# Triple unit

https://github.com/heptagons/meccano/units/triple

**Abstract**

Triple unit is a group of five meccano [1] strips $a, b, c, d, e$ intended to build regular polygons three consecutive perimeter sides. This unit has three angles equal to the polygon internal angle $\theta$. Triple unis has been using to build the pentagon type 2 mentioned in pentagons paper[2].

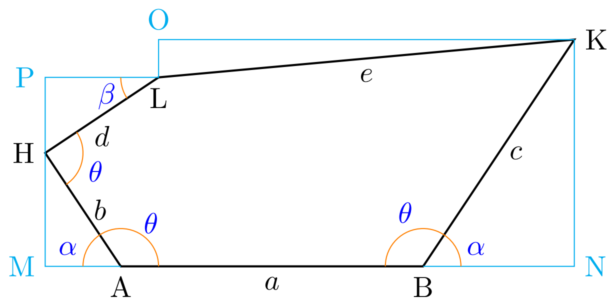


Figure 1: Triple unit has five strips $a, b, c, d, e$

## 1 Algebra

From nodes $A$ and $B$ of fig 1 we get $\alpha$ from $\theta$ ($\pi = 180°$):

$$\begin{aligned} \theta &= \pi - \alpha \\ \alpha &= \pi - \theta \end{aligned} \tag{1}$$

And from node $H$ we get $\beta$ from $\theta$:

$$\begin{aligned} &\theta = \alpha + \beta \\ &\beta = \theta - \alpha = \theta - (\pi - \theta) = 2\theta - \pi \end{aligned} \tag{2}$$

We calculate horizontal segment $\overline{OK}$:

$$\begin{aligned} \overline{OK} &= \overline{MA} + a + \overline{BN} - \overline{PL} \\ &= b\cos\alpha + a + c\cos\alpha - d\cos\beta \\ &= a + (b+c)\cos\alpha - d\cos\beta \\ &= a + (b+c)\cos(\pi-\theta) - d\cos(2\theta-\pi) \\ &= a - (b+c)\cos\theta + d\cos(2\theta) \end{aligned} \tag{3}$$

[1] Meccano mathematics by ‘t Hooft
[2] Meccano pentagons

And vertical segment $\overline{OL}$:

$$\begin{aligned}
\overline{OL} &= \overline{KN} - \overline{PH} - \overline{HM} \\
&= c \sin\alpha - d \sin\beta - b \sin\alpha \\
&= (c-b)\sin\alpha - d\sin\beta \\
&= (c-b)\sin(\pi-\theta) - d\sin(2\theta-\pi) \\
&= (c-b)\sin\theta + d\sin(2\theta) \qquad (4)
\end{aligned}$$

So we can express $e$ in function of $a, b, c, d$ and angle $\theta$:

$$\begin{aligned}
e^2 &= (\overline{OK})^2 + (\overline{OL})^2 \\
&= (a - (b+c)\cos\theta + d\cos(2\theta))^2 + ((c-b)\sin\theta + d\sin(2\theta))^2 \\
&= a^2 + (b^2+2bc+c^2)\cos^2\theta + d^2\cos^2(2\theta) + (c^2-2cb+b^2)sin^2\theta + d^2\sin^2(2\theta) \\
&\quad - 2a(b+c)\cos\theta + 2ad\cos(2\theta) - 2(b+c)d\cos\theta\cos(2\theta) \\
&\quad + 2(c-b)d\sin\theta\sin(2\theta) \\
&= a^2+b^2+c^2+d^2+2bc\cos^2\theta - 2bc\sin^2\theta \\
&\quad - 2a(b+c)\cos\theta + 2ad\cos(2\theta) \\
&\quad - 2d((b+c)\cos\theta\cos(2\theta) + (b-c)\sin\theta\sin(2\theta)) \qquad (5) \\
&= a^2+b^2+c^2+d^2+2bc(\cos^2\theta - \sin^2\theta) - 2a(b+c)\cos\theta + 2ad\cos(2\theta) \\
&\quad - 2d(b(\cos\theta\cos(2\theta)+\sin\theta\sin(2\theta)) + c(\cos\theta\cos(2\theta) - \sin\theta\sin(2\theta))) \\
&= a^2+b^2+c^2+d^2+2bc\cos(2\theta) - 2a(b+c)\cos\theta + 2ad\cos(2\theta) \\
&\quad - 2d(b\cos(\theta-2\theta) + c\cos(\theta+2\theta)) \\
&= a^2+b^2+c^2+d^2+2(bc+ad)\cos(2\theta) - 2a(b+c)\cos\theta - 2d(b\cos\theta + c\cos(3\theta)) \\
&= a^2+b^2+c^2+d^2+2(bc+ad)\cos(2\theta) - 2(ab+ac)\cos\theta - 2(bd\cos\theta + cd\cos(3\theta)) \\
&= \boxed{a^2+b^2+c^2+d^2 - 2(ab+ac+bd)\cos\theta + 2(bc+ad)\cos(2\theta) - 2cd\cos(3\theta)} \qquad (6)
\end{aligned}$$

