Meccano pentagons gallery

https://github.com/heptagons/meccano/penta/gallery

2023/12/8

Abstract

We build rigid meccano regular pentagons from sides 12 to 3. We restrict all internal strips to remain inside the pentagon's perimeter and don't permit they overlap with others. We follow three steps. 1) We calculate distances between selected strips holes from the regular pentagons perimeter. 2) We run some programs available in this repo to look for rigid strips clusters which contains the distance. 3) We simplify or reduce the cluster to fit inside the pentagon. We prove the correctness of the cluster distance applied to check the software. We try each construction is relevant for the pentagon size.

1 Pentagons of size 12

1.1 Size 12 with 4 internal strips

A program found that side 12 is the smallest pentagon that can be made rigid with a rhoumbus and two strips as diagonals so need only 4 strips as diagonals. We show two cases.

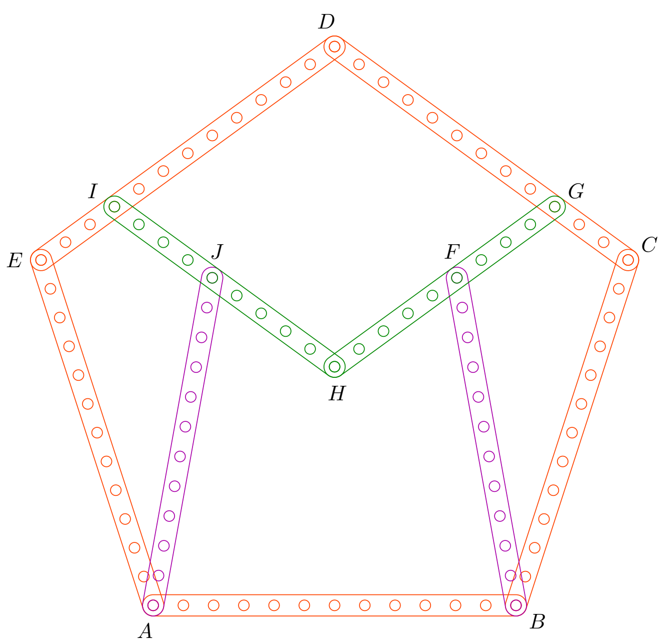

Figure 1: Regular pentagon size 12 (case a) made rigid with only four internal strips.

Figure 1 show a regular pentagon A, B, C, D, E of side 12 with a rhombus D, I, H, G of side 9. We prove strips AJ, BF are correct. First we calculate the abscissas going through vertices A, E, I, J substracting when we move to the left and adding when we move to the right:

$$\begin{aligned}
AJ_x &= AE_x + EI_x + IJ_x \\
&= -\overline{AE}\cos\left(\frac{2\pi}{5}\right) + \overline{EI}\cos\left(\frac{\pi}{5}\right) + \overline{IJ}\cos\left(\frac{\pi}{5}\right) \\
&= -12\left(\frac{\sqrt{5}-1}{4}\right) + 3\left(\frac{1+\sqrt{5}}{4}\right) + 4\left(\frac{1+\sqrt{5}}{4}\right) = \frac{19-5\sqrt{5}}{4}
\end{aligned} \tag{1}$$

Then we calculate the ordinates going to the same order of vertices adding when we go up and substracting when we go down:

$$\begin{aligned}
AJ_y &= -AE_y + EI_y + IJ_y \\
&= \overline{AE}\sin\left(\frac{2\pi}{5}\right) + \overline{EI}\sin\left(\frac{\pi}{5}\right) - \overline{IJ}\sin\left(\frac{\pi}{5}\right) \\
&= 12\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right) + 3\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right) - 4\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right) = \frac{\sqrt{1450+190\sqrt{5}}}{4}
\end{aligned} \tag{2}$$

Finally we calculate the distance $\overline{AJ}$ which coincides with strip size 11:

$$\begin{aligned}
\overline{AJ} &= \sqrt{(AJ_x)^2 + (AJ_y)^2} \\
&= \sqrt{\left(\frac{19-5\sqrt{5}}{4}\right)^2 + \frac{1450+190\sqrt{5}}{16}} = \sqrt{121} = 11
\end{aligned} \tag{3}$$

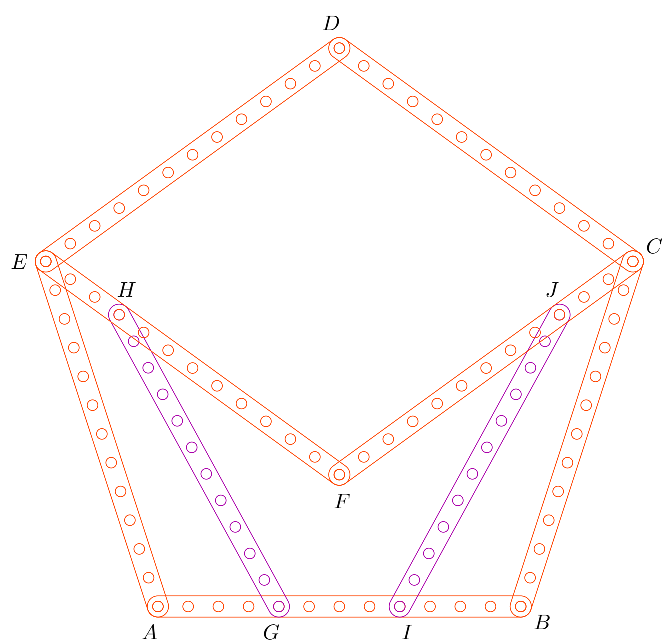

Figure 2: Regular pentagon size 12 case (b) made rigid with only four internal strips.

Figure 2 show a regular pentagon A, B, C, D, E of size 12 with a rhombus D, I, H, G of size 12. We prove strips GH, IJ are correct. First we calculate the abscissas going through vertices G, A, E, H substracting when we move to the left and adding when we move to the right:

$$\begin{aligned} GH_x &= -GA_x - AE_x + EH_x \\ &= -\overline{GA} - \overline{AE}\cos\left(\frac{2\pi}{5}\right) + \overline{EH}\cos\left(\frac{\pi}{5}\right) \\ &= -4 - 12\left(\frac{\sqrt{5}-1}{4}\right) + 3\left(\frac{1+\sqrt{5}}{4}\right) = \frac{-1-9\sqrt{5}}{4} \end{aligned} \tag{4}$$

Then we calculate the ordinates going to the same order of vertices adding when we go up and substracting when we go down:

$$\begin{aligned} GH_y &= AG_y + AE_y - EH_y \\ &= 0 + \overline{AE}\sin\left(\frac{2\pi}{5}\right) - \overline{EH}\sin\left(\frac{\pi}{5}\right) \\ &= 12\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right) - 3\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right) = \frac{\sqrt{1530-18\sqrt{5}}}{4} \end{aligned} \tag{5}$$

Finally we calculate the distance $\overline{GH}$ which coincides with strip size 11:

$$\begin{aligned} \overline{GH} &= \sqrt{(GH_x)^2 + (GH_y)^2} \\ &= \sqrt{\left(\frac{-1-9\sqrt{5}}{4}\right)^2 + \frac{1530-18\sqrt{5}}{16}} = \sqrt{121} = 11 \end{aligned} \tag{6}$$

1.2 Size 12 with 6 internal strips

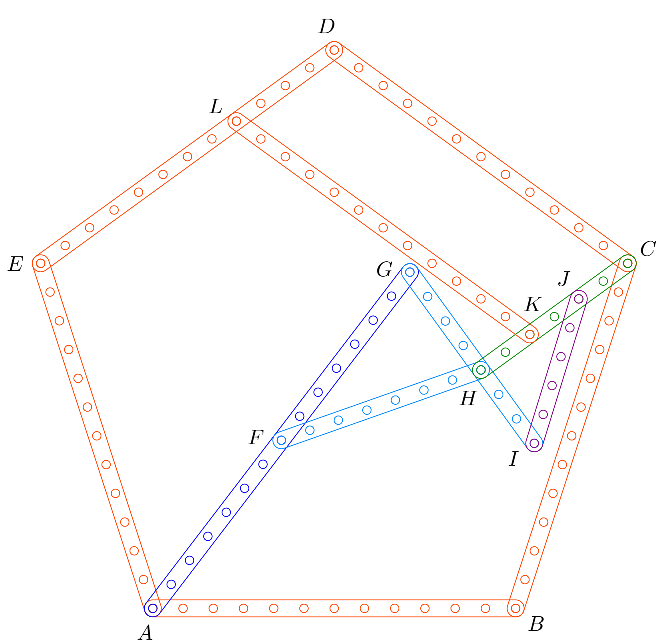

Figure 3: Regular pentagon size 12 made rigid with 6 internal strips. In the text we prove $\overline{AC} = 6 + 6\sqrt{5}$.

