Meccano pentagons

https://github.com/heptagons/meccano/penta

1 Regular pentagon type 1

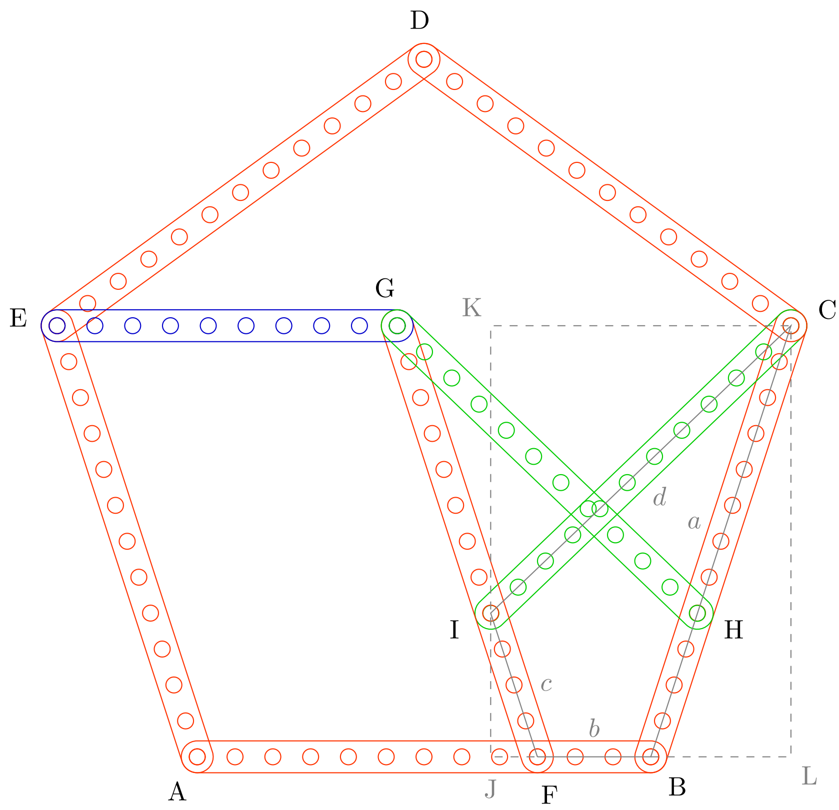

Figure 1: Pentagon of type 1.

1.1 Type 1 equations

Figure 1 show the layout of the meccano regular pentagon of type 1. Let define the side of the pentagon as a and define other three variables b, c and d:

$$a = \overline{BC}$$
$$b = \overline{BF}$$
$$c = \overline{FI}$$
$$d = \overline{CI}$$

Angles $\angle LBC$ and $\angle JFI$ are equal to $\frac{2\pi}{5}$ so:

$$\begin{aligned}
\alpha &= \frac{2\pi}{5} \\
\overline{BL} &= a\cos\alpha \\
\overline{CL} &= a\sin\alpha \\
\overline{FJ} &= c\cos\alpha \\
\overline{IJ} &= c\sin\alpha
\end{aligned}$$

Let calculate d in function of a, b and c:

$$\begin{aligned}
d^2 &= (\overline{CI})^2 \\
&= (\overline{CK})^2 + (\overline{IK})^2 \\
&= (\overline{BL} + \overline{BF} + \overline{FJ})^2 + (\overline{CL} - \overline{IJ})^2 \\
&= (a\cos\alpha + b + c\cos\alpha)^2 + (a\sin\alpha - c\sin\alpha)^2 \\
&= ((a+c)\cos\alpha + b)^2 + ((a-c)\sin\alpha)^2 \\
&= (a+c)^2\cos^2\alpha + 2(a+c)b\cos\alpha + b^2 + (a-c)^2\sin^2\alpha \\
&= (a^2+c^2)(cos^2\alpha + sin^2\alpha) + 2ac(cos^2\alpha - \sin^2\alpha) + 2(a+c)b\cos\alpha + b^2 \\
&= (a^2+c^2) + 2ac(cos^2\alpha - \sin^2\alpha) + 2(a+c)b\cos\alpha + b^2
\end{aligned}$$

