Meccano four frame

https://github.com/heptagons/meccano/frames/four

Abstract

Four frame is a group of four rigid meccano [1] strips.

1 Four frame

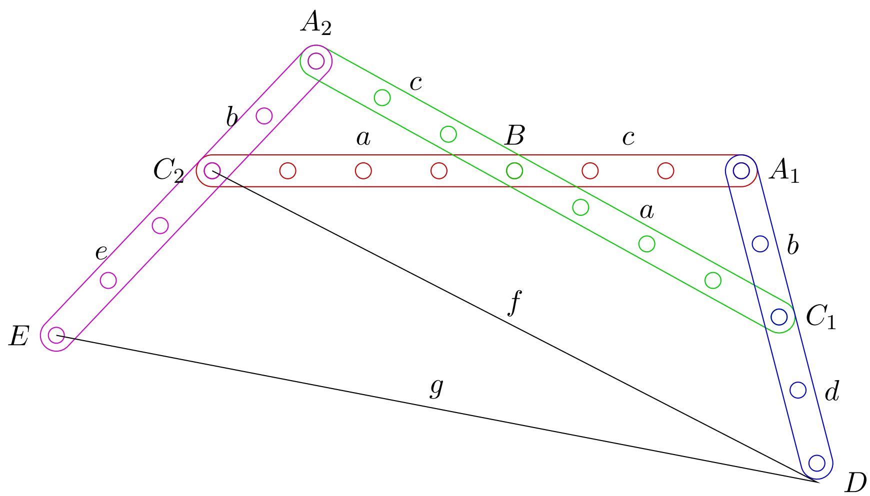

Figure 1: Antisymmetric four frame.

Figure 1 show the antisymmetric four-strips frame. From the figure we define $\alpha \equiv \angle BA_1C_1$ and define integers $m = b^2 + c^2 - a^2$ and $n = 2bc$ using the law of cosines, then we calculate $\cos\alpha$ and $\sin\alpha$:

$$(\alpha, m, n) \equiv (\angle BA_1C_1, b^2 + c^2 - a^2, 2bc) \tag{1}$$

$$\cos\alpha = \frac{m}{n} \tag{2}$$

$$\sin\alpha = \frac{\sqrt{n^2 - m^2}}{n} \tag{3}$$

From the figure 1 we define $\gamma \equiv \angle BA_2C_2$ and define integers $s = a^2 + b^2 - c^2$ and $t = 2ab$ and calculate $\cos\gamma$ and $\sin\gamma$:

$$(\gamma, s, t) \equiv (\angle BA_2C_2, a^2 + b^2 - c^2, 2ab) \tag{4}$$

$$\cos\gamma = \frac{s}{t} \tag{5}$$

$$\sin\gamma = \frac{\sqrt{t^2 - s^2}}{t} \tag{6}$$

[1] Meccano mathematics by 't Hooft

We calculate the distance $f = \overline{C_2D}$ with the law of cosines using angle α and defining integers $x = a+c$ and $y = b+d$:

$$\begin{aligned}
(x, y) &\equiv (a+c, b+d) && (7)\\
f^2 &= (a+c)^2 + (b+d)^2 - 2(a+c)(b+d)\cos\alpha && (8)\\
&= x^2 + y^2 - \frac{2mxy}{n} && (9)\\
f &= \frac{\sqrt{n^2(x^2+y^2) - 2mnxy}}{n} && (10)
\end{aligned}$$

We define a new integer $z \equiv n^2(x^2+y^2) - 2mnxy$ so we have:

$$\begin{aligned}
z &\equiv n^2(x^2+y^2) - 2mnxy && (11)\\
f &= \frac{\sqrt{z}}{n} && (12)
\end{aligned}$$

We define angle $\theta \equiv \angle A_1C_2D$ and calculate $\cos\theta$ and $\sin\theta$:

$$\begin{aligned}
\theta &\equiv \angle A_1C2D && (13)\\
\cos\theta &= \frac{(a+c)^2 + f^2 - (b+d)^2}{2(a+c)f}\\
&= \frac{x^2 + f^2 - y^2}{2xf}\\
&= \frac{x^2 + x^2 + y^2 - \frac{2mxy}{n} - y^2}{2x\frac{\sqrt{z}}{n}}\\
&= \frac{nx - my}{\sqrt{z}} && (14)\\
\sin^2\theta &= 1 - \cos^2\theta = \frac{z - (nx-my)^2}{z}\\
&= \frac{n^2(x^2+y^2) - 2mnxy - (nx-my)^2}{z}\\
&= \frac{n^2(x^2+y^2) - 2mnxy - n^2x^2 + 2nxmy - m^2y^2}{z}\\
\sin\theta &= \frac{y\sqrt{n^2-m^2}}{\sqrt{z}} && (15)
\end{aligned}$$

