Meccano pentagons gallery

https://github.com/heptagons/meccano/penta/gallery

2023/12/18

Abstract

We build rigid meccano [1] regular pentagons from sides 3 to 12. We restrict all internal strips to remain inside the pentagon's perimeter and don't permit they overlap with others. We follow three steps. 1) We calculate distances between selected strips holes from the regular pentagon perimeter assuming is regular. 2) We run some programs available in this repo to look for rigid clusters of strips which contains the distance. 3) We simplify or reduce the cluster to fit inside the pentagon. We prove the correctness of the cluster distance applied to check the software. We try each construction is relevant for the pentagon size.

1 Pentagons of size 3

1.1 Size 3 with 10 internal strips *build*$(3:3:1)$

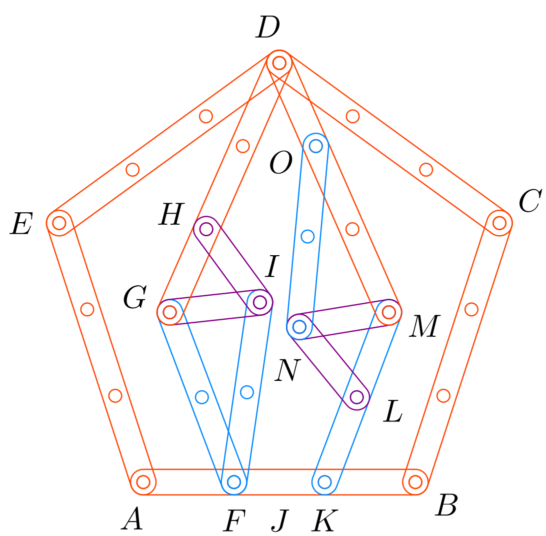

Figure 1: Pentagon of size 3 with 10 internal strips. $\overline{DE}:\overline{EA}:\overline{AF} = 3:3:1$. $\overline{DF} = \overline{DK} = \dfrac{\sqrt{46+18\sqrt{5}}}{2}$.

[1] Meccano mathematics by 't Hooft

Figure 1 show the regular pentagon A, B, C, D, E of size 3. We know the regular pentagon height is side length times $\frac{\sqrt{5+2\sqrt{5}}}{2}$ so in this case $\overline{DJ} = \frac{3\sqrt{5+2\sqrt{5}}}{2}$ and we can calculate $\overline{DF}$:

$$\begin{aligned}\overline{DF} &= \sqrt{(\overline{DJ})^2 + (\overline{FJ})^2} \\ &= \sqrt{\left(\frac{3\sqrt{5+2\sqrt{5}}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{46+18\sqrt{5}}}{2}\end{aligned} \tag{1}$$

1.1.1 Rigid distance $\frac{\sqrt{46+18\sqrt{5}}}{2}$

Our five-strips software found several clusters and we use two different which fit inside the pentagon. We identify two angles $\alpha = \angle HGI = \angle LMN$ of equilateral triangles and $\beta = \angle FGI = \angle NMO$ of isoscelles'. Adding the angles we get angles $\angle DGF = \angle DMK = (\alpha + \beta)$. From equilateral triangle $\triangle HGI$ we calculate α and from isoscelles triangle $\triangle FGI$ we calculate β:

$$\cos\alpha = \frac{\overline{GI}/2}{\overline{GH}} = \frac{1/2}{1} = \frac{1}{2} \tag{2}$$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2} \tag{3}$$

$$\cos\beta = \frac{\overline{GI}/2}{\overline{GF}} = \frac{1/2}{2} = \frac{1}{4} \tag{4}$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4} \tag{5}$$

Finally we calculate $\cos(\alpha+\beta)$ with the sum identity and using the law of cosines we verify distance $\overline{DF}$:

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{15}}{4}\right) = \frac{1-3\sqrt{5}}{8}\end{aligned} \tag{6}$$

$$\begin{aligned}\overline{DF} &= \sqrt{(\overline{DG})^2 + (\overline{FG}^2) - 2(\overline{DG})(\overline{FG})\cos(\alpha+\beta)} \\ &= \sqrt{3^2 + 2^2 - 2(3)(2)\left(\frac{1-3\sqrt{5}}{8}\right)} = \frac{\sqrt{46+18\sqrt{5}}}{2} \quad \blacksquare\end{aligned} \tag{7}$$

1.2 Size 3 with 14 internal strips $build(3:1)$

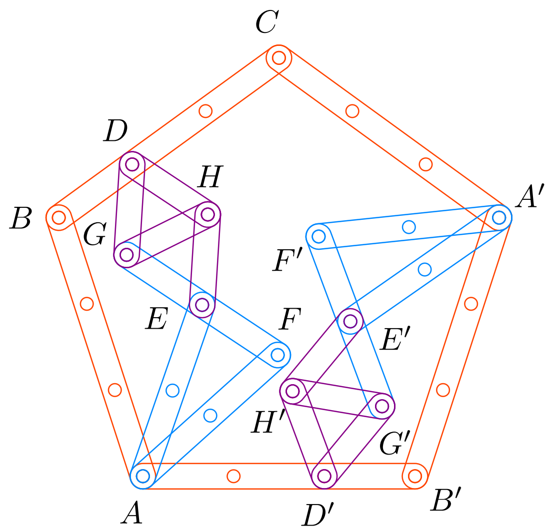

Figure 2: Pentagon of size 3 with 14 internal strips. $\overline{AB} : \overline{BD} = 3 : 1$. $\overline{AD} = \dfrac{\sqrt{34+5\sqrt{5}}}{2}$.

Figure 2 show the regular pentagon A, B', A', C, B of size 3. We know the internal angle of regular pentagon is $\theta \equiv \angle ABD = \dfrac{3\pi}{5}$ and $\cos\theta = \dfrac{1-\sqrt{5}}{4}$ so with the law of cosines we can calculate $\overline{AD}$ and the angle $\delta \equiv \angle BAD$:

$$\begin{aligned}\overline{AD} &= \sqrt{(\overline{AB})^2 + (\overline{BD})^2 - 2(\overline{AB})(\overline{BD})\cos\theta} \\ &= \sqrt{3^2 + 1^2 - 2(3)(1)\left(\frac{1-\sqrt{5}}{4}\right)} = \frac{\sqrt{34+6\sqrt{5}}}{2}\end{aligned} \tag{8}$$

$$\begin{aligned}\cos\delta &= \frac{(\overline{AD})^2 + (\overline{AB})^2 - (\overline{BD})^2}{2(\overline{AD})(\overline{AB})} \\ &= \frac{\frac{34+6\sqrt{5}}{4} + 3^2 - 1^2}{2\left(\frac{\sqrt{34+6\sqrt{5}}}{2}\right)(3)} = \frac{11+\sqrt{5}}{2\sqrt{34+6\sqrt{5}}}\end{aligned} \tag{9}$$

1.2.1 Rigid distance $\dfrac{\sqrt{34+6\sqrt{5}}}{2}$

Our five-strips software found several options to make the distance but we add manually a sixth strip in order to make a cluster narrow enough to fit two times inside the pentagon. The result is shown as cluster with vertices $DEFGH$ of figure 2. We prove the cluster's distance $\overline{AD}$ matches the distance alredy calculated.

We have a pair of adjacent equilateral triangles $\triangle DGH$ and $\triangle EGH$ so angle $\gamma \equiv \angle DGE = 2\pi/3$ also

$\cos\gamma = -1/2$ so we can calculate $\overline{DE}$:

$$\begin{aligned}\overline{DE} &= \sqrt{(\overline{DG})^2 + (\overline{GE})^2 - 2(\overline{DG})(\overline{GE})\cos\gamma} \\ &= \sqrt{1^2 + 1^2 - 2(1)(1)\left(-\frac{1}{2}\right)} = \sqrt{3} \end{aligned} \tag{10}$$

We define angle $\alpha = \angle GED$ and calculate it using $\overline{DE}$:

$$\cos\alpha = \frac{(\overline{GE})^2 + (\overline{DE})^2 - (\overline{DG})^2}{2(\overline{GE})(\overline{DE})} = \frac{3 + 1^2 - 1^2}{2(\sqrt{3})(1)} = \frac{\sqrt{3}}{2} \tag{11}$$

$$\sin\alpha = \sqrt{1 - \cos^2\alpha} = \sqrt{1 - \left(\frac{\sqrt{3}}{4}\right)^2} = \frac{1}{2} \tag{12}$$

From the isoscelles triangle $\triangle AEF$ we define angle $\epsilon \equiv \angle AEF$ noting $\cos\epsilon = \dfrac{\overline{EF}/2}{\overline{AE}} = \dfrac{1/2}{2} = \dfrac{1}{4}$ and we define angle $\beta \equiv \angle AEG$ the supplementary of ϵ and we get:

$$\begin{aligned}\beta &= \pi - \epsilon \\ \cos\beta &= -\cos\epsilon = -\frac{1}{4} \end{aligned} \tag{13}$$

$$\sin\beta = \sqrt{1 - \cos^2\beta} = \sqrt{1 - \left(\frac{-1}{4}\right)^2} = \frac{\sqrt{15}}{4} \tag{14}$$

Finally we calculate the angle $\alpha + \beta$ with the sum identity plugin the last sines and cosines and use it to verify $\overline{AD}$ with the law of cosines:

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{4}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{15}}{4}\right) = -\frac{\sqrt{3}+\sqrt{15}}{8} \end{aligned} \tag{15}$$

$$\begin{aligned}\overline{AD} &= \sqrt{(\overline{DE})^2 + (\overline{EA})^2 - 2(\overline{DE})(\overline{EA})\cos(\alpha+\beta)} \\ &= \sqrt{3 + 2^2 - 2(\sqrt{3})(2)\left(-\frac{\sqrt{3}+\sqrt{15}}{8}\right)} = \frac{\sqrt{34 + 6\sqrt{5}}}{2} \quad \blacksquare \end{aligned} \tag{16}$$

2 Pentagons of size 4

2.1 Size 4 with 8 internal strips *build*$(2:4:1:2)$

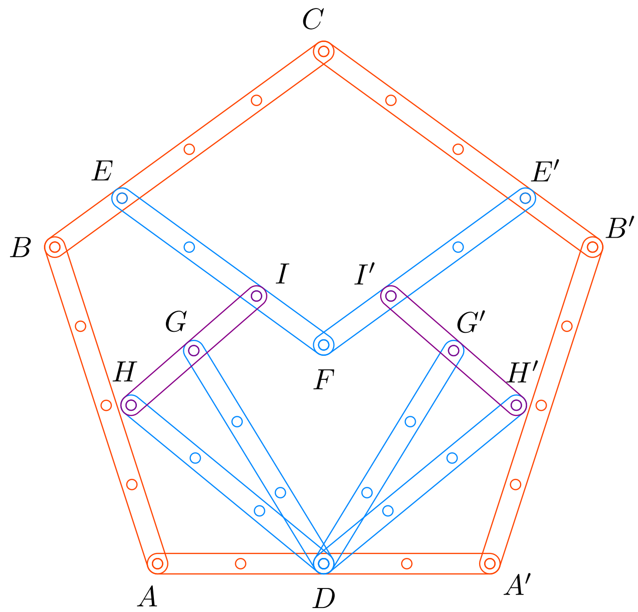

Figure 3: Pentagon of size 4 with 6 internal strips. $\overline{DA}:\overline{AB}:\overline{BE}:\overline{EI} = 2:4:1:2$. $\overline{DI} = \sqrt{11}$.

Figure 3 show the regular pentagon $AA'B'CB$ of size 4. We calculate the distance $\overline{DI}$ assuming vertex D is at origin and calculating abscissa and ordinate of vertex I and knowing $\alpha = \angle ADB = 3\pi/5$ and $\beta = \angle B'BE = \pi/5$. Adding and substracting through vertices $DABEI$ we get:

$$\begin{aligned} I_x &= -\overline{DA} - \overline{AB}|\cos\alpha| + \overline{BE}\cos\beta + \overline{EI}\cos\beta \\ &= -2 - (4)\left(\frac{\sqrt{5}-1}{4}\right) + (1+2)\left(\frac{\sqrt{5}+1}{4}\right) = -\frac{1+\sqrt{5}}{4} \end{aligned} \tag{17}$$

$$\begin{aligned} I_y &= \overline{AB}\sin\alpha + \overline{BE}\sin\beta - \overline{BI}\sin\beta \\ &= (4)\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right) + (1-2)\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right) = \frac{4\sqrt{10+2\sqrt{5}} - \sqrt{10-2\sqrt{5}}}{4} \end{aligned} \tag{18}$$

$$\begin{aligned} \overline{DI} &= \sqrt{(I_x - D_x)^2 + (I_y - I_y)^2} \\ &= \frac{\sqrt{(1+\sqrt{5})^2 + (4\sqrt{10+2\sqrt{5}} - \sqrt{10-2\sqrt{5}})^2}}{4} = \sqrt{11} \end{aligned} \tag{19}$$

2.1.1 Rigid distance $\sqrt{11}$

Our three-strips software found several clusters for distance $\sqrt{11}$. We prove the selected cluster $DHGI$ inside the pentagon of figure 3 matches the expected distance. First we calculate the angle $\alpha \equiv \angle DHG$

with the law of cosines and use the value to finally verify the distance $\overline{DI}$ with again the law of cosines:

$$\begin{aligned}\cos\alpha &= \frac{(\overline{HD})^2 + (\overline{HG})^2 - (\overline{DG})^2}{2(\overline{HD})(\overline{HG})} \\ &= \frac{3^2 + 1^2 - 3^2}{2(3)(1)} = \frac{1}{6} \end{aligned} \tag{20}$$

$$\begin{aligned}\overline{DI} &= \sqrt{(\overline{HD})^2 + (\overline{HI})^2 - 2(\overline{HD})(\overline{HD})\cos\alpha} \\ &= \sqrt{3^2 + 2^2 - 2(3)(2)\left(\frac{1}{6}\right)} = \sqrt{11} \quad \blacksquare \end{aligned} \tag{21}$$

2.2 Size 4 with 10 internal strips *build*(3 : 4)

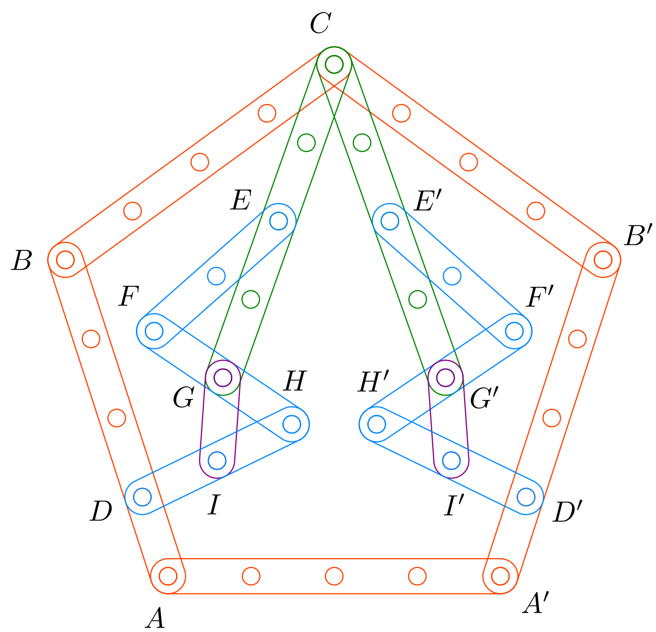

Figure 4: Pentagon of size 4 with 10 internal strips. $\overline{CB} : \overline{BD} = 4 : 3$. $\overline{CD} = \sqrt{19 + 6\sqrt{5}}$.

Figure 4 show the regular pentagon A, A', B', C, B of size 4. We know the internal angle of pentagon is $\theta = \angle CBD = \frac{3\pi}{5}$ and $\cos\theta = \frac{1-\sqrt{5}}{4}$ so with the law of cosines we can calculate $\overline{CD}$ and the angle $\delta = \angle BDC$:

$$\begin{aligned}\overline{CD} &= \sqrt{(\overline{BC})^2 + (\overline{BD})^2 - 2(\overline{BC})(\overline{BD})\cos\theta} \\ &= \sqrt{4^2 + 3^2 - 2(4)(3)\left(\frac{1-\sqrt{5}}{4}\right)} = \sqrt{19 + 6\sqrt{5}} \end{aligned} \tag{22}$$

$$\begin{aligned}\cos\delta &= \frac{(\overline{CD})^2 + (\overline{BC})^2 - (\overline{BD})^2}{2(\overline{CD})(\overline{BC})} \\ &= \frac{(19 + 6\sqrt{5}) + 4^2 - 3^2}{2\left(\sqrt{19 + 6\sqrt{5}}\right)(4)} = \frac{13 + 3\sqrt{5}}{4\sqrt{19 + 6\sqrt{5}}} \end{aligned} \tag{23}$$

2.2.1 Rigid distance $\sqrt{19+6\sqrt{5}}$

Our five-strips software found several clusters for distance $\sqrt{19+6\sqrt{5}}$. We prove selected cluster $CEFGHID$ show in the figure 4 matches pentagon's distace $\overline{CD}$. Set the cluster in the coordinate plane such that vertice G is at the origin and vertices F at $(-1,0)$ and vertice H at $(+1,0)$. Since triangle $\triangle EFG$ is isoscelles and $\overline{CG}$ is the double of $\overline{GE}$ we know angle $\angle CFG = \pi/2$ and we can calculate the abscissa and ordinate of vertex C:

$$C_x = -\overline{FG} = -1 \tag{24}$$

$$C_y = \sqrt{(\overline{CG})^2 - (\overline{FG})^2} = \sqrt{4^2 - 1^2} = \sqrt{15} \tag{25}$$

Since triangle $\triangle GHI$ is equilateral and $\overline{HD}$ is the double of $\overline{HI}$ we know angle $\angle DGH = \pi/2$ and we can calculate the abscissa and ordinate of vertex D:

$$D_x = 0 \tag{26}$$

$$D_y = -\sqrt{(\overline{HD})^2 - (\overline{GH})^2} = -\sqrt{2^2 - 1^2} = -\sqrt{3} \tag{27}$$

Finally we verify the distance $\overline{CD}$

$$\begin{aligned}\overline{CD} &= \sqrt{(C_x - D_x)^2 + (C_y - D_y)^2} \\ &= \sqrt{(-1-0)^2 + (\sqrt{15}+\sqrt{3})^2} = \sqrt{19+6\sqrt{5}} \quad \blacksquare \end{aligned} \tag{28}$$

3 Pentagons of size 5

3.1 Size 5 with 10 internal strips *build*(3, 5)

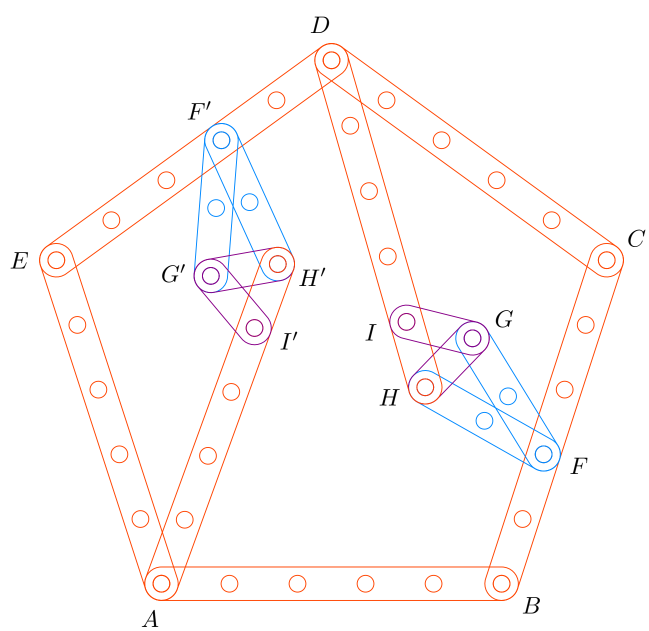

Figure 5: Pentagon of size 5 with 10 internal strips. $\overline{FC} : \overline{CD} = 3:5$. $\overline{FD} = \dfrac{\sqrt{106+30\sqrt{5}}}{2}$.

