

Meccano frames

<https://github.com/heptagons/meccano/frames>

Abstract

Meccano frames are groups of rigid meccano¹ strips. Can be used as internal diagonals of polygons to be rigid. The lengths of such diagonals are algebraic numbers of the form $\frac{B+C\sqrt{D}}{A}$ or $\frac{E\sqrt{F+H\sqrt{G}}}{A}$.

1 Triangular frame

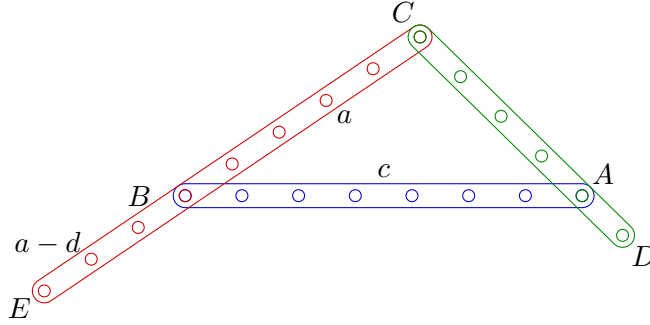


Figure 1: Triangular frame.

Figure 1 shows a triangular frame. With three strips we form the triangle $\triangle ABC$. At least we extend one of the two strips \overline{CB} and \overline{CA} to become \overline{CE} and \overline{CD} . The new vertices D and E distance is rigid of the form $\frac{p\sqrt{s}}{q}$, where $p, q, s \in \mathbb{Z}^+$.

First we identify five integer distances a, b, c, d, e :

$$a \equiv \overline{CB}, \quad b \equiv \overline{CA}, \quad c \equiv \overline{AB}, \quad c < a + b \quad (1)$$

$$d \equiv \overline{CB} + \overline{BE} = \overline{CE} \geq a \quad (2)$$

$$e \equiv \overline{CA} + \overline{AD} = \overline{CD} \geq b \quad (3)$$

We calculate the cosine of $\angle BCA$:

$$\theta \equiv \angle BCA \quad (4)$$

$$\cos \theta = \frac{a^2 + b^2 - c^2}{2ab} \quad (5)$$

Then we apply the cosine to the triangle $\triangle CED$ to get the extensions distance \overline{DE} :

$$\begin{aligned} \overline{DE}^2 &= \overline{CD}^2 + \overline{CE}^2 - 2\overline{CD} \times \overline{CE} \cos \theta \\ &= d^2 + e^2 - 2de \cos \theta \\ &= d^2 + e^2 - de \left(\frac{a^2 + b^2 - c^2}{ab} \right) \end{aligned} \quad (6)$$

¹ Meccano mathematics by 't Hooft

We extract the square root:

$$\begin{aligned}
\overline{DE} &= \sqrt{d^2 + e^2 - de \left(\frac{a^2 + b^2 - c^2}{ab} \right)} \\
&= \frac{\sqrt{a^2 b^2 (d^2 + e^2) - abde(a^2 + b^2 - c^2)}}{ab} \\
&= \frac{\sqrt{ab((ad - be)(bd - ae) + c^2 de)}}{ab}
\end{aligned} \tag{7}$$