Figure 3 show a regular pentagon A, B, C, D, E of side 12. We know the regular pentagon diagonal for size 12 is $\overline{AC} = 12\left(\frac{1+\sqrt{5}}{2}\right) = 6 + 6\sqrt{5}$. We show the five strips $\overline{GH}, \overline{GI}, \overline{HF}, \overline{HC}, \overline{IJ}$ make the diagonal rigid which makes rigid the angle $\angle ABC$ of the pentagon. We have an isoscelles triangle $\triangle FGH$ and $\overline{AG}$ is two times $\overline{FG}$ so we have a right angle $\angle AHG = \frac{\pi}{2}$ and we can calculate $\overline{AH} = \sqrt{(\overline{AG})^2 - (\overline{GH})^2} = \sqrt{14^2 - 4^2} = 6\sqrt{5}$. Now we have another right angle $\angle IHC = \frac{\pi}{2}$ because the Pythagoras triangle $\triangle HIJ$. Since G, H, I are collinear then we have another right angle $\angle GHC = \frac{\pi}{2}$. Both right angles $\angle AHG, \angle CHG$ guaranty vertices A, H, C are collinear and we can calculate $\overline{AC} = \overline{AH} + \overline{HC} = 6 + 6\sqrt{5}$. Finally we add a sixth strip $\overline{KL}$ parallel to $\overline{CD}$ to make rigid the last three perimeter strips $\overline{CD}, \overline{DE}, \overline{EA}$ of the pentagon.

1.3 Size 12 with 8 internal strips

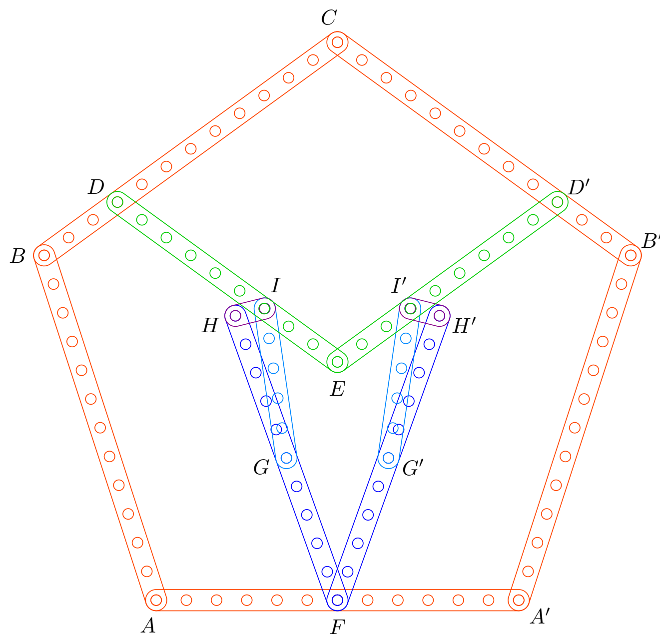

Figure 4: Regular pentagon size 12 made rigid with 8 internal strips. In the text we prove $\overline{FI} = 3\sqrt{11}$

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Figure 4 show a regular pentagon $AA'B'CBA$ of side 12. First we calculate the distance $\overline{FI}$ using the abscissas and ordinates following the vertices F, A, B, D, I for a regular pentagon angles $\alpha = \dfrac{3\pi}{5}, \beta = \dfrac{\pi}{5}$:

$$\begin{aligned}
FI_x &= -\overline{AF} - \overline{AB}\cos\alpha + (\overline{BD} + \overline{DI})\cos\beta \\
&= -6 + (12)\frac{1-\sqrt{5}}{4} + (3+6)\frac{\sqrt{5}+1}{4} = -\frac{3+3\sqrt{5}}{4} \qquad (7) \\
FI_y &= \overline{AB}\sin\alpha + (\overline{BD} - \overline{DI})\sin\beta \\
&= (12)\frac{\sqrt{10+2\sqrt{5}}}{4} + (3-6)\frac{\sqrt{10-2\sqrt{5}}}{4} = \frac{12\sqrt{10+2\sqrt{5}} - 3\sqrt{10-2\sqrt{5}}}{4} \qquad (8) \\
\overline{FI} &= \sqrt{(FI_x)^2 + (FI_y)^2} \\
&= \frac{\sqrt{(-3-3\sqrt{5})^2 + (12\sqrt{10+2\sqrt{5}} - 3\sqrt{10-2\sqrt{5}})^2}}{4} = \frac{\sqrt{1584}}{4} = 3\sqrt{11} \qquad (9)
\end{aligned}$$

Finally we calculate the distance $\overline{FI}$ made rigid by cluster F, G, H, I. We have an isoscelles triangle

$\triangle GHI$ and $\overline{FH} = 2\overline{GH}$ so we have a right triangle $\angle FIH = \frac{\pi}{2}$ so:

$$\begin{aligned}\overline{FI} &= \sqrt{(\overline{FH})^2 - (\overline{HI})^2} \\ &= \sqrt{10^2 - 1^2} = 3\sqrt{11} \end{aligned} \tag{10}$$

1.4 Size 12 with 10 internal strips

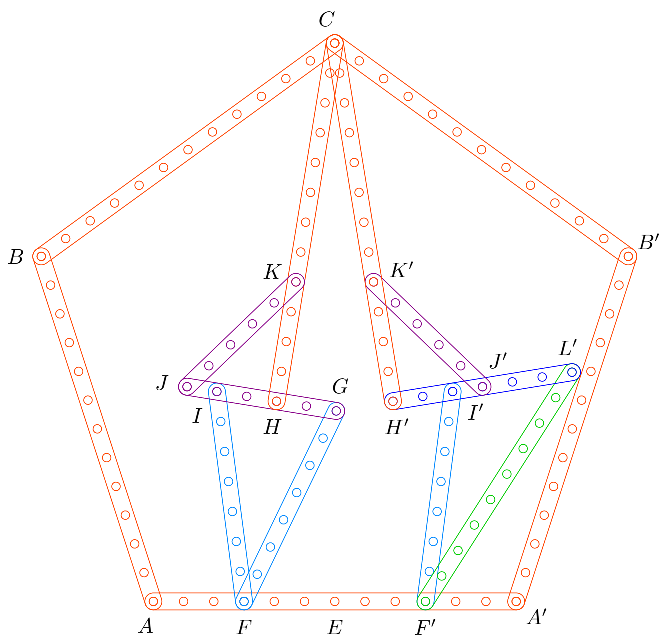

Figure 5: Regular pentagon size 12 made rigid with two internal clusters of five strips each. In the text we prove $\overline{CF} = \overline{CF'} = 12 + 3\sqrt{5}$.

Figure 5 show a regular pentagon $AA'B'CB$ of side 12. We know the regular pentagon height is $\dfrac{\sqrt{5+2\sqrt{5}}}{2}$ times the side. So here we have $\overline{CE} = 12\frac{\sqrt{5+2\sqrt{5}}}{2} = 6\sqrt{5+2\sqrt{5}}$ and we can calculate $\overline{CF}$:

$$\begin{aligned}\overline{CF} &= \sqrt{\overline{CE}^2 + \overline{EF}^2} \\ &= \sqrt{36(5+2\sqrt{5}) + 3^2} = 3\sqrt{21 + 8\sqrt{5}} = 3(4 + \sqrt{5}) \end{aligned} \tag{11}$$

After testing $\overline{AA'} \leq 1800$ a program found that the last denesting is somehow special since other fractions $\dfrac{\overline{AF}}{\overline{AA'}} \neq \dfrac{1}{4}$ generated $\overline{CF}$s that can't be denested.

We have the Pythagorean triangle $\triangle HJK$ and the isoscelles $\triangle FGI$ so vertices FHC are collinear. First we calculate $\overline{FH} = \sqrt{\overline{FG}^2 - \overline{GH}^2} = \sqrt{7^2 - 2^2} = 3\sqrt{5}$ and then $\overline{FC} = \overline{FH} + \overline{HC} = 3\sqrt{5} + 12$ matching last calculation. Finally we prove angle $\angle F'H'L' = \frac{\pi}{2}$ noting $\overline{F'H'} = \sqrt{(\overline{F'L'})^2 - (\overline{H'L'})^2} = \sqrt{9^2 - 6^2} = 3\sqrt{5}$ matching $\overline{FH}$.

