

Meccano pentagons gallery

<https://github.com/heptagons/meccano/penta/gallery>

2023/12/11

Abstract

We build rigid meccano ¹ regular pentagons from sides 3 to 12. We restrict all internal strips to remain inside the pentagon's perimeter and don't permit they overlap with others. We follow three steps. 1) We calculate distances between selected strips holes from the regular pentagon perimeter assuming is regular. 2) We run some programs available in this repo to look for rigid clusters of strips which contains the distance. 3) We simplify or reduce the cluster to fit inside the pentagon. We prove the correctness of the cluster distance applied to check the software. We try each construction is relevant for the pentagon size.

1 Pentagons of size 3

1.1 Size 3 with 10 internal strips

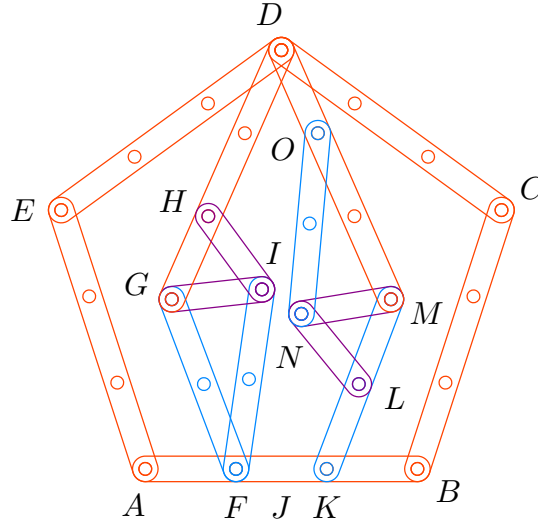


Figure 1: Pentagon of size 3 with 10 internal strips. $\overline{DE} : \overline{EA} : \overline{AF} = 3 : 3 : 1$. $\overline{DF} = \overline{DK} = \frac{\sqrt{46 + 18\sqrt{5}}}{2}$.

¹ Meccano mathematics by 't Hooft

Figure 1 show the regular pentagon A, B, C, D, E of size 3. We know the regular pentagon height is side length times $\frac{\sqrt{5+2\sqrt{5}}}{2}$ so in this case $\overline{DJ} = \frac{3\sqrt{5+2\sqrt{5}}}{2}$ and we can calculate \overline{DF} :

$$\begin{aligned}\overline{DF} &= \sqrt{(\overline{DJ})^2 + (\overline{FJ})^2} \\ &= \sqrt{\left(\frac{3\sqrt{5+2\sqrt{5}}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{46+18\sqrt{5}}}{2}\end{aligned}\tag{1}$$

1.1.1 Rigid distance $\frac{\sqrt{46+18\sqrt{5}}}{2}$

Our software found several five strips clusters and we use two different which fit inside the pentagon.

We identify two angles $\alpha = \angle HGI = \angle LMN$ of equilateral triangles and $\beta = \angle FGI = \angle NMO$ of isoscelles'. Adding the angles we get angles $\angle DGF = \angle DMK = (\alpha + \beta)$. From equilateral triangle $\triangle HGI$ we calculate $\cos \alpha$ and $\sin \alpha$:

$$\cos \alpha = \frac{\frac{\overline{GI}}{2}}{\overline{GH}} = \frac{\frac{1}{2}}{1} = \frac{1}{2}\tag{2}$$

$$\sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}\tag{3}$$

From isoscelles triangle $\triangle FGI$ we calculate $\cos \beta$ and $\sin \beta$:

$$\cos \beta = \frac{\frac{\overline{GI}}{2}}{\overline{GF}} = \frac{\frac{1}{2}}{2} = \frac{1}{4}\tag{4}$$

$$\sin \beta = \sqrt{1 - \cos^2 \beta} = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}\tag{5}$$

Now we calculate $\cos(\alpha + \beta)$ with the sum identity:

$$\begin{aligned}\cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ &= \left(\frac{1}{2}\right) \left(\frac{1}{4}\right) - \left(\frac{\sqrt{3}}{2}\right) \left(\frac{\sqrt{15}}{4}\right) = \frac{1 - 3\sqrt{5}}{8}\end{aligned}\tag{6}$$

Finally using the law of cosines we verify distance \overline{DF} :

$$\begin{aligned}\overline{DF} &= \sqrt{(\overline{DG})^2 + (\overline{FG})^2 - 2(\overline{DG})(\overline{FG})\cos(\alpha + \beta)} \\ &= \sqrt{3^2 + 2^2 - 2(3)(2) \left(\frac{1 - 3\sqrt{5}}{8}\right)} = \frac{\sqrt{46+18\sqrt{5}}}{2} \blacksquare\end{aligned}\tag{7}$$

1.2 Size 3 with 14 internal strips

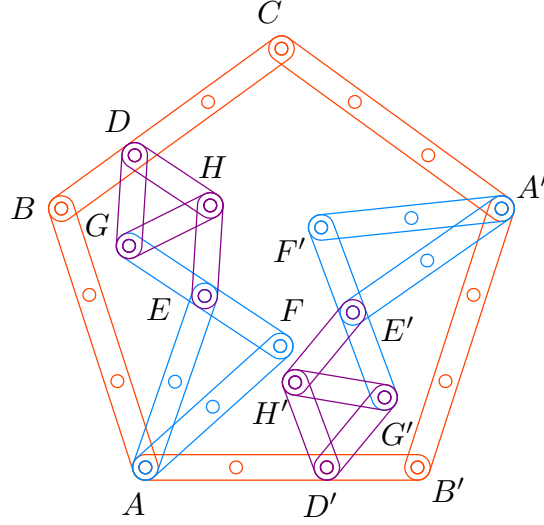


Figure 2: Pentagon of size 3 with 14 internal strips. $\overline{AB} : \overline{BD} = 3 : 1$. $\overline{AD} = \frac{\sqrt{34 + 5\sqrt{5}}}{2}$.

Figure 2 show the regular pentagon A, B', A', C, B of size 3. We know the internal angle of pentagon is $\theta = \frac{3\pi}{5}$ and $\cos \theta = \frac{1 - \sqrt{5}}{4}$ so with the law of cosines we can calculate \overline{AD} and the angle $\angle BAD$:

$$\begin{aligned} \overline{AD} &= \sqrt{(\overline{AB})^2 + (\overline{BD})^2 - 2(\overline{AB})(\overline{BD}) \cos \theta} \\ &= \sqrt{3^2 + 1^2 - 2(3)(1) \left(\frac{1 - \sqrt{5}}{4} \right)} = \frac{\sqrt{34 + 6\sqrt{5}}}{2} \end{aligned} \quad (8)$$

$$\begin{aligned} \cos(\angle BAD) &= \frac{(\overline{AD})^2 + (\overline{AB})^2 - (\overline{BD})^2}{2(\overline{AD})(\overline{AB})} \\ &= \frac{\frac{34 + 6\sqrt{5}}{4} + 3^2 - 1^2}{2 \left(\frac{\sqrt{34 + 6\sqrt{5}}}{2} \right) (3)} = \frac{11 + \sqrt{5}}{2\sqrt{34 + 6\sqrt{5}}} \end{aligned} \quad (9)$$

1.2.1 Rigid distance $\frac{\sqrt{34 + 6\sqrt{5}}}{2}$

Our software found several options to make the distance with five strips but we have to manually add a sixth strip in order to make a cluster narrow enough to fit two times inside the pentagon of size 3. The result show as the duplicated cluster with vertices D, E, F, G, H of the figure. We are going to prove the cluster's distance \overline{AD} matches the distance already calculated.

Noting the pair of adjacent equilateral triangles $\triangle DGH$ and $\triangle EGH$ we know angle $\angle DGE = \frac{2\pi}{3}$ and

also $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$ so we can calculate \overline{DE} and the angle $\alpha \equiv \angle GDE$:

$$\overline{DE} = \sqrt{(\overline{DG})^2 + (\overline{GE})^2 - 2(\overline{DG})(\overline{GE})\cos(\angle DGE)} = \sqrt{1^2 + 1^2 - 2(1)(1)\left(-\frac{1}{2}\right)} = \sqrt{3} \quad (10)$$

$$\begin{aligned} \alpha &\equiv \angle GED \\ \cos \alpha &= \frac{(\overline{GE})^2 + (\overline{DE})^2 - (\overline{DG})^2}{2(\overline{GE})(\overline{DE})} = \frac{3 + 1^2 - 1^2}{2(\sqrt{3})(1)} = \frac{\sqrt{3}}{2} \end{aligned} \quad (11)$$

$$\sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{2} \quad (12)$$

From the isoscelles triangle $\triangle AEF$ we can calculate angle $\angle AEF$ noting $\cos(\angle AEF) = \frac{\overline{EF}/2}{\overline{AE}} = \frac{1/2}{2} = \frac{1}{4}$ so we define angle $\beta \equiv \angle AEG$ the supplementary of $\angle AEF$ and we get:

$$\begin{aligned} \beta &\equiv \angle AEG = \pi - \angle AEF \\ \cos \beta &= -\cos \angle AEF = -\frac{1}{4} \end{aligned} \quad (13)$$

$$\sin \beta = \sqrt{1 - \cos^2 \beta} = \sqrt{1 - \left(-\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4} \quad (14)$$

Now we can calculate the angle $\angle AED = \alpha + \beta$ with the sum identity and plugin the last sines and cosines:

$$\begin{aligned} \cos(\angle AED) &= \cos(\alpha + \beta) \\ &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ &= \left(\frac{\sqrt{3}}{2}\right) \left(-\frac{1}{4}\right) - \left(\frac{1}{2}\right) \left(\frac{\sqrt{15}}{4}\right) = -\frac{\sqrt{3} + \sqrt{15}}{8} \end{aligned} \quad (15)$$

Finally we calculate \overline{AD} with the law of cosines:

$$\begin{aligned} \overline{AD} &= \sqrt{(\overline{DE})^2 + (\overline{EA})^2 - 2(\overline{DE})(\overline{EA})\cos(\angle AED)} \\ &= \sqrt{3 + 2^2 - 2(\sqrt{3})(2)\left(-\frac{\sqrt{3} + \sqrt{15}}{8}\right)} = \frac{\sqrt{34 + 6\sqrt{5}}}{2} \quad \blacksquare \end{aligned} \quad (16)$$

2 Pentagons of size 4

2.1 Size 4 with 8 internal strips

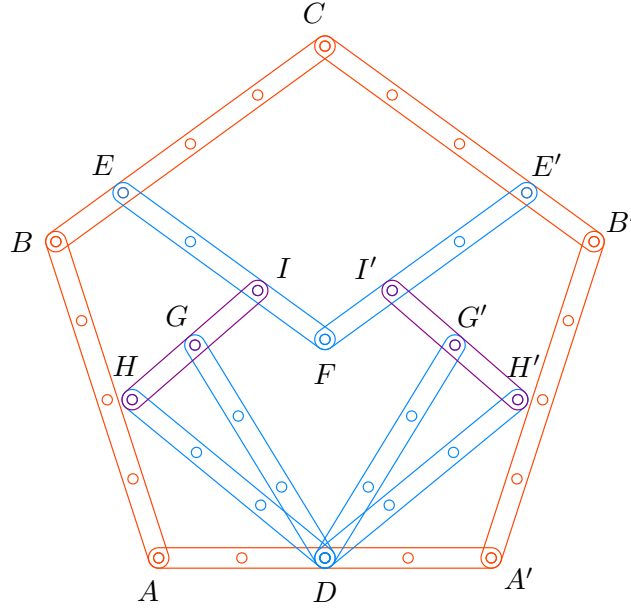


Figure 3: Pentagon of size 4 with 6 internal strips. $\overline{DA} : \overline{AB} : \overline{BE} : \overline{EI} = 2 : 4 : 1 : 2$. $\overline{DI} = \sqrt{11}$.

