Meccano hexagons gallery

https://github.com/heptagons/meccano/hexa/gallery

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Abstract

We build meccano [1] rigid regular hexagons from sides 4 to 24. The perimeter has six equal strips and for their rigidity we add some internal strips as diagonals under three conditions: Will remain totally inside the perimeter, don't overlap with any other and must not be parallel to any external strip. With algebra and then sofware, we produce `hexagonal triplets` to make triangles having an internal angle of 120° used as the base of the internal strips.

1 Hexagonal triplets

We look strips that can make rigid two consecutive internal sides the regular hexagon. Figure 1 show the first four cases found. From any figure we have the internal hexagon angle is $\theta \equiv \angle GBC = 2\pi/3$. First we define the hexagon side as $p \equiv \overline{BC}$. From the triangle $\triangle GBC$ we define the other two sides as $b \equiv \overline{GB}$ and $c = \overline{GC}$. By the law of cosines we know that:

$$\begin{aligned} c &= \sqrt{b^2 + p^2 - 2bp\cos\theta} \\ &= \sqrt{b^2 + p^2 - 2bp\left(-\frac{1}{2}\right)} \\ &= \sqrt{b^2 + p^2 + bp} \end{aligned} \tag{1}$$

Then we defie $a \equiv p + b$ and we get:

$$c = \sqrt{a^2 + b^2 - ab} \quad \texttt{where}\ c > a > b \tag{2}$$

a	b	c	p
8	3	7	5
15	7	13	8
21	5	19	16
35	11	31	24
40	7	37	33
48	13	43	35

Table 1: Hexagonal triplets $c > p > b$ as the sides of a triangle with an internal angle $2\pi/3$.

A software iterates first $0 < a < max$ and then $1 < b < a$ and record all c that is an integer. The first cases of such triangles with sides $c > p > b$ are shown table 1 and we call them `Hexagonal triplets`.

Figure 1 shows hexagons of sizes $p = \{5, 8, 16, 24\}$ with perimeter strips in orange made rigid adding three internal green strips of lenght $c = \{7, 13, 19, 31\}$. In the figure we have also an equilateral triangle

