1 Algebraic F 32 bits numbers

Lets define the algebraic F numbers as the next irreducible equation:

$$F \equiv \frac{\pm a\sqrt{c \pm e\sqrt{f}} \pm d}{b}$$

when we are limited to 32-bit integers:

$$0 \leq a, c, d, e, f \leq 4,294,967,295$$
$$1 \leq b \leq 4,294,967,295$$

1.1 Simplifications

When $e = 0$ and $a, c, d > 0$ we have an algebraic D. When $d, e = 0$ and $a, c > 0$ we have an algebraic C. When $c, d, e = 0$ and $a > 0$ we have a rational B.

$$D \equiv \frac{\pm a\sqrt{c} \pm d}{b} \qquad e = 0$$
$$C \equiv \frac{\pm a\sqrt{c}}{b} \qquad d = e = 0$$
$$B \equiv \frac{\pm a}{b} \qquad c = d = e = 0$$

2 Operations

2.1 $B_1 + B_2 = B_3$

$$\begin{aligned} B_1 + B_2 &= \frac{\pm a_1}{b_1} + \frac{\pm a_2}{b_2} \\ &= \frac{\pm a_1 b_2 \pm a_2 b_1}{b_1 b_2} \\ &= \frac{\pm a_3}{b_3} \qquad (\pm a_3, b_3) = gcd(\pm a_1 b_2 \pm a_2 b_1, b_1 b_2) \end{aligned}$$

2.2 $B_1 \times B_2 = B_3$

$$\begin{aligned} B_1 \times B_2 &= \frac{\pm a_1}{b_1} \times \frac{\pm a_2}{b_2} \\ &= \frac{\pm a_1 a_2}{b_1 b_2} \\ &= \frac{\pm a_3}{b_3} \qquad (\pm a_3, b_3) = gcd(\pm a_1 a_2, b_1 b_2) \end{aligned}$$

2.3 $1/B_1 = B_2, a > 0$

$$\begin{aligned} \frac{1}{B_1} &= \frac{1}{\frac{\pm a_1}{b_1}} \\ &= \frac{\pm b_1}{a_1} \\ &= \frac{\pm a_2}{b_2} \end{aligned}$$

2.4 $\sqrt{B_1} = C_2$

$$\begin{aligned}
\sqrt{B_1} &= \sqrt{\frac{a_1}{b_1}} \\
&= \frac{\sqrt{a_1 b_1}}{b_1} \\
&= \frac{m\sqrt{c_2}}{b_1} && a_1 b_1 = m^2 c_2 \\
&= \frac{a_2\sqrt{c_2}}{b_2} && (a_2, b_2) = gcd(m, b_1)
\end{aligned}$$

2.5 $B_1 + C_2 = D_3$

$$\begin{aligned}
B_1 + C_2 &= \frac{\pm a_1}{b_1} + \frac{\pm a_2\sqrt{c_2}}{b_2} \\
&= \frac{\pm a_2 b_1\sqrt{c_2} \pm a_1 b_2}{b_1 b_2} \\
&= \frac{\pm a_3\sqrt{c_2} \pm d_3}{b_3} && (\pm a_3, b_3, \pm d_3) = gcd(\pm a_2 b_1, b_1 b_2, \pm a_1 b_2)
\end{aligned}$$

2.6 $B_1 + D_2 = D_3$

$$\begin{aligned}
B_1 + D_2 &= \frac{\pm a_1}{b_1} + \frac{\pm a_2\sqrt{c_2} \pm d_2}{b_2} \\
&= \frac{\pm a_2 b_1\sqrt{c_2} \pm a_1 b_2 \pm d_2 b_1}{b_1 b_2} \\
&= \frac{\pm a_3\sqrt{c_2} \pm d_3}{b_3} && (\pm a_3, b_3, \pm d_3) = gcd(\pm a_2 b_1, b_1 b_2, \pm a_1 b_2 \pm d_2 b_1)
\end{aligned}$$

2.7 $B_1 + F_2 = F_3$

$$\begin{aligned}
B_1 + F_2 &= \frac{\pm a_1}{b_1} + \frac{\pm a_2\sqrt{c_2 \pm e_2\sqrt{f_2}} \pm d_2}{b_2} \\
&= \frac{\pm a_2 b1\sqrt{c_2 \pm e_2\sqrt{f_2}} \pm a_1 b_2 \pm d_2 b_1}{b_1 b_2} \\
&= \frac{\pm a_3\sqrt{c_2 \pm e_2\sqrt{f_2}} \pm d_3}{b_3} && (\pm a_3, b_3, \pm d_3) = gcd(\pm a_2 b1, b_1 b_2, \pm a_1 b_2 \pm d_2 b_1)
\end{aligned}$$

