Triple unit

https://github.com/heptagons/meccano/units/triple

Abstract

A **Triple unit** is a group of **five** meccano [1] strips a, b, c, d, e forming **three equal angles** θ intended to build three consecutive perimeter sides of some regular polygons. We look for integer values of strip e in function of integer values of sides a, b, c, d and a particular angle θ. We confirm a generic equation found matches the one used to build pentagons of type 2 [2]. Here we found a lot of hexagons and filter some not trivial solutions. We look for octagons, decagons and dodecagons.

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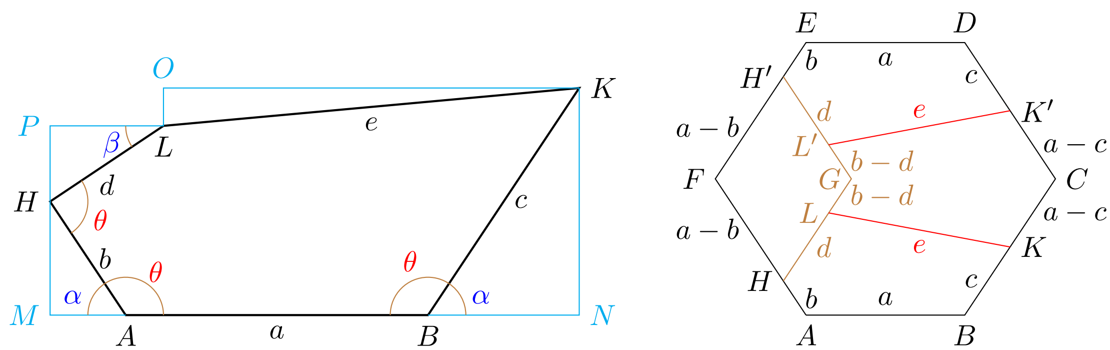

Figure 1: At the left we have the Triple unit (three angles θ) with the strips a, b, c, d, e. At the right we use two units to build a regular polygon of side a extending strips b, c, d to fix everthing.

1 Algebra

From nodes A and B of fig 1 we get α from θ ($\pi = 180°$):

$$\begin{aligned}\theta &= \pi - \alpha \\ \alpha &= \pi - \theta \end{aligned} \tag{1}$$

And from node H we get β from θ:

$$\begin{aligned}\theta &= \alpha + \beta \\ \beta &= \theta - \alpha = \theta - (\pi - \theta) = 2\theta - \pi \end{aligned} \tag{2}$$

We calculate horizontal segment $\overline{OK}$:

$$\begin{aligned}\overline{OK} &= \overline{MA} + a + \overline{BN} - \overline{PL} \\ &= b\cos\alpha + a + c\cos\alpha - d\cos\beta \\ &= a + (b+c)\cos\alpha - d\cos\beta \\ &= a + (b+c)\cos(\pi - \theta) - d\cos(2\theta - \pi) \\ &= a - (b+c)\cos\theta + d\cos(2\theta) \end{aligned} \tag{3}$$

[1] Meccano mathematics by ‘t Hooft

[2] Meccano pentagons

And vertical segment $\overline{OL}$:

$$\begin{aligned}
\overline{OL} &= \overline{KN} - \overline{PH} - \overline{HM} \\
&= c \sin \alpha - d \sin \beta - b \sin \alpha \\
&= (c - b) \sin \alpha - d \sin \beta \\
&= (c - b) \sin (\pi - \theta) - d \sin (2\theta - \pi) \\
&= (c - b) \sin \theta + d \sin (2\theta) \qquad (4)
\end{aligned}$$

So we can express e in function of a, b, c, d and angle θ:

$$\begin{aligned}
e^2 &= (\overline{OK})^2 + (\overline{OL})^2 \\
&= (a - (b + c) \cos \theta + d \cos(2\theta))^2 + ((c - b) \sin \theta + d \sin(2\theta))^2 \\
&= a^2 + (b^2 + 2bc + c^2) \cos^2 \theta + d^2 \cos^2(2\theta) + (c^2 - 2cb + b^2) sin^2\theta + d^2 \sin^2(2\theta) \\
&\quad - 2a(b + c) \cos \theta + 2ad \cos(2\theta) - 2(b + c)d \cos \theta \cos(2\theta) \\
&\quad + 2(c - b)d \sin \theta \sin(2\theta) \\
&= a^2 + b^2 + c^2 + d^2 + 2bc \cos^2 \theta - 2bc \sin^2 \theta \\
&\quad - 2a(b + c) \cos \theta + 2ad \cos(2\theta) \\
&\quad - 2d((b + c) \cos \theta \cos(2\theta) + (b - c) \sin \theta \sin(2\theta)) \qquad (5) \\
&= a^2 + b^2 + c^2 + d^2 + 2bc(\cos^2 \theta - \sin^2 \theta) - 2a(b + c) \cos \theta + 2ad \cos(2\theta) \\
&\quad - 2d(b(\cos \theta \cos(2\theta) + \sin \theta \sin(2\theta)) + c(\cos \theta \cos(2\theta) - \sin \theta \sin(2\theta))) \\
&= a^2 + b^2 + c^2 + d^2 + 2bc \cos(2\theta) - 2a(b + c) \cos \theta + 2ad \cos(2\theta) \\
&\quad - 2d(b \cos(\theta - 2\theta) + c \cos(\theta + 2\theta)) \\
&= a^2 + b^2 + c^2 + d^2 + 2(bc + ad) \cos(2\theta) - 2a(b + c) \cos \theta - 2d(b \cos \theta + c \cos(3\theta)) \\
&= a^2 + b^2 + c^2 + d^2 + 2(bc + ad) \cos(2\theta) - 2(ab + ac) \cos \theta - 2(bd \cos \theta + cd \cos(3\theta)) \\
&= \boxed{a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd) \cos \theta + 2(bc + ad) \cos(2\theta) - 2cd \cos(3\theta)} \qquad (6)
\end{aligned}$$

2 Regular polygons

Polygon	θ	$\cos \theta$	$\cos(2\theta)$	$\cos(3\theta)$
Pentagon	$\frac{3\pi}{5}$	$\frac{1-\sqrt{5}}{4}$	$\frac{-1-\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$
Hexagon	$\frac{2\pi}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1
Octagon	$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$
Decagon	$\frac{4\pi}{5}$	$\frac{-1-\sqrt{5}}{4}$	$\frac{-1+\sqrt{5}}{4}$	$\frac{-1+\sqrt{5}}{4}$
Dodecagon	$\frac{5\pi}{6}$			

Table 1: Regular polygons internal angles and cosines.

We will test last equation into several polygons. Table 1 show the possible constructions and the angles and cosines. Only when we'll get e integer we'll have a solution.

3 Equilateral pentagons

We replace the cosines for pentagon in table 1 in e^2 equation:

$$\begin{aligned}
e^2 &= a^2+b^2+c^2+d^2-2(ab+ac+bd)\cos\theta+2(bc+ad)\cos(2\theta)-2cd\cos(3\theta)\\
&= a^2+b^2+c^2+d^2-2(ab+ac+bd)\left(\frac{1-\sqrt{5}}{4}\right)+2(bc+ad)\left(\frac{-1-\sqrt{5}}{4}\right)-2cd\left(\frac{1+\sqrt{5}}{4}\right)\\
&= a^2+b^2+c^2+d^2-\frac{ab+ac+bd+bc+ad+cd}{2}+\frac{ab+ac+bd-bc-ad-cd}{2}\sqrt{5} \qquad (7)
\end{aligned}$$

e cannot to be and integer if the factor of $\sqrt{5}$ is not zero so we force this factor to be zero:

$$\begin{aligned}
ab+ac+bd-bc-ad-cd &= 0\\
ab+ac+bd &= bc+ad+cd \qquad (8)\\
ab+ac-bc &= (a-b+c)d \qquad (9)
\end{aligned}$$

We replace $ab+ac+bd$ by $bc+ad+cd$ in the e^2 equation to get:

$$\begin{aligned}
e^2 &= a^2+b^2+c^2+d^2-\frac{(bc+ad+cd)+bc+ad+cd}{2}+\frac{0}{2}\sqrt{5}\\
&= a^2+b^2+c^2+d^2-bc-ad-cd\\
e &= \sqrt{a^2+b^2+c^2+d^2-bc-(a+c)d} \quad \texttt{iff} \quad ab+ac-bc=(a-b+c)d \qquad (10)
\end{aligned}$$

The last formula matches the formula used in the paper Meccano pentagons which finds several pentagons of type 2.