Figure 3 show the regular pentagon $AA'B'CB$ of size 4. We calculate the distance \overline{DI} assuming vertex D is at origin and calculating abscissa and ordinate of vertex I and knowing $\alpha = \angle ADB = \frac{3\pi}{5}$ and $\beta = \angle B'BE = \frac{\pi}{5}$. Adding and subtracting through vertices D, A, B, E, I we get:

$$\begin{aligned} I_x &= -\overline{DA} - \overline{AB} \cos \alpha + \overline{BE} \cos \beta + \overline{EI} \cos \beta \\ &= -2 - (4) \left| -\frac{\sqrt{5}-1}{4} \right| + (1+2) \left(\frac{\sqrt{5}+1}{4} \right) = -\frac{1+\sqrt{5}}{4} \end{aligned} \quad (17)$$

$$\begin{aligned} I_y &= \overline{AB} \sin \alpha + \overline{BE} \sin \beta - \overline{BI} \sin \beta \\ &= (4) \left(\frac{\sqrt{10+2\sqrt{5}}}{4} \right) + (1-2) \left(\frac{\sqrt{10-2\sqrt{5}}}{4} \right) = \frac{4\sqrt{10+2\sqrt{5}} - \sqrt{10-2\sqrt{5}}}{4} \end{aligned} \quad (18)$$

$$\begin{aligned} \overline{DI} &= \sqrt{(I_x - D_x)^2 + (I_y - I_y)^2} \\ &= \frac{\sqrt{(1+\sqrt{5})^2 + (4\sqrt{10+2\sqrt{5}} - \sqrt{10-2\sqrt{5}})^2}}{4} = \sqrt{11} \end{aligned} \quad (19)$$

2.1.1 Rigid distance $\sqrt{11}$

Our software found several three strips clusters for rigid distance $\sqrt{11}$. We prove the selected cluster $DHGI$ inside the pentagon matches the expected distance. First we calculate the angle $\angle DHG$ with the

law of cosines and use the value to finally calculate the distance \overline{DI} with again the law of cosines:

$$\begin{aligned}\cos(\angle DHG) &= \frac{(\overline{HD})^2 + (\overline{HG})^2 - (\overline{DG})^2}{2(\overline{HD})(\overline{HG})} \\ &= \frac{3^2 + 1^2 - 3^2}{2(3)(1)} = \frac{1}{6}\end{aligned}\tag{20}$$

$$\begin{aligned}\overline{DI} &= \sqrt{(\overline{HD})^2 + (\overline{HI})^2 - 2(\overline{HD})(\overline{HI})\cos(\angle DHG)} \\ &= \sqrt{3^2 + 2^2 - 2(3)(2)\left(\frac{1}{6}\right)} = \sqrt{11} \quad \blacksquare\end{aligned}\tag{21}$$

2.2 Size 4 with 10 internal strips

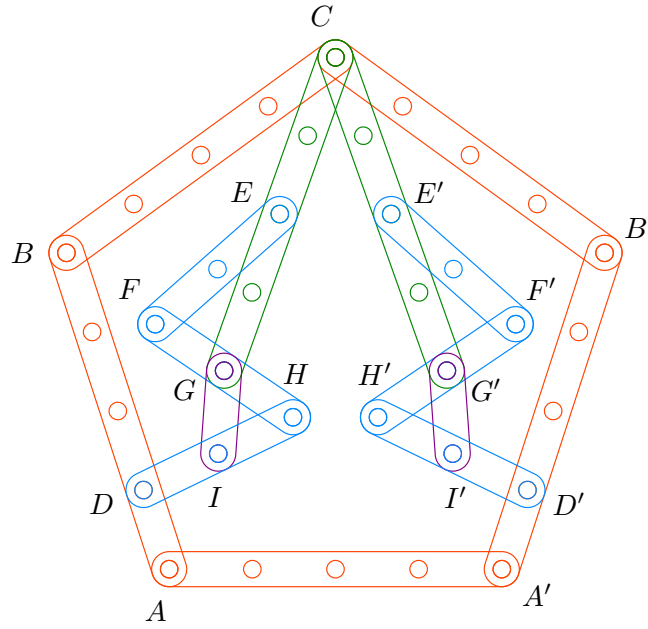


Figure 4: Pentagon of size 4 with 10 internal strips. $\overline{CB} : \overline{BD} = 4 : 3$. $\overline{CD} = \sqrt{19 + 6\sqrt{5}}$.

Figure 4 show the regular pentagon A, A', B', C, B of size 4. We know the internal angle of pentagon is $\theta = \angle CBD = \frac{3\pi}{5}$ and $\cos \theta = \frac{1 - \sqrt{5}}{4}$ so with the law of cosines we can calculate \overline{CD} and the angle $\angle DCB$:

$$\begin{aligned}\overline{CD} &= \sqrt{(\overline{BC})^2 + (\overline{BD})^2 - 2(\overline{BC})(\overline{BD})\cos \theta} \\ &= \sqrt{4^2 + 3^2 - 2(4)(3)\left(\frac{1 - \sqrt{5}}{4}\right)} = \sqrt{19 + 6\sqrt{5}}\end{aligned}\tag{22}$$

$$\begin{aligned}\cos(\angle DCB) &= \frac{(\overline{CD})^2 + (\overline{BC})^2 - (\overline{BD})^2}{2(\overline{CD})(\overline{BC})} \\ &= \frac{(19 + 6\sqrt{5}) + 4^2 - 3^2}{2\left(\sqrt{19 + 6\sqrt{5}}\right)(4)} = \frac{13 + 3\sqrt{5}}{4\sqrt{19 + 6\sqrt{5}}}\end{aligned}\tag{23}$$

2.2.1 Rigid distance $\sqrt{19 + 6\sqrt{5}}$

Our software found several five strips clusters for rigid distance $\sqrt{19 + 6\sqrt{5}}$. We prove selected cluster $CEFGHID$ show in the figure inside the pentagon matches the expected distance. Set the cluster in the coordinate plane such that vertex G is at the origin and vertices F at $(-1, 0)$ and vertex H at $(+1, 0)$. Since triangle $\triangle EFG$ is isoscelles and \overline{CG} is the double of \overline{GE} we know angle $\angle CFG = \frac{\pi}{2}$ and we can calculate the abscissa and ordinate of vertex C :

$$C_x = -\overline{FG} = -1 \quad (24)$$

$$C_y = \sqrt{(\overline{CG})^2 - (\overline{FG})^2} = \sqrt{4^2 - 1^2} = \sqrt{15} \quad (25)$$

Since triangle $\triangle GHI$ is equilateral and \overline{HD} is the double of \overline{HI} we know angle $\angle DGH = \frac{\pi}{2}$ and we can calculate the abscissa and ordinate of vertex D :

$$D_x = 0 \quad (26)$$

$$D_y = -\sqrt{(\overline{HD})^2 - (\overline{GH})^2} = -\sqrt{2^2 - 1^2} = -\sqrt{3} \quad (27)$$

Finally we calculate the distance \overline{CD}

$$\begin{aligned} \overline{CD} &= \sqrt{(C_x - D_x)^2 + (C_y - D_y)^2} \\ &= \sqrt{(-1 - 0)^2 + (\sqrt{15} + \sqrt{3})^2} = \sqrt{19 + 6\sqrt{5}} \quad \blacksquare \end{aligned} \quad (28)$$

3 Pentagons of size 5

3.1 Size 5 with 10 internal strips

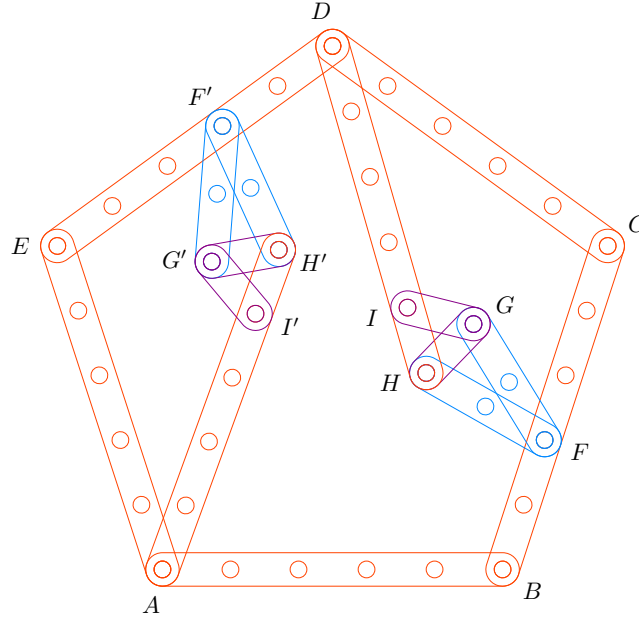


Figure 5: Pentagon of size 5 with 10 internal strips. $\overline{FC} : \overline{CD} = 3 : 5$. $\overline{FD} = \frac{\sqrt{106 + 30\sqrt{5}}}{2}$.

3.1.1 Rigid distance $\frac{\sqrt{106 + 30\sqrt{5}}}{2}$

3.2 Size 5 with 12 internal strips

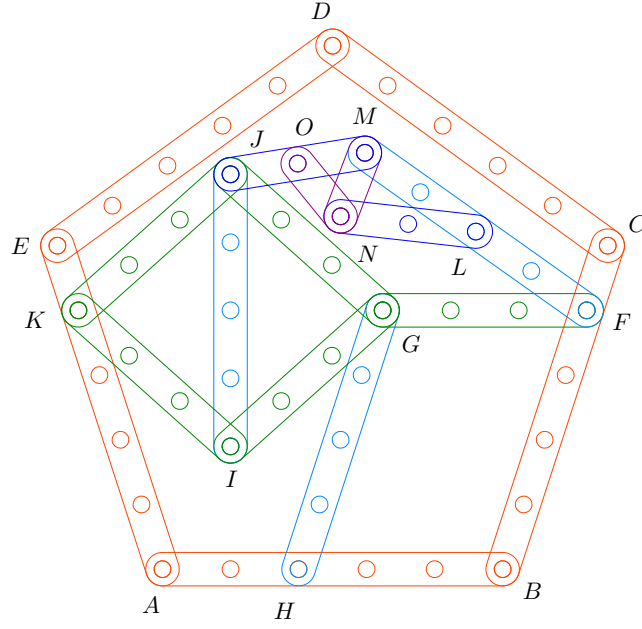


Figure 6: Pentagon of size 5 with 12 internal strips. $\overline{FB} : \overline{BA} : \overline{AK} = 4 : 5 : 4$ and $\overline{FK} = 3 + 2\sqrt{5}$. $\overline{FJ} = \sqrt{18 + 6\sqrt{5}} = \sqrt{3} + \sqrt{15}$.

3.2.1 Rigid distances $3 + 2\sqrt{5}$ and $\sqrt{18 + 6\sqrt{5}} = \sqrt{3} + \sqrt{15}$

4 Pentagons of size 6

4.1 Size 6 with 6 internal strips

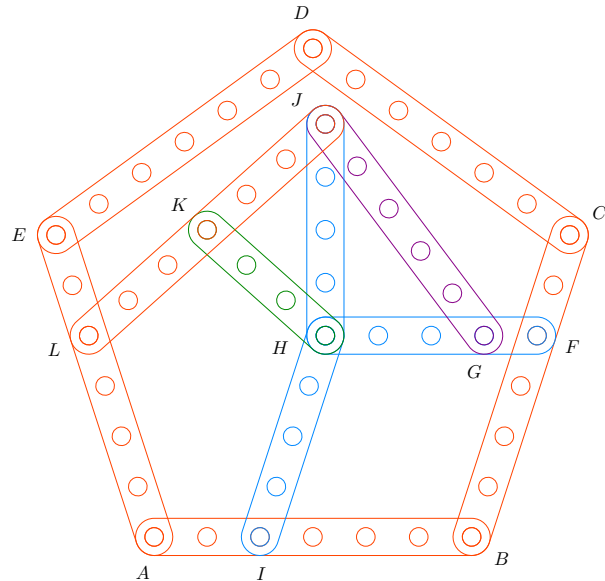


Figure 7: Pentagon of size 6 with 6 internal strips. $\overline{FB} : \overline{BA} : \overline{AL} = 4 : 6 : 4$. $\overline{FL} = 4 + 2\sqrt{5}$.

4.1.1 Rigid distance $4 + 2\sqrt{5}$

4.2 Size 6 with 8 internal strips

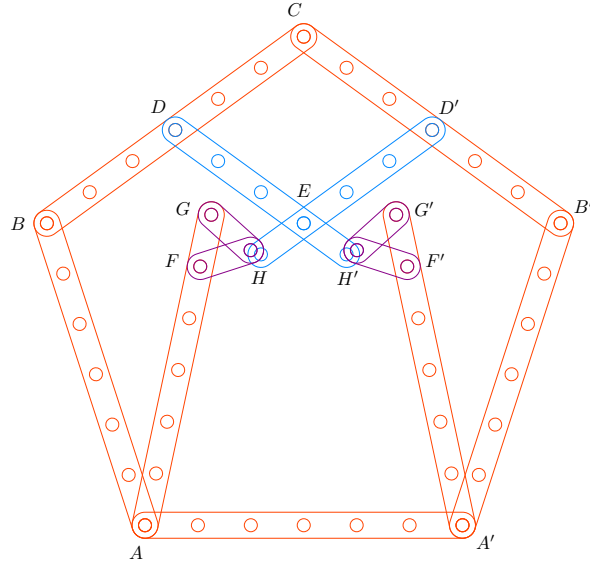


Figure 8: Pentagon of size 6 with 8 internal strips. $\overline{A'A} : \overline{AB} : \overline{BD} : \overline{DF'} = 6 : 6 : 3 : 4$ and $\overline{A'F'} = \sqrt{31}$.

4.2.1 Rigid distance $\sqrt{31}$

4.3 Size 6 with 10 internal strips

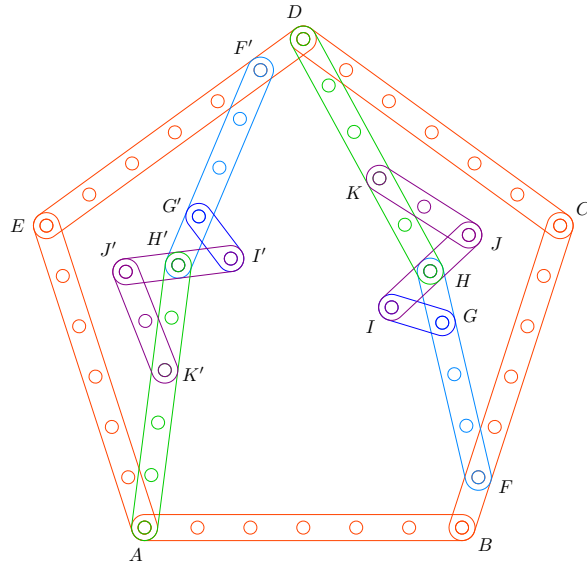


Figure 9: Pentagon of size 6 with 10 internal strips. $\overline{FC} : \overline{CD} = 5 : 6$. $\overline{FD} = \sqrt{46 + 15\sqrt{5}}$.

