Meccano frames

https://github.com/heptagons/meccano/frames

Abstract

Meccano frames are groups of rigid meccano [1] strips. Can be used as internal diagonals of polygons to be rigid. The lengths of such diagonals are algebraic numbers of the form $B + \frac{C\sqrt{D}}{A}$ or $\frac{\sqrt{F+H\sqrt{G}}}{A}$.

1 Triangular frame

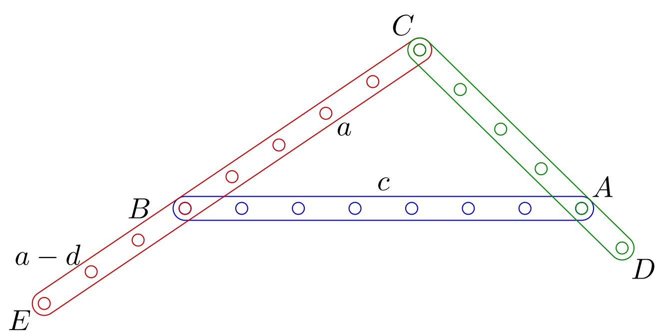

Figure 1: Triangular frame.

Figure 1 shows a triangular frame. With three strips we form the triangle $\triangle ABC$. At least we extend one of the two strips $\overline{CB}$ and $\overline{CA}$ to become $\overline{CE}$ and $\overline{CD}$. The new vertices D and E distance is rigid of the form $\dfrac{p\sqrt{s}}{q}$, where $p, q, s \in \mathbb{Z}^+$.

First we identify five integer distances a, b, c, d, e:

$$a \equiv \overline{CB}, \quad b \equiv \overline{CA}, \quad c \equiv \overline{AB}, \quad c < a + b \tag{1}$$

$$d \equiv \overline{CB} + \overline{BE} = \overline{CE} \geq a \tag{2}$$

$$e \equiv \overline{CA} + \overline{AD} = \overline{CD} \geq b \tag{3}$$

We calculate the cosine of $\angle BCA$:

$$\theta \equiv \angle BCA \tag{4}$$

$$\cos\theta = \frac{a^2 + b^2 - c^2}{2ab} \tag{5}$$

Then we apply the cosine to the triangle $\triangle CED$ to get the extensions distance $\overline{DE}$:

$$\begin{aligned} \overline{DE}^2 &= \overline{CD}^2 + \overline{CE}^2 - 2\overline{CD} \times \overline{CE}\cos\theta \\ &= d^2 + e^2 - 2de\cos\theta \\ &= d^2 + e^2 - de\left(\frac{a^2 + b^2 - c^2}{ab}\right) \end{aligned} \tag{6}$$

[1] Meccano mathematics by ‘t Hooft

We extract the square root:

$$
\begin{aligned}
\overline{DE} &= \sqrt{d^2 + e^2 - de\left(\frac{a^2+b^2-c^2}{ab}\right)} \\
&= \frac{\sqrt{a^2b^2(d^2+e^2) - abde(a^2+b^2-c^2)}}{ab} \\
&= \frac{\sqrt{ab((ad-be)(bd-ae)+c^2de)}}{ab}
\end{aligned} \tag{7}
$$

1.1 Software

We write a software to report all the triangle frames with specific surd $\sqrt{s}$ for a given maximum strips length. We can reject cases $q \neq 1$ and s not square-free. Next list show all the triangles with $q = 1$ and $s = \sqrt{7}$ where $c < a + b$, $a \leq d \leq max$, $b \leq e \leq max$, $c \leq max$:

```
=== RUN   TestFramesTriangleSurds
NewFrames().TriangleSurds surd=7 max=15
  1) a=1 e=1+2 c=1 cos=1/2
  2) d=1+1 e=1+2 c=1 cos=1/2
  3) d=1+2 b=1 c=1 cos=1/2
  4) d=1+2 e=1+1 c=1 cos=1/2
  5) a=2 e=2+1 c=2 cos=1/2
  6) d=2+1 b=2 c=2 cos=1/2
  7) a=3 e=2+2 c=2 cos=3/4 CED=pi/2
  8) d=3+1 e=2+1 c=2 cos=3/4 CDE=pi/2
  9) d=4+2 e=4+4 c=1 cos=31/32
 10) d=4+4 e=4+2 c=1 cos=31/32
 11) a=7 e=5+1 c=3 cos=13/14
 12) a=7 e=5+2 c=3 cos=13/14
```