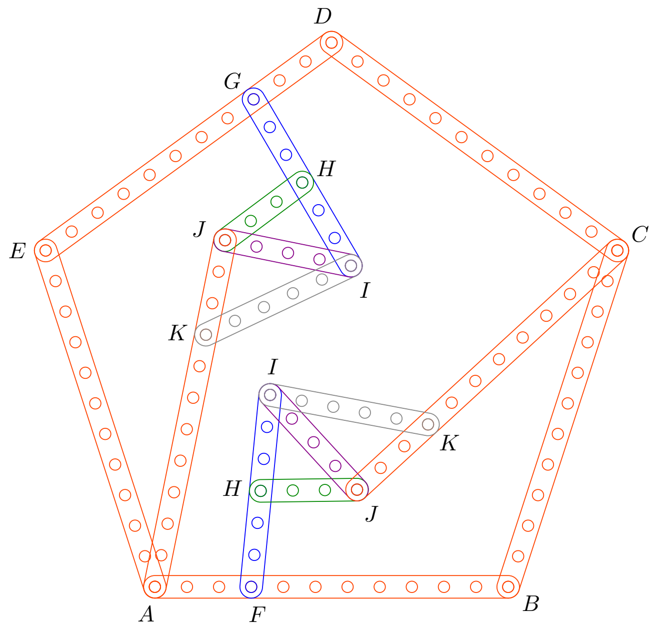

Figure 6: Regular pentagon size 11 made rigid with two internal clusters of five strips each. In the text we prove $\overline{CF} = \overline{AG} = 11 + 2\sqrt{5}$.

2 Pentagon of size 11

2.1 Size 11 with 10 internal strips

Figure 6 show a rigid regular pentagon A, B, C, D, E of size 11. A program found this is the smallest pentagon having a consecutive sides diagonal of the form $\dfrac{z_2 + z_3\sqrt{5}}{z_1}$ instead of the nested form $\dfrac{z_2\sqrt{z_3 + z_4\sqrt{5}}}{z_1}$ where z_i are integers. The mentioned diagonal is the distance $\overline{CF}$ in the figure which can be calculated with the law of cosines knowing angle $\angle CBF = \dfrac{3\pi}{5}$ and denesting the result. We calculate the angle $\angle CFB$ for the drawing:

$$\overline{CF}^2 = \overline{BC}^2 + \overline{BF}^2 - 2(\overline{BC})(\overline{BF})\cos\left(\frac{3\pi}{5}\right)$$

$$= 11^2 + 8^2 - 2(11)(8)\left(\frac{1-\sqrt{5}}{4}\right) = 141 + 44\sqrt{5}$$

$$\overline{CF} = \sqrt{141 + 44\sqrt{5}} = 11 + 2\sqrt{5} \tag{12}$$

$$\cos(\angle CFB) = \frac{\overline{CF}^2 + \overline{BF}^2 - \overline{BC}^2}{2(\overline{CF})(\overline{BF})} = \frac{141 + 44\sqrt{5} + 8^2 - 11^2}{2(11+2\sqrt{5})(8)} = \frac{21 + 11\sqrt{5}}{44 + 8\sqrt{5}} = \frac{121 + 79\sqrt{5}}{404} \tag{13}$$

2.1.1 Distance $11 + 2\sqrt{5}$

A five strips cluster can create a rigid distance like $11+2\sqrt{5}$. In the figure, three strips $\overline{FI} = 2\overline{HJ}, \overline{FI} > \overline{IJ}$ builds a right angle $\angle FJI = \frac{\pi}{2}$, since triangle $\triangle IJH$ is isosceles ($\overline{FH} = \overline{HI} = \overline{JH}$). These three strips also build a distance $\overline{FJ} = \sqrt{\overline{FI}^2 - \overline{IJ}^2} = \sqrt{6^2 - 4^2} = 2\sqrt{5}$. Now we attach strip $\overline{CJ}$ making a second right triangle $\angle CJI = \frac{\pi}{2}$ using strip $\overline{IK} = 5$ as pythagorean diagonal ($\overline{JK} = 3, \overline{IJ} = 4$). We have two right triangles at vertice J so vertices F, J, C are collinear, so we can calculate the distance $\overline{FC} = \overline{CJ} + \overline{JF} = 11 + 2\sqrt{5}$. We repeat the five-strips cluster between vertices A, G preventing overlaps of any strips. Since the clusters are rigid we formed two rigid triangles $\triangle ABC, \triangle DEA$ so the pentagon is rigid.

The program found the next pentagon of this type is a lot bigger: $\overline{BC} = 246, \overline{BF} = 70, \overline{CF} = 41 + 105\sqrt{5}$.

3 Pentagon of size 10

3.1 Size 10 with 10 internal strips

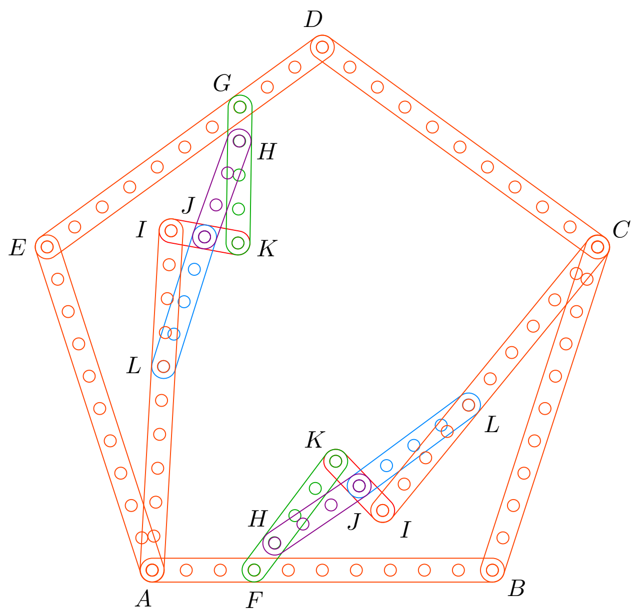

Figure 7: Regular pentagon size 10 made rigid with two internal clusters of five strips each. In the text we prove $\overline{CF} = \overline{AG} = \sqrt{114 + 35\sqrt{5}}$.

Figure 7 show a rigid regular pentagon A, B, C, D, E of size 10. We calculate a diagonal joining two consecutive sides relative primes to have something exclusive to the size 10, we choose $\overline{BF} : \overline{BC} = 7 : 10$. With the law of cosines we calculate $\overline{CF}$. We calculate the angle $\angle CFB$ for the drawing:

$$\overline{CF}^2 = \overline{BC}^2 + \overline{BF}^2 - 2(\overline{BC})(\overline{BF}) \cos\left(\frac{3\pi}{5}\right)$$

$$= 10^2 + 7^2 - 2(10)(7)\left(\frac{1-\sqrt{5}}{4}\right) = 114 + 35\sqrt{5}$$

$$\overline{CF} = \sqrt{114 + 35\sqrt{5}} \tag{14}$$

$$\cos(\angle CFB) = \frac{\overline{CF}^2 + \overline{BF}^2 - \overline{BC}^2}{2(\overline{CF})(\overline{BF})} = \frac{114 + 35\sqrt{5} + 7^2 - 10^2}{2(\sqrt{114 + 35\sqrt{5}})(7)} = \frac{9 + 5\sqrt{5}}{2\sqrt{114 + 35\sqrt{5}}} \tag{15}$$

3.1.1 Distance $\sqrt{114 + 35\sqrt{5}}$

Number $\sqrt{114 + 35\sqrt{5}}$ cannot be denested so we need to solve this with a cluster of strips. A program found a lot of solutions for this distance using five strips, so we choose one narrow enough to fit inside the pentagon.