For $\alpha = \frac{2\pi}{5}$ we have these regular pentagon identities:

$$\begin{aligned}
cos\alpha &= \frac{-1+\sqrt{5}}{4} \\
cos^2\alpha &= \frac{3-\sqrt{5}}{8} \\
sin^2\alpha &= \frac{5+\sqrt{5}}{8} \\
cos^2\alpha - sin^2\alpha &= -\frac{1+\sqrt{5}}{4}
\end{aligned}$$

Applying the identities to the last equation of d we get:

$$\begin{aligned}
d^2 &= a^2 + c^2 - (\frac{1+\sqrt{5}}{2})ac + (\frac{-1+\sqrt{5}}{2})(a+c)b + b^2 \\
&= a^2 + c^2 - \frac{ac}{2} - \frac{(a+c)b}{2} + b^2 + [-\frac{ac}{2} + \frac{(a+c)b}{2}]\sqrt{5} \\
&= a^2 + b^2 + c^2 - \frac{ac + (a+c)b}{2} + [\frac{-ac + (a+c)b}{2}]\sqrt{5}
\end{aligned}$$

Let define two variables p and q such that $d^2 = p + q\sqrt{5}$ so we have:

$$\begin{aligned}
d^2 &= p + q\sqrt{5} \\
q &= \frac{-ac + (a+c)b}{2} \\
p &= a^2 + b^2 + c^2 - \frac{ac + (a+c)b}{2} \\
&= a^2 + b^2 + c^2 - \frac{-ac + (a+c)b}{2} - ac \\
&= a^2 + b^2 + c^2 - q - ac
\end{aligned}$$

For a meccano pentagon we need d to be an integer. If we let the integer $q > 0$ then $d = \sqrt{p + q\sqrt{5}}$ will never be an integer for p and q integers. If we force q to be zero then $d = \sqrt{p}$ has possibilities to be an integer. So before calculating d we force the condition that $q = 0$ or that is the same $-ac + (a + c)b = 0$:

$$
\begin{aligned}
a &>= b \\
a &>= c \\
ac &= (a + c)b \\
d &= \sqrt{a^2 + b^2 + c^2 + ac}
\end{aligned}
$$

1.1.1 Type 1 program

Next **go** program iterate over three variables $a <= max$, $b <= a$, $c <= a$ (lines 30,31,32). The $q = 0$ condition is tested (line 33) and only when valid we check the d is an integer (call in line 34, function in line 20). Only when d is an integer we call function add (call in line 26, function in line 5) to print and store a solution without repetitions by scaling.

```
1  func pentagons_type_1(max int) {
2
3    sols := make([][]int, 0)
4
5    add := func(a, b, c, d int) {
6      for _, s := range sols {
7        if a % s[0] != 0 { continue }
8        // new a is a factor of previous a
9        f := a / s[0]
10       if t := b % s[1] == 0 && b / s[1] == f; !t { continue }
11       if t := c % s[2] == 0 && c / s[2] == f; !t { continue }
12       if t := d % s[3] == 0 && d / s[3] == f; !t { continue }
13       return // scaled solution already found (reject)
14     }
15     // solution!
16     sols = append(sols, []int{ a, b, c, d })
17     fmt.Printf("%3d a=%2d b=%2d c=%2d d=%2d\n", len(sols), a, b, c, d)
18   }
19
20   check := func(a, b, c int) {
21     f := float64(a*a + b*b + c*c - a*c)
22     if f < 0 {
23       return
24     }
25     if d := int(math.Sqrt(f)); math.Pow(float64(d), 2) == f {
26       add(a, b, c, d)
27     }
28   }
29
30   for a := 1; a < max; a++ {
31     for b := 1; b <= a; b++ {
32       for c := 0; c <= a; c++ {
33         if a*c == (a + c)*b {
34           check(a, b, c)
35         }
36       }
37     }
38   }
39 }
```