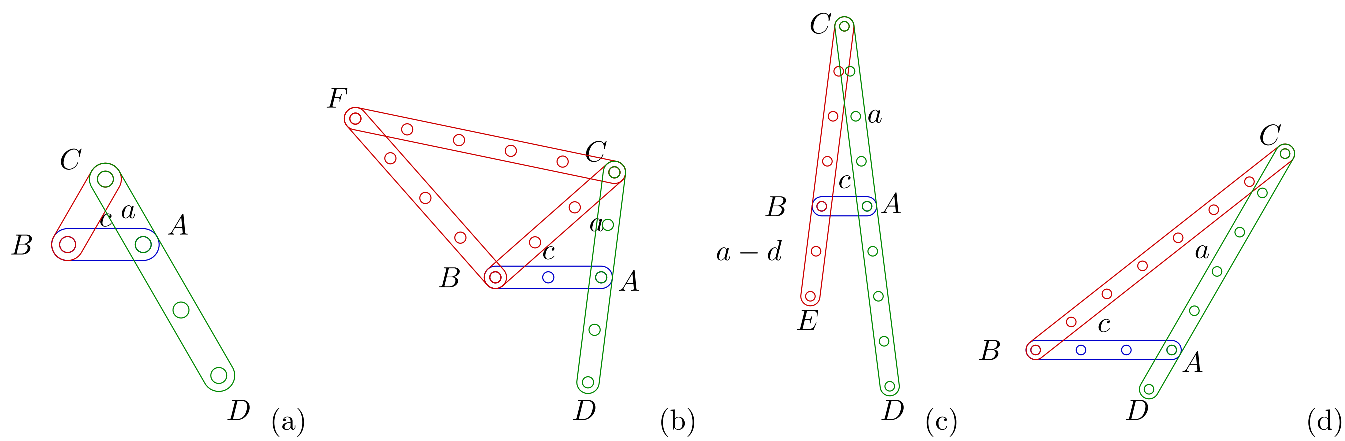

Figure 2: Some triangular frames with rigid distance $\overline{DE} = \sqrt{7}$ found by the software.

Figure 2 show four cases of this list. The code is in the folder `github.com/heptagons/meccano/frames`.

1.2 Triangular distance of the form $\sqrt{s} + f$

In the figure 2, the particular case (b), was reported with the angle $CED = \pi/2$ which means we can append two extra strips to make a pythagorean triangle $\triangle CEF$ where angle $CEF = \pi/2$, which makes the three vertices D, E, F collinear, so the rigid distance $\overline{DF} = \sqrt{7} + 4$ is an algebraic number.

1.3 Another rigid distances $\sqrt{s}+h$

We explore a more complicated frame to get additional cases of distances $\sqrt{s}+h$ without relying in an explicit pythagorean triangle as we saw in case (b) of figure 2.

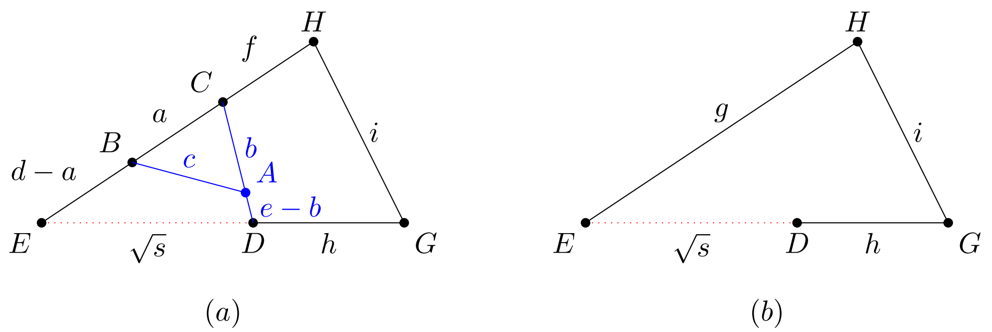

Figure 3: The five strips intented to form an algebraic distance $\overline{EG}=\sqrt{s}+h$.

