Meccano hexagons

https://github.com/heptagons/meccano/hexa

Abstract

We construct meccano[1] regular hexagons. We use six equal strips to build the polygon perimeter and then we attach **internal diagonals** to make the polygon regular and rigid. Common diagonals called **regular diagonals** are aligned with the common equilateral triangular grid and **irregular diagonals** are not and are more interesting. To find the irregular diagonals we develop algebraic formulas and run a program to do the search. Basically the problem is to find triangles with the three integer sides and one angle exactly of 120°.

1 Meccano hexagons

1.1 Regular diagonals

A meccano hexagon can be build easily attaching sufficient equilateral triangles as small as one unit side. Also joining six strips to form a perimeter and using two more strips as **regular diagonals**, which means both diagonals are aligned along the triangular grid. Regular diagonals join opposite hexagon sides.

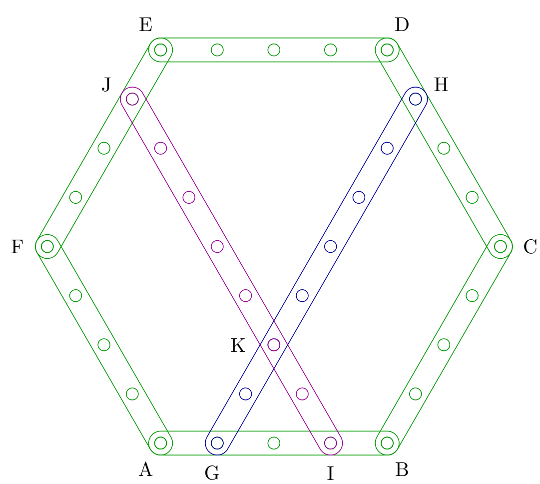

Figure 1: Hexagon of size 4 with two **regular diagonals** of size 7.

Consider figure 1. Start with strip $\overline{AB}$ and add two strips $\overline{GH}$ and $\overline{IJ}$ to form a triangle with three bolts at points G, I and K. At this moment, perimeter points A, B, H and G are rigid.

Then add perimeter strips $\overline{BC}$ and $\overline{CD}$ with a bolt at H. In the same way add perimeter strips $\overline{AF}$ and $\overline{EF}$ with a bolt at J. Finally add strip $\overline{DE}$ with bolts at D and E.

[1] Meccano mathematics by ‘t Hooft

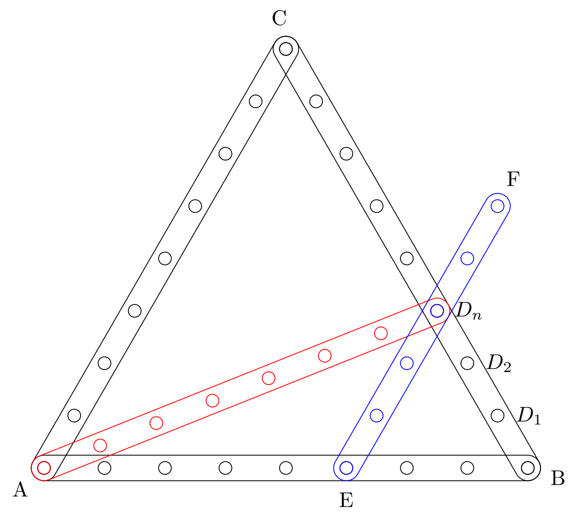

Figure 2: The red strip is an irregular diagonal, with integer length and joining two adjacent hexagon sides $\overline{AE}$ and $\overline{EF}$.

1.2 Irregular diagonals

While regular diagonals are aligned along the triangular grid, **irregular diagonals** don't. Irregular diagonals join two adjacent hexagon sides, making (rigid) irregular triangles.

