Meccano fox-surd frame

https://github.com/heptagons/meccano/frames/fox-surd

Abstract

Meccano [1] fox-surd frame is a generalization of fox-frame[2] where at least one of the frame's strips size is no longer an integer but a surd.

1 Pentagons fox-surd

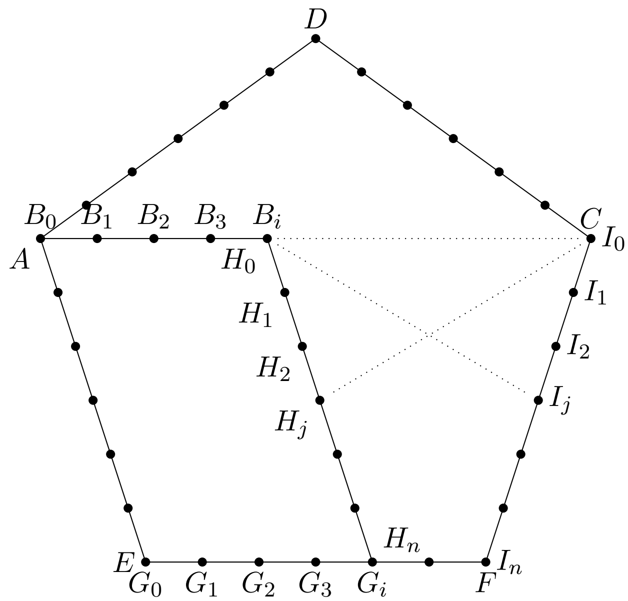

Figure 1: Pentagon of size n where each segment separated by circles represents a unit. We have a surd frame formed by the six points: B_i, I_0, H_j, I_j, H_n and I_n. By iterating the values $i, j = 0, ..., n$ we'll get diverse frames.

From figure 1 the fox-surd frame has three real strips of integer size:

- $\overline{B_iG_i}$ of size n.
- $\overline{G_iI_n}$ of size $n - i$, where $i = 0, ..., n$.
- $\overline{I_0I_n}$ of size n.

The other two strips are generic in the sense the sizes can be surds:

- $\overline{B_iI_j}$ of size to be determined $f(n, i, j)$, where $i, j = 0, ..., n$.
- $\overline{H_jI_0}$ of equal size of $\overline{B_iI_j}$.

[1] Meccano mathematics by 't Hooft
[2] Meccano fox frame

From the regular pentagon we know the main diagonal $\overline{AC}$ equals $\frac{1+\sqrt{5}}{4}n$ where n is the pentagon side size. We can calculate different segments of the main diagonal iterating $i = 0, ..., n$:

$$\begin{aligned}
B_0 &\equiv A \\
\overline{B_0C} &= \frac{1+\sqrt{5}}{4}n \qquad (1) \\
\overline{B_iC} &= \frac{1+\sqrt{5}}{4}n - i \\
&= \frac{n-4i}{4} + \frac{\sqrt{5}}{4}, \quad i = 0, ..., n \\
&= \frac{x_i}{4} + \frac{\sqrt{5}}{4}, \quad x_i = n - 4i \qquad (2)
\end{aligned}$$

From the regular pentagon we know the angle B_iCH_j equals $2\pi/5$ so we have:

$$\theta \equiv \angle B_iCH_j \qquad (3)$$

$$\cos\theta = \cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4} \qquad (4)$$

1.1 Pentagon surds sizes

Using the law of cosines we can calculate one of the frame surds $s_{ij} \equiv \overline{B_iI_j}$. We notice the value of $\overline{CI_j}$ equals j, and we'll use the values of $\overline{B_iC}$ from equation 2, and the cosine value from equation 4 to get:

$$\begin{aligned}
s_{ij}^2 &\equiv \overline{B_iI_j}^2 \qquad (5) \\
&= \overline{CI_j}^2 + \overline{B_iC}^2 - 2\overline{CI_j} \times \overline{B_iC}\cos\theta \\
&= j^2 + \left(\frac{x_i}{4} + \frac{\sqrt{5}}{4}\right)^2 - 2j\left(\frac{x_i}{4} + \frac{\sqrt{5}}{4}\right)\left(\frac{-1+\sqrt{5}}{4}\right) \qquad (6) \\
&= j^2 + \frac{1}{16}\left(x_i + \sqrt{5}\right)^2 - \frac{2j}{16}\left(x_i + \sqrt{5}\right)\left(-1+\sqrt{5}\right) \qquad (7)
\end{aligned}$$

We multiply both sides by 16:

$$\begin{aligned}
(4s_{ij})^2 &= 16j^2 + x_i^2 + 2x_i\sqrt{5} + 5 - 2j\left(x_i + \sqrt{5}\right)\left(-1+\sqrt{5}\right) \qquad (8) \\
&= 16j^2 + x_i^2 + 2x_i\sqrt{5} + 5 - 2j(-x_i + 5 + (x_i - 1)\sqrt{5}) \qquad (9) \\
&= 16j^2 + x_i^2 + 5 + 2x_ij - 10j + 2(x_i - x_ij + j)\sqrt{5} \qquad (10)
\end{aligned}$$

In order to have a simpler $(4s_i)^2 = u + v\sqrt{5}$ we define two variables u and v. We replace again $x_i = n - 4i$ defined in equation 2:

$$\begin{aligned}
u &\equiv 16j^2 + x_i^2 + 5 + 2x_ij - 10j \\
&= 16j^2 + (n-4i)^2 + 5 + 2(n-4i)j - 10j \\
&= 16j^2 + n^2 - 8ni + 16i^2 + 5 - 2ni - 8i^2 \\
&= 16j^2 + n^2 - 10ni + 8i^2 + 5 \\
&= (n-5i)^2 + 16j^2 - 17i^2 + 5 \\
&= (n-5i)^2 + 16(j^2 - i^2) + 5 - i^2 \qquad (11) \\
v &\equiv 2(x_i - ix_i + i) \\
&= 2(n - 4i - i(n-4i) + i) \\
&= 2(n - 3i - ni + 4i^2) \\
&= 2((2i-1)^2 - (n-1)(i-1)) \qquad (12)
\end{aligned}$$

Finally we have $s_i j$ in function of n the side:

$$\begin{aligned} s_{ij} &= \frac{\sqrt{u + v\sqrt{5}}}{4} \\ &= \frac{\sqrt{(n-5i)^2 + 16(j^2 - i^2) + 5 - i^2 + 2((2i-1)^2 - (n-1)(i-1))\sqrt{5}}}{4} \end{aligned} \tag{13}$$

1.2 Pentagons surds simplification

If value v from equation 12 is zero s_{ij} simplifies to $\frac{\sqrt{u}}{4}$:

$$\begin{aligned} v &= 0 \\ 2((2i-1)^2 - (n-1)(i-1)) &= 0 \\ (2i-1)^2 &= (n-1)(i-1) \end{aligned} \tag{14}$$