1 32 bits algebraic integers

Let A_0, A_1, A_2 and A_3 algebraic integers with levels 0, 1, 2 and 3:

$$A_0 = \pm b \tag{1.1}$$

$$A_1 = \pm c\sqrt{\pm d} \tag{1.2}$$

$$A_2 = \pm e\sqrt{f \pm g\sqrt{\pm h}} \tag{1.3}$$

$$A_3 = \pm i\sqrt{j \pm k\sqrt{l \pm m\sqrt{\pm n}}} \tag{1.4}$$

We will use fourteen different 32-bit natural numbers, where a goes in the denominators and $b, ...n$ in the numerators.

$$1 \leq a \leq 2^{32} - 1 \tag{1.5}$$

$$0 \leq b, c, d, e, f, g, h, i, j, k, l, m, n \leq 2^{32} - 1 \tag{1.6}$$

The signs are managed appart as extra boolean variables and there is one for each of the seven variables b, c, e, g, i, k and m.

1.1 N32, I32 and AI32

```
1  type N32 uint32 // range 0 - 0xffffffff
2  
3  type I32 struct {
4    s bool // sign: true means negative
5    n N32  // positive value
6  }
7  
8  type AI32 struct {
9    o *I32  // outside radical
10   i *I32  // inside radical square-free
11   e *AI32 // inside radical extension
12 }
```

In this list we define three 32 bit numbers in Golang code.
In line 1 we define the natural number $N32$ with a range of $0 < n \leq 2^{32} - 1$.
In line 3 we define the integer number $I32$, the number sign is negative if s is true and the number value always is a positive. If $I32$ is nil, then we assume the number is zero.
In line 8 we define the algebraci integer number $AI32$. The number is recursive with a value of

$$\pm o\sqrt{\pm i \pm e.o\sqrt{\pm e.i \pm e.e.o...}} \tag{1.7}$$

where each sign $\pm$ corresponds to its integer sign s of the values of integers o and i.

1.2 Reductions

Reductions simplify and standarize the numbers representations. Are applied to the inputs and outputs numbers in operations like copy, addition, multiplication, inversion and square roots extraction. Any complicated abstract integer is reduced from the inside to the outside.

$$\begin{aligned}
A_3 &= \pm i\sqrt{\pm j \pm k\sqrt{\pm l \pm m\sqrt{\pm n}}} && (1.8)\\
&\quad m_1, n_1 = reduceOI(m, n) \\
&= \pm i\sqrt{\pm j \pm k\sqrt{\pm l \pm m_1\sqrt{\pm n_1}}} && (1.9)\\
&\quad k_1, l_1, m_2 = reduceOII(k, l, m_1) \\
&= \pm i\sqrt{\pm j \pm k_1\sqrt{\pm l_1 \pm m_2\sqrt{\pm n_1}}} && (1.10)\\
&\quad i_1, j_1, k_2 = reduceOII(i, j, k_1) \\
&= \pm i_2\sqrt{\pm j_2 \pm k_3\sqrt{\pm l_1 \pm m_2\sqrt{\pm n_1}}} && (1.11)
\end{aligned}$$

1.3 Reduction *roi*

This reduction is done for $AI32$ numbers **without** extension e. This is the case of part $\pm c\sqrt{\pm d}$ of A_1, part $\pm g\sqrt{\pm h}$ of A_2 and part $\pm m\sqrt{\pm n}$ of A_3. Example of reducing A_1:

$$A_1 = \pm c\sqrt{\pm d} \tag{1.12}$$

$$d = p^2 d_1 \qquad \textbf{From } d \textbf{ find } p, d_1 \textbf{ where } d_1 \textbf{ is square-free or } 1 \tag{1.13}$$

$$roi(c,d) = \begin{cases} 0 & \text{case 1: if } c = 0 \text{ or } d = 0 \\ \pm cp & \text{case 2: if } d_1 = +1 \\ \pm c\sqrt{\pm d} & \text{case 3: if } p = 1 \\ \pm cp\sqrt{\pm d_1} & \text{case 4: otherwise} \end{cases} \tag{1.14}$$

For cases 1 and 2 we got A_1 degenerated into A_0. For case 3 values remain the same because the number was irreducible. For case 4 we got reduced A_1 with new values $c_1 = cp$ and d_1.