1.1 Software

We write a software to report all the triangle frames with specific surd \sqrt{s} for a given maximum strips length. We can reject cases $q \neq 1$ and s not square-free. Next list show all the triangles with $q = 1$ and $s = \sqrt{7}$ where $c < a + b$, $a \leq d \leq \max$, $b \leq e \leq \max$, $c \leq \max$:

```

1  === RUN    TestFramesTriangleSurds
2  NewFrames().TriangleSurds surd=7 max=15
3      1) a=1 e=1+2 c=1 cos=1/2
4      2) d=1+1 e=1+2 c=1 cos=1/2
5      3) d=1+2 b=1 c=1 cos=1/2
6      4) d=1+2 e=1+1 c=1 cos=1/2
7      5) a=2 e=2+1 c=2 cos=1/2
8      6) d=2+1 b=2 c=2 cos=1/2
9      7) a=3 e=2+2 c=2 cos=3/4 CED=pi/2
10     8) d=3+1 e=2+1 c=2 cos=3/4 CDE=pi/2
11     9) d=4+2 e=4+4 c=1 cos=31/32
12    10) d=4+4 e=4+2 c=1 cos=31/32
13    11) a=7 e=5+1 c=3 cos=13/14
14    12) a=7 e=5+2 c=3 cos=13/14

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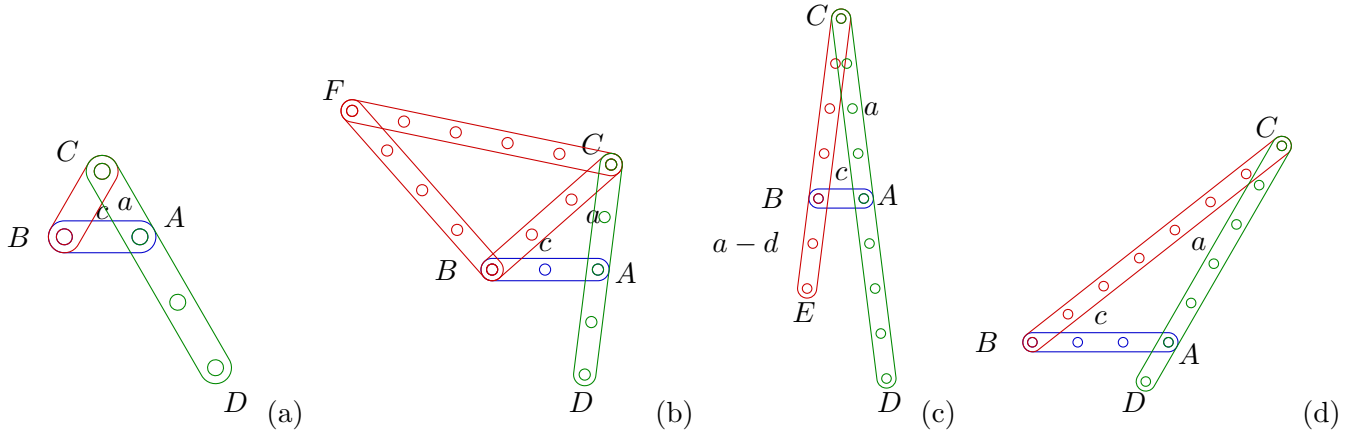


Figure 2: Some triangular frames with rigid distance $\overline{DE} = \sqrt{7}$ found by the software.

Figure 2 show four cases of this list. The code is in the folder github.com/heptagons/meccano/frames.

1.2 Triangular distance of the form $\sqrt{s} + f$

In the figure 2, the particular case (b), was reported with the angle $CED = \pi/2$ which means we can append two extra strips to make a pythagorean triangle $\triangle CEF$ where angle $CEF = \pi/2$, which makes the three vertices D, E, F collinear, so the rigid distance $\overline{DF} = \sqrt{7} + 4$ is an algebraic number.

1.3 Another rigid distances $\sqrt{s} + h$

We explore a more complicated frame to get additional cases of distances $\sqrt{s} + h$ without relying in an explicit pythagorean triangle as we saw in case (b) of figure 2.

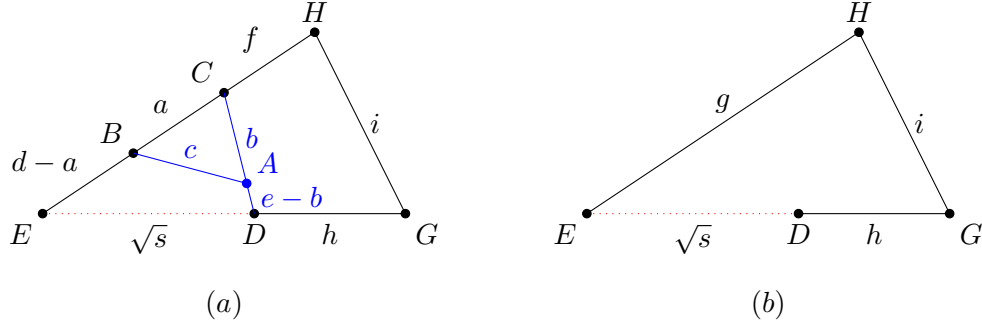


Figure 3: The five strips intended to form an algebraic distance $\overline{EG} = \sqrt{s} + h$.

From figure 3 (a) we know \sqrt{s} distance between nodes E and D is produced by the three strips frame $a + d$, $b + e$ and c . Using the law of cosines we calculate the angle $\theta = \angle CED$ in terms of \sqrt{s} :

$$\begin{aligned} \cos \theta &= \frac{d^2 + (\sqrt{s})^2 - e^2}{2d\sqrt{s}} \\ &= \frac{(d^2 + s - e^2)\sqrt{s}}{2ds} \end{aligned} \tag{8}$$

$$= \frac{m\sqrt{s}}{n} \tag{9}$$