4.3.1 Rigid distance $\sqrt{46 + 15\sqrt{5}}$

4.4 Size 6 with 12 internal strips

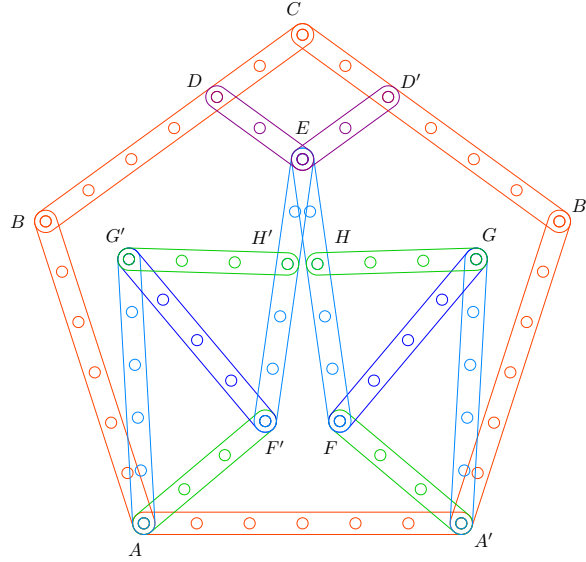


Figure 10: Pentagon of size 6 with 12 internal strips. $\overline{AA} : \overline{AB} : \overline{BD} : \overline{DE} = 0 : 6 : 4 : 2$ and $AE = \sqrt{34 + 10\sqrt{5}}$

4.4.1 Rigid distance $\sqrt{34 + 10\sqrt{5}}$

5 Pentagons of size 7

5.1 Size 7 with 6 internal strips

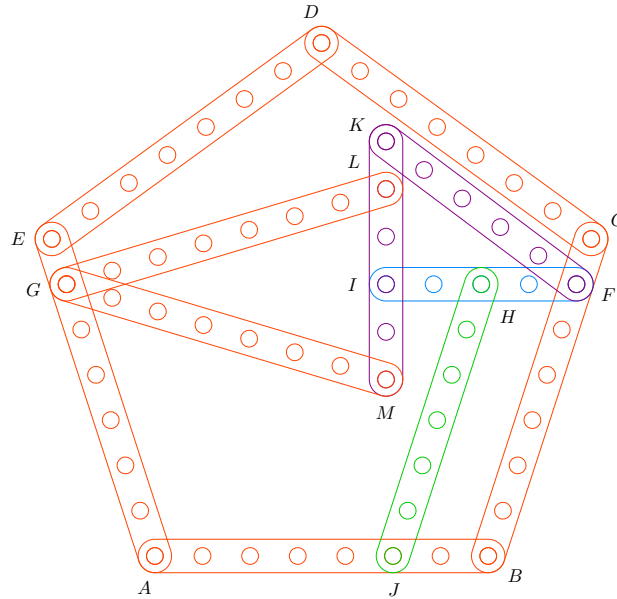


Figure 11: Pentagon of size 7 with 6 internal strips. $\overline{FC} : \overline{CD} = 6 : 7$. $\overline{FD} = 4 + 3\sqrt{5}$.

5.1.1 Rigid distance $4 + 3\sqrt{5}$

5.2 Size 7 with 10 internal strips

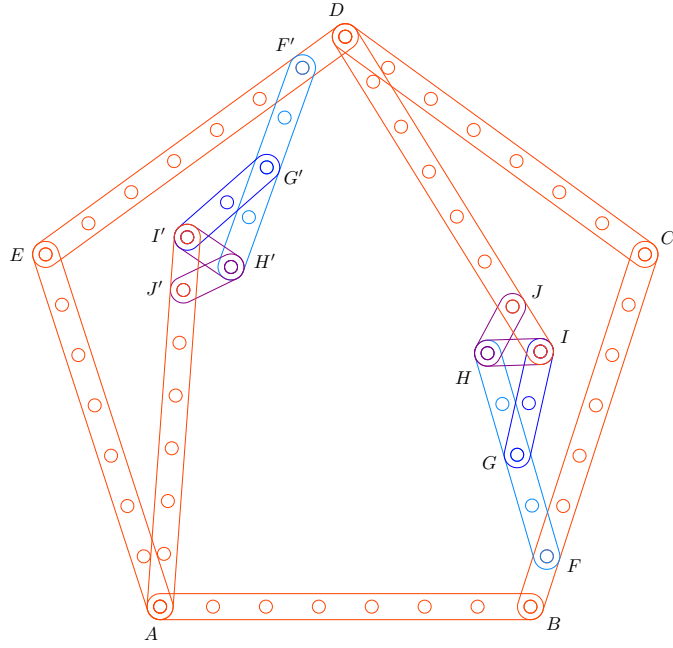


Figure 12: Pentagon of size 7 with 10 internal strips. $\overline{FC} : \overline{CD} = 6 : 7$. $\overline{FD} = \sqrt{64 + 21\sqrt{5}}$.

5.2.1 Rigid distance $\sqrt{64 + 21\sqrt{5}}$

5.3 Size 7 with 14 internal strips

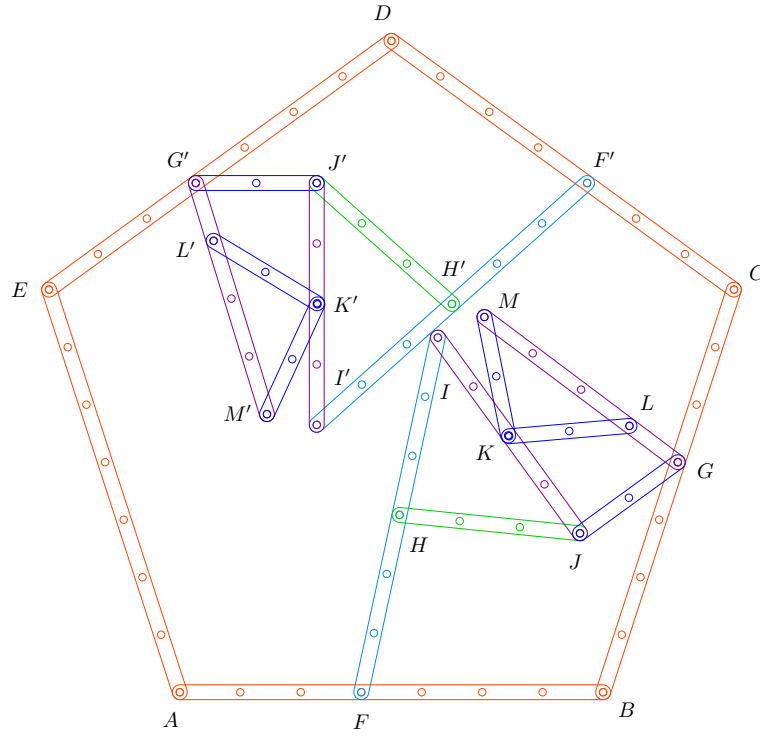


Figure 13: Pentagon of size 7 with 14 internal strips. $\overline{FB} : \overline{BG} = 4 : 4$ and $\overline{FG} = 2 + 2\sqrt{5}$.

5.3.1 Rigid distance $2 + 2\sqrt{5}$

6 Pentagons of size 8

6.1 Size 8 with 6 internal strips

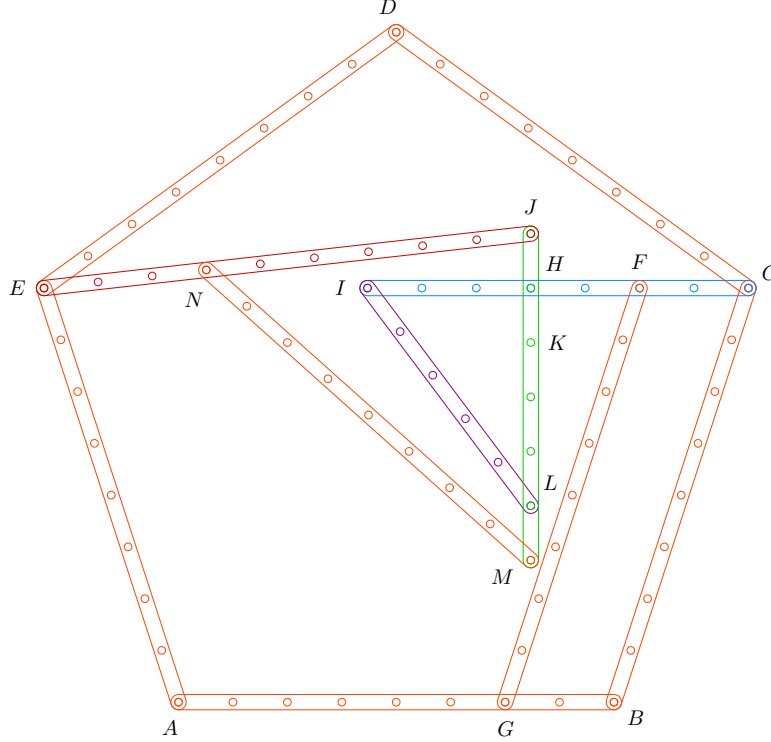


Figure 14: Pentagon of size 8 with 6 internal strips. $\overline{CD} : \overline{DE} = 8 : 8$. $\overline{FD} = 4 + 4\sqrt{5}$.

Figure 14 show regular pentagon $ABCDE$ of size 8. We know the pentagon width like \overline{CE} is $\frac{1+\sqrt{5}}{2}$ times the side of the pentagon so here we have $\overline{CE} = 4 + 4\sqrt{5}$.

6.1.1 Rigid distance $4 + 4\sqrt{5}$

From the figure consider triangle $\triangle NJM$ and calculate with the law of cosines angle $\theta = \angle NJM$:

$$\cos \theta = \frac{(\overline{JM})^2 + (\overline{JN})^2 - (\overline{MN})^2}{2(\overline{JM})(\overline{JN})} = \frac{6^2 + 6^2 - 8^2}{2(6)(6)} = \frac{1}{9} \quad (29)$$

Since $\frac{\overline{EH}}{\overline{EJ}} = \sin \theta$ we have:

$$\overline{EH} = (\overline{EJ}) \sin \theta = 9\sqrt{1 - \cos^2 \theta} = 9\sqrt{1 - \left(\frac{1}{9}\right)^2} = 4\sqrt{5} \quad (30)$$

Since $(\overline{EJ})^2 = (\overline{EH})^2 + (\overline{HJ})^2 \mapsto 9^2 = 16(5) + 1$ we conclude we have the right angle $\angle EHJ = \frac{\pi}{2}$. By inspection we have a second right angle $\angle IHL = \frac{\pi}{2}$ formed by the Pythagorean triangle $\triangle HIL$ and a third one $\angle LHC = \frac{\pi}{2}$ since is supplementary of $\angle IHL$. With the right angles we conclude vertices E, H, C are colinear so the distance $\overline{EH} = \overline{EH} + \overline{HC} = 4\sqrt{5} + 4$ ■. The strip \overline{FG} keeps rigid the vertices A and B .

6.2 Size 8 with 8 internal strips

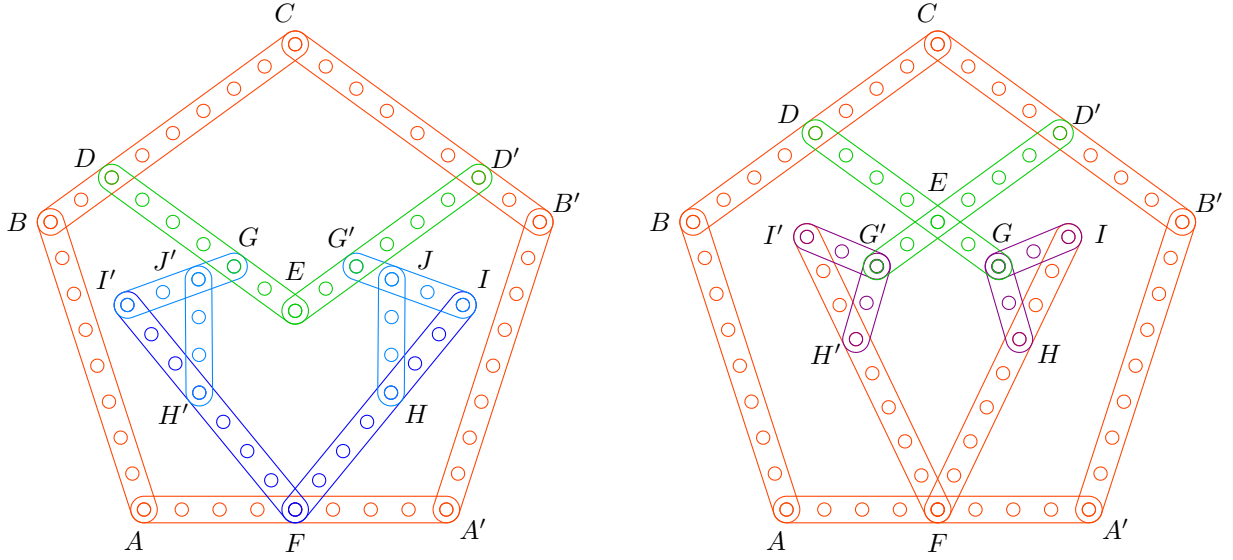


Figure 15: Pentagons of size 8 with 8 internal strips. For the pentagon variation 1 at the left $\overline{FA} : \overline{AB} : \overline{BD} : \overline{DG} = 4 : 8 : 2 : 4$ and $\overline{FG} = 2\sqrt{11}$. For the pentagon variation 2 at the right $\overline{AF} : \overline{AB} : \overline{BD} : \overline{DG} = 4 : 8 : 4 : 6$ and $\overline{FG} = 2\sqrt{11}$.