1.4 Reduction *roie*

This reduction is done for $AI32$ numbers **with** extension e. This is the case of part $\pm e\sqrt{\pm f + g\sqrt{\ldots}}$ of A_2, part $\pm i\sqrt{\pm j + k\sqrt{\ldots}}$ of A_3 and part $\pm k\sqrt{\pm l + m\sqrt{\ldots}}$ of A_3. Example of reducing A_2. First we reduce part $\pm g\sqrt{\pm h}$ with reduceOII function:

$$A_1 = \pm g\sqrt{\pm h} \tag{1.15}$$

$$roi(g,h) = \begin{cases} \pm g_1 & \text{reduceOII cases 1 or 2} \\ \pm g_2\sqrt{\pm h_2} & \text{reduceOII cases 3 or 4} \end{cases} \tag{1.16}$$

$$A_2 = \begin{cases} 0 & case1 : \text{if } e = 0 \\ \pm e\sqrt{\pm f \pm g_1} & case2 : \text{if } h_2 = +1 \\ \pm e\sqrt{\pm f \pm g_2\sqrt{\pm h_2}} & case3 : \text{otherwise} \end{cases} \tag{1.17}$$

For case 1 we have that A_2 degenerated into a A_0 so we finish. For case 2 we have that A_2 degenerated into a A_1 with values $\pm e$ and $\pm f + g_1$ and is then reduced with function $reduceOI$.

and 3 we have that A_2 degenerated into A_1, so we proceed to go to reduce further this new A_1 as in previous section. For cases 4 and 5 we rewrite the A_2 with reduced values g_1 and square-free h_1:

$$\tag{1.18}$$

$$A_2 = \pm e\sqrt{\pm f \pm g_1\sqrt{\pm h_1}} \tag{1.19}$$

$$g_1 = r^2 g_2 \qquad \textbf{From } g_1 \textbf{ found } r, g_2 \textbf{ where } r \textbf{ matches with next equation's} \tag{1.20}$$

$$f = r^2 f_1 \qquad \textbf{From } f \textbf{ found } r, f_1 \textbf{ where } r \textbf{ matches with previous equation's} \tag{1.21}$$

$$A_2 = \begin{cases} \pm e\sqrt{\pm f \pm g_1\sqrt{\pm h_1}} & case6 : \text{if } r = 1 \text{ nothing changed} \\ \pm er\sqrt{\pm f_1 \pm g_2\sqrt{\pm h_1}} & case7 : \text{otherwise} \end{cases} \tag{1.22}$$

1.5 B, D, H, N

We define four numbers of increasing complexity:

$$B \equiv \frac{A_0}{a} \tag{1.23}$$

$$D \equiv \frac{A_0 + A_1}{a} \tag{1.24}$$

$$H \equiv \frac{A_0 + A_1 + A_2}{a} \tag{1.25}$$

$$N \equiv \frac{A_0 + A_1 + A_2 + A_3}{a} \tag{1.26}$$

2 functions

Each of the radicals $r_0, ..., r_3$ has a function to read their corresponding signs and integers variables:

$$f_0 \equiv f(\pm b) \tag{2.1}$$
$$f_1 \equiv f(\pm c, d) \tag{2.2}$$
$$f_2 \equiv f(\pm e, f, \pm g, h) \tag{2.3}$$
$$f_3 \equiv f(\pm i, j, \pm k, l, \pm m, n) \tag{2.4}$$

Each $f_0, ...f_4$ reduces the values with gcd and root simplifications.

Each of the algebraic numbers B, D, H and N has a function to read their radicals functions as inputs:

$$f_B \equiv f(f_0(...), a) \tag{2.5}$$
$$f_D \equiv f(f_0(...), f_1(...), a) \tag{2.6}$$
$$f_H \equiv f(f_0(...), f_1(...), f_2(...), a) \tag{2.7}$$
$$f_N \equiv f(f_0(...), f_1(...), f_2(...), f_3(...), a) \tag{2.8}$$

Each $f_B, ...f_N$ adds the radicals reducing once more the variables with gcd root simplifications and now considering the denominator a.

3 Examples

3.1 f_B examples

$$\cos 0 = 1 \implies f_B(f_0(1), 1) \tag{3.1}$$
$$\sin\frac{\pi}{6} = \frac{1}{2} \implies f_B(f_0(1), 2) \tag{3.2}$$

3.2 f_D examples

$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \implies f_D(\emptyset, f_1(1, 2), 2) \tag{3.3}$$
$$\sin\frac{\pi}{10} = \frac{-1+\sqrt{5}}{4} \implies f_D(f_0(-1), f_1(1, 5), 4) \tag{3.4}$$

