Meccano four frame

https://github.com/heptagons/meccano/frames/four

Abstract

Four frame is a group of four rigid meccano [1] strips.

1 Four frame

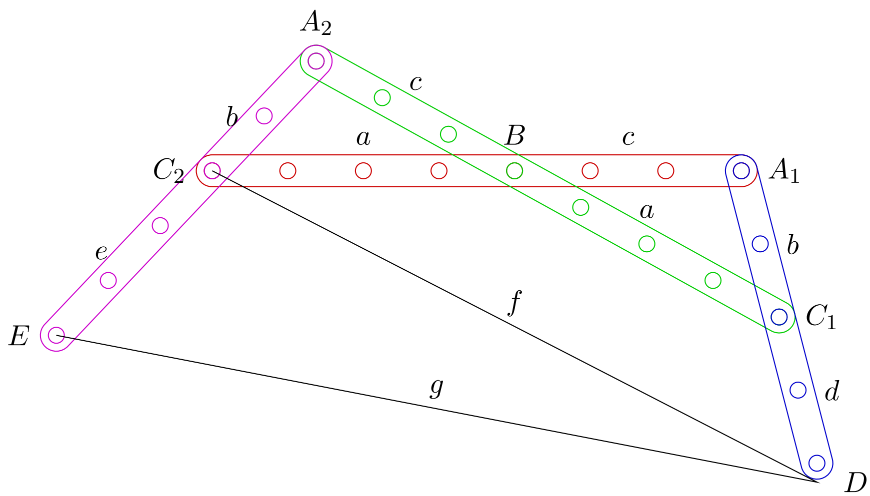

Figure 1: Antisymmetric four frame.

Figure 1 show the antisymmetric four-strips frame. From the figure we define $\alpha \equiv \angle BA_1C_1$ and define integers $m = b^2 + c^2 - a^2$ and $n = 2bc$ using the law of cosines, then we calculate $\cos\alpha$ and $\sin\alpha$:

$$(\alpha, m, n) \equiv (\angle BA_1C_1, b^2 + c^2 - a^2, 2bc) \tag{1}$$

$$\cos\alpha = \frac{m}{n} \tag{2}$$

$$\sin\alpha = \frac{\sqrt{n^2 - m^2}}{n} \tag{3}$$

From the figure 1 we define $\gamma \equiv \angle BA_2C_2$ and define integers $s = a^2 + b^2 - c^2$ and $t = 2ab$ and calculate $\cos\gamma$ and $\sin\gamma$:

$$(\gamma, s, t) \equiv (\angle BA_2C_2, a^2 + b^2 - c^2, 2ab) \tag{4}$$

$$\cos\gamma = \frac{s}{t} \tag{5}$$

$$\sin\gamma = \frac{\sqrt{t^2 - s^2}}{t} \tag{6}$$

[1] Meccano mathematics by ‘t Hooft

We calculate the distance $f = \overline{C_2 D}$ with the law of cosines using angle α and defining integers $x = a+c$ and $y = b+d$:

$$\begin{align}
(x, y) &\equiv (a+c, b+d) \tag{7}\\
f^2 &= (a+c)^2 + (b+d)^2 - 2(a+c)(b+d)\cos\alpha \tag{8}\\
&= x^2 + y^2 - \frac{2mxy}{n} \tag{9}\\
f &= \frac{\sqrt{n^2(x^2+y^2) - 2mnxy}}{n} \tag{10}
\end{align}$$

We define a new integer $z \equiv n^2(x^2+y^2) - 2mnxy$ so we have:

$$\begin{align}
z &\equiv n^2(x^2+y^2) - 2mnxy \tag{11}\\
f &= \frac{\sqrt{z}}{n} \tag{12}
\end{align}$$

We define angle $\theta \equiv \angle A_1 C_2 D$ and calculate $\cos\theta$ and $\sin\theta$:

$$\begin{align}
\theta &\equiv \angle A_1 C2D \tag{13}\\
\cos\theta &= \frac{(a+c)^2 + f^2 - (b+d)^2}{2(a+c)f} \nonumber\\
&= \frac{x^2 + f^2 - y^2}{2xf} \nonumber\\
&= \frac{x^2 + x^2 + y^2 - \frac{2mxy}{n} - y^2}{2x\frac{\sqrt{z}}{n}} \nonumber\\
&= \frac{nx - my}{\sqrt{z}} \tag{14}\\
\sin^2\theta &= 1 - \cos^2\theta = \frac{z - (nx-my)^2}{z} \nonumber\\
&= \frac{n^2(x^2+y^2) - 2mnxy - (nx-my)^2}{z} \nonumber\\
&= \frac{n^2(x^2+y^2) - 2mnxy - n^2x^2 + 2nxmy - m^2y^2}{z} \nonumber\\
\sin\theta &= \frac{y\sqrt{n^2-m^2}}{\sqrt{z}} \tag{15}
\end{align}$$

We define angle $\phi \equiv \angle A_2 C_2 D$ and we note is the sum of angles $\theta + \gamma$ and we calculate $\cos\phi$:

$$\begin{align}
\phi &\equiv \angle A_2 C_2 D \tag{16}\\
&= \theta + \gamma \tag{17}\\
\cos\phi &= \cos(\theta+\gamma) \tag{18}\\
&= \cos\theta\cos\gamma - \sin\theta\sin\gamma \nonumber\\
&= \frac{(nx-my)s}{\sqrt{z}t} - \frac{(y\sqrt{n^2-m^2})\sqrt{t^2-s^2}}{\sqrt{z}t} \nonumber\\
&= \frac{s(nx-my) - y\sqrt{(n^2-m^2)(t^2-s^2)}}{\sqrt{z}t} \tag{19}
\end{align}$$