Consider figure 2. Start with an equilateral triangle ABC of side $\overline{AB}$. Test one by one the irregular diagonals from the point A to the points D_1, D_2..., D_n which are over the strip $\overline{BC}$. Define the tree variables to use:

$$a = \overline{AB}$$
$$b = \overline{BD_n}$$
$$d = \overline{AD_n}$$

Acording to the cosines law and knowing the $\angle EBD_n = 60°$, calculate d:

$$\begin{aligned} d &= \sqrt{a^2 + b^2 - 2ab\cos\frac{\pi}{3}} \\ &= \sqrt{a^2 + b^2 - ab} \\ &= \sqrt{(a-b)^2 + ab} \end{aligned}$$

Reject any non-integer diagonal d, since any meccano strip length should be an integer. For the valid diagonal such as $\overline{AD_n}$, locate a point E over the strip $\overline{AB}$ such so the distance $\overline{BE}$ equals the distance $\overline{BD_n}$. From the point E create a new strip $\overline{EF}$ passing over the point D_n (blue strip in the figure). Finally we got a valid **irregular diagonal** d for the pair of adjacent hexagon sides $\overline{AE}$ and $\overline{EF}$.

1.3 Irregular diagonals program

We need a program to iterate over integer a, then over integer b to test whether d value is an integer too. Next golang program find the diagonals. We iterate from $a = 1$ to a given maximum (line 2). Then we iterate over $1 < b \leq a/2$ (line 3), to avoid repeating symmetric values. In order to reject repetitions by scaling we check for greatest commond divisor of a and b to be 1 (line 4). Then we calculate the diagonal using the formula $d^2 = (a-b)^2 + ab$ (line 5) and report only the case when the diagonal is a square number (line 8).

```
1  func triangle_diagonals(max int) {
2    for a := 1; a < max; a++ {
3      for b := 1; b <= a/2; b++ {
4        if gcd(a, b) == 1 {
5          diag := (a-b)*(a-b) + a*b
6          cd := math.Sqrt(float64(diag))
7          d := int(cd)
8          if cd == float64(d) {
9            num := float64(diag + a*a - b*b)
10           den := 2.0 * cd * float64(a)
11           angle := 180*math.Acos(num/den)/math.Pi
12           fmt.Printf("a=%3d b=%3d d=%3d angle=%8.4f\n", a, b, d, angle)
13         }
14       }
15     }
16   }
17 }
18 func gcd(a, b int) int { // greatest common divisor
19   if b == 0 {
20     return a
21   }
22   return gcd(b, a % b)
23 }
```

1.4 Irregular diagonals results

The program found 13 distinct irregular diagonals for sides $a \leq 100$. Next table show the results including the angle EAD_n needed for the latex drawing scripts.

```
1  a=  8 b=  3 d=  7 angle= 21.7868
2  a= 15 b=  7 d= 13 angle= 27.7958
3  a= 21 b=  5 d= 19 angle= 13.1736
4  a= 35 b= 11 d= 31 angle= 17.8966
5  a= 40 b=  7 d= 37 angle=  9.4300
6  a= 48 b= 13 d= 43 angle= 15.1782
7  a= 55 b= 16 d= 49 angle= 16.4264
8  a= 65 b=  9 d= 61 angle=  7.3410
9  a= 77 b= 32 d= 67 angle= 24.4327
10 a= 80 b= 17 d= 73 angle= 11.6351
11 a= 91 b= 40 d= 79 angle= 26.0078
12 a= 96 b= 11 d= 91 angle=  6.0090
13 a= 99 b= 19 d= 91 angle= 10.4174
```

1.5 Examples of result 1

Result 1 reports $a = 8$, $b = 3$ and $d = 7$, so the diagonal is of length 7 and the minimum hexagon size is $a - b = 5$. Figure 3 shows the smallest hexagon with irregular diagonals. In figure 4, the side is incremented to 6 and in figure 5 the side is incremented to 7 so all hexagon's strips are of the same length. Finally in figure 6, the size is incremented to 8 and we see two hexagons at the same time of sizes 7 and 8.