From figure 3 (a) we know $\sqrt{s}$ distance between nodes E and D is produced by the three strips frame $a+d$, $b+e$ and c. Using the law of cosines we calculate the angle $\theta = \angle CED$ in terms of $\sqrt{s}$:

$$\begin{aligned}
\cos\theta &= \frac{d^2+(\sqrt{s})^2-e^2}{2d\sqrt{s}} \\
&= \frac{(d^2+s-e^2)\sqrt{s}}{2ds} && (8)\\
&= \frac{m\sqrt{s}}{n} && (9)\\
m &= d^2+s-e^2 && (10)\\
n &= 2ds && (11)
\end{aligned}$$

From figure 3 (a) we notice two sets of points are collinear: $\{E,B,C,H\}$ and $\{E,D,G\}$. Using the law of cosines we calculate the angle $\theta = \angle HEG$ in terms of distances $g, \sqrt{s}+h, i$:

$$\begin{aligned}
\cos\theta &= \frac{g^2+(\sqrt{s}+h)^2-i^2}{2g(\sqrt{s}+h)} \\
&= \frac{g^2+s+2\sqrt{s}h+h^2-i^2}{2g(\sqrt{s}+h)} \\
&= \frac{g^2+s+h^2-i^2+2\sqrt{s}h}{2g(\sqrt{s}+h)} && (12)
\end{aligned}$$

We multiply both numerator and denominator by $\sqrt{s} - h$ to eliminate the surd from denominator:

$$\begin{aligned}
\cos\theta &= \frac{(s + g^2 + h^2 - i^2)(\sqrt{s} - h) + 2\sqrt{s}h(\sqrt{s} - h)}{2g(\sqrt{s} + h)(\sqrt{s} - h)} \\
&= \frac{(s + g^2 + h^2 - i^2)(\sqrt{s} - h) + 2sh - 2\sqrt{s}h^2}{2g(s - h^2)} \\
&= \frac{-h(s + g^2 + h^2 - i^2 - 2s) + (s + g^2 + h^2 - i^2 - 2h^2)\sqrt{s}}{2g(s - h^2)} \\
&= \frac{h(s - g^2 - h^2 + i^2) + (s + g^2 - h^2 - i^2)\sqrt{s}}{2g(s - h^2)} \\
&= \frac{o + p\sqrt{s}}{q} \qquad (13) \\
o &= h(s - g^2 - h^2 + i^2) \qquad (14) \\
p &= s + g^2 - h^2 - i^2 \qquad (15) \\
q &= 2g(s - h^2) \qquad (16)
\end{aligned}$$

We compare both cosines equations 9 and 13:

$$\frac{m\sqrt{s}}{n} = \frac{o + p\sqrt{s}}{q} \qquad (17)$$

Since all variables are integers we need two conditions. First o should be zero. And second $\frac{m}{n} = \frac{p}{q}$.
For condition 1, we force o to be zero:

$$\begin{aligned}
o &= 0 \\
h(s - g^2 - h^2 + i^2) &= 0 \\
s &= g^2 + h^2 - i^2 \qquad (18)
\end{aligned}$$

For condition2, we force m, n, p, q as:

$$\begin{aligned}
\frac{m}{n} &= \frac{p}{q} \\
\frac{d^2 + s - e^2}{2ds} &= \frac{s + g^2 - h^2 - i^2}{2g(s - h^2)}
\end{aligned} \qquad (19)$$