Figure 5 show the regular pentagon $ABCDE$ of size 5. We know the internal angle of pentagon $\theta = \angle FCD = \frac{3\pi}{5}$ and $\cos\theta = \frac{1-\sqrt{5}}{4}$ so with the law of cosines we can calculate $\overline{FD}$ and the angle $\delta = \angle CFD$:

$$\begin{aligned}\overline{FD} &= \sqrt{(\overline{FC})^2 + (\overline{CD})^2 - 2(\overline{FC})(\overline{CD})\cos\theta} \\ &= \sqrt{3^2 + 5^2 - 2(3)(5)\left(\frac{1-\sqrt{5}}{4}\right)} = \frac{\sqrt{106+30\sqrt{5}}}{2}\end{aligned} \tag{29}$$

$$\begin{aligned}\cos\delta &= \frac{(\overline{FD})^2 + (\overline{FC})^2 - (\overline{CD})^2}{2(\overline{FD})(\overline{FC})} \\ &= \frac{\left(\frac{106+30\sqrt{5}}{4}\right) + 3^2 - 5^2}{2\left(\frac{\sqrt{106+30\sqrt{5}}}{2}\right)(3)} = \frac{7+5\sqrt{5}}{2\sqrt{106+30\sqrt{5}}}\end{aligned} \tag{30}$$

3.1.1 Rigid distance $\frac{\sqrt{106+30\sqrt{5}}}{2}$

Our five-strips software found the cluster $FGHID$ of figure 5. We calculate two angles, angle $\alpha \equiv \angle FHG$ within isoscelles triangle $\triangle FHG$ and angle $\beta \equiv \angle GHI$ within equilateral triangle $\triangle GHI$:

$$\cos\alpha = \frac{\overline{HG}/2}{\overline{FH}} = \frac{1/2}{2} = \frac{1}{4} \tag{31}$$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4} \tag{32}$$

$$\cos\beta = \frac{\overline{HG}/2}{\overline{HI}} = \frac{1/2}{1} = \frac{1}{2} \tag{33}$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2} \tag{34}$$

Finally we calculte angle $\angle FHD = \alpha+\beta$ with the cosines sum identity and use it to verify the distance $\overline{FD}$ using the law of cosines:

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) - \left(\frac{\sqrt{15}}{4}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{1-3\sqrt{5}}{8}\end{aligned} \tag{35}$$

$$\begin{aligned}\overline{FD} &= \sqrt{(\overline{FH})^2 + (\overline{HD})^2 - 2(\overline{FH})(\overline{HD})\cos(\alpha+\beta)} \\ &= \sqrt{2^2 + 5^2 - 2(2)(5)\left(\frac{1-3\sqrt{5}}{8}\right)} = \frac{\sqrt{106+30\sqrt{5}}}{2} \quad \blacksquare\end{aligned} \tag{36}$$

3.2 Size 5 with 12 internal strips $build(4:5:4)$

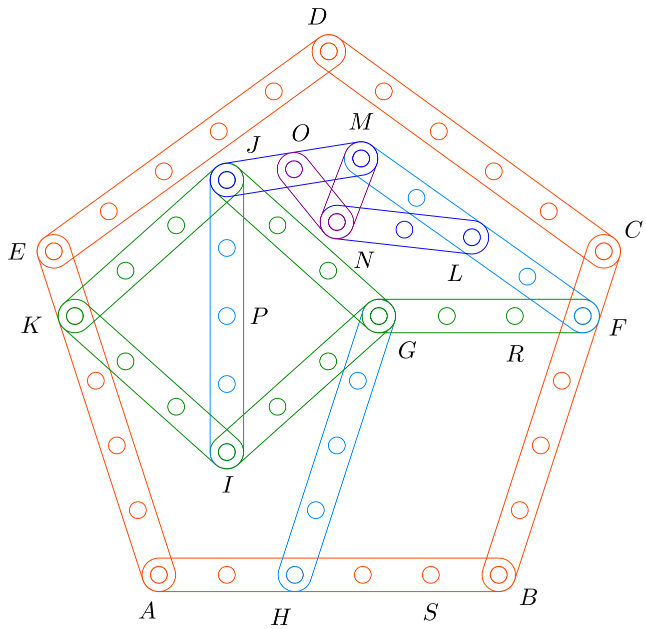

Figure 6: Pentagon of size 5 with 12 internal strips. $\overline{FB} : \overline{BA} : \overline{AK} = 4 : 5 : 4$ and $\overline{FK} = 3 + 2\sqrt{5}$. $\overline{FJ} = \sqrt{18 + 6\sqrt{5}} = \sqrt{3} + \sqrt{15}$.

Figure 6 show pentagon $ABCDE$ of size 5. For this construction we need vertices K, G, F to be collinear. First we note the three consecutive sides: $\overline{KA}$, $\overline{AS}$ and $\overline{SR}$ of an (incomplete) sub-pentagon and we know its **width** $\overline{KR}$ equals $\frac{1+\sqrt{5}}{2}$ times the side $\overline{KA}$ so here $\overline{KR} = 2 + 2\sqrt{5}$ and we have:

$$\overline{KF} = \overline{KR} + \overline{RF} = 3 + 2\sqrt{5} \tag{37}$$

Strip $\overline{GH}$ is parallel and of the same size of $\overline{FB}$ so makes rigid vertices A and B once colliner vertices K, G, F are rigid which we will show next.

3.2.1 Rigid distance $3 + 2\sqrt{5}$

In figure 6 we have an isoscelles triangle $\triangle GIJ$ and we calculate $\overline{PG} = \sqrt{(\overline{GI})^2 - (\overline{PI})^2} = \sqrt{3^2 - 2^2} = \sqrt{5}$. Similarly the other isoscelles triange $\triangle IJK$ gives $\overline{PK} = \sqrt{5}$ so we have (iff K, P, G, F were collinear):

$$\overline{KF} = \overline{KP} + \overline{PG} + \overline{GF} = 3 + 2\sqrt{5} \quad \blacksquare \tag{38}$$

Vertices K, P, G already are collinear. For P, G, F collinearity we need two angles $\alpha \equiv \angle PGJ$ and $\beta \equiv \overline{FGJ}$ with the condition $\alpha + \beta = \pi$. Within isoscelles triangle $\triangle GIJ$ we calculate α and then β (its supplement) and use β to calculate the distance $\overline{FJ}$ with the law of cosines:

$$\cos\alpha = \frac{\overline{PG}}{\overline{GJ}} = \frac{\sqrt{5}}{3} \tag{39}$$

$$\cos\beta = -\cos\alpha = -\frac{\sqrt{5}}{3} \tag{40}$$

$$\begin{aligned}\overline{FJ} &= \sqrt{(\overline{FG})^2 + (\overline{GJ})^2 - 2(\overline{FG})(\overline{GJ})\cos\beta} \\ &= \sqrt{3^2 + 3^2 - 2(3)(3)\left(-\frac{\sqrt{5}}{3}\right)} = \sqrt{18+6\sqrt{5}}\end{aligned} \tag{41}$$

With extra five-strips cluster $FLMNOJ$ distance FJ is made rigid as we show in next section.

3.2.2 Rigid distance $\sqrt{18+6\sqrt{5}} = \sqrt{3} + \sqrt{15}$

Our five-strips software generated among other solutions the cluster $FLMNOJ$ of figure 6. This cluster makes rigid the other cluster $FGJIK$. Here we validate the five-strips cluster. We have the equilateral triangle $\triangle MNO$ and $\overline{MJ}$ is the double of $\overline{MO}$ so angle $\angle MNJ = \pi/2$. Similarly we have the isoscelles triangle $\triangle MNL$ and $\overline{MF}$ is the double of $\overline{ML}$ so angle $\angle MNF = \pi/2$. With the two right angles vertices J, N, F are collinear so we can calculate easily $\overline{FJ}$:

$$\begin{aligned}\overline{FJ} &= \overline{JN} + \overline{NF} \\ &= \sqrt{(\overline{JM})^2 - (\overline{MN})^2} + \sqrt{(\overline{FM})^2 - (\overline{MN})^2} \\ &= \sqrt{2^2-1^2} + \sqrt{4^2-1^2} = \sqrt{3} + \sqrt{15} = \sqrt{18+6\sqrt{5}} \quad \blacksquare\end{aligned} \tag{42}$$

4 Pentagons of size 6

4.1 Size 6 with 6 internal strips *build*(4 : 6 : 4)

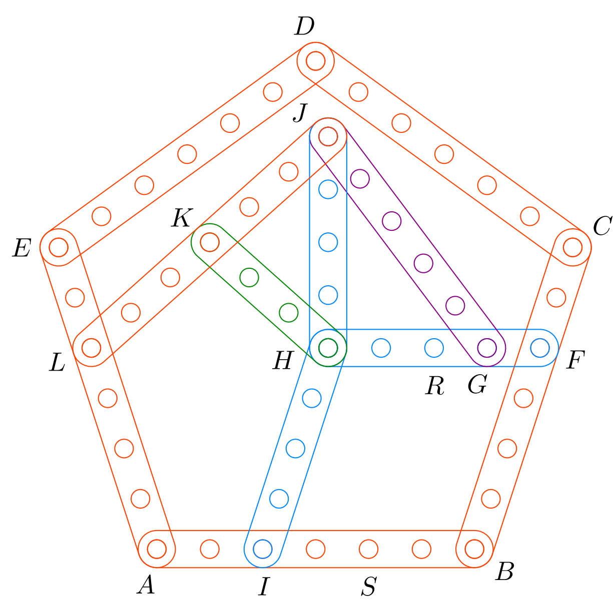

Figure 7: Pentagon of size 6 with 6 internal strips. $\overline{FB} : \overline{BA} : \overline{AL} = 4 : 6 : 4$. $\overline{FL} = 4 + 2\sqrt{5}$.

Figure 7 show pentagon $ABCDE$ of size 5. For this construction we need vertices L, H, F to be collinear. First we note the three consecutive sides: $\overline{LA}$, $\overline{AS}$ and $\overline{SR}$ of a (incomplete) sub-pentagon and we know its **width** $\overline{LR}$ equals $\dfrac{1+\sqrt{5}}{2}$ times the side $\overline{LA}$ so here $\overline{LR} = 2 + 2\sqrt{5}$ and we can have:

$$\overline{LF} = \overline{LR} + \overline{RF} = 4 + 2\sqrt{5} \tag{43}$$

Strip $\overline{HI}$ is parallel and of the same size of $\overline{FB}$ so makes rigid vertices A and B.

4.1.1 Rigid distance $4 + 2\sqrt{5}$

In figure 7 we have two right angles. First angle $\angle LHJ = \pi/2$ because triangle $\triangle HJK$ is isoscelles and $\overline{JL}$ is the double of $\overline{JK}$. Second angle $\angle FHJ = \pi/2$ because $\triangle GHJ$ is Pythagorean. Then vertices L, H, F are colliear and we can calculate $\overline{LH}$:

$$\begin{aligned} \overline{LF} &= \overline{LH} + \overline{HF} \\ &= \sqrt{(\overline{LJ})^2 - (\overline{JH})^2} + 4 = \sqrt{6^2 - 4^2} + 4 = 4 + 2\sqrt{5} \quad \blacksquare \end{aligned} \tag{44}$$

4.2 Size 6 with 8 internal strips $build(6:6:3:4)$

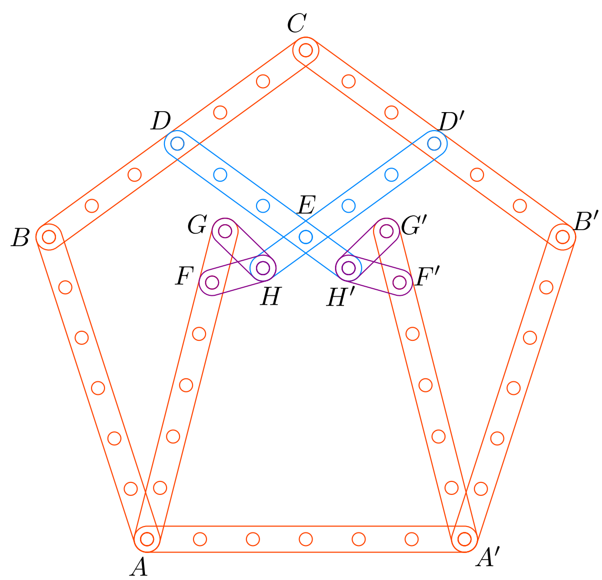

Figure 8: Pentagon of size 6 with 8 internal strips. $\overline{A'A} : \overline{AB} : \overline{BD} : \overline{DH'} = 6:6:3:4$ and $\overline{A'H'} = \sqrt{31}$.

Figure 8 show the regular pentagon $AA'B'CB$ of size 6. We calculate the distance $\overline{A'H'}$ assuming vertex A' is at origin and calculating abscissa and ordinate of vertex H'. We define $\alpha \equiv \angle ADB = 3\pi/5$ and

$\beta \equiv \angle B'BE = \pi/5$ and add and substract through vertices A', A, B, D, H' to get:

$$\begin{aligned} H'_x &= -\overline{A'A} - \overline{AB}|\cos\alpha| + \overline{BD}\cos\beta + \overline{DH}\cos\beta \\ &= -6 - (6)\left(\frac{\sqrt{5}-1}{4}\right) + (3+4)\left(\frac{\sqrt{5}+1}{4}\right) = \frac{-11+\sqrt{5}}{4} \end{aligned} \tag{45}$$

$$\begin{aligned} H'_y &= \overline{AB}\sin\alpha + \overline{BD}\sin\beta - \overline{DH'}\sin\beta \\ &= (6)\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right) + (3-4)\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right) = \frac{6\sqrt{10+2\sqrt{5}} - \sqrt{10-2\sqrt{5}}}{4} \end{aligned} \tag{46}$$

$$\begin{aligned} \overline{A'H'} &= \sqrt{(H'_x)^2 + (H'_y)^2} \\ &= \frac{\sqrt{(-11+\sqrt{5})^2 + (6\sqrt{10+2\sqrt{5}} - \sqrt{10-2\sqrt{5}})^2}}{4} = \sqrt{31} \end{aligned} \tag{47}$$

4.2.1 Rigid distance $\sqrt{31}$

Our three-strips software found the cluster $AFGH$ in the figure which fits inside this pentagon. We define angle $\theta \equiv \angle FGH = \pi/3$ since triangle $\triangle FGH$ is equilateral so $\cos\theta = 1/2$ and we can verify $\overline{AH}$ with the law of cosines:

$$\begin{aligned} \overline{AH} &= \sqrt{(\overline{GH})^2 + (\overline{AG})^2 - 2(\overline{GH})(\overline{AG})\cos\theta} \\ &= \sqrt{1^2 + 6^2 - 2(1)(6)\left(\frac{1}{2}\right)} = \sqrt{31} \quad \blacksquare \end{aligned} \tag{48}$$

4.3 Size 6 with 8 internal strips *build*$(3:6:4:6)$

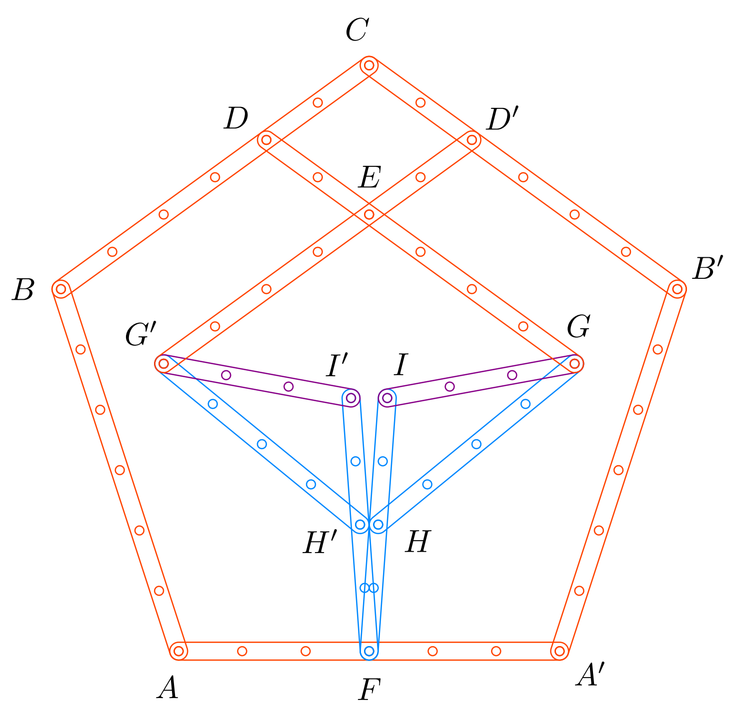

Figure 9: Pentagon of size 6 with 8 internal strips. $\overline{FA} : \overline{AB} : \overline{BD} : \overline{DG} = 3:6:4:6$ and $\overline{FG} = \sqrt{31}$.

