1 32 bits algebraic numbers

Let r_0, r_1, r_2 and r_3 irreducibles radicals with nesting 0, 1, 2 and 3:

$$r_0 = \pm b \tag{1.1}$$
$$r_1 = \pm c\sqrt{d} \tag{1.2}$$
$$r_2 = \pm e\sqrt{f \pm g\sqrt{h}} \tag{1.3}$$
$$r_3 = \pm i\sqrt{j \pm k\sqrt{l \pm m\sqrt{n}}} \tag{1.4}$$

We will use fourteen different 32-bit natural numbers, where a goes in the denominators and $b, ...n$ in the numerators.

$$1 \leq a \leq 2^{32} - 1 \tag{1.5}$$
$$0 \leq b, c, d, e, f, g, h, i, j, k, l, m, n \leq 2^{32} - 1 \tag{1.6}$$

The signs are managed appart as extra boolean variables and there is one for each of the seven variables b, c, e, g, i, k and m.

We define four numbers of increasing complexity:

$$B \equiv \frac{r_0}{a} \tag{1.7}$$
$$D \equiv \frac{r_0 + r_1}{a} \iff c, d > 0 \tag{1.8}$$
$$H \equiv \frac{r_0 + r_1 + r_2}{a} \iff e, f, g, h > 0 \tag{1.9}$$
$$N \equiv \frac{r_0 + r_1 + r_2 + r_3}{a} \iff i, j, k, l, m, n > 0 \tag{1.10}$$

2 functions

Each of the radicals $r_0, ..., r_3$ has a function to read their corresponding signs and integers variables:

$$f_0 \equiv f(\pm b) \tag{2.1}$$
$$f_1 \equiv f(\pm c, d) \tag{2.2}$$
$$f_2 \equiv f(\pm e, f, \pm g, h) \tag{2.3}$$
$$f_3 \equiv f(\pm i, j, \pm k, l, \pm m, n) \tag{2.4}$$

Each $f_0, ...f_4$ reduces the values with gcd and root simplifications.

Each of the algebraic numbers B, D, H and N has a function to read their radicals functions as inputs:

$$f_B \equiv f(f_0(...), a) \tag{2.5}$$
$$f_D \equiv f(f_0(...), f_1(...), a) \tag{2.6}$$
$$f_H \equiv f(f_0(...), f_1(...), f_2(...), a) \tag{2.7}$$
$$f_N \equiv f(f_0(...), f_1(...), f_2(...), f_3(...), a) \tag{2.8}$$

Each $f_B, ...f_N$ adds the radicals reducing once more the variables with gcd root simplifications and now considering the denominator a.

3 Examples

3.1 f_B examples

$$\cos 0 = 1 \implies f_B(f_0(1), 1) \tag{3.1}$$
$$\sin \frac{\pi}{6} = \frac{1}{2} \implies f_B(f_0(1), 2) \tag{3.2}$$

3.2 f_D examples

$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \implies f_D(\emptyset, f_1(1,2), 2) \tag{3.3}$$
$$\sin\frac{\pi}{10} = \frac{-1+\sqrt{5}}{4} \implies f_D(f_0(-1), f_1(1,5), 4) \tag{3.4}$$

3.3 f_H examples

$$\sin\frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4} \implies f_H(\emptyset, \emptyset, f_2(1,10,-2,5), 4) \tag{3.5}$$
$$\sin\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4} \implies f_H(\emptyset, f_1(1,6), f_2(1,2,0,0), 4)* \tag{3.6}$$
$$\sin\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2} \implies f_H(\emptyset, \emptyset, f_2(1,2,1,3), 2) \tag{3.7}$$
$$\cos\frac{\pi}{15} = \frac{1+\sqrt{5}+\sqrt{30-6\sqrt{5}}}{8} \implies f_E(f_0(1), f_1(1,5), f_2(1,30,-6,5), 8) \tag{3.8}$$