Figure 15 show two rigid regular pentagons A, A', B', C, B of size 9. For the variation 1 at the left and assuming vertex F is at the origin we calculate the distance \overline{FG} using the abscissas and ordinates following the vertices F, A, B, D, G for a regular pentagon angles $3\pi/5$, $\beta = \pi/5$:

$$\begin{aligned} FI_x &= -\overline{AF} - \overline{AB} \cos \alpha + (\overline{BD} + \overline{DI}) \cos \beta \\ &= -4 + (8) \frac{1 - \sqrt{5}}{4} + (2 + 4) \frac{\sqrt{5} + 1}{4} = -\frac{2 + 2\sqrt{5}}{4} \end{aligned} \quad (31)$$

$$\begin{aligned} FI_y &= \overline{AB} \sin \alpha + (\overline{BD} - \overline{DI}) \sin \beta \\ &= (8) \frac{\sqrt{10 + 2\sqrt{5}}}{4} + (2 - 4) \frac{\sqrt{10 - 2\sqrt{5}}}{4} = \frac{8\sqrt{10 + 2\sqrt{5}} - 2\sqrt{10 - 2\sqrt{5}}}{4} \end{aligned} \quad (32)$$

$$\begin{aligned} \overline{FI} &= \sqrt{(FI_x)^2 + (FI_y)^2} \\ &= \frac{\sqrt{(2 + 2\sqrt{5})^2 + (8\sqrt{10 + 2\sqrt{5}} - 2\sqrt{10 - 2\sqrt{5}})^2}}{4} = \frac{\sqrt{704}}{4} = 2\sqrt{11} \end{aligned} \quad (33)$$

For the variation 2 at the right vertices G, G' switch positions of those of variation 1 and in both variations we have $\overline{FG} = \overline{FG'} = 2\sqrt{11}$.

6.2.1 Rigid distances $2\sqrt{11}$

Our software found several options. For the variation 1 at the left of the figure consider the cluster $FHIJG'$. Within the isoscelles triangle $\triangle HIJ$ we calculate the cosine of angle $\theta = \angle JIH$ and use it with the law of

cosines to calculate $\overline{FG'}$:

$$\begin{aligned}
\cos \theta &= \frac{\overline{IJ}/2}{\overline{IH}} = \frac{1}{3} \\
\overline{FG'} &= \sqrt{(\overline{FI})^2 + (\overline{IG'})^2 - 2(\overline{FI})(\overline{IG'}) \cos \theta} \\
&= \sqrt{7^2 + 3^2 - 2(7)(3)\frac{1}{3}} = 2\sqrt{11} \quad \blacksquare
\end{aligned} \tag{34}$$

For the variation 2 at the right of the figure consider the cluster $FHIG$. Within the isoscelles triangle GIH we calculate the cosine of angle $\phi = \angle HIG$ and use it with the law of cosines to calculate \overline{FG} :

$$\begin{aligned}
\cos \phi &= \frac{\overline{HI}/2}{\overline{GI}} = \frac{3}{4} \\
\overline{FH} &= \sqrt{(\overline{FI})^2 + (\overline{GI})^2 - 2(\overline{FI})(\overline{GI}) \cos \phi} \\
&= \sqrt{8^2 + 2^2 - 2(8)(2)\frac{3}{4}} = 2\sqrt{11} \quad \blacksquare
\end{aligned} \tag{35}$$

6.3 Size 8 with 10 internal strips $build(4 : 8)$

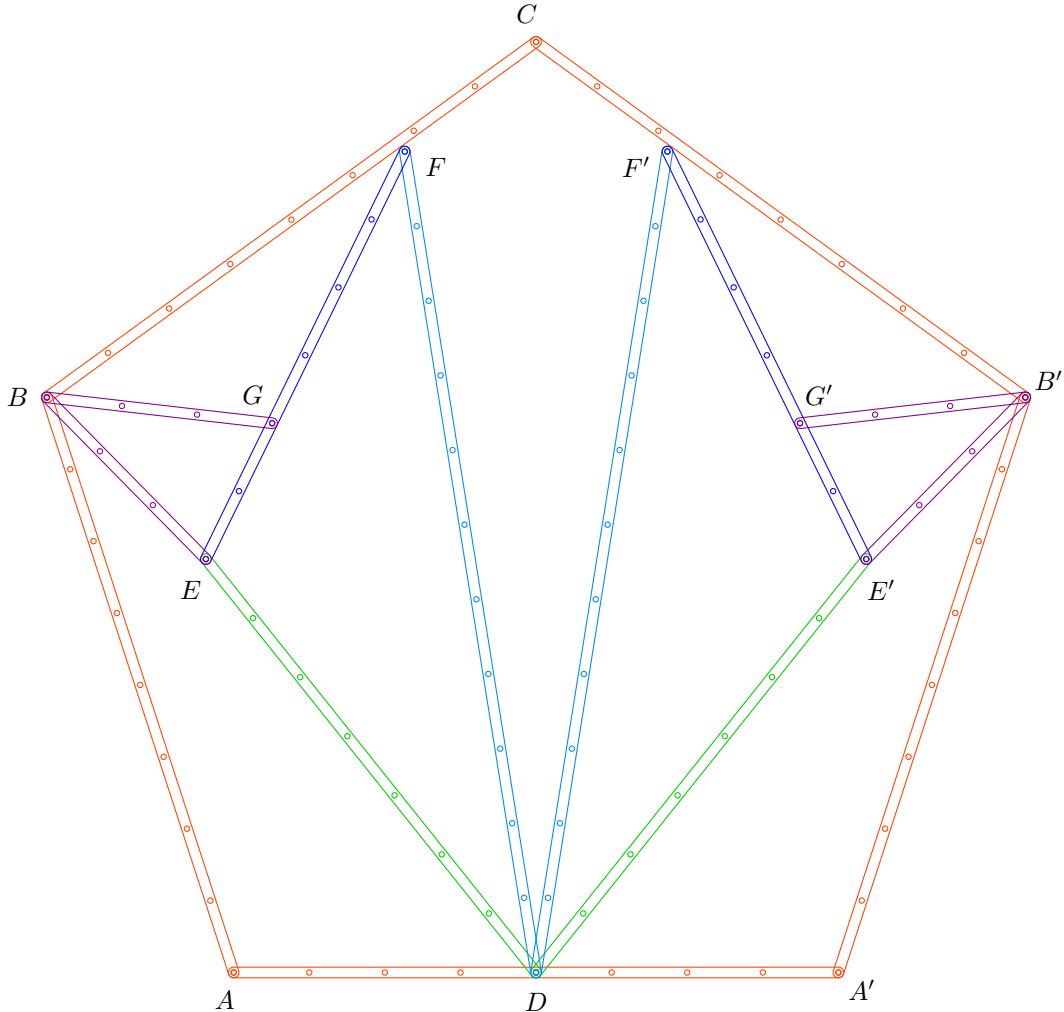


Figure 16: Pentagon of size 8 with 10 internal strips. $\overline{DA} : \overline{AB} = 4 : 8$ and $\overline{DB} = 4\sqrt{4 + \sqrt{5}}$

Figure 16 show pentagon $AA'B'CB$ of size 8. We use the law of cosines to calculate distance \overline{DB} knowing $\theta = \angle DAB = \frac{3\pi}{5}$ and $\cos \theta = \frac{1-\sqrt{5}}{4}$:

$$\begin{aligned}\overline{DB} &= \sqrt{(\overline{AD})^2 + (\overline{AB})^2 - 2(\overline{AD})(\overline{AB}) \cos \theta} \\ &= \sqrt{4^2 + 8^2 - 2(4)(8) \frac{1-\sqrt{5}}{4}} = 4\sqrt{4 + \sqrt{5}}\end{aligned}\quad (36)$$

6.3.1 Rigid distance $4\sqrt{4 + \sqrt{5}}$

Our software found few solutions but the cluster $DEBGF$ shown in the figure fit (hardly) inside the pentagon. We calculate angles around vertex E $\alpha = \angle DEF$ with the law of cosines and $\beta = \angle BEG$ whitin the isoscelles triangle $\triangle BEG$:

$$\cos \alpha = \frac{(\overline{DE})^2 + (\overline{EF})^2 - (\overline{DF})^2}{2(\overline{DE})(\overline{EF})} = \frac{7^2 + 6^2 - 11^2}{2(7)(6)} = -\frac{3}{7}\quad (37)$$

$$\sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - \left(-\frac{3}{7}\right)^2} = \frac{2\sqrt{10}}{7}\quad (38)$$

$$\cos \beta = \frac{\frac{\overline{EG}}{2}}{\frac{\overline{EB}}{2}} = \frac{\frac{2}{2}}{\frac{3}{2}} = \frac{1}{3}\quad (39)$$

$$\sin \beta = \sqrt{1 - \sin^2 \beta} = \sqrt{1 - \left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3}\quad (40)$$

Now we calculate the angle $\angle DEB = \gamma = \alpha + \beta$ using the cosines sum identity:

$$\begin{aligned}\cos \gamma &= \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ &= \left(-\frac{3}{7}\right) \left(\frac{1}{3}\right) - \left(\frac{2\sqrt{10}}{7}\right) \left(\frac{2\sqrt{2}}{3}\right) = -\frac{3 + 8\sqrt{5}}{21}\end{aligned}\quad (41)$$

Finally we calculate the distance \overline{DB} using previous angle γ :

$$\begin{aligned}\overline{DB} &= \sqrt{(\overline{DE})^2 + (\overline{EB})^2 - 2(\overline{DE})(\overline{EB}) \cos \gamma} \\ &= \sqrt{7^2 + 3^2 - 2(7)(3) \left(-\frac{3 + 8\sqrt{5}}{21}\right)} = 4\sqrt{4 + \sqrt{5}} \quad \blacksquare\end{aligned}\quad (42)$$

6.4 Size 8 with 10 internal strips $build(7 : 8)$

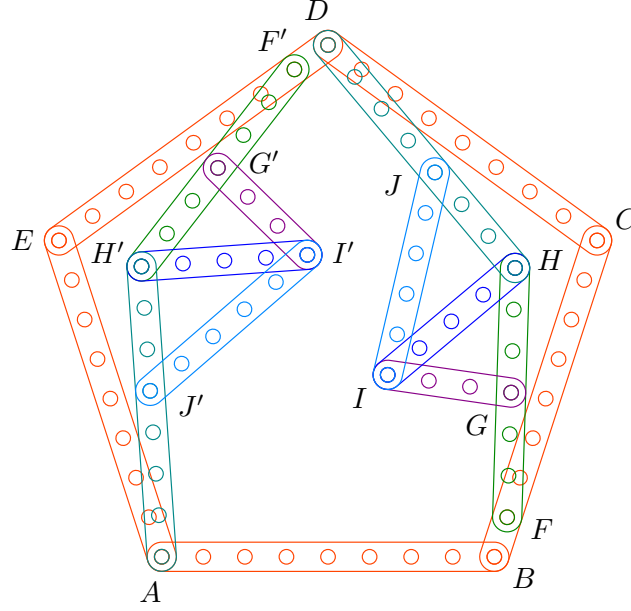


Figure 17: Pentagon of size 8 with 10 internal strips. $\overline{FC} : \overline{CD} = 7 : 8$ and $\overline{CD} = \sqrt{85 + 28\sqrt{5}}$

Figure 17 show pentagon $ABCDE$ of size 8. We use the law of cosines to calculate distance \overline{FD} knowing $\theta = \angle FCD = \frac{3\pi}{5}$ and $\cos \theta = \frac{1-\sqrt{5}}{4}$:

$$\begin{aligned} \overline{FC} &= \sqrt{(\overline{FC})^2 + (\overline{CD})^2 - 2(\overline{FC})(\overline{CD}) \cos \theta} \\ &= \sqrt{7^2 + 8^2 - 2(7)(8) \frac{1-\sqrt{5}}{4}} = \sqrt{85 + 28\sqrt{5}} \end{aligned} \quad (43)$$

6.4.1 Rigid distance $\sqrt{85 + 28\sqrt{5}}$

Our software found several solutions, consider the cluster $FGHIJD$ in the figure. Assume vertex I is at the origin and vertex H is at coordinate $(4, 0)$. Since triangle $\triangle HIG$ is isoscelles and \overline{HF} is the double of \overline{IG} then $\overline{IF} = \sqrt{(\overline{HF})^2 - (\overline{IH})^2} = \sqrt{6^2 - 4^2} = 2\sqrt{5}$ and the coordinates of vertex $F(x, y)$ are $(0, \overline{IF}) = (0, -2\sqrt{5})$. Since the Pythagorean triangle $\triangle HIJ$ we have a right angle $\angle IHD = \frac{\pi}{2}$ and then the coordinates of vertex $D(x, y)$ are $(\overline{IH}, \overline{HD}) = (4, 7)$. Finally we calculat the distance \overline{FD} :

$$\begin{aligned} \overline{FD} &= \sqrt{(F_x - D_x)^2 + (F_y - D_y)^2} \\ &= \sqrt{(0 - 4)^2 + (-2\sqrt{5} - 7)^2} = \sqrt{85 + 28\sqrt{5}} \quad \blacksquare \end{aligned} \quad (44)$$

7 Pentagons of size 9

7.1 Size 9 with 6 internal strips

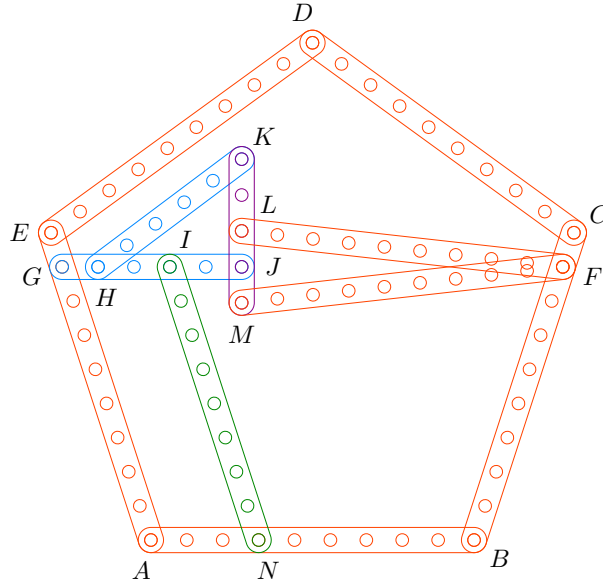


Figure 18: Pentagon size 9 with six internal strips. $\overline{FB} : \overline{BA} : \overline{AG} = 8 : 9 : 8$. $\overline{FG} = 5 + 4\sqrt{5}$.