3.3 f_H examples

$$\sin\frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4} \implies f_H(\emptyset, \emptyset, f_2(1, 10, -2, 5), 4) \tag{3.5}$$
$$\sin\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4} \implies f_H(\emptyset, f_1(1, 6), f_2(1, 2, 0, 0), 4)* \tag{3.6}$$
$$\sin\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2} \implies f_H(\emptyset, \emptyset, f_2(1, 2, 1, 3), 2) \tag{3.7}$$
$$\cos\frac{\pi}{15} = \frac{1+\sqrt{5}+\sqrt{30-6\sqrt{5}}}{8} \implies f_E(f_0(1), f_1(1, 5), f_2(1, 30, -6, 5), 8) \tag{3.8}$$

3.4 f_N examples

$$\cos\frac{\pi}{16} = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \tag{3.9}$$
$$\implies f_N(\emptyset,\emptyset,\emptyset,f_3(1,2,1,2,1,2),2)$$

$$\cos\frac{\pi}{24} = \frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2} \tag{3.10}$$
$$\implies f_N(\emptyset,\emptyset,\emptyset,f_3(1,2,1,2,1,3),2)$$

$$\cos\frac{2\pi}{17} = \frac{-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}+2\sqrt{17+3\sqrt{17}-\sqrt{170+38\sqrt{17}}}}{16} \tag{3.11}$$
$$\implies f_N(f_0(-1),f_1(1,17),f_2(1,34,-2,17),f_3(2,17,3,17,-1,170,+38,17),16) \tag{3.12}$$

4 Operations with result B

4.1 NewB $B = B_1$

$$B_1 = \frac{\pm b_1}{a_1} \tag{4.1}$$
$$\textbf{Reduce } \{a,b\} = \{a_1/G, b_1/G\} \iff G = \gcd\{a_1,b_1\} > 1$$
$$= \frac{\pm b}{a} \tag{4.2}$$

4.2 AddBB $B = B_2 + B_3$

$$B_2 + B_3 = \frac{\pm b_2}{a_2} + \frac{\pm b_3}{a_3} \tag{4.3}$$
$$= \frac{\pm b_2 a_3 \pm b_3 a_2}{a_2 a_3} = \frac{q}{p} \tag{4.4}$$
$$\textbf{Reduce } \{a_1,b_1\} = \{p/G, q/G\} \iff G = \gcd\{p,q\} > 1$$
$$= \frac{\pm b_1}{a_1} \textbf{ Solve as NewB} \tag{4.5}$$

4.3 MulBB $B = B_2 \times B_3$

$$B_2 \times B_3 = \frac{\pm b_2}{a_2} \times \frac{\pm b_3}{a_3} \tag{4.6}$$
$$= \frac{\pm b_2 b_3}{a_2 a_3} = \frac{q}{p} \tag{4.7}$$
$$\textbf{Reduce } \{a_1,b_1\} = \{p/G, q/G\} \iff G = \gcd\{p,q\} > 1$$
$$= \frac{\pm b_1}{a_1} \textbf{ Solve as NewB} \tag{4.8}$$

4.4 InvB $B = 1/B_2$

$$\begin{aligned}
\frac{1}{B_2} &= \frac{1}{\pm b_2/a_2} && (4.9)\\
&= \frac{\pm a_2}{b_2} = \frac{q}{p} && (4.10)\\
&\quad \textbf{Reduce } \{a_1, b_1\} = \{p/G, q/G\} \iff G = \gcd\{p, q\} > 1 \\
&= \frac{\pm b_1}{a_1} \textbf{ Solve as NewB} && (4.11)
\end{aligned}$$

5 Operations with result D

5.1 NewD $D = D_1$

$$\begin{aligned}
D_1 &= \frac{\pm b_1 \pm c_1\sqrt{d_1}}{a_1} && (5.1)\\
&\quad \textbf{Reduce } \{p, q, r\} = \{a_1/G, b_1/G, c_1/G\} \iff G = \gcd\{a_1, b_1, c_1\} > 1 \\
&= \frac{\pm q \pm r\sqrt{d_1}}{p} && (5.2)\\
&\quad \textbf{Reduce } \{d\} = s^2 d_1 \iff s > 1 \\
&= \frac{\pm q \pm rs\sqrt{d}}{p} && (5.3)\\
&\quad \textbf{Reduce } \{a, b, c\} = \{p/G, q/G, rs/G\} \iff G = \gcd\{p, q, rs\} \\
&= \frac{\pm b \pm c\sqrt{d}}{a} && (5.4)
\end{aligned}$$

5.2 SqrtB $D = \sqrt{B_2}$

$$\begin{aligned}
\sqrt{B_2} &= \sqrt{\frac{\pm b_2}{a_2}} && (5.5)\\
&= \frac{\sqrt{a_2 b_2}}{a_2} && (5.6)\\
&\quad \textbf{Set } \{a_1, b_1, c_1, d_1\} = \{a_2, 0, 1, a_2 b_2\} \\
&= \frac{\pm b_1 \pm c_1\sqrt{d_1}}{a_1} \textbf{ Solve as NewD} && (5.7)
\end{aligned}$$

