1 32 bits algebra

Define one rational number B and three algebraic numbers C, E and G all irreducibles:

$$B \equiv \frac{\pm b_1}{a}$$

$$C \equiv \frac{\pm b_1 \pm b_2\sqrt{c_1}}{a}$$

$$E \equiv \frac{\pm b_1 \pm b_2\sqrt{c_1} \pm b_3\sqrt{c_2 \pm d_1\sqrt{e_1}}}{a}$$

$$G \equiv \frac{\pm b_1 \pm b_2\sqrt{c_1} \pm b_3\sqrt{c_2 \pm d_1\sqrt{e_1}} \pm b_4\sqrt{c_3 \pm d_2\sqrt{e_2 \pm f_1\sqrt{g_1}}}}{a}$$

for 32-bit natural numbers:

$$1 \le a \le 4,294,967,295$$
$$0 \le b, c, d, e, f, g \le 4,294,967,295$$

1.1 functions

$$f_1 = f(\pm b_1)$$
$$f_2 = f(\pm b_2, c_1)$$
$$f_3 = f(\pm b_3, c_2, \pm d_1, e_1)$$
$$f_4 = f(\pm b_4, c_3, \pm d_2, e_2, \pm f_1, g_1)$$

$$f_B = f(f_1(), a)$$
$$f_C = f(f_1(), f_2(), a)$$
$$f_E = f(f_1(), f_2(), f_3(), a)$$
$$f_G = f(f_1(), f_2(), f_3(), f_4(), a)$$

f_B examples:

$$\cos 0 = 1 \implies f_B(f_1(1), 1)$$
$$\sin \frac{\pi}{6} = \frac{1}{2} \implies f_B(f_1(1), 2)$$

f_C examples:

$$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \implies f_C(\emptyset, f_2(1, 2), 2)$$
$$\sin \frac{\pi}{10} = \frac{-1 + \sqrt{5}}{4} \implies f_C(f_1(-1), f_2(1, 5), 4)$$

f_E examples:

$$\sin \frac{\pi}{5} = \frac{\sqrt{10 - 2\sqrt{5}}}{4} \implies f_E(\emptyset, \emptyset, f_3(1, 10, -2, 5), 4)$$
$$\sin \frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4} \implies f_E(\emptyset, f_1(1, 6), f_3(1, 2, 0, 0), 4)*$$
$$\sin \frac{\pi}{12} = \frac{\sqrt{2 + \sqrt{3}}}{2} \implies f_E(\emptyset, \emptyset, f_3(1, 2, 1, 3), 2)$$
$$\cos \frac{\pi}{15} = \frac{1 + \sqrt{5} + \sqrt{30 - 6\sqrt{5}}}{8} \implies f_E(f_1(1), f_2(1, 5), f_3(1, 30, -6, 5), 8)$$

f_G examples:

$$\cos\frac{\pi}{16} = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$$
$$\implies f_G(\emptyset, \emptyset, \emptyset, f_4(1,2,1,2,1,2), 2)$$

$$\cos\frac{\pi}{24} = \frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2}$$
$$\implies f_G(\emptyset, \emptyset, \emptyset, f_4(1,2,1,2,1,3), 2)$$

$$\cos\frac{2\pi}{17} = \frac{-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}+2\sqrt{17+3\sqrt{17}-\sqrt{170+38\sqrt{17}}}}{16}$$
$$\implies f_G(f_1(-1), f_2(1,17), f_3(1,34,-2,17), f_4(2,17,3,17,-1,170,+38,17), 16)$$

2 Operations

2.1 $B_1 \times B_2 = B_3$

$$B_1 \times B_2 = \frac{\pm a_1}{b_1} \times \frac{\pm a_2}{b_2}$$
$$= \frac{\pm a_1 a_2}{b_1 b_2}$$
$$= \frac{\pm a_3}{b_3} \qquad\qquad (\pm a_3, b_3) = \gcd(\pm a_1 a_2, b_1 b_2)$$

2.2 $1/B_1 = B_2, a > 0$

$$\frac{1}{B_1} = \frac{1}{\pm a_1/b_1}$$
$$= \frac{\pm b_1}{a_1}$$
$$= \frac{\pm a_2}{b_2}$$

2.3 $B_1 + B_2 = B_3$

$$B_1 + B_2 = \frac{\pm a_1}{b_1} + \frac{\pm a_2}{b_2}$$
$$= \frac{\pm a_1 b_2 \pm a_2 b_1}{b_1 b_2}$$
$$= \frac{\pm a_3}{b_3} \qquad\qquad (\pm a_3, b_3) = \gcd(\pm a_1 b_2, \pm a_2 b_1, b_1 b_2)$$

2.4 $\sqrt{B_1} = C_2$

$$\sqrt{B_1} = \sqrt{\frac{a_1}{b_1}}$$
$$= \frac{\sqrt{a_1 b_1}}{b_1}$$
$$= \frac{m\sqrt{c_2}}{b_1} \qquad\qquad a_1 b_1 = m^2 c_2$$
$$= \frac{a_2\sqrt{c_2}}{b_2} \qquad\qquad (a_2, b_2) = gcd(m, b_1)$$

