Meccano pentagons gallery

https://github.com/heptagons/meccano/penta/gallery

2023/12/9

Abstract

We build rigid meccano [1] regular pentagons from sides 3 to 12. We restrict all internal strips to remain inside the pentagon's perimeter and don't permit they overlap with others. We follow three steps. 1) We calculate distances between selected strips holes from the regular pentagon perimeter assuming is regular. 2) We run some programs available in this repo to look for rigid clusters of strips which contains the distance. 3) We simplify or reduce the cluster to fit inside the pentagon. We prove the correctness of the cluster distance applied to check the software. We try each construction is relevant for the pentagon size.

1 Pentagons of size 3

1.1 Size 3 with 10 internal strips

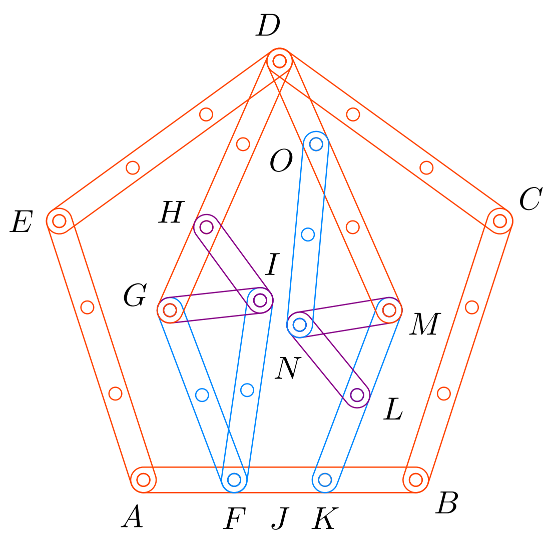

Figure 1: Regular pentagon of size 3 made rigid with 10 internal strips. $\overline{DE} : \overline{EA} : \overline{AF} = 3 : 3 : 1$. $\overline{DF} = \overline{DK} = \dfrac{\sqrt{46+18\sqrt{5}}}{2}$.

[1] Meccano mathematics by 't Hooft

Figure 1 show the regular pentagon A, B, C, D, E of size 3. We know the regular pentagon height is side length times $\dfrac{\sqrt{5+2\sqrt{5}}}{2}$ so in this case $\overline{DJ} = \dfrac{3\sqrt{5+2\sqrt{5}}}{2}$ and we can calculate $\overline{DF}$:

$$\begin{aligned}\overline{DF} &= \sqrt{(\overline{DJ})^2 + (\overline{FJ})^2} \\ &= \sqrt{\left(\frac{3\sqrt{5+2\sqrt{5}}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{46+18\sqrt{5}}}{2} \end{aligned} \tag{1}$$

1.1.1 Rigid distance $\frac{\sqrt{46+18\sqrt{5}}}{2}$

Our software found several five strips clusters and we use two different which fit inside the pentagon.

We identify two angles $\alpha = \angle HGI = \angle LMN$ of equilateral triangles and $\beta = \angle FGI = \angle NMO$ of isoscelles'. Adding the angles we get angles $\angle DGF = \angle DMK = (\alpha + \beta)$. From equilateral triangle $\triangle HGI$ we calculate $\cos\alpha$ and $\sin\alpha$:

$$\cos\alpha = \frac{\frac{\overline{GI}}{2}}{\overline{GH}} = \frac{\frac{1}{2}}{1} = \frac{1}{2} \tag{2}$$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2} \tag{3}$$

From isoscelles triangle $\triangle FGI$ we calculate $\cos\beta$ and $\sin\beta$:

$$\cos\beta = \frac{\frac{\overline{GI}}{2}}{\overline{GF}} = \frac{\frac{1}{2}}{2} = \frac{1}{4} \tag{4}$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4} \tag{5}$$

Now we calculate $\cos(\alpha+\beta)$ with the sum identity:

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{15}}{4}\right) = \frac{1-3\sqrt{5}}{8}\end{aligned} \tag{6}$$

Finally using the law of cosines we verify distance $\overline{DF}$:

$$\begin{aligned}\overline{DF} &= \sqrt{(\overline{DG})^2 + (\overline{FG}^2) - 2(\overline{DG})(\overline{FG})\cos(\alpha+\beta)} \\ &= \sqrt{3^2 + 2^2 - 2(3)(2)\left(\frac{1-3\sqrt{5}}{8}\right)} = \frac{\sqrt{46+18\sqrt{5}}}{2} \quad \blacksquare\end{aligned} \tag{7}$$

1.2 Size 3 with 14 internal strips

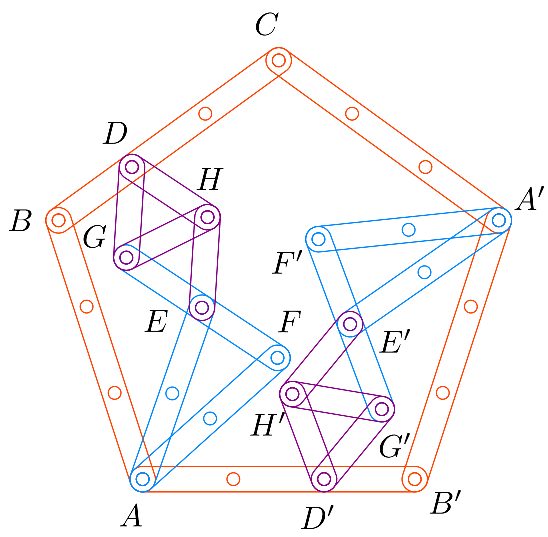

Figure 2: Regular pentagon of size 3 made rigid with 14 internal strips. $\overline{AB} : \overline{BD} = 3 : 1$. $\overline{AD} = \dfrac{\sqrt{34+5\sqrt{5}}}{2}$.

Figure 2 show the regular pentagon A, B', A', C, B of size 3. We know the internal angle of pentagon is $\theta = \dfrac{3\pi}{5}$ and $\cos\theta = \dfrac{1-\sqrt{5}}{4}$ so with the law of cosines we can calculate $\overline{AD}$ and the angle $\angle BAD$:

$$
\begin{aligned}
\overline{AD} &= \sqrt{(\overline{AB})^2 + (\overline{BD})^2 - 2(\overline{AB})(\overline{BD})\cos\theta} \\
&= \sqrt{3^2 + 1^2 - 2(3)(1)\left(\frac{1-\sqrt{5}}{4}\right)} = \frac{\sqrt{34+6\sqrt{5}}}{2} \qquad (8) \\
\cos(\angle BAD) &= \frac{(\overline{AD})^2 + (\overline{AB})^2 - (\overline{BD})^2}{2(\overline{AD})(\overline{AB})} \\
&= \frac{\dfrac{34+6\sqrt{5}}{4} + 3^2 - 1^2}{2\left(\dfrac{\sqrt{34+6\sqrt{5}}}{2}\right)(3)} = \frac{11+\sqrt{5}}{2\sqrt{34+6\sqrt{5}}} \qquad (9)
\end{aligned}
$$

1.2.1 Rigid distance $\dfrac{\sqrt{34+6\sqrt{5}}}{2}$

Our software found several options to make the distance with five strips but we have to manually add a sixth strip in order to make a cluster narrow enough to fit two times inside the pentagon of size 3. The result show as the duplicated cluster with vertices D, E, F, G, H of the figure. We are going to prove the cluster's distance $\overline{AD}$ matches the distance alredy calculated.

Noting the pair of adjacent equilateral triangles $\triangle DGH$ and $\triangle EGH$ we know angle $\angle DGE = \dfrac{2\pi}{3}$ and also $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$ so we can calculate $\overline{DE}$ and the angle $\alpha \equiv \angle GDE$:

$$\overline{DE} = \sqrt{(\overline{DG})^2 + (\overline{GE})^2 - 2(\overline{DG})(\overline{GE})\cos(\angle DGE)} = \sqrt{1^2 + 1^2 - 2(1)(1)\left(-\frac{1}{2}\right)} = \sqrt{3} \tag{10}$$

$$\alpha \equiv= \angle GED$$

$$\cos\alpha = \frac{(\overline{GE})^2 + (\overline{DE})^2 - (\overline{DG})^2}{2(\overline{GE})(\overline{DE})} = \frac{3 + 1^2 - 1^2}{2(\sqrt{3})(1)} = \frac{\sqrt{3}}{2} \tag{11}$$

$$\sin\alpha = \sqrt{1 - \cos^2\alpha} = \sqrt{1 - \left(\frac{\sqrt{3}}{4}\right)^2} = \frac{1}{2} \tag{12}$$

From the isoscelles triangle $\triangle AEF$ we can calculate angle $\angle AEF$ noting $\cos(\angle AEF) = \frac{\overline{EF}/2}{\overline{AE}} = \frac{1/2}{2} = \frac{1}{4}$ so we define angle $\beta \equiv \angle AEG$ the supplementary of $\angle AEF$ and we get:

$$\beta \equiv \angle AEG = \pi - \angle AEF$$

$$\cos\beta = -\cos\angle AEF = -\frac{1}{4} \tag{13}$$

$$\sin\beta = \sqrt{1 - \cos^2\beta} = \sqrt{1 - \left(\frac{-1}{4}\right)^2} = \frac{\sqrt{15}}{4} \tag{14}$$

Now we can calculate the angle $\angle AED = \alpha + \beta$ with the sum identity and plugin the last sines and cosines:

$$\begin{aligned}\cos(\angle AED) &= \cos(\alpha + \beta)\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ &= \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{4}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{15}}{4}\right) = -\frac{\sqrt{3} + \sqrt{15}}{8}\end{aligned} \tag{15}$$

Finally we calculate $\overline{AD}$ with the law of cosines:

$$\begin{aligned}\overline{AD} &= \sqrt{(\overline{DE})^2 + (\overline{EA})^2 - 2(\overline{DE})(\overline{EA})\cos(\angle AED)}\\ &= \sqrt{3 + 2^2 - 2(\sqrt{3})(2)\left(-\frac{\sqrt{3} + \sqrt{15}}{8}\right)} = \frac{\sqrt{34 + 6\sqrt{5}}}{2} \quad \blacksquare\end{aligned} \tag{16}$$

2 Pentagons of size 4

2.1 Size 4 with 8 internal strips

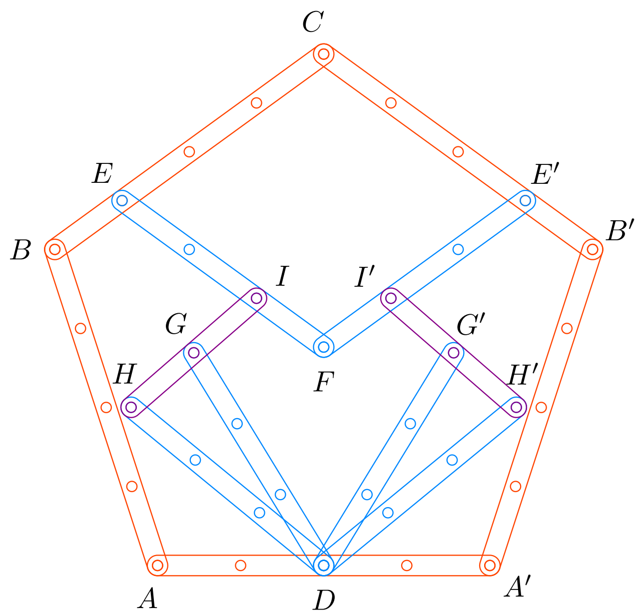

Figure 3: Regular pentagon of size 4 made rigid with 6 internal strips. $\overline{DA} : \overline{AB} : \overline{BE} : \overline{EI} = 2 : 4 : 1 : 2$. $\overline{DI} = \sqrt{11}$.

