Meccano frames

https://github.com/heptagons/meccano/frames

Abstract

Meccano frames are groups of rigid meccano [1] strips. Can be used as internal diagonals of polygons we want to be rigid. The lengths of such diagonals are algebraic numbers of the form $\frac{z_2\sqrt{z_3+z_4\sqrt{z_5}}}{z_1}$ which in some cases can be denested as $\frac{z_2+z_3\sqrt{z_4}}{z_1}$ where $z_i \in \mathbb{Z}$.

1 Triangular frame

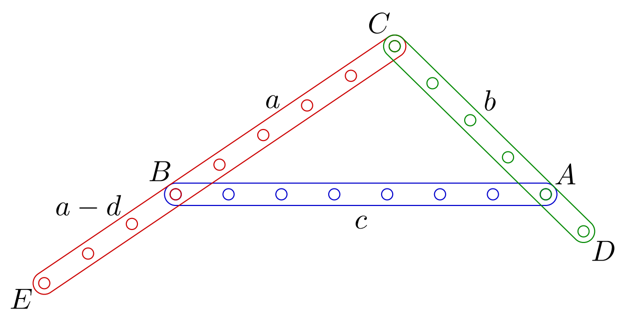


Figure 1: Triangular frame.

Figure 1 shows a triangular frame. With three strips we form the triangle $\triangle ABC$. At least we extend one of the two strips $\overline{CB}$ and $\overline{CA}$ to become $\overline{CE}$ and $\overline{CD}$. The new vertices D and E distance is rigid as the triangle with the form $\frac{z_2\sqrt{z_3}}{z_1}$, where $z_i \in \mathbb{Z}^+$.

First we identify five integer distances a, b, c, d, e:

$$a \equiv \overline{CB}, \quad b \equiv \overline{CA}, \quad c \equiv \overline{AB}, \quad c < a + b \tag{1}$$

$$d \equiv \overline{CB} + \overline{BE} = \overline{CE} \geq a \tag{2}$$

$$e \equiv \overline{CA} + \overline{AD} = \overline{CD} \geq b \tag{3}$$

We calculate the cosine of $\angle BCA$:

$$\theta \equiv \angle BCA \tag{4}$$

$$\cos\theta = \frac{a^2 + b^2 - c^2}{2ab} \tag{5}$$

We define $g \equiv \overline{DE}$ the triangular frame main diagonal. We calculate the diagonal with the law of

[1] Meccano mathematics by 't Hooft

cosines:

$$
\begin{aligned}
g^2 &= \overline{DE}^2 \\
&= \overline{CD}^2 + \overline{CE}^2 - 2\overline{CD} \times \overline{CE} \cos\theta \\
&= d^2 + e^2 - 2de\cos\theta \\
&= d^2 + e^2 - de\left(\frac{a^2+b^2-c^2}{ab}\right) \\
g &= \sqrt{d^2 + e^2 - de\left(\frac{a^2+b^2-c^2}{ab}\right)} \\
&= \frac{\sqrt{a^2b^2(d^2+e^2) - abde(a^2+b^2-c^2)}}{ab} \\
&= \frac{\sqrt{ab((ad-be)(bd-ae)+c^2de)}}{ab} \qquad (6)
\end{aligned}
$$

1.1 Triangular frame software

From the last equation of diagonal g we identify two `input` integers i_1, i_2 which are used to get $g(i)$. Then the nested radicals software will return square-free `output` integers z_1, z_2, z_3 as $g(z)$:

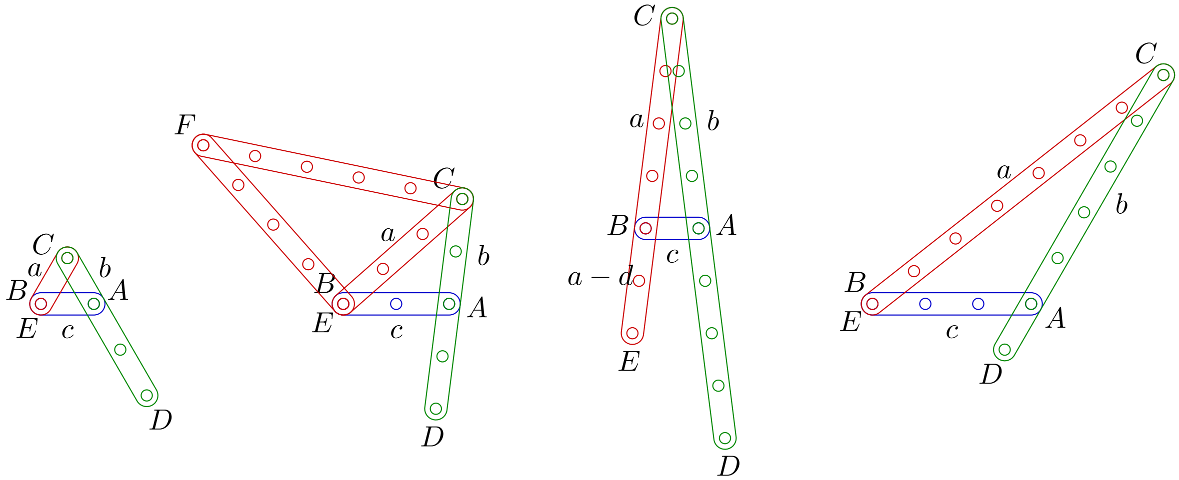
$$
\begin{aligned}
i_1 &= ab \qquad (7) \\
i_2 &= ab((ad-be)(bd-ae)+c^2de) \qquad (8) \\
g(i) &= \frac{\sqrt{i_2}}{i_1} \qquad (9) \\
g(z) &= \frac{z_2\sqrt{z_3}}{z_1} \qquad (10)
\end{aligned}
$$

We request a software report for all the triangle frames with specific distance $\sqrt{z_3}$ for a given maximum strips length. This report reject the triangles where $z_1, z_2 \neq 1$. Next example report list all the triangles $z_3 = 7, max = 15$ so the software filters are $c < a + b$, $a \leq d \leq max$, $b \leq e \leq max$, $c \leq max$:

```
=== RUN   TestFramesTriangleSurds
NewFrames().TriangleSurds surd=7 max=15
  1) a=1 e=1+2 c=1 cos=1/2
  2) d=1+1 e=1+2 c=1 cos=1/2
  3) d=1+2 b=1 c=1 cos=1/2
  4) d=1+2 e=1+1 c=1 cos=1/2
  5) a=2 e=2+1 c=2 cos=1/2
  6) d=2+1 b=2 c=2 cos=1/2
  7) a=3 e=2+2 c=2 cos=3/4 CED=pi/2
  8) d=3+1 e=2+1 c=2 cos=3/4 CDE=pi/2
  9) d=4+2 e=4+4 c=1 cos=31/32
 10) d=4+4 e=4+2 c=1 cos=31/32
 11) a=7 e=5+1 c=3 cos=13/14
 12) a=7 e=5+2 c=3 cos=13/14
```

The code is in the folder `github.com/heptagons/meccano/frames`.

Figure 2 show four triangles from the mentioned report.



(a) (b) (c) (d)

Figure 2: Some triangular frames with distances $g = \overline{DE} = \sqrt{7}$ found by the software.

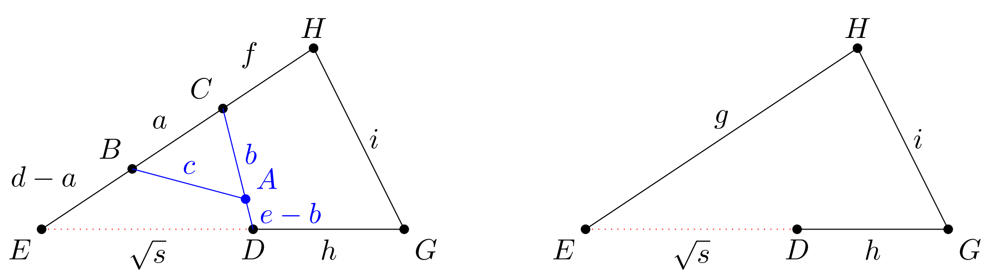
1.2 Triangular frame distance of the form $\sqrt{z_3} + z_4$

