Triple unit

https://github.com/heptagons/meccano/units/triple

Abstract

A **Triple unit** is a group of **five** meccano [1] strips a, b, c, d, e forming **three equal angles** θ intended to build three consecutive perimeter sides of some regular polygons. We look for integer values of strip e in function of integer values of sides a, b, c, d and a particular angle θ. We confirm a generic equation found matches the one used to build pentagons of type 2 [2]. Here we found a lot of hexagons and filter some not trivial solutions. We look for octagons, decagons and dodecagons.

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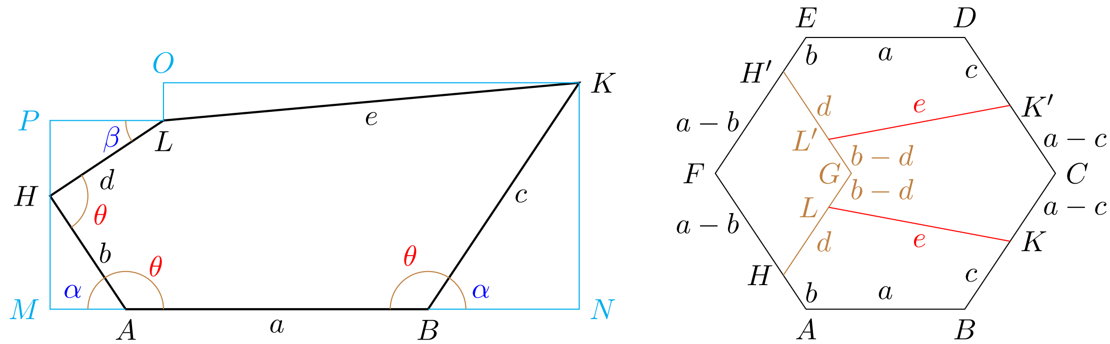

Figure 1: At the left we have the Triple unit (three angles θ) with the strips a, b, c, d, e. At the right we use two units to build a regular polygon of side a extending strips b, c, d to fix everthing.

1 Algebra

From nodes A and B of fig 1 we get α from θ ($\pi = 180°$):

$$\begin{aligned} \theta &= \pi - \alpha \\ \alpha &= \pi - \theta \end{aligned} \tag{1}$$

And from node H we get β from θ:

$$\begin{aligned} \theta &= \alpha + \beta \\ \beta &= \theta - \alpha = \theta - (\pi - \theta) = 2\theta - \pi \end{aligned} \tag{2}$$

We calculate horizontal segment $\overline{OK}$:

$$\begin{aligned} \overline{OK} &= \overline{MA} + a + \overline{BN} - \overline{PL} \\ &= b\cos\alpha + a + c\cos\alpha - d\cos\beta \\ &= a + (b+c)\cos\alpha - d\cos\beta \\ &= a + (b+c)\cos(\pi - \theta) - d\cos(2\theta - \pi) \\ &= a - (b+c)\cos\theta + d\cos(2\theta) \end{aligned} \tag{3}$$

[1] Meccano mathematics by ‘t Hooft
[2] Meccano pentagons

And vertical segment $\overline{OL}$:

$$\begin{aligned}
\overline{OL} &= \overline{KN} - \overline{PH} - \overline{HM} \\
&= c\sin\alpha - d\sin\beta - b\sin\alpha \\
&= (c-b)\sin\alpha - d\sin\beta \\
&= (c-b)\sin(\pi-\theta) - d\sin(2\theta-\pi) \\
&= (c-b)\sin\theta + d\sin(2\theta) \qquad (4)
\end{aligned}$$