We replace the value of s of last equation RHS with the value of equation 18 of condition 1:

$$\begin{aligned}
\frac{d^2 - e^2 + s}{ds} &= \frac{s + g^2 - h^2 - i^2}{g(s - h^2)} \\
&= \frac{g^2 + h^2 - i^2 + g^2 - h^2 - i^2}{g(g^2 + h^2 - i^2 - h^2)} \\
&= \frac{2(g^2 - i^2)}{g(g^2 - i^2)} \\
&= \frac{2}{g} \\
(d^2 - e^2 + s)g &= 2ds \qquad (20)
\end{aligned}$$

$\boxed{TODO: Examples!!!}$

2 Triangle pair frame

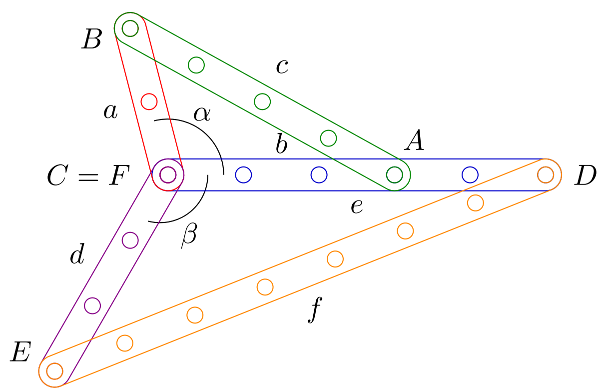

Figure 4: Triangle pair frame. We join triangles $\triangle ABC$ and $\triangle DEF$ in such a way vertices C and F coincide and vertices A, C, D, E be collinear. The result is a five strips frame. We are interested in the distance $\overline{BE}$.

Figure 4 shows a triangle pair frame. The triangles share a strip which contains four of the vertices. The remaining two vertices are separated by distances of the form $\frac{\sqrt{F+G\sqrt{H}}}{A}$. With only five strips this frame is small and useful to make up the diagonals inside polygons we want to be rigid.

2.1 Triangle pair algebra

First we calculate the cosines:

$$\cos\alpha = \frac{a^2+b^2-c^2}{2ab}$$
$$\cos\beta = \frac{d^2+e^2-f^2}{2de}$$

We define integers m, n, o, p to simplify cosines and get sines:

$$(m,n) \equiv (a^2+b^2-c^2, 2ab) \quad |m| \leq n \tag{21}$$
$$(o,p) \equiv (d^2+e^2-f^2, 2de) \quad |o| \leq p \tag{22}$$
$$\cos\alpha = \frac{m}{n} \tag{23}$$
$$\cos\beta = \frac{o}{p} \tag{24}$$
$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \frac{\sqrt{n^2-m^2}}{n} \tag{25}$$
$$\sin\beta = \sqrt{1-\cos^2\beta} = \frac{\sqrt{p^2-o^2}}{p} \tag{26}$$

Then, we use the cosines sum identity:

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(\frac{m}{n}\right)\left(\frac{o}{p}\right) - \left(\frac{\sqrt{n^2-m^2}}{n}\right)\left(\frac{\sqrt{p^2-o^2}}{p}\right) \\ &= \frac{mo-\sqrt{(n^2-m^2)(p^2-o^2)}}{np}\end{aligned} \tag{27}$$

Finally we can calculate the distance $g \equiv \overline{BE}$ using the law of cosines:

$$\begin{aligned}
g &\equiv \overline{BE} \\
&= \sqrt{a^2 + d^2 - 2ad\cos(\alpha + \beta)} \\
&= \sqrt{a^2 + d^2 - 2ad\left(\frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{np}\right)} \\
&= \sqrt{a^2 + d^2 - 2ad\left(\frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{4abde}\right)} \\
&= \sqrt{a^2 + d^2 - \frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{2be}} \\
&= \frac{\sqrt{4b^2e^2(a^2 + d^2) - 2bemo + 2be\sqrt{(n^2 - m^2)(p^2 - o^2)}}}{2be} \qquad (28)
\end{aligned}$$

For the software we can define integers A, F, G, H to calculate and reduce g:

$$A \equiv 2be \qquad (29)$$

$$F \equiv A^2(a^2 + d^2) - Amo \qquad (30)$$

$$G \equiv A \qquad (31)$$

$$H \equiv (n^2 - m^2)(p^2 - o^2) \qquad (32)$$

$$g = \frac{\sqrt{F + G\sqrt{H}}}{A} \qquad (33)$$

2.2 Triangle pairs software

We run a program to inspect triangle pairs having a given distance g. The software iterates over the two triangles sides (a, b, c) and (d, e, f) up to a maximum strip length.

