Fox-face unit

https://github.com/heptagons/meccano/fox-face

Abstract

Fox-face is a group of five meccano [1] strips not forming implicit triangles but a fox-faced figure used to build a regular pentagon. Here, we'll look for other angles but not only pentagon's $\cos 2\pi/5$.

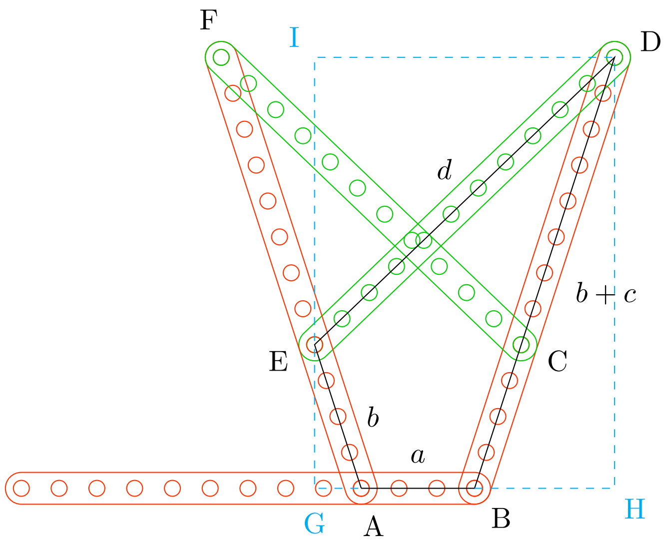

Figure 1: Fox-figure

Figure 1 show the so called fox-face unit. Has five strips of three types:

- Single $\overline{AB}$ of length a.
- Pair { $\overline{BD}$, $\overline{AF}$ } of length $b+c$.
- Pair { $\overline{DE}$, $\overline{CF}$ } of length d.

In other words we have four different distances:

- a distance of segment $\overline{AB}$.
- b distance of segments $\overline{BC}$ and $\overline{AE}$.
- c distance of segments $\overline{CD}$ and $\overline{EF}$.
- d distance of segments $\overline{DE}$ and $\overline{CF}$.

We are going to test several values of (a,b,c,d) and calculate the angle $\angle HBD$. First we'll calculate a formula and then we'll run a program iterating integer values.

[1] Meccano mathematics by 't Hooft

1 Algebra

From figure 1 we define $\theta = \angle HBD$ and figure out sines and cosines:

$$\theta \equiv \angle HBD = \angle GAE \tag{1}$$
$$\overline{BH} = (b+c)\cos\theta \tag{2}$$
$$\overline{DH} = (b+c)\sin\theta \tag{3}$$
$$\overline{AG} = b\cos\theta \tag{4}$$
$$\overline{EG} = b\sin\theta \tag{5}$$

We calculate d in function of (a, b, c):

$$\begin{aligned}
d^2 &= (\overline{DE})^2 \\
&= (\overline{DI})^2 + (\overline{EI})^2 \\
&= (\overline{AG} + \overline{AB} + \overline{BH})^2 + (\overline{DH} - \overline{EG})^2 && (6)\\
&= (b\cos\theta + a + (b+c)\cos\theta)^2 + ((b+c)\sin\theta - b\sin\theta)^2 && (7)\\
&= (a + (2b+c)\cos\theta)^2 + (c\sin\theta)^2) \\
&= a^2 + 2a(2b+c)\cos\theta + (2b+c)^2\cos^2\theta + c^2\sin^2\theta \\
&= a^2 + 2a(2b+c)\cos\theta + (4b^2+4bc+c^2)\cos^2\theta + c^2\sin^2\theta \\
&= a^2 + 2a(2b+c)\cos\theta + (4b^2+4bc)\cos^2\theta + c^2 \\
&= 4b(b+c)\cos^2\theta + 2a(2b+c)\cos\theta + a^2 + c^2 && (8)
\end{aligned}$$

Let do $X = \cos^2\theta$ so last equation can be written as:

$$4b(b+c)X^2 + 2a(2b+c)X + a^2 + c^2 - d^2 = 0 \tag{9}$$

So we can calculate $X = \cos^2\theta$ with the quadratic formula:

$$\begin{aligned}
\cos\theta &= \frac{-2a(2b+c) \pm \sqrt{4a^2(2b+c)^2 - 16b(b+c)(a^2+c^2-d^2)}}{8b(b+c)} \\
&= \frac{-a(2b+c) \pm \sqrt{a^2c^2 + 4b(b+c)(d^2-c^2)}}{4b(b+c)} && (10)
\end{aligned}$$

1.1 Test pentagon known case

Fox-face unit appears in the single solution found of the meccano regular pentagon type 1 construction [2].

In this case we have $a = 3$, $b = 4$, $c = 8$ and $d = 11$. Applying these values in the last equation we have:

$$\begin{aligned}
\cos\theta &= \frac{-48 \pm \sqrt{11520}}{192} \\
&= \frac{-1 \pm \sqrt{5}}{4} && (11)
\end{aligned}$$

Since $\cos 2\pi/5 = (\sqrt{5} - 1)/4$ the equation passes one test. In the next section, a program will find more angles changing the distances a, b, c, d.

2 Program

[2] Meccano pentagons