Figure 3 show the regular pentagon $AA'B'CB$ of size 4. We calculate the distance $\overline{DI}$ assuming vertex D is at origin and calculating abscissa and ordinate of vertex I and knowing $\alpha = \angle ADB = \frac{3\pi}{5}$ and $\beta = \angle B'BE = \frac{\pi}{5}$. Adding and substracting through vertices D, A, B, E, I we get:

$$\begin{aligned} I_x &= -\overline{DA} - \overline{AB}|\cos\alpha| + \overline{BE}\cos\beta + \overline{EI}\cos\beta \\ &= -2 - (4)\left|-\frac{\sqrt{5}-1}{4}\right| + (1+2)\left(\frac{\sqrt{5}+1}{4}\right) = -\frac{1+\sqrt{5}}{4} \end{aligned} \tag{17}$$

$$\begin{aligned} I_y &= \overline{AB}\sin\alpha + \overline{BE}\sin\beta - \overline{BI}\sin\beta \\ &= (4)\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right) + (1-2)\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right) = \frac{4\sqrt{10+2\sqrt{5}} - \sqrt{10-2\sqrt{5}}}{4} \end{aligned} \tag{18}$$

$$\begin{aligned} \overline{DI} &= \sqrt{(I_x - D_x)^2 + (I_y - I_y)^2} \\ &= \frac{\sqrt{(1+\sqrt{5})^2 + (4\sqrt{10+2\sqrt{5}} - \sqrt{10-2\sqrt{5}})^2}}{4} = \sqrt{11} \end{aligned} \tag{19}$$

2.1.1 Rigid distance $\sqrt{11}$

Our software found several three strips clusters for rigid distance $\sqrt{11}$. We prove the selected cluster $DHGI$ inside the pentagon matches the expected distance. First we calculate the angle $\angle DHG$ with the law of cosines and use the value to finally calculate the distance $\overline{DI}$ with again the law of cosines:

$$\begin{aligned}\cos(\angle DHG) &= \frac{(\overline{HD})^2 + (\overline{HG})^2 - (\overline{DG})^2}{2(\overline{HD})(\overline{HG})} \\ &= \frac{3^2 + 1^2 - 3^2}{2(3)(1)} = \frac{1}{6} \qquad (20)\\ \overline{DI} &= \sqrt{(\overline{HD})^2 + (\overline{HI})^2 - 2(\overline{HD})(\overline{HD})\cos(\angle DHG)} \\ &= \sqrt{3^2 + 2^2 - 2(3)(2)\left(\frac{1}{6}\right)} = \sqrt{11} \quad \blacksquare \qquad (21)\end{aligned}$$

2.2 Size 4 with 10 internal strips

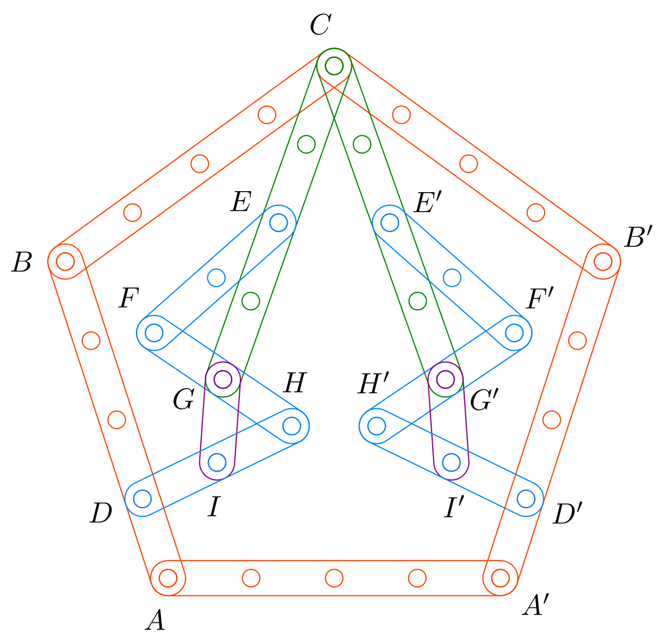

Figure 4: Regular pentagon of size 4 made rigid with 10 internal strips. $\overline{CB} : \overline{BD} = 4 : 3$. $\overline{CD} = \sqrt{19 + 6\sqrt{5}}$.

Figure 4 show the regular pentagon A, A', B', C, B of size 4. We know the internal angle of pentagon is $\theta = \angle CBD = \dfrac{3\pi}{5}$ and $\cos\theta = \dfrac{1-\sqrt{5}}{4}$ so with the law of cosines we can calculate $\overline{CD}$ and the angle $\angle DCB$:

$$\begin{aligned}\overline{CD} &= \sqrt{(\overline{BC})^2 + (\overline{BD})^2 - 2(\overline{BC})(\overline{BD})\cos\theta} \\ &= \sqrt{4^2 + 3^2 - 2(4)(3)\left(\frac{1-\sqrt{5}}{4}\right)} = \sqrt{19 + 6\sqrt{5}} \qquad (22)\\ \cos(\angle DCB) &= \frac{(\overline{CD})^2 + (\overline{BC})^2 - (\overline{BD})^2}{2(\overline{CD})(\overline{BC})} \\ &= \frac{(19 + 6\sqrt{5}) + 4^2 - 3^2}{2\left(\sqrt{19 + 6\sqrt{5}}\right)(4)} = \frac{13 + 3\sqrt{5}}{4\sqrt{19 + 6\sqrt{5}}} \qquad (23)\end{aligned}$$

2.2.1 Rigid distance $\sqrt{19+6\sqrt{5}}$

Our software found several five strips clusters for rigid distance $\sqrt{19+6\sqrt{5}}$. We prove selected cluster $CEFGHID$ show in the figure inside the pentagon matches the expected distance. Set the cluster in the coordinate plane such that vertice G is at the origin and vertices F at $(-1,0)$ and vertice H at $(+1,0)$. Since triangle $\triangle EFG$ is isoscelles and $\overline{CG}$ is the double of $\overline{GE}$ we know angle $\angle CFG = \frac{\pi}{2}$ and we can calculate the abscissa and ordinate of vertex C:

$$C_x = -\overline{FG} = -1 \tag{24}$$

$$C_y = \sqrt{(\overline{CG})^2 - (\overline{FG})^2} = \sqrt{4^2 - 1^2} = \sqrt{15} \tag{25}$$

Since triangle $\triangle GHI$ is equilateral and $\overline{HD}$ is the double of $\overline{HI}$ we know angle $\angle DGH = \frac{\pi}{2}$ and we can calculate the abscissa and ordinate of vertex D:

$$D_x = 0 \tag{26}$$

$$D_y = -\sqrt{(\overline{HD})^2 - (\overline{GH})^2} = -\sqrt{2^2 - 1^2} = -\sqrt{3} \tag{27}$$

Finally we calculate the distance $\overline{CD}$

$$\begin{aligned}\overline{CD} &= \sqrt{(C_x - D_x)^2 + (C_y - D_y)^2} \\ &= \sqrt{(-1-0)^2 + (\sqrt{15}+\sqrt{3})^2} = \sqrt{19+6\sqrt{5}} \quad \blacksquare\end{aligned} \tag{28}$$

3 Pentagons of size 5

3.1 Size 5 with 10 internal strips

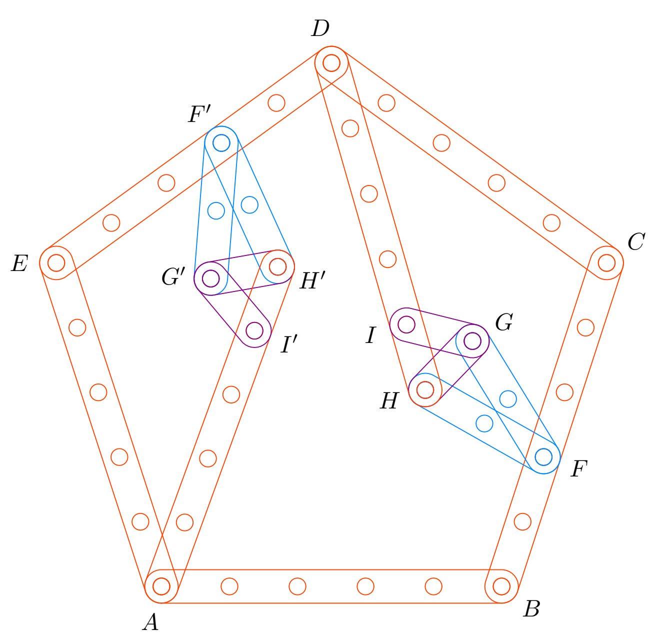

Figure 5: Regular pentagon of size 5 made rigid with 10 internal strips. $\overline{FC} : \overline{CD} = 3:5$. $\overline{FD} = \dfrac{\sqrt{106+30\sqrt{5}}}{2}$.

