Meccano four frame

https://github.com/heptagons/meccano/frames/four

Abstract

Four frame is a group of four rigid meccano [1] strips.

1 Four frame

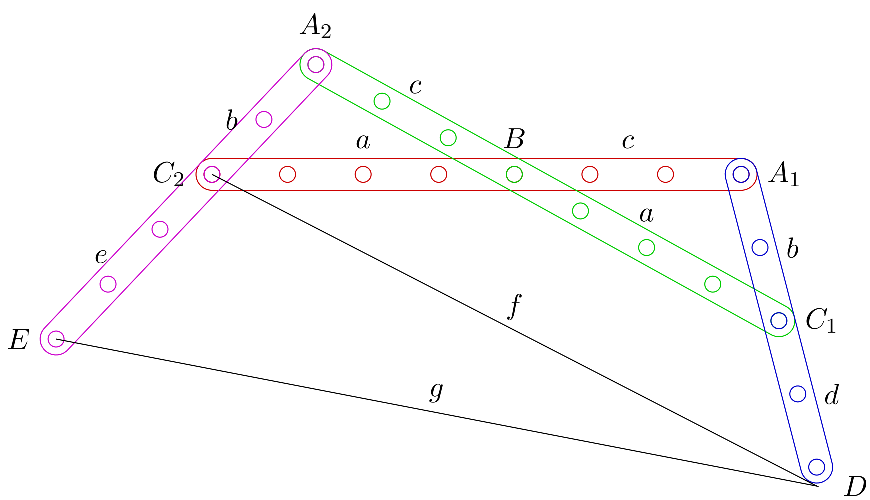

Figure 1: Antisymmetric four frame.

Figure 1 show the antisymmetric four-strips frame. From the figure we define $\alpha \equiv \angle BA_1C_1$ and define integers $m = b^2 + c^2 - a^2$ and $n = 2bc$ using the law of cosines, then we calculate $\cos\alpha$ and $\sin\alpha$:

$$(\alpha, m, n) \equiv (\angle BA_1C_1, b^2 + c^2 - a^2, 2bc) \tag{1}$$

$$\cos\alpha = \frac{m}{n} \tag{2}$$

$$\sin\alpha = \frac{\sqrt{n^2 - m^2}}{n} \tag{3}$$

We calculate the distance $f = \overline{C_2D}$ with the law of cosines using angle α and defining integers $x = a+c$ and $y = b + d$:

$$(x, y) \equiv (a + c, b + d) \tag{4}$$

$$f^2 = (a + c)^2 + (b + d)^2 - 2(a + c)(b + d)\cos\alpha \tag{5}$$

$$= x^2 + y^2 - \frac{2mxy}{n} \tag{6}$$

$$f = \frac{\sqrt{n^2(x^2 + y^2) - 2mnxy}}{n} \tag{7}$$

[1] Meccano mathematics by ‘t Hooft

We define a new integer $z \equiv n^2(x^2 + y^2) - 2mnxy$ so we have:

$$z \equiv n^2(x^2 + y^2) - 2mnxy \tag{8}$$
$$f = \frac{\sqrt{z}}{n} \tag{9}$$

We define angle $\theta \equiv \angle A_1C_2D$ and calculate $\cos\theta$ and $\sin\theta$:

$$\theta \equiv \angle A_1C2D \tag{10}$$
$$\begin{aligned}
\cos\theta &= \frac{(a+c)^2 + f^2 - (b+d)^2}{2(a+c)f} \\
&= \frac{x^2 + f^2 - y^2}{2xf} \\
&= \frac{x^2 + x^2 + y^2 - \dfrac{2mxy}{n} - y^2}{2x\dfrac{\sqrt{z}}{n}}
\end{aligned}$$
$$= \frac{nx - my}{\sqrt{z}} \tag{11}$$
$$\begin{aligned}
\sin^2\theta = 1 - \cos^2\theta &= \frac{z - (nx - my)^2}{z} \\
&= \frac{n^2(x^2 + y^2) - 2mnxy - (nx - my)^2}{z} \\
&= \frac{n^2(x^2 + y^2) - 2mnxy - n^2x^2 + 2nxmy - m^2y^2}{z}
\end{aligned}$$
$$\sin\theta = \frac{y\sqrt{n^2 - m^2}}{\sqrt{z}} \tag{12}$$