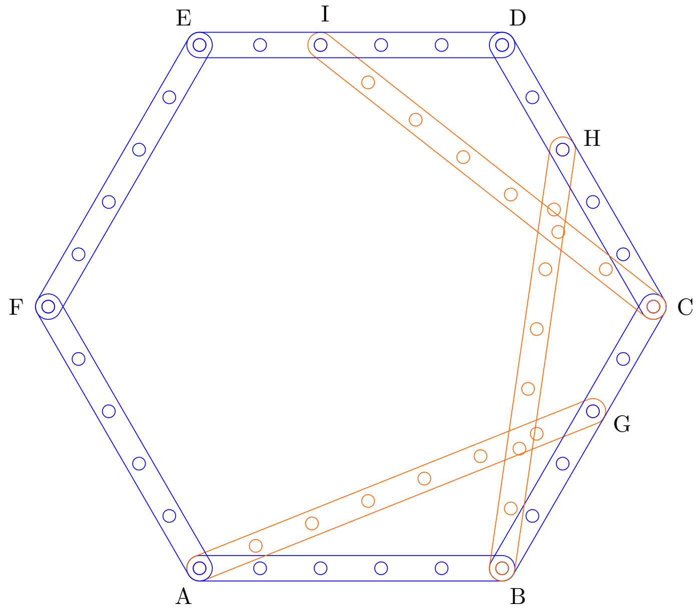

Figure 3: Result 1. Hexagon side lenght: 5, diagonal length: 7. This is the smallest hexagon of the results. The number of bolts is at minimum, 9.

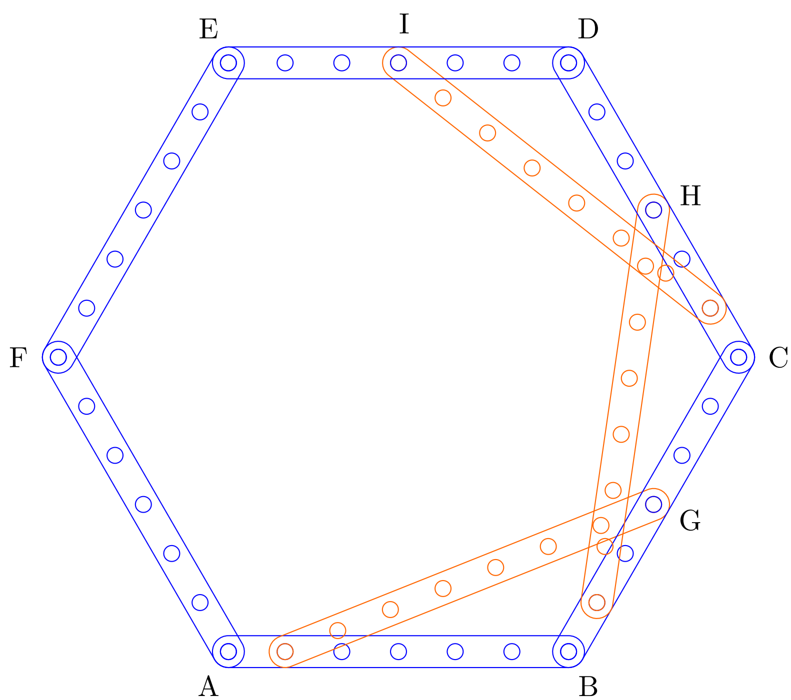

Figure 4: Result 1. Hexagon side length: $5 + 1 = 6$, diagonal length: 7. Has the same diagonals of figure 3 but the perimeter was increased by one. We need here 12 bolts.