3.1.1 Rigid distance $\dfrac{\sqrt{106+30\sqrt{5}}}{2}$

3.2 Size 5 with 12 internal strips

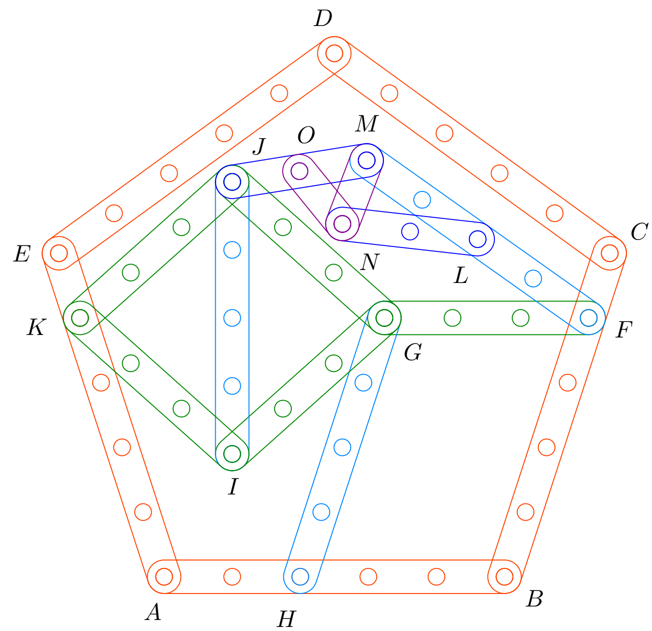

Figure 6: Regular pentagon of size 5 made rigid with 12 internal strips. $\overline{FB} : \overline{BA} : \overline{AK} = 4 : 5 : 4$ and $\overline{FK} = 3 + 2\sqrt{5}$. $\overline{FJ} = \sqrt{18 + 6\sqrt{5}} = \sqrt{3} + \sqrt{15}$.

3.2.1 Rigid distances $3 + 2\sqrt{5}$ and $\sqrt{18 + 6\sqrt{5}} = \sqrt{3} + \sqrt{15}$

4 Pentagons of size 6

4.1 Size 6 with 6 internal strips

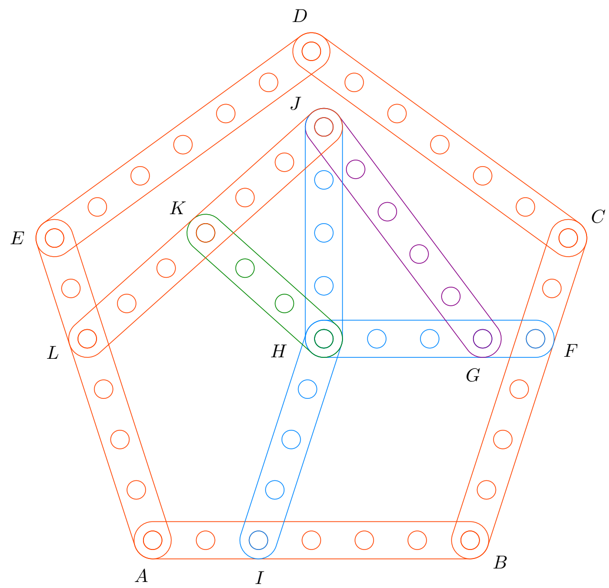

Figure 7: Regular pentagon of size 6 made rigid with 6 internal strips. $\overline{FB} : \overline{BA} : \overline{AL} = 4 : 6 : 4$. $\overline{FL} = 4 + 2\sqrt{5}$.

4.1.1 Rigid distance $4 + 2\sqrt{5}$

4.2 Size 6 with 8 internal strips

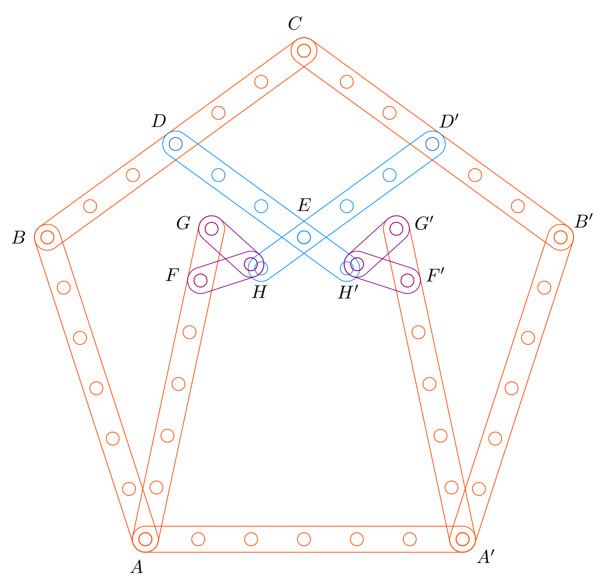

Figure 8: Regular pentagon of size 6 made rigid with 8 internal strips. $\overline{A'A} : \overline{AB} : \overline{BD} : \overline{DF'} = 6 : 6 : 3 : 4$ and $\overline{A'F'} = \sqrt{31}$.

4.2.1 Rigid distance $\sqrt{31}$

4.3 Size 6 with 10 internal strips

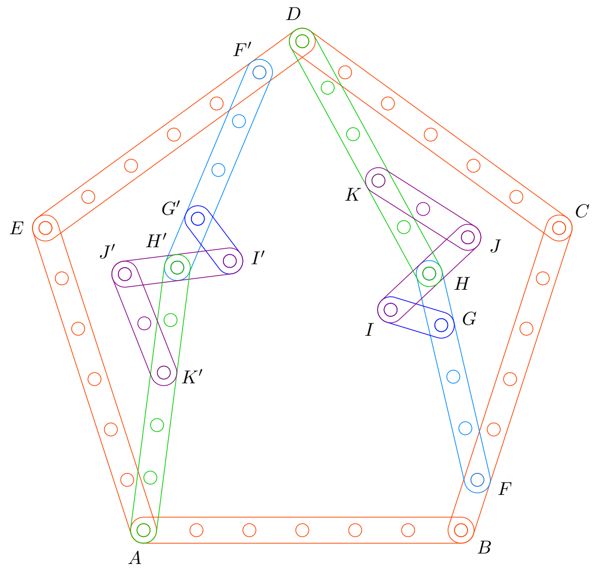

Figure 9: Regular pentagon of size 6 made rigid with 10 internal strips. $\overline{FC} : \overline{CD} = 5 : 6$. $\overline{FD} = \sqrt{46 + 15\sqrt{5}}$.

4.3.1 Rigid distance $\sqrt{46 + 15\sqrt{5}}$

4.4 Size 6 with 12 internal strips

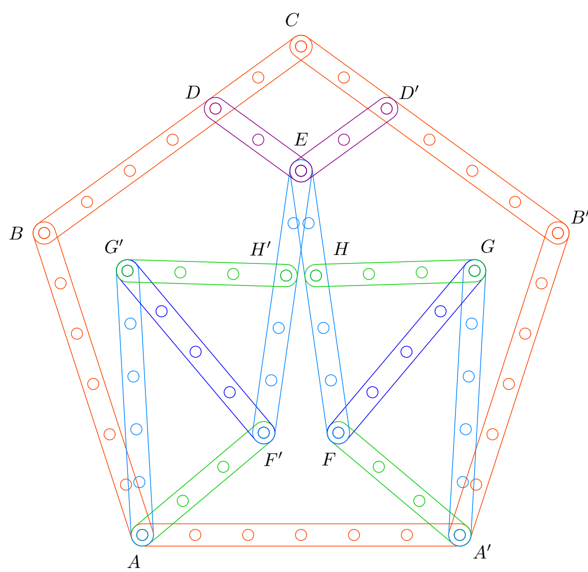

Figure 10: Regular pentagon of size 6 made rigid with 12 internal strips. $\overline{AA} : \overline{AB} : \overline{BD} : \overline{DE} = 0 : 6 : 4 : 2$ and $AE = \sqrt{34 + 10\sqrt{5}}$

4.4.1 Rigid distance $\sqrt{34+10\sqrt{5}}$

5 Pentagons of size 7

6 Pentagons of size 8

7 Pentagons of size 9

7.1 Size 9 with 6 internal strips

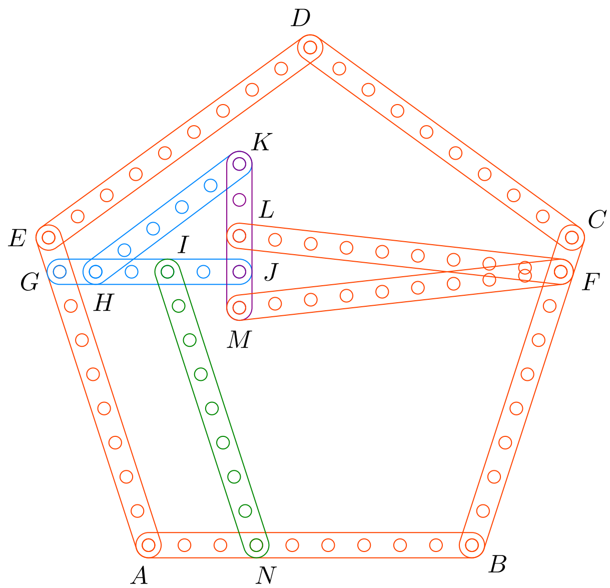

Figure 11: Regular pentagon size 9 made rigid with six internal strips. $\overline{FB} : \overline{BA} : \overline{AG} = 8 : 9 : 8$. $\overline{FG} = 5 + 4\sqrt{5}$.

Figure 11 show a rigid regular pentagon A, B, C, D, E of size 9. The regular pentagon distance $\overline{CE}$ is called width and equals $\dfrac{1+\sqrt{5}}{2}\overline{AB}$. Is easy to note the distance $\overline{FG}$ equals the width of smaller pentagon size $9 - 1 = 8$ plus 1. So we have:

$$\begin{aligned}\overline{FG} &= \frac{1+\sqrt{5}}{2}(\overline{BC} - \overline{FC}) + \overline{FC} \\ &= \frac{1+\sqrt{5}}{2}(9-1) + 1 = 5 + 4\sqrt{5} \qquad (29)\end{aligned}$$

7.1.1 Rigid distance $5 + 4\sqrt{5}$

From the figure we see two right angles. Angle $\angle GJK = \frac{\pi}{2}$ because we have a Pythagorean triangle $\triangle HJK$. Angle $\angle FJM = \frac{\pi}{2}$ because we have an isosceles triangle $\triangle FLM$. The two right angles share vertice J so vertices G, J, F are collinear. First we calculate the distance $\overline{JF} = \sqrt{(LF)^2 - (LJ)^2} = \sqrt{9^2 - 1^2} = 4\sqrt{5}$ and finally the distance $\overline{GF} = \overline{GJ} + \overline{JF} = 5 + 4\sqrt{5}$ which matches the value in last equation above $\blacksquare$. To make rigid the pentagon we add strip $\overline{IN}$ parallel to side $\overline{GA}$.

