Meccano frames

https://github.com/heptagons/meccano/frames

Abstract

Meccano frames are groups of meccano [1] strips intended to be a base to build diverse meccano larger objects.

1 Triangular frame

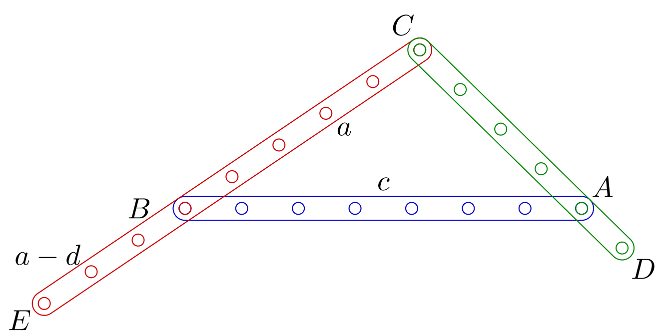

Figure 1: Triangular frame.

Figure 1 shows a triangular frame. We three strips we form the triangle $\triangle ABC$. At least we extend one of the two strips $\overline{CB}$ and $\overline{CA}$ to become $\overline{CE}$ and $\overline{CD}$. The new vertices D and E distance is rigid as the triangle and we'll calculate the distance between them which is of the form $\frac{q\sqrt{r}}{p}$, where p, q, r are integers.

First we identify five integer distances a, b, c, d, e:

$$a \equiv \overline{CB}, \quad b \equiv \overline{CA}, \quad c \equiv \overline{AB}, \quad c < a + b \tag{1}$$

$$d \equiv \overline{CB} + \overline{BE} = \overline{CE} \geq a \tag{2}$$

$$e \equiv \overline{CA} + \overline{AD} = \overline{CD} \geq b \tag{3}$$

We calculate the cosine of $\angle BCA$:

$$\theta \equiv \angle BCA \tag{4}$$

$$\cos\theta = \frac{a^2 + b^2 - c^2}{2ab} \tag{5}$$

Then we apply the cosine to the triangle $\triangle CED$ to get the extensions distance $\overline{DE}$:

$$\begin{aligned}\overline{DE}^2 &= \overline{CD}^2 + \overline{CE}^2 - 2\overline{CD} \times \overline{CE}\cos\theta \\ &= d^2 + e^2 - 2de\cos\theta \\ &= d^2 + e^2 - de\left(\frac{a^2 + b^2 - c^2}{ab}\right)\end{aligned} \tag{6}$$

[1] Meccano mathematics by ‘t Hooft

We expect at most a value of the form $\sqrt{s}/t$ where $s, t \in \mathbb{Z}^+$ so we define the surd as:

$$
\begin{aligned}
\overline{DE} &= \sqrt{d^2 + e^2 - de\left(\frac{a^2 + b^2 - c^2}{ab}\right)} \\
&= \frac{\sqrt{a^2b^2(d^2 + e^2) - abde(a^2 + b^2 - c^2)}}{ab} \\
&= \frac{\sqrt{ab((ad - be)(bd - ae) + c^2de)}}{ab} \\
&= \frac{\sqrt{s}}{t} && (7)
\end{aligned}
$$

