4 & 9 & 16 & 25 & \cdots \\ & & & 0 & 1 & 4 & 9 & 16 & \cdots \\ & & & & 0 & 1 & 4 & 9 & \cdots \\ & & & & & 0 & 1 & 4 & \cdots \\ & & & & & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array} \tag{44}$$

We remove cells except those satisfying the condition $(a-b)^2 < \gamma$:

$$(a_m - b_n)^2 = \begin{array}{cc} & \begin{array}{ccccccccc} a=5 & a=6 & a=7 & a=8 & a=9 & a=10 & a=11 & a=12 & \cdots \end{array} \\ \begin{array}{c} b=5 \\ b=6 \\ b=7 \\ b=8 \\ b=9 \\ b=10 \\ \end{array} & \begin{pmatrix} 0 & 1 & 4 & 9 & 16 & \times & \times & \times & \cdots \\ & 0 & 1 & 4 & 9 & 16 & \times & \times & \cdots \\ & & 0 & 1 & 4 & 9 & 16 & \times & \cdots \\ & & & 0 & 1 & 4 & 9 & 16 & \cdots \\ & & & & 0 & 1 & 4 & 9 & \cdots \\ & & & & & 0 & 1 & 4 & \cdots \\ & & & & & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array} \tag{45}$$

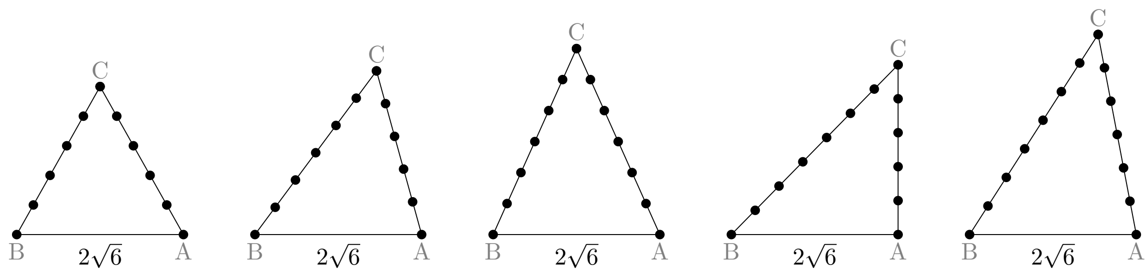

Figure 6: Some triangles with sides $a \geq b > c = 2\sqrt{6}$

So we found that infinite triangles are valid, the smaller ones are:

$$(a, b, 2\sqrt{6}) = \begin{array}{cc} & \begin{array}{ccc} \cos A & \cos B & \cos C \end{array} \\ \begin{array}{c} (5,5,2\sqrt{6}) \\ (6,5,2\sqrt{6}) \\ (6,6,2\sqrt{6}) \\ (7,5,2\sqrt{6}) \\ (7,6,2\sqrt{6}) \\ (7,7,2\sqrt{6}) \\ \cdots \end{array} & \begin{pmatrix} \frac{\sqrt{6}}{5} & \frac{\sqrt{6}}{5} & \frac{13}{25} \\ \frac{13\sqrt{6}}{120} & \frac{35\sqrt{6}}{144} & \frac{37}{60} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{2}{3} \\ 0 & \frac{2\sqrt{6}}{7} & \frac{5}{7} \\ \frac{11\sqrt{6}}{144} & \frac{37\sqrt{6}}{168} & \frac{61}{84} \\ \frac{\sqrt{6}}{7} & \frac{\sqrt{6}}{7} & \frac{37}{49} \\ \cdots & \cdots & \cdots \end{pmatrix} \end{array} \tag{46}$$

Figure 6 show some triangles $(a, b, 2\sqrt{6})$. The cosines are calculated by code at:

`github.com/heptagons/meccano/nest/t_test.go TestTslursC`

5 Triangles pairs

We can attach two triangles to share a common side and vertex to get more angles and diagonals.

5.1 Triangles pairs rational cosines angles

When we sum two angles $Z = X + Y$, where $\cos X \equiv x_n/x_d$ and $\cos Y \equiv y_n/y_d$ we have:

$$\begin{aligned}
\cos Z &= \cos X \cos Y - \sin X \sin Y && (47)\\
&= \cos X \cos Y - \sqrt{1-\cos^2 X}\sqrt{1-\cos^2 Y} && (48)\\
&= \frac{x_n y_n}{x_d y_d} - \sqrt{\frac{x_d^2 - x_n^2}{x_d^2}}\sqrt{\frac{y_d^2 - y_n^2}{y_d^2}} && (49)\\
&= \frac{x_n y_n - \sqrt{(x_d^2 - x_n^2)(y_d^2 - y_n^2)}}{x_d y_d} \qquad \equiv \frac{b_1 + c_1\sqrt{d_1}}{a_1} && (50)
\end{aligned}$$

So we can calculate new diagonals from one triangle side to another triangle side:

$$\begin{aligned}
\delta &= \sqrt{m^2 + n^2 - 2mn\cos Z} && (51)\\
&= \sqrt{m^2 + n^2 - 2mn\frac{b_1 + c_1\sqrt{d_1}}{a_1}} && (52)\\
&= \frac{\sqrt{a_1^2(m^2+n^2) - 2mn(b_1 + c_1\sqrt{d_1})}}{a_1} && (53)\\
&= \frac{\sqrt{a_1^2(m^2+n^2) - 2b_1mn - 2c_1mn\sqrt{d_1}}}{a_1} \qquad \equiv \frac{b_2 + c_2\sqrt{d_2 + e_2\sqrt{f_2}}}{a_2} && (54)
\end{aligned}$$

5.2 Triangles pairs surds angles

When we sum two algebraic angles $W = U + V$ when $\cos U = \sqrt{u_n}/u_d$ and $\cos V = \sqrt{v_n}/v_d$ we have:

$$\begin{aligned}
\cos W &= \cos U \cos V - \sin U \sin V && (55)\\
&= \cos U \cos V - \sqrt{1-\cos^2 U}\sqrt{1-\cos^2 V} && (56)\\
&= \frac{\sqrt{u_n v_n}}{u_d v_d} - \sqrt{\frac{u_d^2 - u_n}{u_d^2}}\sqrt{\frac{v_d^2 - v_n}{v_d^2}} && (57)\\
&= \frac{\sqrt{u_n v_n} - \sqrt{(u_d^2 - u_n)(v_d^2 - v_n)}}{u_d v_d} && (58)
\end{aligned}$$