Triple frame

https://github.com/heptagons/meccano/frames/triple

Abstract

Triple frame is a group of **five** meccano [1] strips a, b, c, d, e forming **three equal angles** θ intended to build three consecutive sides of some regular polygons perimeter. We look for integer values of strip e in function of integer values of sides a, b, c, d and a particular angle θ. We confirm a generic equation found matches the one used to build pentagons of type 2 [2]. Here we found a lot of hexagons and filter some not trivial solutions. We look for octagons, decagons and dodecagons.

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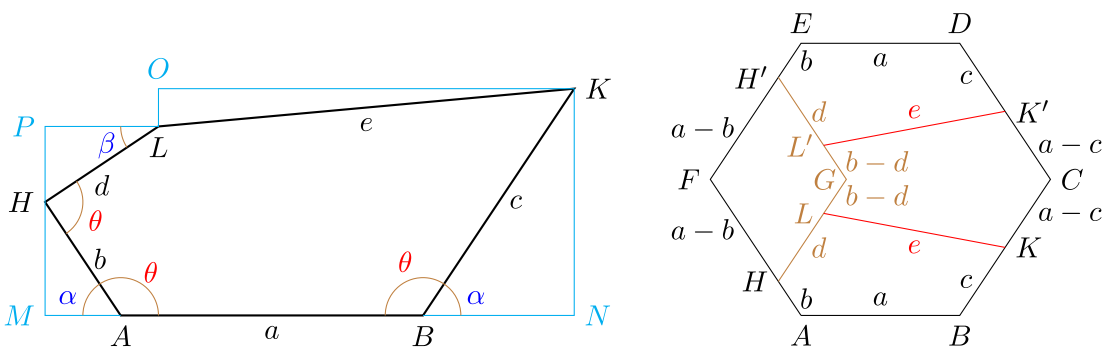

Figure 1: At the left we have the triple frame (three angles θ) with the strips a, b, c, d, e. At the right we use two frames to build a regular polygon of side a extending strips b, c, d to fix everthing. This construction is possible only when $a > b, c$.

1 Algebra

From nodes A and B of fig 1 we get α from θ ($\pi = 180°$):

$$\begin{aligned} \theta &= \pi - \alpha \\ \alpha &= \pi - \theta \end{aligned} \tag{1}$$

And from node H we get β from θ:

$$\begin{aligned} \theta &= \alpha + \beta \\ \beta &= \theta - \alpha = \theta - (\pi - \theta) = 2\theta - \pi \end{aligned} \tag{2}$$

[1] Meccano mathematics by 't Hooft
[2] Meccano pentagons

We calculate horizontal segment $\overline{OK}$:

$$\begin{aligned}\overline{OK} &= \overline{MA} + a + \overline{BN} - \overline{PL} \\ &= b\cos\alpha + a + c\cos\alpha - d\cos\beta \\ &= a + (b+c)\cos\alpha - d\cos\beta \\ &= a + (b+c)\cos(\pi - \theta) - d\cos(2\theta - \pi) \\ &= a - (b+c)\cos\theta + d\cos(2\theta) \end{aligned} \tag{3}$$

And vertical segment $\overline{OL}$:

$$\begin{aligned}\overline{OL} &= \overline{KN} - \overline{PH} - \overline{HM} \\ &= c\sin\alpha - d\sin\beta - b\sin\alpha \\ &= (c-b)\sin\alpha - d\sin\beta \\ &= (c-b)\sin(\pi - \theta) - d\sin(2\theta - \pi) \\ &= (c-b)\sin\theta + d\sin(2\theta) \end{aligned} \tag{4}$$