$$m = d^2 + s - e^2 \tag{10}$$

$$n = 2ds \tag{11}$$

From figure 3 (a) we notice two sets of points are collinear: $\{E, B, C, H\}$ and $\{E, D, G\}$. Using the law of cosines we calculate the angle $\theta = \angle HEG$ in terms of distances g , $\sqrt{s} + h$, i :

$$\begin{aligned} \cos \theta &= \frac{g^2 + (\sqrt{s} + h)^2 - i^2}{2g(\sqrt{s} + h)} \\ &= \frac{g^2 + s + 2\sqrt{s}h + h^2 - i^2}{2g(\sqrt{s} + h)} \\ &= \frac{g^2 + s + h^2 - i^2 + 2\sqrt{s}h}{2g(\sqrt{s} + h)} \end{aligned} \tag{12}$$

We multiply both numerator and denominator by $\sqrt{s} - h$ to eliminate the surd from denominator:

$$\begin{aligned}
\cos \theta &= \frac{(s + g^2 + h^2 - i^2)(\sqrt{s} - h) + 2\sqrt{s}h(\sqrt{s} - h)}{2g(\sqrt{s} + h)(\sqrt{s} - h)} \\
&= \frac{(s + g^2 + h^2 - i^2)(\sqrt{s} - h) + 2sh - 2\sqrt{s}h^2}{2g(s - h^2)} \\
&= \frac{-h(s + g^2 + h^2 - i^2 - 2s) + (s + g^2 + h^2 - i^2 - 2h^2)\sqrt{s}}{2g(s - h^2)} \\
&= \frac{h(s - g^2 - h^2 + i^2) + (s + g^2 - h^2 - i^2)\sqrt{s}}{2g(s - h^2)} \\
&= \frac{o + p\sqrt{s}}{q}
\end{aligned} \tag{13}$$

$$o = h(s - g^2 - h^2 + i^2) \tag{14}$$

$$p = s + g^2 - h^2 - i^2 \tag{15}$$

$$q = 2g(s - h^2) \tag{16}$$

We compare both cosines equations 9 and 13:

$$\frac{m\sqrt{s}}{n} = \frac{o + p\sqrt{s}}{q} \tag{17}$$

Since all variables are integers we need two conditions. First o should be zero. And second $\frac{m}{n} = \frac{p}{q}$.

For condition 1, we force o to be zero:

$$\begin{aligned}
o &= 0 \\
h(s - g^2 - h^2 + i^2) &= 0 \\
s &= g^2 + h^2 - i^2
\end{aligned} \tag{18}$$

For condition2, we force m, n, p, q as:

$$\begin{aligned}
\frac{m}{n} &= \frac{p}{q} \\
\frac{d^2 + s - e^2}{2ds} &= \frac{s + g^2 - h^2 - i^2}{2g(s - h^2)}
\end{aligned} \tag{19}$$

We replace the value of s of last equation RHS with the value of equation 18 of condition 1:

$$\begin{aligned}
\frac{d^2 - e^2 + s}{ds} &= \frac{s + g^2 - h^2 - i^2}{g(s - h^2)} \\
&= \frac{g^2 + h^2 - i^2 + g^2 - h^2 - i^2}{g(g^2 + h^2 - i^2 - h^2)} \\
&= \frac{2(g^2 - i^2)}{g(g^2 - i^2)} \\
&= \frac{2}{g} \\
(d^2 - e^2 + s)g &= 2ds
\end{aligned} \tag{20}$$

TODO : Examples!!!

2 Triangle pair frame

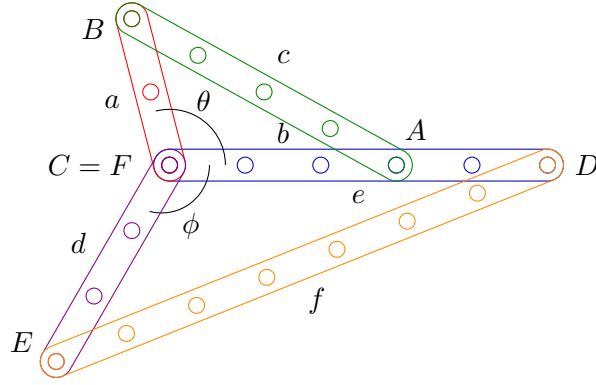


Figure 4: Triangle pair frame.