[1] Meccano mathematics by ‘t Hooft

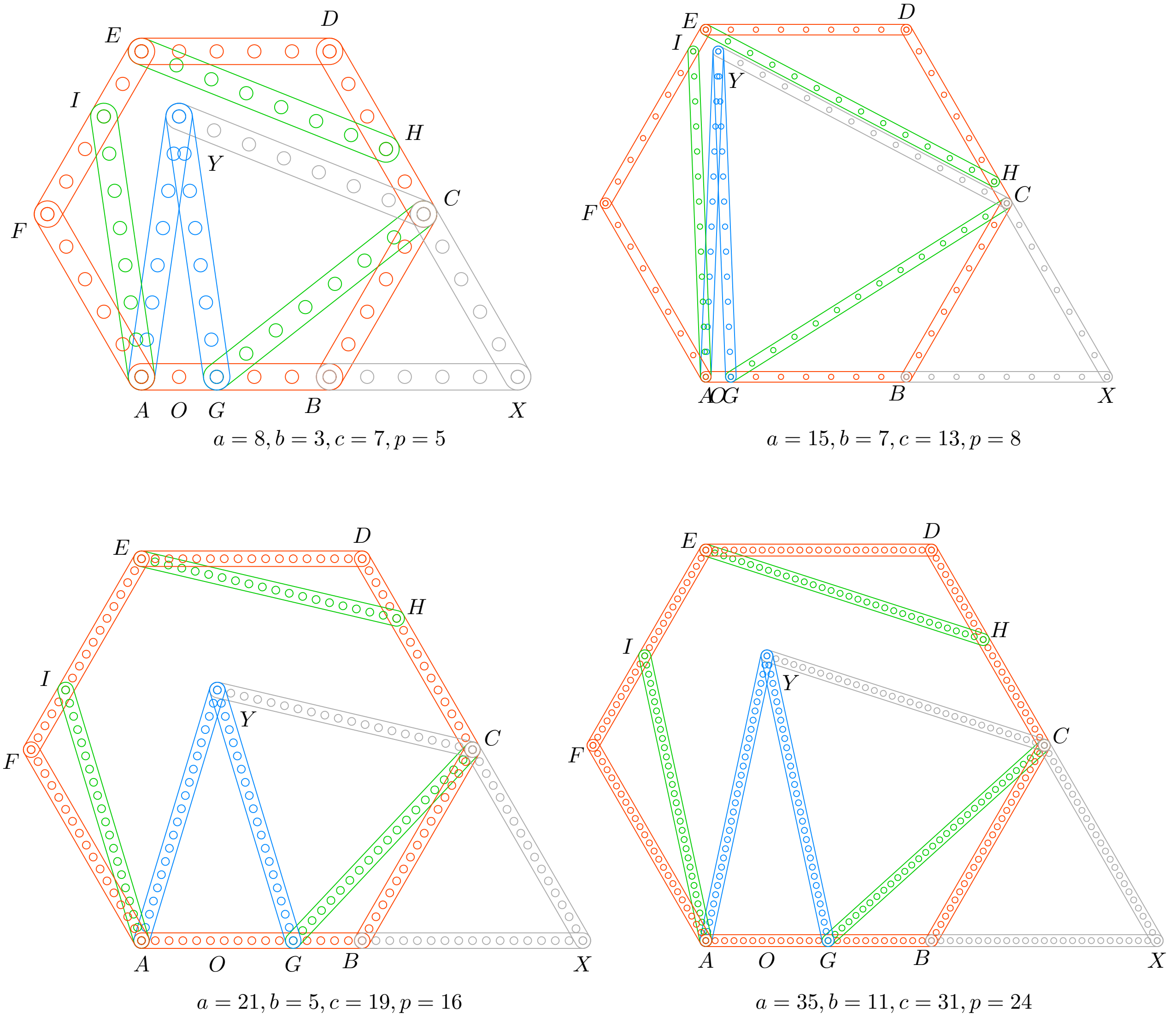

Figure 1: First four cases where internal strip $c = \overline{GC}$ is an integer and makes rigid two consecutive regular hexagon sides $p = \overline{AB} = \overline{BC}$. Our software inspect two integers $a > b$ and looks for $c = \sqrt{a^2 + b^2 - ab}$ to be an integer. In the figures $b = \overline{GB}$ and $a = \overline{GX} = p + b$.

$\triangle GCY$ and an isosceles triangle $\triangle AGY$. The base of the isosceles triangle is $x \equiv \overline{AG} = \overline{AB} - \overline{GB} = p - b$ and the equals sides are $\overline{AY} = \overline{GY} = c$. So we can calculate the height $y \equiv \overline{OY}$ substituting c using equation 1:

$$\begin{aligned} y &= \sqrt{(\overline{GY})^2 - (\overline{AO})^2} \\ &= \sqrt{c^2 - \left(\frac{p-b}{2}\right)^2} \\ &= \sqrt{b^2 + p^2 + bp - \left(\frac{p-b}{2}\right)^2} = \frac{(p+b)\sqrt{3}}{2} = \frac{a\sqrt{3}}{2} \end{aligned} \tag{3}$$

