Triple unit

https://github.com/heptagons/meccano/units/triple

Abstract

Triple unit is a group of five meccano [1] strips a, b, c, d, e intended to build regular polygons three consecutive perimeter sides. This unit has three angles equal to the polygon internal angle θ. Triple unis has been using to build the pentagon type 2 mentioned in pentagons paper[2].

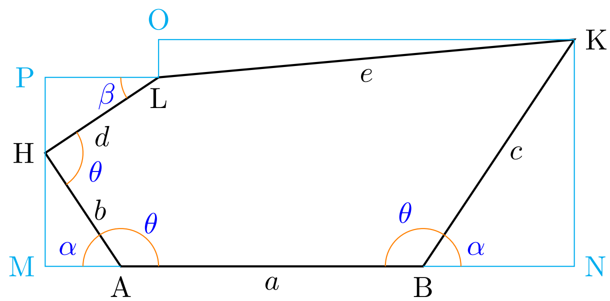

Figure 1: Triple unit has five strips a, b, c, d, e

1 Algebra

From nodes A and B of fig 1 we get α from θ ($\pi = 180°$):

$$\begin{aligned}\theta &= \pi - \alpha \\ \alpha &= \pi - \theta \end{aligned} \tag{1}$$

And from node H we get β from θ:

$$\begin{aligned}\theta &= \alpha + \beta \\ \beta &= \theta - \alpha = \theta - (\pi - \theta) = 2\theta - \pi \end{aligned} \tag{2}$$

We calculate horizontal segment $\overline{OK}$:

$$\begin{aligned}\overline{OK} &= \overline{MA} + a + \overline{BN} - \overline{PL} \\ &= b\cos\alpha + a + c\cos\alpha - d\cos\beta \\ &= a + (b+c)\cos\alpha - d\cos\beta \\ &= a + (b+c)\cos(\pi - \theta) - d\cos(2\theta - \pi) \\ &= a - (b+c)\cos\theta + d\cos(2\theta) \end{aligned} \tag{3}$$

[1] Meccano mathematics by ‘t Hooft
[2] Meccano pentagons

And vertical segment $\overline{OL}$:

$$\begin{aligned}
\overline{OL} &= \overline{KN} - \overline{PH} - \overline{HM} \\
&= c\sin\alpha - d\sin\beta - b\sin\alpha \\
&= (c-b)\sin\alpha - d\sin\beta \\
&= (c-b)\sin(\pi-\theta) - d\sin(2\theta-\pi) \\
&= (c-b)\sin\theta + d\sin(2\theta) \qquad (4)
\end{aligned}$$

So we can express e in function of a, b, c, d and angle θ:

$$\begin{aligned}
e^2 &= (\overline{OK})^2 + (\overline{OL})^2 \\
&= (a-(b+c)\cos\theta + d\cos(2\theta))^2 + ((c-b)\sin\theta + d\sin(2\theta))^2 \\
&= a^2 + (b^2+2bc+c^2)\cos^2\theta + d^2\cos^2(2\theta) - 2a(b+c)\cos\theta + 2ad\cos(2\theta) - 2(b+c)d\cos\theta\cos(2\theta) \\
&\quad (c^2-2bc+b^2)\sin^2\theta + 2(c-b)d\sin\theta\sin(2\theta) + d^2\sin^2(2\theta) \\
&= a^2 + (b^2+c^2)(cos^2\theta + sin^2\theta) + d^2(cos^2(2\theta) + sin^2(2\theta)) \\
&\quad + 2bc\cos^2\theta - 2a(b+c)\cos\theta + 2ad\cos(2\theta) - 2(b+c)d\cos\theta\cos(2\theta) - 2bc\sin^2\theta + 2(c-b)d\sin\theta\sin(2\theta) \\
&= a^2 + b^2 + c^2 + d^2 + 2bc(\cos^2\theta - \sin^2\theta) - 2a(b+c)\cos\theta + 2ad\cos(2\theta) \\
&\quad - 2(b+c)d\cos\theta\cos(2\theta) + 2(c-b)d\sin\theta\sin(2\theta) \\
&= a^2 + b^2 + c^2 + d^2 + 2bc\cos(2\theta) - 2a(b+c)\cos\theta + 2ad\cos(2\theta) \\
&\quad - 2bd(\cos\theta\cos(2\theta) - sin\theta\sin(2\theta)) - 2cd(\cos\theta\cos(2\theta) - sin\theta\sin(2\theta)) \\
&= a^2 + b^2 + c^2 + d^2 + 2(bc+ad)\cos(2\theta) - 2a(b+c)\cos\theta \\
&\quad - 2bd\cos(\theta+2\theta) - 2cd\cos(\theta+2\theta) \\
&= a^2 + b^2 + c^2 + d^2 + 2(bc+ad)\cos(2\theta) - 2(ab+ac)\cos\theta - 2(bd+cd)\cos(3\theta) \\
&= a^2 + b^2 + c^2 + d^2 - 2a(b+c)\cos\theta + 2(ad+bc)\cos(2\theta) - 2d(b+c)\cos(3\theta) \\
&= \boxed{a^2 + b^2 + c^2 + d^2 - 2(b+c)(a\cos\theta + d\cos(3\theta)) + 2(ad+bc)\cos(2\theta)} \qquad (5)
\end{aligned}$$

2 Regular polygons

Polygon	θ	$\cos\theta$	$\cos(2\theta)$	$\cos(3\theta)$
Pentagon	$\frac{3\pi}{5}$	$\frac{1-\sqrt{5}}{4}$	$\frac{-1-\sqrt{5}}{4}$	$\frac{1+\sqrt{5}}{4}$
Hexagon	$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\frac{1}{2}$	1
Heptagon	$\frac{5\pi}{7}$			
Octagon	$\frac{3\pi}{4}$			
Decagon	$\frac{4\pi}{5}$			
Dodecagon	$\frac{5\pi}{6}$			

Table 1: Regular polygons internal angles and cosines.

2.1 Equilateral pentagon

We replace the cosines for pentagon in table 1 in e^2 equation:

$$\begin{aligned}
e^2 &= a^2+b^2+c^2+d^2-2(b+c)(a\cos\theta+d\cos(3\theta))+2(ad+bc)\cos(2\theta)\\
&= a^2+b^2+c^2+d^2-2(b+c)\left(a\left(\frac{1-\sqrt{5}}{4}\right)+d\left(\frac{1+\sqrt{5}}{4}\right)\right)+2(ad+bc)\left(\frac{-1-\sqrt{5}}{4}\right)\\
&= a^2+b^2+c^2+d^2-\frac{(b+c)(a+d)+ad+bc}{2}+\frac{(b+c)(a-d)-ad-bc}{2}\sqrt{5} \qquad (6)
\end{aligned}$$

e cannot to be and integer if $\sqrt{5}$ multiplicand is not zero so we force it to be zero:

$$\begin{aligned}
(b+c)(a-d)-ad-bc &= 0\\
ad+bc &= (b+c)(a-d) \qquad (7)
\end{aligned}$$

We apply the condition $ad+bc=(b+c)(a-d)$ in the last equation of e^2 and get:

$$\begin{aligned}
e^2 &= a^2+b^2+c^2+d^2-\frac{(b+c)(a+d)+(b+c)(a-d)}{2}\\
&= \boxed{a^2+b^2+c^2+d^2-a(b+c)} \qquad (8)
\end{aligned}$$