2.5 $B_1 + C_2 = D_3$

$$\begin{aligned}
B_1 + C_2 &= \frac{\pm a_1}{b_1} + \frac{\pm a_2\sqrt{c_2}}{b_2}\\
&= \frac{\pm a_2 b_1 \sqrt{c_2} \pm a_1 b_2}{b_1 b_2}\\
&= \frac{\pm a_3 \sqrt{c_2} \pm d_3}{b_3} \qquad\qquad (\pm a_3, b_3, \pm d_3) = gcd \pm a_2 b_1 b_1 b_2 \pm a_1 b_2
\end{aligned}$$

2.6 $B_1 + D_2 = D_3$

$$\begin{aligned}
B_1 + D_2 &= \frac{\pm a_1}{b_1} + \frac{\pm a_2\sqrt{c_2} \pm d_2}{b_2}\\
&= \frac{\pm a_2 b_1 \sqrt{c_2} \pm a_1 b_2 \pm d_2 b_1}{b_1 b_2}\\
&= \frac{\pm a_3 \sqrt{c_2} \pm d_3}{b_3} \qquad\qquad (\pm a_3, b_3, \pm d_3) = gcd(\pm a_2 b_1, b_1 b_2, \pm a_1 b_2 \pm d_2 b_1)
\end{aligned}$$

2.7 $B_1 + F_2 = F_3$

$$\begin{aligned}
B_1 + F_2 &= \frac{\pm a_1}{b_1} + \frac{\pm a_2\sqrt{c_2 \pm e_2\sqrt{f_2}} \pm d_2}{b_2}\\
&= \frac{\pm a_2 b1 \sqrt{c_2 \pm e_2\sqrt{f_2}} \pm a_1 b_2 \pm d_2 b_1}{b_1 b_2}\\
&= \frac{\pm a_3 \sqrt{c_2 \pm e_2\sqrt{f_2}} \pm d_3}{b_3} \qquad\qquad (\pm a_3, b_3, \pm d_3) = gcd(\pm a_2 b1, b_1 b_2, \pm a_1 b_2 \pm d_2 b_1)
\end{aligned}$$

2.8 $C_1 + C_2 = F_3$

$$\begin{aligned}
C_1 + C_2 &= \frac{\pm a_1\sqrt{c_1}}{b_1} + \frac{\pm a_2\sqrt{c_2}}{b_2}\\
&= \frac{\pm a_1 b_2 \sqrt{c_1} \pm a_2 b_1 \sqrt{c_2}}{b_1 b_2}\\
&= \frac{\pm m\sqrt{c_1} \pm n\sqrt{c_2}}{o}\\
&= \frac{\pm\sqrt{m^2 c_1 + n^2 c_2 \pm 2mn\sqrt{c_1 c_2}}}{o}\\
&= \frac{\pm\sqrt{q \pm 2mnp\sqrt{f_3}}}{o} \qquad\qquad c_1 c_2 = p^2 f_3\\
&= \frac{\pm r\sqrt{c_3 \pm e_3\sqrt{f_3}}}{o} \qquad\qquad q = r^2 c_3,\ 2mnp = r^2 e_3\\
&= \frac{\pm a_3 \sqrt{c_3 \pm e_3\sqrt{f_3}}}{b_3}
\end{aligned}$$

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2.9 $C_1 \times C_2 = C_3$

$$C_1 \times C_2 = \frac{\pm a_1 \sqrt{c_1}}{b_1} \times \frac{\pm a_2 \sqrt{c_2}}{b_2}$$

$$= \frac{\pm a_1 a_2 \sqrt{c_1 c_2}}{b_1 b_2}$$

$$= \frac{\pm a_1 a_2 m \sqrt{c_3}}{b_1 b_2} \qquad\qquad c_1 c_2 = m^2 c_3$$

$$= \frac{\pm a_3 \sqrt{c_3}}{b_3} \qquad\qquad (\pm a_3, b_3) = gcd(\pm a_1 a_2 m, b_1 b_2)$$

2.10 $1/C_1 = C_2$

$$1/C_1 = \frac{1}{\frac{\pm a_1 \sqrt{c_1}}{b_1}}$$

$$= \frac{b_1}{\pm a_1 \sqrt{c_1}}$$

$$= \frac{\pm b_1 \sqrt{c_1}}{c_1}$$

$$= \frac{\pm a_2 \sqrt{c_1}}{b_2} \qquad\qquad (\pm a_2, b_2) = gcd(\pm b_1, c1)$$

2.11 $\sqrt{C_1} = F_2$

$$\sqrt{C_1} = \sqrt{\frac{a_1 \sqrt{c_1}}{b_1}}$$

$$= \frac{\sqrt{a_1 b_1 \sqrt{c_1}}}{b_1}$$

$$= \frac{m \sqrt{e_2 \sqrt{c_1}}}{b_1} \qquad\qquad a_1 b_1 = m^2 e_2$$

$$= \frac{a_2 \sqrt{e_2 \sqrt{c_1}}}{b_2} \qquad\qquad (a_2, b_2) = gcd(m, b_1)$$