3.4 f_N examples

$$\cos\frac{\pi}{16} = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \implies f_N(\emptyset, \emptyset, \emptyset, f_3(1,2,1,2,1,2), 2) \tag{3.9}$$
$$\cos\frac{\pi}{24} = \frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2} \implies f_N(\emptyset, \emptyset, \emptyset, f_3(1,2,1,2,1,3), 2) \tag{3.10}$$
$$\cos\frac{2\pi}{17} = \frac{-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}+2\sqrt{17+3\sqrt{17}-\sqrt{170+38\sqrt{17}}}}{16} \tag{3.11}$$
$$\implies f_N(f_0(-1), f_1(1,17), f_2(1,34,-2,17), f_3(2,17,3,17,-1,170,+38,17), 16) \tag{3.12}$$

4 B operations

4.1 New $B_1 \mapsto B$

$$B1 = \frac{\pm b_1}{a_1} \tag{4.1}$$
$$= \frac{\pm b}{a} \implies \{\pm b, a\} = \gcd\{\pm b_1, a_1\} \tag{4.2}$$

4.2 Add $B_1 + B_2 \mapsto B$

$$B_1 + B_2 = \frac{\pm b_1}{a_1} + \frac{\pm b_2}{a_2} \tag{4.3}$$
$$= \frac{\pm b_1 a_2 \pm b_2 a_1}{a_1 a_2} \tag{4.4}$$
$$= \frac{\pm b}{a} \implies (\pm b, a) = \gcd(\pm b_1 a_2 \pm b_2 a_1, a_1 a_2) \tag{4.5}$$

4.3 Mul $B_1 \times B_2 \mapsto B$

$$
\begin{aligned}
B_1 \times B_2 &= \frac{\pm b_1}{a_1} \times \frac{\pm b_2}{a_2} && (4.6)\\
&= \frac{\pm b_1 b_2}{a_1 a_2} && (4.7)\\
&= \frac{\pm b}{a} \implies (\pm b, a) = \gcd(\pm b_1 b_2, a_1 a_2) && (4.8)
\end{aligned}
$$

4.4 Inv $1/B_1 \mapsto B \iff a_1 > 0$

$$
\begin{aligned}
\frac{1}{B_1} &= \frac{1}{\pm b_1/a_1} && (4.9)\\
&= \frac{\pm a_1}{b_1} && (4.10)\\
&= \frac{\pm b}{a} \implies \{b\} = a_1, \{a\} = b_1 && (4.11)
\end{aligned}
$$

5 D operations

5.1 SqrtB $\sqrt{B_1} \mapsto D \iff b_1 > 0$

$$
\begin{aligned}
\sqrt{B_1} &= \sqrt{\frac{+b_1}{a_1}} && (5.1)\\
&= \frac{\sqrt{a_1 b_1}}{a_1} && (5.2)\\
&= \frac{x\sqrt{d}}{a_1} \implies \{x^2 d\} = a_1 b_1 && (5.3)\\
&= \frac{c\sqrt{d}}{a} \implies \{a, c\} = gcd\{a_1, x\} && (5.4)
\end{aligned}
$$

5.2 Mul $D_1 \times D_2 \mapsto D$

$$
\begin{aligned}
D_1 \times D_2 &= \frac{\pm a_1 \sqrt{c_1}}{b_1} \times \frac{\pm a_2 \sqrt{c_2}}{b_2}\\
&= \frac{\pm a_1 a_2 \sqrt{c_1 c_2}}{b_1 b_2}\\
&= \frac{\pm a_1 a_2 m \sqrt{c_3}}{b_1 b_2} && c_1 c_2 = m^2 c_3\\
&= \frac{\pm a_3 \sqrt{c_3}}{b_3} && (\pm a_3, b_3) = gcd(\pm a_1 a_2 m, b_1 b_2)
\end{aligned}
$$