So we can express e in function of a, b, c, d and angle θ:

$$\begin{aligned}e^2 &= (\overline{OK})^2 + (\overline{OL})^2 \\ &= (a - (b+c)\cos\theta + d\cos(2\theta))^2 + ((c-b)\sin\theta + d\sin(2\theta))^2 \\ &= a^2 + (b^2 + 2bc + c^2)\cos^2\theta + d^2\cos^2(2\theta) + (c^2 - 2cb + b^2)sin^2\theta + d^2\sin^2(2\theta) \\ &\quad - 2a(b+c)\cos\theta + 2ad\cos(2\theta) - 2(b+c)d\cos\theta\cos(2\theta) \\ &\quad + 2(c-b)d\sin\theta\sin(2\theta) \\ &= a^2 + b^2 + c^2 + d^2 + 2bc\cos^2\theta - 2bc\sin^2\theta \\ &\quad - 2a(b+c)\cos\theta + 2ad\cos(2\theta) \\ &\quad - 2d((b+c)\cos\theta\cos(2\theta) + (b-c)\sin\theta\sin(2\theta)) \end{aligned} \tag{5}$$

$$\begin{aligned} &= a^2 + b^2 + c^2 + d^2 + 2bc(\cos^2\theta - \sin^2\theta) - 2a(b+c)\cos\theta + 2ad\cos(2\theta) \\ &\quad - 2d(b(\cos\theta\cos(2\theta) + \sin\theta\sin(2\theta)) + c(\cos\theta\cos(2\theta) - \sin\theta\sin(2\theta))) \\ &= a^2 + b^2 + c^2 + d^2 + 2bc\cos(2\theta) - 2a(b+c)\cos\theta + 2ad\cos(2\theta) \\ &\quad - 2d(b\cos(\theta - 2\theta) + c\cos(\theta + 2\theta)) \\ &= a^2 + b^2 + c^2 + d^2 + 2(bc + ad)\cos(2\theta) - 2a(b+c)\cos\theta - 2d(b\cos\theta + c\cos(3\theta)) \\ &= a^2 + b^2 + c^2 + d^2 + 2(bc + ad)\cos(2\theta) - 2(ab + ac)\cos\theta - 2(bd\cos\theta + cd\cos(3\theta)) \end{aligned} \tag{6}$$

$$\boxed{e^2 = a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd)\cos\theta + 2(bc + ad)\cos(2\theta) - 2cd\cos(3\theta)} \tag{7}$$

2 Regular polygons

We will test last equation into several polygons. Table 1 show the possible constructions and the angles and cosines. Only when we'll get e integer we'll have a solution.

Polygon	θ	$\cos\theta$	$\cos(2\theta)$	$\cos(3\theta)$	Notes
Pentagon	$\frac{3\pi}{5}$	$\frac{1-\sqrt{5}}{4}$	$\frac{-1-\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$	
Hexagon	$\frac{2\pi}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	
Octagon	$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	
Decagon	$\frac{4\pi}{5}$	$\frac{-1-\sqrt{5}}{4}$	$\frac{-1+\sqrt{5}}{4}$	$\frac{-1+\sqrt{5}}{4}$	
Dodecagon	$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$			

Table 1: Regular polygons internal angles and cosines.

3 Equilateral pentagons

We replace the cosines for pentagon in table 1 in equation 7:

$$\begin{aligned}
e^2 &= a^2+b^2+c^2+d^2-2(ab+ac+bd)\cos\theta+2(bc+ad)\cos(2\theta)-2cd\cos(3\theta) \\
&= a^2+b^2+c^2+d^2-2(ab+ac+bd)\left(\frac{1-\sqrt{5}}{4}\right)+2(bc+ad)\left(\frac{-1-\sqrt{5}}{4}\right)-2cd\left(\frac{1+\sqrt{5}}{4}\right) \\
&= a^2+b^2+c^2+d^2-\frac{ab+ac+bd+bc+ad+cd}{2}+\frac{ab+ac+bd-bc-ad-cd}{2}\sqrt{5} \qquad (8)
\end{aligned}$$

e cannot to be and integer if the factor of $\sqrt{5}$ is not zero so we force this factor to be zero:

$$\begin{aligned}
ab+ac+bd-bc-ad-cd &= 0 \\
ab+ac+bd &= bc+ad+cd \qquad (9) \\
ab+ac-bc &= (a-b+c)d \qquad (10)
\end{aligned}$$

We replace $ab+ac+bd$ by $bc+ad+cd$ in the e^2 equation to get:

$$\begin{aligned}
e^2 &= a^2+b^2+c^2+d^2-\frac{(bc+ad+cd)+bc+ad+cd}{2}+\frac{0}{2}\sqrt{5} \\
&= a^2+b^2+c^2+d^2-bc-ad-cd \qquad (11)
\end{aligned}$$

$$\boxed{e=\sqrt{a^2+b^2+c^2+d^2-bc-(a+c)d} \iff ab+ac-bc=(a-b+c)d} \qquad (12)$$

The last formula matches the formula used in the paper Meccano pentagons which finds several pentagons of type 2.