$$t = ab \tag{8}$$

$$s = t((ad - be)(bd - ae) + c^2de) \tag{9}$$

1.1 Rigid distances $\sqrt{s}$ and $\sqrt{s} + f$

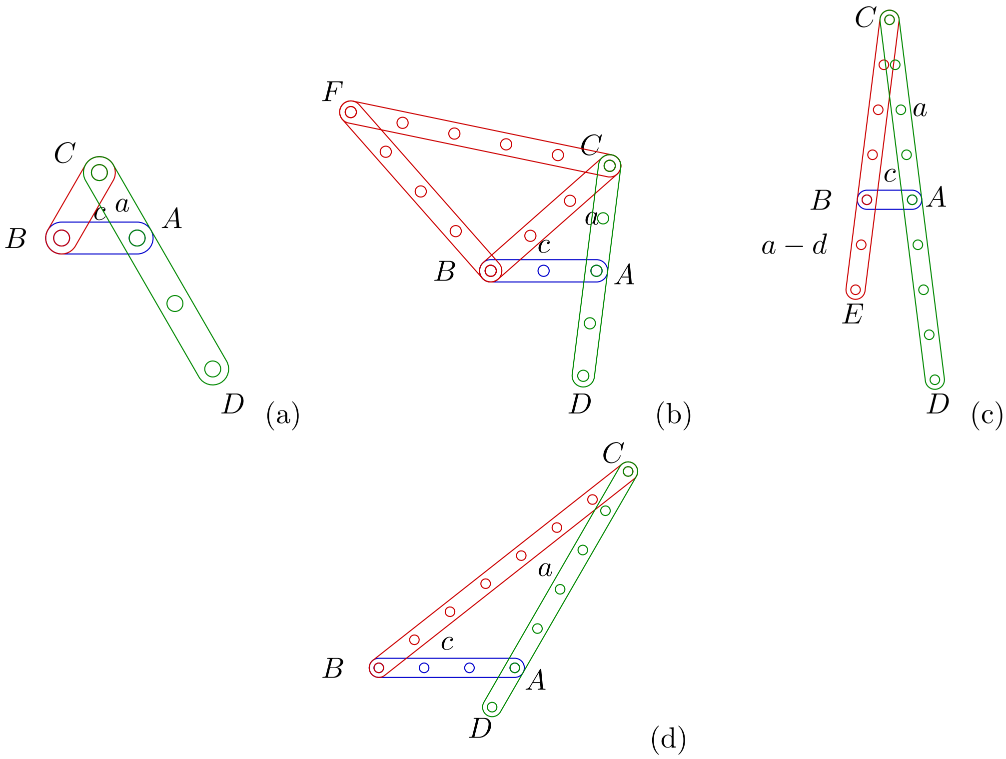

Figure 2: Four frames with rigid distance $\overline{DE} = \sqrt{7}$ reported by the factory software mentioned in this section. The particular case (b), was reported with the angle $CED = \pi/2$ which means we can append two extra strips to make a pythagorean triangle $\triangle CEF$ where angle $CEF = \pi/2$, which makes the three vertices D, E, F collinear, so the rigid distance $\overline{DF} = \sqrt{7} + 4$ is an algebraic number.

We write a factory to report all the triangles with specific surd $\sqrt{s}$ for a given maximum strips length. We reject $t \neq 1$ and s as not square-free, which includes pythagorean triangles. Next list show all the triangles with $s = \sqrt{7}, t = 1$ where $c < a + b$, $a \leq d \leq max$, $b \leq e \leq max$, $c \leq max$:

```
=== RUN   TestFramesTriangleSurds
NewFrames().TriangleSurds surd=7 max=15
  1) a=1 e=1+2 c=1 cos=1/2
```

```
  2) d=1+1 e=1+2 c=1 cos=1/2
  3) d=1+2 b=1 c=1 cos=1/2
  4) d=1+2 e=1+1 c=1 cos=1/2
  5) a=2 e=2+1 c=2 cos=1/2
  6) d=2+1 b=2 c=2 cos=1/2
  7) a=3 e=2+2 c=2 cos=3/4 CED=pi/2
  8) d=3+1 e=2+1 c=2 cos=3/4 CDE=pi/2
  9) d=4+2 e=4+4 c=1 cos=31/32
 10) d=4+4 e=4+2 c=1 cos=31/32
 11) a=7 e=5+1 c=3 cos=13/14
 12) a=7 e=5+2 c=3 cos=13/14
```

Figure 2 show four cases of this list. The code is in the folder `github.com/heptagons/meccano/frames`.

1.2 Another rigid distances $\sqrt{s}+h$

We explore a more complicated frame to get additional cases of distances $\sqrt{s}+h$ without relying in an explicit pythagorean triangle as we saw in case (b) of figure 2.