5.3 Inv $1/D_1 \mapsto D$

$$
\begin{aligned}
1/D_1 &= \frac{1}{\frac{\pm a_1 \sqrt{c_1}}{b_1}}\\
&= \frac{b_1}{\pm a_1 \sqrt{c_1}}\\
&= \frac{\pm b_1 \sqrt{c_1}}{c_1}\\
&= \frac{\pm a_2 \sqrt{c_1}}{b_2} && (\pm a_2, b_2) = gcd(\pm b_1, c1)
\end{aligned}
$$

6 H operations

6.1 $D_1 + D_2 \mapsto H$ ¡¡¡¡¡

$$\begin{aligned}
D_1 + D_2 &= \frac{\pm b_1 \pm c_1\sqrt{d_1}}{a_1} + \frac{\pm b_2 \pm c_2\sqrt{d_2}}{a_2} && (6.1)\\
&= \frac{(\pm a_2 b_1 \pm a_1 b_2) \pm a_2 c_1\sqrt{d_1} \pm a_1 c_2\sqrt{d_2}}{a_1 a_2} && (6.2)\\
&= \frac{\pm q \pm r\sqrt{d_1} \pm s\sqrt{d_2}}{p} && (6.3)\\
&\textbf{where } \{p, q, r, s\} = \gcd\{a_1 a_2, (\pm a_2 b_1 \pm a_1 b_2), \pm a_2 c_1, \pm a_1 c_2\}\\
&= \frac{\pm q \pm \sqrt{r^2 d_1 + s^2 d_2 \pm 2rs\sqrt{d_1 d_2}}}{p} && (6.4)\\
&= \frac{\pm q \pm \sqrt{t \pm 2rsu\sqrt{h}}}{p} && (6.5)\\
&\textbf{where } \{t\} = r^2 d_1 + s^2 d_2 \textbf{ and } \{u^2 h\} = d_1 d_2\\
&= \frac{\pm q \pm v\sqrt{f \pm g\sqrt{h}}}{p} && (6.6)\\
&\textbf{where } \{v^2 f\} = t \textbf{ and } \{v^2 g\} = 2rsu\\
&= \frac{\pm d \pm e\sqrt{f \pm g\sqrt{h}}}{a} && (6.7)\\
&\textbf{where } \{a, d, e\} = \gcd\{p, \pm q, \pm qv\}\\
& && (6.8)
\end{aligned}$$

6.2 $\sqrt{C_1} = F_2$

$$\begin{aligned}
\sqrt{C_1} &= \sqrt{\frac{a_1\sqrt{c_1}}{b_1}}\\
&= \frac{\sqrt{a_1 b_1\sqrt{c_1}}}{b_1}\\
&= \frac{m\sqrt{e_2\sqrt{c_1}}}{b_1} && a_1 b_1 = m^2 e_2\\
&= \frac{a_2\sqrt{e_2\sqrt{c_1}}}{b_2} && (a_2, b_2) = gcd(m, b_1)
\end{aligned}$$

6.3 $C_1 + D_2 = F_3$

$$\begin{aligned}
C_1 + D_2 &= \frac{\pm a_1\sqrt{c_1}}{b_1} + \frac{\pm a_2\sqrt{c_2} \pm d_2}{b_2} \\
&= \frac{\pm a_1 b_2\sqrt{c_1} \pm a_2 b_1\sqrt{c_2} \pm d_2 b_1}{b_1 b_2} \\
&= \frac{\pm m\sqrt{c_1} \pm n\sqrt{c_2} \pm p}{o} && (\pm m, \pm n, \pm p, o) = gcd(\pm a_1 b_2, \pm a_2 b_1, \pm d_2 b_1, b_1 b_2) \\
&= \frac{\sqrt{m^2 c_1 + n^2 c_2 \pm 2mn\sqrt{c1 c_2}} \pm p}{o} \\
&= \frac{\sqrt{q \pm 2mnr\sqrt{f_3}} \pm p}{o} && q = m^2 c_1 + n^2 c_2, c_1 c_2 = r^2 f_3 \\
&= \frac{s\sqrt{c_3 \pm e_3\sqrt{f_3}} \pm p}{o} && q = s^2 c_3, 2mnr = s^2 e_3 \\
&= \frac{a_3\sqrt{c_3 \pm e_3\sqrt{f_3}} \pm d_3}{b_3} && (a_3, b_3, \pm d_3) = gcd(s, \pm p, o)
\end{aligned}$$