From the figure 1 we define $\gamma \equiv \angle BA_2C_2$ and define integers $s = a^2 + b^2 - c^2$ and $t = 2ab$ and calculate $\cos\gamma$ and $\sin\gamma$:

$$(\gamma, s, t) \equiv (\angle BA_2C_2, a^2 + b^2 - c^2, 2ab) \tag{13}$$
$$\cos\gamma = \frac{s}{t} \tag{14}$$
$$\sin\gamma = \frac{\sqrt{t^2 - s^2}}{t} \tag{15}$$

We define angle $\phi \equiv \angle A_2C_2D$ and we note is the sum of angles $\theta + \gamma$ and we calculate $\cos\phi$:

$$\phi \equiv \angle A_2C_2D \tag{16}$$
$$= \theta + \gamma \tag{17}$$
$$\cos\phi = \cos(\theta + \gamma) \tag{18}$$
$$\begin{aligned}
&= \cos\theta\cos\gamma - \sin\theta\sin\gamma \\
&= \frac{(nx - my)s}{\sqrt{z}t} - \frac{(y\sqrt{n^2 - m^2})\sqrt{t^2 - s^2}}{\sqrt{z}t}
\end{aligned}$$
$$= \frac{(nx - my)s - y\sqrt{(n^2 - m^2)(t^2 - s^2)}}{\sqrt{z}t} \tag{19}$$

From the figure we define angle $\psi \equiv \angle DC_2E$ and we note equals angle $\pi - \phi$, so we have:

$$\begin{aligned}
\psi &\equiv \angle DC_2E && (20)\\
&= \pi - \phi && (21)\\
\cos\psi &= \cos(\pi - \phi) && (22)\\
&= -\cos\phi \\
&= \frac{-(nx - my)s + y\sqrt{(n^2 - m^2)(t^2 - s^2)}}{\sqrt{z}t} && (23)
\end{aligned}$$

Finally with $\cos\psi$, e and f we can calculate distance $g = \overline{ED}$:

$$\begin{aligned}
g^2 &= e^2 + f^2 - 2ef\cos\psi && (24)\\
&= e^2 + x^2 + y^2 - \frac{2mxy}{n} - 2e\left(\frac{\sqrt{z}}{n}\right)\left(\frac{-(nx - my)s + y\sqrt{(n^2 - m^2)(t^2 - s^2)}}{\sqrt{z}t}\right) && (25)\\
&= e^2 + x^2 + y^2 - \frac{2mxy}{n} - 2e\left(\frac{-(nx - my)s + y\sqrt{(n^2 - m^2)(t^2 - s^2)}}{nt}\right) && (26)\\
&= \frac{(e^2 + x^2 + y^2)nt - 2mxyt + 2es(nx - my) - 2ey\sqrt{(n^2 - m^2)(t^2 - s^2)}}{nt} \\
g &= \frac{\sqrt{(e^2 + x^2 + y^2)n^2t^2 - 2mnxyt^2 + 2esnt(nx - my) - 2eynt\sqrt{(n^2 - m^2)(t^2 - s^2)}}}{nt} && (27)
\end{aligned}$$

1.1 Antisymmetric four frame software

From the last equation of g we identify three `input` integers i_1, i_2, i_3 which are used to get $g(i)$. Then the nested radicals software will return square-free `output` integers z_1, z_2, z_3, z_4, z_5 as $g(z)$:

$$\begin{aligned}
i_1 &= nt && (28)\\
i_2 &= (e^2 + x^2 + y^2)i_1^2 - 2mxyti_1 + 2esi_1(nx - my) && (29)\\
i_3 &= -2eyi_1 && (30)\\
i_4 &= (n^2 - m^2)(t^2 - s^2) && (31)\\
g(i) &= \frac{\sqrt{i_2 + i_3\sqrt{i_4}}}{i_1} && (32)\\
g(z) &= \frac{z_2\sqrt{z_3 + z_4\sqrt{z_5}}}{z_1} && (33)
\end{aligned}$$

where m, n are calculated with equations 1, x, y are calculated with equations 4 and s, t are calculated with equations 13.