Meccano pentagons diagonals

https://github.com/heptagons/meccano/penta

Abstract

We construct meccano [1] regular pentagons internal diagonals.

1 Regular pentagon diagonals

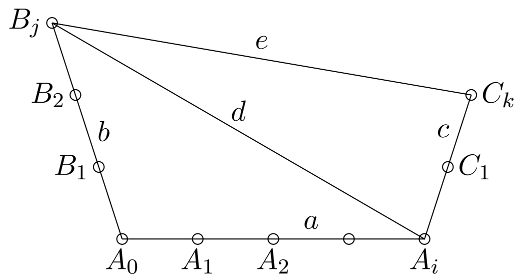

Figure 1: Regular pentagon basic diagonals d and e from sides segments $a \geq b \geq c$.

From figure 1 we know the regular internal pentagons angle is $\theta = 3\pi/5$:

$$\begin{aligned}
\alpha &= \angle A_0 A_i B_j && (1)\\
\beta &= \angle B_j A_i C_k && (2)\\
\theta &= \angle B_j A_0 A_i && (3)\\
&= \alpha + \beta && (4)\\
\cos\theta &= \frac{1-\sqrt{5}}{4} && (5)\\
\sin\theta &= \frac{\sqrt{10+2\sqrt{5}}}{4} && (6)
\end{aligned}$$

We use the cosines sum identity to express $\cos\beta$ in function of the rest of variables:

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos\theta && (7)\\
&= \cos\alpha\cos\beta - \sin\alpha\sin\beta && (8)\\
\sin\beta &= \frac{\cos\alpha\cos\beta - \cos\theta}{\sin\alpha} && (9)\\
\sin^2\beta &= \frac{(\cos\alpha\cos\beta - \cos\theta)^2}{\sin^2\alpha} && (10)\\
1-\cos^2\beta &= \frac{\cos^2\alpha\cos^2\beta - 2\cos\alpha\cos\beta\cos\theta + \cos^2\theta}{sin^2\alpha} && (11)
\end{aligned}$$

[1] Meccano mathematics by ‘t Hooft

We set $X = \cos\beta$ and rearrange the last equation to get:

$$\sin^2\alpha X^2 - 2\cos\alpha\cos\theta X + \cos^2\alpha\cos^2\beta + cos^2\theta - \sin^2\alpha = 0 \tag{12}$$

And solve the quadratic equation $AX^2 + BX + C = 0$ to get $\cos\beta$:

$$\begin{aligned}\cos\beta &= \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \\ &= \frac{2\cos\alpha\cos\theta \pm \sqrt{(2\cos\alpha\cos\theta)^2 - 4\sin^2\alpha(\cos^2\alpha\cos^2\beta + cos^2\theta - \sin^2\alpha)}}{2\sin^2\alpha} \\ &= \frac{\cos\alpha\cos\theta \pm \sqrt{\cos^2\alpha\cos^2\theta - \sin^2\alpha(\cos^2\alpha\cos^2\beta + cos^2\theta - \sin^2\alpha)}}{\sin^2\alpha}\end{aligned} \tag{13}$$

From the law of cosines we calculate distance d from integers a, b which equal respectively to iterators i, j:

$$\begin{aligned}d &= \sqrt{a^2 + b^2 - 2ab\cos\theta} \\ &= \sqrt{a^2 + b^2 - 2ab\left(\frac{1-\sqrt{5}}{4}\right)} \\ &= \frac{\sqrt{4a^2 + 4b^2 - 2ab + 2ab\sqrt{5}}}{2}\end{aligned} \tag{14}$$

Using the law of cosines we calculate the angles $\alpha = \angle A_0 A_i B_j$ and $\beta = \angle B_j A_i C_k$:

$$\cos\alpha = \frac{a^2 + d^2 - b^2}{2ad} \tag{15}$$

$$\begin{aligned}&= \frac{a^2 + (a^2 + b^2 - 2ab\cos\theta) - b^2}{2ad} \\ &= \frac{a - b\cos\theta}{d}\end{aligned} \tag{16}$$

$$\begin{aligned}\cos\beta &= \frac{c^2 + d^2 - e^2}{2cd} \\ &= \frac{c^2 + (a^2 + b^2 - 2ab\cos\theta) - e^2}{2cd} \\ &= \frac{a^2 + b^2 + c^2 - e^2 - 2ab\cos\theta}{2cd}\end{aligned} \tag{17}$$

We define new variable f to simplify $\cos\beta$ to obtain:

$$f \equiv \frac{a^2 + b^2 + c^2 - e^2}{2} \tag{18}$$

$$\cos\beta = \frac{f - ab\cos\theta}{cd} \tag{19}$$

We calculate $\sin\alpha = \sqrt{1 - \cos^2\alpha}$:

$$\begin{aligned}
\sin\alpha &= \sqrt{1 - \frac{(a - b\cos\theta)^2}{d^2}} \\
&= \frac{\sqrt{d^2 - a^2 + 2ab\cos\theta - b^2\cos\theta^2}}{d} \\
&= \frac{\sqrt{(a^2 + b^2 - 2ab\cos\theta) - a^2 + 2ab\cos\theta - b^2\cos^2\theta}}{d} \\
&= \frac{\sqrt{b^2(1 - \cos^2\theta)}}{d} \\
&= \frac{b\sin\theta}{d} \qquad (20)
\end{aligned}$$

$$(21)$$