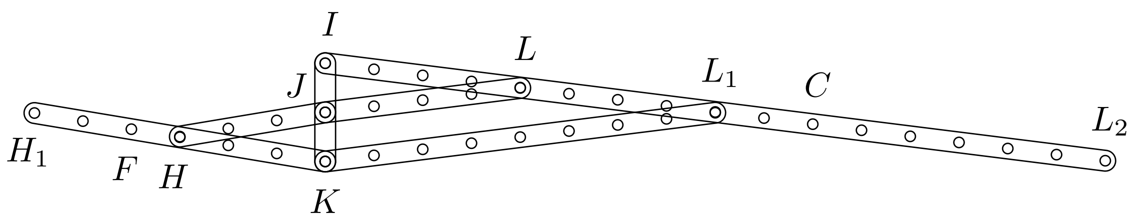

Figure 8: Construction of distance $\overline{CF} = \sqrt{114 + 35\sqrt{5}}$

Figure 8 shows how to prove the cluster selected is correct. In the figure we have two isoscelles triangles $\triangle IKL_1$ and $\triangle JKH$. The sides IL_1 and KH are extended to double the original size to the vertices L_2 and H_1 building two right angles $\angle IKL_2$ and $\angle KJH_1$. The right triangles permit the calculation of the abscissas and ordinates of vertices C and F to calculate their distance.

From the figure we calculate $\overline{KL_2}$ and $\overline{JH_1}$ from their respective right triangles:

$$\overline{KL_2} = \sqrt{(\overline{IL_2})^2 - (\overline{IK})^2} = \sqrt{16^2 - 2^2} = 6\sqrt{7} \tag{16}$$

$$\overline{JH_1} = \sqrt{(\overline{KH_1})^2 - (\overline{KJ})^2} = \sqrt{6^2 - 1^2} = \sqrt{35} \tag{17}$$

Assuming vertice K is at the origin we can calculate the abscissas C_x, F_x and ordinates C_y, F_y of vertices C and F using as factors $c = \frac{\overline{IC}}{\overline{IL_2}} = \frac{10}{16} = \frac{5}{8}$ and $f = \frac{\overline{KF}}{\overline{KH1}} = \frac{4}{6} = \frac{2}{3}$:

$$C_x = +c(\overline{KL_2}) = \frac{5}{8}(6\sqrt{7}) = \frac{15}{4}\sqrt{7} \tag{18}$$

$$F_x = -f(\overline{JH_1}) = -\frac{2}{3}\sqrt{35} \tag{19}$$

$$C_y = +(\overline{KI}) - c(\overline{KI}) = 2 - \frac{5}{8}(2) = \frac{3}{4} \tag{20}$$

$$F_y = +f(\overline{KJ}) = \frac{2}{3}(1) = \frac{2}{3} \tag{21}$$

Finally we calculate the distance $\overline{CF}$:

$$\begin{aligned}\overline{CF} &= \sqrt{(C_x - F_x)^2 + (C_y - F_y)^2} \\ &= \sqrt{\left(\frac{15}{4}\sqrt{7} + \frac{2}{3}\sqrt{35}\right)^2 + \left(\frac{3}{4} - \frac{2}{3}\right)^2} = \sqrt{114 + 35\sqrt{5}}\end{aligned} \tag{22}$$

A minimal part with five strips of the construction of figure 8 including only vertices F, H, I, J, K, L, C is used twice to make rigid the pentagon of side 10 as show in figure 7.

4 Pentagons of size 9

4.1 Size 9 with 6 internal strips

Figure 9 show a rigid regular pentagon A, B, C, D, E of size 9. The regular pentagon distance $\overline{CE}$ is called width and equals $\frac{1+\sqrt{5}}{2}\overline{AB}$. Is easy to note the distance $\overline{FG}$ equals the width of smaller pentagon size

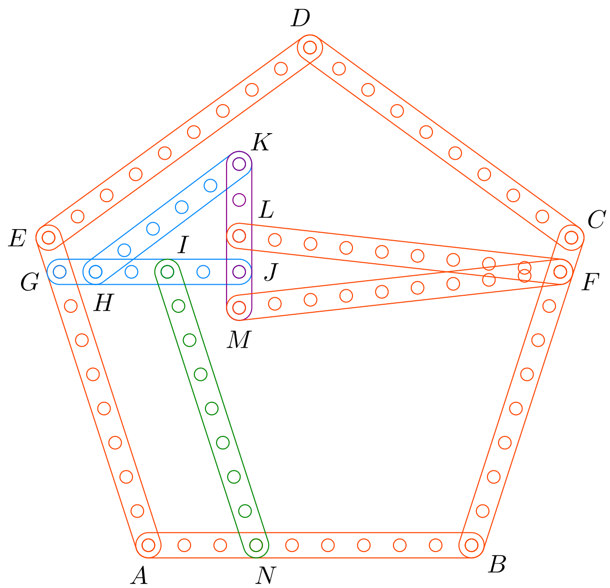

Figure 9: Regular pentagon size 9 made rigid with six internal strips. In the text we prove $\overline{FG} = 5 + 4\sqrt{5}$.

$9 - 1 = 8$ plus 1. So we have:

$$\begin{aligned} \overline{FG} &= \frac{1+\sqrt{5}}{2}(\overline{BC} - \overline{FC}) + \overline{FC} \\ &= \frac{1+\sqrt{5}}{2}(9-1) + 1 = 5 + 4\sqrt{5} \end{aligned} \tag{23}$$

From the figure we see two right angles. Angle $\angle GJK = \frac{\pi}{2}$ because we have a Pythagorean triangle $\triangle HJK$. Angle $\angle FJM = \frac{\pi}{2}$ because we have an isosceles triangle $\triangle FLM$. The two right angles share vertice J so vertices G, J, F are collinear. First we calculate the distance $\overline{JF} = \sqrt{(LF)^2 - (LJ)^2} = \sqrt{9^2 - 1^2} = 4\sqrt{5}$ and finally the distance $\overline{GF} = \overline{GJ} + \overline{JF} = 5 + 4\sqrt{5}$ which matches the value in last equation above. To make rigid the pentagon we add strip $\overline{IN}$ parallel to side $\overline{GA}$.

4.2 Size 9 with 8 internal strips

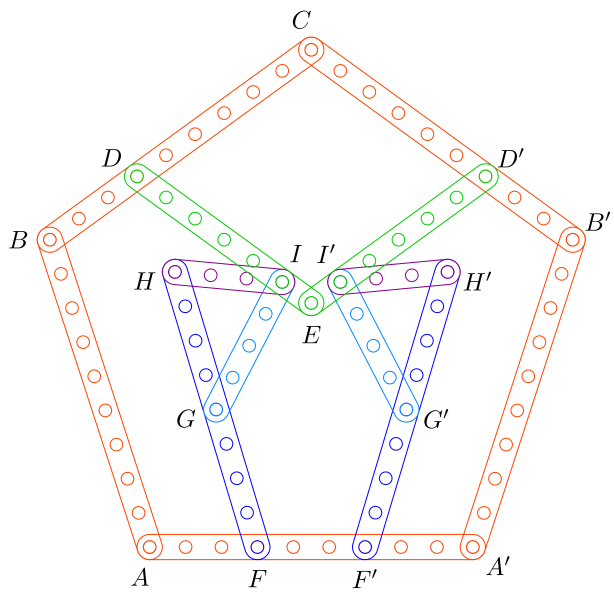

Figure 10: Regular pentagon size 9 made rigid with 8 strips variation 1. In the text we prove $\overline{FI} = \sqrt{55}$.

Figure 10 show a rigid regular pentagon A, A', B', C, B of size 9. First we calculate the distance $\overline{FI}$ using the abscissas and ordinates following the vertices F, A, B, D, I for a regular pentagon angles $\alpha = \dfrac{3\pi}{5}, \beta = \dfrac{\pi}{5}$:

$$\begin{aligned} FI_x &= -\overline{AF} - \overline{AB}\cos\alpha + (\overline{BD} + \overline{DI})\cos\beta \\ &= -3 + (9)\frac{1-\sqrt{5}}{4} + (3+5)\frac{\sqrt{5}+1}{4} = \frac{5-\sqrt{5}}{4} \end{aligned} \tag{24}$$

$$\begin{aligned} FI_y &= \overline{AB}\sin\alpha + (\overline{BD} - \overline{DI})\sin\beta \\ &= (9)\frac{\sqrt{10+2\sqrt{5}}}{4} + (3-5)\frac{\sqrt{10-2\sqrt{5}}}{4} = \frac{9\sqrt{10+2\sqrt{5}} - 2\sqrt{10-2\sqrt{5}}}{4} \end{aligned} \tag{25}$$

$$\begin{aligned} \overline{FI} &= \sqrt{(FI_x)^2 + (FI_y)^2} \\ &= \frac{\sqrt{(5-\sqrt{5})^2 + (9\sqrt{10+2\sqrt{5}} - 2\sqrt{10-2\sqrt{5}})^2}}{4} = \frac{\sqrt{880}}{4} = \sqrt{55} \end{aligned} \tag{26}$$