Figure 9 show the regular pentagon $AA'B'CB$ of size 6. We calculate the distance $\overline{FG}$ assuming vertex F is at origin and calculating abscissa and ordinate of vertex G. We define $\alpha \equiv \angle A'AB = 3\pi/5$ and $\beta \equiv \angle B'BD = \pi/5$ and add and substract ortogonal distances through vertices F, A, B, D, G to get:

$$\begin{aligned}
G_x &= -\overline{FA} - \overline{AB}|\cos\alpha| + \overline{BD}\cos\beta + \overline{DG}\cos\beta \\
&= -3 - (6)\left(\frac{\sqrt{5}-1}{4}\right) + (4+6)\left(\frac{\sqrt{5}+1}{4}\right) = \frac{4+4\sqrt{5}}{4} \qquad (49)\\
G_y &= \overline{AB}\sin\alpha + \overline{BD}\sin\beta - \overline{DG}\sin\beta \\
&= (6)\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right) + (4-6)\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right) = \frac{6\sqrt{10+2\sqrt{5}} - 2\sqrt{10-2\sqrt{5}}}{4} \qquad (50)\\
\overline{FG} &= \sqrt{(G_x)^2 + (G_y)^2} \\
&= \frac{\sqrt{(4+4\sqrt{5})^2 + (6\sqrt{10+2\sqrt{5}} - 2\sqrt{10-2\sqrt{5}})^2}}{4} = \sqrt{31} \qquad (51)
\end{aligned}$$

4.3.1 Rigid distance $\sqrt{31}$

Our three-strips software found the cluster $FGHI$ that fits (hardly) inside the pentagon in the figure. With the law of cosines we calculate angle $\theta = \angle IHG$ and then we use the supplement (cosine negative) as $\angle FHG$ to verify $\overline{FG}$, again with the law of cosines:

$$\begin{aligned}
\cos\theta &= \frac{(\overline{HG})^2 + (\overline{HI})^2 - (\overline{IG})^2}{2(\overline{HG})(\overline{HI})} \\
&= \frac{4^2 + 2^2 - 3^2}{2(4)(2)} = \frac{11}{16} \qquad (52)\\
\overline{FG} &= \sqrt{(\overline{FH})^2 + (\overline{HG})^2 - 2(\overline{FH})(\overline{HG})(-\cos\theta)} \\
&= \sqrt{2^2 + 4^2 - 2(2)(4)\left(-\frac{11}{16}\right)} = \sqrt{31} \quad \blacksquare \qquad (53)
\end{aligned}$$

4.4 Size 6 with 10 internal strips $build(5:6)$

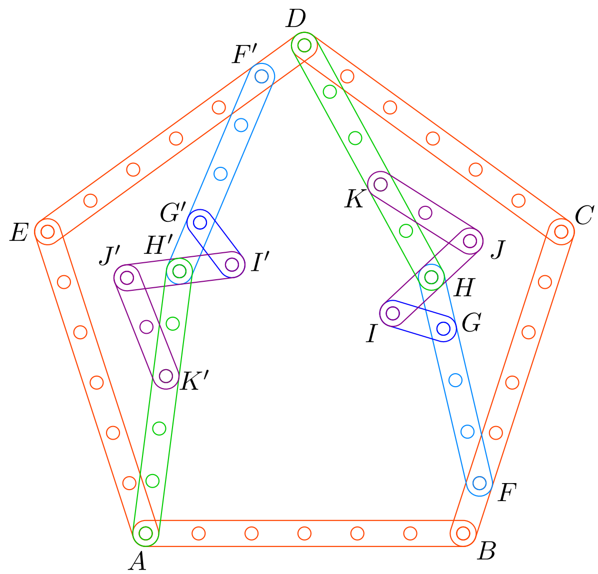

Figure 10: Pentagon of size 6 with 10 internal strips. $\overline{FC} : \overline{CD} = 5:6$. $\overline{FD} = \sqrt{46+15\sqrt{5}}$.

Figure 10 show the regular pentagon $ABCDE$ of size 6. We know the internal angle of pentagon $\theta = \angle FCD = \dfrac{3\pi}{5}$ and $\cos\theta = \dfrac{1-\sqrt{5}}{4}$ so with the law of cosines we can calculate $\overline{FD}$ and the angle $\delta = \angle CFD$:

$$\begin{aligned}
\overline{FD} &= \sqrt{(\overline{FC})^2 + (\overline{CD})^2 - 2(\overline{FC})(\overline{CD})\cos\theta} \\
&= \sqrt{5^2 + 6^2 - 2(5)(6)\left(\frac{1-\sqrt{5}}{4}\right)} = \sqrt{46+15\sqrt{5}} \qquad (54)
\end{aligned}$$

$$\begin{aligned}
\cos\delta &= \frac{(\overline{FD})^2 + (\overline{FC})^2 - (\overline{CD})^2}{2(\overline{FD})(\overline{FC})} \\
&= \frac{(46+15\sqrt{5}) + 5^2 - 6^2}{2\left(\sqrt{46+15\sqrt{5}}\right)(5)} = \frac{7+3\sqrt{5}}{2\sqrt{46+15\sqrt{5}}} \qquad (55)
\end{aligned}$$

4.4.1 Rigid distance $\sqrt{46+15\sqrt{5}}$

Our five-strips software found cluster $FGHIJKD$ shown in figure 10. Suppose vertice H is at the origin while vertice I is at $\{-1,0\}$ and vertice J is at $\{+1,0\}$. Triangle $\triangle HJK$ is isoscelles so we can found the coordinates of vertice K and vertice D which is a scaled coordinate of K by a factor $\overline{DH}/\overline{KH}$:

$$\begin{aligned} K\{x,y\} &= \left\{\frac{\overline{HJ}}{2}, \sqrt{(\overline{HK})^2 - \left(\frac{\overline{HJ}}{2}\right)^2}\right\} \\ &= \left\{\frac{1}{2}, \sqrt{2^2 - \left(\frac{1}{2}\right)^2}\right\} = \left\{\frac{1}{2}, \frac{\sqrt{15}}{2}\right\} \qquad (56) \\ D\{x,y\} &= \left(\frac{\overline{DH}}{\overline{KH}}\right) K\{x,y\} \\ &= \left(\frac{5}{2}\right)\left\{\frac{1}{2}, \frac{\sqrt{15}}{2}\right\} = \left\{\frac{5}{4}, \frac{5\sqrt{15}}{4}\right\} \qquad (57) \end{aligned}$$

Triangle $\triangle GHI$ is equilateral so we can found the coordinates of vertice G and vertice F which is a scaled coordinate of G by factor of $\overline{HF}/\overline{GF}$:

$$\begin{aligned} G\{x,y\} &= \left\{-\frac{\overline{HI}}{2}, -\sqrt{(\overline{HG})^2 - \left(\frac{\overline{HI}}{2}\right)^2}\right\} \\ &= \left\{-\frac{1}{2}, -\sqrt{1^2 - \left(\frac{1}{2}\right)^2}\right\} = \left\{-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right\} \qquad (58) \\ F\{x,y\} &= \left(\frac{\overline{HF}}{\overline{GF}}\right) G\{x,y\} \\ &= (4)\left\{-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right\} = \left\{-2, -2\sqrt{3}\right\} \qquad (59) \end{aligned}$$

Finally we verify the distance $\overline{FD}$:

$$\begin{aligned} \overline{FD} &= \sqrt{(D_x - F_x)^2 + (D_y - F_y)^2} \\ &= \sqrt{\left(\frac{5}{4} - (-2)\right)^2 + \left(\frac{5\sqrt{15}}{4} - (-2\sqrt{3})\right)^2} = \sqrt{46 + 15\sqrt{5}} \quad \blacksquare \qquad (60) \end{aligned}$$

4.5 Size 6 with 12 internal strips $build(0:6:4:2)$

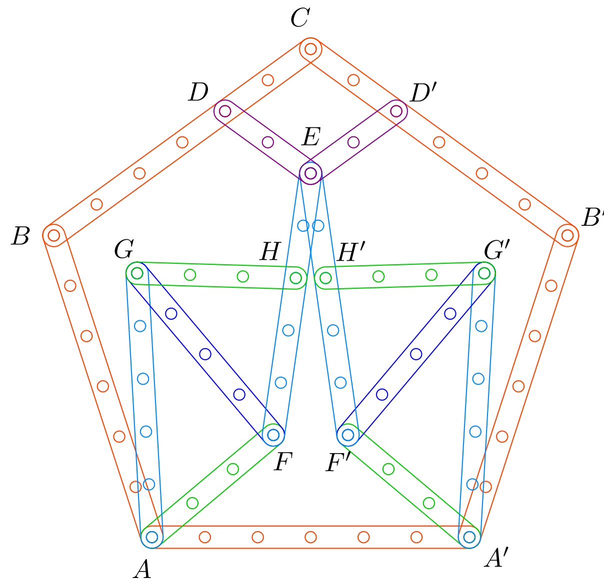

Figure 11: Pentagon of size 6 with 12 internal strips. $\overline{AA}$: $\overline{AB}$: $\overline{BD}$: $\overline{DE}$ = 0 : 6 : 4 : 2 and $AE = \sqrt{34+10\sqrt{5}}$

Figure 11 show the regular pentagon $AA'B'CB$ of size 6. We calculate the distance $\overline{AE}$ assuming vertex A is at origin and calculating abscissa and ordinate of vertex E. We define $\alpha \equiv \angle A'AB = 3\pi/5$ and $\beta \equiv \angle B'BC = \pi/5$ and add and substract orthogonal distances through vertices $ABDE$ to get:

$$\begin{aligned} E_x &= -\overline{AB}|\cos\alpha| + \overline{BD}\cos\beta + \overline{DE}\cos\beta \\ &= -(6)\left(\frac{\sqrt{5}-1}{4}\right) + (4+2)\left(\frac{\sqrt{5}+1}{4}\right) = \frac{12}{4} \qquad (61) \end{aligned}$$

$$\begin{aligned} E_y &= \overline{AB}\sin\alpha + \overline{BD}\sin\beta - \overline{DE}\sin\beta \\ &= (6)\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right) + (4-2)\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right) = \frac{6\sqrt{10+2\sqrt{5}}+2\sqrt{10-2\sqrt{5}}}{4} \qquad (62) \end{aligned}$$

$$\begin{aligned} \overline{AE} &= \sqrt{(E_x)^2+(E_y)^2} \\ &= \frac{\sqrt{(12)^2+(6\sqrt{10+2\sqrt{5}}+2\sqrt{10-2\sqrt{5}})^2}}{4} = \sqrt{34+10\sqrt{5}} \qquad (63) \end{aligned}$$

4.5.1 Rigid distance $\sqrt{34+10\sqrt{5}}$

Our five-strips sofware found the cluster $AFGHE$ shown in figure 11. We define two angles $\alpha \equiv \angle AFG = \pi/2$ within the Pythagorean triangle $\triangle AFG$ and $\beta \equiv \angle GFH$ within isoscelles triangle $\triangle GFH$ so we have:

$$\cos\alpha = 0 \qquad (64)$$

$$\sin\alpha = 1 \qquad (65)$$

$$\cos\beta = \frac{\overline{GF}/2}{\overline{FH}} = \frac{4/2}{3} = \frac{2}{3} \qquad (66)$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\left(\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3} \qquad (67)$$

Finally we calculate $\angle AFE = \alpha + \beta$ with the cosine sum identity and use it to verify the distance $\overline{AE}$ with the law of cosines:

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= (0)\left(\frac{2}{3}\right) - (1)\left(\frac{\sqrt{5}}{3}\right) = -\frac{\sqrt{5}}{3} \qquad (68)\\ \overline{AE} &= \sqrt{(\overline{AF})^2 + (\overline{FE})^2 - 2(\overline{AF})(\overline{FE})\cos(\alpha+\beta)} \\ &= \sqrt{3^2 + 5^2 - 2(3)(5)\left(-\frac{\sqrt{5}}{3}\right)} = \sqrt{34 + 10\sqrt{5}} \quad \blacksquare \qquad (69)\end{aligned}$$

5 Pentagons of size 7

5.1 Size 7 with 6 internal strips *build*$(6:7:6)$

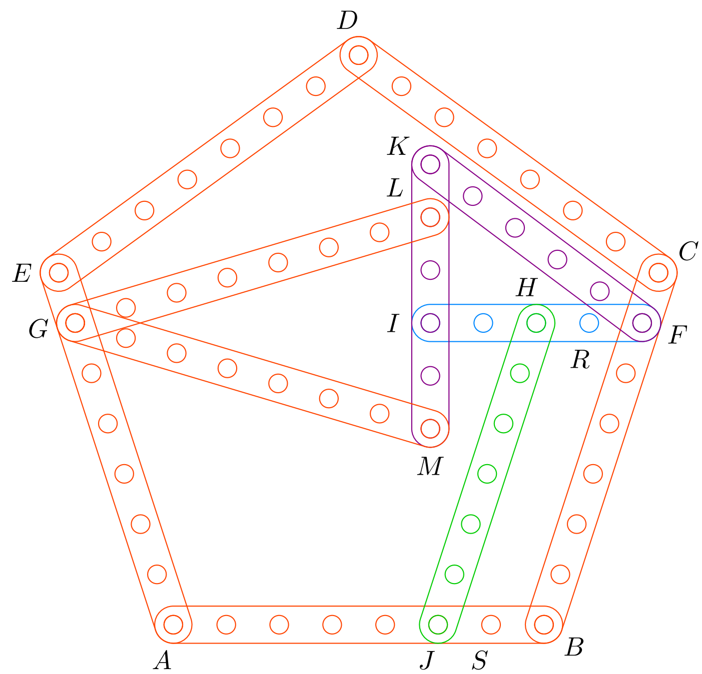

Figure 12: Pentagon of size 7 with 6 internal strips. $\overline{FC} : \overline{CD} : \ = 6:7:6$. $\overline{FD} = 4 + 3\sqrt{5}$.

Figure 12 show pentagon $ABCDE$ of size 7. For this construction we need vertices G, I, F to be collinear. First we note the three consecutive sides: $\overline{GA}$, $\overline{AS}$ and $\overline{SR}$ of an (incomplete) sub-pentagon of size 6. We know its **width** $\overline{GR}$ equals $\dfrac{1+\sqrt{5}}{2}$ times the side $\overline{GA}$ so here $\overline{GR} = 3 + 3\sqrt{5}$ and then:

$$\overline{GF} = \overline{GR} + \overline{RF} = 4 + 3\sqrt{5} \qquad (70)$$

Strip $\overline{HJ}$ is parallel and of the same size of $\overline{FB}$ so makes rigid vertices A and B once vertices G, I, F were rigid.

5.1.1 Rigid distance $4 + 3\sqrt{5}$

In figure 12 triangle $\triangle GLM$ is isoscelles so angle $\angle GIL = \pi/2$ and triangle $\triangle FIK$ is Pythagorean so angle $\angle FIK = \pi/2$. With the two right angles the vertices G, I, F are collinear and we can easily verify distance $\overline{GF}$:

$$\begin{aligned}\overline{GF} &= \overline{GI} + \overline{IF} \\ &= \sqrt{(\overline{GL})^2 - (\overline{LI})^2} + 4 = \sqrt{7^2 - 2^2} + 4 = 4 + 3\sqrt{5} \quad \blacksquare \end{aligned} \tag{71}$$

5.2 Size 7 with 10 internal strips $build(6, 7)$

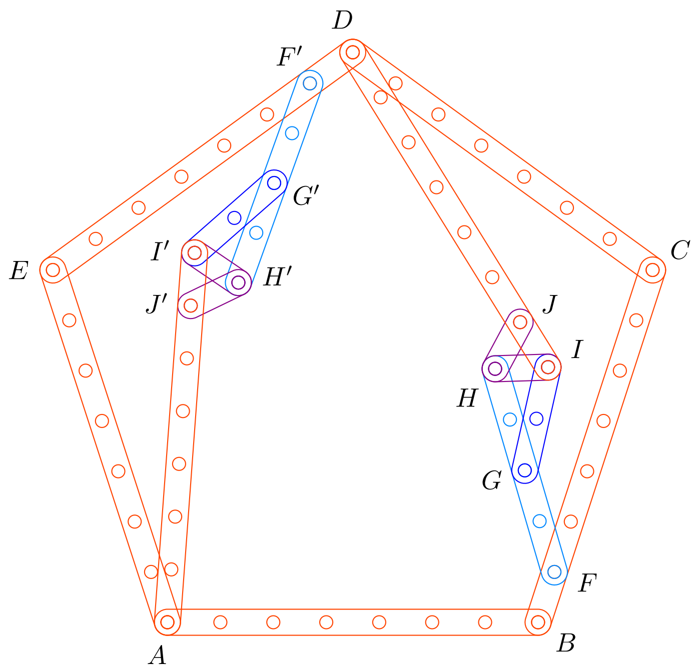

Figure 13: Pentagon of size 7 with 10 internal strips. $\overline{FC} : \overline{CD} = 6 : 7$. $\overline{FD} = \sqrt{64 + 21\sqrt{5}}$.