Figure 4 shows a triangle pair frame. The pair joins triangles $\triangle ABC$ and $\triangle DEF$ in such a way vertices C and F coincide and vertices A, C, D, F be collinear. With only five strips this frame is small and useful to make up the rigid polygons diagonals of the form $\overline{BE} = \frac{z_2\sqrt{z_3 + z_4\sqrt{z_5}}}{z_1}, z_i \in \mathbb{Z}$.

2.1 Triangle pair algebra

Using the law of cosines we calculate the angle $\theta = \angle ACB$ with defined variables m, n and the angle $\phi = \angle DFE$ with defined variables o, p :

$$(m, n) \equiv (a^2 + b^2 - c^2, 2ab), \quad |m| \leq n, \quad m, n \in \mathbb{Z} \quad (21)$$

$$\cos \theta = \frac{m}{n} \quad (22)$$

$$\sin \theta = \sqrt{1 - \cos^2 \theta} = \frac{\sqrt{n^2 - m^2}}{n} \quad (23)$$

$$(o, p) \equiv (d^2 + e^2 - f^2, 2de), \quad |o| \leq p, \quad o, p \in \mathbb{Z} \quad (24)$$

$$\cos \phi = \frac{o}{p} \quad (25)$$

$$\sin \phi = \sqrt{1 - \cos^2 \phi} = \frac{\sqrt{p^2 - o^2}}{p} \quad (26)$$

Then, we use the cosines sum identity:

$$\begin{aligned} \cos(\theta + \phi) &= \cos \theta \cos \phi - \sin \theta \sin \phi \\ &= \left(\frac{m}{n}\right) \left(\frac{o}{p}\right) - \left(\frac{\sqrt{n^2 - m^2}}{n}\right) \left(\frac{\sqrt{p^2 - o^2}}{p}\right) \\ &= \frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{np} \end{aligned} \quad (27)$$

Finally we can calculate the distance $g \equiv \overline{BE}$ using the law of cosines:

$$\begin{aligned}
g &\equiv \overline{BE} \\
&= \sqrt{a^2 + d^2 - 2ad \cos(\theta + \phi)} \\
&= \sqrt{a^2 + d^2 - 2ad \left(\frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{np} \right)} \\
&= \sqrt{a^2 + d^2 - 2ad \left(\frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{4abde} \right)} \\
&= \sqrt{a^2 + d^2 - \frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{2be}} \\
&= \frac{\sqrt{4b^2e^2(a^2 + d^2) - 2bem o + 2be\sqrt{(n^2 - m^2)(p^2 - o^2)}}}{2be}
\end{aligned} \tag{28}$$

2.2 Triangle pairs software

From the last equation of g we identify three **input** integers i_1, i_2, i_3 which are used to get $g(i)$. Then the nested radicals software returns **output** integers z_1, z_2, z_3, z_4, z_5 as $g(z)$:

$$i_1 \equiv 2be \tag{29}$$

$$i_2 \equiv i_1^2(a^2 + d^2) - i_1mo \tag{30}$$

$$i_3 \equiv (n^2 - m^2)(p^2 - o^2) \tag{31}$$

$$g(i) = \frac{\sqrt{i_2 + i_1\sqrt{i_3}}}{i_1} \tag{32}$$