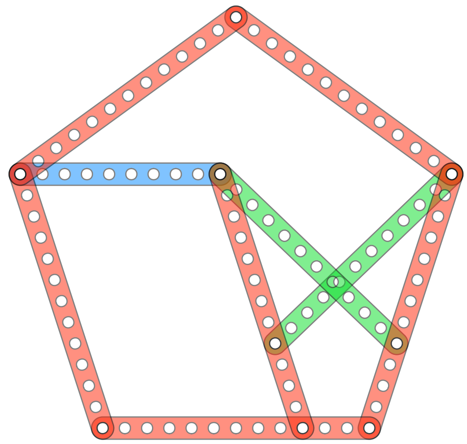

Figure 2: The smallest and maybe unique (?) of pentagons of type 1. Is composed of 6 rods of length $a = 12$ in color red, two rods of length $d = 11$ in green and one rod of size $a - b = 9$ in blue.

1.1.2 Type 1 results

After serching for values of $a <= 5000$ we found a single result:

```
a=12 b=3 c=4 d=11
```

Figure 2 shows the first (unique?) pentagon of type 1 with values $a = 12$, $b = 3$, $c = 4$ and $d = 11$.

2 Regular pentagon type 2

2.1 Type 2 equations

Figure 3 show the layout of the meccano regular pentagon of type 2. Let define the side of the pentagon as a and define other four variables b, c, d and e:

$$a = \overline{AB}$$
$$b = \overline{AH}$$
$$c = \overline{BK}$$
$$d = \overline{HL}$$
$$e = \overline{KL}$$

Angles $\angle NBC$ and $\angle MAH$ are equal to $\frac{2\pi}{5}$ so:

$$\alpha = \frac{2\pi}{5}$$
$$\overline{BN} = b\cos\alpha$$
$$\overline{KN} = b\sin\alpha$$
$$\overline{AM} = c\cos\alpha$$
$$\overline{HM} = c\sin\alpha$$

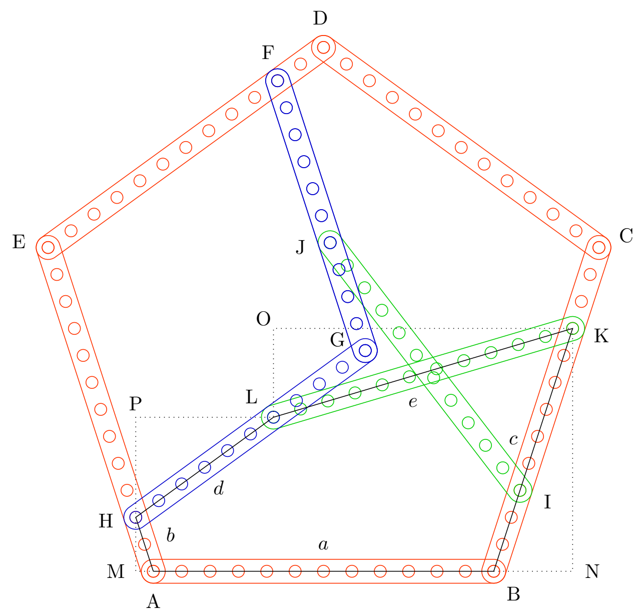

Figure 3: Pentagon of type 2.

Angle $\angle PLH$ is equal to $\frac{\pi}{5}$ so:

$$\begin{aligned} \beta &= \frac{\pi}{5} \\ \overline{LP} &= d\cos\beta \\ \overline{HP} &= d\sin\beta \end{aligned}$$

Our goal is to find e as integer as funcion of variables a, b, c and d. e^2 equals $(\overline{KO})^2 + (\overline{LO})^2$ so we first calculate $\overline{KO}$ and $\overline{LO}$. From figure 3:

$$\begin{aligned} \overline{KO} &= \overline{AM} + \overline{AB} + \overline{BN} - \overline{LP} \\ &= b\cos\alpha + a + c\cos\alpha - d\cos\beta \\ &= (b+c)\cos\alpha + a - d\cos\beta \\ \overline{LO} &= \overline{KN} - \overline{HM} - \overline{HP} \\ &= c\sin\alpha - b\sin\alpha - d\sin\beta \\ &= (c-b)\sin\alpha - d\sin\beta \end{aligned}$$