We define angle $\phi \equiv \angle A_2C_2D$ and we note is the sum of angles $\theta + \gamma$ and we calculate $\cos\phi$:

$$\begin{aligned}
\phi &\equiv \angle A_2C_2D && (16)\\
&= \theta + \gamma && (17)\\
\cos\phi &= \cos(\theta+\gamma) && (18)\\
&= \cos\theta\cos\gamma - \sin\theta\sin\gamma\\
&= \frac{(nx-my)s}{\sqrt{z}t} - \frac{(y\sqrt{n^2-m^2})\sqrt{t^2-s^2}}{\sqrt{z}t}\\
&= \frac{(nx-my)s - y\sqrt{(n^2-m^2)(t^2-s^2)}}{\sqrt{z}t} && (19)
\end{aligned}$$

We simplify $\sqrt{(n^2-m^2)(t^2-s^2)}$:

$$
\begin{aligned}
(n^2-m^2)(t^2-s^2) &= (n-m)(n+m)(t-s)(t+s) \\
&= (2bc-(b^2+c^2-a^2))(2bc+b^2+c^2-a^2)(2ab-(a^2+b^2-c^2))(2ab+a^2+b^2-c^2) \\
&= (a^2-(b-c)^2)((b+c)^2-a^2)(c^2-(a-b)^2)((a+b)^2-c^2) \\
&= (a+b-c)(a-b+c)(b+c+a)(b+c-a)(c+a-b)(c-a+b)(a+b+c)(a+b-c) \\
&= (a+b+c)^2(a+b-c)^2(a-b+c)^2(-a+b+c)^2 \\
\sqrt{(n^2-m^2)(t^2-s^2)} &= (a+b+c)(a+b-c)(a-b+c)(-a+b+c) \\
&= ((a+b)^2-c^2)(c^2-(a-b)^2) \\
&= (s+t)(m+n) \qquad (20)
\end{aligned}
$$

We substitute equation 20 into equation 19 and we get:

$$\cos\phi = \frac{(nx-my)s-y(m+n)(s+t)}{\sqrt{z}t} \qquad (21)$$

From the figure we define angle $\psi \equiv \angle DC_2E$ and we note equals angle $\pi-\phi$, so we have:

$$
\begin{aligned}
\psi &\equiv \angle DC_2E \qquad (22)\\
&= \pi-\phi \qquad (23)\\
\cos\psi &= \cos(\pi-\phi) \qquad (24)\\
&= -\cos\phi \\
&= \frac{-(nx-my)s+y(m+n)(s+t)}{\sqrt{z}t} \qquad (25)
\end{aligned}
$$

Finally with $\cos\psi$, e and f we can calculate distance $g=\overline{ED}$:

$$
\begin{aligned}
g^2 &= e^2+f^2-2ef\cos\psi \qquad (26)\\
&= e^2+x^2+y^2-\frac{2mxy}{n}-2e\left(\frac{\sqrt{z}}{n}\right)\left(\frac{-(nx-my)s+y(m+n)(s+t)}{\sqrt{z}t}\right) \qquad (27)\\
&= e^2+x^2+y^2-\frac{2mxy}{n}+2e\frac{(nx-my)s-y(m+n)(s+t)}{nt} \qquad (28)\\
&= \frac{(e^2+x^2+y^2)nt-2mxyt+2es(nx-my)-2ey(m+n)(s+t)}{nt} \\
g &= \frac{\sqrt{(e^2+x^2+y^2)n^2t^2-2mnxyt^2+2esnt(nx-my)-2eynt(m+n)(s+t)}}{nt} \qquad (29)
\end{aligned}
$$

1.1 Antisymmetric four frame software

From the last equation of g we identify three `input` integers i_1, i_2, i_3 which are used to get $g(i)$. Then the nested radicals software will return square-free `output` integers z_1, z_2, z_3, z_4, z_5 as $g(z)$:

$$
\begin{aligned}
i_1 &= nt \qquad (30)\\
i_2 &= i_1^2(e^2+x^2+y^2)-2i_1mxyt+2i_1es(nx-my) \qquad (31)\\
i_3 &= -2i_1ey \qquad (32)\\
i_4 &= (n^2-m^2)(t^2-s^2) \qquad (33)\\
g(i) &= \frac{\sqrt{i_2+i_3\sqrt{i_4}}}{i_1} \qquad (34)\\
g(z) &= \frac{z_2\sqrt{z_3+z_4\sqrt{z_5}}}{z_1} \qquad (35)
\end{aligned}
$$

where m, n are calculated with equations 1, x, y are calculated with equations 7 and s, t are calculated with equations 4.