Finally we calculate the distance $\overline{FI}$ made rigid by cluster F, G, H, I. We have an isoscelles triangle $\triangle GHI$ and $\overline{FH} = 2\overline{GH}$ so we have a right triangle $\angle FIH = \frac{\pi}{2}$ so:

$$\begin{aligned} \overline{FI} &= \sqrt{(\overline{FH})^2 - (\overline{HI})^2} \\ &= \sqrt{8^2 - 3^2} = \sqrt{55} \end{aligned} \tag{27}$$

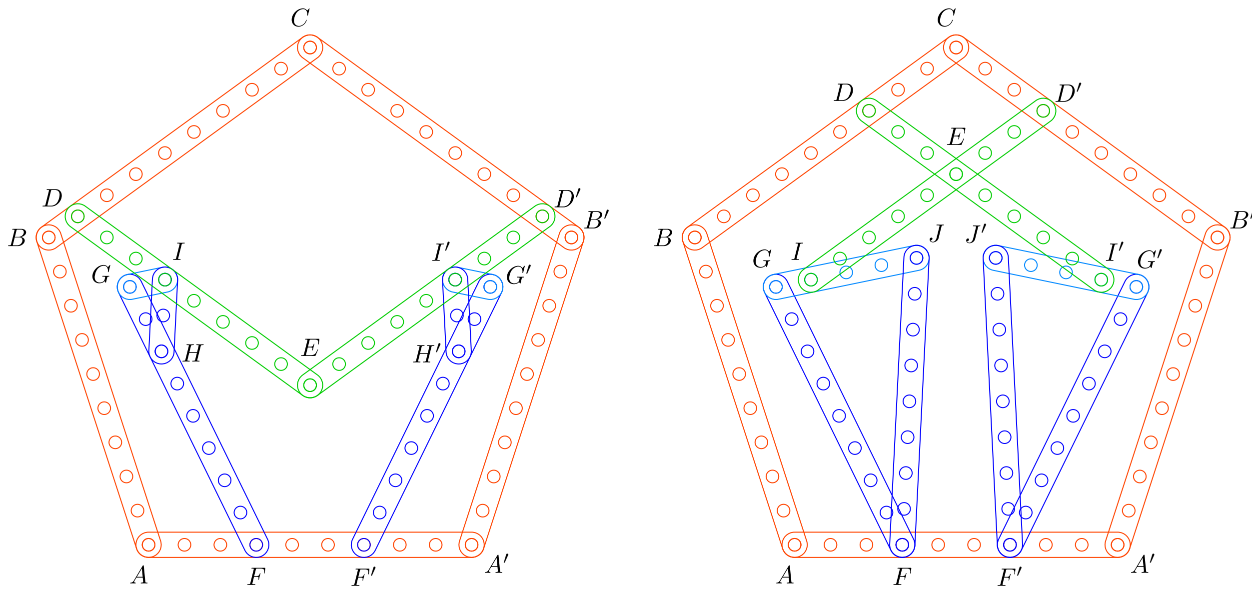

Figure 11: Regular pentagons size 9 made rigid with 8 strips variations 2 and 3. In the text we prove $\overline{FI} = \sqrt{61}$.

Figure 11 show two rigid pentagons A, A', B', C, B' of size 9. The pentagon at the left is called variation 2 and the right one variation 3. Both variations have the vertices I, I' at the same positions and the same distance $\overline{FI}$ which first we calculate using the abscissas and ordinates following the vertices F, A, B, D, I of the variation 2 for a regular pentagon angles $\alpha = \dfrac{3\pi}{5}, \beta = \dfrac{\pi}{5}$:

$$
\begin{aligned}
FI_x &= -\overline{AF} - \overline{AB}\cos\alpha + (\overline{BD} + \overline{DI})\cos\beta \\
&= -3 + (9)\frac{1-\sqrt{5}}{4} + (1+3)\frac{\sqrt{5}+1}{4} = \frac{1-5\sqrt{5}}{4} && (28)\\
FI_y &= \overline{AB}\sin\alpha + (\overline{BD} - \overline{DI})\sin\beta \\
&= (9)\frac{\sqrt{10+2\sqrt{5}}}{4} + (1-3)\frac{\sqrt{10-2\sqrt{5}}}{4} = \frac{9\sqrt{10+2\sqrt{5}} - 2\sqrt{10-2\sqrt{5}}}{4} && (29)\\
\overline{FI} &= \sqrt{(FI_x)^2 + (FI_y)^2} \\
&= \frac{\sqrt{(1-\sqrt{5})^2 + (9\sqrt{10+2\sqrt{5}} - 2\sqrt{10-2\sqrt{5}})^2}}{4} = \frac{\sqrt{976}}{4} = \sqrt{61} && (30)
\end{aligned}
$$

Finally we calculate the distance $\overline{FI}$ made rigid by clusters F, G, H, I or F, G, I, J since in both variations we have the same $\overline{GF}$ and same angles $\angle FGI = \angle FJG$. With the law of cosines first we calculate $\cos(\angle FJG)$ and then $\overline{FI}$:

$$
\begin{aligned}
\cos(\angle FJG) &= \frac{\overline{FJ}^2 + \overline{JG}^2 - \overline{GF}^2}{2(\overline{FJ})(\overline{JG})} = \frac{8^2 + 4^2 - 8^2}{2(8)(4)} = \frac{1}{4} \\
\overline{FI} &= \sqrt{\overline{IJ}^2 + \overline{FJ}^2 - 2(\overline{IJ})(\overline{FJ})\cos(\angle FJG)} = \sqrt{3^2 + 8^2 - 2(3)(8)\left(\frac{1}{4}\right)} = \sqrt{61} && (31)
\end{aligned}
$$

4.3 Size 9 with 10 internal strips

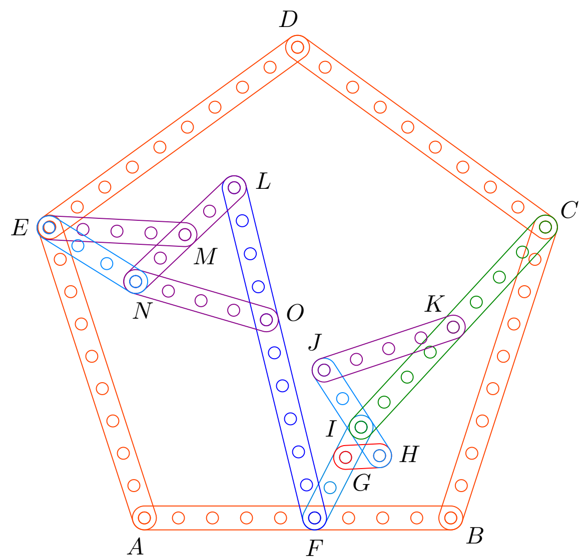

Figure 12: Regular pentagon size 9 (case b) made rigid with two internal clusters of five strips each. In the text we prove $\overline{FC} = \sqrt{79+18\sqrt{5}}$ and $\overline{FE} = \dfrac{\sqrt{334+90\sqrt{5}}}{2}$.

Figure 12 show a regular pentagon A, B, C, D, E of size 9 made it rigid with the help of clusters fixing the distances $\overline{CF}$ and $\overline{EF}$. Pentagon size 9 is the smaller one with diagonals where consecutive side segments fractions are $\overline{BF}/\overline{BC} = \dfrac{4}{9}$ and $\overline{AF}/\overline{AE} = \dfrac{5}{9}$. We calculate the diagonals $\overline{CF}$, $\overline{EF}$ and the angles to side $\overline{AB}$ using the law of cosines and the internal pentagon angle $\theta = \angle FBC = \angle FAE = \dfrac{3\pi}{5}$ where $\cos\theta = \dfrac{1-\sqrt{5}}{4}$:

$$\overline{CF} = \sqrt{\overline{BC}^2 + \overline{BF}^2 - 2(\overline{BC})(\overline{BF})\cos\theta} = \sqrt{9^2+4^2-2(9)(4)\left(\frac{1-\sqrt{5}}{4}\right)} = \sqrt{79+18\sqrt{5}} \quad (32)$$

$$\cos(\angle CFB) = \frac{\overline{CF}^2 + \overline{BF}^2 - \overline{BC}^2}{2(\overline{CF})(\overline{BF})} = \frac{79+18\sqrt{5}+4^2-9^2}{2(\sqrt{79+18\sqrt{5}})(4)} = \frac{7+9\sqrt{5}}{4\sqrt{79+18\sqrt{5}}} \quad (33)$$

$$\overline{EF} = \sqrt{\overline{AE}^2 + \overline{AF}^2 - 2(\overline{AE})(\overline{AF})\cos\theta} = \sqrt{9^2+5^2-2(9)(5)\left(\frac{1-\sqrt{5}}{4}\right)} = \frac{\sqrt{334+90\sqrt{5}}}{2} \quad (34)$$

$$\cos(\angle EFA) = \frac{\overline{EF}^2 + \overline{AF}^2 - \overline{EA}^2}{2(\overline{EF})(\overline{AF})} = \frac{\dfrac{334+90\sqrt{5}}{4} + 5^2 - 9^2}{2\left(\dfrac{\sqrt{334+90\sqrt{5}}}{2}\right)(5)} = \frac{11+9\sqrt{5}}{2\sqrt{334+90\sqrt{5}}} \quad (35)$$

Our software found several options with five strips to build distances $\sqrt{79+18\sqrt{5}}$ and $\dfrac{\sqrt{334+90\sqrt{5}}}{2}$.