2.12 $C_1 + D_2 = F_3$

$$C_1 + D_2 = \frac{\pm a_1 \sqrt{c_1}}{b_1} + \frac{\pm a_2 \sqrt{c_2} \pm d_2}{b_2}$$

$$= \frac{\pm a_1 b_2 \sqrt{c_1} \pm a_2 b_1 \sqrt{c_2} \pm d_2 b_1}{b_1 b_2}$$

$$= \frac{\pm m \sqrt{c_1} \pm n \sqrt{c_2} \pm p}{o} \qquad (\pm m, \pm n, \pm p, o) = gcd(\pm a_1 b_2, \pm a_2 b_1, \pm d_2 b_1, b_1 b_2)$$

$$= \frac{\sqrt{m^2 c_1 + n^2 c_2 \pm 2mn \sqrt{c1 c_2}} \pm p}{o}$$

$$= \frac{\sqrt{q \pm 2mnr \sqrt{f_3}} \pm p}{o} \qquad\qquad q = m^2 c_1 + n^2 c_2, c_1 c_2 = r^2 f_3$$

$$= \frac{s \sqrt{c_3 \pm e_3 \sqrt{f_3}} \pm p}{o} \qquad\qquad q = s^2 c_3, 2mnr = s^2 e_3$$

$$= \frac{a_3 \sqrt{c_3 \pm e_3 \sqrt{f_3}} \pm d_3}{b_3} \qquad\qquad (a_3, b_3, \pm d_3) = gcd(s, \pm p, o)$$

2.13 $1/D_1 = D_2$

$$1/D_1 = \frac{b_1}{\pm a_1\sqrt{c_1} \pm d_1}$$
$$= \frac{\pm a_1 b_1 \sqrt{c_1} \mp b_1 d_1}{a_1^2 c_1 - d_1^2}$$
$$= \frac{a_2\sqrt{c_1} \pm d_2}{b_2} \qquad\qquad (a_2, b_2, d_2) = gcd(\pm a_1 b_1, \mp b_1 d_1, a_1^2 c_1 - d_1^2)$$

2.14 $\sqrt{D_1} = F_2$ **editing...**

$$\sqrt{D_1} = \sqrt{\frac{\pm a_1\sqrt{c_1} \pm d_1}{b_1}}$$
$$= \frac{\sqrt{\pm b_1 d_1 \pm a_1 b_1 \sqrt{f_2}}}{b_1} \qquad\qquad f_2 = c_1$$
$$= \frac{m\sqrt{c_2 \pm e_2\sqrt{f_2}}}{b_1} \qquad\qquad \pm b_1 d_1 = m^2 c_2, \pm a_1 b_1 = m^2 e_2$$
$$= \frac{a_2\sqrt{c_2 \pm e_2\sqrt{f_2}}}{b_2} \qquad\qquad (a_2, b_2) = gcd(m, b_1)$$

2.15 $D_1 + D_2 = F_3$

$$D_1 + D_2 = \frac{\pm a_1\sqrt{c_1} \pm d_1}{b_1} + \frac{\pm a_2\sqrt{c_2} \pm d_2}{b_2}$$
$$= \frac{\pm a_1 b_2 \sqrt{c_1} \pm a_2 b_1 \sqrt{c_2} \pm d_1 b_2 \pm d_2 b_1}{b_1 b_2}$$
$$= \frac{\pm m\sqrt{c_1} \pm n\sqrt{c_2} \pm p}{o} \qquad (\pm m, \pm n, \pm p, o) = gcd(\pm a_1 b_2, \pm a_2 b_1, \pm d_1 b_2 \pm d_2 b_1, b_1 b_2)$$
$$= \frac{\sqrt{m^2 c_1 + n^2 c_2 \pm 2mn\sqrt{c_1 c_2}} \pm p}{o}$$
$$= \frac{\sqrt{q \pm 2mnr\sqrt{f_3}} \pm p}{o} \qquad\qquad q = m^2 c_1 + n^2 c_2, c_1 c_2 = r^2 f_3$$
$$= \frac{s\sqrt{c_3 \pm e_3\sqrt{f_3}} \pm p}{o} \qquad\qquad q = s^2 c_3, 2mnr = s^2 e_3$$
$$= \frac{a_3\sqrt{c_3 \pm e_3\sqrt{f_3}} \pm d_3}{b_3} \qquad\qquad (a_3, b_3, \pm d_3) = gcd(s, \pm p, o)$$

2.16 $D_1 \times D_2 = F_3$

$$D_1 \times D_2 = \frac{\pm a_1\sqrt{c_1} \pm d_1}{b_1} \times \frac{\pm a_2\sqrt{c_2} \pm d_2}{b_2}$$
$$= \frac{\pm a_1 a_2 \sqrt{c_1 c_2} \pm a_1 d_2 \sqrt{c_1} \pm a_2 d_1 \sqrt{c_2} \pm d_1 d_2}{b_1 b_2}$$