Figure 18 show a rigid regular pentagon A, B, C, D, E of size 9. The regular pentagon distance \overline{CE} is called width and equals $\frac{1 + \sqrt{5}}{2} \overline{AB}$. Is easy to note the distance \overline{FG} equals the width of smaller pentagon size $9 - 1 = 8$ plus 1. So we have:

$$\begin{aligned} \overline{FG} &= \frac{1 + \sqrt{5}}{2} (\overline{BC} - \overline{FC}) + \overline{FC} \\ &= \frac{1 + \sqrt{5}}{2} (9 - 1) + 1 = 5 + 4\sqrt{5} \end{aligned} \quad (45)$$

7.1.1 Rigid distance $5 + 4\sqrt{5}$

From the figure we see two right angles. Angle $\angle GJK = \frac{\pi}{2}$ because we have a Pythagorean triangle $\triangle HJK$. Angle $\angle FJM = \frac{\pi}{2}$ because we have an isosceles triangle $\triangle FLM$. The two right angles share vertex J so vertices G, J, F are collinear. First we calculate the distance $\overline{JF} = \sqrt{(LF)^2 - (LJ)^2} = \sqrt{9^2 - 1^2} = 4\sqrt{5}$ and finally the distance $\overline{GF} = \overline{GJ} + \overline{JF} = 5 + 4\sqrt{5}$ which matches the value in last equation above ■. To make rigid the pentagon we add strip \overline{IN} parallel to side \overline{GA} .

7.2 Size 9 with 8 internal strips

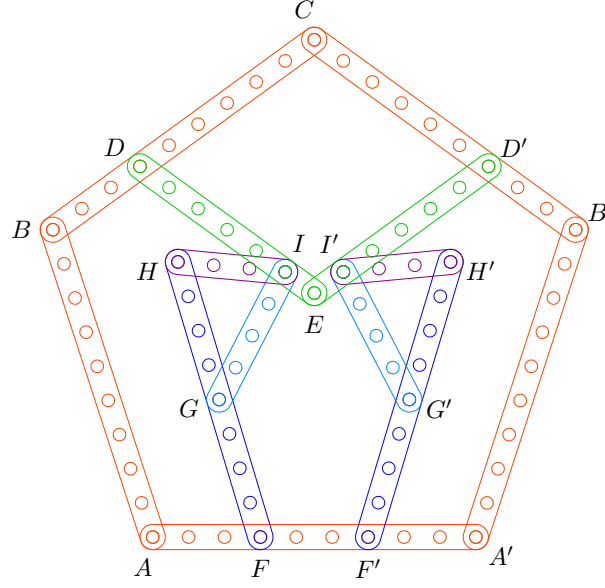


Figure 19: Pentagon size 9 with 8 strips variation 1. $\overline{FA} : \overline{AB} : \overline{BD} : \overline{DI} = 3 : 9 : 3 : 5$. $\overline{FI} = \sqrt{55}$.

Figure 19 show a rigid regular pentagon A, A', B', C, B of size 9. First we calculate the distance \overline{FI} using the abscissas and ordinates following the vertices F, A, B, D, I for a regular pentagon angles $\alpha = \frac{3\pi}{5}, \beta = \frac{\pi}{5}$:

$$\begin{aligned} FI_x &= -\overline{AF} - \overline{AB} \cos \alpha + (\overline{BD} + \overline{DI}) \cos \beta \\ &= -3 + (9) \frac{1 - \sqrt{5}}{4} + (3 + 5) \frac{\sqrt{5} + 1}{4} = \frac{5 - \sqrt{5}}{4} \end{aligned} \quad (46)$$

$$\begin{aligned} FI_y &= \overline{AB} \sin \alpha + (\overline{BD} - \overline{DI}) \sin \beta \\ &= (9) \frac{\sqrt{10 + 2\sqrt{5}}}{4} + (3 - 5) \frac{\sqrt{10 - 2\sqrt{5}}}{4} = \frac{9\sqrt{10 + 2\sqrt{5}} - 2\sqrt{10 - 2\sqrt{5}}}{4} \end{aligned} \quad (47)$$

$$\begin{aligned} \overline{FI} &= \sqrt{(FI_x)^2 + (FI_y)^2} \\ &= \frac{\sqrt{(5 - \sqrt{5})^2 + (9\sqrt{10 + 2\sqrt{5}} - 2\sqrt{10 - 2\sqrt{5}})^2}}{4} = \frac{\sqrt{880}}{4} = \sqrt{55} \end{aligned} \quad (48)$$

7.2.1 Rigid distance $\sqrt{55}$

By software we find several options for the distance and we use the one shown in the figure. We calculate the distance \overline{FI} made rigid by cluster F, G, H, I . We have an isoscelles triangle $\triangle GHI$ and $\overline{FH} = 2\overline{GH}$ so we have a right triangle $\angle FIH = \frac{\pi}{2}$ so:

$$\begin{aligned} \overline{FI} &= \sqrt{(\overline{FH})^2 - (\overline{HI})^2} \\ &= \sqrt{8^2 - 3^2} = \sqrt{55} \quad \blacksquare \end{aligned} \quad (49)$$

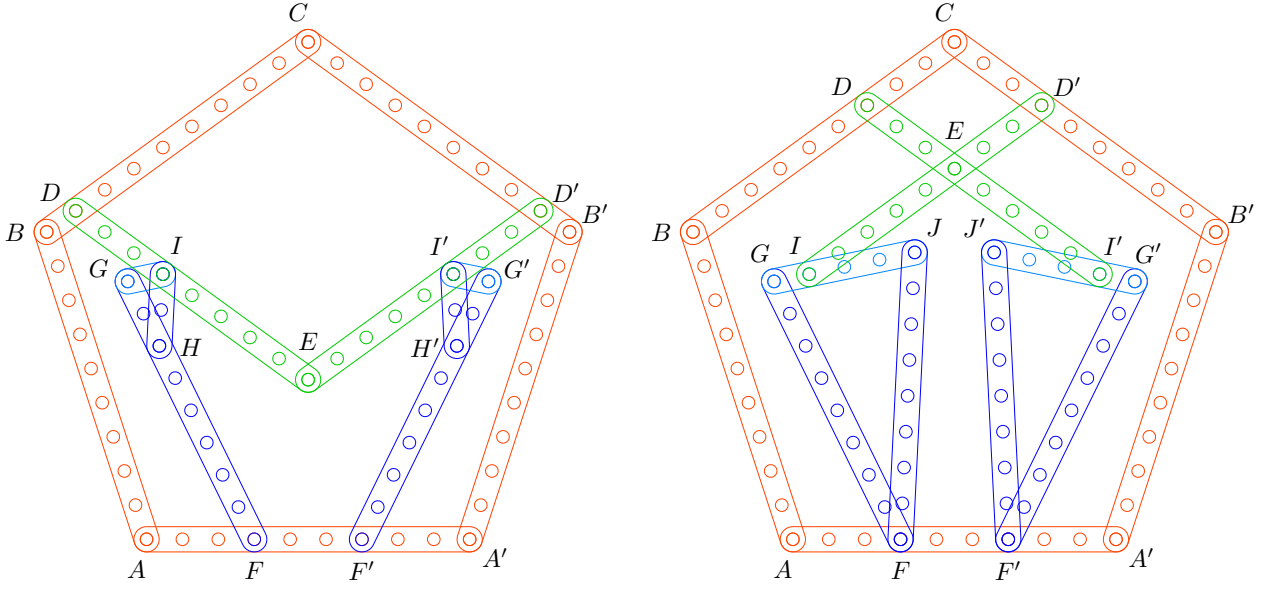


Figure 20: Regular pentagons size 9 made rigid with 8 strips variations 2 and 3. At the left we have $\overline{FA} : \overline{AB} : \overline{BD} : \overline{DI} = 3 : 9 : 1 : 4$ and $\overline{FI} = \sqrt{61}$. At the right we have $\overline{F'A} : \overline{AB} : \overline{BD} : \overline{DI'} = 6 : 9 : 6 : 8$ and $\overline{F'I'} = \sqrt{61}$.

Figure 20 show two rigid pentagons A, A', B', C, B' of size 9. The pentagon at the left is called variation 2 and the right one variation 3. Both variations have the vertices I, I' at the same positions and the same distance \overline{FI} which first we calculate using the abscissas and ordinates following the vertices F, A, B, D, I of the variation 2 for a regular pentagon angles $\alpha = \frac{3\pi}{5}, \beta = \frac{\pi}{5}$:

$$\begin{aligned} FI_x &= -\overline{AF} - \overline{AB} \cos \alpha + (\overline{BD} + \overline{DI}) \cos \beta \\ &= -3 + (9) \frac{1 - \sqrt{5}}{4} + (1 + 3) \frac{\sqrt{5} + 1}{4} = \frac{1 - 5\sqrt{5}}{4} \end{aligned} \quad (50)$$

$$\begin{aligned} FI_y &= \overline{AB} \sin \alpha + (\overline{BD} - \overline{DI}) \sin \beta \\ &= (9) \frac{\sqrt{10 + 2\sqrt{5}}}{4} + (1 - 3) \frac{\sqrt{10 - 2\sqrt{5}}}{4} = \frac{9\sqrt{10 + 2\sqrt{5}} - 2\sqrt{10 - 2\sqrt{5}}}{4} \end{aligned} \quad (51)$$

$$\begin{aligned} \overline{FI} &= \sqrt{(FI_x)^2 + (FI_y)^2} \\ &= \frac{\sqrt{(1 - \sqrt{5})^2 + (9\sqrt{10 + 2\sqrt{5}} - 2\sqrt{10 - 2\sqrt{5}})^2}}{4} = \frac{\sqrt{976}}{4} = \sqrt{61} \end{aligned} \quad (52)$$

7.2.2 Rigid distance $\sqrt{61}$

Our software found several clusters and we use two different for each variation. We calculate the distance \overline{FI} made rigid by clusters F, G, H, I or F, G, I, J since in both variations we have the same \overline{GF} and same angles $\angle FGI = \angle FJG$. With the law of cosines first we calculate $\cos(\angle FJG)$ and then \overline{FI} :

$$\begin{aligned} \cos(\angle FJG) &= \frac{\overline{FJ}^2 + \overline{JG}^2 - \overline{GF}^2}{2(\overline{FJ})(\overline{JG})} = \frac{8^2 + 4^2 - 8^2}{2(8)(4)} = \frac{1}{4} \\ \overline{FI} &= \sqrt{\overline{IJ}^2 + \overline{FJ}^2 - 2(\overline{IJ})(\overline{FJ}) \cos(\angle FJG)} = \sqrt{3^2 + 8^2 - 2(3)(8) \left(\frac{1}{4}\right)} = \sqrt{61} \quad \blacksquare \end{aligned} \quad (53)$$

7.3 Size 9 with 10 internal strips

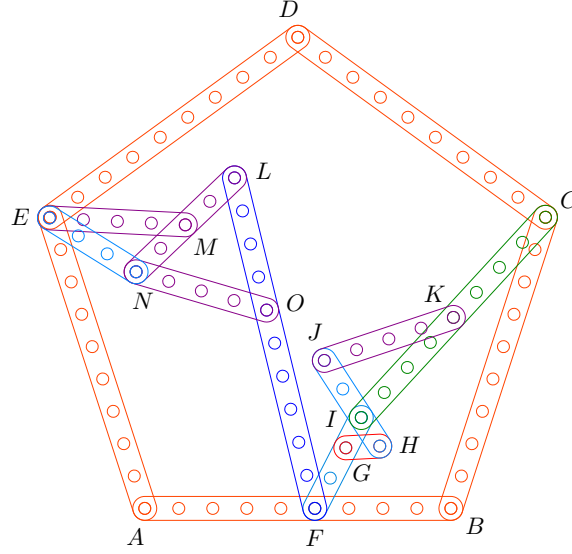


Figure 21: Pentagons size 9 with two 10 internal strips. For the cluster of the left we have $\overline{FA} : \overline{AE} = 5 : 9$ and $\overline{FE} = \frac{\sqrt{334 + 90\sqrt{5}}}{2}$. For the cluster at the right we have $\overline{FB} : \overline{BC} = 4 : 9$ and $\overline{FC} = \sqrt{79 + 18\sqrt{5}}$.