7.2 Size 9 with 8 internal strips

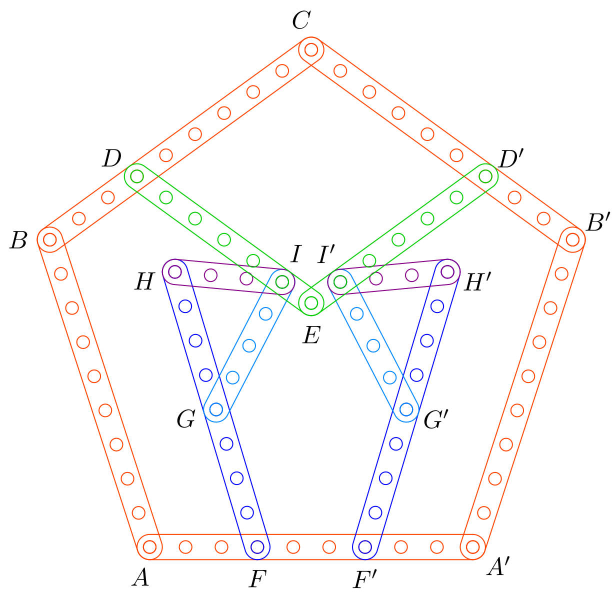

Figure 12: Regular pentagon size 9 made rigid with 8 strips variation 1. $\overline{FA}:\overline{AB}:\overline{BD}:\overline{DI}=3:9:3:5$. $\overline{FI}=\sqrt{55}$.

Figure 12 show a rigid regular pentagon A, A', B', C, B of size 9. First we calculate the distance $\overline{FI}$ using the abscissas and ordinates following the vertices F, A, B, D, I for a regular pentagon angles $\alpha=\frac{3\pi}{5}, \beta=\frac{\pi}{5}$:

$$
\begin{aligned}
FI_x &= -\overline{AF}-\overline{AB}\cos\alpha+(\overline{BD}+\overline{DI})\cos\beta \\
&= -3+(9)\frac{1-\sqrt{5}}{4}+(3+5)\frac{\sqrt{5}+1}{4}=\frac{5-\sqrt{5}}{4} && (30)\\
FI_y &= \overline{AB}\sin\alpha+(\overline{BD}-\overline{DI})\sin\beta \\
&= (9)\frac{\sqrt{10+2\sqrt{5}}}{4}+(3-5)\frac{\sqrt{10-2\sqrt{5}}}{4}=\frac{9\sqrt{10+2\sqrt{5}}-2\sqrt{10-2\sqrt{5}}}{4} && (31)\\
\overline{FI} &= \sqrt{(FI_x)^2+(FI_y)^2} \\
&= \frac{\sqrt{(5-\sqrt{5})^2+(9\sqrt{10+2\sqrt{5}}-2\sqrt{10-2\sqrt{5}})^2}}{4}=\frac{\sqrt{880}}{4}=\sqrt{55} && (32)
\end{aligned}
$$

7.2.1 Rigid distance $\sqrt{55}$

By software we find several options for the distance and we use the one shown in the figure. We calculate the distance $\overline{FI}$ made rigid by cluster F, G, H, I. We have an isoscelles triangle $\triangle GHI$ and $\overline{FH}=2\overline{GH}$ so we have a right triangle $\angle FIH=\frac{\pi}{2}$ so:

$$
\begin{aligned}
\overline{FI} &= \sqrt{(\overline{FH})^2-(\overline{HI})^2} \\
&= \sqrt{8^2-3^2}=\sqrt{55} \quad \blacksquare && (33)
\end{aligned}
$$

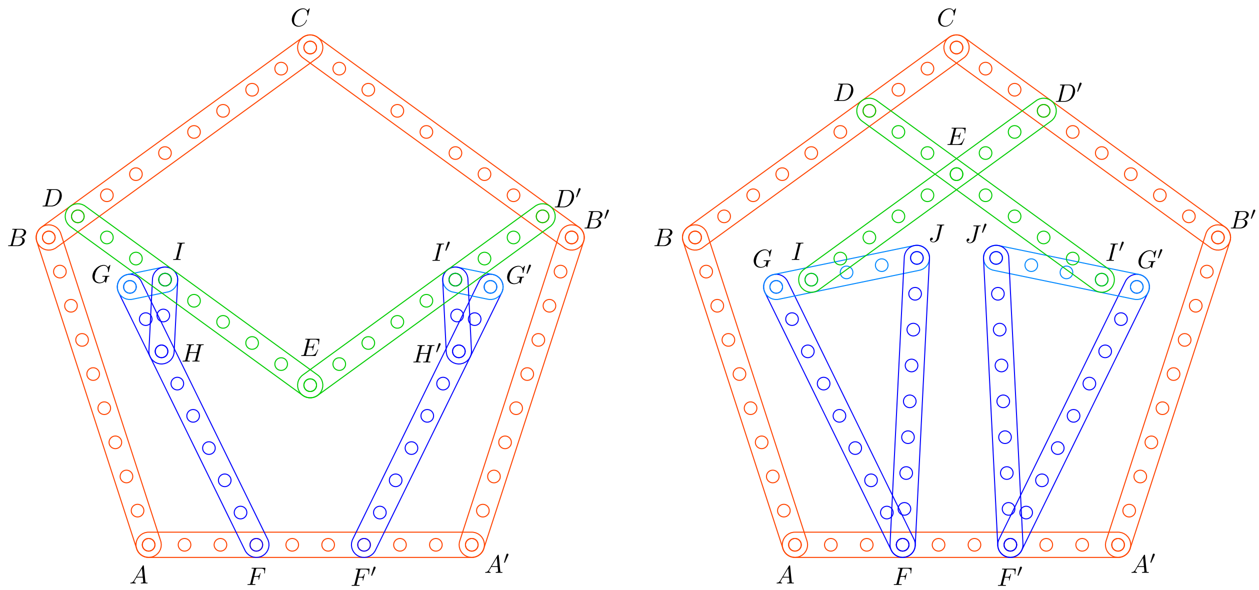

Figure 13: Regular pentagons size 9 made rigid with 8 strips variations 2 and 3. At the left we have $\overline{FA}:\overline{AB}:\overline{BD}:\overline{DI}=3:9:1:4$ and $\overline{FI}=\sqrt{61}$. At the right we have $\overline{F'A}:\overline{AB}:\overline{BD}:\overline{DI'}=6:9:6:8$ and $\overline{F'I'}=\sqrt{61}$.

Figure 13 show two rigid pentagons A, A', B', C, B' of size 9. The pentagon at the left is called variation 2 and the right one variation 3. Both variations have the vertices I, I' at the same positions and the same distance $\overline{FI}$ which first we calculate using the abscissas and ordinates following the vertices F, A, B, D, I of the variation 2 for a regular pentagon angles $\alpha = \dfrac{3\pi}{5}, \beta = \dfrac{\pi}{5}$:

$$\begin{aligned} FI_x &= -\overline{AF} - \overline{AB}\cos\alpha + (\overline{BD} + \overline{DI})\cos\beta \\ &= -3 + (9)\frac{1-\sqrt{5}}{4} + (1+3)\frac{\sqrt{5}+1}{4} = \frac{1-5\sqrt{5}}{4} \end{aligned} \tag{34}$$

$$\begin{aligned} FI_y &= \overline{AB}\sin\alpha + (\overline{BD} - \overline{DI})\sin\beta \\ &= (9)\frac{\sqrt{10+2\sqrt{5}}}{4} + (1-3)\frac{\sqrt{10-2\sqrt{5}}}{4} = \frac{9\sqrt{10+2\sqrt{5}} - 2\sqrt{10-2\sqrt{5}}}{4} \end{aligned} \tag{35}$$

$$\begin{aligned} \overline{FI} &= \sqrt{(FI_x)^2 + (FI_y)^2} \\ &= \frac{\sqrt{(1-\sqrt{5})^2 + (9\sqrt{10+2\sqrt{5}} - 2\sqrt{10-2\sqrt{5}})^2}}{4} = \frac{\sqrt{976}}{4} = \sqrt{61} \end{aligned} \tag{36}$$

7.2.2 Rigid distance $\sqrt{61}$

Our software found several clusters and we use two different for each variation. We calculate the distance $\overline{FI}$ made rigid by clusters F, G, H, I or F, G, I, J since in both variations we have the same $\overline{GF}$ and same angles $\angle FGI = \angle FJG$. With the law of cosines first we calculate $\cos(\angle FJG)$ and then $\overline{FI}$:

$$\begin{aligned} \cos(\angle FJG) &= \frac{\overline{FJ}^2 + \overline{JG}^2 - \overline{GF}^2}{2(\overline{FJ})(\overline{JG})} = \frac{8^2 + 4^2 - 8^2}{2(8)(4)} = \frac{1}{4} \\ \overline{FI} &= \sqrt{\overline{IJ}^2 + \overline{FJ}^2 - 2(\overline{IJ})(\overline{FJ})\cos(\angle FJG)} = \sqrt{3^2 + 8^2 - 2(3)(8)\left(\frac{1}{4}\right)} = \sqrt{61} \quad \blacksquare \end{aligned} \tag{37}$$

7.3 Size 9 with 10 internal strips

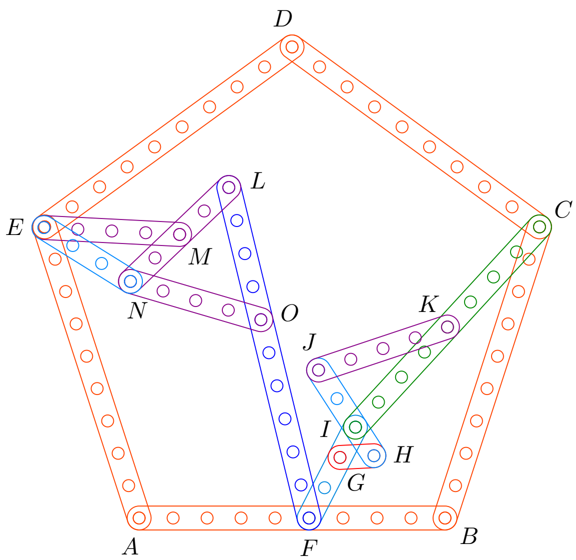

Figure 14: Regular pentagon size 9 made rigid with two internal clusters of five strips each. For the cluster of the left we have $\overline{FA} : \overline{AE} = 5 : 9$ and $\overline{FE} = \dfrac{\sqrt{334 + 90\sqrt{5}}}{2}$. For the cluster at the right we have $\overline{FB} : \overline{BC} = 4 : 9$ and $\overline{FC} = \sqrt{79 + 18\sqrt{5}}$.

