Meccano pentagons gallery

https://github.com/heptagons/meccano/penta/gallery

Abstract

We show constructions of meccano rigid regular pentagons from side 12 to 3. We restrict all internal strips, we call diagonals, to remain inside the pentagon's perimeter and that don't overlap. Several programs found the solutions and we show some alternatives and prove the claimed values are exact.

1 Pentagons of size 12

A program found that side 12 is the smallest pentagon that can be made rigid with a rhoumbus and two strips as diagonals so need only 4 strips as diagonals. We show two cases.

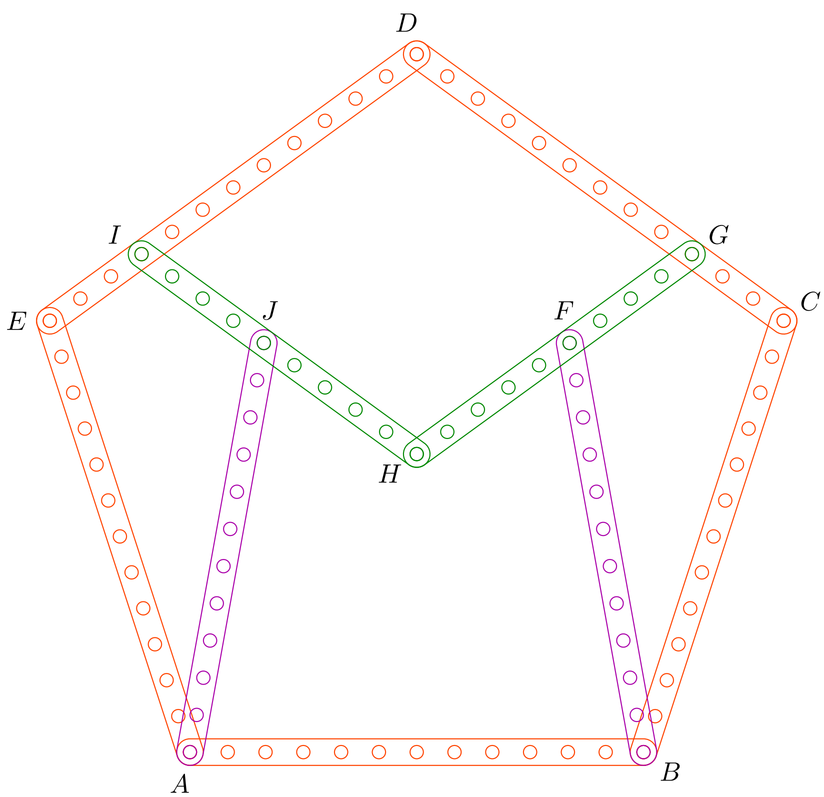

Figure 1: Pentagon size 12 (case a).

Figure 1 show a regular pentagon A, B, C, D, E of side 12 with a rhombus D, I, H, G of side 9. We prove strips AJ, BF are correct. First we calculate the abscissas going through vertices A, E, I, J substracting when we move to the left and adding when we move to the right:

$$\begin{aligned} AJ_x &= AE_x + EI_x + IJ_x \qquad (1) \\ &= -\overline{AE}\cos\left(\frac{2\pi}{5}\right) + \overline{EI}\cos\left(\frac{\pi}{5}\right) + \overline{IJ}\cos\left(\frac{\pi}{5}\right) \\ &= -12\left(\frac{\sqrt{5}-1}{4}\right) + 3\left(\frac{1+\sqrt{5}}{4}\right) + 4\left(\frac{1+\sqrt{5}}{4}\right) = \frac{19-5\sqrt{5}}{4} \qquad (2) \end{aligned}$$

Then we calculate the ordinates going to the same order of vertices adding when we go up and substracting when we go down:

$$\begin{aligned} AJ_y &= -AE_y + EI_y + IJ_y \qquad (3) \\ &= \overline{AE}\sin\left(\frac{2\pi}{5}\right) + \overline{EI}\sin\left(\frac{\pi}{5}\right) - \overline{IJ}\sin\left(\frac{\pi}{5}\right) \\ &= 12\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right) + 3\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right) - 4\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right) \\ &= \frac{12\sqrt{10+2\sqrt{5}} - \sqrt{10-2\sqrt{5}}}{4} = \frac{\sqrt{1450+190\sqrt{5}}}{4} \qquad (4) \end{aligned}$$