Figure 21 show a regular pentagon A, B, C, D, E of size 9 made it rigid with the help of clusters fixing the distances \overline{CF} and \overline{EF} . Pentagon size 9 is the smaller one with diagonals where consecutive side segments fractions are $\overline{BF}/\overline{BC} = \frac{4}{9}$ and $\overline{AF}/\overline{AE} = \frac{5}{9}$. We calculate the diagonals \overline{CF} , \overline{EF} and the angles to side \overline{AB} using the law of cosines and the internal pentagon angle $\theta = \angle FBC = \angle FAE = \frac{3\pi}{5}$ where $\cos \theta = \frac{1 - \sqrt{5}}{4}$:

$$\overline{CF} = \sqrt{\overline{BC}^2 + \overline{BF}^2 - 2(\overline{BC})(\overline{BF}) \cos \theta} = \sqrt{9^2 + 4^2 - 2(9)(4) \left(\frac{1 - \sqrt{5}}{4} \right)} = \sqrt{79 + 18\sqrt{5}} \quad (54)$$

$$\cos(\angle CFB) = \frac{\overline{CF}^2 + \overline{BF}^2 - \overline{BC}^2}{2(\overline{CF})(\overline{BF})} = \frac{79 + 18\sqrt{5} + 4^2 - 9^2}{2(\sqrt{79 + 18\sqrt{5}})(4)} = \frac{7 + 9\sqrt{5}}{4\sqrt{79 + 18\sqrt{5}}} \quad (55)$$

$$\overline{EF} = \sqrt{\overline{AE}^2 + \overline{AF}^2 - 2(\overline{AE})(\overline{AF}) \cos \theta} = \sqrt{9^2 + 5^2 - 2(9)(5) \left(\frac{1 - \sqrt{5}}{4} \right)} = \frac{\sqrt{334 + 90\sqrt{5}}}{2} \quad (56)$$

$$\cos(\angle EFA) = \frac{\overline{EF}^2 + \overline{AF}^2 - \overline{EA}^2}{2(\overline{EF})(\overline{AF})} = \frac{\frac{334 + 90\sqrt{5}}{4} + 5^2 - 9^2}{2 \left(\frac{\sqrt{334 + 90\sqrt{5}}}{2} \right) (5)} = \frac{11 + 9\sqrt{5}}{2\sqrt{334 + 90\sqrt{5}}} \quad (57)$$

7.3.1 Rigid distance $\sqrt{79 + 18\sqrt{5}}$

Our software found several options with five strips to build distance $\sqrt{79 + 18\sqrt{5}}$.

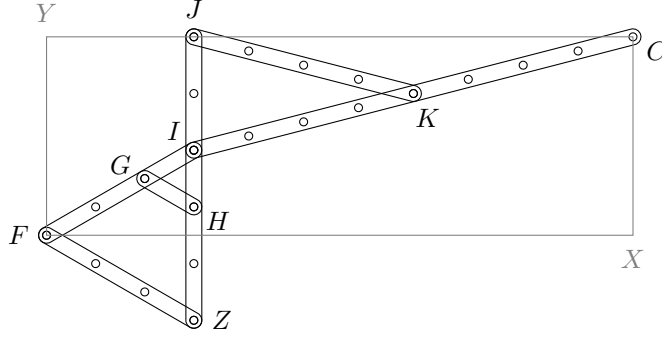


Figure 22: Construction of distance $\overline{FC} = \sqrt{79 + 18\sqrt{5}}$

Figure 22 show one of several ways to build the distance $\sqrt{79 + 18\sqrt{5}}$. Equilateral triangle $\triangle FIZ$ and isosceles $\triangle IJK$ share vertex I and the base \overline{JZ} which help to form rectangle $FXYC$ with base \overline{FX} and height \overline{FY} useful to calculate the diagonal \overline{FC} :

$$\begin{aligned}
 \overline{FX} &= \overline{YJ} + \overline{JC} \\
 &= \sqrt{\overline{FI}^2 - \left(\frac{\overline{IZ}}{2}\right)^2} + \sqrt{\overline{IC}^2 - \overline{IJ}^2} = \sqrt{3^2 - \left(\frac{3}{2}\right)^2} + \sqrt{8^2 - 2^2} = \frac{3\sqrt{3}}{2} + 2\sqrt{15} \\
 \overline{FY} &= \overline{JI} + \frac{\overline{IZ}}{2} = 2 + \frac{3}{2} = \frac{7}{2} \\
 \overline{FC} &= \sqrt{\overline{FX}^2 + \overline{FY}^2} = \sqrt{\left(\frac{3\sqrt{3}}{2} + 2\sqrt{15}\right)^2 + \left(\frac{7}{2}\right)^2} = \sqrt{79 + 18\sqrt{5}} \quad \blacksquare \tag{58}
 \end{aligned}$$

We use a smaller part of this construction, the five strips with vertices F, G, H, I, J, K, C , as a cluster to made rigid the consecutive strips $\overline{AB}, \overline{BC}$ of the pentagon of side 9 of figure 21.

7.3.2 Rigid distance $\frac{\sqrt{334 + 90\sqrt{5}}}{2}$

Our software found several options with five strips to build distance $\frac{\sqrt{334 + 90\sqrt{5}}}{2}$.

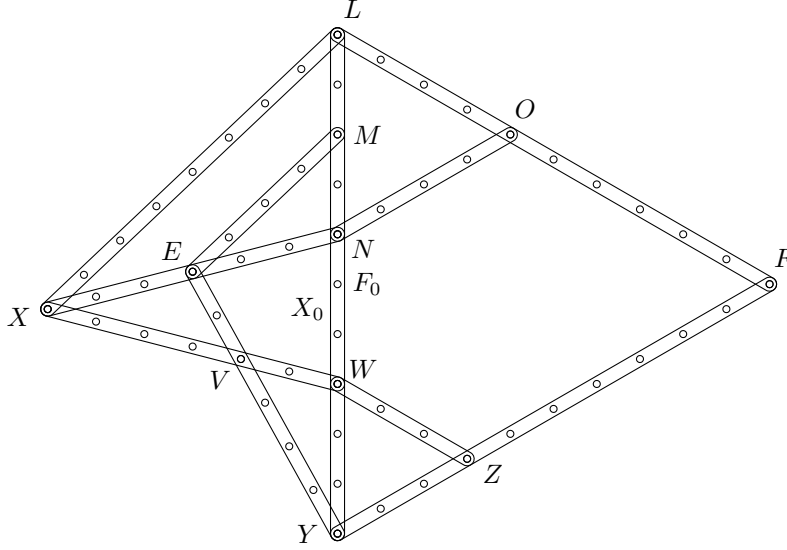


Figure 23: Construction of distance $\overline{EF} = \frac{\sqrt{334 + 90\sqrt{5}}}{2}$

Figure 23 show equilateral triangle $\triangle FLY$ and isoscelles triangle $\triangle NXW$ sharing strip \overline{LY} which helps to calculate abscissas and ordinates of vertices E, F to calculate distance \overline{EF} . Vertice Y is located at the origin so:

$$E_x = -\left(\frac{\overline{NE}}{\overline{NX}}\right) \overline{XX_0} = -\frac{3}{6} \sqrt{\overline{NX}^2 - \overline{NX_0}^2} = -\frac{1}{2} \sqrt{6^2 - \left(\frac{3}{2}\right)^2} = -\frac{3\sqrt{15}}{4} \quad (59)$$

$$E_y = \overline{YN} - \left(\frac{\overline{NE}}{\overline{NX}}\right) \overline{NX_0} = 6 - \left(\frac{3}{6}\right) \left(\frac{3}{2}\right) = \frac{21}{4} \quad (60)$$

$$F_x = \overline{F_0F} = \sqrt{\overline{YF}^2 - \overline{YF_0}^2} = \sqrt{10^2 - 5^2} = 5\sqrt{3} \quad (61)$$

$$F_y = \overline{YF_0} = 5 \quad (62)$$

$$\overline{EF} = \sqrt{(E_x - F_x)^2 + (E_y - F_y)^2} = \sqrt{\left(-\frac{3\sqrt{15}}{4} - 5\sqrt{3}\right)^2 + \left(\frac{21}{4} - 5\right)^2} = \frac{\sqrt{334 + 90\sqrt{5}}}{2} \blacksquare \quad (63)$$

We form a cluster from the last construction to be applied in the pentagon of side 9. We choose the five strips with vertices E, N, M, L, O, F . Is easy to prove strip \overline{EM} is correct in the cluster comparing equal cosines at vertice Y for triangles $\triangle YVW, \triangle YEN, \triangle YEM$ using the law of cosines for each triangle.

8 Pentagon of size 10

8.1 Size 10 with 10 internal strips

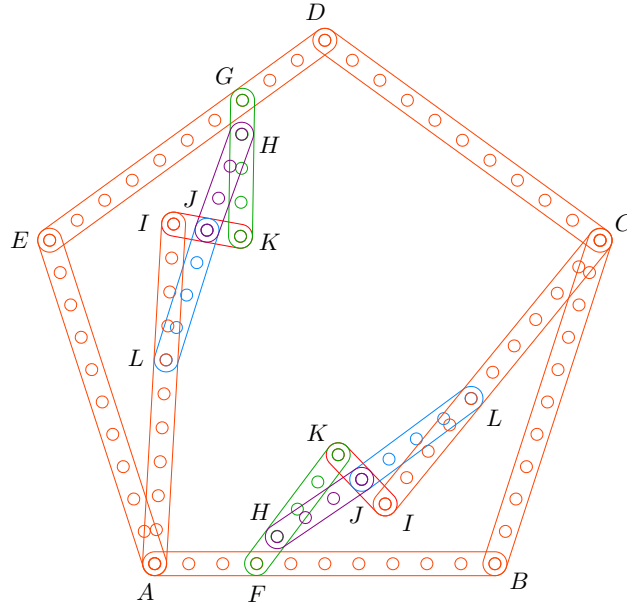


Figure 24: Pentagon size 10 with 10 internal strips. $\overline{FB} : \overline{BC} = 7 : 10$ and $\overline{CF} = \sqrt{114 + 35\sqrt{5}}$.

Figure 24 show a rigid regular pentagon A, B, C, D, E of size 10. We calculate a diagonal joining two consecutive sides relative primes to have something exclusive to the size 10, we choose $\overline{BF} : \overline{BC} = 7 : 10$. With the law of cosines we calculate \overline{CF} . We calculate the angle $\angle CFB$ for the drawing:

$$\begin{aligned}\overline{CF}^2 &= \overline{BC}^2 + \overline{BF}^2 - 2(\overline{BC})(\overline{BF})\cos\left(\frac{3\pi}{5}\right) \\ &= 10^2 + 7^2 - 2(10)(7)\left(\frac{1-\sqrt{5}}{4}\right) = 114 + 35\sqrt{5} \\ \overline{CF} &= \sqrt{114 + 35\sqrt{5}}\end{aligned}\tag{64}$$

$$\cos(\angle CFB) = \frac{\overline{CF}^2 + \overline{BF}^2 - \overline{BC}^2}{2(\overline{CF})(\overline{BF})} = \frac{114 + 35\sqrt{5} + 7^2 - 10^2}{2(\sqrt{114 + 35\sqrt{5}})(7)} = \frac{9 + 5\sqrt{5}}{2\sqrt{114 + 35\sqrt{5}}}\tag{65}$$

8.1.1 Rigid distance $\sqrt{114 + 35\sqrt{5}}$

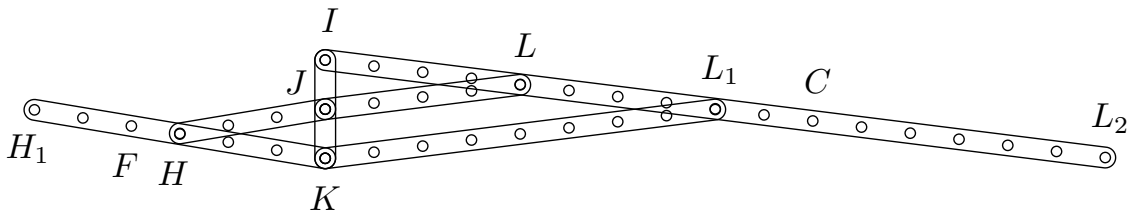


Figure 25: Construction of distance $\overline{CF} = \sqrt{114 + 35\sqrt{5}}$

Our software found several solutions for this distance using five strips, and we choose one narrow enough to fit inside the pentagon.

Figure 25 shows how to prove the cluster selected is correct. In the figure we have two isoscelles triangles $\triangle IKL_1$ and $\triangle JKH$. The sides IL_1 and KH are extended to double the original size to the vertices L_2 and H_1 building two right angles $\angle IKL_2$ and $\angle KJH_1$. The right triangles permit the calculation of the abscissas and ordinates of vertices C and F to calculate their distance.