Figure 14 show a regular pentagon A, B, C, D, E of size 9 made it rigid with the help of clusters fixing the distances $\overline{CF}$ and $\overline{EF}$. Pentagon size 9 is the smaller one with diagonals where consecutive side segments fractions are $\overline{BF}/\overline{BC} = \dfrac{4}{9}$ and $\overline{AF}/\overline{AE} = \dfrac{5}{9}$. We calculate the diagonals $\overline{CF}$, $\overline{EF}$ and the angles to side $\overline{AB}$ using the law of cosines and the internal pentagon angle $\theta = \angle FBC = \angle FAE = \dfrac{3\pi}{5}$ where $\cos\theta = \dfrac{1-\sqrt{5}}{4}$:

$$\overline{CF} = \sqrt{\overline{BC}^2 + \overline{BF}^2 - 2(\overline{BC})(\overline{BF})\cos\theta} = \sqrt{9^2 + 4^2 - 2(9)(4)\left(\frac{1-\sqrt{5}}{4}\right)} = \sqrt{79 + 18\sqrt{5}} \quad (38)$$

$$\cos(\angle CFB) = \frac{\overline{CF}^2 + \overline{BF}^2 - \overline{BC}^2}{2(\overline{CF})(\overline{BF})} = \frac{79 + 18\sqrt{5} + 4^2 - 9^2}{2(\sqrt{79 + 18\sqrt{5}})(4)} = \frac{7 + 9\sqrt{5}}{4\sqrt{79 + 18\sqrt{5}}} \quad (39)$$

$$\overline{EF} = \sqrt{\overline{AE}^2 + \overline{AF}^2 - 2(\overline{AE})(\overline{AF})\cos\theta} = \sqrt{9^2 + 5^2 - 2(9)(5)\left(\frac{1-\sqrt{5}}{4}\right)} = \frac{\sqrt{334 + 90\sqrt{5}}}{2} \quad (40)$$

$$\cos(\angle EFA) = \frac{\overline{EF}^2 + \overline{AF}^2 - \overline{EA}^2}{2(\overline{EF})(\overline{AF})} = \frac{\dfrac{334 + 90\sqrt{5}}{4} + 5^2 - 9^2}{2\left(\dfrac{\sqrt{334 + 90\sqrt{5}}}{2}\right)(5)} = \frac{11 + 9\sqrt{5}}{2\sqrt{334 + 90\sqrt{5}}} \quad (41)$$

7.3.1 Rigid distance $\sqrt{79 + 18\sqrt{5}}$

Our software found several options with five strips to build distance $\sqrt{79 + 18\sqrt{5}}$.

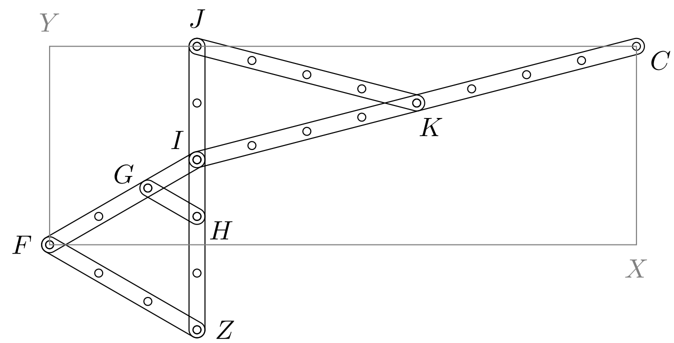

Figure 15: Construction of distance $\overline{FC} = \sqrt{79 + 18\sqrt{5}}$

Figure 15 show one of several ways to build the distance $\sqrt{79 + 18\sqrt{5}}$. Equilateral triangle $\triangle FIZ$ and isosceles $\triangle IJK$ share vertice I and the base $\overline{JZ}$ which help to form rectangle $FXCY$ with base $\overline{FX}$ and height $\overline{FY}$ useful to calculate the diagonal $\overline{FC}$:

$$\begin{aligned}
\overline{FX} &= \overline{YJ} + \overline{JC} \\
&= \sqrt{\overline{FI}^2 - \left(\frac{\overline{IZ}}{2}\right)^2} + \sqrt{\overline{IC}^2 - \overline{IJ}^2} = \sqrt{3^2 - \left(\frac{3}{2}\right)^2} + \sqrt{8^2 - 2^2} = \frac{3\sqrt{3}}{2} + 2\sqrt{15} \\
\overline{FY} &= \overline{JI} + \frac{\overline{IZ}}{2} = 2 + \frac{3}{2} = \frac{7}{2} \\
\overline{FC} &= \sqrt{\overline{FX}^2 + \overline{FY}^2} = \sqrt{\left(\frac{3\sqrt{3}}{2} + 2\sqrt{15}\right)^2 + \left(\frac{7}{2}\right)^2} = \sqrt{79 + 18\sqrt{5}} \quad \blacksquare
\end{aligned} \tag{42}$$

We use a smaller part of this construction, the five strips with vertices F, G, H, I, J, K, C, as a cluster to made rigid the consecutive strips $\overline{AB}, \overline{BC}$ of the pentagon of side 9 of figure 14.

7.3.2 Rigid distance $\frac{\sqrt{334 + 90\sqrt{5}}}{2}$

Our software found several options with five strips to build distance $\frac{\sqrt{334 + 90\sqrt{5}}}{2}$.

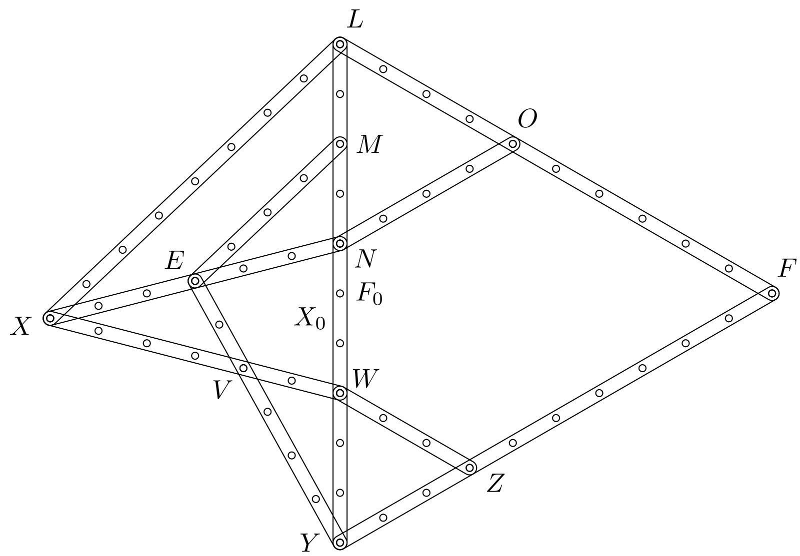

Figure 16: Construction of distance $\overline{EF} = \dfrac{\sqrt{334+90\sqrt{5}}}{2}$

Figure 16 show equilateral triangle $\triangle FLY$ and isoscelles triangle $\triangle NXW$ sharing strip $\overline{LY}$ which helps to calculate abscissas and ordinates of vertices E, F to calculate distance $\overline{EF}$. Vertice Y is located at the origin so:

$$E_x = -\left(\frac{\overline{NE}}{\overline{NX}}\right)\overline{XX_0} = -\frac{3}{6}\sqrt{\overline{NX}^2 - \overline{NX_0}^2} = -\frac{1}{2}\sqrt{6^2 - \left(\frac{3}{2}\right)^2} = -\frac{3\sqrt{15}}{4} \tag{43}$$

$$E_y = \overline{YN} - \left(\frac{\overline{NE}}{\overline{NX}}\right)\overline{NX_0} = 6 - \left(\frac{3}{6}\right)\left(\frac{3}{2}\right) = \frac{21}{4} \tag{44}$$

$$F_x = \overline{F_0F} = \sqrt{\overline{YF}^2 - \overline{YF_0}^2} = \sqrt{10^2 - 5^2} = 5\sqrt{3} \tag{45}$$

$$F_y = \overline{YF_0} = 5 \tag{46}$$

$$\overline{EF} = \sqrt{(E_x - F_x)^2 + (E_y - F_y)^2} = \sqrt{\left(-\frac{3\sqrt{15}}{4} - 5\sqrt{3}\right)^2 + \left(\frac{21}{4} - 5\right)^2} = \frac{\sqrt{334+90\sqrt{5}}}{2} \quad \blacksquare \tag{47}$$

We form a cluster from the last construction to be applied in the pentagon of side 9. We choose the five strips with vertices E, N, M, L, O, F. Is easy to prove strip $\overline{EM}$ is correct in the cluster comparing equal cosines at vertice Y for triangles $\triangle YVW, \triangle YEN, \triangle YEM$ using the law of cosines for each triangle.

8 Pentagon of size 10

8.1 Size 10 with 10 internal strips

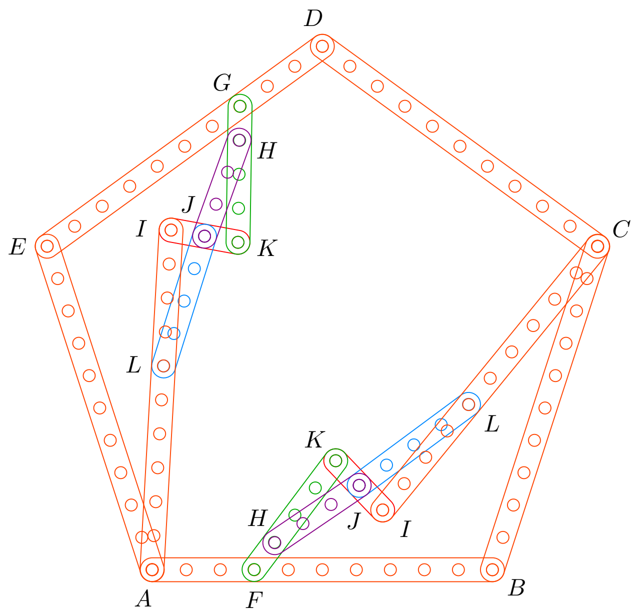

Figure 17: Regular pentagon size 10 made rigid with two internal clusters of five strips each. $\overline{FB} : \overline{BC} = 7 : 10$ and $\overline{CF} = \sqrt{114 + 35\sqrt{5}}$.