Next example request distances of the form $\sqrt{46 + 18\sqrt{5}}$ up to strip length 10:

$Folder$: `github.com/heptagons/meccano/frames`
$Call$: `NewFrames().TrianglePairsTex(10, [46 18 5])`

$(a,b,c) \oplus (d,e,f) \mapsto g$
$(2,1,2) \oplus (3,3,3) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(2,1,2) \oplus (3,8,7) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(2,2,2) \oplus (3,6,6) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(2,3,4) \oplus (3,5,7) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(2,4,4) \oplus (3,8,7) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(3,3,3) \oplus (2,4,4) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(4,2,4) \oplus (6,6,6) \mapsto \sqrt{46+18\sqrt{5}}$
$(4,4,4) \oplus (6,7,8) \mapsto \sqrt{46+18\sqrt{5}}$
$(6,3,6) \oplus (4,4,4) \mapsto \sqrt{46+18\sqrt{5}}$
$(6,3,6) \oplus (9,9,9) \mapsto \dfrac{3\sqrt{46+18\sqrt{5}}}{2}$
$(6,6,6) \oplus (4,8,8) \mapsto \sqrt{46+18\sqrt{5}}$
$(6,7,8) \oplus (9,9,9) \mapsto \dfrac{3\sqrt{46+18\sqrt{5}}}{2}$

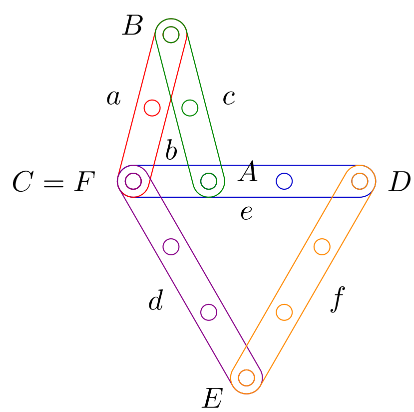

Figure 5: Triangle pair frame $(2,1,2) \oplus (3,3,3)$ makes $\overline{BE} = \dfrac{\sqrt{46+18\sqrt{5}}}{2}$.

