Meccano frames

https://github.com/heptagons/meccano/frames

Abstract

Meccano frames are groups of meccano [1] strips intended to be a base to build diverse meccano larger objects.

1 Triangle frame with extensions

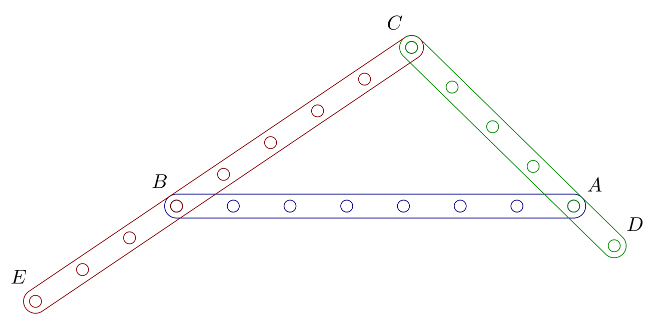


Figure 1: Triangle frame. We have three strips to form the triangle $\triangle ABC$. At least we extend one of two strips $\overline{CB}$ and $\overline{CA}$ to become $\overline{CE}$ and $\overline{CD}$. The new nodes D and E are rigid and the distance between them can be a surd, that we can use to be part of more complicated constructions.

Figure 1 is a triangle with extentions. We'll calculate the distance $\overline{DE}$ to be used as a surd. First we define integer distances a, b, c, d, e and calculate the cosine of $\angle BCA$:

$$a = \overline{CB}, \quad b = \overline{CA} \quad c = \overline{AB} \quad d = \overline{BE}, \quad e = \overline{AD} \tag{1}$$

$$\theta \equiv \angle BCA \tag{2}$$

$$\cos\theta = \frac{a^2 + b^2 - c^2}{2ab} \tag{3}$$

Then we apply the cosine to the triangle $\triangle CED$:

$$\begin{aligned} \overline{ED}^2 &= \overline{CD}^2 + \overline{CE}^2 - 2\overline{CD} \times \overline{CE}\cos\theta \\ &= (a+d)^2 + (b+e)^2 - 2(a+d)(b+e)\cos\theta \end{aligned} \tag{4}$$

So we define the surd $s = \overline{DE}$ as:

$$\begin{aligned} s &= \sqrt{(a+d)^2 + (b+e)^2 - 2(a+d)(b+e)\cos\theta} \\ &= \sqrt{(a+d)^2 + (b+e)^2 - 2(a+d)(b+e)\left(\frac{a^2+b^2-c^2}{2ab}\right)} \end{aligned} \tag{5}$$

[1] Meccano mathematics by ‘t Hooft

1.1 Software

We write a software to find all the triangles with extensions for a particular surd and for a maximum sides $a+d, b+w, c$. For example next list show all the $s=\sqrt{7}$ for the three strips $a+d, b+e, c \leq 10$:

```
sqrt{7} max=10
  1) a=1 b+e=1+2 c=1 cos=1/2
  2) a+d=1+1 b+e=1+2 c=1 cos=1/2
  3) a+d=1+2 b=1 c=1 cos=1/2
  4) a+d=1+2 b+e=1+1 c=1 cos=1/2
  5) a=2 b+e=2+1 c=2 cos=1/2
  6) a+d=2+1 b=2 c=2 cos=1/2
  7) a=3 b+e=2+2 c=2 cos=3/4 E=pi/2
  8) a+d=3+1 b+e=2+1 c=2 cos=3/4 D=pi/2
  9) a+d=4+2 b+e=4+4 c=1 cos=31/32
 10) a+d=4+4 b+e=4+2 c=1 cos=31/32
 11) a=7 b+e=5+1 c=3 cos=13/14
 12) a=7 b+e=5+2 c=3 cos=13/14
```

The code is in file `github.com/heptagons/meccano/frames/frames.go` and function `func (t *Frames) SurdsInt(surd Z, max N32, frame func(a *FrameSurd))`

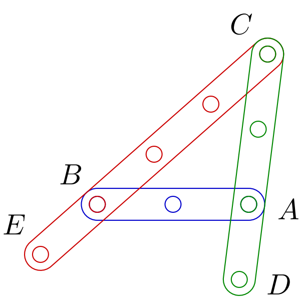
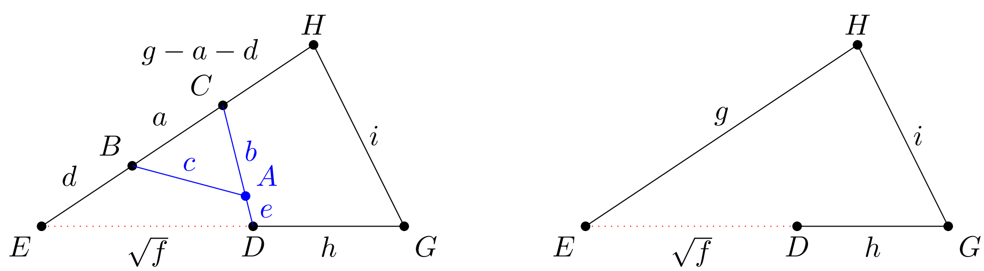


Figure 2: Solution for $\overline{DE}=\sqrt{7}$ when $a+d=3+1, b+e=2+1, c=2$.