In the figure 2, the particular triangle at (b) was reported with the angle $\angle CED = \pi/2$. For such triangle, if we add a triangle $\triangle CBF$ where angle $\angle CBF = \pi/2$ also, then we'll have vertices D, E, F collinear. With two extra strips we can form a pythagorean triangle sharing the strip a. The figure (b) shows the pythagorean triangle with sides $3, 4, 5$. This five-strips frame has a new distance:

$$\begin{aligned} h &= \overline{DF} \\ &= \overline{DB} + \overline{BF} \\ &= \sqrt{7} + 4. \end{aligned}$$

1.3 Another rigid distances $\sqrt{z_3} + z_4$

We explore a more complicated frame to get additional cases of distances $\sqrt{s} + h$ without relying in an explicit pythagorean triangle as we saw in case (b) of figure 2.



(a) (b)

Figure 3: The five strips intented to form an algebraic distance $\overline{EG} = \sqrt{s} + h$.

From figure 3 (a) we know $\sqrt{s}$ distance between nodes E and D is produced by the three strips frame $a + d$, $b + e$ and c. Using the law of cosines we calculate the angle $\theta = \angle CED$ in terms of $\sqrt{s}$:

$$
\begin{aligned}
\cos\theta &= \frac{d^2 + (\sqrt{s})^2 - e^2}{2d\sqrt{s}} \\
&= \frac{(d^2 + s - e^2)\sqrt{s}}{2ds} && (11) \\
&= \frac{m\sqrt{s}}{n} && (12) \\
m &= d^2 + s - e^2 && (13) \\
n &= 2ds && (14)
\end{aligned}
$$

From figure 3 (a) we notice two sets of points are collinear: $\{E, B, C, H\}$ and $\{E, D, G\}$. Using the law of cosines we calculate the angle $\theta = \angle HEG$ in terms of distances $g, \sqrt{s} + h, i$:

$$
\begin{aligned}
\cos\theta &= \frac{g^2 + (\sqrt{s} + h)^2 - i^2}{2g(\sqrt{s} + h)} \\
&= \frac{g^2 + s + 2\sqrt{s}h + h^2 - i^2}{2g(\sqrt{s} + h)} \\
&= \frac{g^2 + s + h^2 - i^2 + 2\sqrt{s}h}{2g(\sqrt{s} + h)} && (15)
\end{aligned}
$$

We multiply both numerator and denominator by $\sqrt{s} - h$ to eliminate the surd from denominator:

$$
\begin{aligned}
\cos\theta &= \frac{(s + g^2 + h^2 - i^2)(\sqrt{s} - h) + 2\sqrt{s}h(\sqrt{s} - h)}{2g(\sqrt{s} + h)(\sqrt{s} - h)} \\
&= \frac{(s + g^2 + h^2 - i^2)(\sqrt{s} - h) + 2sh - 2\sqrt{s}h^2}{2g(s - h^2)} \\
&= \frac{-h(s + g^2 + h^2 - i^2 - 2s) + (s + g^2 + h^2 - i^2 - 2h^2)\sqrt{s}}{2g(s - h^2)} \\
&= \frac{h(s - g^2 - h^2 + i^2) + (s + g^2 - h^2 - i^2)\sqrt{s}}{2g(s - h^2)} \\
&= \frac{o + p\sqrt{s}}{q} && (16) \\
o &= h(s - g^2 - h^2 + i^2) && (17) \\
p &= s + g^2 - h^2 - i^2 && (18) \\
q &= 2g(s - h^2) && (19)
\end{aligned}
$$

We compare both cosines equations 12 and 16:

$$
\frac{m\sqrt{s}}{n} = \frac{o + p\sqrt{s}}{q} \tag{20}
$$

Since all variables are integers we need two conditions. First o should be zero. And second $\frac{m}{n} = \frac{p}{q}$.