So by adding the squares we get:

$$\begin{aligned} e^2 &= (\overline{KO})^2 + (\overline{LO})^2 \\ &= ((b+c)\cos\alpha)^2 + 2(b+c)\cos\alpha(a - d\cos\beta) + (a - d\cos\beta)^2 \\ &\quad + ((c-b)\sin\alpha)^2 - 2(c-b)\sin\alpha d\sin\beta + (d\sin\beta)^2 \\ &= (b^2+c^2)(cos^2\alpha + sin^2\alpha) + 2bc(cos^2\alpha - sin^2\alpha) \\ &\quad + 2a(b+c)cos\alpha - 2(b+c)d\cos\alpha\cos\beta - 2(c-b)d\sin\alpha\sin\beta \\ &\quad + a^2 - 2ad\cos\beta + d^2(\cos^2\beta + \sin^2\beta) \end{aligned}$$

Calculate the α and β identities that appear in the last equation:

$$\begin{aligned}
\cos^2\alpha - \sin^2\alpha &= -\frac{1+\sqrt{5}}{4} \\
\cos\alpha &= \frac{-1+\sqrt{5}}{4} \\
\cos\alpha\cos\beta &= \frac{1}{4} \\
\sin\alpha\sin\beta &= \frac{\sqrt{5}}{4} \\
\cos\beta &= \frac{1+\sqrt{5}}{4}
\end{aligned}$$

Replace the identities:

$$\begin{aligned}
e^2 &= (b^2+c^2)(1) + 2bc(-\frac{1+\sqrt{5}}{4}) \\
&\quad + 2a(b+c)(\frac{-1+\sqrt{5}}{4}) - 2(b+c)d(\frac{1}{4}) - 2(c-b)d(\frac{\sqrt{5}}{4}) \\
&\quad + a^2 - 2ad(\frac{1+\sqrt{5}}{4}) + d^2(1) \\
&= b^2 + c^2 - bc(\frac{1+\sqrt{5}}{2}) \\
&\quad + a(b+c)(\frac{-1+\sqrt{5}}{2}) - (b+c)d(\frac{1}{2}) - (c-b)d(\frac{\sqrt{5}}{2}) \\
&\quad + a^2 - ad(\frac{1+\sqrt{5}}{2}) + d^2 \\
&= a^2 + b^2 + c^2 + d^2 - (b+c)d(\frac{1}{2}) \\
&\quad - (ad+bc)(\frac{1+\sqrt{5}}{2}) + a(b+c)(\frac{-1+\sqrt{5}}{2}) - (c-b)d(\frac{\sqrt{5}}{2}) \\
&= a^2 + b^2 + c^2 + d^2 - \frac{(b+c)d}{2} \\
&\quad - \frac{(ad+bc)(1+\sqrt{5})}{2} + \frac{a(b+c)(-1+\sqrt{5})}{2} - \frac{(c-b)d\sqrt{5}}{2}
\end{aligned}$$

Let define two variables p and q such that $e^2 = p + q\sqrt{5}$:

$$\begin{aligned}
p &= a^2 + b^2 + c^2 + d^2 - \frac{(b+c)d}{2} - \frac{ad+bc}{2} + \frac{-a(b+c)}{2} \\
&= a^2 + b^2 + c^2 + d^2 - \frac{bd + cd + ad + bc + ab + ac}{2} \\
&= a^2 + b^2 + c^2 + d^2 - \frac{(a+b)(c+d) + ab + cd}{2} \\
q &= -\frac{ad+bc}{2} + \frac{a(b+c)}{2} - \frac{(c-b)d}{2} \\
&= \frac{-ad - bc + ab + ac - cd + bd}{2} \\
&= \frac{(a-b)(c-d) + ab - cd}{2}
\end{aligned}$$