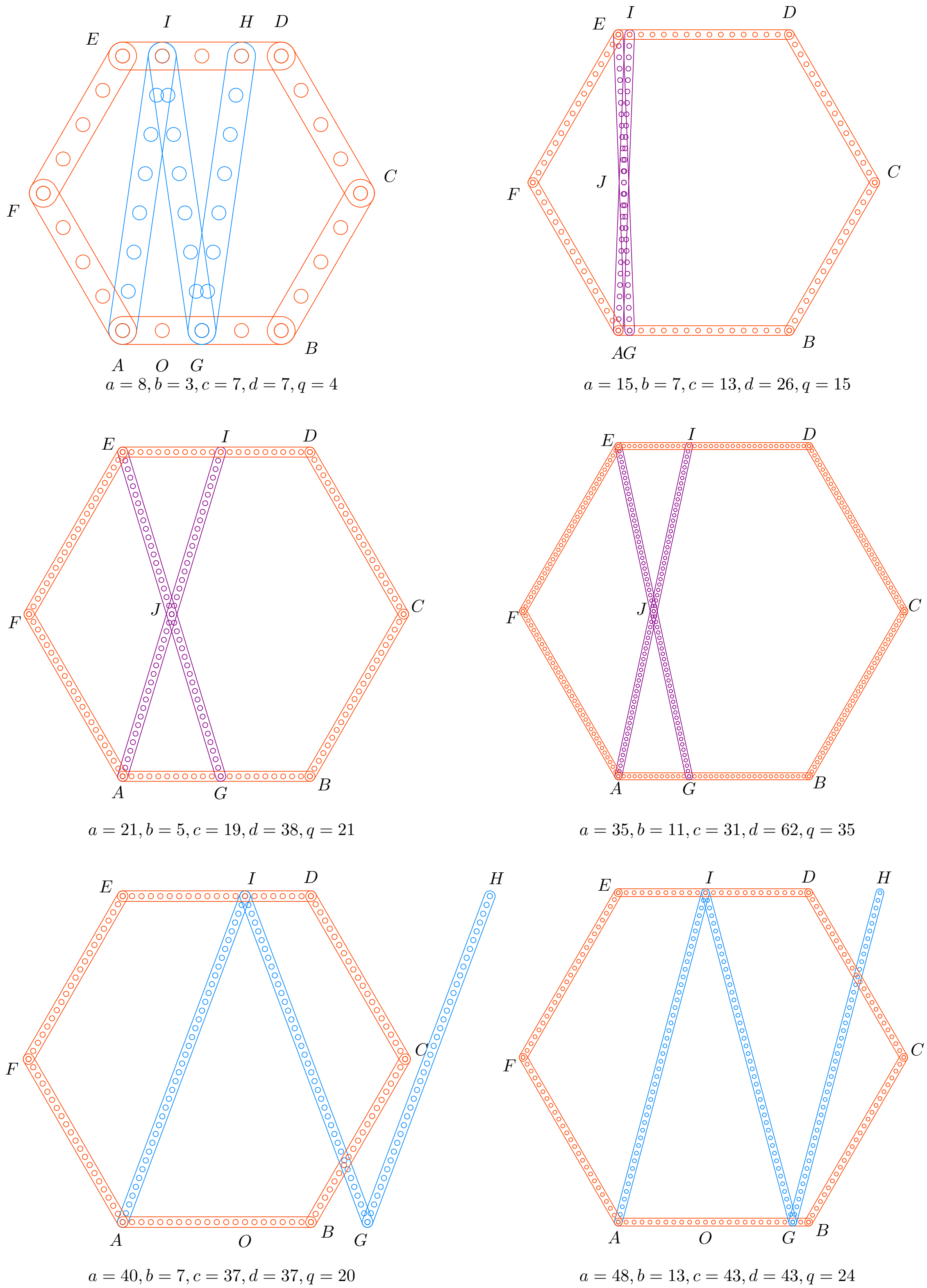

Figure 2: First six cases of integral distances c. When the distance $p - b = \overline{AG}$ is even, we use the strips $d = c$ (in blue) to join opposites sides of hexagons of side $q = a/2$. When is odd, we use the strips $d = 2c$ (in purple) to join opposites sides of hexagons of side $q = a$.

We know $\dfrac{a\sqrt{3}}{2}$ is the height of the regular hexagon of side $\dfrac{a}{2}$ so we can use the blue strips to connect opposite sides. Figure 2 show the smaller hexagons that have integer strips connecting opposites sides.

Through the gallery we will use the green, blue and purple strips and their scaled copies as internal diagonals to make rigid regular hexagons from size 4 to 24. We prioritize minimum number of strips and bolts and the largest strips sizes as possible.