4 Equilateral hexagons

We replace the cosines for hexagon in table 1 in equation 7:

$$
\begin{aligned}
e^2 &= a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd)\cos\theta + 2(bc + ad)\cos(2\theta) - 2cd\cos(3\theta) \\
&= a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd)\left(-\frac{1}{2}\right) + 2(bc + ad)\left(-\frac{1}{2}\right) - 2cd(1) \\
&= a^2 + b^2 + c^2 + d^2 + ab + ac + bd - bc - ad - 2cd \\
&= (a + b)^2 + (c - d)^2 - ab + ac + bd - bc - ad \\
&= (a + b)^2 + (c - d)^2 + (c - d)(a - b) - ab \\
&= (a + b)^2 + (c - d)(a - b + c - d) - ab
\end{aligned} \tag{13}
$$

$$
\boxed{e = \sqrt{(a + b)^2 + (c - d)(a - b + c - d) - ab}} \tag{14}
$$

4.1 Hexagons software

We wrote software code to look for hexagons using the formula for e and set several filters to prevent trivial solutions. We say an hexagon is nice when $e \leq a$. Next is a partial list of nice hexagons:

```
  1   a=  7  b=  1  c=  2  d=  6  e=  7
  2   a=  7  b=  1  c=  4  d=  6  e=  7
  3   a= 13  b=  2  c=  5  d= 11  e= 13
  4   a= 13  b=  2  c=  6  d= 11  e= 13
  5   a= 14  b=  1  c=  6  d= 13  e= 13
  6   a= 14  b=  1  c=  7  d= 13  e= 13
  7   a= 15  b=  1  c=  5  d= 14  e= 14
  8   a= 15  b=  1  c=  9  d= 14  e= 14
  9   a= 19  b=  2  c=  3  d= 17  e= 19
 10   a= 19  b=  2  c= 14  d= 17  e= 19
 11   a= 20  b=  1  c=  4  d= 19  e= 19
 12   a= 20  b=  1  c= 15  d= 19  e= 19
      ...
105   a= 58  b=  5  c= 10  d= 53  e= 57
106   a= 58  b=  5  c= 43  d= 53  e= 57
107   a= 59  b=  1  c= 27  d= 58  e= 52
108   a= 59  b=  1  c= 31  d= 58  e= 52
109   a= 59  b=  4  c= 11  d= 55  e= 57
110   a= 59  b=  4  c= 44  d= 55  e= 57
111   a= 59  b=  5  c= 19  d= 54  e= 56
112   a= 59  b=  5  c= 35  d= 54  e= 56
--- PASS: TestHexagonsNice (0.01s)
```

Results from `github.com/heptagons/meccano/frames/triple/triple_test.go TestHexagonsNice`

4.2 Hexagons examples

The nice hexagons results has related pairs and there are several ways to build each case. Figure 2 show different ways to build a nice hexagon.