3 Triangle pair extended frame

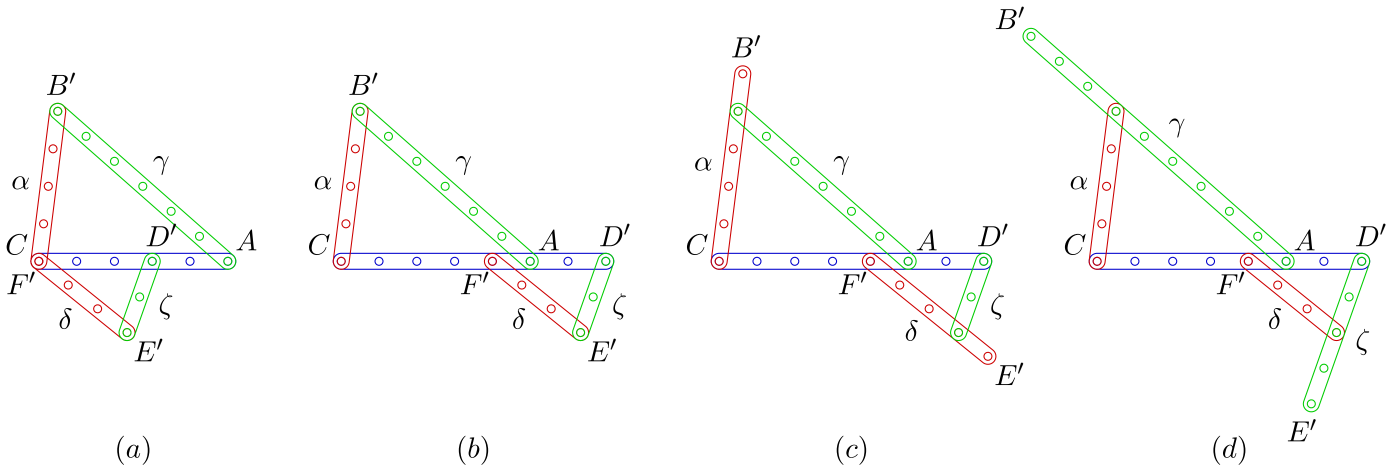

Figure 6: Triangle pair extended frame. Is like previous triangle pair frame except we can extend strips α or γ, δ or ζ, and we can separate vertices C and F'. Vertices A, C, D', F' remain collinear and we are interested in the distance $\overline{B'E'}$. We show four examples: (a) is the original triangle pair, (b) has moved the $\triangle D'E'F'$ to the right, (c) also extends strips α and δ and (d) extends strips γ and ζ.

The triangle pair extended frame is shown in figure 6. As with figure 4 we also have two triangles with five strips, but we can do up to three transformations:

1. Separate nodes C and F which moves $\triangle D'E'F'$.
2. Extends strip $a \to \alpha$ or strip $c \to \gamma$ but not both.
3. Extend strip $d \to \delta$ or strip $f \to \zeta$ but not both.

We define three integers x, y_1, y_2 to do the transformations:

$$x = \begin{cases} 0 & C, F \text{ vertices remain joined} \\ \geq 0 & \triangle DEF \text{ is moved to the right a distance equal to } x \end{cases} \tag{34}$$

$$y_1 = \begin{cases} 0 & \alpha = a, \quad \gamma = c \\ > 0 & \alpha = a + y_1, \quad \gamma = c \\ < 0 & \alpha = a, \quad \gamma = c + |y_1| \end{cases} \tag{35}$$

$$y_2 = \begin{cases} 0 & \delta = d, \quad \zeta = f \\ > 0 & \delta = d + y_2, \quad \zeta = f \\ < 0 & \delta = d, \quad \zeta = f + |y_2| \end{cases} \tag{36}$$

Let define $M(a, b, c)$ the triangle above, $N(d, e, f)$ the triangle below and $T(x, y_1, y_2)$ the transformations. Then we can describe the cases $(a) - (d)$ of figure 6 as operations:

$$(a) : M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(0, 0, 0)$$
$$(b) : M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(4, 0, 0)$$
$$(c) : M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(4, +2, +1)$$
$$(d) : M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(4, -3, -2)$$

3.1 Triangle pair extended frame algebra

We are going to calculate the distance $\overline{B'E'}$ of the triangle pair extended using the M, N, T values. We start setting the vertex C at the origin of the standard two-dimensional graph and defining (B_x, By) the abscissa and ordinate of vertex B' and (E_x, E_y) the abscissa and ordinate of vertex E'.

For the triangle above $M(a, b, c)$ we make $m = |y_1|$ and obtain (B_x, B_y):

$$a_1 \equiv b^2 + c^2 - a^2, \quad a_2 \equiv 2bc, \quad \text{iff } |a_1| \le a_2 \tag{37}$$

$$c_1 \equiv a^2 + b^2 - c^2, \quad c_2 \equiv 2ab, \quad \text{iff } |c_1| \le c_2 \tag{38}$$

$$\cos A = \frac{a_1}{a_2}, \quad \sin A = \frac{\sqrt{a_2^2 - a_1^2}}{a_2} \tag{39}$$

$$\cos C = \frac{c_1}{c_2}, \quad \sin C = \frac{\sqrt{c_2^2 - c_1^2}}{c_2} \tag{40}$$

$$\alpha = a + m \tag{41}$$

$$\gamma = c + m \tag{42}$$

$$B_x = \begin{cases} y_1 \ge 0 & \alpha \cos C \\ y_1 < 0 & b - \gamma \cos A \end{cases} \tag{43}$$

$$B_y = \begin{cases} y_1 \ge 0 & \alpha \sin C \\ y_1 < 0 & \gamma \sin A \end{cases} \tag{44}$$