Figure 13 show the regular pentagon $ABCDE$ of size 7. We know the internal angle of pentagon $\theta = \angle FCD = \frac{3\pi}{5}$ and $\cos\theta = \frac{1-\sqrt{5}}{4}$ so with the law of cosines we can calculate $\overline{FD}$ and the angle $\delta = \angle CFD$:

$$\begin{aligned}\overline{FD} &= \sqrt{(\overline{FC})^2 + (\overline{CD})^2 - 2(\overline{FC})(\overline{CD})\cos\theta} \\ &= \sqrt{6^2 + 7^2 - 2(6)(7)\left(\frac{1-\sqrt{5}}{4}\right)} = \sqrt{64 + 21\sqrt{5}} \end{aligned} \tag{72}$$

$$\begin{aligned}\cos\delta &= \frac{(\overline{FD})^2 + (\overline{FC})^2 - (\overline{CD})^2}{2(\overline{FD})(\overline{FC})} \\ &= \frac{(64 + 21\sqrt{5}) + 6^2 - 7^2}{2\left(\sqrt{64 + 21\sqrt{5}}\right)(6)} = \frac{17 + 7\sqrt{5}}{4\sqrt{64 + 21\sqrt{5}}} \end{aligned} \tag{73}$$

5.2.1 Rigid distance $\sqrt{64 + 21\sqrt{5}}$

Our five-strips software produced the cluster $FGHIJD$ shown in figure 13. Consider vertice I at origin $(0, 0)$ and vertice H at coordinate $(-1, 0)$. Since triangle $\triangle GHI$ is isoscelles and $\overline{HF}$ is the double of FG

we have angle $\angle HIF = \pi/2$ and we can calculate the coordinates of vertice F as:

$$F_x = 0 \tag{74}$$

$$\begin{aligned} F_y &= \overline{IF} \\ &= -\sqrt{(\overline{HF})^2 - (\overline{HI})^2} = -\sqrt{4^2 - 1^2} = -\sqrt{15} \end{aligned} \tag{75}$$

Since triangle $\triangle HIJ$ is equilateral we have this coordinates for vertice J:

$$J_x = -\overline{IJ}\cos(\pi/3) = -(1)\frac{1}{2} = -\frac{1}{2} \tag{76}$$

$$J_y = +\overline{IJ}\sin(\pi/3) = (1)\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \tag{77}$$

We notice vertice D is a coordinate scaled from vertice J by a factor of $\overline{DI}/\overline{JI} = 7$ so $D_x = -7/2$ and $D_y = 7\sqrt{3}/2$. Finally we verify the distance $\overline{FD}$ using the coordinates values of vertices F, D:

$$\begin{aligned} \overline{FD} &= \sqrt{(D_x - F_x)^2 + (D_y - F_y)^2} \\ &= \sqrt{\left(-\frac{7}{2} - 0\right)^2 + \left(\frac{7\sqrt{3}}{2} - (-\sqrt{15})\right)^2} = \sqrt{64 + 21\sqrt{5}} \quad \blacksquare \end{aligned} \tag{78}$$

5.3 Size 7 with 14 internal strips

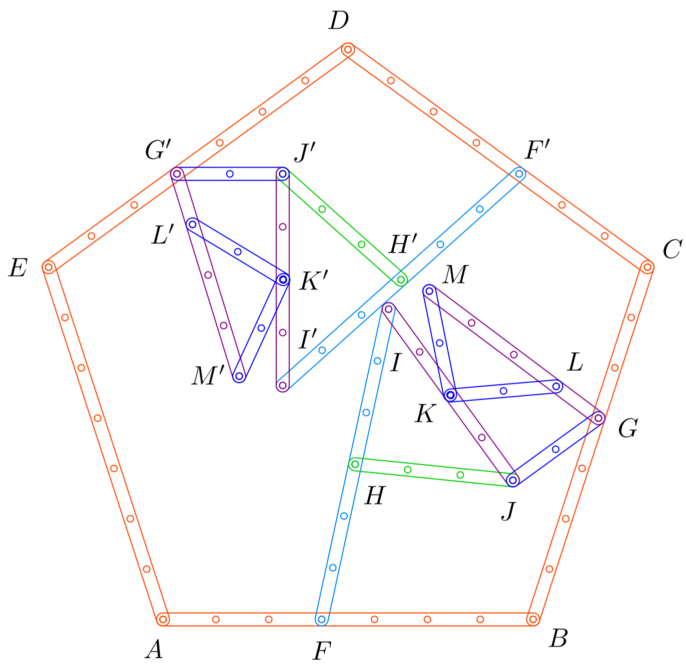

Figure 14: Pentagon of size 7 with 14 internal strips. $\overline{FB} : \overline{BG} = 4 : 4$ and $\overline{FG} = 2 + 2\sqrt{5}$.

Figure 14 show pentagon $ABCDE$ of size 7. We notice an (incomplete) sub-pentagon of size 4 with two consecutive sides $\overline{FB}$ and $\overline{BG}$. We know the **widht** of this sub-pentagon is $\dfrac{1+\sqrt{5}}{2}$ times the sub-side so here:

$$\overline{FG} = (4)\frac{1+\sqrt{5}}{2} = 2 + 2\sqrt{5} \tag{79}$$

5.3.1 Rigid distance $2 + 2\sqrt{5}$

Consider the cluster $GLMK$ in figure 14. First we calculate angle $\theta \equiv \angle LMK$ with the law of cosines and use this angle to calculate $\overline{KG}$ with the law of cosines:

$$\cos\theta = \frac{(\overline{ML})^2 + (\overline{MK})^2 - (\overline{KL})^2}{2(\overline{ML})(\overline{ML})} = \frac{3^2 + 2^2 - 2^2}{2(3)(2)} = \frac{3}{4} \tag{80}$$

$$\begin{aligned}\overline{KG} &= \sqrt{(\overline{GM})^2 + (\overline{MK})^2 - 2(\overline{GM})(\overline{MK})\cos\theta} \\ &= \sqrt{4^2 + 2^2 - (2)(4)(2)\frac{3}{4}} = 2\sqrt{2}\end{aligned} \tag{81}$$

Now since $\overline{KG} = 2\sqrt{2}$ and $\overline{GJ} = \overline{JK} = 2$ we conclude we have a right angle $\angle GJK = \pi/2$. And we have a second right angle $\angle FJI = \pi/2$ since triangle $\triangle HIJ$ is isoscelles and $\overline{IF}$ is the double of $\overline{IH}$. From the two right angles at vertice J we conclude vertices F, J, G are collinear and we verify distance $\overline{FG}$:

$$\begin{aligned}\overline{FG} &= \overline{FJ} + \overline{JG} \\ &= \sqrt{(\overline{IF})^2 - (\overline{IJ})^2} + 2 = \sqrt{6^2 - 4^2} + 2 = 2 + 2\sqrt{5} \quad \blacksquare\end{aligned} \tag{82}$$

6 Pentagons of size 8

6.1 Size 8 with 6 internal strips $build(8:8)$

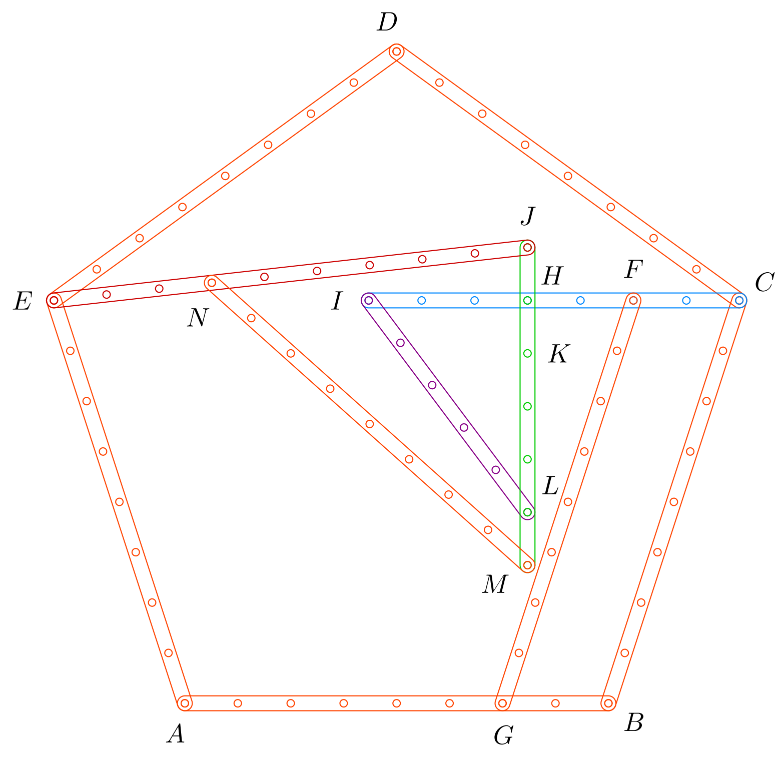

Figure 15: Pentagon of size 8 with 6 internal strips. $\overline{CD} : \overline{DE} = 8 : 8$. $\overline{FD} = 4 + 4\sqrt{5}$.

Figure 15 show regular pentagon $ABCDE$ of size 8. We know the pentagon width like $\overline{CE}$ is $\frac{1+\sqrt{5}}{2}$ times the side of the pentagon so here we have:

$$\overline{CE} = 8\left(\frac{1+\sqrt{5}}{2}\right) = 4 + 4\sqrt{5} \tag{83}$$

We expect vertices E, H, C to be collinear so strip $\overline{FG}$ which is parallel and of the same size that strip $\overline{CB}$ makes rigid the vertices A and B.

6.1.1 Rigid distance $4 + 4\sqrt{5}$

From the figure 15 consider triangle $\triangle NJM$ and calculate with the law of cosines angle $\theta \equiv \angle NJM$:

$$\cos\theta = \frac{(\overline{JM})^2 + (\overline{JN})^2 - (\overline{MN})^2}{2(\overline{JM})(\overline{JN})} = \frac{6^2 + 6^2 - 8^2}{2(6)(6)} = \frac{1}{9} \tag{84}$$

Since $\dfrac{\overline{EH}}{\overline{EJ}} = \sin\theta$ we have:

$$\overline{EH} = (\overline{EJ})\sin\theta = 9\sqrt{1-\cos^2\theta} = 9\sqrt{1-\left(\frac{1}{9}\right)^2} = 4\sqrt{5} \tag{85}$$

Since $(\overline{EJ})^2 = (\overline{EH})^2 + (\overline{HJ})^2 \mapsto 9^2 = 16(5) + 1$ we found a first right angle $\angle EHJ = \pi/2$. By inspection we have a second right angle $\angle IHL = \pi/2$ formed by the Pythagorean triangle $\triangle HIL$ and a third one $\angle LHC = \pi/2$ since is supplementary of $\angle IHL$. From the right angles we conclude vertices E, H, C are colinear so we verify the distance $\overline{EH}$:

$$\overline{EH} = \overline{EH} + \overline{HC} = 4 + 4\sqrt{5} \quad \blacksquare \tag{86}$$

6.2 Size 8 with 8 internal strips *build*$(4:8:2:4)$ and *build*$(4:8:4:6)$

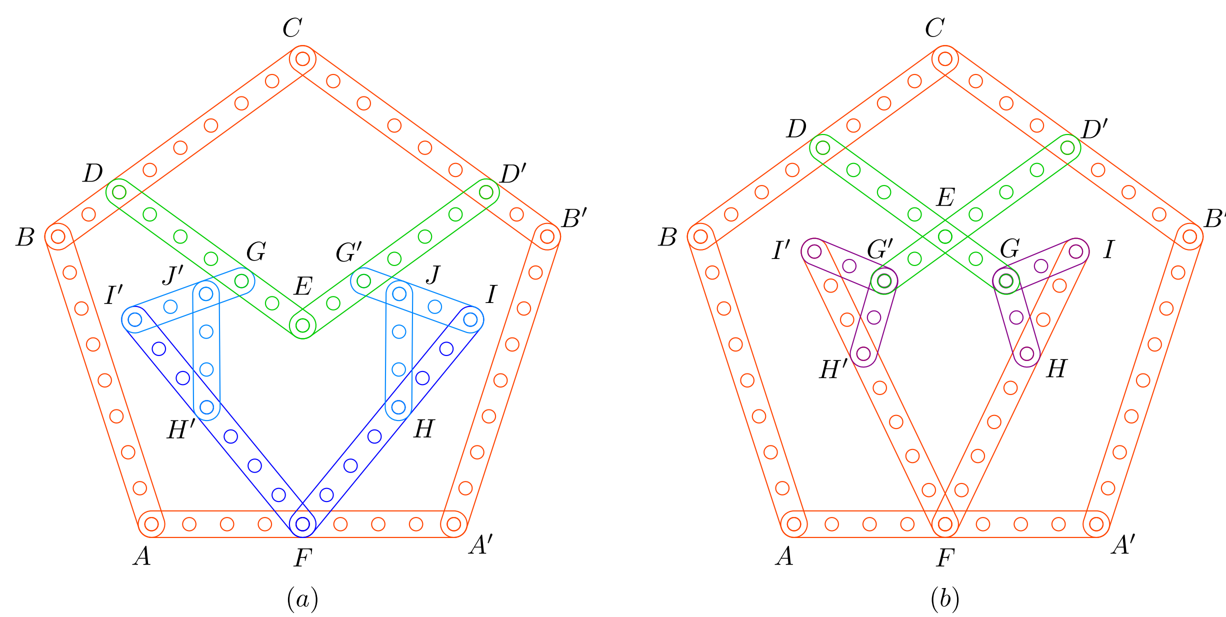

Figure 16: Pentagons of size 8 with 8 internal strips. For pentagon at (a) $\overline{FA} : \overline{AB} : \overline{BD} : \overline{DG} = 4:8:2:4$ and $\overline{FG} = 2\sqrt{11}$. For pentagon at (b) $\overline{AF} : \overline{AB} : \overline{BD} : \overline{DG} = 4:8:4:6$ and $\overline{FG} = 2\sqrt{11}$.

Figure 16 show two rigid regular pentagons A, A', B', C, B of size 9. For the pentagon at (a) we assume vertice F is at the origin and calculate the distance $\overline{FG}$ using the abscissas and ordinates following the vertices $FABDG$ for a regular pentagon angles $\alpha = 3\pi/5$, $\beta = \pi/5$:

$$\begin{aligned}
FI_x &= -\overline{AF} - \overline{AB}|\cos\alpha| + (\overline{BD} + \overline{DI})\cos\beta \\
&= -4 - (8)\frac{\sqrt{5}-1}{4} + (2+4)\frac{\sqrt{5}+1}{4} = -\frac{2+2\sqrt{5}}{4} \qquad (87)\\
FI_y &= \overline{AB}\sin\alpha + (\overline{BD} - \overline{DI})\sin\beta \\
&= (8)\frac{\sqrt{10+2\sqrt{5}}}{4} + (2-4)\frac{\sqrt{10-2\sqrt{5}}}{4} = \frac{8\sqrt{10+2\sqrt{5}} - 2\sqrt{10-2\sqrt{5}}}{4} \qquad (88)\\
\overline{FI} &= \sqrt{(FI_x)^2 + (FI_y)^2} \\
&= \frac{\sqrt{(2+2\sqrt{5})^2 + (8\sqrt{10+2\sqrt{5}} - 2\sqrt{10-2\sqrt{5}})^2}}{4} = \frac{\sqrt{704}}{4} = 2\sqrt{11} \qquad (89)
\end{aligned}$$

For the pentagon at (b) the vertices G, G' switch positions with those of pentagon at (a). In both pentagons we have $\overline{FG} = \overline{FG'} = 2\sqrt{11}$.

6.2.1 Rigid distances $2\sqrt{11}$

Our three-strips software found several options for the clusters of pentagons of figure 16. For the pentagon at (a) consider the cluster $FHIJG'$. Within the isoscelles triangle $\triangle HIJ$ we calculate the cosine of angle $\theta \equiv \angle JIH$ and use it with the law of cosines to verify $\overline{FG'}$:

$$\begin{aligned}
\cos\theta &= \frac{\overline{IJ}/2}{\overline{IH}} = \frac{1}{3} \\
\overline{FG'} &= \sqrt{(\overline{FI})^2 + (\overline{IG'})^2 - 2(\overline{FI})(\overline{IG'})\cos\theta} \\
&= \sqrt{7^2 + 3^2 - 2(7)(3)\frac{1}{3}} = 2\sqrt{11} \quad \blacksquare \qquad (90)
\end{aligned}$$

For the pentagon at (b) consider the cluster $FHIG$. Within the isoscelles triangle GIH we calculate the cosine of angle $\phi = \angle HIG$ and use it with the law of cosines to verify $\overline{FG}$:

$$\begin{aligned}
\cos\phi &= \frac{\overline{HI}/2}{\overline{GI}} = \frac{3}{4} \\
\overline{FH} &= \sqrt{(\overline{FI})^2 + (\overline{GI})^2 - 2(\overline{FI})(\overline{GI})\cos\phi} \\
&= \sqrt{8^2 + 2^2 - 2(8)(2)\frac{3}{4}} = 2\sqrt{11} \quad \blacksquare \qquad (91)
\end{aligned}$$