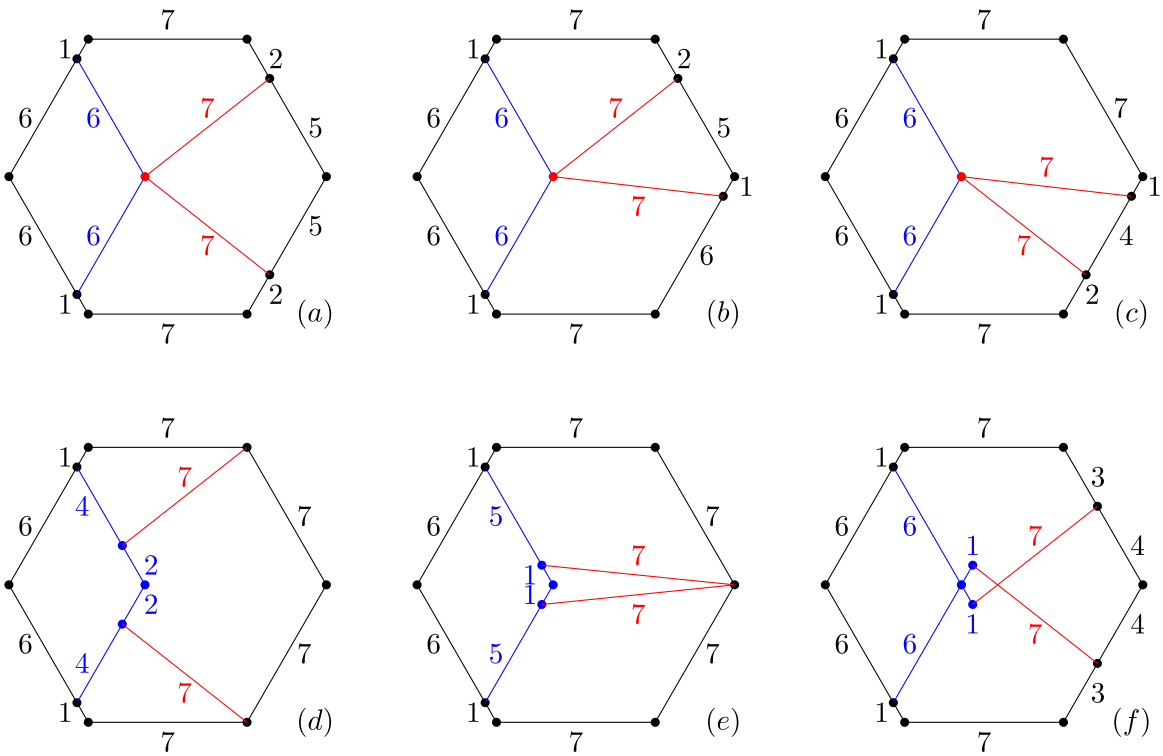

Figure 2: Constructions options of the nice hexagon side $a = 7, b = 1, e = 7$. Cases $(a) - (e)$ requires only eleven bolts. Case (f) has the 10 strips of size 7.

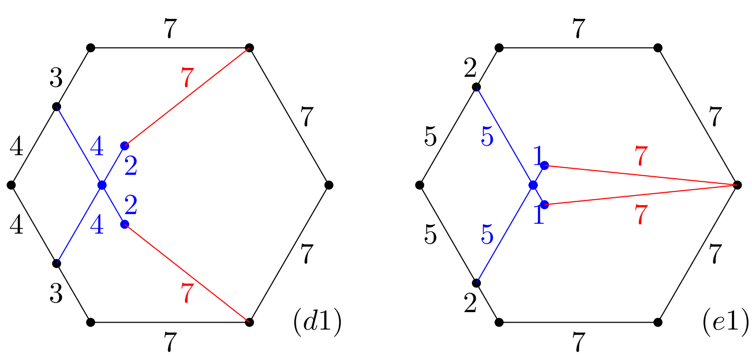

Figure 3: Variations of constructions of the nice hexagon side $a = 7, b = 1, e = 7$. Cases $(d1)$ and $(e1)$ are adaptations of cases (d) and (e) of figure 2 where only the blue strips are displaced. Such changes mantain the internal bolts, red strips and perimeter the same. The original **triple frame** a, b, c, d, e irregular pentagon is replaced by an irregular hexagon clearly visible in case $(e1)$.