2 Algebraic distance not right



(a) (b)

Figure 3: The five strips intented to form an algebraic distance $\sqrt{f}+h$.

From figure 3 (a) we know $\sqrt{f}$ distance between nodes E and D is produced by the three strips frame $a+d$, $b+e$ and c. Using the law of cosines we calculate the angle $\theta = \angle CED$ in terms of $\sqrt{f}$:

$$\begin{aligned}
\cos\theta &= \frac{(a+d)^2 + (\sqrt{f})^2 - (b+e)^2}{2(a+d)\sqrt{f}} \\
&= \frac{((a+d)^2 + f - (b+e)^2)\sqrt{f}}{2(a+d)f} && (6) \\
&= \frac{m\sqrt{f}}{n} && (7) \\
m &= (a+d)^2 + f - (b+e)^2 && (8) \\
n &= 2(a+d)f && (9)
\end{aligned}$$

From figure 3 (a) we notice two sets of points are collinear: $\{E, B, C, H\}$ and $\{E, D, G\}$. Using the law of cosines we calculate the angle $\theta = \angle HEG$ in terms of distances g, h, i:

$$\begin{aligned}
\cos\theta &= \frac{g^2 + (\sqrt{f}+h)^2 - i^2}{2g(\sqrt{f}+h)} \\
&= \frac{g^2 + f + 2\sqrt{f}h + h^2 - i^2}{2g(\sqrt{f}+h)} \\
&= \frac{g^2 + f + h^2 - i^2 + 2\sqrt{f}h}{2g(\sqrt{f}+h)} && (10)
\end{aligned}$$

We multiply both numerator and denominator by $\sqrt{f} - h$ to eliminate the surd from denominator:

$$\begin{aligned}
\cos\theta &= \frac{(f + g^2 + h^2 - i^2)(\sqrt{f}-h) + 2\sqrt{f}h(\sqrt{f}-h)}{2g(\sqrt{f}+h)(\sqrt{f}-h)} \\
&= \frac{(f + g^2 + h^2 - i^2)(\sqrt{f}-h) + 2fh - 2\sqrt{f}h^2}{2g(f-h^2)} \\
&= \frac{-h(f + g^2 + h^2 - i^2 - 2f) + (f + g^2 + h^2 - i^2 - 2h^2)\sqrt{f}}{2g(f-h^2)} \\
&= \frac{h(f - g^2 - h^2 + i^2) + (f + g^2 - h^2 - i^2)\sqrt{f}}{2g(f-h^2)} \\
&= \frac{o + p\sqrt{f}}{q} && (11) \\
o &= h(f - g^2 - h^2 + i^2) && (12) \\
p &= f + g^2 - h^2 - i^2 && (13) \\
q &= 2g(f - h^2) && (14)
\end{aligned}$$

We compare both cosines equations 7 and 11:

$$\frac{m\sqrt{f}}{n} = \frac{o + p\sqrt{f}}{q} \tag{15}$$

Since all variables are integers we need two conditions. First o should be zero. And second $\frac{m}{n} = \frac{p}{q}$.

For condition 1, we force o to be zero:

$$\begin{aligned}
o &= 0 \\
h(f - g^2 - h^2 + i^2) &= 0 \\
f &= g^2 + h^2 - i^2 && (16)
\end{aligned}$$

For condition2, we force m, n, p, q as:

$$\begin{aligned}\frac{m}{n} &= \frac{p}{q} \\ \frac{(a+d)^2 + f - (b+e)^2}{2(a+d)f} &= \frac{f + g^2 - h^2 - i^2}{2g(f - h^2)}\end{aligned} \tag{17}$$

We replace the value of f of last equation RHS with the value of equation 16 of condition 1:

$$\begin{aligned}\frac{(a+d)^2 - (b+e)^2 + f}{(a+d)f} &= \frac{f + g^2 - h^2 - i^2}{g(f - h^2)} \\ &= \frac{g^2 + h^2 - i^2 + g^2 - h^2 - i^2}{g(g^2 + h^2 - i^2 - h^2)} \\ &= \frac{2(g^2 - i^2)}{g(g^2 - i^2)} \\ &= \frac{2}{g} \\ ((a+d)^2 - (b+e)^2 + f)g &= 2(a+d)f\end{aligned} \tag{18}$$