

Meccano frames

<https://github.com/heptagons/meccano/frames>

Abstract

Meccano frames are groups of meccano ¹ strips intended to be a base to build diverse meccano larger objects.

1 Triangular frame

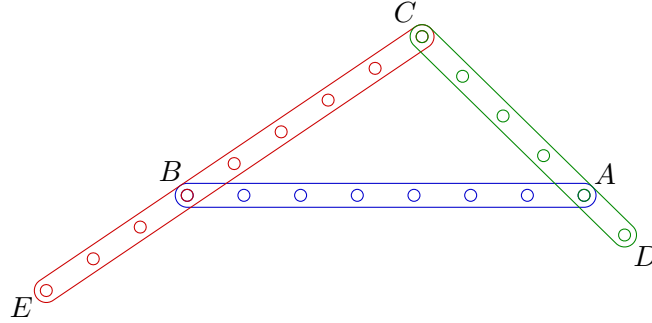


Figure 1: Triangular frame. With three strips we form the triangle $\triangle ABC$. At least we extend one of two strips \overline{CB} and \overline{CA} to become \overline{CE} and \overline{CD} . The new vertices D and E are rigid as the triangle and we'll calculate the distance between them.

Figure 1 shows a triangular frame with strips with extentions. First we define five integer distances a, b, c, d, e and calculate the cosine of $\angle BCA$:

$$a \equiv \overline{CB}, \quad b \equiv \overline{CA} \quad c \equiv \overline{AB} \quad d \equiv \overline{BE}, \quad e \equiv \overline{AD} \quad (1)$$

$$\theta \equiv \angle BCA \quad (2)$$

$$\cos \theta = \frac{a^2 + b^2 - c^2}{2ab} \quad (3)$$

Then we apply the cosine to the triangle $\triangle CED$ to get the extensions distance \overline{DE} :

$$\begin{aligned} \overline{DE}^2 &= \overline{CD}^2 + \overline{CE}^2 - 2\overline{CD} \times \overline{CE} \cos \theta \\ &= (a + d)^2 + (b + e)^2 - 2(a + d)(b + e) \cos \theta \\ &= (a + d)^2 + (b + e)^2 - (a + d)(b + e) \left(\frac{a^2 + b^2 - c^2}{ab} \right) \end{aligned} \quad (4)$$

We expect at most a value of the form \sqrt{s}/t where $s, t \in \mathbb{Z}$ so we define the surd as:

$$\overline{DE} = \frac{\sqrt{s}}{t} = \sqrt{(a + d)^2 + (b + e)^2 - (a + d)(b + e) \left(\frac{a^2 + b^2 - c^2}{ab} \right)} \quad (5)$$

¹ Meccano mathematics by 't Hooft

1.1 Software for distance \sqrt{s}

We write a factory to build all the triangles with a given surd \sqrt{s} for a given maximum limit for the distances $a + d, b + e, c$. We reject $t \neq 1$ and s as not square-free, which includes pythagorean triangles. Next list show all the triangles with $s = \sqrt{7}, t = 1$ where $a + d, b + e, c \leq 10$:

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1 NewFrames().TriangleSurd surd=7 max=10
2 1) a=1 b+e=1+2 c=1 cos=1/2
3 2) a+d=1+1 b+e=1+2 c=1 cos=1/2
4 3) a+d=1+2 b=1 c=1 cos=1/2
5 4) a+d=1+2 b+e=1+1 c=1 cos=1/2
6 5) a=2 b+e=2+1 c=2 cos=1/2
7 6) a+d=2+1 b=2 c=2 cos=1/2
8 7) a=3 b+e=2+2 c=2 cos=3/4 CED=pi/2
9 8) a+d=3+1 b+e=2+1 c=2 cos=3/4 CDE=pi/2
10 9) a+d=4+2 b+e=4+4 c=1 cos=31/32
11 10) a+d=4+4 b+e=4+2 c=1 cos=31/32
12 11) a=7 b+e=5+1 c=3 cos=13/14
13 12) a=7 b+e=5+2 c=3 cos=13/14

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The code is in file github.com/heptagons/meccano/frames/frames.go.

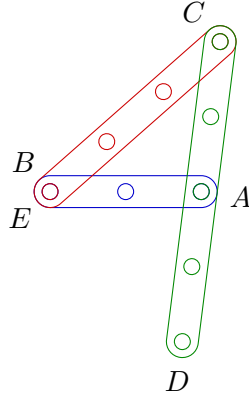


Figure 2: Triangle with $\overline{DE} = \sqrt{7}$ where $a = 3, b + e = 2 + 2, c = 2, \cos = \frac{3}{4}, CED = \pi/2$.