4.3.1 Distance $\sqrt{79+18\sqrt{5}}$

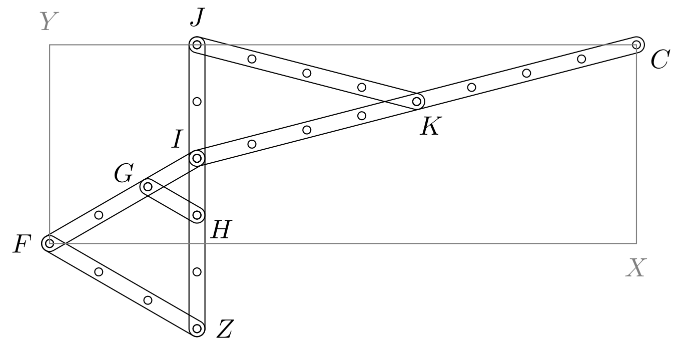

Figure 13: Construction of distance $\overline{FC} = \sqrt{79+18\sqrt{5}}$

Figure 13 show one of several ways to build the distance $\sqrt{79+18\sqrt{5}}$. Equilateral triangle $\triangle FIZ$ and isosceles $\triangle IJK$ share vertice I and the base $\overline{JZ}$ which help to form rectangle $FXCY$ with base $\overline{FX}$ and height $\overline{FY}$ useful to calculate the diagonal $\overline{FC}$:

$$\begin{aligned}
\overline{FX} &= \overline{YJ} + \overline{JC} \\
&= \sqrt{\overline{FI}^2 - \left(\frac{\overline{IZ}}{2}\right)^2} + \sqrt{\overline{IC}^2 - \overline{IJ}^2} = \sqrt{3^2 - \left(\frac{3}{2}\right)^2} + \sqrt{8^2 - 2^2} = \frac{3\sqrt{3}}{2} + 2\sqrt{15} \\
\overline{FY} &= \overline{JI} + \frac{\overline{IZ}}{2} = 2 + \frac{3}{2} = \frac{7}{2} \\
\overline{FC} &= \sqrt{\overline{FX}^2 + \overline{FY}^2} = \sqrt{\left(\frac{3\sqrt{3}}{2} + 2\sqrt{15}\right)^2 + \left(\frac{7}{2}\right)^2} = \sqrt{79+18\sqrt{5}} \qquad (36)
\end{aligned}$$

We use a smaller part of this construction, the five strips with vertices F, G, H, I, J, K, C, as a cluster to made rigid the consecutive strips $\overline{AB}, \overline{BC}$ of the pentagon of side 9 of figure 12.

4.3.2 Distance $\dfrac{\sqrt{334+90\sqrt{5}}}{2}$

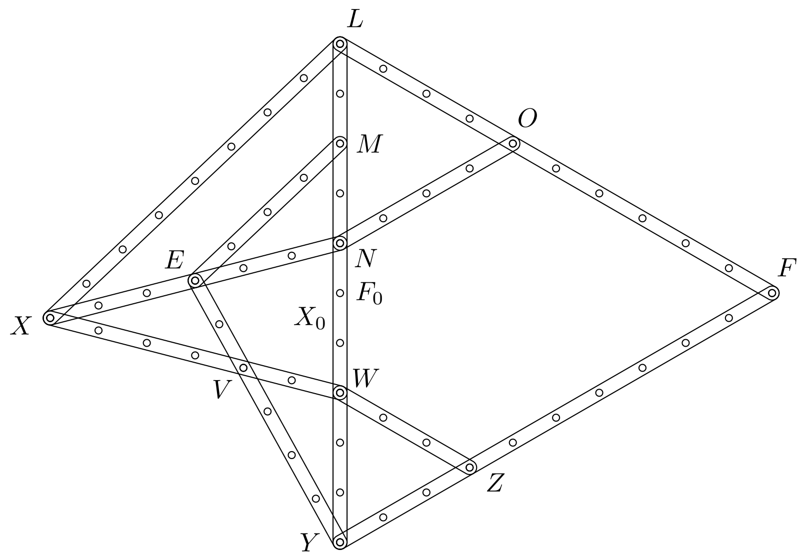

Figure 14: Construction of distance $\overline{EF} = \dfrac{\sqrt{334+90\sqrt{5}}}{2}$

Figure 14 show equilateral triangle $\triangle FLY$ and isoscelles triangle $\triangle NXW$ sharing strip $\overline{LY}$ which helps to calculate abscissas and ordinates of vertices E, F to calculate distance $\overline{EF}$. Vertice Y is located at the origin so:

$$E_x = -\left(\frac{\overline{NE}}{\overline{NX}}\right)\overline{XX_0} = -\frac{3}{6}\sqrt{\overline{NX}^2 - \overline{NX_0}^2} = -\frac{1}{2}\sqrt{6^2 - \left(\frac{3}{2}\right)^2} = -\frac{3\sqrt{15}}{4} \tag{37}$$

$$E_y = \overline{YN} - \left(\frac{\overline{NE}}{\overline{NX}}\right)\overline{NX_0} = 6 - \left(\frac{3}{6}\right)\left(\frac{3}{2}\right) = \frac{21}{4} \tag{38}$$

$$F_x = \overline{F_0F} = \sqrt{\overline{YF}^2 - \overline{YF_0}^2} = \sqrt{10^2 - 5^2} = 5\sqrt{3} \tag{39}$$

$$F_y = \overline{YF_0} = 5 \tag{40}$$

$$\overline{EF} = \sqrt{(E_x - F_x)^2 + (E_y - F_y)^2} = \sqrt{\left(-\frac{3\sqrt{15}}{4} - 5\sqrt{3}\right)^2 + \left(\frac{21}{4} - 5\right)^2} = \frac{\sqrt{334+90\sqrt{5}}}{2} \tag{41}$$

We form a cluster from the last construction to be applied in the pentagon of side 9. We choose the five strips with vertices E, N, M, L, O, F. Is easy to prove strip $\overline{EM}$ is correct in the cluster comparing equal cosines at vertice Y for triangles $\triangle YVW, \triangle YEN, \triangle YEM$ using the law of cosines for each triangle.

5 Pentagons of size 8

6 Pentagons of size 7

7 Pentagons of size 6

8 Pentagons of size 5

9 Pentagons of size 4

9.1 Size 4 with 8 internal strips

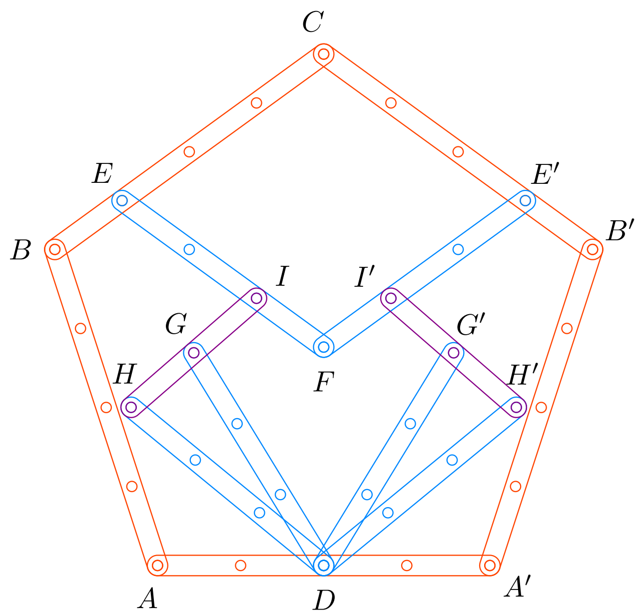

Figure 15: Regular pentagon of size 4 made rigid with 6 internal strips. $\overline{DI} = \sqrt{11}$.