For a meccano pentagon we need e to be an integer. If we let the integer $q > 0$ then $e = \sqrt{p + q\sqrt{5}}$ will never be an integer for p and q integers. If we force q to be zero then $e = \sqrt{p}$ has possibilities to be an integer. So before calculating e we force the condition that $q = 0$ or that is the same $cd = (a-b)(c-d)+ab$:

$$
\begin{aligned}
a &>= b \\
a &>= c \\
cd &= (a-b)(c-d) + ab
\end{aligned}
$$

From the condition $q = 0$ we know replace $cd = (a-b)(c-d)+ab$, replacing cd in in the equation for p we get:

$$
\begin{aligned}
p &= a^2 + b^2 + c^2 + d^2 - \frac{(a+b)(c+d) + ab + cd}{2} \\
&= a^2 + b^2 + c^2 + d^2 - \frac{(a+b)(c+d) + ab + (a-b)(c-d) + ab}{2} \\
&= a^2 + b^2 + c^2 + d^2 - ac - bd - ab
\end{aligned}
$$

So finally, when $q = 0$ we calculate $e = \sqrt{p}$ expecting to be an integer:

$$
\begin{aligned}
e &= \sqrt{a^2 + b^2 + c^2 + d^2 - ac - bd - ab} \\
&= \sqrt{a^2 + b^2 + c^2 + d^2 - ad - bc - cd}
\end{aligned}
$$

2.1.1 Type 2 program

With last equations, another program, for the pentagon type 2, can iterate over the integer values of rods a, b, c and d to discover a rod e with integer length too. Next javascript program was run and found 40 different pentagons with rods length <= 183.

```
func pentagons_type_2(max int) {

  sols := make([][]int, 0)

  add := func(a, b, c, d, e int) {
    for _, s := range sols {
      if a % s[0] != 0 { continue }
      // new a is a factor of previous a
      f := a / s[0]
      if t := b % s[1] == 0 && b / s[1] == f; !t { continue }
      if t := c % s[2] == 0 && c / s[2] == f; !t { continue }
      if t := d % s[3] == 0 && d / s[3] == f; !t { continue }
      if t := e % s[4] == 0 && e / s[4] == f; !t { continue }
      return // scaled solution already found (reject)
    }
    // solution!
    sols = append(sols, []int{ a, b, c, d, e })
    fmt.Printf("%3d a=%3d b=%3d c=%3d d=%3d e=%3d\n", len(sols), a, b, c, d, e)
  }

  check := func(a, b, c, d int) {
    f := float64(a*a + b*b + c*c + d*d - a*d - b*c - c*d)
      if f < 0 {
        return
      }
    if e := int(math.Sqrt(f)); math.Pow(float64(e), 2) == f {
      add(a, b, c, d, e)
    }
```

```
  }

    for a := 1 ; a < max; a++ {
      for b := 1; b < a; b++ {
          for c := 1; c < a; c++ {
              for d := 1; d < a; d++ {
                if ((a - b)*(c - d) + a*b == c*d) {
                    check(a, b, c, d)
                  }
                }
            }
        }
    }
}
```

2.2 Type 2 results

The program found as much as 124 pentagons of type 2 for $a <= 488$.

```
  1 a= 12 b=  2 c=  9 d=  6 e= 11
  2 a= 12 b=  6 c=  3 d= 10 e= 11
  3 a= 31 b=  4 c= 28 d= 16 e= 31
  4 a= 31 b= 15 c=  3 d= 27 e= 31
  5 a= 38 b= 12 c= 18 d= 21 e= 31
  6 a= 38 b= 17 c= 20 d= 26 e= 31
  7 a= 48 b=  8 c= 24 d= 21 e= 41
  8 a= 48 b= 12 c=  9 d= 20 e= 41
  9 a= 48 b= 27 c= 24 d= 40 e= 41
 10 a= 48 b= 28 c= 39 d= 36 e= 41
 11 a= 72 b= 21 c= 48 d= 40 e= 61
 12 a= 72 b= 24 c= 16 d= 39 e= 61
 13 a= 72 b= 32 c= 24 d= 51 e= 61
 14 a= 72 b= 33 c= 56 d= 48 e= 61
 15 a= 78 b= 27 c=  4 d= 42 e= 71
 16 a= 78 b= 36 c= 74 d= 51 e= 71
 . . .
 . . .
119 a=488 b= 72 c= 15 d= 96 e=451
120 a=488 b=132 c=423 d=276 e=451
121 a=488 b=152 c=269 d=272 e=401
122 a=488 b=212 c= 65 d=356 e=451
123 a=488 b=216 c=219 d=336 e=401
124 a=488 b=392 c=473 d=416 e=451
```