For condition 1, we force o to be zero:

$$
\begin{aligned}
o &= 0 \\
h(s - g^2 - h^2 + i^2) &= 0 \\
s &= g^2 + h^2 - i^2 && (21)
\end{aligned}
$$

For condition2, we force m, n, p, q as:

$$\begin{aligned}\frac{m}{n} &= \frac{p}{q} \\ \frac{d^2+s-e^2}{2ds} &= \frac{s+g^2-h^2-i^2}{2g(s-h^2)}\end{aligned} \tag{22}$$

We replace the value of s of last equation RHS with the value of equation 21 of condition 1:

$$\begin{aligned}\frac{d^2-e^2+s}{ds} &= \frac{s+g^2-h^2-i^2}{g(s-h^2)} \\ &= \frac{g^2+h^2-i^2+g^2-h^2-i^2}{g(g^2+h^2-i^2-h^2)} \\ &= \frac{2(g^2-i^2)}{g(g^2-i^2)} \\ &= \frac{2}{g} \\ (d^2-e^2+s)g &= 2ds\end{aligned} \tag{23}$$

$\boxed{TODO: Examples!!!}$

2 Triangle pair frame

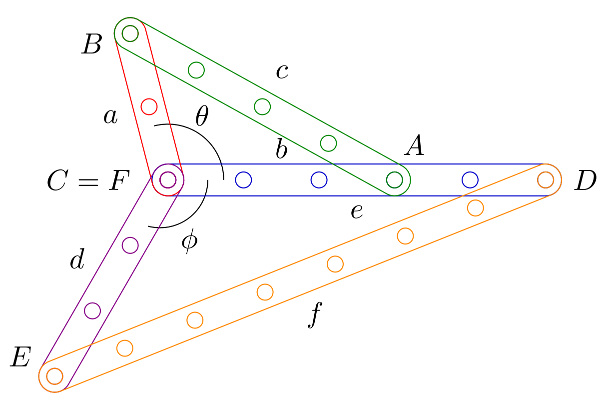


Figure 4: Triangle pair frame.

Figure 4 shows a triangle pair frame. The pair joins triangles $\triangle ABC$ and $\triangle DEF$ in such a way vertices C and F coincide and vertices A, C, D, F be collinear. With only five strips this frame is small and useful to make up the rigid polygons diagonals of the form $g = \overline{BE} = \frac{z_2\sqrt{z_3+z_4\sqrt{z_5}}}{z_1}, z_i \in \mathbb{Z}$. In some cases the diagonal can be denested to the form $g = \frac{z_2+z_3\sqrt{z_4}}{z_1}$.

2.1 Triangle pair algebra

Using the law of cosines we calculate the angle $\theta = \angle ACB$ with defined variables m, n and the angle $\phi = \angle DFE$ with defined variables o, p:

$$(\theta, m, n) \equiv (\angle ACB, a^2 + b^2 - c^2, 2ab), \quad |m| \leq n, \quad m, n \in \mathbb{Z} \tag{24}$$

$$\cos\theta = \frac{m}{n} \tag{25}$$

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \frac{\sqrt{n^2 - m^2}}{n} \tag{26}$$

$$(\phi, o, p) \equiv (\angle DFE, d^2 + e^2 - f^2, 2de), \quad |o| \leq p, \quad o, p \in \mathbb{Z} \tag{27}$$

$$\cos\phi = \frac{o}{p} \tag{28}$$

$$\sin\phi = \sqrt{1 - \cos^2\phi} = \frac{\sqrt{p^2 - o^2}}{p} \tag{29}$$

Then, we use the cosines sum identity:

$$\begin{aligned}
\cos(\theta + \phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi \\
&= \left(\frac{m}{n}\right)\left(\frac{o}{p}\right) - \left(\frac{\sqrt{n^2 - m^2}}{n}\right)\left(\frac{\sqrt{p^2 - o^2}}{p}\right) \\
&= \frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{np}
\end{aligned} \tag{30}$$