For the triangle below $N(d, e, f)$ we make $n = |y_2|$ and obtain (E_x, E_y):

$$d_1 \equiv e^2 + f^2 - d^2, \quad d_2 \equiv 2ef \tag{45}$$

$$f_1 \equiv d^2 + e^2 - f^2, \quad f_2 \equiv 2de \tag{46}$$

$$\cos D = \frac{d_1}{d_2}, \quad \sin D = \frac{\sqrt{d_2^2 - d_1^2}}{d_2} \tag{47}$$

$$\cos F = \frac{f_1}{f_2}, \quad \sin F = \frac{\sqrt{f_2^2 - f_1^2}}{f_2} \tag{48}$$

$$\delta = d + n \tag{49}$$

$$\zeta = f + n \tag{50}$$

$$E_x = \begin{cases} y_2 \ge 0 & x + \delta \cos F \\ y_2 < 0 & x + e - \zeta \cos D \end{cases} \tag{51}$$

$$E_y = \begin{cases} y_2 \ge 0 & -\delta \sin F \\ y_2 < 0 & -\zeta \sin D \end{cases} \tag{52}$$

With the four components B_x, B_y, E_x, E_y we can calculate $\overline{B'E'}$:

$$\overline{B'E'} = \sqrt{(B_x + E_x)^2 + (B_y + E_y)^2} \tag{53}$$

$$= \sqrt{(B_x^2 + B_y^2) + (E_x^2 + E_y^2) + 2B_xE_x + 2B_yE_y} \tag{54}$$

For $y_1 \geq 0$ and $y_2 \geq 0$ we have $m = y_1, n = y_2$:

$$\alpha = a + m, \qquad \delta = d + n \tag{55}$$

$$\begin{aligned} B_x^2 + B_y^2 &= \alpha^2 \cos^2 C + \alpha^2 \sin^2 C \\ &= \alpha^2 \end{aligned} \tag{56}$$

$$\begin{aligned} E_x^2 + E_y^2 &= (x + \delta \cos F)^2 + (-\delta \sin F)^2 \\ &= x^2 + 2x\delta \cos F + \delta^2 \cos^2 F + \delta^2 \sin^2 F \\ &= x^2 + 2x\delta \cos F + \delta^2 \\ &= \frac{f_2 x^2 + 2x\delta f_1 + f_2\delta^2}{f_2} \end{aligned} \tag{57}$$

$$\begin{aligned} B_x E_x &= (\alpha \cos C)(x + \delta \cos F) \\ &= \frac{\alpha c_1 (f_2 x + \delta f_1)}{c_2 f_2} \end{aligned} \tag{58}$$

$$\begin{aligned} B_y E_y &= (\alpha \sin C)(-\delta \sin F) \\ &= -\frac{\alpha\delta\sqrt{(c_2^2 - c_1^2)(f_2^2 - f_1^2)}}{c_2 f_2} \end{aligned} \tag{59}$$

$$\begin{aligned} \overline{B'E'} &= \sqrt{\alpha^2 + \frac{f_2 x^2 + 2x\delta f_1 + f_2\delta^2}{f_2} + \frac{2\alpha c_1 (f_2 x + \delta f_1)}{c_2 f_2} - \frac{2\alpha\delta\sqrt{(c_2^2 - c_1^2)(f_2^2 - f_1^2)}}{c_2 f_2}} \\ &= \sqrt{\frac{c_2 f_2 \alpha^2 + c_2(f_2 x^2 + 2x\delta f_1 + f_2\delta^2) + 2\alpha c_1 (f_2 x + \delta f_1) - 2\alpha\delta\sqrt{(c_2^2 - c_1^2)(f_2^2 - f_1^2)}}{c_2 f_2}} \end{aligned} \tag{60}$$