6.3 Size 8 with 10 internal strips $build(4:8)$

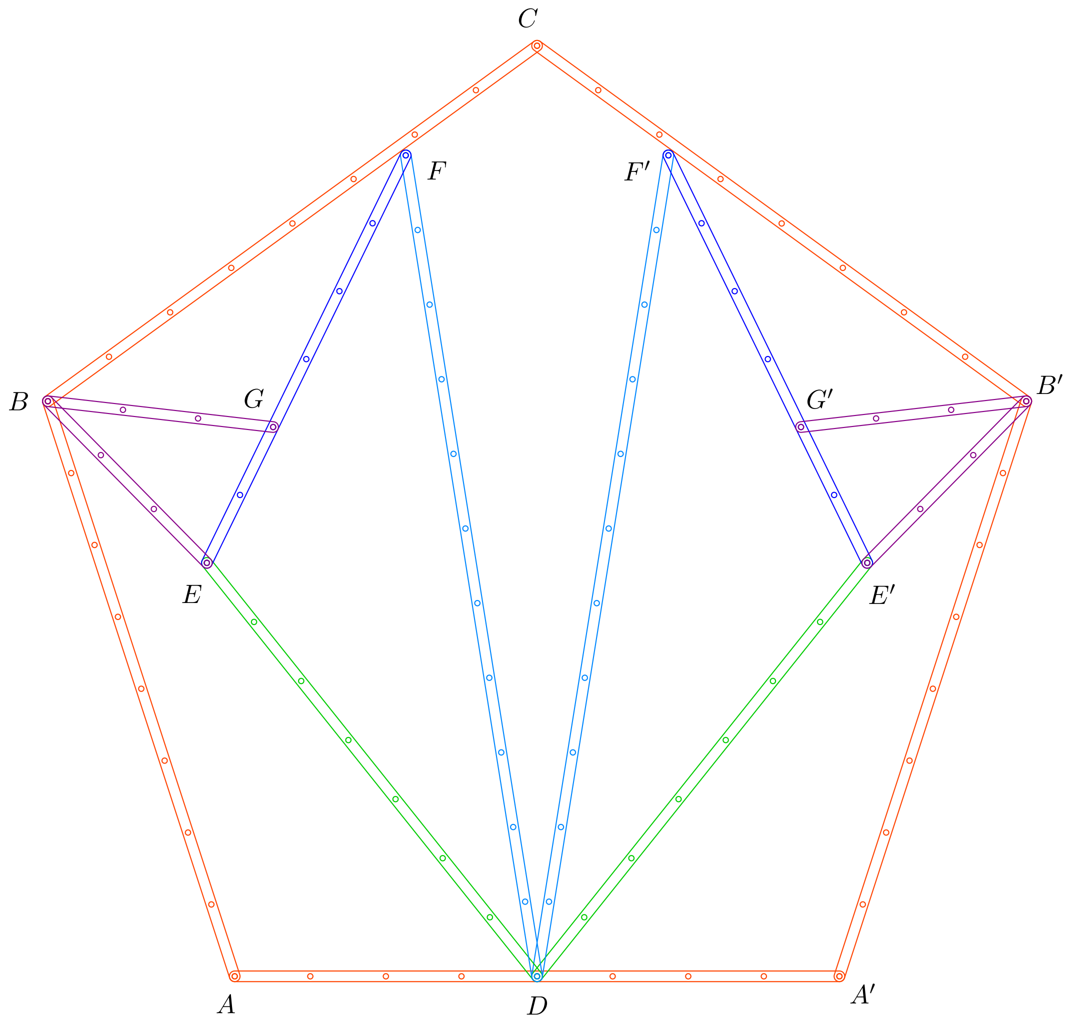

Figure 17: Pentagon of size 8 with 10 internal strips. $\overline{DA} : \overline{AB} = 4 : 8$ and $\overline{DB} = 4\sqrt{4+\sqrt{5}}$

Figure 17 show pentagon $AA'B'CB$ of size 8. We use the law of cosines to calculate distance $\overline{DB}$. We know $\theta \equiv \angle DAB = \dfrac{3\pi}{5}$ and $\cos\theta = \dfrac{1-\sqrt{5}}{4}$ so we have:

$$\begin{aligned} \overline{DB} &= \sqrt{(\overline{AD})^2 + (\overline{AB})^2 - 2(\overline{AD})(\overline{AB})\cos\theta} \\ &= \sqrt{4^2 + 8^2 - 2(4)(8)\frac{1-\sqrt{5}}{4}} = 4\sqrt{4+\sqrt{5}} \end{aligned} \tag{92}$$

6.3.1 Rigid distance $4\sqrt{4+\sqrt{5}}$

Our software found few solutions and the cluster $DEBGF$ shown in the figure 17 fit (hardly) inside the pentagon. We calculate two angles around vertice E. Angle $\alpha \equiv \angle DEF$ whitin the scalene triangle $\triangle DEF$

with the law of cosines and angle $\beta \equiv \angle BEG$ whitin the isoscelles triangle $\triangle BEG$:

$$\cos\alpha = \frac{(\overline{DE})^2 + (\overline{EF})^2 - (\overline{DF})^2}{2(\overline{DE})(\overline{EF})} = \frac{7^2 + 6^2 - 11^2}{2(7)(6)} = -\frac{3}{7} \tag{93}$$

$$\sin\alpha = \sqrt{1 - \cos^2\alpha} = \sqrt{1 - \left(-\frac{3}{7}\right)^2} = \frac{2\sqrt{10}}{7} \tag{94}$$

$$\cos\beta = \frac{\frac{\overline{EG}}{2}}{\overline{EB}} = \frac{\frac{2}{2}}{3} = \frac{1}{3} \tag{95}$$

$$\sin\beta = \sqrt{1 - \sin^2\beta} = \sqrt{1 - \left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3} \tag{96}$$

Finally we calculate the angle $\angle DEB = \alpha + \beta$ using the cosines sum identity and use it to verify the distance $\overline{DB}$:

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(-\frac{3}{7}\right)\left(\frac{1}{3}\right) - \left(\frac{2\sqrt{10}}{7}\right)\left(\frac{2\sqrt{2}}{3}\right) = -\frac{3 + 8\sqrt{5}}{21} \end{aligned} \tag{97}$$

$$\begin{aligned} \overline{DB} &= \sqrt{(\overline{DE})^2 + (\overline{EB})^2 - 2(\overline{DE})(\overline{EB})\cos(\alpha+\beta)} \\ &= \sqrt{7^2 + 3^2 - 2(7)(3)\left(-\frac{3 + 8\sqrt{5}}{21}\right)} = 4\sqrt{4 + \sqrt{5}} \quad \blacksquare \end{aligned} \tag{98}$$

6.4 Size 8 with 10 internal strips *build*$(7:8)$

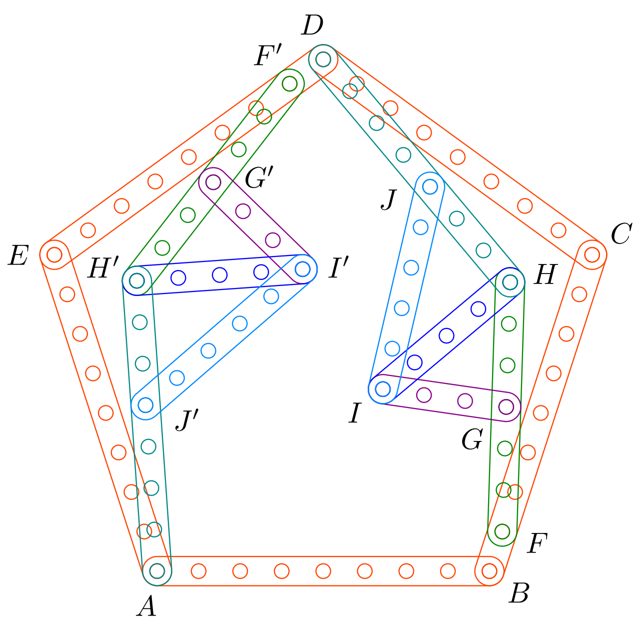

Figure 18: Pentagon of size 8 with 10 internal strips. $\overline{FC} : \overline{CD} = 7 : 8$ and $\overline{FD} = \sqrt{85 + 28\sqrt{5}}$

Figure 18 show pentagon $ABCDE$ of size 8. We use the law of cosines to calculate distance $\overline{FD}$ knowing $\theta = \angle FCD = \frac{3\pi}{5}$ and $\cos\theta = \frac{1-\sqrt{5}}{4}$ and calculate angle $\delta \equiv \angle CFD$:

$$\overline{FD} = \sqrt{(\overline{FC})^2 + (\overline{CD})^2 - 2(\overline{FC})(\overline{CD})\cos\theta}$$

$$= \sqrt{7^2 + 8^2 - 2(7)(8)\frac{1-\sqrt{5}}{4}} = \sqrt{85 + 28\sqrt{5}} \tag{99}$$

$$\cos\delta = \frac{(\overline{FD})^2 + (\overline{FC})^2 - (\overline{CD})^2}{2(\overline{FD})(\overline{FC})}$$

$$= \frac{85 + 28\sqrt{5} + 7^2 - 8^2}{2\sqrt{85+28\sqrt{5}}(7)} = \frac{5 + 2\sqrt{5}}{\sqrt{85+28\sqrt{5}}} \tag{100}$$

6.4.1 Rigid distance $\sqrt{85 + 28\sqrt{5}}$

Our five-strips software found several solutions, consider the cluster $FGHIJD$ in the figure 18. Assume vertice I is at the origin and vertice H is at coordinate $(4, 0)$. Since triangle $\triangle HIG$ is isoscelles and $\overline{HF}$ is the double of $\overline{IG}$ then $\overline{IF} = \sqrt{(\overline{HF})^2 - (\overline{IH})^2} = \sqrt{6^2 - 4^2} = 2\sqrt{5}$ and the coordinates of vertice F are $F(x, y) = (0, -\overline{IF}) = (0, -2\sqrt{5})$. Since the triangle $\triangle HIJ$ is Pythagorean we have a right angle $\angle IHD = \pi/2$ and then the coordinates of vertice D are $D(x, y) = (\overline{IH}, \overline{HD}) = (4, 7)$. Finally we verify the distance $\overline{FD}$:

$$\overline{FD} = \sqrt{(F_x - D_x)^2 + (F_y - D_y)^2}$$

$$= \sqrt{(0-4)^2 + (-2\sqrt{5} - 7)^2} = \sqrt{85 + 28\sqrt{5}} \quad \blacksquare \tag{101}$$

6.5 Size 8 with 10 internal strips $build(8:8)$ and $build(4:8:4)$

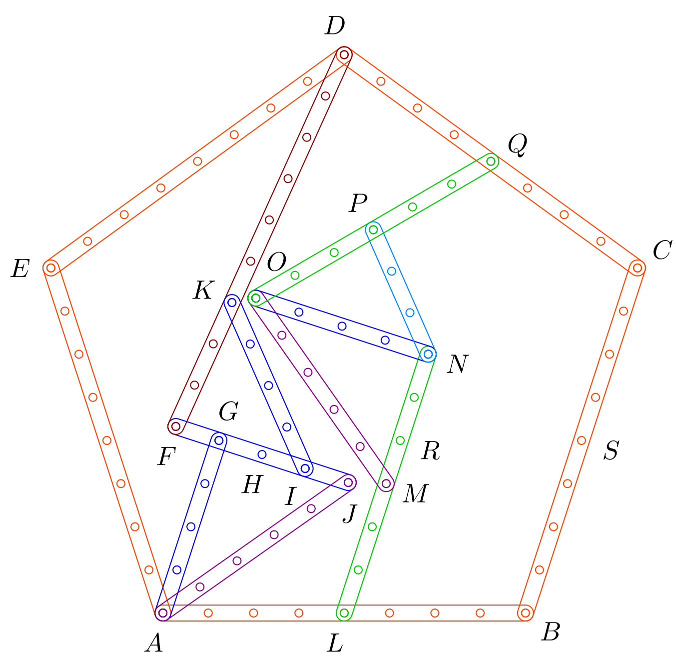

Figure 19: Pentagon of size 8 with 10 internal strips. $\overline{AE} : \overline{ED} = 8 : 8$ and $\overline{AD} = 4 + 4\sqrt{5}$. $\overline{LB} : \overline{BC} : \overline{CQ} = 4 : 8 : 4$ and $\overline{LQ} = 6 + 2\sqrt{5}$.

Figure 19 show pentagon $ABCDE$ of size 8. We have a width $\overline{AD}$ that we know equals the side times $\frac{1+\sqrt{5}}{2}$ so we have here:

$$\overline{AD} = 8\left(\frac{1+\sqrt{5}}{2}\right) = 4 + 4\sqrt{5} \tag{102}$$

We have a partial sub-pentagon of side 4 with a semi-perimeter $\overline{RS}, \overline{SC}, \overline{CQ}$ with width $\overline{RS} = 4\left(\frac{1+\sqrt{5}}{2}\right) = 2 + 2\sqrt{5}$ so we have:

$$\overline{LQ} = \overline{LR} + \overline{RQ} = 4 + 2 + 2\sqrt{5} = 6 + 2\sqrt{5} \tag{103}$$

6.5.1 Rigid distances $4 + 4\sqrt{5}$ and $6 + 2\sqrt{5}$

From the figure 19 we have a (partial) isoscelles triangle $\triangle FHD$ where $\alpha = \angle DFG = \dfrac{\overline{FG}}{\overline{FD}} = \dfrac{1}{9}$. The (missing) strip $\overline{HD}$ is substituted by strip $\overline{IK}$ since triangle $\triangle FIK$ internal angle $\angle IFK$ matches α after using the law of cosines:

$$\cos(\angle IFK) = \frac{(\overline{FK})^2 + (\overline{FI})^2 - (\overline{IK})^2}{2(\overline{FK})(\overline{FI})} = \frac{3^2 + 3^2 - 4^2}{2(3)(3)} = \frac{1}{9} \quad \blacksquare \tag{104}$$

Angle $\angle HGD = \pi/2$ since $\triangle FHD$ is isoscelles and also $\angle AGJ = \pi/2$ since triangle $\triangle AGJ$ is Pythagorean so vertices A, G, D are collinear and then we verify distance $\overline{AD}$:

$$\begin{aligned}\overline{AD} &= \overline{AG} + \overline{GD} \\ &= 4 + \sqrt{(\overline{FD})^2 - (\overline{FG})^2} = 4 + \sqrt{9^2 - 1^2} = 4 + 4\sqrt{5} \quad \blacksquare\end{aligned} \tag{105}$$

From the figure 19 we see triangle $\triangle NOP$ is isoscelles and that strip $\overline{OQ}$ is the double of strip $\overline{PN}$ so angle $\angle ONQ = \pi/2$ and we can calculate $\overline{NQ} = \sqrt{(\overline{OQ})^2 - (\overline{ON})^2} = \sqrt{6^2 - 4^2} = 2\sqrt{5}$. Also angle $\angle MNO = \pi/2$ since triangle $\triangle MNO$ is Pythagorean so vertices M, N, Q are collinear and we verify distance $\overline{LQ}$:

$$\begin{aligned}\overline{LQ} &= \overline{LN} + \overline{NQ} \\ &= 6 + 2\sqrt{5} \quad \blacksquare\end{aligned} \tag{106}$$

7 Pentagons of size 9

7.1 Size 9 with 6 internal strips $build(8:9:8)$

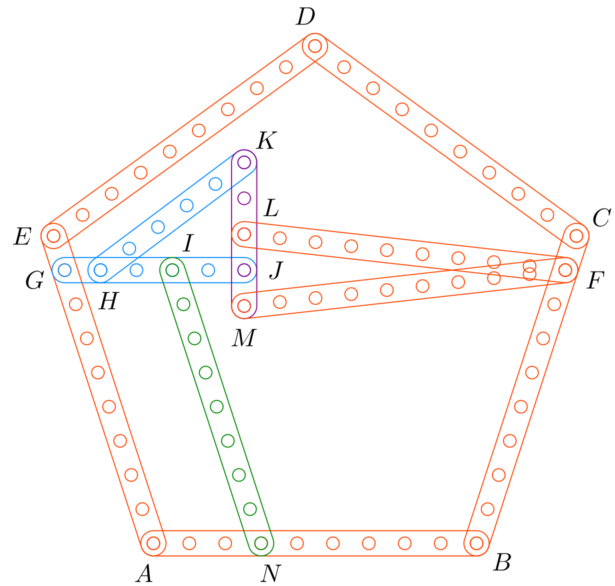

Figure 20: Pentagon size 9 with six internal strips. $\overline{FB} : \overline{BA} : \overline{AG} = 8 : 9 : 8$. $\overline{FG} = 5 + 4\sqrt{5}$.

Figure 20 show a rigid regular pentagon A, B, C, D, E of size 9. The regular pentagon distance $\overline{CE}$ is called width and equals $\frac{1+\sqrt{5}}{2}\overline{AB}$. Is easy to note the distance $\overline{FG}$ equals the width of smaller pentagon size $9 - 1 = 8$ plus 1. So we have:

$$\begin{aligned} \overline{FG} &= \frac{1+\sqrt{5}}{2}(\overline{BC} - \overline{FC}) + \overline{FC} \\ &= \frac{1+\sqrt{5}}{2}(9-1) + 1 = 5 + 4\sqrt{5} \end{aligned} \tag{107}$$

7.1.1 Rigid distance $5 + 4\sqrt{5}$

From the figure we see two right angles. Angle $\angle GJK = \frac{\pi}{2}$ because we have a Pythagorean triangle $\triangle HJK$. Angle $\angle FJM = \frac{\pi}{2}$ because we have an isosceles triangle $\triangle FLM$. The two right angles share vertice J so vertices G, J, F are collinear. First we calculate the distance $\overline{JF} = \sqrt{(LF)^2 - (LJ)^2} = \sqrt{9^2 - 1^2} = 4\sqrt{5}$ and finally the distance $\overline{GF} = \overline{GJ} + \overline{JF} = 5 + 4\sqrt{5}$ which matches the value in last equation above $\blacksquare$. To make rigid the pentagon we add strip $\overline{IN}$ parallel to side $\overline{GA}$.

7.2 Size 9 with 8 internal strips $build(3, 9, 3, 5)$

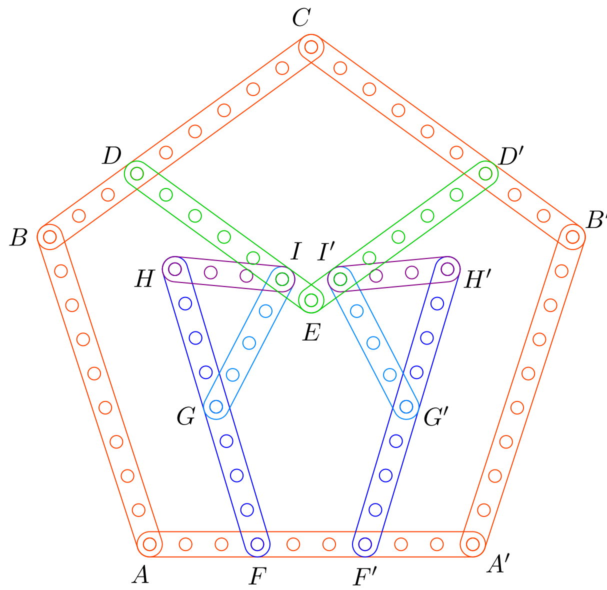

Figure 21: Pentagon size 9 with 8 strips variation 1. $\overline{FA} : \overline{AB} : \overline{BD} : \overline{DI} = 3 : 9 : 3 : 5$. $\overline{FI} = \sqrt{55}$.