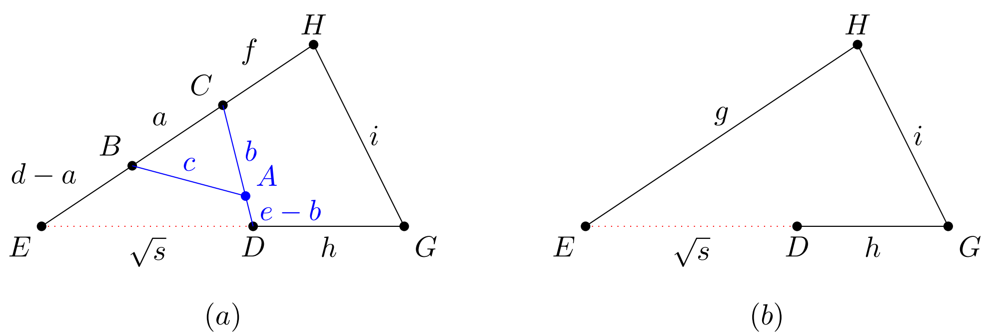

Figure 3: The five strips intented to form an algebraic distance $\overline{EG}=\sqrt{s}+h$.

From figure 3 (a) we know $\sqrt{s}$ distance between nodes E and D is produced by the three strips frame $a+d$, $b+e$ and c. Using the law of cosines we calculate the angle $\theta=\angle CED$ in terms of $\sqrt{s}$:

$$\begin{aligned}
\cos\theta &= \frac{d^2+(\sqrt{s})^2-e^2}{2d\sqrt{s}} \\
&= \frac{(d^2+s-e^2)\sqrt{s}}{2ds} && (10) \\
&= \frac{m\sqrt{s}}{n} && (11) \\
m &= d^2+s-e^2 && (12) \\
n &= 2ds && (13)
\end{aligned}$$

From figure 3 (a) we notice two sets of points are collinear: $\{E,B,C,H\}$ and $\{E,D,G\}$. Using the law of cosines we calculate the angle $\theta=\angle HEG$ in terms of distances $g,\sqrt{s}+h,i$:

$$
\begin{aligned}
\cos\theta &= \frac{g^2 + (\sqrt{s}+h)^2 - i^2}{2g(\sqrt{s}+h)} \\
&= \frac{g^2 + s + 2\sqrt{s}h + h^2 - i^2}{2g(\sqrt{s}+h)} \\
&= \frac{g^2 + s + h^2 - i^2 + 2\sqrt{s}h}{2g(\sqrt{s}+h)} \qquad (14)
\end{aligned}
$$

We multiply both numerator and denominator by $\sqrt{s} - h$ to eliminate the surd from denominator:

$$
\begin{aligned}
\cos\theta &= \frac{(s + g^2 + h^2 - i^2)(\sqrt{s}-h) + 2\sqrt{s}h(\sqrt{s}-h)}{2g(\sqrt{s}+h)(\sqrt{s}-h)} \\
&= \frac{(s + g^2 + h^2 - i^2)(\sqrt{s}-h) + 2sh - 2\sqrt{s}h^2}{2g(s-h^2)} \\
&= \frac{-h(s + g^2 + h^2 - i^2 - 2s) + (s + g^2 + h^2 - i^2 - 2h^2)\sqrt{s}}{2g(s-h^2)} \\
&= \frac{h(s - g^2 - h^2 + i^2) + (s + g^2 - h^2 - i^2)\sqrt{s}}{2g(s-h^2)} \\
&= \frac{o + p\sqrt{s}}{q} \qquad (15) \\
o &= h(s - g^2 - h^2 + i^2) \qquad (16) \\
p &= s + g^2 - h^2 - i^2 \qquad (17) \\
q &= 2g(s - h^2) \qquad (18)
\end{aligned}
$$

We compare both cosines equations 11 and 15:

$$
\frac{m\sqrt{s}}{n} = \frac{o + p\sqrt{s}}{q} \qquad (19)
$$

Since all variables are integers we need two conditions. First o should be zero. And second $\frac{m}{n} = \frac{p}{q}$.