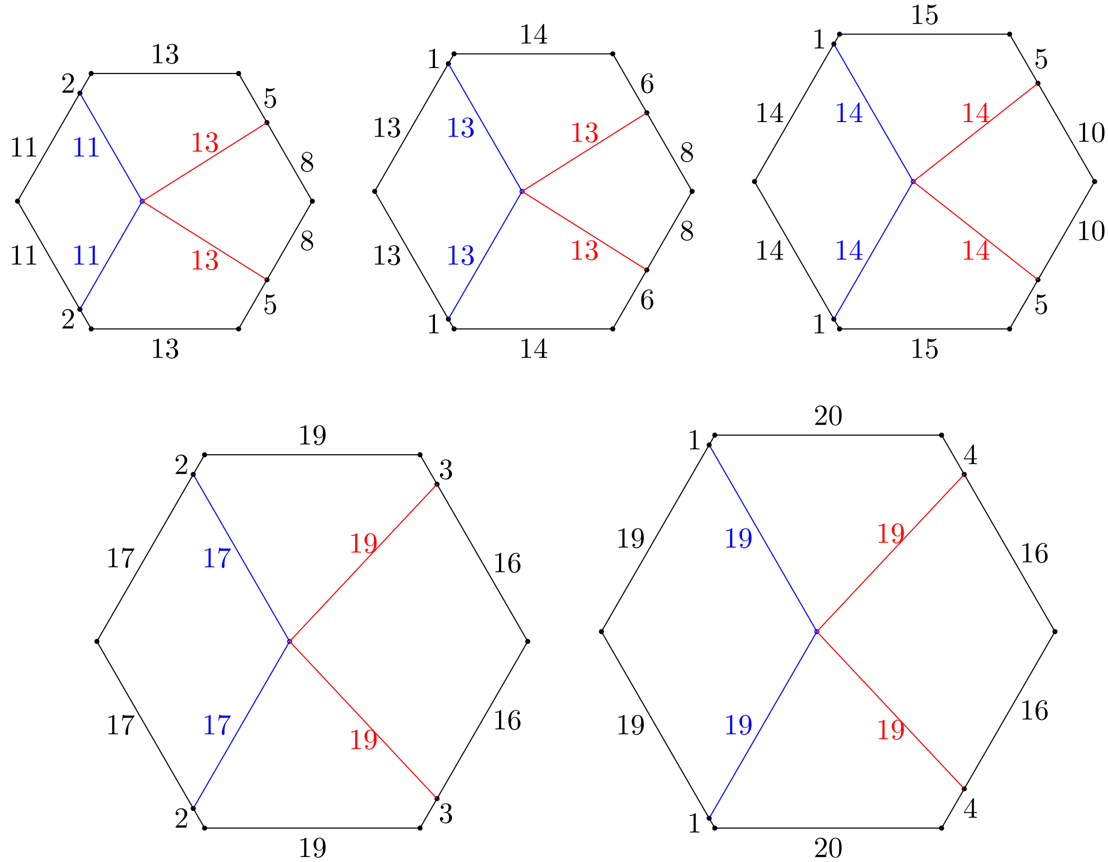

Figure 4: More nice hexagons from sizes $13 - 20$.

5 Regular octagons

We replace the cosines for octagon in table 1 in e^2 equation:

$$\begin{aligned} e^2 &= a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd)\cos\theta + 2(bc + ad)\cos(2\theta) - 2cd\cos(3\theta) \\ &= a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd)\left(-\frac{\sqrt{2}}{2}\right) + 2(bc + ad)\,(0) - 2cd\left(\frac{\sqrt{2}}{2}\right) \\ &= a^2 + b^2 + c^2 + d^2 + (ab + ac + bd - cd)\sqrt{2} \end{aligned} \tag{15}$$

e cannot to be and integer if the factor of $\sqrt{2}$ is not zero, so we force this factor to be zero:

$$ab + ac + bd - cd = 0$$
$$a(b + c) = (c - b)d$$
$$e^2 = a^2 + b^2 + c^2 + d^2 + (0)\sqrt{2}$$

$$\boxed{e = \sqrt{a^2 + b^2 + c^2 + d^2} \quad \iff \quad a(b + c) = (c - b)d} \tag{16}$$

5.1 Octagons examples

Conjecture: No possible octagons formed with triple frame.

6 Equilateral decagons

We replace the cosines for decagon in table 1 in e^2 equation:

$$
\begin{aligned}
e^2 &= a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd)\cos\theta + 2(bc + ad)\cos(2\theta) - 2cd\cos(3\theta) \\
&= a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd)\left(\frac{-1-\sqrt{5}}{4}\right) + 2(bc + ad)\left(\frac{-1+\sqrt{5}}{4}\right) - 2cd\left(\frac{-1+\sqrt{5}}{4}\right) \\
&= a^2 + b^2 + c^2 + d^2 + \frac{ab + ac + bd - bc - ad + cd}{2} + \frac{ab + ac + bd + bc + ad - cd}{2}\sqrt{5} \qquad (17)
\end{aligned}
$$

e cannot to be and integer if the factor of $\sqrt{5}$ is not zero so we force this factor to be zero:

$$
\begin{aligned}
ab + ac + bd + bc + ad - cd = 0& \\
ab + ac + bd &= cd - bc - ad \qquad (18) \\
ab + ac + bc &= (c - a - b)d \qquad (19)
\end{aligned}
$$