$$g(z) = \frac{z_2\sqrt{z_3 + z_4\sqrt{z_5}}}{z_1} \tag{33}$$

We run a program to print a list of triangle pairs with sides $1 < a, b, c, d, e, f \leq max$ having a given distance $\overline{BE} = g$ or particular z_3, z_4, z_5 . Next example request a pairs list with $g = z_2\sqrt{46 + 18\sqrt{5}}/z_1$ up to strip length 10 so we set as limits $max = 10, z_3 = 46, z_4 = 18, z_5 = 5$ to get next report (text in blue):

Folder : github.com/heptagons/meccano/frames

Call : `NewFrames().TrianglePairsTex(10, [46 18 5])`

$$\begin{aligned}
(a, b, c) \oplus (d, e, f) &\mapsto g \\
(2, 1, 2) \oplus (3, 3, 3) &\mapsto \frac{\sqrt{46 + 18\sqrt{5}}}{2} \\
(2, 1, 2) \oplus (3, 8, 7) &\mapsto \frac{\sqrt{46 + 18\sqrt{5}}}{2} \\
(2, 2, 2) \oplus (3, 6, 6) &\mapsto \frac{\sqrt{46 + 18\sqrt{5}}}{2} \\
(2, 3, 4) \oplus (3, 5, 7) &\mapsto \frac{\sqrt{46 + 18\sqrt{5}}}{2} \\
(2, 4, 4) \oplus (3, 8, 7) &\mapsto \frac{\sqrt{46 + 18\sqrt{5}}}{2} \\
(3, 3, 3) \oplus (2, 4, 4) &\mapsto \frac{\sqrt{46 + 18\sqrt{5}}}{2} \\
(4, 2, 4) \oplus (6, 6, 6) &\mapsto \sqrt{46 + 18\sqrt{5}} \\
(4, 4, 4) \oplus (6, 7, 8) &\mapsto \sqrt{46 + 18\sqrt{5}} \\
(6, 3, 6) \oplus (4, 4, 4) &\mapsto \sqrt{46 + 18\sqrt{5}} \\
(6, 3, 6) \oplus (9, 9, 9) &\mapsto \frac{3\sqrt{46 + 18\sqrt{5}}}{2} \\
(6, 6, 6) \oplus (4, 8, 8) &\mapsto \sqrt{46 + 18\sqrt{5}} \\
(6, 7, 8) \oplus (9, 9, 9) &\mapsto \frac{3\sqrt{46 + 18\sqrt{5}}}{2}
\end{aligned}$$

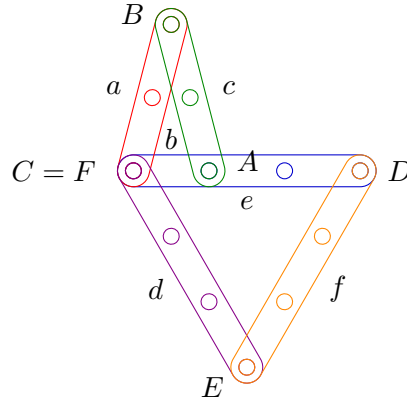


Figure 5: Triangle pair frame $(2, 1, 2) \oplus (3, 3, 3)$ makes $\overline{BE} = \frac{\sqrt{46 + 18\sqrt{5}}}{2}$.

In figure 5 we build a triangular pair following one of the last report results, when $abc = (2, 1, 2)$ and $def = (3, 3, 3)$.

3 Triangle pair extended frame

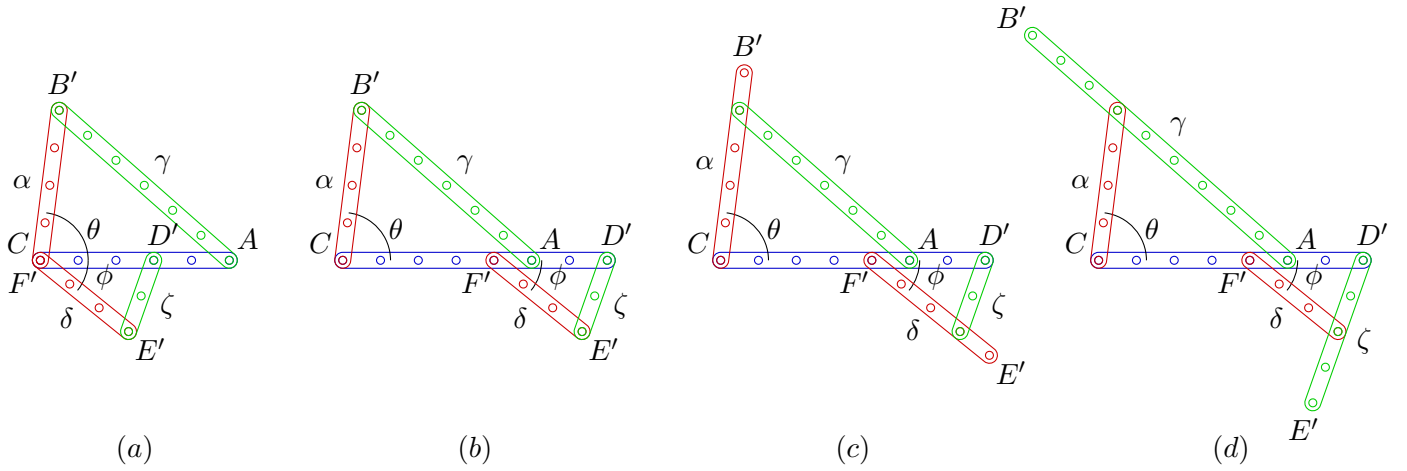


Figure 6: Triangle pair extended frame. Is like previous triangle pair frame except we can extend strips α or γ , δ or ζ , and we can separate vertices C and F' . Vertices A, C, D', F' remain collinear and we are interested in the distance $B'E'$. We show four examples: (a) is the original triangle pair, (b) has moved the $\triangle D'E'F'$ to the right, (c) also extends strips α and δ and (d) extends strips γ and ζ .

The triangle pair extended frame is shown in figure 6. As with figure 4 we also have two triangles with five strips, but we can do up to three transformations:

1. Separate nodes C and F which moves $\triangle D'E'F'$.
2. Extends strip $a \rightarrow \alpha$ or strip $c \rightarrow \gamma$ but not both.
3. Extend strip $d \rightarrow \delta$ or strip $f \rightarrow \zeta$ but not both.

We define three integers x, y_1, y_2 to do the transformations:

$$x = \begin{cases} 0 & C, F \text{ vertices remain joined} \\ \geq 0 & \triangle DEF \text{ is moved to the right a distance equal to } x \end{cases} \quad (34)$$

$$y_1 = \begin{cases} 0 & \alpha = a, \quad \gamma = c \\ > 0 & \alpha = a + y_1, \quad \gamma = c \\ < 0 & \alpha = a, \quad \gamma = c + |y_1| \end{cases} \quad (35)$$

$$y_2 = \begin{cases} 0 & \delta = d, \quad \zeta = f \\ > 0 & \delta = d + y_2, \quad \zeta = f \\ < 0 & \delta = d, \quad \zeta = f + |y_2| \end{cases} \quad (36)$$

Let define $M(a, b, c)$ the triangle above, $N(d, e, f)$ the triangle below and $T(x, y_1, y_2)$ the transformations. Then we can describe the cases (a) – (d) of figure 6 as operations:

$$\begin{aligned} (a) &: M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(0, 0, 0) \\ (b) &: M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(4, 0, 0) \\ (c) &: M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(4, +2, +1) \\ (d) &: M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(4, -3, -2) \end{aligned}$$

3.1 Triangle pair extended frame algebra

We are going to calculate the distance $\overline{B'E'}$ of the triangle pair extended using the M, N, T values. We start setting the vertex C at the origin of the standard two-dimensional graph and defining (B_x, B_y) the abscissa and ordinate of vertex B' and (E_x, E_y) the abscissa and ordinate of vertex E' .

3.1.1 (B_x, B_y)

We use the triangle above $M(a, b, c)$ and make the extension $u = |y_1|$ and obtain (B_x, B_y) . We already calculated angle $\theta = \angle ACB$ for the original triangle pair using the variables m, n of equation 21.

$$a_1 \equiv b^2 + c^2 - a^2, \quad a_2 \equiv 2bc, \quad \text{iff } |a_1| \leq a_2 \quad (37)$$

$$\cos A = \frac{a_1}{a_2}, \quad \sin A = \frac{\sqrt{a_2^2 - a_1^2}}{a_2} \quad (38)$$

$$\alpha = a + u \quad (39)$$

$$\gamma = c + u \quad (40)$$

$$B_x = \begin{cases} y_1 \geq 0 & \alpha \cos \theta \\ y_1 < 0 & b - \gamma \cos A \end{cases} \quad (41)$$