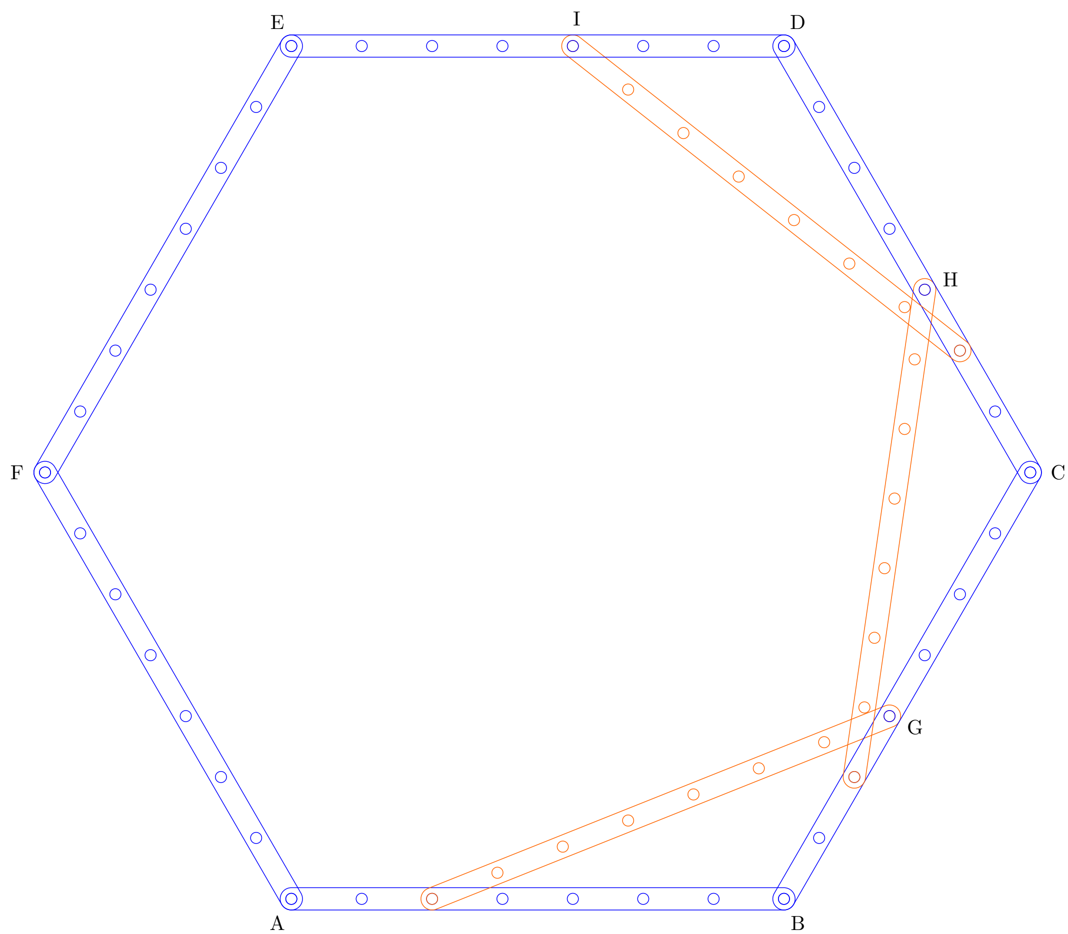

Figure 5: Result 1: Hexagon sides $5+2=7$, diagonals 7. This is an interesting puzzle. Given nine strips of length 7, build a rigid regular hexagon. Notice how separated the holes should be in the strips to prevent bolts conflicts near points G and H.

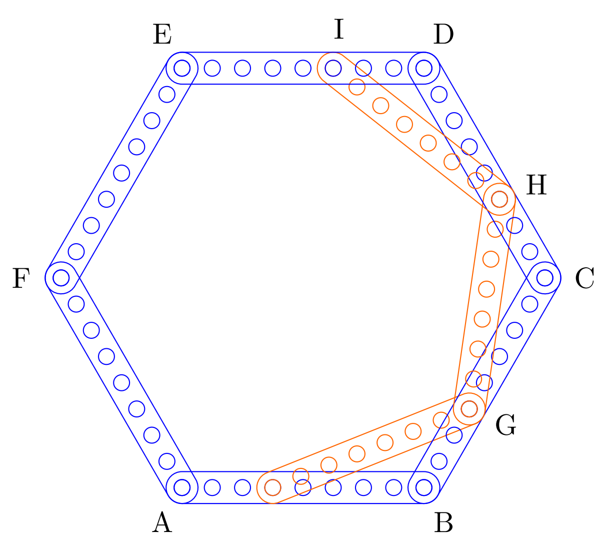

Figure 6: Result 1: Hexagon sides $5+3=8$, diagonals 7. This case is interesting because adding three more orange strips we can make two rigid hexagons only when both share bolts.