So we can express e in function of a, b, c, d and angle θ:

$$\begin{aligned}
e^2 &= (\overline{OK})^2 + (\overline{OL})^2 \\
&= (a-(b+c)\cos\theta + d\cos(2\theta))^2 + ((c-b)\sin\theta + d\sin(2\theta))^2 \\
&= a^2 + (b^2+2bc+c^2)\cos^2\theta + d^2\cos^2(2\theta) + (c^2-2cb+b^2)sin^2\theta + d^2\sin^2(2\theta) \\
&\quad - 2a(b+c)\cos\theta + 2ad\cos(2\theta) - 2(b+c)d\cos\theta\cos(2\theta) \\
&\quad + 2(c-b)d\sin\theta\sin(2\theta) \\
&= a^2+b^2+c^2+d^2 + 2bc\cos^2\theta - 2bc\sin^2\theta \\
&\quad - 2a(b+c)\cos\theta + 2ad\cos(2\theta) \\
&\quad - 2d((b+c)\cos\theta\cos(2\theta) + (b-c)\sin\theta\sin(2\theta)) \qquad (5) \\
&= a^2+b^2+c^2+d^2 + 2bc(\cos^2\theta - \sin^2\theta) - 2a(b+c)\cos\theta + 2ad\cos(2\theta) \\
&\quad - 2d(b(\cos\theta\cos(2\theta) + \sin\theta\sin(2\theta)) + c(\cos\theta\cos(2\theta) - \sin\theta\sin(2\theta))) \\
&= a^2+b^2+c^2+d^2 + 2bc\cos(2\theta) - 2a(b+c)\cos\theta + 2ad\cos(2\theta) \\
&\quad - 2d(b\cos(\theta-2\theta) + c\cos(\theta+2\theta)) \\
&= a^2+b^2+c^2+d^2 + 2(bc+ad)\cos(2\theta) - 2a(b+c)\cos\theta - 2d(b\cos\theta + c\cos(3\theta)) \\
&= a^2+b^2+c^2+d^2 + 2(bc+ad)\cos(2\theta) - 2(ab+ac)\cos\theta - 2(bd\cos\theta + cd\cos(3\theta)) \\
&= \boxed{a^2+b^2+c^2+d^2 - 2(ab+ac+bd)\cos\theta + 2(bc+ad)\cos(2\theta) - 2cd\cos(3\theta)} \qquad (6)
\end{aligned}$$

2 Regular polygons

Polygon	θ	$\cos\theta$	$\cos(2\theta)$	$\cos(3\theta)$
Pentagon	$\frac{3\pi}{5}$	$\frac{1-\sqrt{5}}{4}$	$\frac{-1-\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$
Hexagon	$\frac{2\pi}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1
Heptagon	$\frac{5\pi}{7}$			
Octagon	$\frac{3\pi}{4}$			
Decagon	$\frac{4\pi}{5}$			
Dodecagon	$\frac{5\pi}{6}$			

Table 1: Regular polygons internal angles and cosines.

2.1 Equilateral pentagon

We replace the cosines for pentagon in table 1 in e^2 equation:

$$\begin{aligned}
e^2 &= a^2+b^2+c^2+d^2-2(ab+ac+bd)\cos\theta+2(bc+ad)\cos(2\theta)-2cd\cos(3\theta) \\
&= a^2+b^2+c^2+d^2-2(ab+ac+bd)\left(\frac{1-\sqrt{5}}{4}\right)+2(bc+ad)\left(\frac{-1-\sqrt{5}}{4}\right)-2cd\left(\frac{1+\sqrt{5}}{4}\right) \\
&= a^2+b^2+c^2+d^2-\frac{ab+ac+bd+bc+ad+cd}{2}+\frac{ab+ac+bd-bc-ad-cd}{2}\sqrt{5} \qquad (7)
\end{aligned}$$

e cannot to be and integer if the factor of $\sqrt{5}$ is not zero so we force this factor to be zero:

$$\begin{aligned}
ab+ac+bd-bc-ad-cd &= 0 \\
ab+ac+bd &= bc+ad+cd \qquad (8)
\end{aligned}$$