4 Equilateral hexagons

We replace the cosines for hexagon in table 1 in e^2 equation:

$$\begin{aligned}
e^2 &= a^2+b^2+c^2+d^2-2(ab+ac+bd)\cos\theta+2(bc+ad)\cos(2\theta)-2cd\cos(3\theta)\\
&= a^2+b^2+c^2+d^2-2(ab+ac+bd)\left(-\frac{1}{2}\right)+2(bc+ad)\left(-\frac{1}{2}\right)-2cd(1)\\
&= a^2+b^2+c^2+d^2+ab+ac+bd-bc-ad-2cd\\
&= (a+b)^2+(c-d)^2-ab+ac+bd-bc-ad\\
&= (a+b)^2+(c-d)^2+(c-d)(a-b)-ab\\
&= (a+b)^2+(c-d)(a-b+c-d)-ab\\
e &= \sqrt{(a+b)^2+(c-d)(a-b+c-d)-ab} \qquad (11)
\end{aligned}$$

4.1 Hexagons software

We wrote software code to look for hexagons using the formula for e and set several filters to prevent trivial solutions. We say an hexagon is nice when $e \le a$. Next is a partial list of nice hexagons:

```
  1   a=  7 b=  1 c=  2 d=  6 e=  7
  2   a=  7 b=  1 c=  4 d=  6 e=  7
  3   a= 13 b=  2 c=  5 d= 11 e= 13
  4   a= 13 b=  2 c=  6 d= 11 e= 13
  5   a= 14 b=  1 c=  6 d= 13 e= 13
  6   a= 14 b=  1 c=  7 d= 13 e= 13
```

```
  7   a= 15  b=   1  c=   5  d= 14  e= 14
  8   a= 15  b=   1  c=   9  d= 14  e= 14
  9   a= 19  b=   2  c=   3  d= 17  e= 19
 10   a= 19  b=   2  c= 14  d= 17  e= 19
 11   a= 20  b=   1  c=   4  d= 19  e= 19
 12   a= 20  b=   1  c= 15  d= 19  e= 19
      ...
105   a= 58  b=   5  c= 10  d= 53  e= 57
106   a= 58  b=   5  c= 43  d= 53  e= 57
107   a= 59  b=   1  c= 27  d= 58  e= 52
108   a= 59  b=   1  c= 31  d= 58  e= 52
109   a= 59  b=   4  c= 11  d= 55  e= 57
110   a= 59  b=   4  c= 44  d= 55  e= 57
111   a= 59  b=   5  c= 19  d= 54  e= 56
112   a= 59  b=   5  c= 35  d= 54  e= 56
--- PASS: TestHexagonsNice (0.01s)
```

Results from `github.com/heptagons/meccano/units/triple/triple_test.go TestHexagonsNice`