Figure 17 show a rigid regular pentagon A, B, C, D, E of size 10. We calculate a diagonal joining two consecutive sides relative primes to have something exclusive to the size 10, we choose $\overline{BF} : \overline{BC} = 7 : 10$. With the law of cosines we calculate $\overline{CF}$. We calculate the angle $\angle CFB$ for the drawing:

$$\begin{aligned}
\overline{CF}^2 &= \overline{BC}^2 + \overline{BF}^2 - 2(\overline{BC})(\overline{BF})\cos\left(\frac{3\pi}{5}\right) \\
&= 10^2 + 7^2 - 2(10)(7)\left(\frac{1-\sqrt{5}}{4}\right) = 114 + 35\sqrt{5} \\
\overline{CF} &= \sqrt{114 + 35\sqrt{5}} \qquad (48) \\
\cos(\angle CFB) &= \frac{\overline{CF}^2 + \overline{BF}^2 - \overline{BC}^2}{2(\overline{CF})(\overline{BF})} = \frac{114 + 35\sqrt{5} + 7^2 - 10^2}{2(\sqrt{114 + 35\sqrt{5}})(7)} = \frac{9 + 5\sqrt{5}}{2\sqrt{114 + 35\sqrt{5}}} \qquad (49)
\end{aligned}$$

8.1.1 Rigid distance $\sqrt{114 + 35\sqrt{5}}$

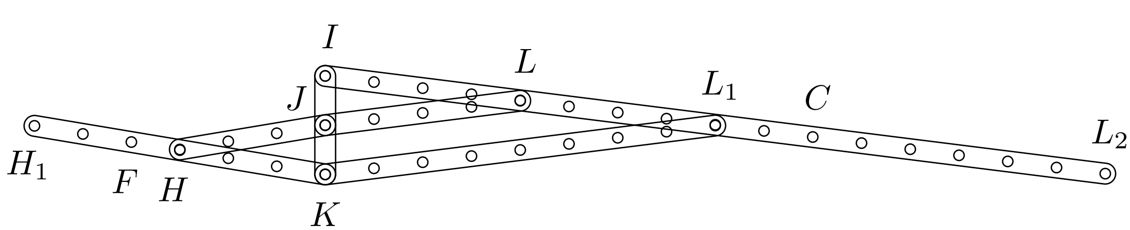

Figure 18: Construction of distance $\overline{CF} = \sqrt{114 + 35\sqrt{5}}$

Our software found several solutions for this distance using five strips, and we choose one narrow enough to fit inside the pentagon.

Figure 18 shows how to prove the cluster selected is correct. In the figure we have two isoscelles triangles $\triangle IKL_1$ and $\triangle JKH$. The sides IL_1 and KH are extended to double the original size to the vertices L_2 and H_1 building two right angles $\angle IKL_2$ and $\angle KJH_1$. The right triangles permit the calculation of the abscissas and ordinates of vertices C and F to calculate their distance.

From the figure we calculate $\overline{KL_2}$ and $\overline{JH_1}$ from their respective right triangles:

$$\overline{KL_2} = \sqrt{(\overline{IL_2})^2 - (IK)^2} = \sqrt{16^2 - 2^2} = 6\sqrt{7} \tag{50}$$

$$\overline{JH_1} = \sqrt{(\overline{KH_1})^2 - (KJ)^2} = \sqrt{6^2 - 1^2} = \sqrt{35} \tag{51}$$

Assuming vertice K is at the origin we can calculate the abscissas C_x, F_x and ordinates C_y, F_y of vertices C and F using as factors $c = \dfrac{\overline{IC}}{\overline{IL_2}} = \dfrac{10}{16} = \dfrac{5}{8}$ and $f = \dfrac{\overline{KF}}{\overline{KH1}} = \dfrac{4}{6} = \dfrac{2}{3}$:

$$C_x = +c(\overline{KL_2}) = \frac{5}{8}(6\sqrt{7}) = \frac{15}{4}\sqrt{7} \tag{52}$$

$$F_x = -f(\overline{JH_1}) = -\frac{2}{3}\sqrt{35} \tag{53}$$

$$C_y = +(\overline{KI}) - c(\overline{KI}) = 2 - \frac{5}{8}(2) = \frac{3}{4} \tag{54}$$

$$F_y = +f(\overline{KJ}) = \frac{2}{3}(1) = \frac{2}{3} \tag{55}$$

Finally we calculate the distance $\overline{CF}$:

$$\begin{aligned} \overline{CF} &= \sqrt{(C_x - F_x)^2 + (C_y - F_y)^2} \\ &= \sqrt{\left(\frac{15}{4}\sqrt{7} + \frac{2}{3}\sqrt{35}\right)^2 + \left(\frac{3}{4} - \frac{2}{3}\right)^2} = \sqrt{114 + 35\sqrt{5}} \quad \blacksquare \end{aligned} \tag{56}$$

A minimal part with five strips of the construction of figure 18 including only vertices F, H, I, J, K, L, C is used twice to make rigid the pentagon of side 10 as show in figure 17.

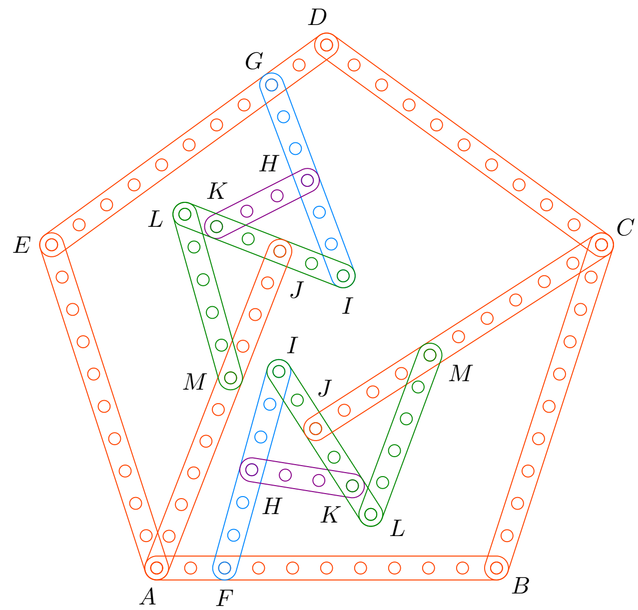

Figure 19: Regular pentagon size 10 made rigid with two internal clusters of five strips each. $\overline{FB} : \overline{BC} = 8 : 10$ and $\overline{CF} = 2\sqrt{31 + 10\sqrt{5}}$.

Figure 19 show a rigid regular pentagon A, B, C, D, E of size 10. We include here the relation $\overline{FB} : \overline{BC} = 8 : 10$ since the relation $4 : 5$ for pentagon size 5 gave clusters that can't fit inside such pentagon. With the law of cosines we calculate $\overline{CF}$ and the angle $\angle CFB$ for the drawing:

$$\overline{CF}^2 = \overline{BC}^2 + \overline{BF}^2 - 2(\overline{BC})(\overline{BF}) \cos\left(\frac{3\pi}{5}\right)$$

$$= 10^2 + 8^2 - 2(10)(8)\left(\frac{1-\sqrt{5}}{4}\right) = 124 + 40\sqrt{5}$$

$$\overline{CF} = 2\sqrt{31 + 10\sqrt{5}} \tag{57}$$

$$\cos(\angle CFB) = \frac{\overline{CF}^2 + \overline{BF}^2 - \overline{BC}^2}{2(\overline{CF})(\overline{BF})} = \frac{124 + 40\sqrt{5} + 8^2 - 10^2}{2(2\sqrt{31+10\sqrt{5}})(8)} = \frac{11 + 5\sqrt{5}}{4\sqrt{31 + 10\sqrt{5}}} \tag{58}$$

8.1.2 Rigid distance $2\sqrt{31 + 10\sqrt{5}}$

One solution from our software is shown in figure as cluster $FHIJKLM$. Assume vertex J is at the origin, vertice I at $(-2, 0)$ and vertice K at $(+2, 0)$. Since triangle $\triangle IKH$ is isoscelles and $\overline{IF}$ is the double of $\overline{IH}$ then angle $\angle FKI = \frac{\pi}{2}$ and we can calculate the abscissa and the ordinate of vertice F:

$$F_x = \overline{JK} = 2 \tag{59}$$

$$F_y = -\sqrt{(\overline{IF})^2 - (\overline{IK})^2} = -\sqrt{6^2 - 4^2} = -2\sqrt{5} \tag{60}$$

Since we have the Pythagorean triangle $\triangle JLM$ is easy to note that the abscissa and ordinate of vertice C are $C_x = 0$ and $C_y = \overline{JC} = 10$, so finally we have:

$$\overline{CF} = \sqrt{(F_x - C_x)^2 + (F_y - C_y)^2}$$

$$= \sqrt{(2-0)^2 + (-2\sqrt{5} - 10)^2} = 2\sqrt{31 + 10\sqrt{5}} \quad \blacksquare \tag{61}$$

9 Pentagon of size 11

9.1 Size 11 with 10 internal strips

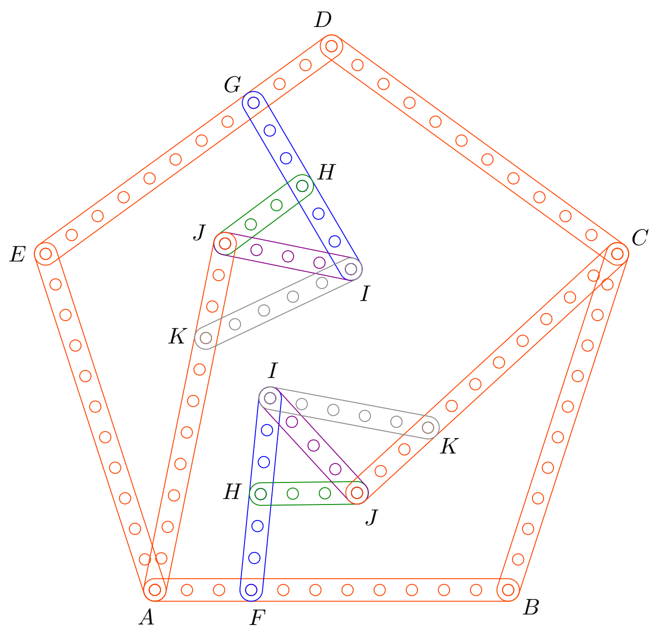

Figure 20: Regular pentagon size 11 made rigid with two internal clusters of five strips each. $\overline{FB} : \overline{BC} = 8 : 11$ and $\overline{CF} = 11 + 2\sqrt{5}$.