For condition 1, we force o to be zero:

$$
\begin{aligned}
o &= 0 \\
h(s - g^2 - h^2 + i^2) &= 0 \\
s &= g^2 + h^2 - i^2 \qquad (20)
\end{aligned}
$$

For condition2, we force m, n, p, q as:

$$
\begin{aligned}
\frac{m}{n} &= \frac{p}{q} \\
\frac{d^2 + s - e^2}{2ds} &= \frac{s + g^2 - h^2 - i^2}{2g(s-h^2)}
\end{aligned}
\qquad (21)
$$

We replace the value of s of last equation RHS with the value of equation 20 of condition 1:

$$\begin{aligned} \frac{d^2 - e^2 + s}{ds} &= \frac{s + g^2 - h^2 - i^2}{g(s - h^2)} \\ &= \frac{g^2 + h^2 - i^2 + g^2 - h^2 - i^2}{g(g^2 + h^2 - i^2 - h^2)} \\ &= \frac{2(g^2 - i^2)}{g(g^2 - i^2)} \\ &= \frac{2}{g} \\ (d^2 - e^2 + s)g &= 2ds \end{aligned} \tag{22}$$

$\boxed{TODO : Examples!!!}$

2 Two triangles frame

Two triangles frame consist of two triangles sharing a side which contains four of the vertices. The remaining two vertices are separated by distances of the form $\dfrac{b + c\sqrt{d} + e\sqrt{f}}{2}$ or $\dfrac{e\sqrt{f + g\sqrt{h}}}{2}$. With only five strips they result smaller and useful as diagonals inside rigid polygons.