5.3 InvD $D = 1/D_2$

$$\begin{aligned}
1/D_2 &= \frac{a_2}{\pm b_2 \pm c_2\sqrt{d_2}} \\
&= \frac{\pm a_2 b_2 \mp a_2 c_2\sqrt{d_2}}{b_2^2 - c_2^2 d_2} \\
&\quad \textbf{Set } \{a_1, b_1, c_1, d_1\} = \{b_2^2 - c_2^2 d_2, \pm a_2 b_2, \mp a_2 c_2, d_2\} \\
&= \frac{\pm b_1 \pm c_1\sqrt{d_1}}{a_1} \textbf{ Solve as NewD}
\end{aligned}$$

6 Operations with result H

6.1 $D_1 + D_2 \mapsto H$ ¡¡¡¡¡

$$\begin{aligned}
D_1 + D_2 &= \frac{\pm b_1 \pm c_1\sqrt{d_1}}{a_1} + \frac{\pm b_2 \pm c_2\sqrt{d_2}}{a_2} && (6.1)\\
&= \frac{(\pm a_2 b_1 \pm a_1 b_2) \pm a_2 c_1\sqrt{d_1} \pm a_1 c_2 \sqrt{d_2}}{a_1 a_2} && (6.2)\\
&= \frac{\pm q \pm r\sqrt{d_1} \pm s\sqrt{d_2}}{p} && (6.3)\\
&\quad \textbf{where } \{p, q, r, s\} = \gcd\{a_1 a_2, (\pm a_2 b_1 \pm a_1 b_2), \pm a_2 c_1, \pm a_1 c_2\}\\
&= \frac{\pm q \pm \sqrt{r^2 d_1 + s^2 d_2 \pm 2rs\sqrt{d_1 d_2}}}{p} && (6.4)\\
&= \frac{\pm q \pm \sqrt{t \pm 2rsu\sqrt{h}}}{p} && (6.5)\\
&\quad \textbf{where } \{t\} = r^2 d_1 + s^2 d_2 \textbf{ and } \{u^2 h\} = d_1 d_2\\
&= \frac{\pm q \pm v\sqrt{f \pm g\sqrt{h}}}{p} && (6.6)\\
&\quad \textbf{where } \{v^2 f\} = t \textbf{ and } \{v^2 g\} = 2rsu\\
&= \frac{\pm d \pm e\sqrt{f \pm g\sqrt{h}}}{a} && (6.7)\\
&\quad \textbf{where } \{a, d, e\} = \gcd\{p, \pm q, \pm qv\}\\
& && (6.8)
\end{aligned}$$

6.2 $\sqrt{C_1} = F_2$

$$\begin{aligned}
\sqrt{C_1} &= \sqrt{\frac{a_1\sqrt{c_1}}{b_1}}\\
&= \frac{\sqrt{a_1 b_1 \sqrt{c_1}}}{b_1}\\
&= \frac{m\sqrt{e_2\sqrt{c_1}}}{b_1} && a_1 b_1 = m^2 e_2\\
&= \frac{a_2\sqrt{e_2\sqrt{c_1}}}{b_2} && (a_2, b_2) = gcd(m, b_1)
\end{aligned}$$

6.3 $C_1 + D_2 = F_3$

$$
\begin{aligned}
C_1 + D_2 &= \frac{\pm a_1\sqrt{c_1}}{b_1} + \frac{\pm a_2\sqrt{c_2} \pm d_2}{b_2} \\
&= \frac{\pm a_1 b_2\sqrt{c_1} \pm a_2 b_1\sqrt{c_2} \pm d_2 b_1}{b_1 b_2} \\
&= \frac{\pm m\sqrt{c_1} \pm n\sqrt{c_2} \pm p}{o} && (\pm m, \pm n, \pm p, o) = gcd(\pm a_1 b_2, \pm a_2 b_1, \pm d_2 b_1, b_1 b_2) \\
&= \frac{\sqrt{m^2 c_1 + n^2 c_2 \pm 2mn\sqrt{c1 c_2}} \pm p}{o} \\
&= \frac{\sqrt{q \pm 2mnr\sqrt{f_3}} \pm p}{o} && q = m^2 c_1 + n^2 c_2, c_1 c_2 = r^2 f_3 \\
&= \frac{s\sqrt{c_3 \pm e_3\sqrt{f_3}} \pm p}{o} && q = s^2 c_3, 2mnr = s^2 e_3 \\
&= \frac{a_3\sqrt{c_3 \pm e_3\sqrt{f_3}} \pm d_3}{b_3} && (a_3, b_3, \pm d_3) = gcd(s, \pm p, o)
\end{aligned}
$$