$$B_y = \begin{cases} y_1 \geq 0 & \alpha \sin \theta \\ y_1 < 0 & \gamma \sin A \end{cases} \quad (42)$$

For the triangle below $N(d, e, f)$ we make $n = |y_2|$ and obtain (E_x, E_y) :

$$d_1 \equiv e^2 + f^2 - d^2, \quad d_2 \equiv 2ef \quad (43)$$

$$f_1 \equiv d^2 + e^2 - f^2, \quad f_2 \equiv 2de \quad (44)$$

$$\cos D = \frac{d_1}{d_2}, \quad \sin D = \frac{\sqrt{d_2^2 - d_1^2}}{d_2} \quad (45)$$

$$\cos F = \frac{f_1}{f_2}, \quad \sin F = \frac{\sqrt{f_2^2 - f_1^2}}{f_2} \quad (46)$$

$$\delta = d + n \quad (47)$$

$$\zeta = f + n \quad (48)$$

$$E_x = \begin{cases} y_2 \geq 0 & x + \delta \cos F \\ y_2 < 0 & x + e - \zeta \cos D \end{cases} \quad (49)$$

$$E_y = \begin{cases} y_2 \geq 0 & -\delta \sin F \\ y_2 < 0 & -\zeta \sin D \end{cases} \quad (50)$$

With the four components B_x, B_y, E_x, E_y we can calculate $\overline{B'E'}$:

$$\overline{B'E'} = \sqrt{(B_x + E_x)^2 + (B_y + E_y)^2} \quad (51)$$

$$= \sqrt{(B_x^2 + B_y^2) + (E_x^2 + E_y^2) + 2B_xE_x + 2B_yE_y} \quad (52)$$

For $y_1 \geq 0$ and $y_2 \geq 0$ we have $m = y_1, n = y_2$:

$$\alpha = a + m, \quad \delta = d + n \quad (53)$$

$$\begin{aligned} B_x^2 + B_y^2 &= \alpha^2 \cos^2 C + \alpha^2 \sin^2 C \\ &= \alpha^2 \end{aligned} \quad (54)$$

$$\begin{aligned} E_x^2 + E_y^2 &= (x + \delta \cos F)^2 + (-\delta \sin F)^2 \\ &= x^2 + 2x\delta \cos F + \delta^2 \cos^2 F + \delta^2 \sin^2 F \\ &= x^2 + 2x\delta \cos F + \delta^2 \\ &= \frac{f_2 x^2 + 2x\delta f_1 + f_2 \delta^2}{f_2} \end{aligned} \quad (55)$$

$$\begin{aligned} B_x E_x &= (\alpha \cos C)(x + \delta \cos F) \\ &= \frac{\alpha c_1 (f_2 x + \delta f_1)}{c_2 f_2} \end{aligned} \quad (56)$$

$$\begin{aligned} B_y E_y &= (\alpha \sin C)(-\delta \sin F) \\ &= -\frac{\alpha \delta \sqrt{(c_2^2 - c_1^2)(f_2^2 - f_1^2)}}{c_2 f_2} \end{aligned} \quad (57)$$

$$\begin{aligned} g &= \overline{B'E'} \\ &= \sqrt{\alpha^2 + \frac{f_2 x^2 + 2x\delta f_1 + f_2 \delta^2}{f_2} + \frac{2\alpha c_1 (f_2 x + \delta f_1)}{c_2 f_2} - \frac{2\alpha \delta \sqrt{(c_2^2 - c_1^2)(f_2^2 - f_1^2)}}{c_2 f_2}} \\ &= \sqrt{\frac{c_2 f_2 \alpha^2 + c_2 (f_2 x^2 + 2x\delta f_1 + f_2 \delta^2) + 2\alpha c_1 (f_2 x + \delta f_1) - 2\alpha \delta \sqrt{(c_2^2 - c_1^2)(f_2^2 - f_1^2)}}{c_2 f_2}} \end{aligned} \quad (58)$$