Figures 4, 5 and 6 show some of the pentagons of type 2 found.

2.3 Another layout for type 2

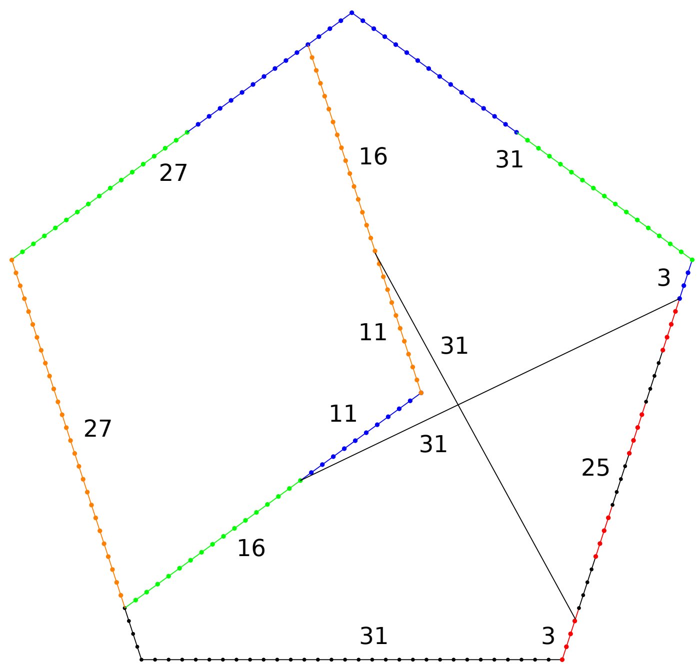

Figure 4: Pentagon of type 2 with $a = 31$. This construction requires 7 rods of length 31 and 2 rods of length 27.