Figure 21 show a rigid regular pentagon A, A', B', C, B of size 9. First we calculate the distance $\overline{FI}$ using the abscissas and ordinates following the vertices F, A, B, D, I for a regular pentagon angles $\alpha = \dfrac{3\pi}{5}, \beta = \dfrac{\pi}{5}$:

$$\begin{aligned} FI_x &= -\overline{AF} - \overline{AB}\cos\alpha + (\overline{BD} + \overline{DI})\cos\beta \\ &= -3 + (9)\frac{1-\sqrt{5}}{4} + (3+5)\frac{\sqrt{5}+1}{4} = \frac{5-\sqrt{5}}{4} \end{aligned} \tag{108}$$

$$\begin{aligned} FI_y &= \overline{AB}\sin\alpha + (\overline{BD} - \overline{DI})\sin\beta \\ &= (9)\frac{\sqrt{10+2\sqrt{5}}}{4} + (3-5)\frac{\sqrt{10-2\sqrt{5}}}{4} = \frac{9\sqrt{10+2\sqrt{5}} - 2\sqrt{10-2\sqrt{5}}}{4} \end{aligned} \tag{109}$$

$$\begin{aligned} \overline{FI} &= \sqrt{(FI_x)^2 + (FI_y)^2} \\ &= \frac{\sqrt{(5-\sqrt{5})^2 + (9\sqrt{10+2\sqrt{5}} - 2\sqrt{10-2\sqrt{5}})^2}}{4} = \frac{\sqrt{880}}{4} = \sqrt{55} \end{aligned} \tag{110}$$

7.2.1 Rigid distance $\sqrt{55}$

By software we find several options for the distance and we use the one shown in the figure. We calculate the distance $\overline{FI}$ made rigid by cluster F, G, H, I. We have an isoscelles triangle $\triangle GHI$ and $\overline{FH} = 2\overline{GH}$ so we have a right triangle $\angle FIH = \frac{\pi}{2}$ so:

$$\begin{aligned} \overline{FI} &= \sqrt{(\overline{FH})^2 - (\overline{HI})^2} \\ &= \sqrt{8^2 - 3^2} = \sqrt{55} \quad \blacksquare \end{aligned} \tag{111}$$

7.3 Size 9 with 8 internal strips $build(3,9,1,4)$ and $build(6,9,6,8)$

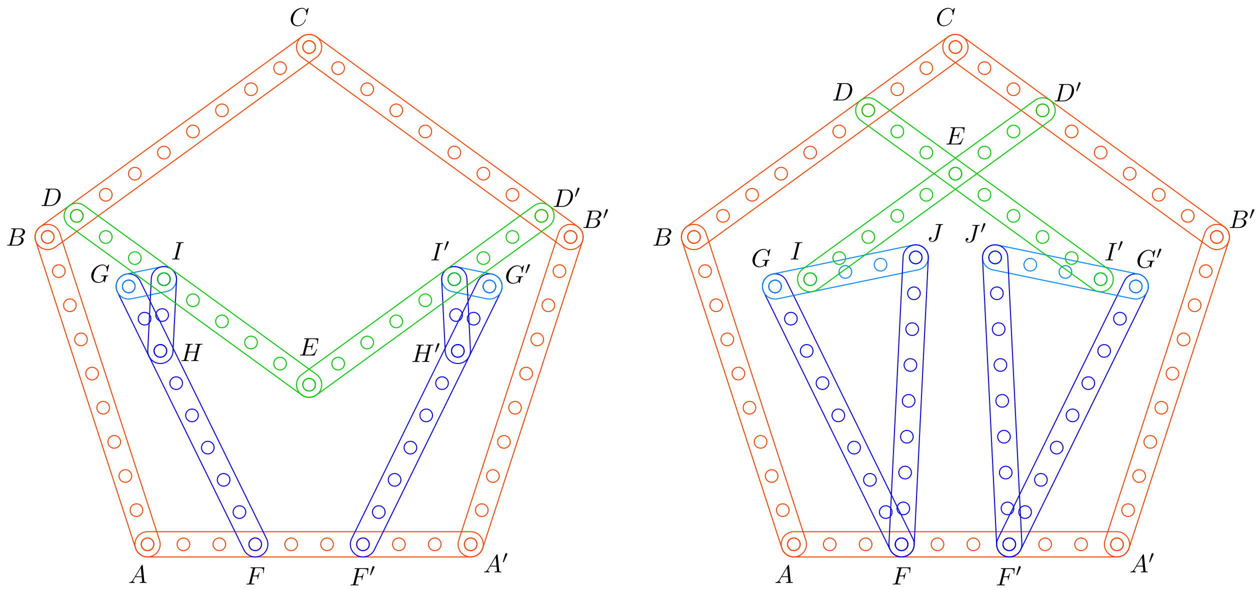

Figure 22: Regular pentagons size 9 made rigid with 8 strips variations 2 and 3. At the left we have $\overline{FA} : \overline{AB} : \overline{BD} : \overline{DI} = 3 : 9 : 1 : 4$ and $\overline{FI} = \sqrt{61}$. At the right we have $\overline{F'A} : \overline{AB} : \overline{BD} : \overline{DI'} = 6 : 9 : 6 : 8$ and $\overline{F'I'} = \sqrt{61}$.

Figure 22 show two rigid pentagons A, A', B', C, B' of size 9. The pentagon at the left is called variation 2 and the right one variation 3. Both variations have the vertices I, I' at the same positions and the same distance $\overline{FI}$ which first we calculate using the abscissas and ordinates following the vertices F, A, B, D, I of the variation 2 for a regular pentagon angles $\alpha = \frac{3\pi}{5}, \beta = \frac{\pi}{5}$:

$$\begin{aligned} FI_x &= -\overline{AF} - \overline{AB}\cos\alpha + (\overline{BD} + \overline{DI})\cos\beta \\ &= -3 + (9)\frac{1-\sqrt{5}}{4} + (1+3)\frac{\sqrt{5}+1}{4} = \frac{1-5\sqrt{5}}{4} \end{aligned} \tag{112}$$

$$\begin{aligned} FI_y &= \overline{AB}\sin\alpha + (\overline{BD} - \overline{DI})\sin\beta \\ &= (9)\frac{\sqrt{10+2\sqrt{5}}}{4} + (1-3)\frac{\sqrt{10-2\sqrt{5}}}{4} = \frac{9\sqrt{10+2\sqrt{5}} - 2\sqrt{10-2\sqrt{5}}}{4} \end{aligned} \tag{113}$$

$$\begin{aligned} \overline{FI} &= \sqrt{(FI_x)^2 + (FI_y)^2} \\ &= \frac{\sqrt{(1-\sqrt{5})^2 + (9\sqrt{10+2\sqrt{5}} - 2\sqrt{10-2\sqrt{5}})^2}}{4} = \frac{\sqrt{976}}{4} = \sqrt{61} \end{aligned} \tag{114}$$

7.3.1 Rigid distance $\sqrt{61}$

Our software found several clusters and we use two different for each variation. We calculate the distance $\overline{FI}$ made rigid by clusters F, G, H, I or F, G, I, J since in both variations we have the same $\overline{GF}$ and same angles $\angle FGI = \angle FJG$. With the law of cosines first we calculate $\cos(\angle FJG)$ and then $\overline{FI}$:

$$\cos(\angle FJG) = \frac{\overline{FJ}^2 + \overline{JG}^2 - \overline{GF}^2}{2(\overline{FJ})(\overline{JG})} = \frac{8^2 + 4^2 - 8^2}{2(8)(4)} = \frac{1}{4}$$

$$\overline{FI} = \sqrt{\overline{IJ}^2 + \overline{FJ}^2 - 2(\overline{IJ})(\overline{FJ})\cos(\angle FJG)} = \sqrt{3^2 + 8^2 - 2(3)(8)\left(\frac{1}{4}\right)} = \sqrt{61} \quad \blacksquare \tag{115}$$

7.4 Size 9 with 10 internal strips *build*(4 : 9) and *build*(5 : 9)

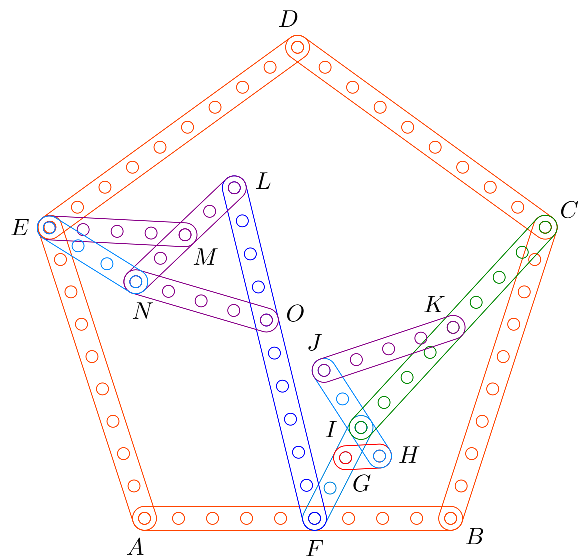

Figure 23: Pentagons size 9 with two 10 internal strips. For the cluster of the left we have $\overline{FA} : \overline{AE} = 5 : 9$ and $\overline{FE} = \dfrac{\sqrt{334 + 90\sqrt{5}}}{2}$. For the cluster at the right we have $\overline{FB} : \overline{BC} = 4 : 9$ and $\overline{FC} = \sqrt{79 + 18\sqrt{5}}$.

Figure 23 show a regular pentagon A, B, C, D, E of size 9 made it rigid with the help of clusters fixing the distances $\overline{CF}$ and $\overline{EF}$. Pentagon size 9 is the smaller one with diagonals where consecutive side segments fractions are $\overline{BF}/\overline{BC} = \dfrac{4}{9}$ and $\overline{AF}/\overline{AE} = \dfrac{5}{9}$. We calculate the diagonals $\overline{CF}$, $\overline{EF}$ and the angles to side $\overline{AB}$ using the law of cosines and the internal pentagon angle $\theta = \angle FBC = \angle FAE = \frac{3\pi}{5}$ where $\cos\theta = \frac{1-\sqrt{5}}{4}$. Diagonal $\overline{CF}$:

$$\begin{aligned}\overline{CF} &= \sqrt{\overline{BC}^2 + \overline{BF}^2 - 2(\overline{BC})(\overline{BF})\cos\theta} \\ &= \sqrt{9^2 + 4^2 - 2(9)(4)\left(\frac{1-\sqrt{5}}{4}\right)} = \sqrt{79 + 18\sqrt{5}}\end{aligned} \tag{116}$$

$$\cos(\angle CFB) = \frac{\overline{CF}^2 + \overline{BF}^2 - \overline{BC}^2}{2(\overline{CF})(\overline{BF})} = \frac{79 + 18\sqrt{5} + 4^2 - 9^2}{2(\sqrt{79 + 18\sqrt{5}})(4)} = \frac{7 + 9\sqrt{5}}{4\sqrt{79 + 18\sqrt{5}}} \tag{117}$$

Diagonal $\overline{EF}$:

$$\overline{EF} = \sqrt{\overline{AE}^2 + \overline{AF}^2 - 2(\overline{AE})(\overline{AF})\cos\theta}$$
$$= \sqrt{9^2 + 5^2 - 2(9)(5)\left(\frac{1-\sqrt{5}}{4}\right)} = \frac{\sqrt{334+90\sqrt{5}}}{2} \tag{118}$$

$$\cos(\angle EFA) = \frac{\overline{EF}^2 + \overline{AF}^2 - \overline{EA}^2}{2(\overline{EF})(\overline{AF})} = \frac{\frac{334+90\sqrt{5}}{4} + 5^2 - 9^2}{2\left(\frac{\sqrt{334+90\sqrt{5}}}{2}\right)(5)} = \frac{11+9\sqrt{5}}{2\sqrt{334+90\sqrt{5}}} \tag{119}$$

7.4.1 Rigid distance $\sqrt{79+18\sqrt{5}}$

Our software found several options with five strips to build distance $\sqrt{79+18\sqrt{5}}$.

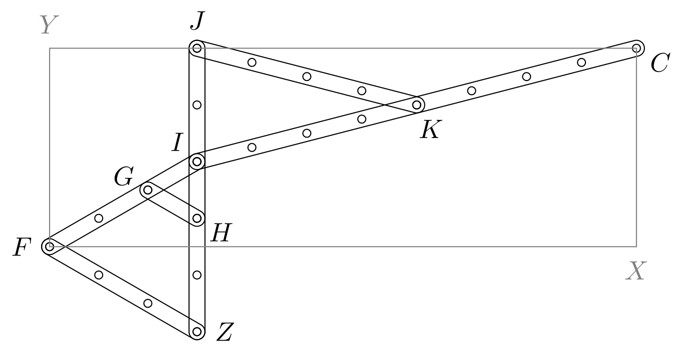

Figure 24: Construction of distance $\overline{FC} = \sqrt{79+18\sqrt{5}}$

Figure 24 show one of several ways to build the distance $\sqrt{79+18\sqrt{5}}$. Equilateral triangle $\triangle FIZ$ and isosceles $\triangle IJK$ share vertice I and the base $\overline{JZ}$ which help to form rectangle $FXCY$ with base $\overline{FX}$ and height $\overline{FY}$ useful to calculate the diagonal $\overline{FC}$:

$$\overline{FX} = \overline{YJ} + \overline{JC}$$
$$= \sqrt{\overline{FI}^2 - \left(\frac{\overline{IZ}}{2}\right)^2} + \sqrt{\overline{IC}^2 - \overline{IJ}^2} = \sqrt{3^2 - \left(\frac{3}{2}\right)^2} + \sqrt{8^2 - 2^2} = \frac{3\sqrt{3}}{2} + 2\sqrt{15}$$
$$\overline{FY} = \overline{JI} + \frac{\overline{IZ}}{2} = 2 + \frac{3}{2} = \frac{7}{2}$$
$$\overline{FC} = \sqrt{\overline{FX}^2 + \overline{FY}^2} = \sqrt{\left(\frac{3\sqrt{3}}{2} + 2\sqrt{15}\right)^2 + \left(\frac{7}{2}\right)^2} = \sqrt{79+18\sqrt{5}} \quad \blacksquare \tag{120}$$

We use a smaller part of this construction, the five strips with vertices F, G, H, I, J, K, C, as a cluster to made rigid the consecutive strips $\overline{AB}, \overline{BC}$ of the pentagon of side 9 of figure 23.

7.4.2 Rigid distance $\frac{\sqrt{334+90\sqrt{5}}}{2}$

Our software found several options with five strips to build distance $\frac{\sqrt{334+90\sqrt{5}}}{2}$.

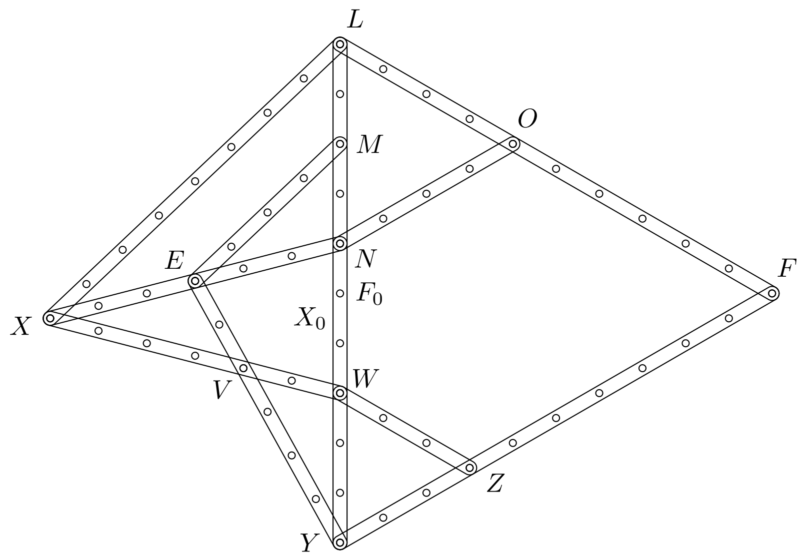

Figure 25: Construction of distance $\overline{EF} = \dfrac{\sqrt{334+90\sqrt{5}}}{2}$

Figure 25 show equilateral triangle $\triangle FLY$ and isoscelles triangle $\triangle NXW$ sharing strip $\overline{LY}$ which helps to calculate abscissas and ordinates of vertices E, F to calculate distance $\overline{EF}$. Vertice Y is located at the origin so:

$$E_x = -\left(\frac{\overline{NE}}{\overline{NX}}\right)\overline{XX_0} = -\frac{3}{6}\sqrt{\overline{NX}^2 - \overline{NX_0}^2} = -\frac{1}{2}\sqrt{6^2 - \left(\frac{3}{2}\right)^2} = -\frac{3\sqrt{15}}{4} \tag{121}$$

$$E_y = \overline{YN} - \left(\frac{\overline{NE}}{\overline{NX}}\right)\overline{NX_0} = 6 - \left(\frac{3}{6}\right)\left(\frac{3}{2}\right) = \frac{21}{4} \tag{122}$$

$$F_x = \overline{F_0F} = \sqrt{\overline{YF}^2 - \overline{YF_0}^2} = \sqrt{10^2 - 5^2} = 5\sqrt{3} \tag{123}$$

$$F_y = \overline{YF_0} = 5 \tag{124}$$

$$\overline{EF} = \sqrt{(E_x - F_x)^2 + (E_y - F_y)^2} = \sqrt{\left(-\frac{3\sqrt{15}}{4} - 5\sqrt{3}\right)^2 + \left(\frac{21}{4} - 5\right)^2} = \frac{\sqrt{334+90\sqrt{5}}}{2} \quad \blacksquare \tag{125}$$

We form a cluster from the last construction to be applied in the pentagon of side 9. We choose the five strips with vertices E, N, M, L, O, F. Is easy to prove strip $\overline{EM}$ is correct in the cluster comparing equal cosines at vertice Y for triangles $\triangle YVW, \triangle YEN, \triangle YEM$ using the law of cosines for each triangle.