Figure 15 show the regular pentagon $AA'B'CB$ of size 4. We calculate the distance $\overline{DI}$ assuming vertex D is at origin and calculating abscissa and ordinate of vertex I and knowing $\alpha = \angle ADB = \dfrac{3\pi}{5}$ and $\beta = \angle B'BE = \dfrac{\pi}{5}$. Adding and substracting through vertices D, A, B, E, I we get:

$$\begin{aligned}
I_x &= -\overline{DA} - \overline{AB}|\cos\alpha| + \overline{BE}\cos\beta + \overline{EI}\cos\beta \\
&= -2 - (4)\left|-\frac{\sqrt{5}-1}{4}\right| + (1+2)\left(\frac{\sqrt{5}+1}{4}\right) = -\frac{1+\sqrt{5}}{4} \qquad (42)\\
I_y &= \overline{AB}\sin\alpha + \overline{BE}\sin\beta - \overline{BI}\sin\beta \\
&= (4)\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right) + (1-2)\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right) = \frac{4\sqrt{10+2\sqrt{5}} - \sqrt{10-2\sqrt{5}}}{4} \qquad (43)\\
\overline{DI} &= \sqrt{(I_x - D_x)^2 + (I_y - I_y)^2} \\
&= \frac{\sqrt{(1+\sqrt{5})^2 + (4\sqrt{10+2\sqrt{5}} - \sqrt{10-2\sqrt{5}})^2}}{4} = \sqrt{11} \qquad (44)
\end{aligned}$$

9.1.1 Rigid distance $\sqrt{11}$

Our software found several three strips clusters for rigid distance $\sqrt{11}$. We prove the selected cluster $DHGI$ inside the pentagon matches the expected distance. First we calculate the angle $\angle DHG$ with the law of cosines and use the value to finally calculate the distance $\overline{DI}$ with again the law of cosines:

$$\begin{aligned}
\cos(\angle DHG) &= \frac{(\overline{HD})^2 + (\overline{HG})^2 - (\overline{DG})^2}{2(\overline{HD})(\overline{HG})} \\
&= \frac{3^2 + 1^2 - 3^2}{2(3)(1)} = \frac{1}{6} \qquad (45) \\
\overline{DI} &= \sqrt{(\overline{HD})^2 + (\overline{HI})^2 - 2(\overline{HD})(\overline{HD})\cos(\angle DHG)} \\
&= \sqrt{3^2 + 2^2 - 2(3)(2)\left(\frac{1}{6}\right)} = \sqrt{11} \quad \blacksquare \qquad (46)
\end{aligned}$$

9.2 Size 4 with 10 internal strips

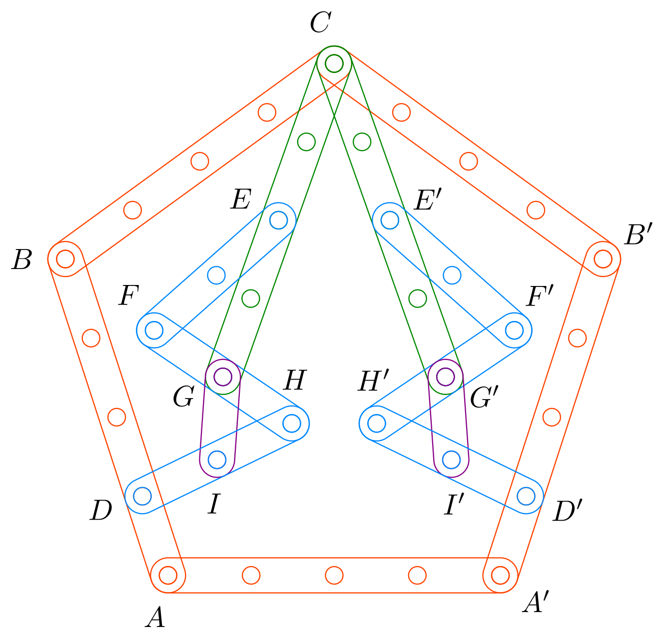

Figure 16: Regular pentagon of size 4 made rigid with 10 internal strips. $\overline{CD} = \sqrt{19 + 6\sqrt{5}}$.

Figure 16 show the regular pentagon A, A', B', C, B of size 4. We know the internal angle of pentagon is $\theta = \angle CBD = \dfrac{3\pi}{5}$ and $\cos\theta = \dfrac{1-\sqrt{5}}{4}$ so with the law of cosines we can calculate $\overline{CD}$ and the angle

$\angle DCB$:

$$\begin{aligned} \overline{CD} &= \sqrt{(\overline{BC})^2 + (\overline{BD})^2 - 2(\overline{BC})(\overline{BD})\cos\theta} \\ &= \sqrt{4^2 + 3^2 - 2(4)(3)\left(\frac{1-\sqrt{5}}{4}\right)} = \sqrt{19+6\sqrt{5}} \end{aligned} \tag{47}$$

$$\begin{aligned} \cos(\angle DCB) &= \frac{(\overline{CD})^2 + (\overline{BC})^2 - (\overline{BD})^2}{2(\overline{CD})(\overline{BC})} \\ &= \frac{(19+6\sqrt{5}) + 4^2 - 3^2}{2\left(\sqrt{19+6\sqrt{5}}\right)(4)} = \frac{13+3\sqrt{5}}{4\sqrt{19+6\sqrt{5}}} \end{aligned} \tag{48}$$

9.2.1 Rigid distance $\sqrt{19+6\sqrt{5}}$

Our software found several five strips clusters for rigid distance $\sqrt{19+6\sqrt{5}}$. We prove selected cluster $CEFGHID$ show in the figure inside the pentagon matches the expected distance. Set the cluster in the coordinate plane such that vertice G is at the origin and vertices F at $(-1, 0)$ and vertice H at $(+1, 0)$. Since triangle $\triangle EFG$ is isoscelles and $\overline{CG}$ is the double of $\overline{GE}$ we know angle $\angle CFG = \frac{\pi}{2}$ and we can calculate the abscissa and ordinate of vertex C:

$$C_x = -\overline{FG} = -1 \tag{49}$$

$$C_y = \sqrt{(\overline{CG})^2 - (\overline{FG})^2} = \sqrt{4^2 - 1^2} = \sqrt{15} \tag{50}$$

Since triangle $\triangle GHI$ is equilateral and $\overline{HD}$ is the double of $\overline{HI}$ we know angle $\angle DGH = \frac{\pi}{2}$ and we can calculate the abscissa and ordinate of vertex D:

$$D_x = 0 \tag{51}$$

$$D_y = -\sqrt{(\overline{HD})^2 - (\overline{GH})^2} = -\sqrt{2^2 - 1^2} = -\sqrt{3} \tag{52}$$

Finally we calculate the distance $\overline{CD}$

$$\begin{aligned} \overline{CD} &= \sqrt{(C_x - D_x)^2 + (C_y - D_y)^2} \\ &= \sqrt{(-1-0)^2 + (\sqrt{15}+\sqrt{3})^2} = \sqrt{19+6\sqrt{5}} \quad \blacksquare \end{aligned} \tag{53}$$

10 Pentagons of size 3

10.1 Size 3 with 10 internal strips

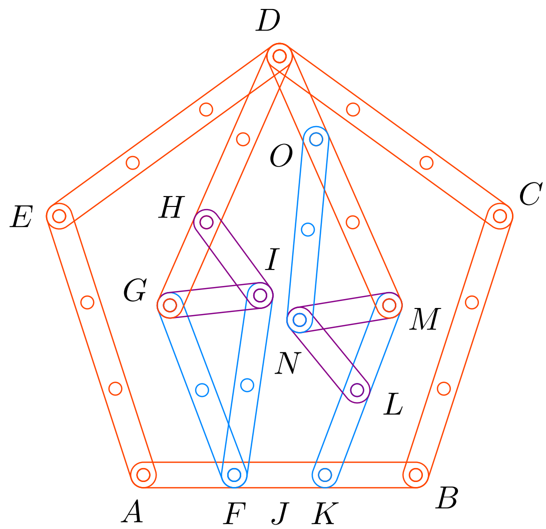

Figure 17: Regular pentagon of size 3 made rigid with 10 internal strips. $\overline{DF} = \overline{DK} = \dfrac{\sqrt{46+18\sqrt{5}}}{2}$.

