

1 32 bits algebraic numbers

Let r_0, r_1, r_2 and r_3 irreducibles radicals with nesting 0, 1, 2 and 3:

$$r_0 = \pm b \quad (1.1)$$

$$r_1 = \pm c\sqrt{d} \quad (1.2)$$

$$r_2 = \pm e\sqrt{f \pm g\sqrt{h}} \quad (1.3)$$

$$r_3 = \pm i\sqrt{j \pm k\sqrt{l \pm m\sqrt{n}}} \quad (1.4)$$

We will use fourteen different 32-bit natural numbers, where a goes in the denominators and b, \dots, n in the numerators.

$$1 \leq a \leq 2^{32} - 1 \quad (1.5)$$

$$0 \leq b, c, d, e, f, g, h, i, j, k, l, m, n \leq 2^{32} - 1 \quad (1.6)$$

The signs are managed appart as extra boolean variables and there is one for each of the seven variables b, c, e, g, i, k and m .

We define four numbers of increasing complexity:

$$B \equiv \frac{r_0}{a} \quad (1.7)$$

$$D \equiv \frac{r_0 + r_1}{a} \iff c, d > 0 \quad (1.8)$$

$$H \equiv \frac{r_0 + r_1 + r_2}{a} \iff e, f, g, h > 0 \quad (1.9)$$

$$N \equiv \frac{r_0 + r_1 + r_2 + r_3}{a} \iff i, j, k, l, m, n > 0 \quad (1.10)$$

2 functions

Each of the radicals r_0, \dots, r_3 has a function to read their corresponding signs and integers variables:

$$f_0 \equiv f(\pm b) \quad (2.1)$$

$$f_1 \equiv f(\pm c, d) \quad (2.2)$$

$$f_2 \equiv f(\pm e, f, \pm g, h) \quad (2.3)$$

$$f_3 \equiv f(\pm i, j, \pm k, l, \pm m, n) \quad (2.4)$$

Each f_0, \dots, f_4 reduces the values with gcd and root simplifications.

Each of the algebraic numbers B, D, H and N has a function to read their radicals functions as inputs:

$$f_B \equiv f(f_0(\dots), a) \quad (2.5)$$

$$f_D \equiv f(f_0(\dots), f_1(\dots), a) \quad (2.6)$$

$$f_H \equiv f(f_0(\dots), f_1(\dots), f_2(\dots), a) \quad (2.7)$$

$$f_N \equiv f(f_0(\dots), f_1(\dots), f_2(\dots), f_3(\dots), a) \quad (2.8)$$

Each f_B, \dots, f_N adds the radicals reducing once more the variables with gcd root simplifications and now considering the denominator a .

3 Examples

3.1 f_B examples

$$\cos 0 = 1 \implies f_B(f_0(1), 1) \quad (3.1)$$

$$\sin \frac{\pi}{6} = \frac{1}{2} \implies f_B(f_0(1), 2) \quad (3.2)$$

3.2 f_D examples

$$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \implies f_D(\emptyset, f_1(1, 2), 2) \quad (3.3)$$

$$\sin \frac{\pi}{10} = \frac{-1 + \sqrt{5}}{4} \implies f_D(f_0(-1), f_1(1, 5), 4) \quad (3.4)$$

3.3 f_H examples

$$\sin \frac{\pi}{5} = \frac{\sqrt{10 - 2\sqrt{5}}}{4} \implies f_H(\emptyset, \emptyset, f_2(1, 10, -2, 5), 4) \quad (3.5)$$

$$\sin \frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4} \implies f_H(\emptyset, f_1(1, 6), f_2(1, 2, 0, 0), 4)* \quad (3.6)$$

$$\sin \frac{\pi}{12} = \frac{\sqrt{2 + \sqrt{3}}}{2} \implies f_H(\emptyset, \emptyset, f_2(1, 2, 1, 3), 2) \quad (3.7)$$

$$\cos \frac{\pi}{15} = \frac{1 + \sqrt{5} + \sqrt{30 - 6\sqrt{5}}}{8} \implies f_E(f_0(1), f_1(1, 5), f_2(1, 30, -6, 5), 8) \quad (3.8)$$