From the figure we calculate $\overline{KL_2}$ and $\overline{JH_1}$ from their respective right triangles:

$$\overline{KL_2} = \sqrt{(\overline{IL_2})^2 - (IK)^2} = \sqrt{16^2 - 2^2} = 6\sqrt{7} \quad (66)$$

$$\overline{JH_1} = \sqrt{(\overline{KH_1})^2 - (KJ)^2} = \sqrt{6^2 - 1^2} = \sqrt{35} \quad (67)$$

Assuming vertice K is at the origin we can calculate the abscissas C_x, F_x and ordinates C_y, F_y of vertices C and F using as factors $c = \frac{\overline{IC}}{\overline{IL_2}} = \frac{10}{16} = \frac{5}{8}$ and $f = \frac{\overline{KF}}{\overline{KH_1}} = \frac{4}{6} = \frac{2}{3}$:

$$C_x = +c(\overline{KL_2}) = \frac{5}{8}(6\sqrt{7}) = \frac{15}{4}\sqrt{7} \quad (68)$$

$$F_x = -f(\overline{JH_1}) = -\frac{2}{3}\sqrt{35} \quad (69)$$

$$C_y = +(\overline{KI}) - c(\overline{KI}) = 2 - \frac{5}{8}(2) = \frac{3}{4} \quad (70)$$

$$F_y = +f(\overline{KJ}) = \frac{2}{3}(1) = \frac{2}{3} \quad (71)$$

Finally we calculate the distance \overline{CF} :

$$\begin{aligned} \overline{CF} &= \sqrt{(C_x - F_x)^2 + (C_y - F_y)^2} \\ &= \sqrt{\left(\frac{15}{4}\sqrt{7} + \frac{2}{3}\sqrt{35}\right)^2 + \left(\frac{3}{4} - \frac{2}{3}\right)^2} = \sqrt{114 + 35\sqrt{5}} \quad \blacksquare \end{aligned} \quad (72)$$

A minimal part with five strips of the construction of figure 25 including only vertices F, H, I, J, K, L, C is used twice to make rigid the pentagon of side 10 as show in figure 24.

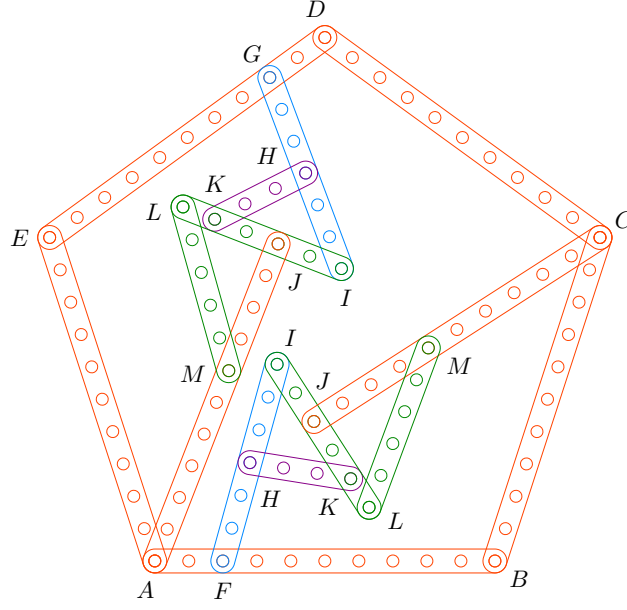


Figure 26: Pentagon size 10 with 10 internal strips. $\overline{FB} : \overline{BC} = 8 : 10$ and $\overline{CF} = 2\sqrt{31 + 10\sqrt{5}}$.

Figure 26 show a rigid regular pentagon A, B, C, D, E of size 10. We include here the relation $\overline{FB} : \overline{BC} = 8 : 10$ since the relation $4 : 5$ for pentagon size 5 gave clusters that can't fit inside such pentagon. With the law of cosines we calculate \overline{CF} and the angle $\angle CFB$ for the drawing:

$$\begin{aligned}\overline{CF}^2 &= \overline{BC}^2 + \overline{BF}^2 - 2(\overline{BC})(\overline{BF}) \cos\left(\frac{3\pi}{5}\right) \\ &= 10^2 + 8^2 - 2(10)(8) \left(\frac{1 - \sqrt{5}}{4}\right) = 124 + 40\sqrt{5} \\ \overline{CF} &= 2\sqrt{31 + 10\sqrt{5}}\end{aligned}\tag{73}$$

$$\cos(\angle CFB) = \frac{\overline{CF}^2 + \overline{BF}^2 - \overline{BC}^2}{2(\overline{CF})(\overline{BF})} = \frac{124 + 40\sqrt{5} + 8^2 - 10^2}{2(2\sqrt{31 + 10\sqrt{5}})(8)} = \frac{11 + 5\sqrt{5}}{4\sqrt{31 + 10\sqrt{5}}}\tag{74}$$

8.1.2 Rigid distance $2\sqrt{31 + 10\sqrt{5}}$

One solution from our software is shown in figure as cluster $FHIJKLM$. Assume vertex J is at the origin, vertex I at $(-2, 0)$ and vertex K at $(+2, 0)$. Since triangle $\triangle IKH$ is isoscelles and \overline{IF} is the double of \overline{IH} then angle $\angle FKI = \frac{\pi}{2}$ and we can calculate the abscissa and the ordinate of vertex F :

$$F_x = \overline{JK} = 2\tag{75}$$

$$F_y = -\sqrt{(\overline{IF})^2 - (\overline{IK})^2} = -\sqrt{6^2 - 4^2} = -2\sqrt{5}\tag{76}$$

Since we have the Pythagorean triangle $\triangle JLM$ is easy to note that the abscissa and ordinate of vertice C are $C_x = 0$ and $C_y = \overline{JC} = 10$, so finally we have:

$$\begin{aligned}\overline{CF} &= \sqrt{(F_x - C_x)^2 + (F_y - C_y)^2} \\ &= \sqrt{(2 - 0)^2 + (-2\sqrt{5} - 10)^2} = 2\sqrt{31 + 10\sqrt{5}} \quad \blacksquare\end{aligned}\tag{77}$$

9 Pentagon of size 11

9.1 Size 11 with 10 internal strips

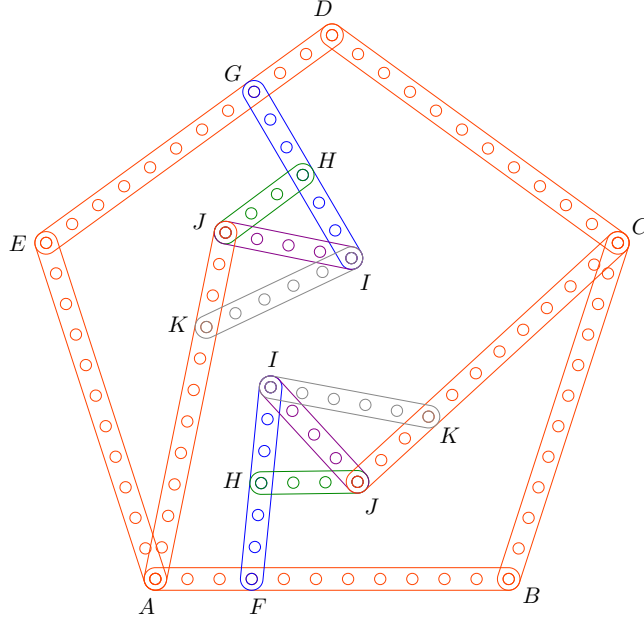


Figure 27: Pentagon size 11 with 10 internal strips. $\overline{FB} : \overline{BC} = 8 : 11$ and $\overline{CF} = 11 + 2\sqrt{5}$.

Figure 27 show a rigid regular pentagon A, B, C, D, E of size 11. Our software found this is the smallest pentagon having a consecutive sides diagonal distance of the form $\frac{z_2 + z_3\sqrt{5}}{z_1}$ instead of the nested form $\frac{z_2\sqrt{z_3 + z_4\sqrt{5}}}{z_1}$ where z_i are integers. The mentioned diagonal is the distance \overline{CF} in the figure which can be calculated with the law of cosines knowing angle $\angle CBF = \frac{3\pi}{5}$ and denesting the result. We calculate the angle $\angle CFB$ for the drawing:

$$\begin{aligned}\overline{CF}^2 &= \overline{BC}^2 + \overline{BF}^2 - 2(\overline{BC})(\overline{BF})\cos\left(\frac{3\pi}{5}\right) \\ &= 11^2 + 8^2 - 2(11)(8)\left(\frac{1 - \sqrt{5}}{4}\right) = 141 + 44\sqrt{5} \\ \overline{CF} &= \sqrt{141 + 44\sqrt{5}} \\ &= 11 + 2\sqrt{5}\end{aligned}\tag{78}$$

$$\cos(\angle CFB) = \frac{\overline{CF}^2 + \overline{BF}^2 - \overline{BC}^2}{2(\overline{CF})(\overline{BF})} = \frac{141 + 44\sqrt{5} + 8^2 - 11^2}{2(11 + 2\sqrt{5})(8)} = \frac{21 + 11\sqrt{5}}{44 + 8\sqrt{5}} = \frac{121 + 79\sqrt{5}}{404}\tag{79}$$

9.1.1 Rigid instance $11 + 2\sqrt{5}$

A five strips cluster can create a rigid distance like $11 + 2\sqrt{5}$. In the figure, three strips $\overline{FI} = 2\overline{HJ}$, $\overline{FI} > \overline{IJ}$ builds a right angle $\angle FJI = \frac{\pi}{2}$, since triangle $\triangle IJH$ is isosceles ($\overline{FH} = \overline{HI} = \overline{JH}$). These three strips also build a distance $\overline{FJ} = \sqrt{\overline{FI}^2 - \overline{IJ}^2} = \sqrt{6^2 - 4^2} = 2\sqrt{5}$. Now we attach strip \overline{CJ} making a second right triangle $\angle CJI = \frac{\pi}{2}$ using strip $\overline{IK} = 5$ as pythagorean diagonal ($\overline{JK} = 3, \overline{IJ} = 4$). We

have two right triangles at vertex J so vertices F, J, C are collinear, so we can calculate the distance $\overline{FC} = \overline{CJ} + \overline{JF} = 11 + 2\sqrt{5}$ ■.

We repeat the five-strips cluster between vertices A, G preventing strips overlaps. Since the clusters are rigid we formed two rigid triangles $\triangle ABC, \triangle DEA$ so the pentagon is rigid.

The software found the next pentagon of this type is a lot bigger: $\overline{BC} = 246, \overline{BF} = 70, \overline{CF} = 41 + 105\sqrt{5}$.

10 Pentagons of size 12

10.1 Size 12 with 4 internal strips

Our software found that side 12 is the smallest pentagon that can be made rigid with a rhombus and two strips as diagonals so need only 4 strips as diagonals. We show two cases.

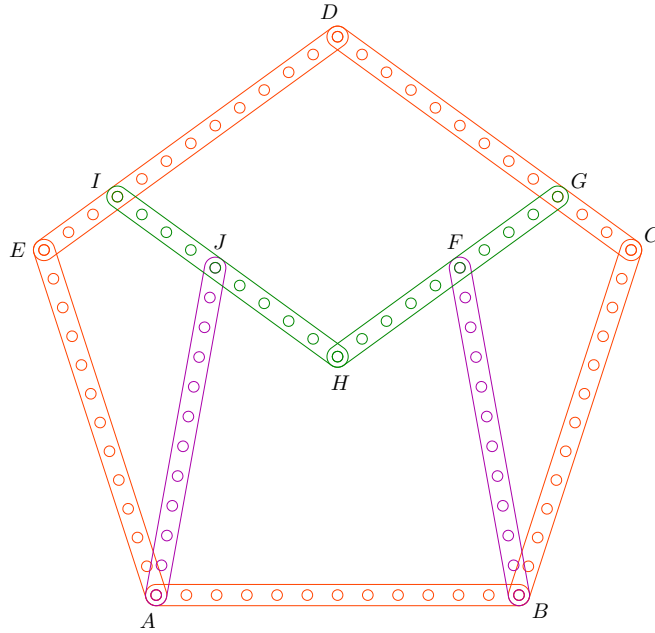


Figure 28: Pentagon size 12 with 4 internal strips. $\overline{AA} : \overline{AE} : \overline{EI} : \overline{IJ} = 0 : 12 : 3 : 4$ and $\overline{AJ} = 11$.

Figure 28 show a regular pentagon A, B, C, D, E of side 12 with a rhombus D, I, H, G of side 9. We prove strips AJ, BF are correct. First we calculate the abscissas going through vertices A, E, I, J subtracting when we move to the left and adding when we move to the right:

$$\begin{aligned}
 AJ_x &= AE_x + EI_x + IJ_x \\
 &= -\overline{AE} \cos\left(\frac{2\pi}{5}\right) + \overline{EI} \cos\left(\frac{\pi}{5}\right) + \overline{IJ} \cos\left(\frac{\pi}{5}\right) \\
 &= -12 \left(\frac{\sqrt{5}-1}{4}\right) + 3 \left(\frac{1+\sqrt{5}}{4}\right) + 4 \left(\frac{1+\sqrt{5}}{4}\right) = \frac{19-5\sqrt{5}}{4}
 \end{aligned} \tag{80}$$

Then we calculate the ordinates going to the same order of vertices adding when we go up and sub-

stracting when we go down:

$$\begin{aligned}
AJ_y &= -AE_y + EI_y + IJ_y \\
&= \overline{AE} \sin\left(\frac{2\pi}{5}\right) + \overline{EI} \sin\left(\frac{\pi}{5}\right) - \overline{IJ} \sin\left(\frac{\pi}{5}\right) \\
&= 12 \left(\frac{\sqrt{10+2\sqrt{5}}}{4} \right) + 3 \left(\frac{\sqrt{10-2\sqrt{5}}}{4} \right) - 4 \left(\frac{\sqrt{10-2\sqrt{5}}}{4} \right) = \frac{\sqrt{1450+190\sqrt{5}}}{4}
\end{aligned} \tag{81}$$

Finally we calculate the distance \overline{AJ} which coincides with strip size 11:

$$\begin{aligned}
\overline{AJ} &= \sqrt{(AJ_x)^2 + (AJ_y)^2} \\
&= \sqrt{\left(\frac{19-5\sqrt{5}}{4} \right)^2 + \frac{1450+190\sqrt{5}}{16}} = \sqrt{121} = 11 \quad \blacksquare
\end{aligned} \tag{82}$$

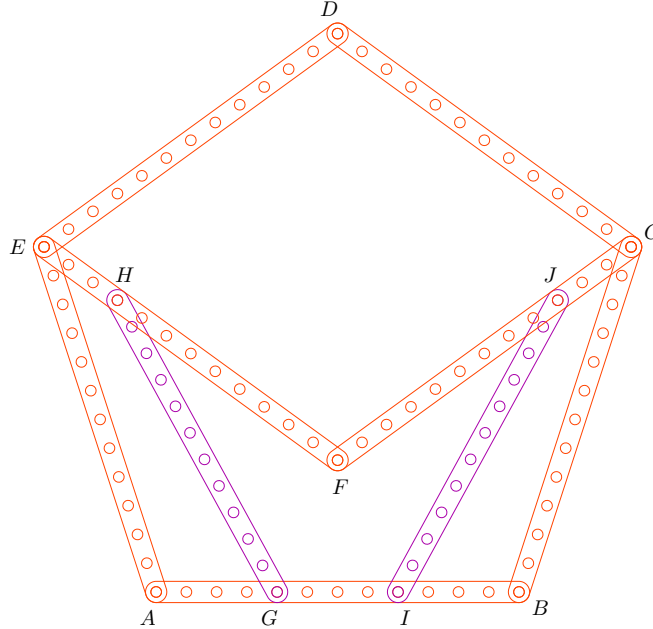


Figure 29: Pentagon size 12 with 4 internal strips. $\overline{GA} : \overline{AE} : \overline{EE} : \overline{EH} = 4 : 12 : 0 : 3$ and $\overline{GH} = 11$.