2 Hexagons of size $s < 10$

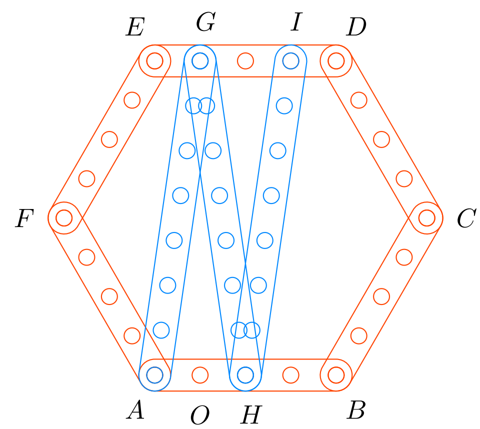

Figure 3: Hexagon of size $s = 4$ with three diagonals $c = \overline{GH} = \overline{HI} = \overline{IG} = 7$.

Figure 3 show a regular hexagon $ABCDEF$ of size 4 with three diagonals of size 7. We confirm the height of the hexagon since the distance $\overline{OG} = \sqrt{(\overline{AG})^2 - (\overline{AO})^2} = \sqrt{7^2 - 1^2} = 4\sqrt{3}$.

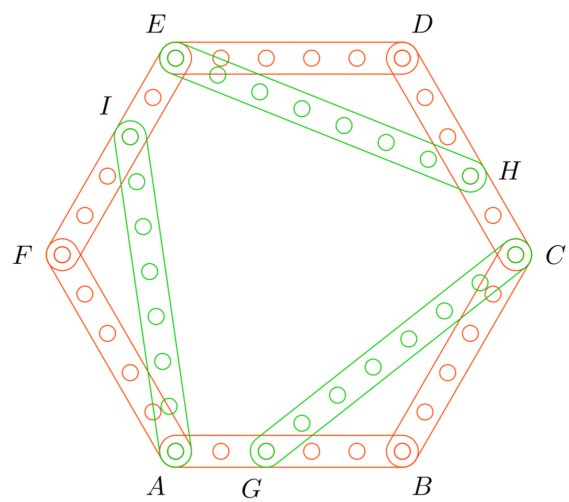

Figure 4: Hexagon of size $s = 5$ with three diagonals $c = \overline{GC} = \overline{HE} = \overline{IA} = 7$.

Figure 4 show a regular hexagon $ABCDEF$ of size 5 with three diagonals of size 7. We confirm the angle of $\alpha \equiv \angle GBC = 120°$ with the law of cosines.

$$\cos\alpha = \frac{(\overline{GB})^2 + (\overline{BC})^2 - (\overline{GC})^2}{2(\overline{GB})(\overline{BC})} = \frac{3^2 + 5^2 - 7^2}{2(3)(5)} = -\frac{1}{2}$$

3 Hexagons of size $10 < s < 20$

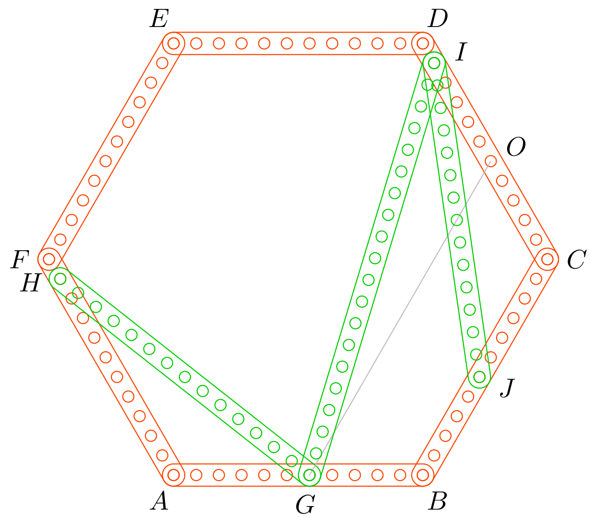

Figure 5: Hexagon of size 11 with three diagonals $\overline{GH} = \overline{IJ} = 14$ and $\overline{IG} = 19$ and four extra bolts in vertices G, H, I, J.