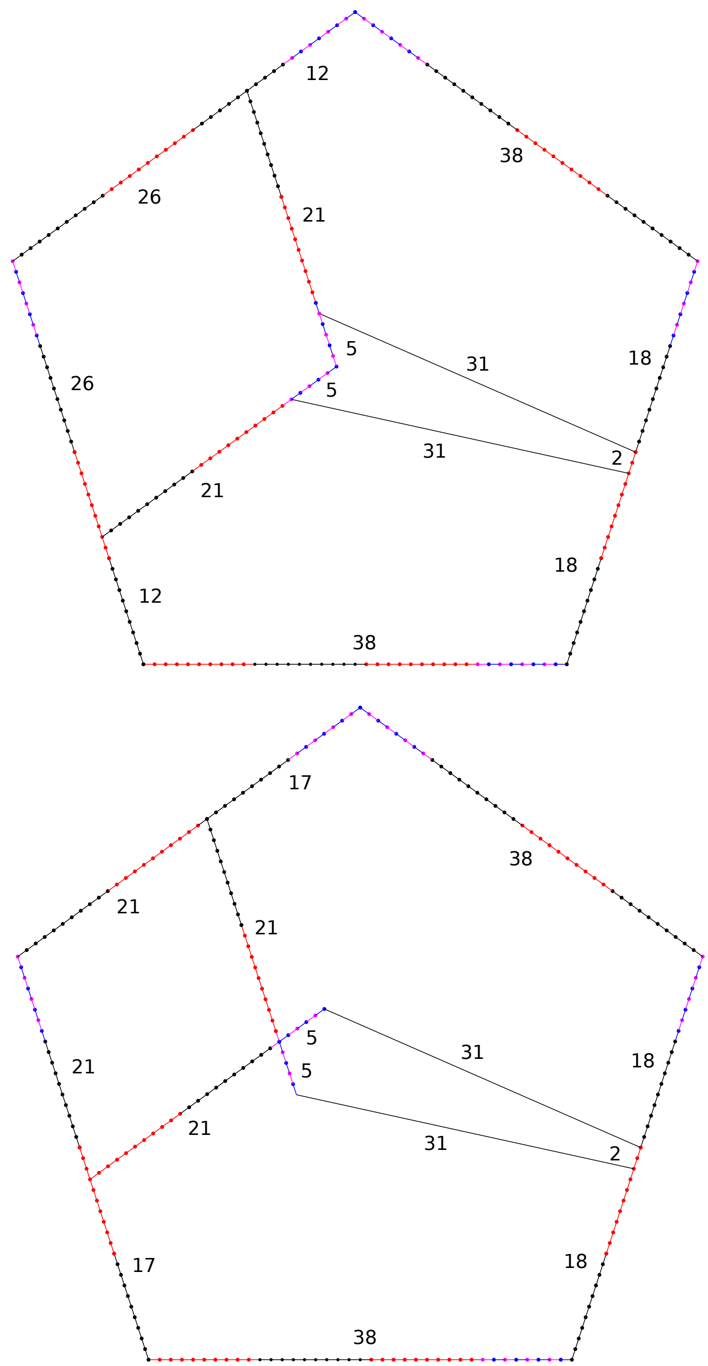

Figure 5: Pentagons of type 2 with $a = 38$. Each construction requires 5 rods of length 38, 2 rods of length 31 and 2 rods of length 26