Finally we can calculate the distance $g \equiv \overline{BE}$ using the law of cosines:

$$\begin{aligned}
g &\equiv \overline{BE} \\
&= \sqrt{a^2 + d^2 - 2ad\cos(\theta + \phi)} \\
&= \sqrt{a^2 + d^2 - 2ad\left(\frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{np}\right)} \\
&= \sqrt{a^2 + d^2 - 2ad\left(\frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{4abde}\right)} \\
&= \sqrt{a^2 + d^2 - \frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{2be}} \\
&= \frac{\sqrt{4b^2e^2(a^2 + d^2) - 2bemo + 2be\sqrt{(n^2 - m^2)(p^2 - o^2)}}}{2be}
\end{aligned} \tag{31}$$

2.2 Triangle pairs software

From the last equation of g we identify three `input` integers i_1, i_2, i_3 which are used to get $g(i)$. Then the nested radicals sofware will return square-free `output` integers z_1, z_2, z_3, z_4, z_5 as $g(z)$:

$$i_1 \equiv 2be \tag{32}$$

$$i_2 \equiv i_1^2(a^2+d^2) - i_1 mo \tag{33}$$

$$i_3 \equiv (n^2-m^2)(p^2-o^2) \tag{34}$$

$$g(i) = \frac{\sqrt{i_2 + i_1\sqrt{i_3}}}{i_1} \tag{35}$$

$$g(z) = \frac{z_2\sqrt{z_3 + z_4\sqrt{z_5}}}{z_1} \tag{36}$$

We run a program to print a list of triangle pairs with sides $1 < a, b, c, d, e, f \leq max$ having a given distance $\overline{BE} = g$ or particular $z3, z4, z5$. Next example request a pairs list with $g = z_2\sqrt{46+18\sqrt{5}}/z_1$ up to strip length 10 so we set as limits $max = 10, z_3 = 46, z_4 = 18, z_5 = 5$ to get next report (text in blue):

$Folder$: `github.com/heptagons/meccano/frames`

$Call$: `NewFrames().TrianglePairsTex(10, [46 18 5])`

$(a,b,c) \oplus (d,e,f) \mapsto g$
$(2,1,2) \oplus (3,3,3) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(2,1,2) \oplus (3,8,7) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(2,2,2) \oplus (3,6,6) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(2,3,4) \oplus (3,5,7) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(2,4,4) \oplus (3,8,7) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(3,3,3) \oplus (2,4,4) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(4,2,4) \oplus (6,6,6) \mapsto \sqrt{46+18\sqrt{5}}$
$(4,4,4) \oplus (6,7,8) \mapsto \sqrt{46+18\sqrt{5}}$
$(6,3,6) \oplus (4,4,4) \mapsto \sqrt{46+18\sqrt{5}}$
$(6,3,6) \oplus (9,9,9) \mapsto \dfrac{3\sqrt{46+18\sqrt{5}}}{2}$
$(6,6,6) \oplus (4,8,8) \mapsto \sqrt{46+18\sqrt{5}}$
$(6,7,8) \oplus (9,9,9) \mapsto \dfrac{3\sqrt{46+18\sqrt{5}}}{2}$

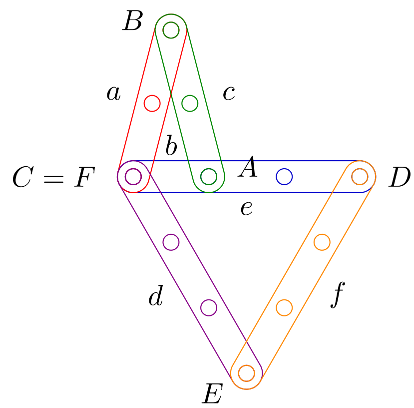


Figure 5: Triangle pair frame $(2,1,2) \oplus (3,3,3)$ makes $\overline{BE} = \dfrac{\sqrt{46 + 18\sqrt{5}}}{2}$.