3.4 f_N examples

$$\cos \frac{\pi}{16} = \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \implies f_N(\emptyset, \emptyset, \emptyset, f_3(1, 2, 1, 2, 1, 2), 2) \quad (3.9)$$

$$\cos \frac{\pi}{24} = \frac{\sqrt{2 + \sqrt{2 + \sqrt{3}}}}{2} \implies f_N(\emptyset, \emptyset, \emptyset, f_3(1, 2, 1, 2, 1, 3), 2) \quad (3.10)$$

$$\cos \frac{2\pi}{17} = \frac{-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + 2\sqrt{17 + 3\sqrt{17} - \sqrt{170 + 38\sqrt{17}}}}{16} \quad (3.11)$$

$$\implies f_N(f_0(-1), f_1(1, 17), f_2(1, 34, -2, 17), f_3(2, 17, 3, 17, -1, 170, +38, 17), 16) \quad (3.12)$$

4 Operations with result B

4.1 NewB $B_1 \mapsto B$

$$B1 = \frac{\pm b_1}{a_1} \quad (4.1)$$

$$= \frac{\pm b}{a} \quad (4.2)$$

where $\{\pm b, a\} = \gcd\{\pm b_1, a_1\}$

4.2 AddBB $B_1 + B_2 \mapsto B$

$$B_1 + B_2 = \frac{\pm b_1}{a_1} + \frac{\pm b_2}{a_2} \quad (4.3)$$

$$= \frac{\pm b_1 a_2 \pm b_2 a_1}{a_1 a_2} \quad (4.4)$$

$$= \frac{\pm b}{a} \quad (4.5)$$

where $\{\pm b, a\} = \gcd\{\pm b_1 a_2 \pm b_2 a_1, a_1 a_2\}$

4.3 MulBB $B_1 \times B_2 \mapsto B$

$$B_1 \times B_2 = \frac{\pm b_1}{a_1} \times \frac{\pm b_2}{a_2} \quad (4.6)$$

$$= \frac{\pm b_1 b_2}{a_1 a_2} \quad (4.7)$$

$$= \frac{\pm b}{a} \quad (4.8)$$

where $\{\pm b, a\} = \gcd\{\pm b_1 b_2, a_1 a_2\}$

4.4 InvB $1/B_1 \mapsto B \iff a_1 > 0$

$$\frac{1}{B_1} = \frac{1}{\pm b_1/a_1} \quad (4.9)$$

$$= \frac{\pm a_1}{b_1} \quad (4.10)$$

$$= \frac{\pm b}{a} \quad (4.11)$$

where $\{\pm b\} = a_1, \{a\} = b_1$

5 Operations with result D

5.1 NewD $D_1 \mapsto D$

$$D_1 = \frac{\pm b_1 \pm c_1 \sqrt{d_1}}{a_1} \quad (5.1)$$

Do $\{p, q, r\} = \{a_1/G, b_1/G, c_1/G\} \iff G = \gcd\{a_1, b_1, c_1\} > 1$

$$= \frac{\pm q \pm r \sqrt{d_1}}{p} \quad (5.2)$$

Do $\{d\} = s^2 d_1 \iff s > 1$

$$= \frac{\pm q \pm rs \sqrt{d}}{p} \quad (5.3)$$

Do $\{a, b, c\} = \{p/G, q/G, rs/G\} \iff G = \gcd\{p, q, rs\}$

$$= \frac{\pm b \pm c \sqrt{d}}{a} \quad (5.4)$$

5.2 SqrtB $\sqrt{B_2} \mapsto D_1$

$$\sqrt{B_2} = \sqrt{\frac{\pm b_2}{a_2}} \quad (5.5)$$

$$= \frac{\sqrt{a_2 b_2}}{a_2} \quad (5.6)$$

Do $\{a_1, b_1, c_1, d_1\} = \{a_2, 0, 1, a_2 b_2\}$

$$= \frac{\pm b_1 \pm c_1 \sqrt{d_1}}{a_1} \quad (5.7)$$

Solve as NewD

5.3 InvD $1/D_2 \mapsto D$

$$\begin{aligned}
1/D_2 &= \frac{a_2}{\pm b_2 \pm c_2 \sqrt{d_2}} \\
&= \frac{\pm a_2 b_2 \mp a_2 c_2 \sqrt{d_2}}{b_2^2 - c_2^2 d_2} \\
&\quad \text{Do } \{a_1, b_1, c_1, d_1\} = \{b_2^2 - c_2^2 d_2, \pm a_2 b_2, \mp a_2 c_2, d_2\} \\
&= \frac{\pm b_1 \pm c_1 \sqrt{d_1}}{a_1} \\
&\quad \text{Solve as NewD}
\end{aligned}$$