Figure 20 show a rigid regular pentagon A, B, C, D, E of size 11. Our software found this is the smallest pentagon having a consecutive sides diagonal distance of the form $\dfrac{z_2 + z_3\sqrt{5}}{z_1}$ instead of the nested form $\dfrac{z_2\sqrt{z_3 + z_4\sqrt{5}}}{z_1}$ where z_i are integers. The mentioned diagonal is the distance $\overline{CF}$ in the figure which can be calculated with the law of cosines knowing angle $\angle CBF = \dfrac{3\pi}{5}$ and denesting the result. We calculate the angle $\angle CFB$ for the drawing:

$$\begin{aligned}
\overline{CF}^2 &= \overline{BC}^2 + \overline{BF}^2 - 2(\overline{BC})(\overline{BF})\cos\left(\frac{3\pi}{5}\right) \\
&= 11^2 + 8^2 - 2(11)(8)\left(\frac{1-\sqrt{5}}{4}\right) = 141 + 44\sqrt{5} \\
\overline{CF} &= \sqrt{141 + 44\sqrt{5}} \\
&= 11 + 2\sqrt{5}
\end{aligned} \tag{62}$$

$$\cos(\angle CFB) = \frac{\overline{CF}^2 + \overline{BF}^2 - \overline{BC}^2}{2(\overline{CF})(\overline{BF})} = \frac{141 + 44\sqrt{5} + 8^2 - 11^2}{2(11 + 2\sqrt{5})(8)} = \frac{21 + 11\sqrt{5}}{44 + 8\sqrt{5}} = \frac{121 + 79\sqrt{5}}{404} \tag{63}$$

9.1.1 Rigid istance $11 + 2\sqrt{5}$

A five strips cluster can create a rigid distance like $11+2\sqrt{5}$. In the figure, three strips $\overline{FI} = 2\overline{HJ}, \overline{FI} > \overline{IJ}$ builds a right angle $\angle FJI = \frac{\pi}{2}$, since triangle $\triangle IJH$ is isosceles ($\overline{FH} = \overline{HI} = \overline{JH}$). These three strips also build a distance $\overline{FJ} = \sqrt{\overline{FI}^2 - \overline{IJ}^2} = \sqrt{6^2 - 4^2} = 2\sqrt{5}$. Now we attach strip $\overline{CJ}$ making a second right triangle $\angle CJI = \frac{\pi}{2}$ using strip $\overline{IK} = 5$ as pythagorean diagonal ($\overline{JK} = 3, \overline{IJ} = 4$). We have two right triangles at vertice J so vertices F, J, C are collinear, so we can calculate the distance $\overline{FC} = \overline{CJ} + \overline{JF} = 11 + 2\sqrt{5} \quad \blacksquare$.

We repeat the five-strips cluster between vertices A, G preventing strips overlaps. Since the clusters are rigid we formed two rigid triangles $\triangle ABC, \triangle DEA$ so the pentagon is rigid.

The software found the next pentagon of this type is a lot bigger: $\overline{BC} = 246, \overline{BF} = 70, \overline{CF} = 41 + 105\sqrt{5}$.

10 Pentagons of size 12

10.1 Size 12 with 4 internal strips

Our sofware found that side 12 is the smallest pentagon that can be made rigid with a rhoumbus and two strips as diagonals so need only 4 strips as diagonals. We show two cases.

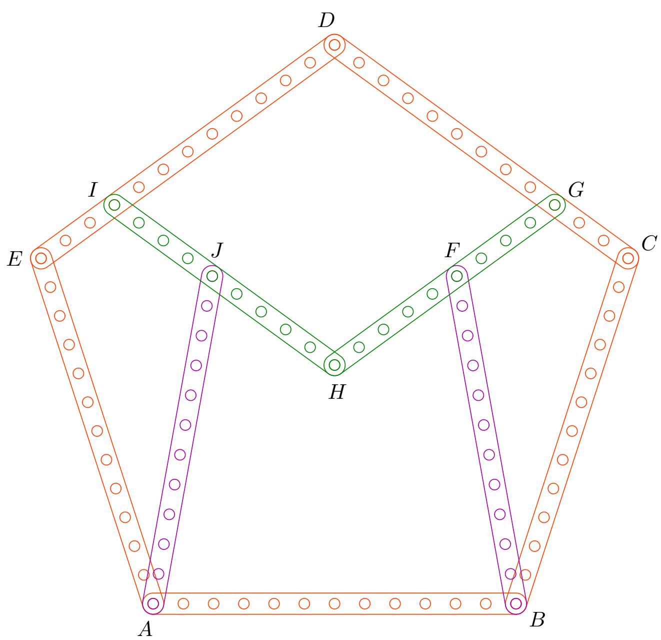

Figure 21: Regular pentagon size 12 (case a) made rigid with only four internal strips. $\overline{AA} : \overline{AE} : \overline{EI} : \overline{IJ} = 0 : 12 : 3 : 4$ and $\overline{AJ} = 11$.

Figure 21 show a regular pentagon A, B, C, D, E of side 12 with a rhombus D, I, H, G of side 9. We prove strips AJ, BF are correct. First we calculate the abscissas going through vertices A, E, I, J substracting when we move to the left and adding when we move to the right:

$$\begin{aligned} AJ_x &= AE_x + EI_x + IJ_x \\ &= -\overline{AE}\cos\left(\frac{2\pi}{5}\right) + \overline{EI}\cos\left(\frac{\pi}{5}\right) + \overline{IJ}\cos\left(\frac{\pi}{5}\right) \\ &= -12\left(\frac{\sqrt{5}-1}{4}\right) + 3\left(\frac{1+\sqrt{5}}{4}\right) + 4\left(\frac{1+\sqrt{5}}{4}\right) = \frac{19-5\sqrt{5}}{4} \end{aligned} \tag{64}$$

Then we calculate the ordinates going to the same order of vertices adding when we go up and substracting when we go down:

$$\begin{aligned}
AJ_y &= -AE_y + EI_y + IJ_y \\
&= \overline{AE}\sin\left(\frac{2\pi}{5}\right) + \overline{EI}\sin\left(\frac{\pi}{5}\right) - \overline{IJ}\sin\left(\frac{\pi}{5}\right) \\
&= 12\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right) + 3\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right) - 4\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right) = \frac{\sqrt{1450+190\sqrt{5}}}{4}
\end{aligned} \tag{65}$$

Finally we calculate the distance $\overline{AJ}$ which coincides with strip size 11:

$$\begin{aligned}
\overline{AJ} &= \sqrt{(AJ_x)^2 + (AJ_y)^2} \\
&= \sqrt{\left(\frac{19-5\sqrt{5}}{4}\right)^2 + \frac{1450+190\sqrt{5}}{16}} = \sqrt{121} = 11 \quad \blacksquare
\end{aligned} \tag{66}$$

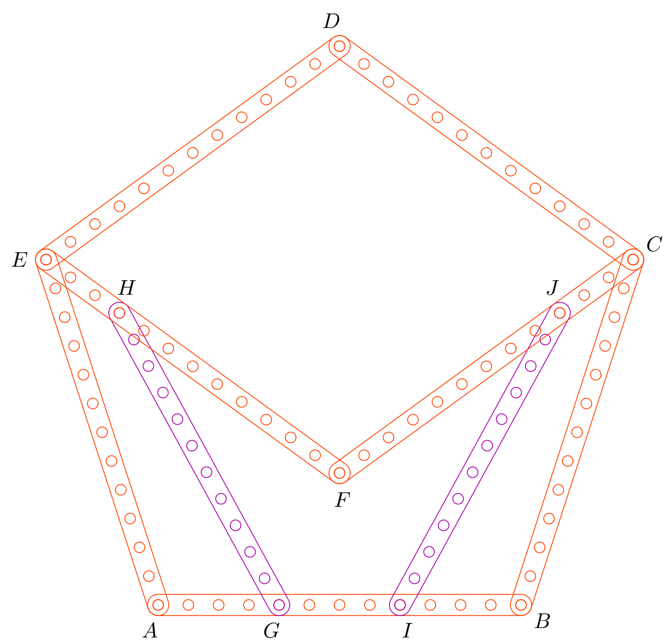

Figure 22: Regular pentagon size 12 case (b) made rigid with only four internal strips. $\overline{GA} : \overline{AE} : \overline{EE} : \overline{EH} = 4 : 12 : 0 : 3$ and $\overline{GH} = 11$.