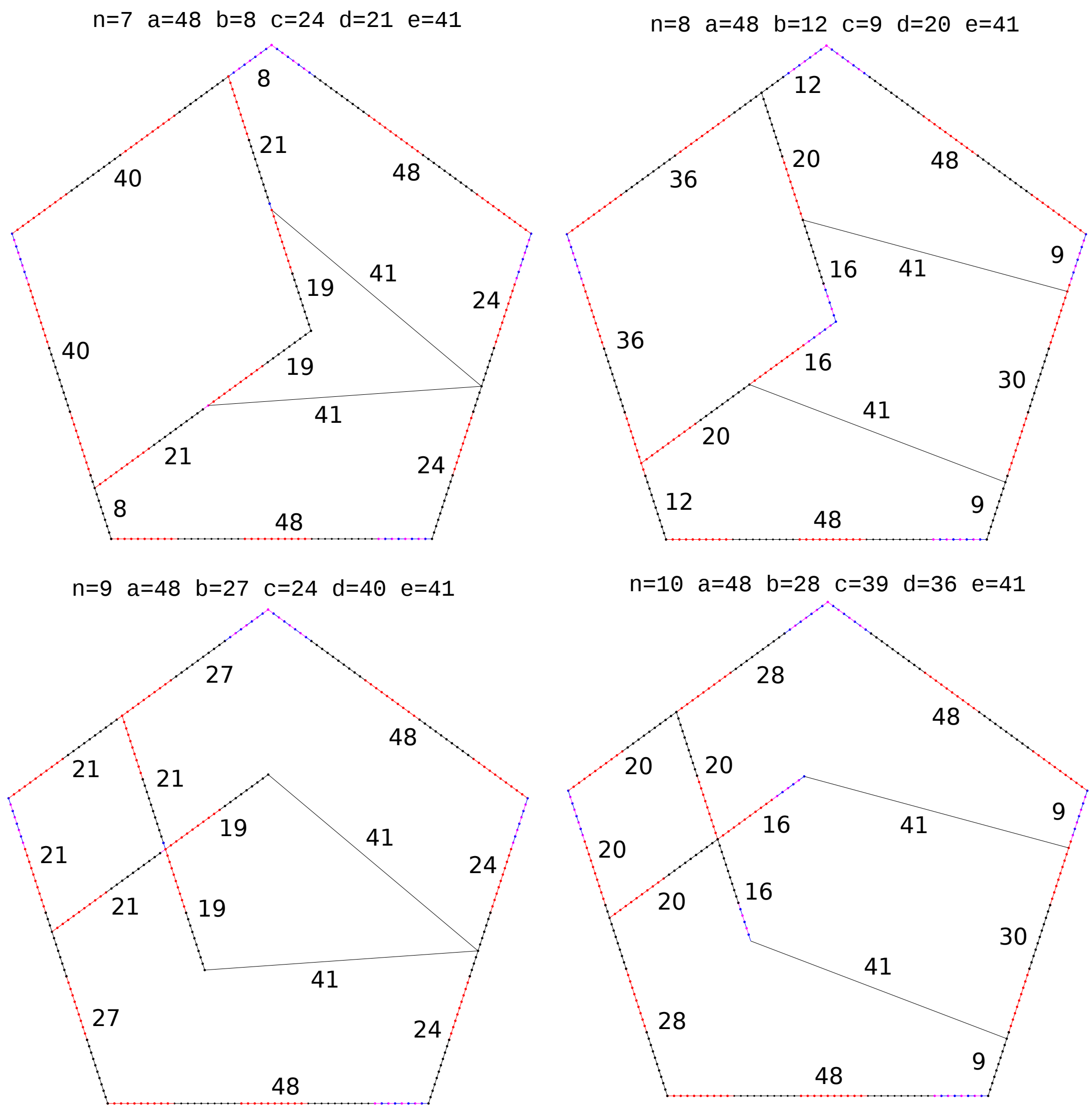

Figure 6: Pentagons of type 2 with $a = 48$

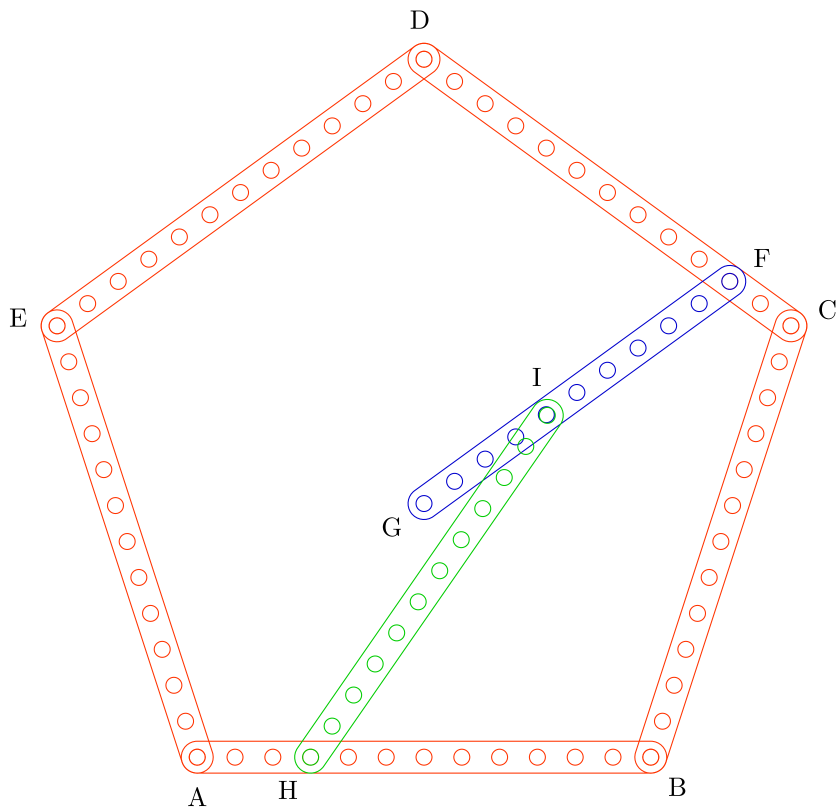

Figure 7: Pentagon of type 2 b.