In figure 5 we build a triangular pair following one of the last report results, when $abc = (2,1,2)$ and $def = (3,3,3)$.

3 Triangle pair extended frame

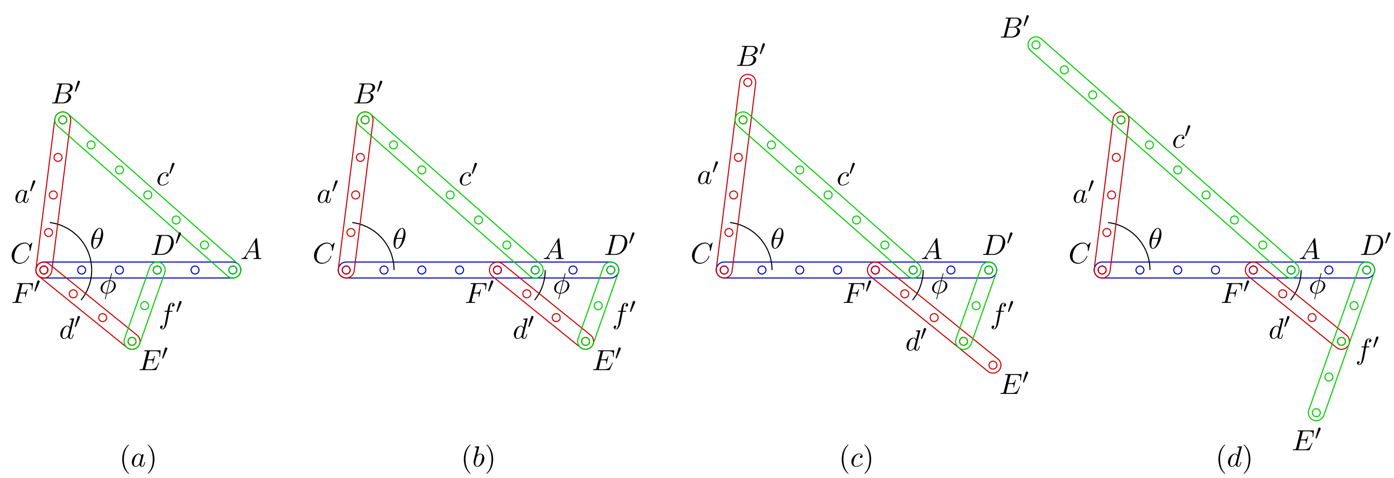


Figure 6: Triangle pair extended frame. Starts like previous triangle pair frame except we can extend strips a' or c', d' or f', and we can separate vertices C and F'. Vertices A, C, D', F' remain collinear and we are interested in the distance $g \equiv \overline{B'E'}$. We show four examples: (a) is the original triangle pair, (b) has moved the $\triangle D'E'F'$ to the right, (c) also extends strips a' and d' and (d) extends strips c' and f'.

We show some triangle pair extended frames in figure 6. As with not-extended triangle pair of figure 4 we also have two triangles with five strips, but we can perform one, two or three transformations on the frame:

1. Separate nodes C and F which moves $\triangle D'E'F'$.
2. Extends strip a' or strip c' but not both.
3. Extend strip d' or strip f' but not both.

For each transformation we define three integers x, y_1, y_2:

$$x = \begin{cases} 0 & C, F \text{ vertices remain joined} \\ \geq 0 & \triangle DEF \text{ is moved to the right a distance equal to } x \end{cases} \tag{37}$$

$$y_1 = \begin{cases} \geq 0 & a' = a + y_1 \quad c' = c \\ < 0 & a' = a \qquad\quad c' = c + |y_1| \end{cases} \tag{38}$$

$$y_2 = \begin{cases} \geq 0 & d' = d + y_2 \quad f' = f \\ < 0 & d' = d \qquad\quad f' = f + |y_2| \end{cases} \tag{39}$$