2 Distance $\sqrt{s} + h$

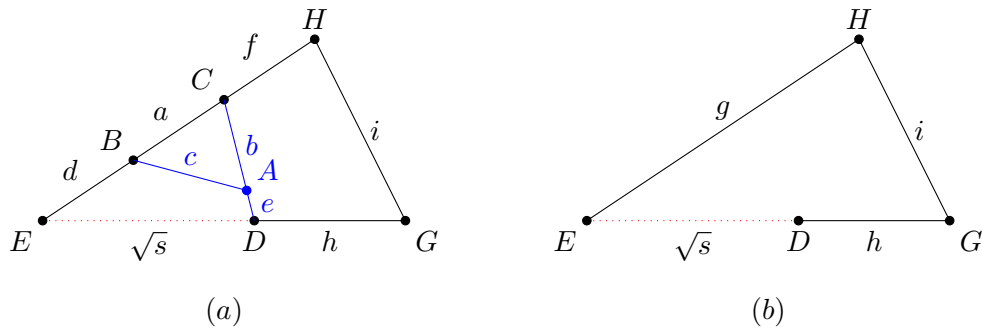


Figure 3: The five strips intended to form an algebraic distance $\overline{EG} = \sqrt{s} + h$.

From figure 3 (a) we know \sqrt{s} distance between nodes E and D is produced by the three strips frame $a + d, b + e$ and c . Using the law of cosines we calculate the angle $\theta = \angle CED$ in terms of \sqrt{s} :

$$\begin{aligned}\cos \theta &= \frac{(a+d)^2 + (\sqrt{s})^2 - (b+e)^2}{2(a+d)\sqrt{s}} \\ &= \frac{((a+d)^2 + s - (b+e)^2)\sqrt{s}}{2(a+d)s}\end{aligned}\tag{6}$$

$$= \frac{m\sqrt{s}}{n}\tag{7}$$

$$m = (a+d)^2 + s - (b+e)^2\tag{8}$$

$$n = 2(a+d)s\tag{9}$$

From figure 3 (a) we notice two sets of points are collinear: $\{E, B, C, H\}$ and $\{E, D, G\}$. Using the law of cosines we calculate the angle $\theta = \angle HEG$ in terms of distances g, h, i :

$$\begin{aligned}\cos \theta &= \frac{g^2 + (\sqrt{s} + h)^2 - i^2}{2g(\sqrt{s} + h)} \\ &= \frac{g^2 + s + 2\sqrt{s}h + h^2 - i^2}{2g(\sqrt{s} + h)} \\ &= \frac{g^2 + s + h^2 - i^2 + 2\sqrt{s}h}{2g(\sqrt{s} + h)}\end{aligned}\tag{10}$$

We multiply both numerator and denominator by $\sqrt{s} - h$ to eliminate the surd from denominator:

$$\begin{aligned}\cos \theta &= \frac{(s + g^2 + h^2 - i^2)(\sqrt{s} - h) + 2\sqrt{s}h(\sqrt{s} - h)}{2g(\sqrt{s} + h)(\sqrt{s} - h)} \\ &= \frac{(s + g^2 + h^2 - i^2)(\sqrt{s} - h) + 2sh - 2\sqrt{s}h^2}{2g(s - h^2)} \\ &= \frac{-h(s + g^2 + h^2 - i^2 - 2s) + (s + g^2 + h^2 - i^2 - 2h^2)\sqrt{s}}{2g(s - h^2)} \\ &= \frac{h(s - g^2 - h^2 + i^2) + (s + g^2 - h^2 - i^2)\sqrt{s}}{2g(s - h^2)} \\ &= \frac{o + p\sqrt{s}}{q}\end{aligned}\tag{11}$$

$$o = h(s - g^2 - h^2 + i^2)\tag{12}$$

$$p = s + g^2 - h^2 - i^2\tag{13}$$

$$q = 2g(s - h^2)\tag{14}$$

We compare both cosines equations 7 and 11:

$$\frac{m\sqrt{s}}{n} = \frac{o + p\sqrt{s}}{q}\tag{15}$$

Since all variables are integers we need two conditions. First o should be zero. And second $\frac{m}{n} = \frac{p}{q}$.

For condition 1, we force o to be zero:

$$\begin{aligned}o &= 0 \\ h(s - g^2 - h^2 + i^2) &= 0 \\ s &= g^2 + h^2 - i^2\end{aligned}\tag{16}$$

For condition2, we force m, n, p, q as:

$$\begin{aligned} \frac{m}{n} &= \frac{p}{q} \\ \frac{(a+d)^2 + s - (b+e)^2}{2(a+d)s} &= \frac{s + g^2 - h^2 - i^2}{2g(s - h^2)} \end{aligned} \tag{17}$$

We replace the value of s of last equation RHS with the value of equation 16 of condition 1:

$$\begin{aligned} \frac{(a+d)^2 - (b+e)^2 + s}{(a+d)s} &= \frac{s + g^2 - h^2 - i^2}{g(s - h^2)} \\ &= \frac{g^2 + h^2 - i^2 + g^2 - h^2 - i^2}{g(g^2 + h^2 - i^2 - h^2)} \\ &= \frac{2(g^2 - i^2)}{g(g^2 - i^2)} \\ &= \frac{2}{g} \\ ((a+d)^2 - (b+e)^2 + s)g &= 2(a+d)s \end{aligned} \tag{18}$$