Figure 5 show hexagon $ABCDEF$ of size 11. Triangles $\triangle GHA$ and $\triangle JIC$ have sides $\{14, 10, 6\}$ which are the hexagonal triangle $\{7, 5, 3\}$ multipled by 2. Triangle $\triangle IGO$ has sides $\{19, 16, 5\}$ which is an hexagonal triangle.

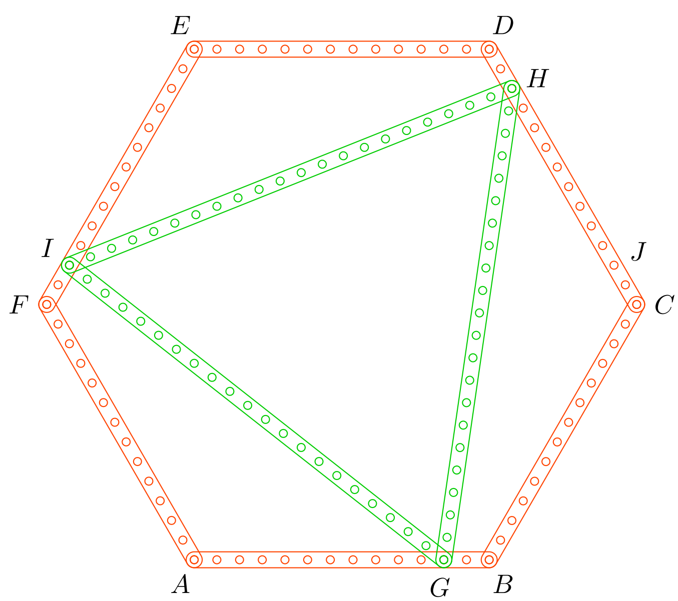

Figure 6: Hexagon of size 13 with three diagonals $\overline{GH} = \overline{HI} = \overline{IG} = 21$ and three bolts at vertices G, H, I.

Figure 6 show hexagon of size $s = 13$. First we detect an offset $o \equiv \overline{DH}$ which we use to calculate the sides p', b', c' of triangle $\triangle GJH$:

$$\begin{aligned}
o &\equiv \overline{DH} = \overline{CJ} = 2 \\
p' &\equiv \overline{GJ} = \overline{BC} + o = 13 + 2 = 15 \\
b' &\equiv \overline{JH} = \overline{CD} - 2o = 13 - 2(2) = 9 \\
c' &\equiv \overline{GH} = 21
\end{aligned} \tag{4}$$

We confirm triplet c', p', b' is three times ($n = 3$) valid hexagonal triplet $c = 7, p = 5, b = 3$. This case has an equilateral triangle $\triangle GHI$ inside the hexagon because is a special of the general case when:

$$\begin{aligned}
nb &= s - 2o \\
np &= s + o \\
nc &= \sqrt{(nb)^2 + (np)^2 - (nb)(np)} \\
&= \sqrt{(s - 2o)^2 + (s + o)^2 + (s - 2o)(s + o)} \\
&= \sqrt{3s^2 - 3so + 3o^2}
\end{aligned} \tag{5}$$

First terms of this case is shown in the table 2

s	o	c	p	b
13	2	21	15	9
23	1	39	24	21
37	11	57	48	15
59	13	93	72	33
73	26	111	99	21
83	22	129	105	39
94	23	147	117	48

Table 2: Equilateral triangles side c inside regular hexagons side s.