6 Operations with result H

6.1 $D_1 + D_2 \mapsto H$ iiii

$$D_1 + D_2 = \frac{\pm b_1 \pm c_1 \sqrt{d_1}}{a_1} + \frac{\pm b_2 \pm c_2 \sqrt{d_2}}{a_2} \quad (6.1)$$

$$= \frac{(\pm a_2 b_1 \pm a_1 b_2) \pm a_2 c_1 \sqrt{d_1} \pm a_1 c_2 \sqrt{d_2}}{a_1 a_2} \quad (6.2)$$

$$= \frac{\pm q \pm r \sqrt{d_1} \pm s \sqrt{d_2}}{p} \quad (6.3)$$

$$\begin{aligned}
&\quad \text{where } \{p, q, r, s\} = \gcd\{a_1 a_2, (\pm a_2 b_1 \pm a_1 b_2), \pm a_2 c_1, \pm a_1 c_2\} \\
&= \frac{\pm q \pm \sqrt{r^2 d_1 + s^2 d_2 \pm 2rs \sqrt{d_1 d_2}}}{p} \quad (6.4)
\end{aligned}$$

$$= \frac{\pm q \pm \sqrt{t \pm 2rsu \sqrt{h}}}{p} \quad (6.5)$$

$$\begin{aligned}
&\quad \text{where } \{t\} = r^2 d_1 + s^2 d_2 \text{ and } \{u^2 h\} = d_1 d_2 \\
&= \frac{\pm q \pm v \sqrt{f \pm g \sqrt{h}}}{p} \quad (6.6)
\end{aligned}$$

$$\begin{aligned}
&\quad \text{where } \{v^2 f\} = t \text{ and } \{v^2 g\} = 2rsu \\
&= \frac{\pm d \pm e \sqrt{f \pm g \sqrt{h}}}{a} \quad (6.7)
\end{aligned}$$

$$\begin{aligned}
&\quad \text{where } \{a, d, e\} = \gcd\{p, \pm q, \pm qv\} \\
&\quad (6.8)
\end{aligned}$$

6.2 $\sqrt{C_1} = F_2$

$$\begin{aligned}
\sqrt{C_1} &= \sqrt{\frac{a_1 \sqrt{c_1}}{b_1}} \\
&= \frac{\sqrt{a_1 b_1 \sqrt{c_1}}}{b_1} \\
&= \frac{m \sqrt{e_2 \sqrt{c_1}}}{b_1} & a_1 b_1 = m^2 e_2 \\
&= \frac{a_2 \sqrt{e_2 \sqrt{c_1}}}{b_2} & (a_2, b_2) = \gcd(m, b_1)
\end{aligned}$$

6.3 $C_1 + D_2 = F_3$

$$\begin{aligned}
C_1 + D_2 &= \frac{\pm a_1 \sqrt{c_1}}{b_1} + \frac{\pm a_2 \sqrt{c_2} \pm d_2}{b_2} \\
&= \frac{\pm a_1 b_2 \sqrt{c_1} \pm a_2 b_1 \sqrt{c_2} \pm d_2 b_1}{b_1 b_2} \\
&= \frac{\pm m \sqrt{c_1} \pm n \sqrt{c_2} \pm p}{o} \\
&= \frac{\sqrt{m^2 c_1 + n^2 c_2 \pm 2mn \sqrt{c_1 c_2}} \pm p}{o} \\
&= \frac{\sqrt{q \pm 2mnr \sqrt{f_3}} \pm p}{o} \\
&= \frac{s \sqrt{c_3} \pm e_3 \sqrt{f_3} \pm p}{o} \\
&= \frac{a_3 \sqrt{c_3} \pm e_3 \sqrt{f_3} \pm d_3}{b_3}
\end{aligned}$$

$$(\pm m, \pm n, \pm p, o) = \gcd(\pm a_1 b_2, \pm a_2 b_1, \pm d_2 b_1, b_1 b_2)$$

$$q = m^2 c_1 + n^2 c_2, c_1 c_2 = r^2 f_3$$

$$q = s^2 c_3, 2mnr = s^2 e_3$$

$$(a_3, b_3, \pm d_3) = \gcd(s, \pm p, o)$$

6.4 $1/D_1 = D_2$

$$\begin{aligned}
1/D_1 &= \frac{b_1}{\pm a_1 \sqrt{c_1} \pm d_1} \\
&= \frac{\pm a_1 b_1 \sqrt{c_1} \mp b_1 d_1}{a_1^2 c_1 - d_1^2} \\
&= \frac{a_2 \sqrt{c_1} \pm d_2}{b_2}
\end{aligned}$$

$$(a_2, b_2, d_2) = \gcd(\pm a_1 b_1, \mp b_1 d_1, a_1^2 c_1 - d_1^2)$$

6.5 $\sqrt{D_1} = F_2$ editing...