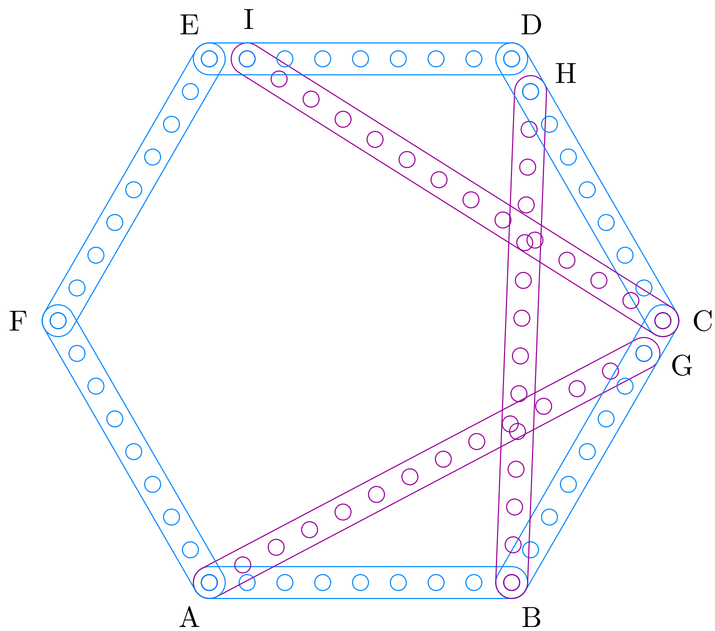

Figure 7: Result 2. Hexagon sides: 8, diagonals: 13.