Finally we calculate the distance $\overline{AJ}$ which coincides with strip size 11:

$$\begin{aligned} \overline{AJ} &= \sqrt{(AJ_x)^2 + (AJ_y)^2} \qquad (5) \\ &= \sqrt{\left(\frac{19-5\sqrt{5}}{4}\right)^2 + \frac{1450+190\sqrt{5}}{16}} \\ &= \sqrt{\frac{486-190\sqrt{5}}{16} + \frac{1450+190\sqrt{5}}{16}} = \sqrt{121} = 11 \qquad (6) \end{aligned}$$

Figure 2 show a regular pentagon A, B, C, D, E of size 12 with a rhombus D, I, H, G of size 12. We prove strips GH, IJ are correct. First we calculate the abscissas going through vertices G, A, E, H substracting when we move to the left and adding when we move to the right:

$$\begin{aligned} GH_x &= -GA_x - AE_x + EH_x \qquad (7) \\ &= -\overline{GA} - \overline{AE}\cos\left(\frac{2\pi}{5}\right) + \overline{EH}\cos\left(\frac{\pi}{5}\right) \\ &= -4 - 12\left(\frac{\sqrt{5}-1}{4}\right) + 3\left(\frac{1+\sqrt{5}}{4}\right) = \frac{-1-9\sqrt{5}}{4} \qquad (8) \end{aligned}$$

Then we calculate the ordinates going to the same order of vertices adding when we go up and substracting when we go down:

$$\begin{aligned} GH_y &= AG_y + AE_y - EH_y \qquad (9) \\ &= 0 + \overline{AE}\sin\left(\frac{2\pi}{5}\right) - \overline{EH}\sin\left(\frac{\pi}{5}\right) \\ &= 12\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right) - 3\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right) \\ &= \frac{12\sqrt{10+2\sqrt{5}} - 3\sqrt{10-2\sqrt{5}}}{4} = \frac{\sqrt{1530-18\sqrt{5}}}{4} \qquad (10) \end{aligned}$$

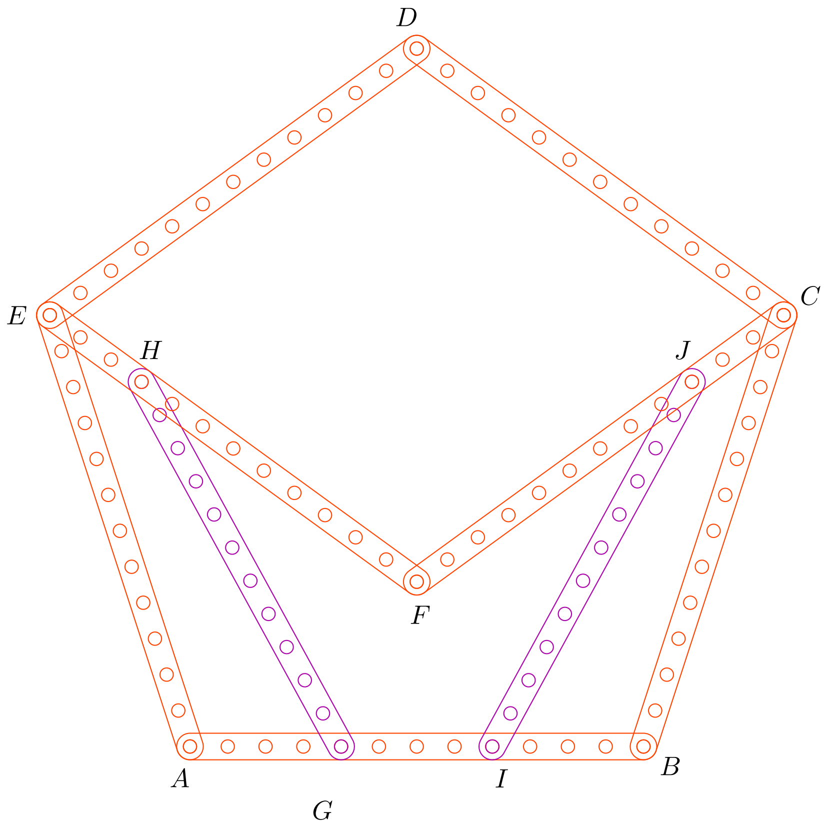

Figure 2: Pentagon size 12 case (b).