Figure 22 show a regular pentagon A, B, C, D, E of size 12 with a rhombus D, I, H, G of size 12. We prove strips GH, IJ are correct. First we calculate the abscissas going through vertices G, A, E, H substracting when we move to the left and adding when we move to the right:

$$\begin{aligned}
GH_x &= -GA_x - AE_x + EH_x \\
&= -\overline{GA} - \overline{AE}\cos\left(\frac{2\pi}{5}\right) + \overline{EH}\cos\left(\frac{\pi}{5}\right) \\
&= -4 - 12\left(\frac{\sqrt{5}-1}{4}\right) + 3\left(\frac{1+\sqrt{5}}{4}\right) = \frac{-1-9\sqrt{5}}{4}
\end{aligned} \tag{67}$$

Then we calculate the ordinates going to the same order of vertices adding when we go up and substracting when we go down:

$$\begin{aligned}GH_y &= AG_y + AE_y - EH_y \\ &= 0 + \overline{AE}\sin\left(\frac{2\pi}{5}\right) - \overline{EH}\sin\left(\frac{\pi}{5}\right) \\ &= 12\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right) - 3\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right) = \frac{\sqrt{1530-18\sqrt{5}}}{4}\end{aligned} \tag{68}$$

Finally we calculate the distance $\overline{GH}$ which coincides with strip size 11:

$$\begin{aligned}\overline{GH} &= \sqrt{(GH_x)^2 + (GH_y)^2} \\ &= \sqrt{\left(\frac{-1-9\sqrt{5}}{4}\right)^2 + \frac{1530-18\sqrt{5}}{16}} = \sqrt{121} = 11 \quad \blacksquare\end{aligned} \tag{69}$$

10.2 Size 12 with 6 internal strips

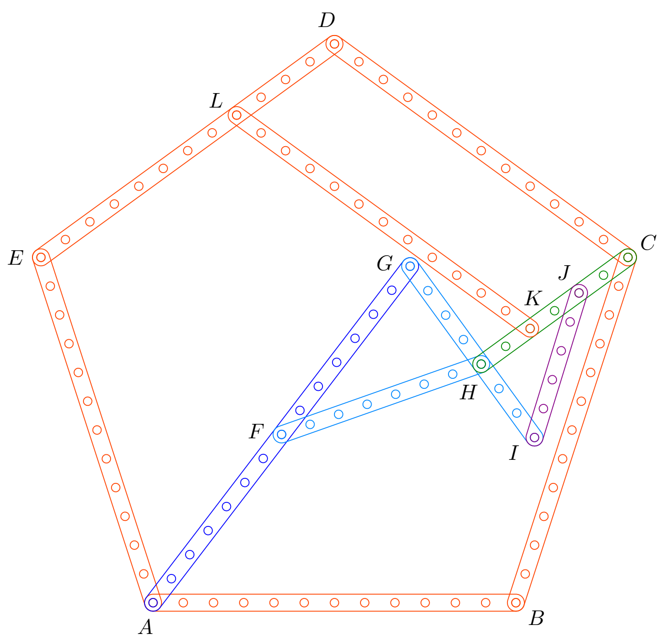

Figure 23: Regular pentagon size 12 made rigid with 6 internal strips. $\overline{AE} : \overline{ED} : \overline{DC} = 12 : 12 : 12$ and $\overline{AC} = 6 + 6\sqrt{5}$

.

Figure 23 show a regular pentagon A, B, C, D, E of side 12. We know the regular pentagon diagonal for size 12 is $\overline{AC} = 12\left(\dfrac{1+\sqrt{5}}{2}\right) = 6 + 6\sqrt{5}$.

10.2.1 Rigid distance $6 + 6\sqrt{5}$

We show the five strips $\overline{GH}, \overline{GI}, \overline{HF}, \overline{HC}, \overline{IJ}$ make the diagonal rigid which makes rigid the angle $\angle ABC$ of the pentagon. We have an isoscelles triangle $\triangle FGH$ and $\overline{AG}$ is two times $\overline{FG}$ so we have a right angle $\angle AHG = \frac{\pi}{2}$ and we can calculate $\overline{AH} = \sqrt{(\overline{AG})^2 - (\overline{GH})^2} = \sqrt{14^2 - 4^2} = 6\sqrt{5}$. Now we have another right angle $\angle IHC = \frac{\pi}{2}$ because the Pythagoras triangle $\triangle HIJ$. Since G, H, I are collinear then we have another right angle $\angle GHC = \frac{\pi}{2}$. Both right angles $\angle AHG, \angle CHG$ guaranty vertices A, H, C are collinear and we can calculate $\overline{AC} = \overline{AH} + \overline{HC} = 6 + 6\sqrt{5}$ ■.

Finally we add a sixth strip $\overline{KL}$ parallel to $\overline{CD}$ to make rigid the last three perimeter strips $\overline{CD}, \overline{DE}, \overline{EA}$ of the pentagon.

10.3 Size 12 with 8 internal strips

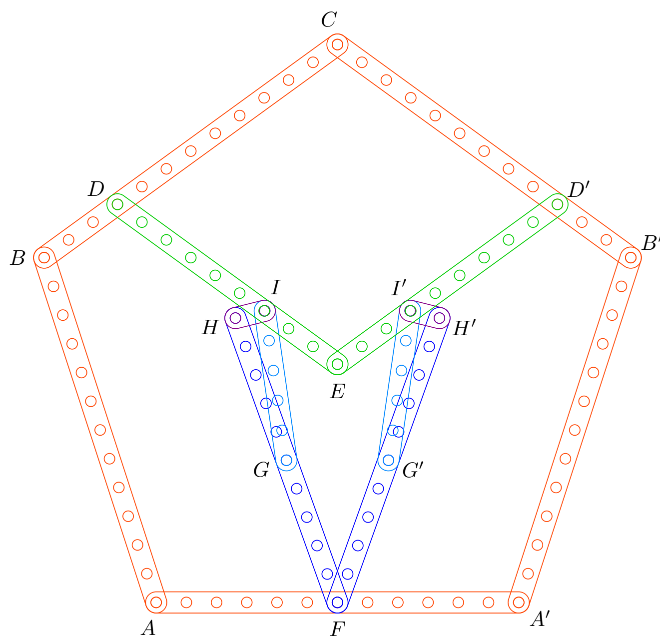

Figure 24: Regular pentagon size 12 made rigid with 8 internal strips. $\overline{FA} : \overline{AB} : \overline{BD} : \overline{DI} = 6 : 12 : 3 : 6$ and $\overline{FI} = 3\sqrt{11}$

.

Figure 24 show a regular pentagon $AA'B'CBA$ of side 12. First we calculate the distance $\overline{FI}$ using the abscissas and ordinates following the vertices F, A, B, D, I for a regular pentagon angles $\alpha = \dfrac{3\pi}{5}, \beta = \dfrac{\pi}{5}$:

$$
\begin{aligned}
FI_x &= -\overline{AF} - \overline{AB}\cos\alpha + (\overline{BD} + \overline{DI})\cos\beta \\
&= -6 + (12)\frac{1-\sqrt{5}}{4} + (3+6)\frac{\sqrt{5}+1}{4} = -\frac{3+3\sqrt{5}}{4} && (70) \\
FI_y &= \overline{AB}\sin\alpha + (\overline{BD} - \overline{DI})\sin\beta \\
&= (12)\frac{\sqrt{10+2\sqrt{5}}}{4} + (3-6)\frac{\sqrt{10-2\sqrt{5}}}{4} = \frac{12\sqrt{10+2\sqrt{5}} - 3\sqrt{10-2\sqrt{5}}}{4} && (71) \\
\overline{FI} &= \sqrt{(FI_x)^2 + (FI_y)^2} \\
&= \frac{\sqrt{(-3-3\sqrt{5})^2 + (12\sqrt{10+2\sqrt{5}} - 3\sqrt{10-2\sqrt{5}})^2}}{4} = \frac{\sqrt{1584}}{4} = 3\sqrt{11} && (72)
\end{aligned}
$$

10.3.1 Rigid distance $3\sqrt{11}$

Finally we calculate the distance $\overline{FI}$ made rigid by cluster F, G, H, I. We have an isoscelles triangle $\triangle GHI$ and $\overline{FH} = 2\overline{GH}$ so we have a right triangle $\angle FIH = \frac{\pi}{2}$ so:

$$\begin{aligned} \overline{FI} &= \sqrt{(\overline{FH})^2 - (\overline{HI})^2} \\ &= \sqrt{10^2 - 1^2} = 3\sqrt{11} \end{aligned} \tag{73}$$

10.4 Size 12 with 10 internal strips

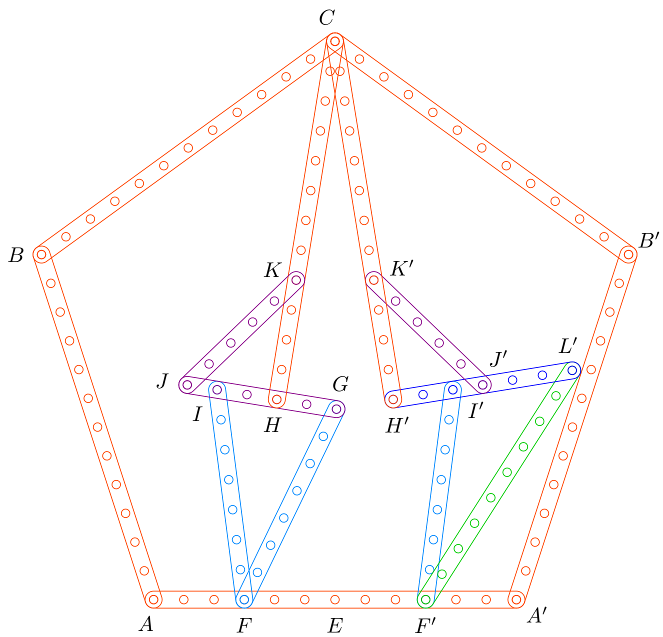

Figure 25: Regular pentagon size 12 made rigid with two internal clusters of five strips each. $\overline{FA} : \overline{AB} : \overline{BC} = 3 : 12 : 12$ and $\overline{CF} = 12 + 3\sqrt{5}$.

Figure 25 show a regular pentagon $AA'B'CB$ of side 12. We know the regular pentagon height is $\dfrac{\sqrt{5+2\sqrt{5}}}{2}$ times the side. So here we have $\overline{CE} = 12\frac{\sqrt{5+2\sqrt{5}}}{2} = 6\sqrt{5+2\sqrt{5}}$ and we can calculate $\overline{CF}$:

$$\begin{aligned} \overline{CF} &= \sqrt{\overline{CE}^2 + \overline{EF}^2} \\ &= \sqrt{36(5+2\sqrt{5}) + 3^2} = 3\sqrt{21+8\sqrt{5}} \\ &= 3(4+\sqrt{5}) \end{aligned} \tag{74}$$

10.4.1 Rigid distance $3(4 + \sqrt{5})$

After testing $\overline{AA'} \leq 1800$ our software found that the last denesting is somehow special since other fractions $\dfrac{\overline{AF}}{\overline{AA'}} \neq \dfrac{1}{4}$ generated $\overline{CF}$s that can't be denested.

We have the Pythagorean triangle $\triangle HJK$ and the isoscelles $\triangle FGI$ so vertices FHC are collinear. First we calculate $\overline{FH} = \sqrt{\overline{FG}^2 - \overline{GH}^2} = \sqrt{7^2 - 2^2} = 3\sqrt{5}$ and then $\overline{FC} = \overline{FH} + \overline{HC} = 3\sqrt{5} + 12$ matching last calculation. Finally we prove angle $\angle F'H'L' = \frac{\pi}{2}$ noting $\overline{F'H'} = \sqrt{(\overline{F'L'})^2 - (\overline{H'L'})^2} = \sqrt{9^2 - 6^2} = 3\sqrt{5}$ matching $\overline{FH}$ ■.