4.2 Hexagons examples

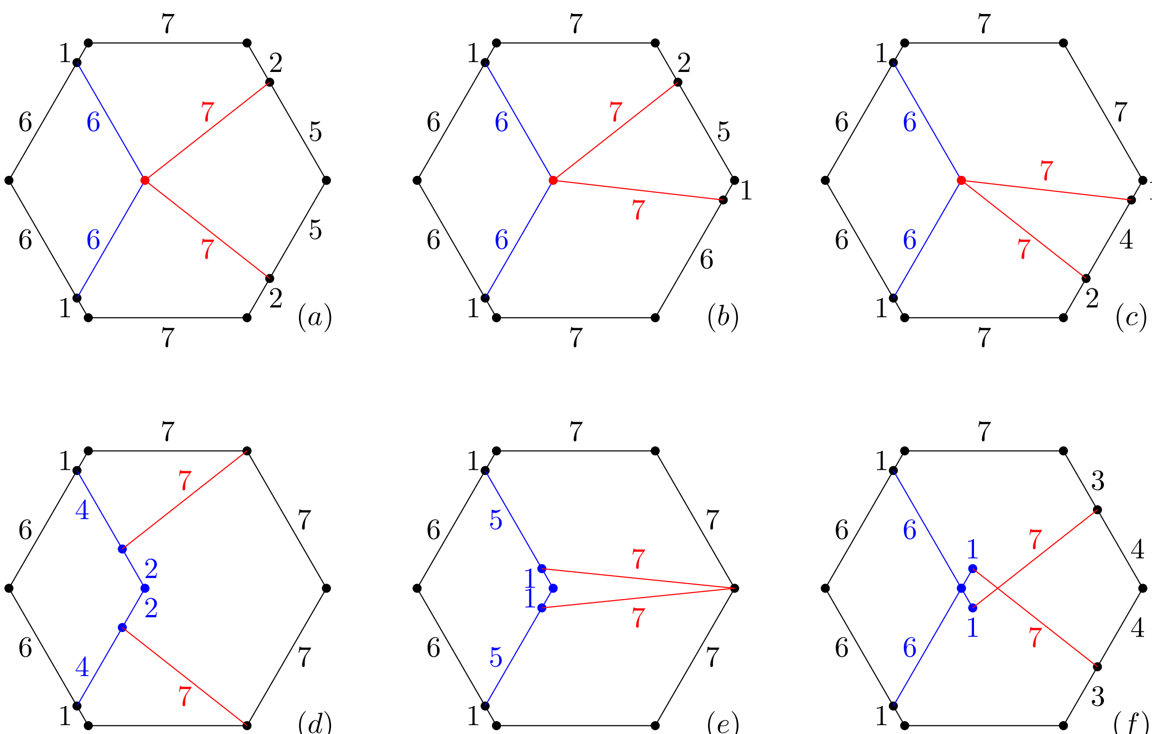

Figure 2: Constructions options of the nice hexagon side $a = 7, b = 1, e = 7$. Cases $(a) - (e)$ requires only eleven bolts. Case (f) has the 10 strips of size 7.

The nice hexagons results has related pairs and there are several ways to build each case. Figure 2 show different ways to build a nice hexagon.

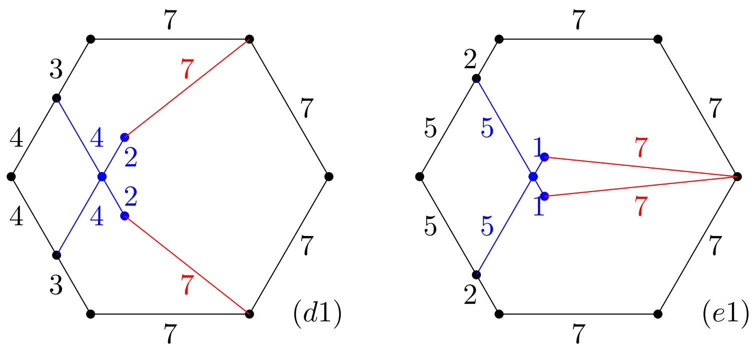

Figure 3: Variations of constructions of the nice hexagon side $a = 7, b = 1, e = 7$. Cases $(d1)$ and $(e1)$ are adaptations of cases (d) and (e) of figure 2 where only the blue strips are displaced. Such changes mantain the internal bolts, red strips and perimeter the same. The original **Triple unit** a, b, c, d, e irregular pentagon is replaced by an irregular hexagon clearly visible in case $(e1)$.