Let define $M(a, b, c)$ the triangle above, $N(d, e, f)$ the triangle below and $T(x, y_1, y_2)$ the transformations. Then we can describe the cases $(a) - (d)$ of figure 6 as operations:

$$\begin{aligned}
(a) &: M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(0, 0, 0) \\
(b) &: M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(4, 0, 0) \\
(c) &: M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(4, +2, +1) \\
(d) &: M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(4, -3, -2)
\end{aligned}$$

3.1 Triangle pair extended frame algebra

We are going to calculate the diagonal $g \equiv \overline{B'E'}$ of the triangle pair extended using the M, N, T values. We start setting the vertex C at the origin of the standard two-dimensional graph and defining (B_x, By) the abscissa and ordinate of vertex B' and (E_x, E_y) the abscissa and ordinate of vertex E'.

For the triangle above $M(a, b, c)$ we have two cases: $y_1 \geq 0$ and $y_1 < 0$. In the not-extended triangle pair we already calculated $\theta = \angle ACB, \cos\theta, \sin\theta$ based in m, n of equation 24. For the case $y_1 < 0$ here, we calculate also $\omega = \angle BAC, \cos\omega, \sin\omega$ using two variables p, q and finally we get (B_x, B_y):

$$(\omega, p, q) \equiv (\angle BAC, b^2 + c^2 - a^2, 2bc), \quad |p| \leq q, \quad p, q \in \mathbb{Z} \tag{40}$$

$$\cos\omega = \frac{p}{q} \tag{41}$$

$$\sin\omega = \sqrt{1 - \cos^2\omega} = \frac{\sqrt{q^2 - p^2}}{q} \tag{42}$$

$$a' = a + |y_1| \tag{43}$$

$$c' = c + |y_1| \tag{44}$$

$$B_x = \begin{cases} y_1 \geq 0 & a' \cos\theta \\ y_1 < 0 & b - c' \cos\omega \end{cases} \tag{45}$$

$$B_y = \begin{cases} y_1 \geq 0 & a' \sin\theta \\ y_1 < 0 & c' \sin\omega \end{cases} \tag{46}$$

For the triangle below $N(d, e, f)$ we have two cases: $y_2 \geq 0$ and $y_2 < 0$. In both cases we will use $x \geq 0$ always for simplicity. In the not-extended triangle pair we already calculated $\phi = \angle DFE, \cos\phi, \sin\phi$ defining o, p in equation 27. For the case $y_2 < 0$ here we calculate also $\psi = \angle EDF, \cos\psi, \sin\psi$ using two variables r, s and finally we get (E_x, E_y):

$$(\psi, r, s) \equiv (\angle EDF, e^2 + f^2 - d^2, 2ef), \quad |r| \leq s, \quad r, s \in \mathbb{Z} \tag{47}$$

$$\cos\psi = \frac{r}{s} \tag{48}$$

$$\sin\psi = \sqrt{1 - \cos^2\psi} = \frac{\sqrt{s^2 - r^2}}{s} \tag{49}$$

$$d' = d + |y_2| \tag{50}$$

$$f' = f + |y_2| \tag{51}$$

$$E_x = \begin{cases} y_2 \geq 0 & x + d'\cos\phi \\ y_2 < 0 & x + e - f'\cos\psi \end{cases} \tag{52}$$

$$E_y = \begin{cases} y_2 \geq 0 & d'\sin\phi \\ y_2 < 0 & f'\sin\psi \end{cases} \tag{53}$$

With the four components B_x, B_y, E_x, E_y we can calculate $g = \overline{B'E'}$. For the abscissas we compute a difference $B_x - E_x$ since they both grow to the same direction: to the right. For the ordinates we compute an addition $B_y + E_y$ since they grow in opposite directions: above and below.

