

Meccano nonagon

<https://github.com/heptagons/meccano/nona>

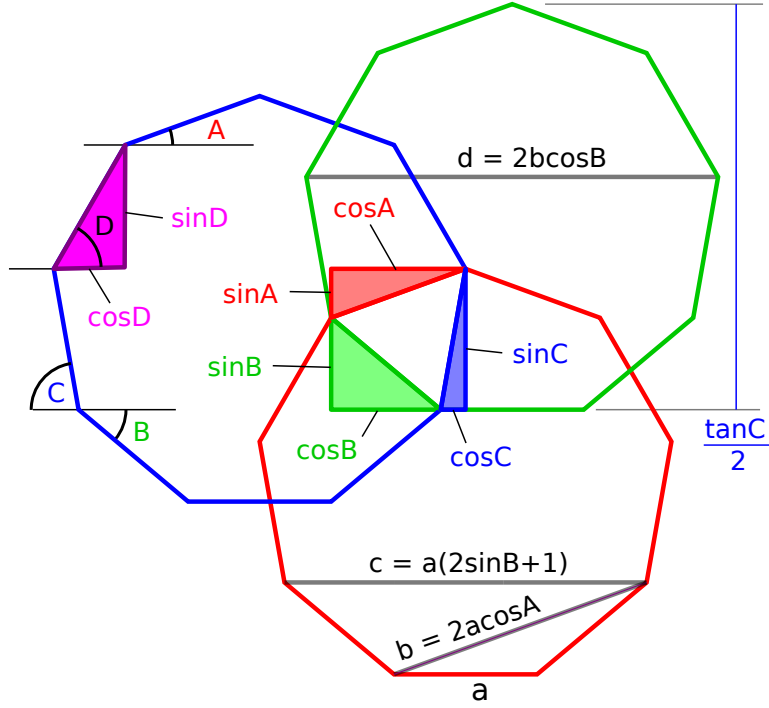


Figure 1: Three regular nonagons connected by an equilateral triangle. We note four angles in the figure A , B , C and D .

1 Algebra

Figure 1 shows three regular nonagons connected by an equilateral triangle. Four angles appear orthogonally in any regular nonagon:

$$A = \pi/9 = 20^\circ \quad (1)$$

$$B = 2\pi/9 = 40^\circ \quad (2)$$

$$C = 3\pi/9 = 60^\circ \quad (3)$$

$$D = 4\pi/9 = 80^\circ \quad (4)$$

$$A + B = D - A = C \quad (5)$$

The relations of angle C are those of equilateral triangle:

$$\cos C = -\frac{1}{2} \quad (6)$$

$$\sin C = \frac{\sqrt{3}}{2} \quad (7)$$

From the figure 1, cosines of angles A , B and D are related as:

$$\cos A = \cos B + \cos D \quad (8)$$

$$= \cos(2A) + \cos(4A)$$

$$= (2\cos^2 A - 1) + (1 - 8\cos^2 A + 8\cos^4 A)$$

$$= 8\cos^4 A - 6\cos^2 A$$

$$1 = 8\cos^3 A - 6\cos A \quad (9)$$

Previous cosines equation is the depressed cubic equation:

$$t^3 + pt + q = 0 \quad (10)$$

$$p = -\frac{3}{4} \quad (11)$$

$$q = -\frac{1}{8} \quad (12)$$

We found the discriminant is negative which means we have three real roots:

$$\Delta = \frac{q^2}{4} + \frac{p^3}{27} = -\frac{3}{64}$$

$$t_k = 2\sqrt{-\frac{p}{3}} \cos\left(\frac{1}{3} \arccos\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right) - k\frac{2\pi}{3}\right) \quad \text{for } k = 0, 1, 2.$$

$$= \cos\left(\frac{1}{3} \arccos\left(\frac{1}{2}\right) - k\frac{2\pi}{3}\right) \quad \text{for } k = 0, 1, 2.$$

$$= \cos\left(\frac{1}{3} \left(\frac{\pi}{3}\right) - k\frac{2\pi}{3}\right) \quad \text{for } k = 0, 1, 2.$$

$$t_0 = \cos\left(\frac{\pi}{9}\right) = \cos A \approx +0.939692 \quad (13)$$

$$t_1 = \cos\left(-\frac{2\pi}{9}\right) = -\cos B \approx -0.766044 \quad (14)$$

$$t_2 = \cos\left(-\frac{4\pi}{9}\right) = -\cos D \approx -0.173648 \quad (15)$$

From equation 10 we know the product of roots squares is $-2p = \frac{3}{2}$:

$$\cos^2 A + \cos^2 B + \cos^2 D = \frac{3}{2} \quad (16)$$

$$1 - \sin^2 A + 1 - \sin^2 B + 1 - \sin^2 D = \frac{3}{2}$$

$$\sin^2 A + \sin^2 B + \sin^2 D = \frac{3}{2} \quad (17)$$

From equation 10 we know the product of roots is $-q = \frac{1}{8}$ matching the “Morrie’s law”:

$$\cos A \cos B \cos D = \frac{1}{8} \quad (18)$$

$$\begin{aligned} (1 - \sin^2 A)(1 - \sin^2 B)(1 - \sin^2 D) &= \frac{1}{64} \\ (\sin A \sin B)^2 + (\sin A \sin D)^2 + (\sin C \sin D)^2 &= \frac{1}{64} - 1 + \sin^2 A + \sin^2 B + \sin^2 D + (\sin A \sin B \sin D)^2 \\ &= \frac{1}{64} - 1 + \frac{3}{2} + \left(\frac{\sqrt{3}}{8}\right)^2 = \left(\frac{9}{4}\right)^2 \end{aligned} \quad (19)$$

From the figure 1, sines of angles A , B and D are related as:

$$\sin A + \sin B = \sin D \quad (20)$$

$$\begin{aligned} &= \sin(2A + B) \\ &= \sin(2A) \cos B + \cos(2A) \sin B \\ &= 2 \sin A \cos A \cos B + (1 - 2 \sin^2 A) \sin B \\ \sin A &= 2 \sin A \cos A \cos B - 2 \sin^2 A \sin B \\ 1 &= 2 \cos A \cos B - 2 \sin A \sin B \\ &= 2 \cos(A + B) = 2 \cos C \end{aligned} \quad (21)$$

Nonagon height is sum of all sines:

$$\sin A + \sin B + \sin C + \sin D = 2 \sin D + \frac{\sqrt{3}}{2} \quad (22)$$

Last equation solves this cubic equation:

$$\begin{aligned} y^3 - \frac{3y}{4} - \frac{3}{8} &= 0 \\ y_1 &= -\sin A \approx -0.342020 \\ y_2 &= -\sin B \approx -0.642787 \\ y_3 &= +\sin C \approx +0.984807 \end{aligned}$$

More sines relations of angles A , B and C are:

$$\begin{aligned} \sin A \sin B \sin C &= \frac{\sqrt{3}}{8} \\ \sin^2 A + \sin^2 B + \sin^2 C &= \frac{3}{2} \end{aligned}$$

Cosines and sines relations are:

$$\begin{aligned} \cos A \cos B - \sin A \sin B &= \frac{1}{2} \\ \frac{1}{\cos C} - \frac{\sqrt{3}}{\sin C} &= 4 \\ \tan C - 4 \sin C &= \sqrt{3} \end{aligned}$$