Finally we calculate the distance $\overline{GH}$ which coincides with strip size 11:

$$\overline{GH} = \sqrt{(GH_x)^2 + (GH_y)^2} \tag{11}$$

$$= \sqrt{\left(\frac{-1-9\sqrt{5}}{4}\right)^2 + \frac{1530-18\sqrt{5}}{16}}$$

$$= \sqrt{\frac{406+18\sqrt{5}}{16} + \frac{1530-18\sqrt{5}}{16}} = \sqrt{121} = 11 \tag{12}$$

2 Pentagon of size 11

Figure 3 show a rigid regular pentagon A, B, C, D, E of size 11. A program found this is the smallest pentagon having a consecutive sides diagonal of the form $\frac{z_2 + z_3\sqrt{5}}{z_1}$ instead of the nested form $\frac{z_2\sqrt{z_3 + z_4\sqrt{5}}}{z_1}$ where z_i are integers (next pentagon of this type has side 246). The mentioned diagonal is the distance $\overline{CF}$ in the figure which can be calculated with the law of cosines knowing angle $\angle CBF = \frac{3\pi}{5}$ and denesting

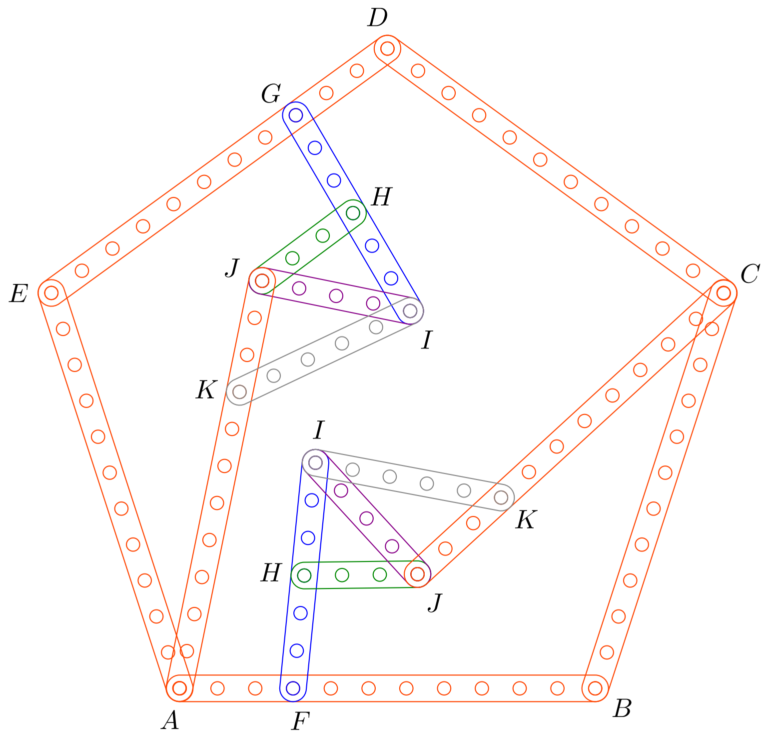

Figure 3: Pentagon size 11.

the result:

$$\overline{CF}^2 = \overline{BC}^2 + \overline{BF}^2 - 2(\overline{BC})(\overline{BF})\cos\left(\frac{3\pi}{5}\right) \tag{13}$$

$$= 11^2 + 8^2 - 2(11)(8)\left(\frac{1-\sqrt{5}}{4}\right) = 141 + 44\sqrt{5}$$

$$\overline{CF} = \sqrt{141 + 44\sqrt{5}} = 11 + 2\sqrt{5} \tag{14}$$

2.1 Five strips builds distance $a + b\sqrt{c}$

A five strips cluster can create a rigid distance like $11+2\sqrt{5}$. In the figure, three strips $\overline{FI} = 2\overline{HJ}, \overline{FI} > \overline{IJ}$ builds a right angle $\angle FJI = \pi$, since triangle $\triangle IJH$ is isosceles ($\overline{FH} = \overline{HI} = \overline{JH}$). These three strips also build a distance $\overline{FJ} = \sqrt{\overline{FI}^2 - \overline{IJ}^2} = \sqrt{6^2 - 4^2} = 2\sqrt{5}$. Now we attach strip $\overline{CJ}$ making a second right triangle $\angle CJI = \pi$ using strip $\overline{IK} = 5$ as pythagorean diagonal ($\overline{JK} = 3, \overline{IJ} = 4$). We have two right triangles at vertice J so vertices F, J, C are collinear, so we can calculate the distance $\overline{FC} = \overline{CJ} + \overline{JF} = 11 + 2\sqrt{5}$. We repeat the five-strips cluster between vertices A, G preventing overlaps of any strips. Since the clusters are rigid we formed two rigid triangles $\triangle ABC, \triangle DEA$ so the pentagon is rigid.