$$g = \sqrt{(B_x - E_x)^2 + (B_y + E_y)^2} \tag{54}$$

$$= \sqrt{(B_x^2 + B_y^2) + (E_x^2 + E_y^2) - 2B_xE_x + 2B_yE_y} \tag{55}$$

We need to calculate separated four types of diagonals $g^{++}, g^{+-}, g^{-+}, g^{--}$ according the signs of y_1 and y_2 as described in the next table:

g	y_1	y_2
g^{++}	≥ 0	≥ 0
g^{+-}	≥ 0	< 0
g^{-+}	< 0	≥ 0
g^{--}	< 0	< 0

3.2 Triangle pair extended g^{++} ($y_1 \geq 0$ and $y_2 \geq 0$)

For g^{++} we calculate sums and products of Bx, By, Ex, Ey when $y_1 \geq 0$ and $y_2 \geq 0$:

$$a' = a + u, \qquad d' = d + v \tag{56}$$

$$\begin{aligned}(B_x^2 + B_y^2)^{++} &= a'^2\cos^2\theta + a'^2\sin^2\theta \\ &= a'^2 \end{aligned}\tag{57}$$

$$\begin{aligned}(E_x^2 + E_y^2)^{++} &= (x + d'\cos\phi)^2 + (d'\sin\phi)^2 \\ &= x^2 + 2xd'\cos\phi + d'^2\cos^2\phi + d'^2\sin^2\phi \\ &= x^2 + 2xd'\cos\phi + d'^2 \\ &= x^2 + d'^2 + \frac{2xd'o}{p}\end{aligned}\tag{58}$$

$$\begin{aligned}(B_xE_x)^{++} &= (a'\cos\theta)(x + d'\cos\phi) \\ &= \frac{a'mx}{n} + \frac{a'md'o}{np}\end{aligned}\tag{59}$$

$$\begin{aligned}(B_yE_y)^{++} &= (a'\sin\theta)(d'\sin\phi) \\ &= \frac{a'd'\sqrt{(n^2 - m^2)(p^2 - o^2)}}{np}\end{aligned}\tag{60}$$

We subsitute the products in equation 55:

$$
\begin{aligned}
g^{++} &= \sqrt{(B_x^2+B_y^2)^{++}+(E_x^2+E_y^2)^{++}-2(B_xE_x)^{++}+2(B_yE_y)^{++}} \\
&= \sqrt{a'^2+\left(x^2+d'^2+\frac{2xd'o}{p}\right)-2\left(\frac{a'mx}{n}+\frac{a'md'o}{np}\right)+2\left(\frac{a'd'\sqrt{(n^2-m^2)(p^2-o^2)}}{np}\right)} \\
&= \sqrt{a'^2+x^2+d'^2+\frac{2xd'o}{p}-\frac{2a'mx}{n}-\frac{2a'md'o}{np}+\frac{2a'd'\sqrt{(n^2-m^2)(p^2-o^2)}}{np}} \\
&= \frac{\sqrt{n^2p^2(a'^2+x^2+d'^2)+2xd'on^2p-2a'mxnp^2-2a'md'onp+2a'd'np\sqrt{(n^2-m^2)(p^2-o^2)}}}{np} \\
&= \frac{\sqrt{n^2p^2(a'^2+x^2+d'^2)+2np(xd'on-a'mxp-a'md'o)+2npa'd'\sqrt{(n^2-m^2)(p^2-o^2)}}}{np} \qquad (61)
\end{aligned}
$$

3.3 Triangle pair extended g^{++} software

From the last equation of g^{++} we identify four `input` integer variables to calculate software $g^{++}(i)$ which will be reduced or even denested as $g^{++}(z)$:

$$
\begin{aligned}
i_1 &= np \qquad (62) \\
i_2 &= i_1^2(a'^2+x^2+d'^2)+2i_1(xd'on-a'mxp-a'md'o) \qquad (63) \\
i_3 &= 2i_1a'd' \\
i_4 &= (n^2-m^2)(p^2-o^2) \qquad (64)
\end{aligned}
$$

$$g^{++}(i) = \frac{\sqrt{i_2+i_3\sqrt{i_4}}}{i_1} \qquad (65)$$

$$g^{++}(z) = \frac{z_2\sqrt{z_3+z_4\sqrt{z_5}}}{z_1} \quad \texttt{or} \quad \frac{z_2+\sqrt{z_3}}{z_1} \qquad (66)$$