$$\begin{aligned}
\sqrt{D_1} &= \sqrt{\frac{\pm a_1 \sqrt{c_1} \pm d_1}{b_1}} \\
&= \frac{\sqrt{\pm b_1 d_1 \pm a_1 b_1 \sqrt{f_2}}}{b_1} \\
&= \frac{m \sqrt{c_2} \pm e_2 \sqrt{f_2}}{b_1} \\
&= \frac{a_2 \sqrt{c_2} \pm e_2 \sqrt{f_2}}{b_2}
\end{aligned}$$

$$f_2 = c_1$$

$$\pm b_1 d_1 = m^2 c_2, \pm a_1 b_1 = m^2 e_2$$

$$(a_2, b_2) = \gcd(m, b_1)$$

6.6 $D_1 + D_2 = F_3$

$$\begin{aligned}
D_1 + D_2 &= \frac{\pm a_1 \sqrt{c_1} \pm d_1}{b_1} + \frac{\pm a_2 \sqrt{c_2} \pm d_2}{b_2} \\
&= \frac{\pm a_1 b_2 \sqrt{c_1} \pm a_2 b_1 \sqrt{c_2} \pm d_1 b_2 \pm d_2 b_1}{b_1 b_2} \\
&= \frac{\pm m \sqrt{c_1} \pm n \sqrt{c_2} \pm p}{o} & (\pm m, \pm n, \pm p, o) = \gcd(\pm a_1 b_2, \pm a_2 b_1, \pm d_1 b_2 \pm d_2 b_1, b_1 b_2) \\
&= \frac{\sqrt{m^2 c_1 + n^2 c_2 \pm 2mn \sqrt{c_1 c_2}} \pm p}{o} \\
&= \frac{\sqrt{q \pm 2mnr \sqrt{f_3}} \pm p}{o} & q = m^2 c_1 + n^2 c_2, c_1 c_2 = r^2 f_3 \\
&= \frac{s \sqrt{c_3 \pm e_3 \sqrt{f_3}} \pm p}{o} & q = s^2 c_3, 2mnr = s^2 e_3 \\
&= \frac{a_3 \sqrt{c_3 \pm e_3 \sqrt{f_3}} \pm d_3}{b_3} & (a_3, b_3, \pm d_3) = \gcd(s, \pm p, o)
\end{aligned}$$

6.7 $D_1 \times D_2 = F_3$

$$\begin{aligned}
D_1 \times D_2 &= \frac{\pm a_1 \sqrt{c_1} \pm d_1}{b_1} \times \frac{\pm a_2 \sqrt{c_2} \pm d_2}{b_2} \\
&= \frac{\pm a_1 a_2 \sqrt{c_1 c_2} \pm a_1 d_2 \sqrt{c_1} \pm a_2 d_1 \sqrt{c_2} \pm d_1 d_2}{b_1 b_2}
\end{aligned}$$

6.8 MulDD $D_1 \times D_2 \mapsto H$???

$$\begin{aligned}
D_1 \times D_2 &= \frac{\pm b_1 \pm c_1 \sqrt{d_1}}{a_1} \times \frac{\pm b_2 \pm c_2 \sqrt{d_2}}{a_2} \\
&= \frac{\pm b_1 b_2 \pm b_1 c_2 \sqrt{d_2} \pm b_2 c_1 \sqrt{d_1} \pm c_1 c_2 \sqrt{d_1 d_2}}{a_1 a_2} \\
&= \frac{\pm a_1 a_2 m \sqrt{c_3}}{b_1 b_2} & c_1 c_2 = m^2 c_3 \\
&= \frac{\pm a_3 \sqrt{c_3}}{b_3} & (\pm a_3, b_3) = \gcd(\pm a_1 a_2 m, b_1 b_2)
\end{aligned}$$