We replace $ab + ac + bd$ by $cd - bc - ad$ in the e^2 equation to get:

$$
\begin{aligned}
e^2 &= a^2 + b^2 + c^2 + d^2 + \frac{(cd - bc - ad) - bc - ad + cd}{2} + \frac{0}{2}\sqrt{5} \\
&= a^2 + b^2 + c^2 + d^2 + cd - bc - ad
\end{aligned}
$$

$$
e = \sqrt{a^2 + b^2 + c^2 + d^2 - bc - (a - c)d} \iff ab + ac + bc = (c - a - b)d \qquad (20)
$$

6.1 Decagons software

Common routine where $a \geq b, c$ doesn't return solutions. But when we change the condition $c \geq a$ we get other type of solutions.

```
func TestDecagonsCBA(t *testing.T) {
    tri := NewTriples()
    tri.DecagonsCBA(500)
}

func (t *Triples) DecagonsCBA(max int) {
  for c := 1; c <= max; c++ {
    for b := 1; b <= c; b++ {
      for a := 1; a <= c; a++ {
        ab_ac_bc := a*b + a*c + b*c
        aa_bb_cc := a*a + b*b + c*c
        for d := 1; d <= max; d++ {
          if ab_ac_bc != (c-a-b)*d {
            continue // condition to reject sqrt{5} from e equation
          }
          if e, ok := t.squareRoot(aa_bb_cc + d*d - b*c -(a-c)*d); ok {
            t.Add(a, b, c, d, e)
          }
        }
      }
    }
  }
}
```

The software solutions are in next listing. As with the case for pentagons, we **conjecture** again the variable e is in the form $10x + 1, x \in \mathbb{Z}$ or simply:

$$e \equiv 1 \mod 10 \tag{21}$$

```
 1   a=  8 b=  4 c= 13 d=188 e=191
 2   a=  3 b=  6 c= 18 d= 20 e= 31
 3   a=  6 b=  3 c= 20 d= 18 e= 31
 4   a= 12 b=  8 c= 36 d= 51 e= 71
 5   a= 24 b=  8 c= 51 d= 96 e=121
 6   a=  8 b= 12 c= 51 d= 36 e= 71
 7   a= 42 b=  7 c= 60 d=294 e=311
 8   a= 20 b= 30 c= 75 d=174 e=211
 9   a= 44 b= 24 c= 84 d=423 e=451
10   a=  2 b= 63 c= 84 d=294 e=341
11   a=  7 b= 57 c= 93 d=219 e=271
12   a=  8 b= 24 c= 96 d= 51 e=121
13   a= 60 b= 15 c=104 d=300 e=341
14   a= 42 b= 36 c=114 d=289 e=341
15   a= 45 b= 24 c=128 d=168 e=241
16   a= 15 b= 57 c=133 d=171 e=251
17   a= 72 b= 39 c=152 d=480 e=541
18   a= 24 b= 84 c=153 d=412 e=491
19   a= 13 b= 83 c=167 d=241 e=341
20   a= 24 b= 45 c=168 d=128 e=241
21   a= 53 b= 55 c=169 d=347 e=431
22   a= 57 b= 15 c=171 d=133 e=251
23   a= 21 b= 91 c=171 d=357 e=451
24   a= 30 b= 20 c=174 d= 75 e=211
25   a=  4 b=  8 c=188 d= 13 e=191
26   a=117 b=  3 c=219 d=269 e=401
27   a= 57 b=  7 c=219 d= 93 e=271
28   a= 28 b= 98 c=221 d=322 e=451
29   a= 34 b= 93 c=228 d=318 e=451
30   a= 83 b= 13 c=241 d=167 e=341
31   a=109 b= 24 c=264 d=288 e=451
32   a= 24 b=144 c=267 d=488 e=641
33   a=  3 b=117 c=269 d=219 e=401
34   a= 36 b= 96 c=276 d=277 e=451
35   a= 96 b= 36 c=277 d=276 e=451
36   a= 24 b=109 c=288 d=264 e=451
37   a= 36 b= 42 c=289 d=114 e=341
38   a= 63 b=  2 c=294 d= 84 e=341
39   a=  7 b= 42 c=294 d= 60 e=311
40   a= 15 b= 60 c=300 d=104 e=341
41   a= 93 b= 34 c=318 d=228 e=451
42   a= 98 b= 28 c=322 d=221 e=451
43   a= 55 b= 53 c=347 d=169 e=431
44   a= 91 b= 21 c=357 d=171 e=451
45   a=105 b= 87 c=363 d=461 e=671
46   a=180 b= 24 c=380 d=465 e=691
47   a=105 b= 90 c=406 d=420 e=671
48   a= 84 b= 24 c=412 d=153 e=491
49   a= 90 b=105 c=420 d=406 e=671
50   a= 24 b= 44 c=423 d= 84 e=451
51   a=222 b= 12 c=454 d=495 e=781
52   a= 87 b=105 c=461 d=363 e=671
53   a= 24 b=180 c=465 d=380 e=691
54   a= 39 b= 72 c=480 d=152 e=541
55   a=144 b= 24 c=488 d=267 e=641
56   a= 12 b=222 c=495 d=454 e=781
--- PASS: TestDecagonsCBA (42.31s)
```