We replace $ab+ac+bd$ by $bc+ad+cd$ in the e^2 equation to get:

$$\begin{aligned}
e^2 &= a^2+b^2+c^2+d^2-\frac{(bc+ad+cd)+bc+ad+cd}{2}+\frac{0}{2}\sqrt{5} \\
&= a^2+b^2+c^2+d^2-bc-ad-cd \\
e &= \sqrt{a^2+b^2+c^2+d^2-bc-ad-cd} \qquad (9)
\end{aligned}$$

The last formula matches the formula used in the paper Meccano pentagons which finds several pentagons of type 2. Only when we get e integer we have a solution.

2.2 Equilateral hexagon

We replace the cosines for hexagon in table 1 in e^2 equation:

$$\begin{aligned}
e^2 &= a^2+b^2+c^2+d^2-2(ab+ac+bd)\cos\theta+2(bc+ad)\cos(2\theta)-2cd\cos(3\theta) \\
&= a^2+b^2+c^2+d^2-2(ab+ac+bd)\left(-\frac{1}{2}\right)+2(bc+ad)\left(-\frac{1}{2}\right)-2cd(1) \\
&= a^2+b^2+c^2+d^2+ab+ac+bd-bc-ad-2cd \\
&= (a+b)^2+(c-d)^2-ab+ac+bd-bc-ad \\
&= (a+b)^2+(c-d)^2+(c-d)(a-b)-ab \\
&= (a+b)^2+(c-d)(a-b+c-d)-ab \\
e &= \sqrt{(a+b)^2+(c-d)(a-b+c-d)-ab} \qquad (10)
\end{aligned}$$

We wrote software code to look for hexagons using the formula for e and set several filters to prevent trivial solutions. We say an hexagon is nice when $e \le a$. Next is a partial list of nice hexagons:

```
  1   a=  7  b=  1  c=  2  d=  6  e=  7
  2   a=  7  b=  1  c=  4  d=  6  e=  7
  3   a= 13  b=  2  c=  5  d= 11  e= 13
  4   a= 13  b=  2  c=  6  d= 11  e= 13
  5   a= 14  b=  1  c=  6  d= 13  e= 13
  6   a= 14  b=  1  c=  7  d= 13  e= 13
  7   a= 15  b=  1  c=  5  d= 14  e= 14
  8   a= 15  b=  1  c=  9  d= 14  e= 14
  9   a= 19  b=  2  c=  3  d= 17  e= 19
 10   a= 19  b=  2  c= 14  d= 17  e= 19
 11   a= 20  b=  1  c=  4  d= 19  e= 19
```

```
 12   a= 20  b=  1  c= 15  d= 19  e= 19
      ...
105   a= 58  b=  5  c= 10  d= 53  e= 57
106   a= 58  b=  5  c= 43  d= 53  e= 57
107   a= 59  b=  1  c= 27  d= 58  e= 52
108   a= 59  b=  1  c= 31  d= 58  e= 52
109   a= 59  b=  4  c= 11  d= 55  e= 57
110   a= 59  b=  4  c= 44  d= 55  e= 57
111   a= 59  b=  5  c= 19  d= 54  e= 56
112   a= 59  b=  5  c= 35  d= 54  e= 56
--- PASS: TestHexagonsNice (0.01s)
```

Results from `github.com/heptagons/meccano/units/triple/triple_test.go TestHexagonsNice`

The results has related pairs and there are several ways to build an hexagon from each pair.

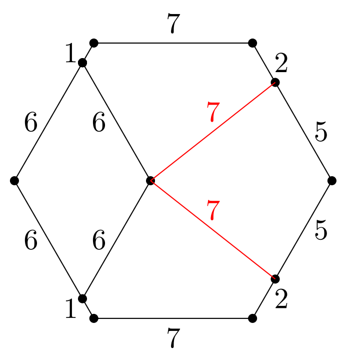

Figure 2: Hexagon.