6.4 $1/D_1 = D_2$

$$
\begin{aligned}
1/D_1 &= \frac{b_1}{\pm a_1\sqrt{c_1} \pm d_1} \\
&= \frac{\pm a_1 b_1\sqrt{c_1} \mp b_1 d_1}{a_1^2 c_1 - d_1^2} \\
&= \frac{a_2\sqrt{c_1} \pm d_2}{b_2} && (a_2, b_2, d_2) = gcd(\pm a_1 b_1, \mp b_1 d_1, a_1^2 c_1 - d_1^2)
\end{aligned}
$$

6.5 $\sqrt{D_1} = F_2$ editing...

$$
\begin{aligned}
\sqrt{D_1} &= \sqrt{\frac{\pm a_1\sqrt{c_1} \pm d_1}{b_1}} \\
&= \frac{\sqrt{\pm b_1 d_1 \pm a_1 b_1\sqrt{f_2}}}{b_1} && f_2 = c_1 \\
&= \frac{m\sqrt{c_2 \pm e_2\sqrt{f_2}}}{b_1} && \pm b_1 d_1 = m^2 c_2, \pm a_1 b_1 = m^2 e_2 \\
&= \frac{a_2\sqrt{c_2 \pm e_2\sqrt{f_2}}}{b_2} && (a_2, b_2) = gcd(m, b_1)
\end{aligned}
$$

6.6 $D_1 + D_2 = F_3$

$$\begin{aligned} D_1 + D_2 &= \frac{\pm a_1\sqrt{c_1} \pm d_1}{b_1} + \frac{\pm a_2\sqrt{c_2} \pm d_2}{b_2} \\ &= \frac{\pm a_1 b_2\sqrt{c_1} \pm a_2 b_1\sqrt{c_2} \pm d_1 b_2 \pm d_2 b_1}{b_1 b_2} \\ &= \frac{\pm m\sqrt{c_1} \pm n\sqrt{c_2} \pm p}{o} && (\pm m, \pm n, \pm p, o) = gcd(\pm a_1 b_2, \pm a_2 b_1, \pm d_1 b_2 \pm d_2 b_1, b_1 b_2) \\ &= \frac{\sqrt{m^2 c_1 + n^2 c_2 \pm 2mn\sqrt{c1 c_2}} \pm p}{o} \\ &= \frac{\sqrt{q \pm 2mnr\sqrt{f_3}} \pm p}{o} && q = m^2 c_1 + n^2 c_2, c_1 c_2 = r^2 f_3 \\ &= \frac{s\sqrt{c_3 \pm e_3\sqrt{f_3}} \pm p}{o} && q = s^2 c_3, 2mnr = s^2 e_3 \\ &= \frac{a_3\sqrt{c_3 \pm e_3\sqrt{f_3}} \pm d_3}{b_3} && (a_3, b_3, \pm d_3) = gcd(s, \pm p, o) \end{aligned}$$

6.7 $D_1 \times D_2 = F_3$

$$\begin{aligned} D_1 \times D_2 &= \frac{\pm a_1\sqrt{c_1} \pm d_1}{b_1} \times \frac{\pm a_2\sqrt{c_2} \pm d_2}{b_2} \\ &= \frac{\pm a_1 a_2\sqrt{c_1 c_2} \pm a_1 d_2\sqrt{c1} \pm a_2 d_1\sqrt{c_2} \pm d_1 d_2}{b_1 b_2} \end{aligned}$$

6.8 MulDD $D_1 \times D_2 \mapsto H$???

$$\begin{aligned} D_1 \times D_2 &= \frac{\pm b_1 \pm c_1\sqrt{d_1}}{a_1} \times \frac{\pm b_2 \pm c_2\sqrt{d_2}}{a_2} \\ &= \frac{\pm b_1 b_2 \pm b_1 c_2\sqrt{d_2} \pm b_2 c_1\sqrt{d_1} \pm c_1 c_2\sqrt{d_1 d_2}}{a_1 a_2} \\ &= \frac{\pm a_1 a_2 m\sqrt{c_3}}{b_1 b_2} && c_1 c_2 = m^2 c_3 \\ &= \frac{\pm a_3\sqrt{c_3}}{b_3} && (\pm a_3, b_3) = gcd(\pm a_1 a_2 m, b_1 b_2) \end{aligned}$$