6.2 Decagons examples

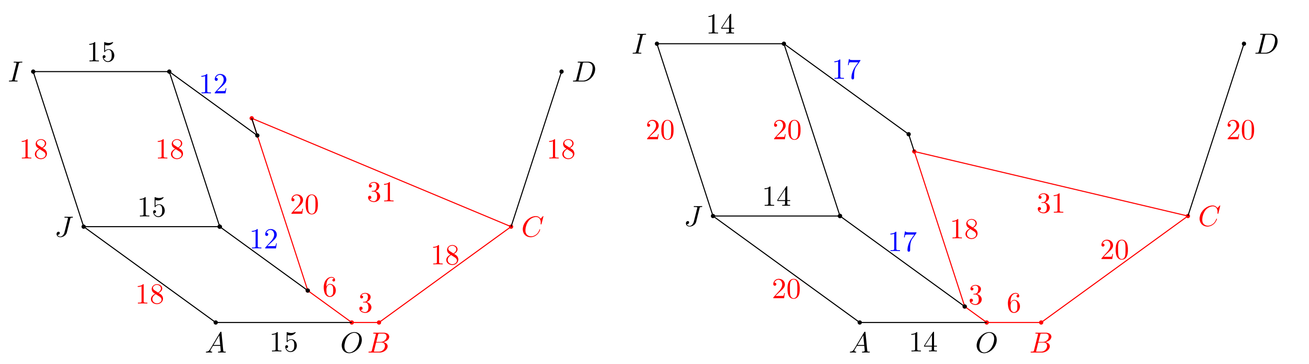

Figure 5: Half of decagons with $e = 31$, sizes 18 and 20.