4 Hexagons of size ≥ 20

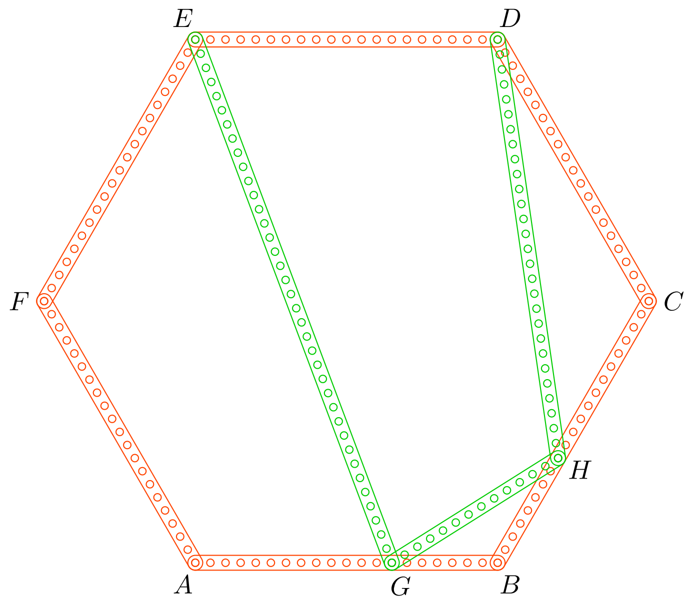

Figure 7: Hexagon of size 20 with three diagonals $\overline{GH} = 13$, $\overline{HD} = 28$ and $\overline{EG} = 37$ and **only** two extra bolts at vertices G and H.

Figure 7 show regular hexagon $ABCDEF$ of size 20. Triangle $\triangle GBH$ sides are $\{13, 8, 7\}$ which is an hexagonal triangle. Triangle $\triangle HCD$ sides are $\{28, 20, 12\}$ which is hexagonal triangle $\{7, 5, 3\}$ scaled by 4. Right triangle $\triangle EAG$ has side $\overline{AE} = \sqrt{(\overline{EG})^2 - (\overline{AG})^2} = \sqrt{37^2 - 13^2} = 20\sqrt{3}$ equals to hexagon height.

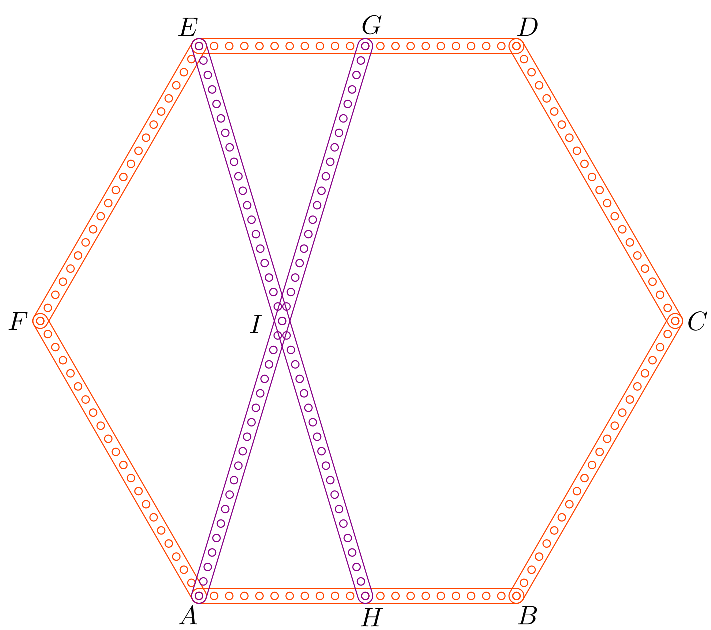

Figure 8: Hexagon of size 21 with **only** two diagonals $\overline{AG} = \overline{EH} = 38$. Segments $\overline{AH} = \overline{EG} = 11$, segment $\overline{AI} = 19$. Three extra bolts at H, G, I.

Figure 8 show regular hexagon $ABCDEF$ of size 21. We confirm the height of the hexagon is $\overline{AE} =$

$\sqrt{(\overline{EH})^2 - (\overline{AH})^2} = \sqrt{38^2 - 11^2} = 21\sqrt{3}$.