Figure 29 show a regular pentagon A, B, C, D, E of size 12 with a rhombus D, I, H, G of size 12. We prove strips GH, IJ are correct. First we calculate the abscissas going through vertices G, A, E, H subtracting when we move to the left and adding when we move to the right:

$$\begin{aligned}
GH_x &= -GA_x - AE_x + EH_x \\
&= -\overline{GA} - \overline{AE} \cos\left(\frac{2\pi}{5}\right) + \overline{EH} \cos\left(\frac{\pi}{5}\right) \\
&= -4 - 12 \left(\frac{\sqrt{5}-1}{4} \right) + 3 \left(\frac{1+\sqrt{5}}{4} \right) = \frac{-1-9\sqrt{5}}{4}
\end{aligned} \tag{83}$$

Then we calculate the ordinates going to the same order of vertices adding when we go up and sub-

stracting when we go down:

$$\begin{aligned}
GH_y &= AG_y + AE_y - EH_y \\
&= 0 + \overline{AE} \sin\left(\frac{2\pi}{5}\right) - \overline{EH} \sin\left(\frac{\pi}{5}\right) \\
&= 12 \left(\frac{\sqrt{10+2\sqrt{5}}}{4} \right) - 3 \left(\frac{\sqrt{10-2\sqrt{5}}}{4} \right) = \frac{\sqrt{1530-18\sqrt{5}}}{4}
\end{aligned} \tag{84}$$

Finally we calculate the distance \overline{GH} which coincides with strip size 11:

$$\begin{aligned}
\overline{GH} &= \sqrt{(GH_x)^2 + (GH_y)^2} \\
&= \sqrt{\left(\frac{-1-9\sqrt{5}}{4}\right)^2 + \frac{1530-18\sqrt{5}}{16}} = \sqrt{121} = 11 \quad \blacksquare
\end{aligned} \tag{85}$$

10.2 Size 12 with 6 internal strips

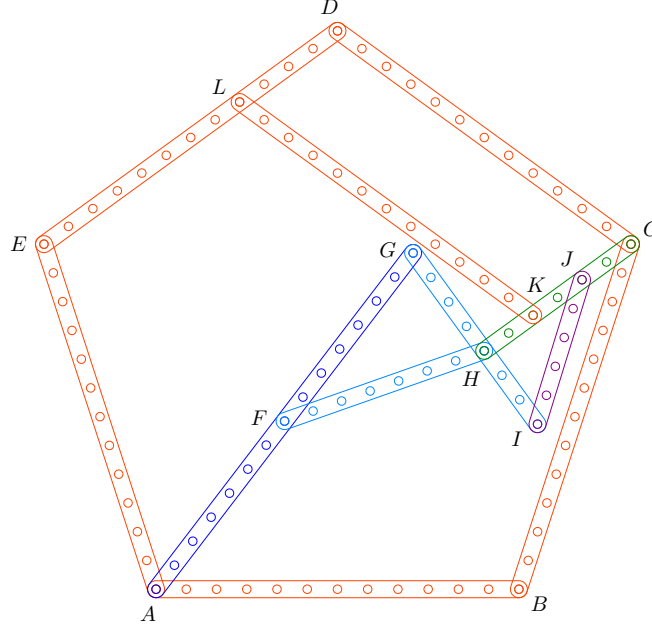


Figure 30: Pentagon size 12 with 6 internal strips. $\overline{AE} : \overline{ED} : \overline{DC} = 12 : 12 : 12$ and $\overline{AC} = 6 + 6\sqrt{5}$

Figure 30 show a regular pentagon A, B, C, D, E of side 12. We know the regular pentagon diagonal for size 12 is $\overline{AC} = 12 \left(\frac{1+\sqrt{5}}{2} \right) = 6 + 6\sqrt{5}$.

10.2.1 Rigid distance $6 + 6\sqrt{5}$

We show the five strips $\overline{GH}, \overline{GI}, \overline{HF}, \overline{HC}, \overline{IJ}$ make the diagonal rigid which makes rigid the angle $\angle ABC$ of the pentagon. We have an isoscelles triangle $\triangle FGH$ and \overline{AG} is two times \overline{FG} so we have a right angle $\angle AHG = \frac{\pi}{2}$ and we can calculate $\overline{AH} = \sqrt{(\overline{AG})^2 - (\overline{GH})^2} = \sqrt{14^2 - 4^2} = 6\sqrt{5}$. Now we have another right angle $\angle IHC = \frac{\pi}{2}$ because the Pythagoras triangle $\triangle HIJ$. Since G, H, I are collinear then we have

another right angle $\angle GHC = \frac{\pi}{2}$. Both right angles $\angle AHG, \angle CHG$ guaranty vertices A, H, C are collinear and we can calculate $\overline{AC} = \overline{AH} + \overline{HC} = 6 + 6\sqrt{5}$ ■.

Finally we add a sixth strip \overline{KL} parallel to \overline{CD} to make rigid the last three perimeter strips $\overline{CD}, \overline{DE}, \overline{EA}$ of the pentagon.

10.3 Size 12 with 8 internal strips

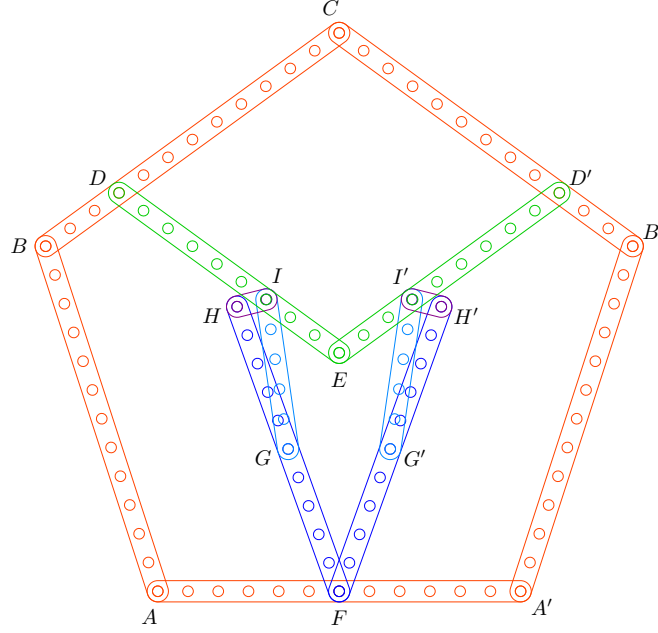


Figure 31: Pentagon size 12 with 8 internal strips. $\overline{FA} : \overline{AB} : \overline{BD} : \overline{DI} = 6 : 12 : 3 : 6$ and $\overline{FI} = 3\sqrt{11}$

Figure 31 show a regular pentagon $AA'B'CB$ of side 12. First we calculate the distance \overline{FI} using the abscissas and ordinates following the vertices F, A, B, D, I for a regular pentagon angles $\alpha = \frac{3\pi}{5}, \beta = \frac{\pi}{5}$:

$$\begin{aligned} FI_x &= -\overline{AF} - \overline{AB} \cos \alpha + (\overline{BD} + \overline{DI}) \cos \beta \\ &= -6 + (12) \frac{1 - \sqrt{5}}{4} + (3 + 6) \frac{\sqrt{5} + 1}{4} = -\frac{3 + 3\sqrt{5}}{4} \end{aligned} \quad (86)$$

$$\begin{aligned} FI_y &= \overline{AB} \sin \alpha + (\overline{BD} - \overline{DI}) \sin \beta \\ &= (12) \frac{\sqrt{10 + 2\sqrt{5}}}{4} + (3 - 6) \frac{\sqrt{10 - 2\sqrt{5}}}{4} = \frac{12\sqrt{10 + 2\sqrt{5}} - 3\sqrt{10 - 2\sqrt{5}}}{4} \end{aligned} \quad (87)$$

$$\begin{aligned} \overline{FI} &= \sqrt{(FI_x)^2 + (FI_y)^2} \\ &= \frac{\sqrt{(-3 - 3\sqrt{5})^2 + (12\sqrt{10 + 2\sqrt{5}} - 3\sqrt{10 - 2\sqrt{5}})^2}}{4} = \frac{\sqrt{1584}}{4} = 3\sqrt{11} \end{aligned} \quad (88)$$

10.3.1 Rigid distance $3\sqrt{11}$

Finally we calculate the distance \overline{FI} made rigid by cluster F, G, H, I . We have an isoscelles triangle $\triangle GHI$ and $\overline{FH} = 2\overline{GH}$ so we have a right triangle $\angle FHI = \frac{\pi}{2}$ so:

$$\begin{aligned}\overline{FI} &= \sqrt{(\overline{FH})^2 - (\overline{HI})^2} \\ &= \sqrt{10^2 - 1^2} = 3\sqrt{11}\end{aligned}\tag{89}$$

10.4 Size 12 with 10 internal strips

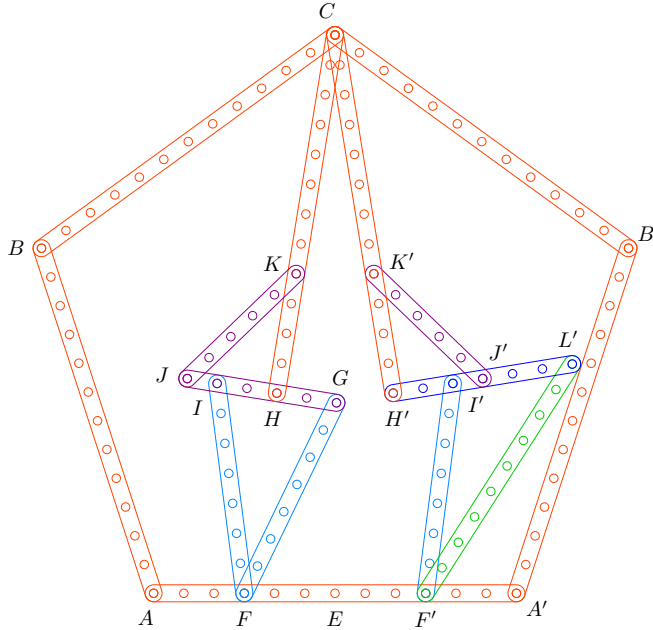


Figure 32: Pentagon size 12 with 10 internal strips. $\overline{FA} : \overline{AB} : \overline{BC} = 3 : 12 : 12$ and $\overline{CF} = 12 + 3\sqrt{5}$.

Figure 32 show a regular pentagon $AA'B'CB$ of side 12. We know the regular pentagon height is $\frac{\sqrt{5+2\sqrt{5}}}{2}$ times the side. So here we have $\overline{CE} = 12 \frac{\sqrt{5+2\sqrt{5}}}{2} = 6\sqrt{5+2\sqrt{5}}$ and we can calculate \overline{CF} :

$$\begin{aligned}\overline{CF} &= \sqrt{\overline{CE}^2 + \overline{EF}^2} \\ &= \sqrt{36(5 + 2\sqrt{5}) + 3^2} = 3\sqrt{21 + 8\sqrt{5}} \\ &= 3(4 + \sqrt{5})\end{aligned}\tag{90}$$

10.4.1 Rigid distance $3(4 + \sqrt{5})$

After testing $\overline{AA'} \leq 1800$ our software found that the last denesting is somehow special since other fractions $\frac{\overline{AF}}{\overline{AA'}} \neq \frac{1}{4}$ generated \overline{CF} s that can't be denested.

We have the Pythagorean triangle $\triangle HJK$ and the isoscelles $\triangle FGI$ so vertices FHC are collinear. First we calculate $\overline{FH} = \sqrt{FG^2 - GH^2} = \sqrt{7^2 - 2^2} = 3\sqrt{5}$ and then $\overline{FC} = \overline{FH} + \overline{HC} = 3\sqrt{5} + 12$ matching last calculation. Finally we prove angle $\angle F'H'L' = \frac{\pi}{2}$ noting $\overline{F'H'} = \sqrt{(\overline{F'L'})^2 - (\overline{H'L'})^2} = \sqrt{9^2 - 6^2} = 3\sqrt{5}$ matching \overline{FH} ■.