8 Pentagon of size 10

8.1 Size 10 with 10 internal strips $build(7:10)$

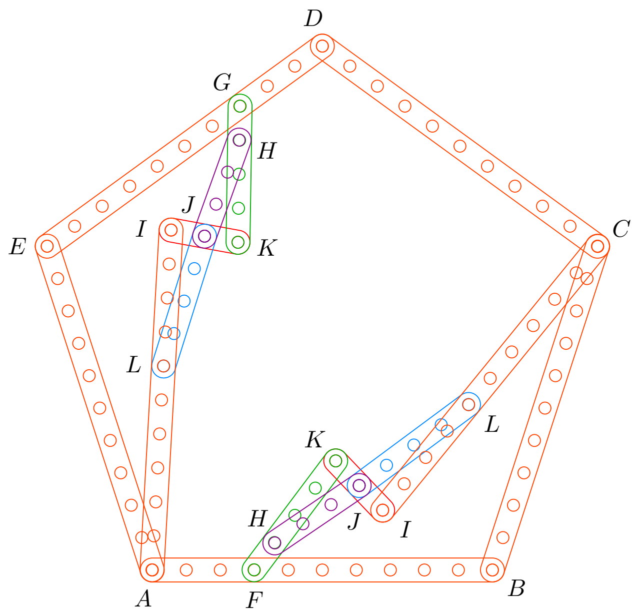

Figure 26: Pentagon size 10 with 10 interal strips. $\overline{FB} : \overline{BC} = 7 : 10$ and $\overline{CF} = \sqrt{114 + 35\sqrt{5}}$.

Figure 26 show a rigid regular pentagon A, B, C, D, E of size 10. We calculate a diagonal joining two consecutive sides relative primes to have something exclusive to the size 10, we choose $\overline{BF} : \overline{BC} = 7 : 10$. With the law of cosines we calculate $\overline{CF}$. We calculate the angle $\angle CFB$ for the drawing:

$$\begin{aligned} \overline{CF}^2 &= \overline{BC}^2 + \overline{BF}^2 - 2(\overline{BC})(\overline{BF}) \cos\left(\frac{3\pi}{5}\right) \\ &= 10^2 + 7^2 - 2(10)(7)\left(\frac{1-\sqrt{5}}{4}\right) = 114 + 35\sqrt{5} \end{aligned}$$

$$\overline{CF} = \sqrt{114 + 35\sqrt{5}} \tag{126}$$

$$\cos(\angle CFB) = \frac{\overline{CF}^2 + \overline{BF}^2 - \overline{BC}^2}{2(\overline{CF})(\overline{BF})} = \frac{114 + 35\sqrt{5} + 7^2 - 10^2}{2(\sqrt{114 + 35\sqrt{5}})(7)} = \frac{9 + 5\sqrt{5}}{2\sqrt{114 + 35\sqrt{5}}} \tag{127}$$

8.1.1 Rigid distance $\sqrt{114 + 35\sqrt{5}}$

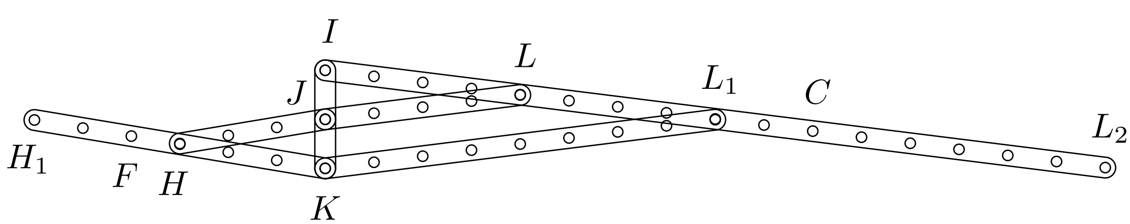

Figure 27: Construction of distance $\overline{CF} = \sqrt{114 + 35\sqrt{5}}$

Our software found several solutions for this distance using five strips, and we choose one narrow enough to fit inside the pentagon.

Figure 27 shows how to prove the cluster selected is correct. In the figure we have two isoscelles triangles $\triangle IKL_1$ and $\triangle JKH$. The sides IL_1 and KH are extended to double the original size to the vertices L_2 and H_1 building two right angles $\angle IKL_2$ and $\angle KJH_1$. The right triangles permit the calculation of the abscissas and ordinates of vertices C and F to calculate their distance.

From the figure we calculate $\overline{KL_2}$ and $\overline{JH_1}$ from their respective right triangles:

$$\overline{KL_2} = \sqrt{(\overline{IL_2})^2 - (IK)^2} = \sqrt{16^2 - 2^2} = 6\sqrt{7} \tag{128}$$

$$\overline{JH_1} = \sqrt{(\overline{KH_1})^2 - (KJ)^2} = \sqrt{6^2 - 1^2} = \sqrt{35} \tag{129}$$

Assuming vertice K is at the origin we can calculate the abscissas C_x, F_x and ordinates C_y, F_y of vertices C and F using as factors $c = \frac{\overline{IC}}{\overline{IL_2}} = \frac{10}{16} = \frac{5}{8}$ and $f = \frac{\overline{KF}}{\overline{KH1}} = \frac{4}{6} = \frac{2}{3}$:

$$C_x = +c(\overline{KL_2}) = \frac{5}{8}(6\sqrt{7}) = \frac{15}{4}\sqrt{7} \tag{130}$$

$$F_x = -f(\overline{JH_1}) = -\frac{2}{3}\sqrt{35} \tag{131}$$

$$C_y = +(\overline{KI}) - c(\overline{KI}) = 2 - \frac{5}{8}(2) = \frac{3}{4} \tag{132}$$

$$F_y = +f(\overline{KJ}) = \frac{2}{3}(1) = \frac{2}{3} \tag{133}$$

Finally we calculate the distance $\overline{CF}$:

$$\begin{aligned} \overline{CF} &= \sqrt{(C_x - F_x)^2 + (C_y - F_y)^2} \\ &= \sqrt{\left(\frac{15}{4}\sqrt{7} + \frac{2}{3}\sqrt{35}\right)^2 + \left(\frac{3}{4} - \frac{2}{3}\right)^2} = \sqrt{114 + 35\sqrt{5}} \quad \blacksquare \end{aligned} \tag{134}$$

A minimal part with five strips of the construction of figure 27 including only vertices F, H, I, J, K, L, C is used twice to make rigid the pentagon of side 10 as show in figure 26.

8.2 Size 10 with 10 internal strips $build(8:10)$

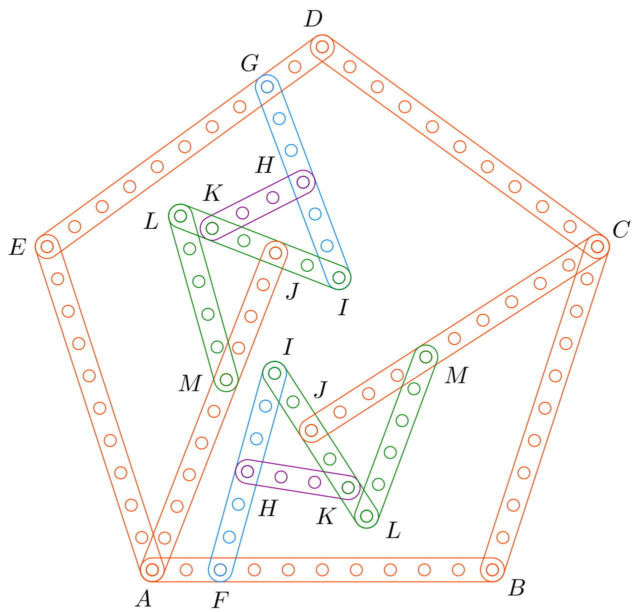

Figure 28: Pentagon size 10 with 10 internal strips. $\overline{FB}:\overline{BC} = 8:10$ and $\overline{CF} = 2\sqrt{31+10\sqrt{5}}$.

Figure 28 show a rigid regular pentagon A, B, C, D, E of size 10. We include here the relation $\overline{FB}:\overline{BC} = 8:10$ since the relation $4:5$ for pentagon size 5 gave clusters that can't fit inside such pentagon. With the law of cosines we calculate $\overline{CF}$ and the angle $\angle CFB$ for the drawing:

$$\overline{CF}^2 = \overline{BC}^2 + \overline{BF}^2 - 2(\overline{BC})(\overline{BF})\cos\left(\frac{3\pi}{5}\right)$$

$$= 10^2 + 8^2 - 2(10)(8)\left(\frac{1-\sqrt{5}}{4}\right) = 124 + 40\sqrt{5}$$

$$\overline{CF} = 2\sqrt{31+10\sqrt{5}} \tag{135}$$

$$\cos(\angle CFB) = \frac{\overline{CF}^2 + \overline{BF}^2 - \overline{BC}^2}{2(\overline{CF})(\overline{BF})} = \frac{124+40\sqrt{5}+8^2-10^2}{2(2\sqrt{31+10\sqrt{5}})(8)} = \frac{11+5\sqrt{5}}{4\sqrt{31+10\sqrt{5}}} \tag{136}$$

8.2.1 Rigid distance $2\sqrt{31+10\sqrt{5}}$

One solution from our software is shown in figure as cluster $FHIJKLM$. Assume vertex J is at the origin, vertice I at $(-2, 0)$ and vertice K at $(+2, 0)$. Since triangle $\triangle IKH$ is isoscelles and $\overline{IF}$ is the double of $\overline{IH}$ then angle $\angle FKI = \frac{\pi}{2}$ and we can calculate the abscissa and the ordinate of vertice F:

$$F_x = \overline{JK} = 2 \tag{137}$$

$$F_y = -\sqrt{(\overline{IF})^2 - (\overline{IK})^2} = -\sqrt{6^2 - 4^2} = -2\sqrt{5} \tag{138}$$

Since we have the Pythagorean triangle $\triangle JLM$ is easy to note that the abscissa and ordinate of vertice C are $C_x = 0$ and $C_y = \overline{JC} = 10$, so finally we have:

$$\overline{CF} = \sqrt{(F_x - C_x)^2 + (F_y - C_y)^2}$$

$$= \sqrt{(2-0)^2 + (-2\sqrt{5} - 10)^2} = 2\sqrt{31+10\sqrt{5}} \quad \blacksquare \tag{139}$$

9 Pentagon of size 11

9.1 Size 11 with 10 internal strips $build(8:11)$

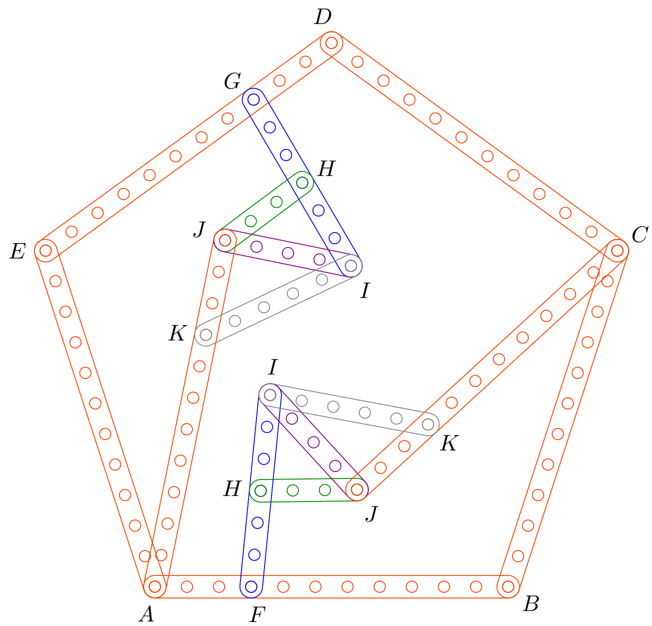

Figure 29: Pentagon size 11 with 10 internal strips. $\overline{FB} : \overline{BC} = 8 : 11$ and $\overline{CF} = 11 + 2\sqrt{5}$.

Figure 29 show a rigid regular pentagon A, B, C, D, E of size 11. Our software found this is the smallest pentagon having a consecutive sides diagonal distance of the form $\dfrac{z_2 + z_3\sqrt{5}}{z_1}$ instead of the nested form $\dfrac{z_2\sqrt{z_3 + z_4\sqrt{5}}}{z_1}$ where z_i are integers. The mentioned diagonal is the distance $\overline{CF}$ in the figure which can be calculated with the law of cosines knowing angle $\angle CBF = \dfrac{3\pi}{5}$ and denesting the result. We calculate the angle $\angle CFB$ for the drawing:

$$
\begin{aligned}
\overline{CF}^2 &= \overline{BC}^2 + \overline{BF}^2 - 2(\overline{BC})(\overline{BF})\cos\left(\frac{3\pi}{5}\right)\\
&= 11^2 + 8^2 - 2(11)(8)\left(\frac{1-\sqrt{5}}{4}\right) = 141 + 44\sqrt{5}\\
\overline{CF} &= \sqrt{141 + 44\sqrt{5}}\\
&= 11 + 2\sqrt{5} \qquad (140)
\end{aligned}
$$

$$
\cos(\angle CFB) = \frac{\overline{CF}^2 + \overline{BF}^2 - \overline{BC}^2}{2(\overline{CF})(\overline{BF})} = \frac{141 + 44\sqrt{5} + 8^2 - 11^2}{2(11 + 2\sqrt{5})(8)} = \frac{21 + 11\sqrt{5}}{44 + 8\sqrt{5}} = \frac{121 + 79\sqrt{5}}{404} \qquad (141)
$$

9.1.1 Rigid istance $11 + 2\sqrt{5}$

A five strips cluster can create a rigid distance like $11+2\sqrt{5}$. In the figure, three strips $\overline{FI} = 2\overline{HJ}, \overline{FI} > \overline{IJ}$ builds a right angle $\angle FJI = \frac{\pi}{2}$, since triangle $\triangle IJH$ is isosceles ($\overline{FH} = \overline{HI} = \overline{JH}$). These three strips also build a distance $\overline{FJ} = \sqrt{\overline{FI}^2 - \overline{IJ}^2} = \sqrt{6^2 - 4^2} = 2\sqrt{5}$. Now we attach strip $\overline{CJ}$ making a second right triangle $\angle CJI = \frac{\pi}{2}$ using strip $\overline{IK} = 5$ as pythagorean diagonal ($\overline{JK} = 3, \overline{IJ} = 4$). We have two right triangles at vertice J so vertices F, J, C are collinear, so we can calculate the distance $\overline{FC} = \overline{CJ} + \overline{JF} = 11 + 2\sqrt{5}$ $\blacksquare$.

We repeat the five-strips cluster between vertices A, G preventing strips overlaps. Since the clusters are rigid we formed two rigid triangles $\triangle ABC, \triangle DEA$ so the pentagon is rigid.

The software found the next pentagon of this type is a lot bigger: $\overline{BC} = 246, \overline{BF} = 70, \overline{CF} = 41 + 105\sqrt{5}$.

10 Pentagons of size 12

10.1 Size 12 with 4 internal strips *build*(0 : 12 : 3 : 4)

Our sofware found that side 12 is the smallest pentagon that can be made rigid with a rhoumbus and two strips as diagonals so need only 4 strips as diagonals.

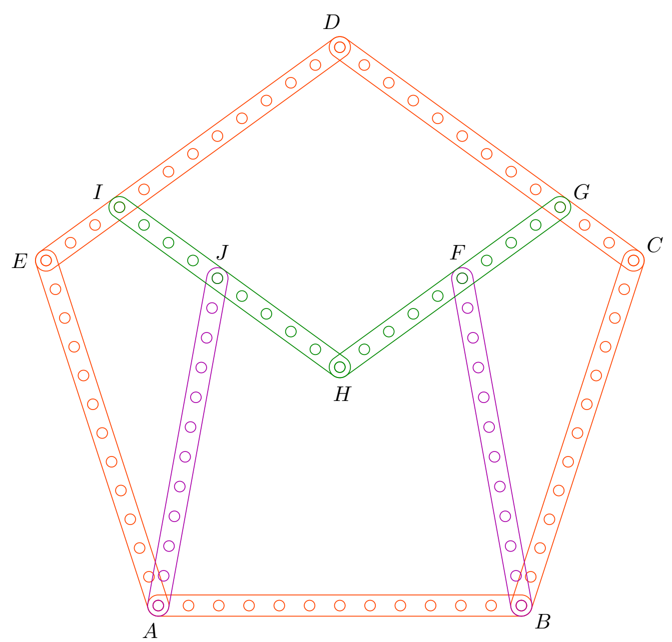

Figure 30: Pentagon size 12 with 4 internal strips. $\overline{AA} : \overline{AE} : \overline{EI} : \overline{IJ} = 0 : 12 : 3 : 4$ and $\overline{AJ} = 11$.

Figure 30 show a regular pentagon A, B, C, D, E of side 12 with a rhombus D, I, H, G of side 9. We prove strips AJ, BF are correct. First we calculate the abscissas going through vertices A, E, I, J substracting when we move to the left and adding when we move to the right:

$$\begin{aligned} AJ_x &= AE_x + EI_x + IJ_x \\ &= -\overline{AE}\cos\left(\frac{2\pi}{5}\right) + \overline{EI}\cos\left(\frac{\pi}{5}\right) + \overline{IJ}\cos\left(\frac{\pi}{5}\right) \\ &= -12\left(\frac{\sqrt{5}-1}{4}\right) + 3\left(\frac{1+\sqrt{5}}{4}\right) + 4\left(\frac{1+\sqrt{5}}{4}\right) = \frac{19-5\sqrt{5}}{4} \end{aligned} \tag{142}$$

Then we calculate the ordinates going to the same order of vertices adding when we go up and substracting when we go down:

$$
\begin{aligned}
AJ_y &= -AE_y + EI_y + IJ_y \\
&= \overline{AE}\sin\left(\frac{2\pi}{5}\right) + \overline{EI}\sin\left(\frac{\pi}{5}\right) - \overline{IJ}\sin\left(\frac{\pi}{5}\right) \\
&= 12\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right) + 3\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right) - 4\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right) = \frac{\sqrt{1450+190\sqrt{5}}}{4} \qquad (143)
\end{aligned}
$$

Finally we calculate the distance $\overline{AJ}$ which coincides with strip size 11:

$$
\begin{aligned}
\overline{AJ} &= \sqrt{(AJ_x)^2 + (AJ_y)^2} \\
&= \sqrt{\left(\frac{19-5\sqrt{5}}{4}\right)^2 + \frac{1450+190\sqrt{5}}{16}} = \sqrt{121} = 11 \quad \blacksquare \qquad (144)
\end{aligned}
$$

10.2 Size 12 with 4 internal strips *build*$(4:12:0:3)$

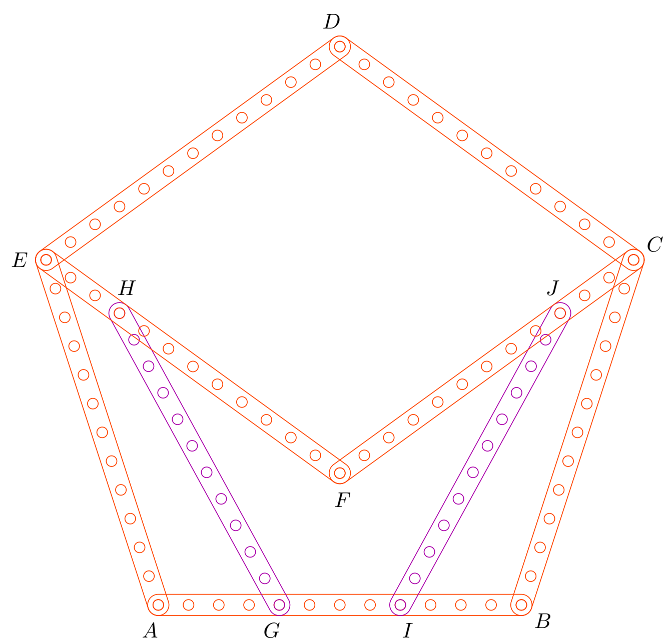

Figure 31: Pentagon size 12 with 4 internal strips. $\overline{GA}:\overline{AE}:\overline{EE}:\overline{EH} = 4:12:0:3$ and $\overline{GH} = 11$.

