Meccano nonagon

https://github.com/heptagons/meccano/nona

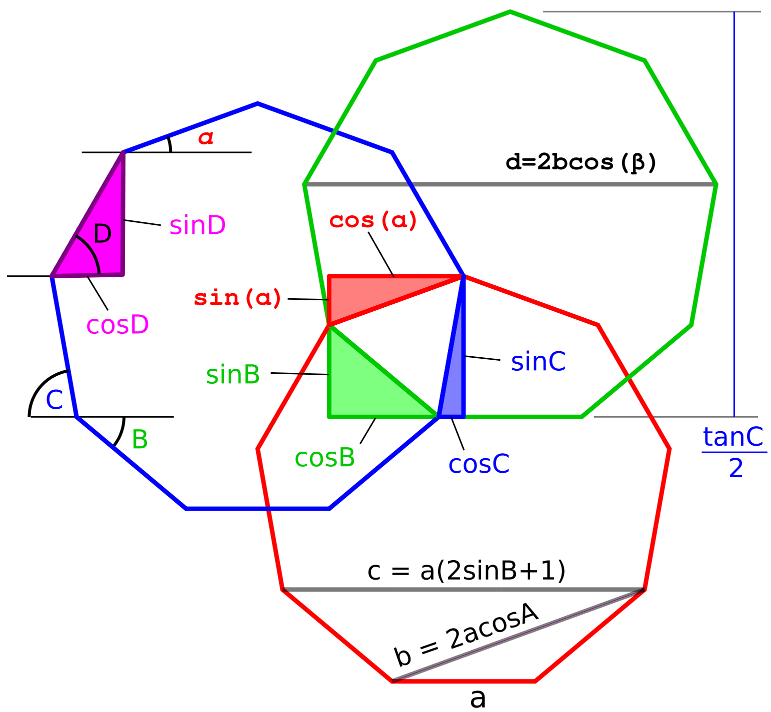

Figure 1: Three regular nonagons connected by an equilateral triangle. We note four angles in the figure A, B, C and D.

1 Algebra

Figure 1 shows three regular nonagons connected by an equilateral triangle. Four angles appear orthogonally in any regular nonagon:

$$\alpha = \pi/9 = 20^\circ \tag{1}$$
$$\beta = 2\pi/9 = 40^\circ \tag{2}$$
$$\gamma = 3\pi/9 = 60^\circ \tag{3}$$
$$\delta = 4\pi/9 = 80^\circ \tag{4}$$
$$\alpha + \beta = \delta - \alpha = \gamma \tag{5}$$

The relations of angle C are those of equilateral triangle:

$$\cos\gamma = -\frac{1}{2} \tag{6}$$
$$\sin\gamma = \frac{\sqrt{3}}{2} \tag{7}$$

From the figure 1, cosines of angles α, β, δ are related as:

$$\begin{aligned}
\cos\alpha &= \cos\beta + \cos\delta &(8)\\
&= \cos(2\alpha) + \cos(4\alpha)\\
&= (2\cos^2\alpha - 1) + (1 - 8\cos^2\alpha + 8\cos^4\alpha)\\
&= 8\cos^4\alpha - 6\cos^2\alpha\\
1 &= 8\cos^3\alpha - 6\cos\alpha &(9)
\end{aligned}$$

Previous cosines equation is the depressed cubic equation with a negative discriminant:

$$t^3 + pt + q = 0 \tag{10}$$
$$p = -\frac{3}{4} \tag{11}$$
$$q = -\frac{1}{8} \tag{12}$$
$$\Delta = \frac{q^2}{4} + \frac{p^3}{27} = -\frac{3}{64}$$

The negative discriminant means we have three real roots which can be found by a geometric interpretation:

$$\begin{aligned}
t_k &= 2\sqrt{-\frac{p}{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right) - k\frac{2\pi}{3}\right) && \texttt{for}\ k = 0,1,2.\\
&= \cos\left(\frac{1}{3}\arccos\left(\frac{1}{2}\right) - k\frac{2\pi}{3}\right) && \texttt{for}\ k = 0,1,2.\\
&= \cos\left(\frac{1}{3}\left(\frac{\pi}{3}\right) - k\frac{2\pi}{3}\right) && \texttt{for}\ k = 0,1,2.\\
t_0 &= \cos\left(\frac{\pi}{9}\right) && = \cos\alpha \approx +0.939692 &(13)\\
t_1 &= \cos\left(-\frac{2\pi}{9}\right) && = -\cos\beta \approx -0.766044 &(14)\\
t_2 &= \cos\left(-\frac{4\pi}{9}\right) && = -\cos\delta \approx -0.173648 &(15)
\end{aligned}$$

From equation 10 we know the product of roots squares is $-2p = \frac{3}{2}$:

$$\begin{aligned}
\cos^2\alpha + \cos^2\beta + \cos^2\delta &= \frac{3}{2} &(16)\\
1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\delta &= \frac{3}{2}\\
\sin^2\alpha + \sin^2\beta + \sin^2\delta &= \frac{3}{2} &(17)
\end{aligned}$$