2.8 $C_1 + C_2 = F_3$

$$\begin{aligned}
C_1 + C_2 &= \frac{\pm a_1\sqrt{c_1}}{b_1} + \frac{\pm a_2\sqrt{c_2}}{b_2} \\
&= \frac{\pm a_1 b_2\sqrt{c_1} \pm a_2 b_1\sqrt{c_2}}{b_1 b_2} \\
&= \frac{\pm m\sqrt{c_1} \pm n\sqrt{c_2}}{o} \\
&= \frac{\pm\sqrt{m^2 c_1 + n^2 c_2 \pm 2mn\sqrt{c_1 c_2}}}{o} \\
&= \frac{\pm\sqrt{q \pm 2mnp\sqrt{f_3}}}{o} && c_1 c_2 = p^2 f_3 \\
&= \frac{\pm r\sqrt{c_3 \pm e_3\sqrt{f_3}}}{o} && q = r^2 c_3, 2mnp = r^2 e_3 \\
&= \frac{\pm a_3\sqrt{c_3 \pm e_3\sqrt{f_3}}}{b_3}
\end{aligned}$$

2.9 $C_1 \times C_2 = C_3$

$$\begin{aligned}
C_1 \times C_2 &= \frac{\pm a_1\sqrt{c_1}}{b_1} \times \frac{\pm a_2\sqrt{c_2}}{b_2} \\
&= \frac{\pm a_1 a_2\sqrt{c_1 c_2}}{b_1 b_2} \\
&= \frac{\pm a_1 a_2 m\sqrt{c_3}}{b_1 b_2} && c_1 c_2 = m^2 c_3 \\
&= \frac{\pm a_3\sqrt{c_3}}{b_3} && (\pm a_3, b_3) = gcd(\pm a_1 a_2 m, b_1 b_2)
\end{aligned}$$

2.10 $1/C_1 = C_2$

$$\begin{aligned}
1/C_1 &= \frac{1}{\frac{\pm a_1\sqrt{c_1}}{b_1}} \\
&= \frac{b_1}{\pm a_1\sqrt{c_1}} \\
&= \frac{\pm b_1\sqrt{c_1}}{c_1} \\
&= \frac{\pm a_2\sqrt{c_1}}{b_2} && (\pm a_2, b_2) = gcd(\pm b_1, c1)
\end{aligned}$$

2.11 $\sqrt{C_1} = F_2$

$$\begin{aligned}
\sqrt{C_1} &= \sqrt{\frac{a_1\sqrt{c_1}}{b_1}} \\
&= \frac{\sqrt{a_1 b_1\sqrt{c_1}}}{b_1} \\
&= \frac{m\sqrt{e_2\sqrt{c_1}}}{b_1} && a_1 b_1 = m^2 e_2 \\
&= \frac{a_2\sqrt{e_2\sqrt{c_1}}}{b_2} && (a_2, b_2) = gcd(m, b_1)
\end{aligned}$$

2.12 $D_1 + D_2 = F_3$???

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\begin{aligned}
D_1 + D_2 &= \frac{\pm a_1\sqrt{c_1} \pm d_1}{b_1} + \frac{\pm a_2\sqrt{c_2} \pm d_2}{b_2} \\
&= \frac{\pm a_1 b_2\sqrt{c_1} \pm a_2 b_1\sqrt{c_2} \pm d_1 b_2 \pm d_2 b_1}{b_1 b_2} \\
&= \frac{\pm m\sqrt{c_1} \pm n\sqrt{c_2} \pm p}{o} && (\pm m, \pm n, \pm p, o) = gcd(\pm a_1 b_2, \pm a_2 b_1, \pm d_1 b_2 \pm d_2 b_1, b_1 b_2) \\
&= \frac{\sqrt{m^2 c_1 + n^2 c_2 \pm 2mn\sqrt{c1c_2}} \pm p}{o} \\
&= \frac{\sqrt{q \pm 2mnr\sqrt{f_3}} \pm p}{o} && q = m^2 c_1 + n^2 c_2, c_1 c_2 = r^2 f_3 \\
&= \frac{s\sqrt{c_3 \pm e_3\sqrt{f_3}} \pm p}{o} && q = s^2 c_3, 2mnr = s^2 e_3 \\
&= \frac{a_3\sqrt{c_3 \pm e_3\sqrt{f_3}} \pm d_3}{b_3} && (a_3, b_3, \pm d_3) = gcd(s, \pm p, o)
\end{aligned}
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