# 2 Regular polygons

| Polygon | $\theta$ | $\cos\theta$ | $\cos(2\theta)$ | $\cos(3\theta)$ |
|---|---|---|---|---|
| Pentagon | $\frac{3\pi}{5}$ | $\frac{1-\sqrt{5}}{4}$ | $\frac{-1-\sqrt{5}}{4}$ | $\frac{1+\sqrt{5}}{4}$ |
| Hexagon | $\frac{2\pi}{3}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 1 |
| Heptagon | $\frac{5\pi}{7}$ | | | |
| Octagon | $\frac{3\pi}{4}$ | | | |
| Decagon | $\frac{4\pi}{5}$ | | | |
| Dodecagon | $\frac{5\pi}{6}$ | | | |

Table 1: Regular polygons internal angles and cosines.

## 2.1 Equilateral pentagon

We replace the cosines for pentagon in table 1 in $e^2$ equation:

$$\begin{aligned}
e^2 &= a^2+b^2+c^2+d^2-2(ab+ac+bd)\cos\theta+2(bc+ad)\cos(2\theta)-2cd\cos(3\theta)\\
&= a^2+b^2+c^2+d^2-2(ab+ac+bd)\left(\frac{1-\sqrt{5}}{4}\right)+2(bc+ad)\left(\frac{-1-\sqrt{5}}{4}\right)-2cd\left(\frac{1+\sqrt{5}}{4}\right)\\
&= a^2+b^2+c^2+d^2-\frac{ab+ac+bd+bc+ad+cd}{2}+\frac{ab+ac+bd-bc-ad-cd}{2}\sqrt{5} \qquad (7)
\end{aligned}$$

$e$ cannot to be and integer if the factor of $\sqrt{5}$ is not zero so we force this factor to be zero:

$$\begin{aligned}
ab+ac+bd-bc-ad-cd &= 0\\
ab+ac+bd &= bc+ad+cd \qquad (8)
\end{aligned}$$

We replace $ab+ac+bd$ by $bc+ad+cd$ in the $e^2$ equation to get:

$$\begin{aligned}
e^2 &= a^2+b^2+c^2+d^2-\frac{(bc+ad+cd)+bc+ad+cd}{2}+\frac{0}{2}\sqrt{5}\\
&= a^2+b^2+c^2+d^2-bc-ad-cd\\
e &= \sqrt{a^2+b^2+c^2+d^2-bc-ad-cd} \qquad (9)
\end{aligned}$$

The last formula matches the formula used in the paper Meccano pentagons which finds several pentagons of type 2. Only when we get $e$ integer we have a solution.

## 2.2 Equilateral hexagon

We replace the cosines for hexagon in table 1 in $e^2$ equation:

$$\begin{aligned}
e^2 &= a^2+b^2+c^2+d^2-2(ab+ac+bd)\cos\theta+2(bc+ad)\cos(2\theta)-2cd\cos(3\theta)\\
&= a^2+b^2+c^2+d^2-2(ab+ac+bd)\left(-\frac{1}{2}\right)+2(bc+ad)\left(-\frac{1}{2}\right)-2cd(1)\\
&= a^2+b^2+c^2+d^2+ab+ac+bd-bc-ad-2cd\\
&= (a+b)^2+(c-d)^2-ab+ac+bd-bc-ad\\
&= (a+b)^2+(c-d)^2+(c-d)(a-b)-ab\\
&= (a+b)^2+(c-d)(a-b+c-d)-ab\\
e &= \sqrt{(a+b)^2+(c-d)(a-b+c-d)-ab} \qquad (10)
\end{aligned}$$