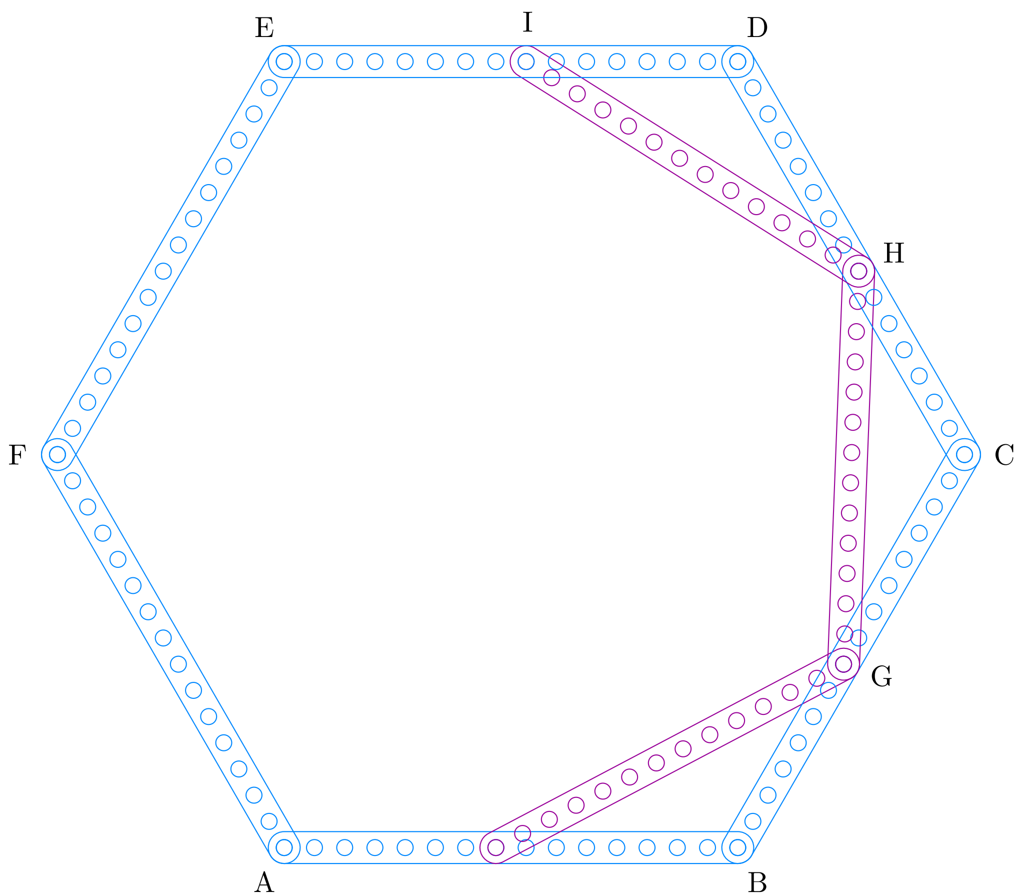

Figure 8: Hexagon sides $8 + 7 = 15$, diagonals 13.

1.6 Examples of result 2

Result 2 reports $a = 15$, $b = 7$ and $d = 13$, so the diagonal is of length 13 and the minimum hexagon size is $a - b = 8$. Figure 7 shows the smallest hexagon with irregular diagonal 13 and figure 8 extends the side from 8 to 15 and we see two hexagons at the same time of sizes 13 and 15.

1.7 Results longer list

This list shows $s = a - b$ which is the side of each hexagon, b the segment of the second side and the diagonal d which is the triangle side opposite to the angle of 120°. $d > s > b$.

```
s=  5 b=  3 d=  7
s=  8 b=  7 d= 13
s= 16 b=  5 d= 19
s= 24 b= 11 d= 31
s= 33 b=  7 d= 37
s= 35 b= 13 d= 43
s= 39 b= 16 d= 49
s= 56 b=  9 d= 61
s= 45 b= 32 d= 67
s= 63 b= 17 d= 73
s= 51 b= 40 d= 79
s= 85 b= 11 d= 91
s= 80 b= 19 d= 91
s= 57 b= 55 d= 97
s= 77 b= 40 d=103
s= 95 b= 24 d=109
s=120 b= 13 d=127
s=120 b= 23 d=133
s= 88 b= 65 d=133
s= 91 b= 69 d=139
s=143 b= 25 d=157
s=115 b= 56 d=151
s=161 b= 15 d=169
s=112 b= 75 d=163
s=175 b= 32 d=193
s=105 b=104 d=181
s=165 b= 56 d=199
s=195 b= 29 d=211
s=208 b= 17 d=217
s=160 b= 87 d=217
s=168 b= 85 d=223
s=224 b= 31 d=241
s=145 b=119 d=229
s=203 b= 72 d=247
s=261 b= 19 d=271
s=187 b= 93 d=247
s=221 b= 64 d=259
s=155 b=144 d=259
s=217 b= 95 d=277
s=279 b= 40 d=301
s=288 b= 35 d=307
s=192 b=133 d=283
s=320 b= 21 d=331
s=209 b=136 d=301
s=247 b=105 d=313
s=323 b= 37 d=343
s=272 b=105 d=337
s=280 b=111 d=349
s=315 b= 88 d=367
s=385 b= 23 d=397
s=231 b=185 d=361
s=273 b=152 d=373
s=259 b=176 d=379
s=357 b= 80 d=403
s=399 b= 41 d=421
s=333 b=115 d=403
s=407 b= 48 d=433
s=304 b=161 d=409
s=352 b=123 d=427
s=456 b= 25 d=469
s=440 b= 43 d=463
s=253 b=240 d=427
s=299 b=205 d=439
s=391 b=129 d=469
s=437 b= 88 d=487
s=287 b=240 d=457
s=369 b=175 d=481
s=336 b=215 d=481
s=451 b=104 d=511
s=533 b= 27 d=547
s=528 b= 47 d=553
s=301 b=275 d=499
s=325 b=264 d=511
s=387 b=208 d=523
s=473 b=135 d=553
s=425 b=184 d=541
s=559 b= 56 d=589
s=475 b=141 d=559
s=575 b= 49 d=601
s=440 b=189 d=559
s=616 b= 29 d=631
s=416 b=235 d=571
s=368 b=297 d=577
s=520 b=147 d=607
s=351 b=329 d=589
s=552 b=145 d=637
```