We simplify $\sqrt{(n^2-m^2)(t^2-s^2)}$:

$$
\begin{aligned}
(n^2-m^2)(t^2-s^2) &= (n-m)(n+m)(t-s)(t+s) \\
&= (2bc-(b^2+c^2-a^2))(2bc+b^2+c^2-a^2)(2ab-(a^2+b^2-c^2))(2ab+a^2+b^2-c^2) \\
&= (a^2-(b-c)^2)((b+c)^2-a^2)(c^2-(a-b)^2)((a+b)^2-c^2) \\
&= (a+b-c)(a-b+c)(b+c+a)(b+c-a)(c+a-b)(c-a+b)(a+b+c)(a+b-c) \\
&= (a+b+c)^2(a+b-c)^2(a-b+c)^2(-a+b+c)^2 \\
\sqrt{(n^2-m^2)(t^2-s^2)} &= (a+b+c)(a+b-c)(a-b+c)(-a+b+c) \\
&= ((a+b)^2-c^2)(c^2-(a-b)^2) \\
&= (s+t)(t-s) \\
&= (t^2-s^2) \qquad (20)
\end{aligned}
$$

We substitute equation 20 into equation 19 and we get:

$$
\cos\phi = \frac{s(nx-my)-y(t^2-s^2)}{\sqrt{z}t} \qquad (21)
$$

From the figure we define angle $\psi \equiv \angle DC_2E$ and we note equals angle $\pi-\phi$, so we have:

$$
\begin{aligned}
\psi &\equiv \angle DC_2E && (22) \\
&= \pi - \phi && (23) \\
\cos\psi &= \cos(\pi-\phi) && (24) \\
&= -\cos\phi \\
&= \frac{-s(nx-my)+y(t^2-s^2)}{\sqrt{z}t} && (25)
\end{aligned}
$$

Finally with $\cos\psi$, e and f we can calculate distance $g = \overline{ED}$:

$$
\begin{aligned}
g^2 &= e^2+f^2-2ef\cos\psi && (26) \\
&= e^2+x^2+y^2-\frac{2mxy}{n}-2e\left(\frac{\sqrt{z}}{n}\right)\left(\frac{-s(nx-my)+y(t^2-s^2)}{\sqrt{z}t}\right) && (27) \\
&= e^2+x^2+y^2-\frac{2mxy}{n}+2e\frac{s(nx-my)-y(t^2-s^2)}{nt} && (28) \\
&= e^2+x^2+y^2-\frac{2mxy}{2bc}+2e\frac{s(nx-my)-y(t^2-s^2)}{4ab^2c} && (29) \\
&= \frac{4a^2b^2c^2(e^2+x^2+y^2)-(2a^2bc)(2mxy)+2aces(nx-my)-2acey(t^2-s^2)}{4a^2b^2c^2} \\
&= \frac{2ac(2ab^2c(e^2+x^2+y^2)-2abmxy+es(nx-my)-ey(t^2-s^2))}{4a^2b^2c^2} \\
g &= \frac{\sqrt{2ac(2ab^2c(e^2+x^2+y^2)-2abmxy+es(nx-my)-ey(t^2-s^2))}}{2abc} && (30)
\end{aligned}
$$

1.1 Antisymmetric four frame software

From the last equation of g we identify two `input` integers i_1, i_2 which are used to get $g(i)$. Then the nested radicals software will return square-free `output` integers z_1, z_2, z_3 as $g(z)$:

$$i_1 = 2abc \tag{31}$$

$$i_2 = 2ac(2ab^2c(e^2 + x^2 + y^2) - 2abmxy + es(nx - my) - ey(t^2 - s^2)) \tag{32}$$

$$g(i) = \frac{\sqrt{i_2}}{i_1} \tag{33}$$

$$g(z) = \frac{z_2\sqrt{z_3}}{z_1} \tag{34}$$

where m, n are calculated with equations 1, x, y are calculated with equations 7 and s, t are calculated with equations 4.

2 Meccano triangles $abc = (2n + 5, 2n + 4, 2n + 3)$

The family of triangles $T_0 = \triangle ABC$ with sides $(2n+5, 2n+4, 2n+3)$ can be decomposed into two meccano subtriangles $T_1 = \triangle ABD$ with sides $(2n + 3, 2n, 2n + 3)$ and $T_2 = \triangle BCD$ with sides $(4, 2n + 3, 2n + 5)$. For the frames with four strips the important thing is that triangles T_0 and T_1 share a common angle α while triangles T_0 and T_2 share a common angle γ.

We demonstrate $\alpha_0 = \alpha_1 =$ and $\gamma_0 = \gamma_2$ with the law of cosines:

$$\cos\alpha_0 = \frac{(2n + 3)^2 + (2n + 4)^2 - (2n + 5)^2}{2(2n + 3)(2n + 4)} = \frac{2n(2n + 4)}{2(2n + 3)(2n + 4)} = \frac{n}{2n + 3} \tag{35}$$

$$\cos\alpha_1 = \frac{(2n + 3)^2 + (2n)^2 - (2n + 3)^2}{2(2n + 3)(2n)} = \frac{4n^2}{4n(2n + 3)} = \frac{n}{2n + 3} \tag{36}$$

$$\cos\gamma_0 = \frac{(2n + 4)^2 + (2n + 5)^2 - (2n + 3)^2}{2(2n + 4)(2n + 5)} = \frac{4(n + 4)(n + 2)}{2(2n + 4)(2n + 5)} = \frac{n + 4}{2n + 5} \tag{37}$$

$$\cos\gamma_2 = \frac{4^2 + (2n + 5)^2 - (2n + 3)^2}{2(4)(2n + 5)} = \frac{8(n + 4)}{8(2n + 5)} = \frac{n + 4}{2n + 5} \tag{38}$$