Figure 31 show a regular pentagon A, B, C, D, E of size 12 with a rhombus D, I, H, G of size 12. We prove strips GH, IJ are correct. First we calculate the abscissas going through vertices G, A, E, H substracting when we move to the left and adding when we move to the right:

$$
\begin{aligned}
GH_x &= -GA_x - AE_x + EH_x \\
&= -\overline{GA} - \overline{AE}\cos\left(\frac{2\pi}{5}\right) + \overline{EH}\cos\left(\frac{\pi}{5}\right) \\
&= -4 - 12\left(\frac{\sqrt{5}-1}{4}\right) + 3\left(\frac{1+\sqrt{5}}{4}\right) = \frac{-1-9\sqrt{5}}{4} \qquad (145)
\end{aligned}
$$

Then we calculate the ordinates going to the same order of vertices adding when we go up and substracting when we go down:

$$\begin{aligned} GH_y &= AG_y + AE_y - EH_y \\ &= 0 + \overline{AE}\sin\left(\frac{2\pi}{5}\right) - \overline{EH}\sin\left(\frac{\pi}{5}\right) \\ &= 12\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right) - 3\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right) = \frac{\sqrt{1530-18\sqrt{5}}}{4} \end{aligned} \tag{146}$$

Finally we calculate the distance $\overline{GH}$ which coincides with strip size 11:

$$\begin{aligned} \overline{GH} &= \sqrt{(GH_x)^2 + (GH_y)^2} \\ &= \sqrt{\left(\frac{-1-9\sqrt{5}}{4}\right)^2 + \frac{1530-18\sqrt{5}}{16}} = \sqrt{121} = 11 \quad \blacksquare \end{aligned} \tag{147}$$

10.3 Size 12 with 6 internal strips *build*(12 : 12)

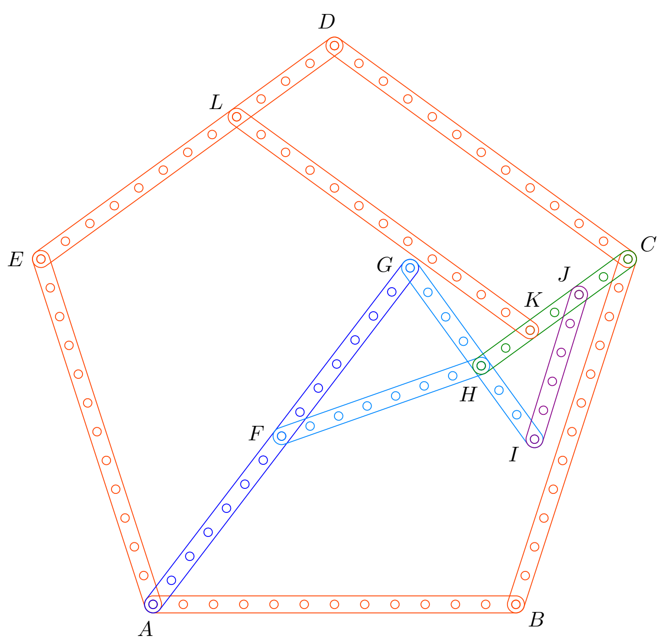

Figure 32: Pentagon size 12 with 6 internal strips. $\overline{AB} : \overline{BC} = 12 : 12$ and $\overline{AC} = 6 + 6\sqrt{5}$

.

Figure 32 show a regular pentagon A, B, C, D, E of side 12. We know the regular pentagon diagonal for size 12 is $\overline{AC} = 12\left(\frac{1+\sqrt{5}}{2}\right) = 6 + 6\sqrt{5}$.

10.3.1 Rigid distance $6 + 6\sqrt{5}$

We show the five strips $\overline{GH}, \overline{GI}, \overline{HF}, \overline{HC}, \overline{IJ}$ make the diagonal rigid which makes rigid the angle $\angle ABC$ of the pentagon. We have an isoscelles triangle $\triangle FGH$ and $\overline{AG}$ is two times $\overline{FG}$ so we have a right angle $\angle AHG = \frac{\pi}{2}$ and we can calculate $\overline{AH} = \sqrt{(\overline{AG})^2 - (\overline{GH})^2} = \sqrt{14^2 - 4^2} = 6\sqrt{5}$. Now we have another right angle $\angle IHC = \frac{\pi}{2}$ because the Pythagoras triangle $\triangle HIJ$. Since G, H, I are collinear then we have another right angle $\angle GHC = \frac{\pi}{2}$. Both right angles $\angle AHG, \angle CHG$ guaranty vertices A, H, C are collinear and we can calculate $\overline{AC} = \overline{AH} + \overline{HC} = 6 + 6\sqrt{5}$ ■.

Finally we add a sixth strip $\overline{KL}$ parallel to $\overline{CD}$ to make rigid the last three perimeter strips $\overline{CD}, \overline{DE}, \overline{EA}$ of the pentagon.

10.4 Size 12 with 8 internal strips *build*(6 : 12 : 3 : 6)

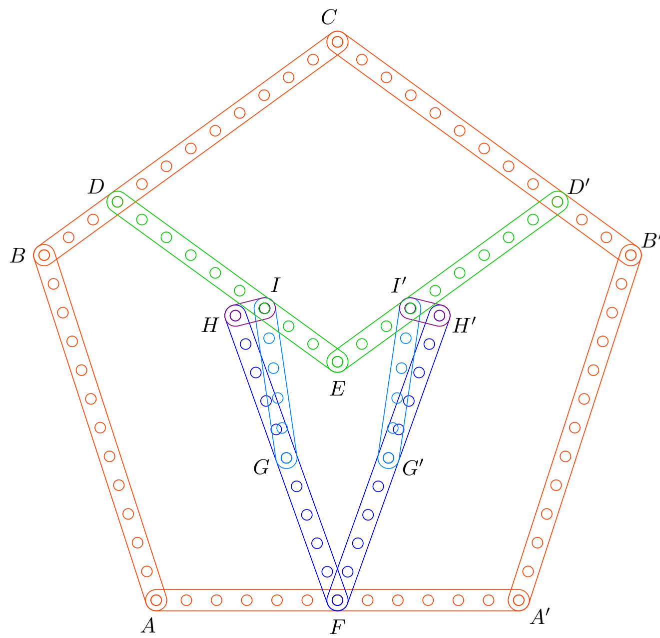

Figure 33: Pentagon size 12 with 8 internal strips. $\overline{FA} : \overline{AB} : \overline{BD} : \overline{DI} = 6 : 12 : 3 : 6$ and $\overline{FI} = 3\sqrt{11}$
.

Figure 33 show a regular pentagon $AA'B'CBA$ of side 12. First we calculate the distance $\overline{FI}$ using the abscissas and ordinates following the vertices F, A, B, D, I for a regular pentagon angles $\alpha = \dfrac{3\pi}{5}, \beta = \dfrac{\pi}{5}$:

$$\begin{aligned} FI_x &= -\overline{AF} - \overline{AB}\cos\alpha + (\overline{BD} + \overline{DI})\cos\beta \\ &= -6 + (12)\frac{1-\sqrt{5}}{4} + (3+6)\frac{\sqrt{5}+1}{4} = -\frac{3+3\sqrt{5}}{4} \qquad (148) \\ FI_y &= \overline{AB}\sin\alpha + (\overline{BD} - \overline{DI})\sin\beta \\ &= (12)\frac{\sqrt{10+2\sqrt{5}}}{4} + (3-6)\frac{\sqrt{10-2\sqrt{5}}}{4} = \frac{12\sqrt{10+2\sqrt{5}} - 3\sqrt{10-2\sqrt{5}}}{4} \qquad (149) \\ \overline{FI} &= \sqrt{(FI_x)^2 + (FI_y)^2} \\ &= \frac{\sqrt{(-3-3\sqrt{5})^2 + (12\sqrt{10+2\sqrt{5}} - 3\sqrt{10-2\sqrt{5}})^2}}{4} = \frac{\sqrt{1584}}{4} = 3\sqrt{11} \qquad (150) \end{aligned}$$

10.4.1 Rigid distance $3\sqrt{11}$

Finally we calculate the distance $\overline{FI}$ made rigid by cluster F, G, H, I. We have an isosceles triangle $\triangle GHI$ and $\overline{FH} = 2\overline{GH}$ so we have a right triangle $\angle FIH = \frac{\pi}{2}$ so:

$$\begin{aligned}\overline{FI} &= \sqrt{(\overline{FH})^2 - (\overline{HI})^2} \\ &= \sqrt{10^2 - 1^2} = 3\sqrt{11}\end{aligned} \tag{151}$$

10.5 Size 12 with 10 internal strips $build(3:12:12)$

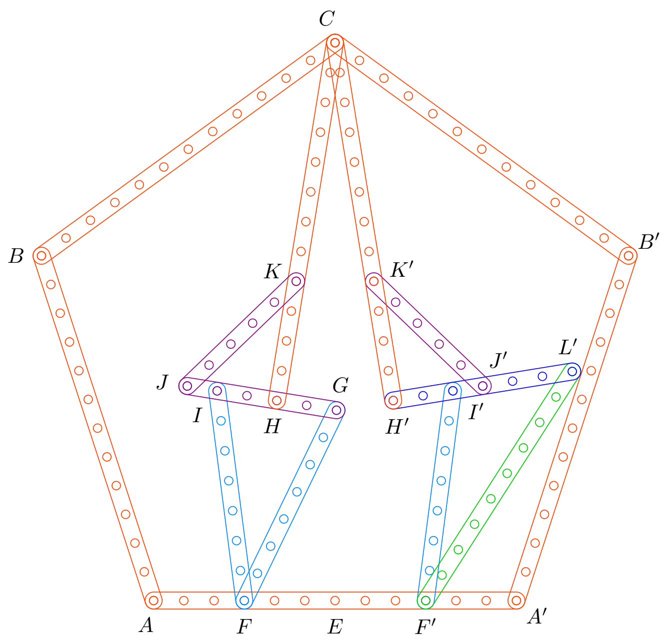

Figure 34: Pentagon size 12 with 10 internal strips. $\overline{FA} : \overline{AB} : \overline{BC} = 3 : 12 : 12$ and $\overline{CF} = 12 + 3\sqrt{5}$.

Figure 34 show a regular pentagon $AA'B'CB$ of side 12. We know the regular pentagon height is $\dfrac{\sqrt{5+2\sqrt{5}}}{2}$ times the side. So here we have $\overline{CE} = 12\frac{\sqrt{5+2\sqrt{5}}}{2} = 6\sqrt{5+2\sqrt{5}}$ and we can calculate $\overline{CF}$:

$$\begin{aligned}\overline{CF} &= \sqrt{\overline{CE}^2 + \overline{EF}^2} \\ &= \sqrt{36(5+2\sqrt{5}) + 3^2} = 3\sqrt{21 + 8\sqrt{5}} \\ &= 3(4 + \sqrt{5})\end{aligned} \tag{152}$$

10.5.1 Rigid distance $3(4 + \sqrt{5})$

After testing $\overline{AA'} \leq 1800$ our software found that the last denesting is somehow special since other fractions $\dfrac{\overline{AF}}{\overline{AA'}} \neq \dfrac{1}{4}$ generated $\overline{CF}$s that can't be denested.

We have the Pythagorean triangle $\triangle HJK$ and the isosceles $\triangle FGI$ so vertices FHC are collinear. First we calculate $\overline{FH} = \sqrt{\overline{FG}^2 - \overline{GH}^2} = \sqrt{7^2 - 2^2} = 3\sqrt{5}$ and then $\overline{FC} = \overline{FH} + \overline{HC} = 3\sqrt{5} + 12$ matching last calculation. Finally we prove angle $\angle F'H'L' = \frac{\pi}{2}$ noting $\overline{F'H'} = \sqrt{(\overline{F'L'})^2 - (\overline{H'L'})^2} = \sqrt{9^2 - 6^2} = 3\sqrt{5}$ matching $\overline{FH}$ ■.

10.6 Size 12 with 10 internal strips $build(8:12)$

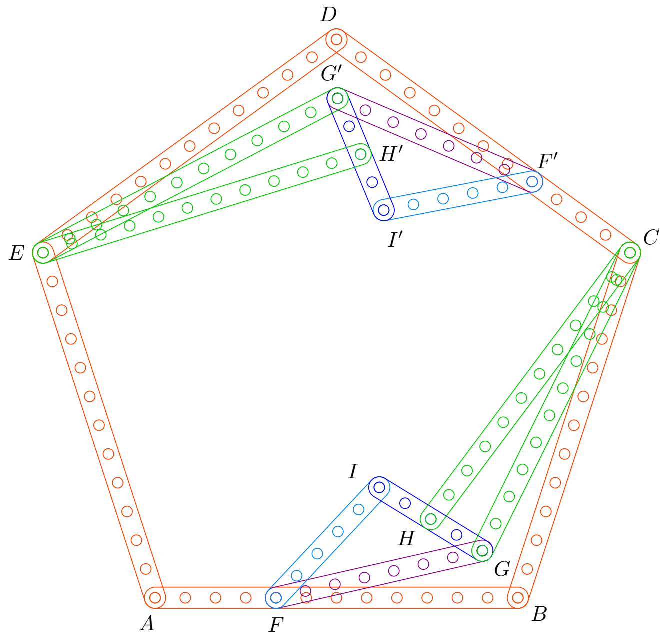

Figure 35: Pentagon size 12 with 10 internal strips. $\overline{FB}:\overline{BC}=8:12$ and $\overline{CF}=4\sqrt{10+3\sqrt{5}}$.

Figure 35 show a pentagon $ABCDE$ of size 12. We calculate the distance $\overline{CF}$ with the law of cosines knowing angle $\theta=\angle CBF=3\pi/5$ and we calculate the angle $\phi=\angle CFB$:

$$\begin{aligned}\overline{CF} &= \sqrt{\overline{BC}^2+\overline{BF}^2-2(\overline{BC})(\overline{BF})\cos\theta)} \\ &= \sqrt{12^2+8^2-2(12)(8)\left(\frac{1-\sqrt{5}}{4}\right)}=4\sqrt{10+3\sqrt{5}} \end{aligned}\tag{153}$$

$$\cos\phi=\frac{\overline{CF}^2+\overline{BF}^2-\overline{BC}^2}{2(\overline{CF})(\overline{BF})}=\frac{16(10+3\sqrt{5})+8^2-12^2}{2(4\sqrt{10+3\sqrt{5}})(8)}=\frac{5+3\sqrt{5}}{4\sqrt{10+3\sqrt{5}}}\tag{154}$$

10.6.1 Rigid distance $4\sqrt{10+3\sqrt{5}}$

Our five-strips software found few options of the form $z_i\sqrt{10+3\sqrt{5}}$. We validate the cluster $FIHGC$ of figure 35. With the law of cosines we calculate $\alpha=\angle FGI$ inside scalene triangle $\triangle FGI$ and angle $\beta=\angle HGC$ inside isoscelles triangle $\triangle HGC$:

$$\cos\alpha=\frac{(\overline{FG})^2+(\overline{GI})^2-(\overline{FI})^2}{2(\overline{FG})(\overline{GI})}=\frac{7^2+4^2-5^2}{2(7)(4)}=\frac{5}{7}\tag{155}$$

$$\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-\left(\frac{5}{7}\right)^2}=\frac{2\sqrt{6}}{7}\tag{156}$$

$$\cos\beta=\frac{\overline{HG}/2}{\overline{GC}}=\frac{1}{11}\tag{157}$$

$$\sin\beta=\sqrt{1-\cos^2\beta}=\sqrt{1-\left(\frac{1}{11}\right)^2}=\frac{2\sqrt{30}}{11}\tag{158}$$

Finally we calculate angle $\gamma = \alpha + \beta$ using the cosines sum identity and use it to calculate distance $\overline{FC}$ with the law of cosines:

$$
\begin{aligned}
\cos\gamma &= \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
&= \left(\frac{5}{7}\right)\left(\frac{1}{11}\right) - \left(\frac{2\sqrt{6}}{7}\right)\left(\frac{2\sqrt{30}}{11}\right) = \frac{5-24\sqrt{5}}{77} && (159) \\
\overline{FC} &= \sqrt{(\overline{FG})^2 + (\overline{GC})^2 - 2(\overline{FG})(\overline{GC})\cos\gamma} \\
&= \sqrt{7^2 + 11^2 - 2(7)(11)\left(\frac{5-24\sqrt{5}}{77}\right)} = 4\sqrt{10+3\sqrt{5}} \quad \blacksquare && (160)
\end{aligned}
$$