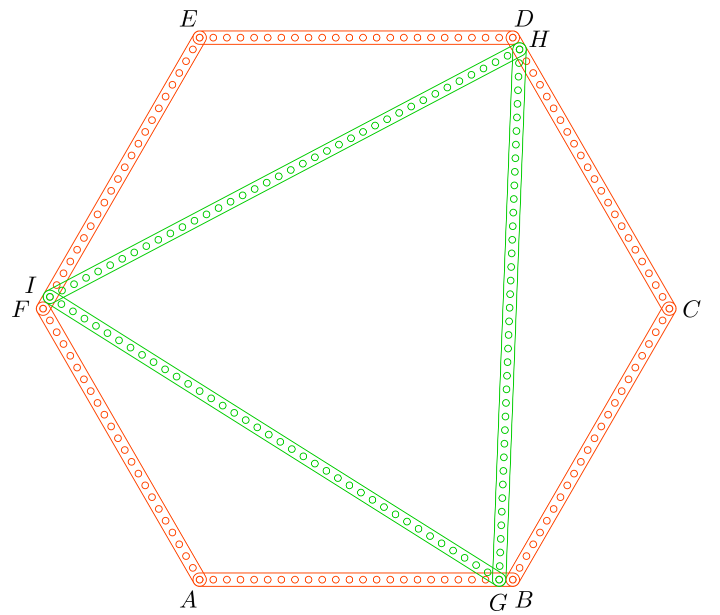

Figure 9: Hexagon of size $s = 23$ with three diagonals $c = \overline{GH} = \overline{HI} = \overline{IG} = 39$.

Figure 9 show a regular hexagon $ABCDEF$ of size $s = 23$. Is the second hexagon having an equilateral triangle inside, in this case of size $c = 39$ and described in table 2.

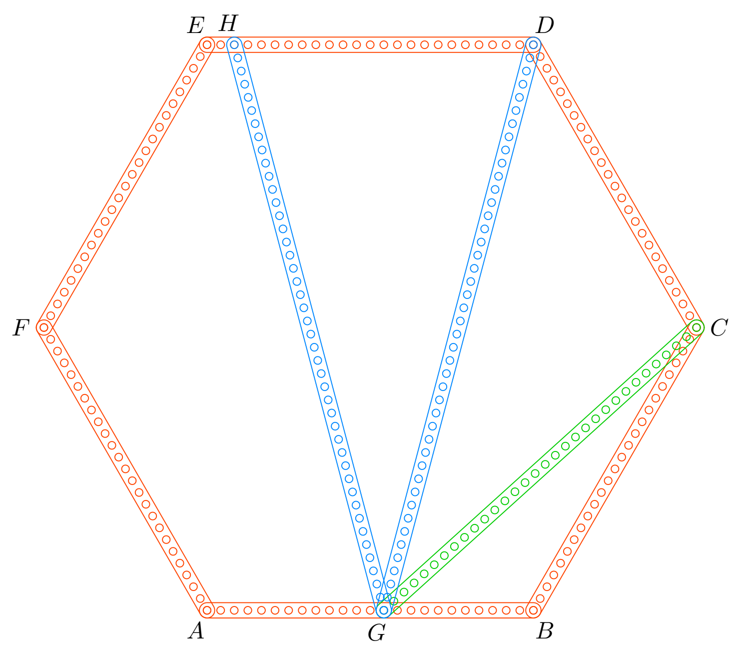

Figure 10: Hexagon of size $s = 24$ with diagonals $\overline{GC} = 31$ and $\overline{GD} = \overline{GH} = 43$ with **only** two extra bolts at vertices G and H. Segments $\overline{GB} = 11$ and $\overline{DH} = 22$.

Figure 10 show regular hexagon $ABCDEF$ of size $s = 24$. Triangle $\triangle GBC$ has sides $\{31, 24, 11\}$ which is the fourth hexagonal triangle in table 1. Triangle $\triangle DHG$ is the isosceles triangle shown in figure 2 case $a = 48$.