6.4 $1/D_1 = D_2$

$$\begin{aligned}
1/D_1 &= \frac{b_1}{\pm a_1\sqrt{c_1} \pm d_1} \\
&= \frac{\pm a_1 b_1\sqrt{c_1} \mp b_1 d_1}{a_1^2 c_1 - d_1^2} \\
&= \frac{a_2\sqrt{c_1} \pm d_2}{b_2} && (a_2, b_2, d_2) = gcd(\pm a_1 b_1, \mp b_1 d_1, a_1^2 c_1 - d_1^2)
\end{aligned}$$

6.5 $\sqrt{D_1} = F_2$ editing...

$$\begin{aligned}
\sqrt{D_1} &= \sqrt{\frac{\pm a_1\sqrt{c_1} \pm d_1}{b_1}} \\
&= \frac{\sqrt{\pm b_1 d_1 \pm a_1 b_1\sqrt{f_2}}}{b_1} && f_2 = c_1 \\
&= \frac{m\sqrt{c_2 \pm e_2\sqrt{f_2}}}{b_1} && \pm b_1 d_1 = m^2 c_2, \pm a_1 b_1 = m^2 e_2 \\
&= \frac{a_2\sqrt{c_2 \pm e_2\sqrt{f_2}}}{b_2} && (a_2, b_2) = gcd(m, b_1)
\end{aligned}$$

6.6 $D_1 + D_2 = F_3$

$$\begin{aligned}
D_1 + D_2 &= \frac{\pm a_1\sqrt{c_1} \pm d_1}{b_1} + \frac{\pm a_2\sqrt{c_2} \pm d_2}{b_2} \\
&= \frac{\pm a_1 b_2\sqrt{c_1} \pm a_2 b_1\sqrt{c_2} \pm d_1 b_2 \pm d_2 b_1}{b_1 b_2} \\
&= \frac{\pm m\sqrt{c_1} \pm n\sqrt{c_2} \pm p}{o} && (\pm m, \pm n, \pm p, o) = gcd(\pm a_1 b_2, \pm a_2 b_1, \pm d_1 b_2 \pm d_2 b_1, b_1 b_2) \\
&= \frac{\sqrt{m^2 c_1 + n^2 c_2 \pm 2mn\sqrt{c1 c_2}} \pm p}{o} \\
&= \frac{\sqrt{q \pm 2mnr\sqrt{f_3}} \pm p}{o} && q = m^2 c_1 + n^2 c_2, c_1 c_2 = r^2 f_3 \\
&= \frac{s\sqrt{c_3 \pm e_3\sqrt{f_3}} \pm p}{o} && q = s^2 c_3, 2mnr = s^2 e_3 \\
&= \frac{a_3\sqrt{c_3 \pm e_3\sqrt{f_3}} \pm d_3}{b_3} && (a_3, b_3, \pm d_3) = gcd(s, \pm p, o)
\end{aligned}$$

6.7 $D_1 \times D_2 = F_3$

$$\begin{aligned}
D_1 \times D_2 &= \frac{\pm a_1\sqrt{c_1} \pm d_1}{b_1} \times \frac{\pm a_2\sqrt{c_2} \pm d_2}{b_2} \\
&= \frac{\pm a_1 a_2\sqrt{c_1 c_2} \pm a_1 d_2\sqrt{c1} \pm a_2 d_1\sqrt{c_2} \pm d_1 d_2}{b_1 b_2}
\end{aligned}$$