3 Two triangles with offsets

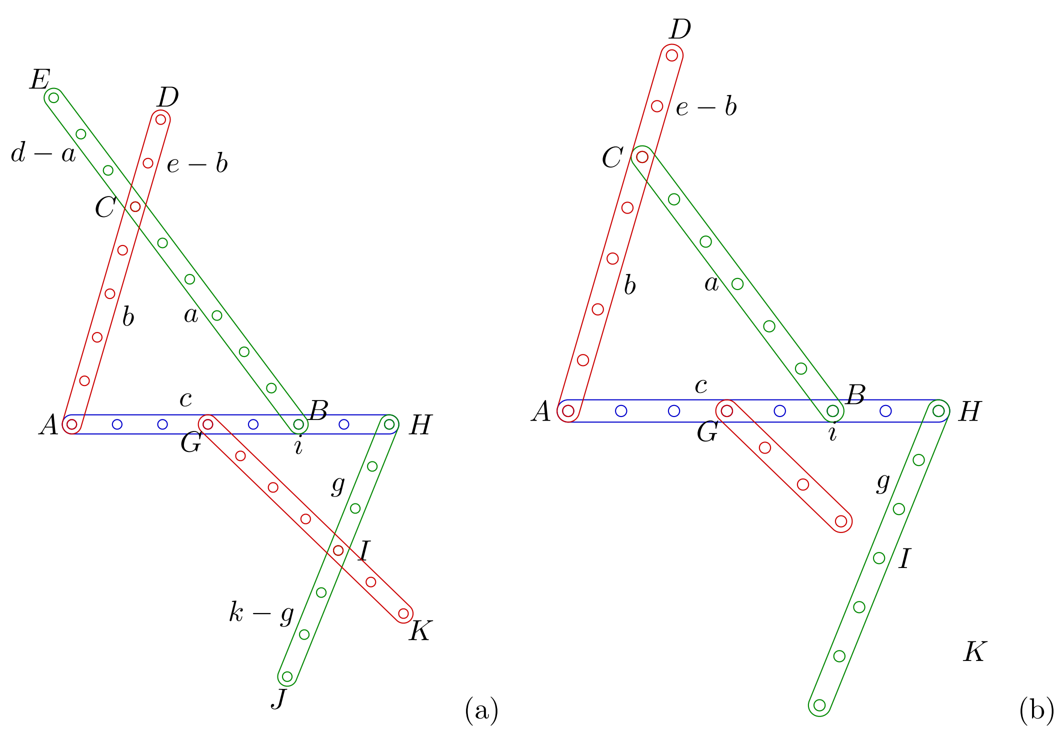

Figure 4: Frame of two triangles with offsets. We construct two triangles $\triangle ABC$ and $\triangle GHI$. Extending the strips we get four vertices E, D, J, K which can form four rigid distances of surd type: $\overline{DJ}, \overline{DK}, \overline{EJ}, \overline{EK}$.

Figure 4 shows a frame with five strips. The frame has eleven variables:

$$a = \overline{BC}, \quad b = \overline{AC}, \quad c = \overline{AB} \tag{23}$$
$$d = \overline{AE}, \quad e = \overline{AD} \tag{24}$$
$$f = \overline{AG} \tag{25}$$
$$g = \overline{HI}, \quad h = \overline{GI}, \quad i = \overline{GH} \tag{26}$$
$$j = \overline{HJ}, \quad k = \overline{HK} \tag{27}$$

Assume vertex A is at the origin. Let $\alpha = \angle BAC$, and D_x, D_y the abscissa and orditate of vertex D so we have:

$$t \equiv b^2 + c^2 - a^2 \tag{28}$$
$$x \equiv 4b^2c^2 - t^2 \tag{29}$$
$$\cos\alpha = \frac{t}{2bc} \tag{30}$$
$$\sin\alpha = \frac{\sqrt{x}}{2bc} \tag{31}$$
$$D_x = d\sin\alpha = \frac{d\sqrt{x}}{2bc} \tag{32}$$
$$D_y = d\cos\alpha = \frac{dt}{2bc} \tag{33}$$
$$D_x^2 + D_y^2 = d^2 \tag{34}$$

Let $\delta = \angle HGI$ and K_x, K_y the abscissa and ordinate of vertex K so we have:

$$v \equiv h^2 + i^2 - g^2 \tag{35}$$
$$y \equiv 4h^2i^2 - v^2 \tag{36}$$
$$\cos\delta = \frac{v}{2hi} \tag{37}$$
$$\sin\delta = \frac{\sqrt{y}}{2hi} \tag{38}$$
$$K_x = f + k\sin\delta = f + \frac{k\sqrt{y}}{2hi} \tag{39}$$
$$K_y = -k\cos\delta = -\frac{kv}{2hi} \tag{40}$$
$$K_x^2 + K_y^2 = f^2 + 2fk\sin\delta + k^2 \tag{41}$$
$$= f^2 + k^2 + \frac{fk\sqrt{y}}{hi} \tag{42}$$

We calculate the distance $\overline{DK}$:

$$\begin{aligned}
\overline{DK}^2 &= (D_x + K_x)^2 + (D_y + K_y)^2 \\
&= D_x^2 + 2D_xK_x + K_x^2 + D_y^2 + 2D_yK_y + K_y^2 \\
&= (D_x^2 + D_y^2) + (K_x^2 + K_y^2) + 2D_xK_x + 2D_yK_y \\
&= d^2 + f^2 + k^2 + \frac{fk\sqrt{y}}{hi} + 2\left(\frac{d\sqrt{x}}{2bc}\right)\left(f + \frac{k\sqrt{y}}{2hi}\right) + 2\left(\frac{dt}{2bc}\right)\left(-\frac{kv}{2hi}\right) \\
&= d^2 + f^2 + k^2 - \frac{dtkv}{2bchi} + \frac{fk\sqrt{y}}{hi} + \frac{df\sqrt{x}}{bc} + \frac{dk\sqrt{xy}}{2bchi}
\end{aligned} \tag{43}$$