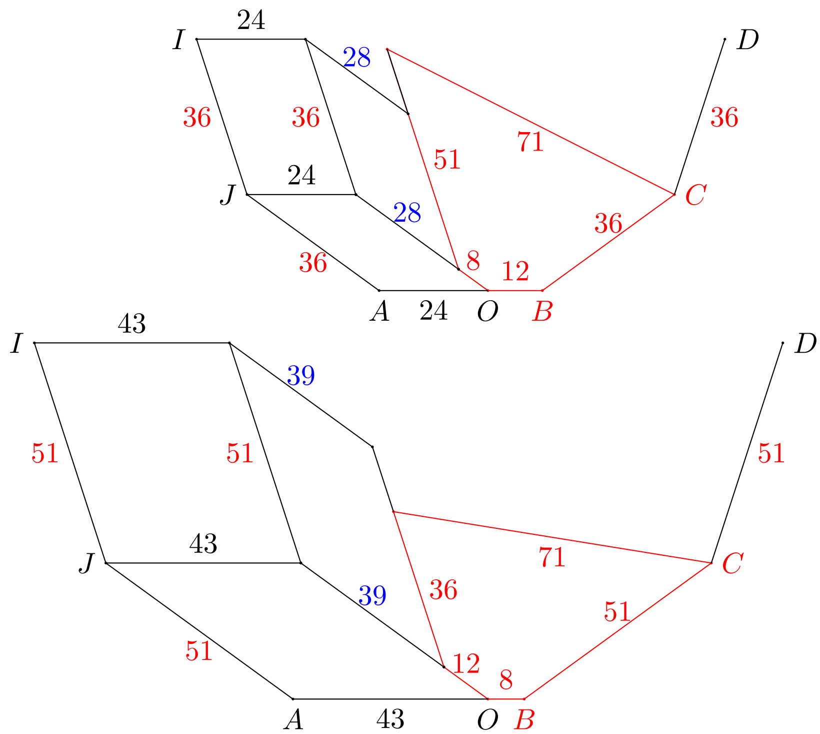

Figure 6: Half of decagons with $e = 71$, sizes 36 and 51.

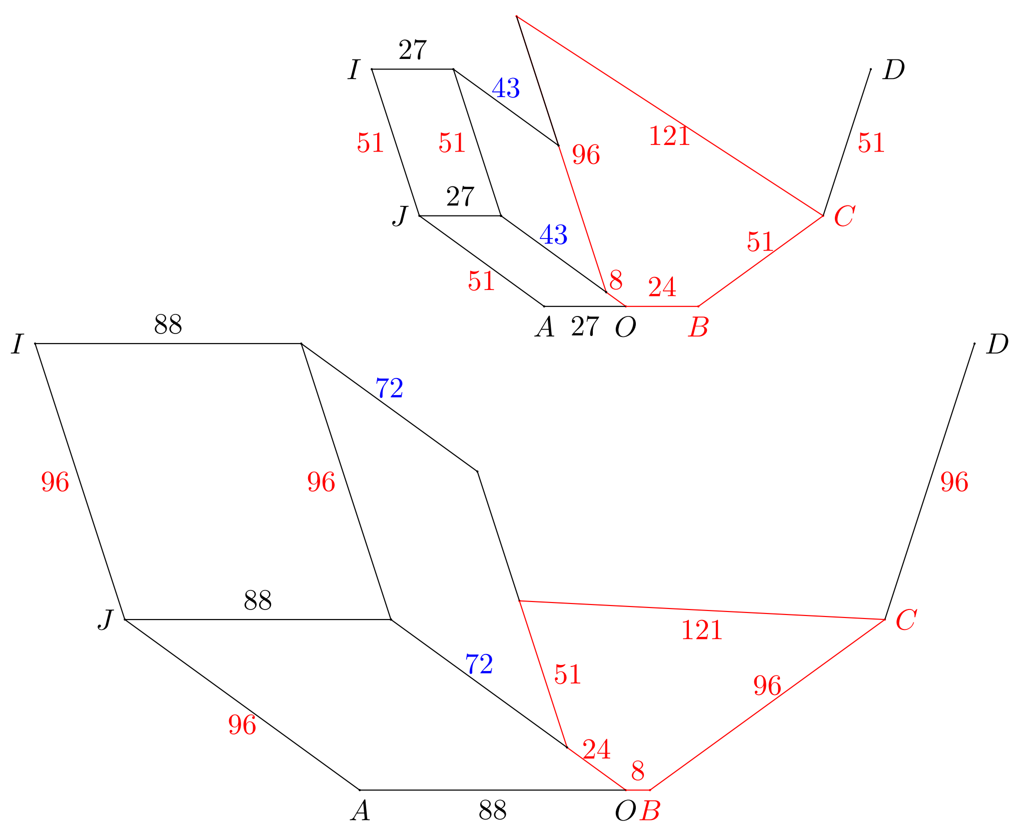

Figure 7: Half of decagons with $e = 121$, sizes 51 and 96.

With the results where $c > a$ we need to add helping rhombi to start building complete decagons. See figures 5, 6 and 7. In red we have the frame, for each pair for the same e the pairs interchage segments a, b and c, d. The complete building and dispositions of helping strips and rigidity is beyond this paper.