From equation 10 we know the product of roots is $-q = \frac{1}{8}$ matching the "Morrie's law":

$$\begin{aligned}
\cos\alpha\cos\beta\cos\delta &= \frac{1}{8} && (18)\\
(1-\sin^2\alpha)(1-\sin^2\beta)(1-\sin^2\delta) &= \frac{1}{64}\\
(\sin\alpha\sin\beta)^2 + (\sin\alpha\sin\delta)^2 + (\sin\beta\sin\delta)^2 &= \frac{1}{64} - 1 + \sin^2\alpha + \sin^2\beta + \sin^2\delta + (\sin\alpha\sin\beta\sin\delta)^2\\
&= \frac{1}{64} - 1 + \frac{3}{2} + \left(\frac{\sqrt{3}}{8}\right)^2 = \left(\frac{9}{4}\right)^2 && (19)
\end{aligned}$$

From the figure 1, sines of angles α, β, δ are related as:

$$\begin{aligned}
\sin\alpha + \sin\beta &= \sin\delta && (20)\\
&= \sin(2\alpha + \beta)\\
&= \sin(2\alpha)\cos\beta + \cos(2\alpha)\sin\beta\\
&= (2\sin\alpha\cos\alpha)\cos\beta + (1 - 2\sin^2\alpha)\sin\beta\\
\sin\alpha &= \sin\alpha(2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta\\
2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta &= 1 && (21)\\
2\cos(\alpha+\beta) &= 2\cos\gamma = 1
\end{aligned}$$

Product of sines of angles α, β, δ is using equations 20 and 21:

$$\begin{aligned}
\sin\alpha\sin\beta\sin\delta &= \frac{1}{2}(2\cos\alpha\cos\beta - 1)(\sin\alpha + \sin\beta)\\
&= \sin(2\alpha)\sin\beta + \sin\alpha\sin(2\beta)\\
&= \frac{\sqrt{3}}{8} && (22)
\end{aligned}$$

Last equation solves this cubic equation:

$$\begin{aligned}
y^3 - \frac{3y}{4} - \frac{3}{8} &= 0\\
y_1 &= -\sin A \approx -0.342020\\
y_2 &= -\sin B \approx -0.642787\\
y_3 &= +\sin C \approx +0.984807
\end{aligned}$$

Cosines and sines relations are:

$$\begin{aligned}
\cos A\cos B - \sin A\sin B &= \frac{1}{2}\\
\frac{1}{\cos C} - \frac{\sqrt{3}}{\sin C} &= 4\\
\tan C - 4\sin C &= \sqrt{3}
\end{aligned}$$

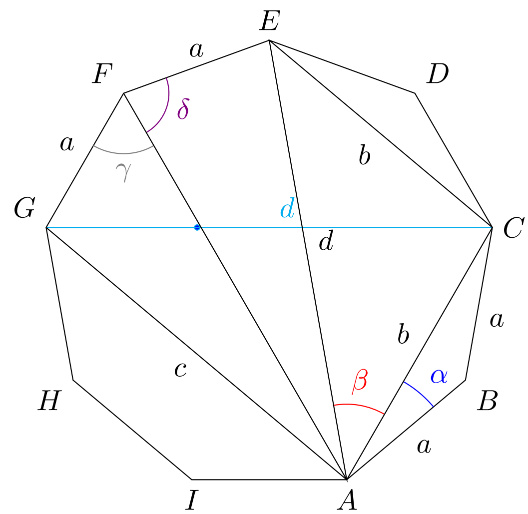

Figure 2: The nonagon perimeter side a and the three internal diagonals b, c, d. Also shown the base angle α and three more $\beta = 2\alpha$, $\gamma = 3\alpha$ and $\delta = 4\alpha$.

From figure 2 we can can calculate $\cos\alpha$ inspecting isosceles $\triangle ABC$ ($\overline{AB} = \overline{AC} = a$):

$$\begin{aligned} a^2 &= a^2 + b^2 - 2ab\cos\alpha \\ b^2 &= 2ab\cos\alpha \implies \boxed{b = 2a\cos\alpha} \end{aligned} \tag{23}$$

We calculate $\cos\beta$ inspecting isosceles $\triangle ACE$ ($\overline{AB} = \overline{AE} = b$):

$$\begin{aligned} b^2 &= b^2 + d^2 - 2bd\cos\beta \\ d^2 &= 2bd\cos\beta \implies \boxed{d = 2b\cos\beta} \end{aligned} \tag{24}$$

We calculate $\cos\delta$ inspecting isosceles $\triangle AEF$ ($\overline{AE} = \overline{AF} =$ d):

$$\begin{aligned} d^2 &= a^2 + d^2 - 2ad\cos\delta \\ a^2 &= 2ad\cos\delta \implies \boxed{a = 2d\cos\delta} \end{aligned} \tag{25}$$

From equations 23, 24 and 25 we have:

$$\cos\begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} = \begin{pmatrix} \dfrac{b}{2a} \\ \dfrac{d}{2b} \\ \dfrac{a}{2d} \end{pmatrix} \tag{26}$$

From equation 8 $\cos\alpha = \cos\beta + \cos\delta$:

$$\frac{b}{2a} = \frac{d}{2b} + \frac{a}{2d} \implies \boxed{\frac{1}{a/b} = \frac{1}{b/d} + \frac{1}{d/a}} \tag{27}$$