Figure 17 show the regular pentagon A, B, C, D, E of size 3. We know the regular pentagon height is side length times $\dfrac{\sqrt{5+2\sqrt{5}}}{2}$ so in this case $\overline{DJ} = \dfrac{3\sqrt{5+2\sqrt{5}}}{2}$ and we can calculate $\overline{DF}$:

$$\begin{aligned} \overline{DF} &= \sqrt{(\overline{DJ})^2 + (\overline{FJ})^2} \\ &= \sqrt{\left(\frac{3\sqrt{5+2\sqrt{5}}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{46+18\sqrt{5}}}{2} \end{aligned} \tag{54}$$

10.1.1 Rigid distance $\dfrac{\sqrt{46+18\sqrt{5}}}{2}$

Our software found several five strips solutions and we use two different which fit inside the pentagon.

We identify two angles $\alpha = \angle HGI = \angle LMN$ of equilateral triangles and $\beta = \angle FGI = \angle NMO$ of isoscelles'. Adding the angles we get angles $\angle DGF = \angle DMK = (\alpha + \beta)$. From equilateral triangle $\triangle HGI$ we calculate $\cos\alpha$ and $\sin\alpha$:

$$\cos\alpha = \frac{\frac{\overline{GI}}{2}}{\overline{GH}} = \frac{\frac{1}{2}}{1} = \frac{1}{2} \tag{55}$$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2} \tag{56}$$

From isoscelles triangle $\triangle FGI$ we calculate $\cos\beta$ and $\sin\beta$:

$$\cos\beta = \frac{\frac{\overline{GI}}{2}}{\overline{GF}} = \frac{\frac{1}{2}}{2} = \frac{1}{4} \tag{57}$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4} \tag{58}$$

Now we calculate $\cos(\alpha+\beta)$ with the sum identity:

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{15}}{4}\right) = \frac{1-3\sqrt{5}}{8}\end{aligned} \tag{59}$$

Finally using the law of cosines we verify distance $\overline{DF}$:

$$\begin{aligned}\overline{DF} &= \sqrt{(\overline{DG})^2 + (\overline{FG}^2) - 2(\overline{DG})(\overline{FG})\cos(\alpha+\beta)} \\ &= \sqrt{3^2+2^2-2(3)(2)\left(\frac{1-3\sqrt{5}}{8}\right)} = \frac{\sqrt{46+18\sqrt{5}}}{2} \quad \blacksquare\end{aligned} \tag{60}$$

10.2 Size 3 with 14 internal strips

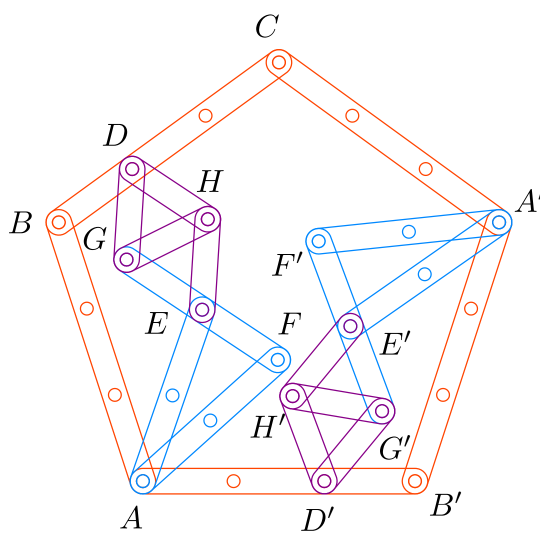

Figure 18: Regular pentagon of size 3 made rigid with 14 internal strips. $\overline{AD} = \dfrac{\sqrt{34+5\sqrt{5}}}{2}$.

Figure 18 show the regular pentagon A, B', A', C, B of size 3. We know the internal angle of pentagon is $\theta = \dfrac{3\pi}{5}$ and $\cos\theta = \dfrac{1-\sqrt{5}}{4}$ so with the law of cosines we can calculate $\overline{AD}$ and the angle $\angle BAD$:

$$\begin{aligned}\overline{AD} &= \sqrt{(\overline{AB})^2 + (\overline{BD})^2 - 2(\overline{AB})(\overline{BD})\cos\theta} \\ &= \sqrt{3^2+1^2-2(3)(1)\left(\frac{1-\sqrt{5}}{4}\right)} = \frac{\sqrt{34+6\sqrt{5}}}{2}\end{aligned} \tag{61}$$

$$\begin{aligned}\cos(\angle BAD) &= \frac{(\overline{AD})^2 + (\overline{AB})^2 - (\overline{BD})^2}{2(\overline{AD})(\overline{AB})} \\ &= \frac{\frac{34+6\sqrt{5}}{4} + 3^2 - 1^2}{2\left(\frac{\sqrt{34+6\sqrt{5}}}{2}\right)(3)} = \frac{11+\sqrt{5}}{2\sqrt{34+6\sqrt{5}}}\end{aligned} \tag{62}$$

10.2.1 Rigid distance $\dfrac{\sqrt{34+6\sqrt{5}}}{2}$

Our software found several options to make the distance with five strips but we have to manually add a sixth strip in order to make a cluster narrow enough to fit two times inside the pentagon of size 3. The result show as the duplicated cluster with vertices D, E, F, G, H of the figure. We are going to prove the cluster's distance $\overline{AD}$ matches the distance alredy calculated.

Noting the pair of adjacent equilateral triangles $\triangle DGH$ and $\triangle EGH$ we know angle $\angle DGE = \dfrac{2\pi}{3}$ and also $\cos\left(\dfrac{2\pi}{3}\right) = -\dfrac{1}{2}$ so we can calculate $\overline{DE}$ and the angle $\alpha \equiv \angle GDE$:

$$\overline{DE} = \sqrt{(\overline{DG})^2 + (\overline{GE})^2 - 2(\overline{DG})(\overline{GE})\cos(\angle DGE)} = \sqrt{1^2 + 1^2 - 2(1)(1)\left(-\frac{1}{2}\right)} = \sqrt{3} \tag{63}$$

$$\alpha \equiv= \angle GED$$

$$\cos\alpha = \frac{(\overline{GE})^2 + (\overline{DE})^2 - (\overline{DG})^2}{2(\overline{GE})(\overline{DE})} = \frac{3 + 1^2 - 1^2}{2(\sqrt{3})(1)} = \frac{\sqrt{3}}{2} \tag{64}$$

$$\sin\alpha = \sqrt{1 - \cos^2\alpha} = \sqrt{1 - \left(\frac{\sqrt{3}}{4}\right)^2} = \frac{1}{2} \tag{65}$$

From the isoscelles triangle $\triangle AEF$ we can calculate angle $\angle AEF$ noting $\cos(\angle AEF) = \dfrac{\overline{EF}/2}{\overline{AE}} = \dfrac{1/2}{2} = \dfrac{1}{4}$ so we define angle $\beta \equiv \angle AEG$ the supplementary of $\angle AEF$ and we get:

$$\beta \equiv \angle AEG = \pi - \angle AEF$$

$$\cos\beta = -\cos\angle AEF = -\frac{1}{4} \tag{66}$$

$$\sin\beta = \sqrt{1 - \cos^2\beta} = \sqrt{1 - \left(\frac{-1}{4}\right)^2} = \frac{\sqrt{15}}{4} \tag{67}$$

Now we can calculate the angle $\angle AED = \alpha + \beta$ with the sum identity and plugin the last sines and cosines:

$$\begin{aligned}\cos(\angle AED) &= \cos(\alpha+\beta)\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ &= \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{4}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{15}}{4}\right) = -\frac{\sqrt{3}+\sqrt{15}}{8}\end{aligned} \tag{68}$$

Finally we calculate $\overline{AD}$ with the law of cosines:

$$\begin{aligned}\overline{AD} &= \sqrt{(\overline{DE})^2 + (\overline{EA})^2 - 2(\overline{DE})(\overline{EA})\cos(\angle AED)}\\ &= \sqrt{3 + 2^2 - 2(\sqrt{3})(2)\left(-\frac{\sqrt{3}+\sqrt{15}}{8}\right)} = \frac{\sqrt{34+6\sqrt{5}}}{2} \quad \blacksquare\end{aligned} \tag{69}$$