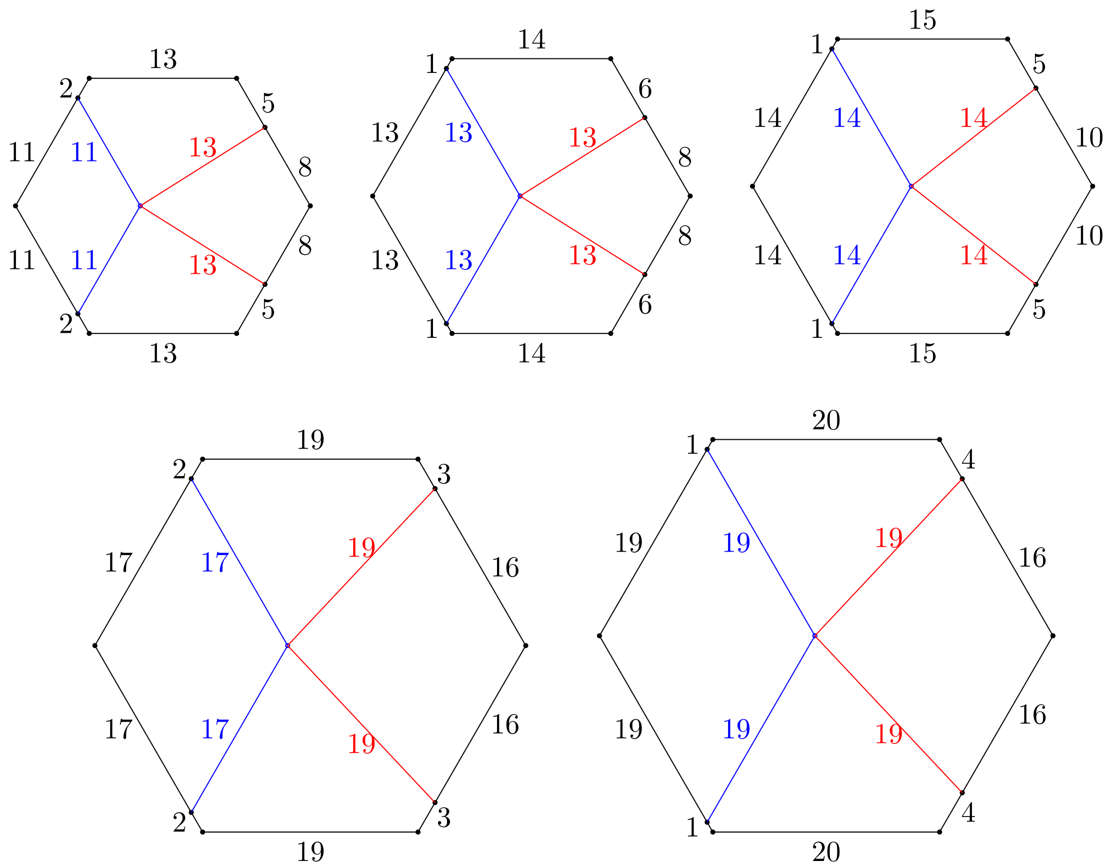

Figure 4: More nice hexagons from sizes $13 - 20$.

5 Regular octagons

We replace the cosines for octagon in table 1 in e^2 equation:

$$\begin{aligned}
e^2 &= a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd)\cos\theta + 2(bc + ad)\cos(2\theta) - 2cd\cos(3\theta) \\
&= a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd)\left(-\frac{\sqrt{2}}{2}\right) + 2(bc + ad)\,(0) - 2cd\left(\frac{\sqrt{2}}{2}\right) \\
&= a^2 + b^2 + c^2 + d^2 + (ab + ac + bd - cd)\sqrt{2} \qquad (12)
\end{aligned}$$

e cannot to be and integer if the factor of $\sqrt{2}$ is not zero, so we force this factor to be zero:

$$\begin{aligned}
ab + ac + bd - cd &= 0 \\
a(b + c) &= (c - b)d \qquad (13) \\
e^2 &= a^2 + b^2 + c^2 + d^2 \qquad (14)
\end{aligned}$$

5.1 Octagons examples

Conjecture: No possible octagons formed with triple unit.

6 Equilateral decagons

We replace the cosines for decagon in table 1 in e^2 equation:

$$\begin{aligned}
e^2 &= a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd)\cos\theta + 2(bc + ad)\cos(2\theta) - 2cd\cos(3\theta) \\
&= a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd)\left(\frac{-1 - \sqrt{5}}{4}\right) + 2(bc + ad)\left(\frac{-1 + \sqrt{5}}{4}\right) - 2cd\left(\frac{-1 + \sqrt{5}}{4}\right) \\
&= a^2 + b^2 + c^2 + d^2 + \frac{ab + ac + bd - bc - ad + cd}{2} + \frac{ab + ac + bd + bc + ad - cd}{2}\sqrt{5} \qquad (15)
\end{aligned}$$

e cannot to be and integer if the factor of $\sqrt{5}$ is not zero so we force this factor to be zero:

$$\begin{aligned}
ab + ac + bd + bc + ad - cd &= 0 \\
ab + ac + bc &= (c - a - b)d \qquad (16)
\end{aligned}$$

We replace $ab + ac + bd$ by $cd - bc - ad$ in the e^2 equation to get:

$$\begin{aligned}
e^2 &= a^2 + b^2 + c^2 + d^2 + \frac{(cd - bc - ad) - bc - ad + cd}{2} + \frac{0}{2}\sqrt{5} \\
&= a^2 + b^2 + c^2 + d^2 + cd - bc - ad \\
e &= \sqrt{a^2 + b^2 + c^2 + d^2 - bc - (a - c)d} \qquad (17)
\end{aligned}$$