Meccano hexagons gallery

https://github.com/heptagons/meccano/hexa/gallery

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Abstract

We build meccano [1] rigid regular hexagons from sides 4 to 24. Hexagon perimeter has 6 equal strips connected with 6 bolts. To make it rigid we add a maximum of 3 internal strips connected with at most 4 extra bolts. The extra strips also must remain totally inside the perimeter, don't overlap with any other and must not be parallel to any external strip. With algebra and then sofware, we produce `hexagonal triangles` which have an internal angle of 120° and we use their sides as the internal strips.

1 Algebra and software

1.1 Hexagon angles

We look strips that can make rigid two consecutive internal sides the regular hexagon. Figure 1 show the first four cases found. From any figure we have the internal hexagon angle is $\theta \equiv \angle GBC = 2\pi/3$. First we define the hexagon side as $p \equiv \overline{BC}$. From the triangle $\triangle GBC$ we define the other two sides as $b \equiv \overline{GB}$ and $c = \overline{GC}$. By the law of cosines we know that:

$$\begin{aligned} c &= \sqrt{b^2 + p^2 - 2bp\cos\theta} \\ &= \sqrt{b^2 + p^2 - 2bp\left(-\frac{1}{2}\right)} = \sqrt{b^2 + p^2 + bp} \end{aligned} \tag{1}$$

Then we defie $a \equiv p + b$ to get:

$$c = \sqrt{a^2 + b^2 - ab} \quad \texttt{where } a > b \tag{2}$$

a	c	p	b
8	7	5	3
15	13	8	7
21	19	16	5
35	31	24	11
40	37	33	7
48	43	35	13

Table 1: **Hexagonal triangles** with sides $c > p > b$ with an internal angle $2\pi/3$.

Our software iterates first $0 < a < max$ and then $1 < b < a$ and record all c that is an integer. The first cases of such triangles with sides $c > p > b$ are shown table 1 and we call them **Hexagonal triangles**.

Figure 1 shows hexagons of sizes $p = \{5, 8, 16, 24\}$ with perimeter strips in orange made rigid adding three internal green strips of lenght $c = \{7, 13, 19, 31\}$.

[1] Meccano mathematics by 't Hooft

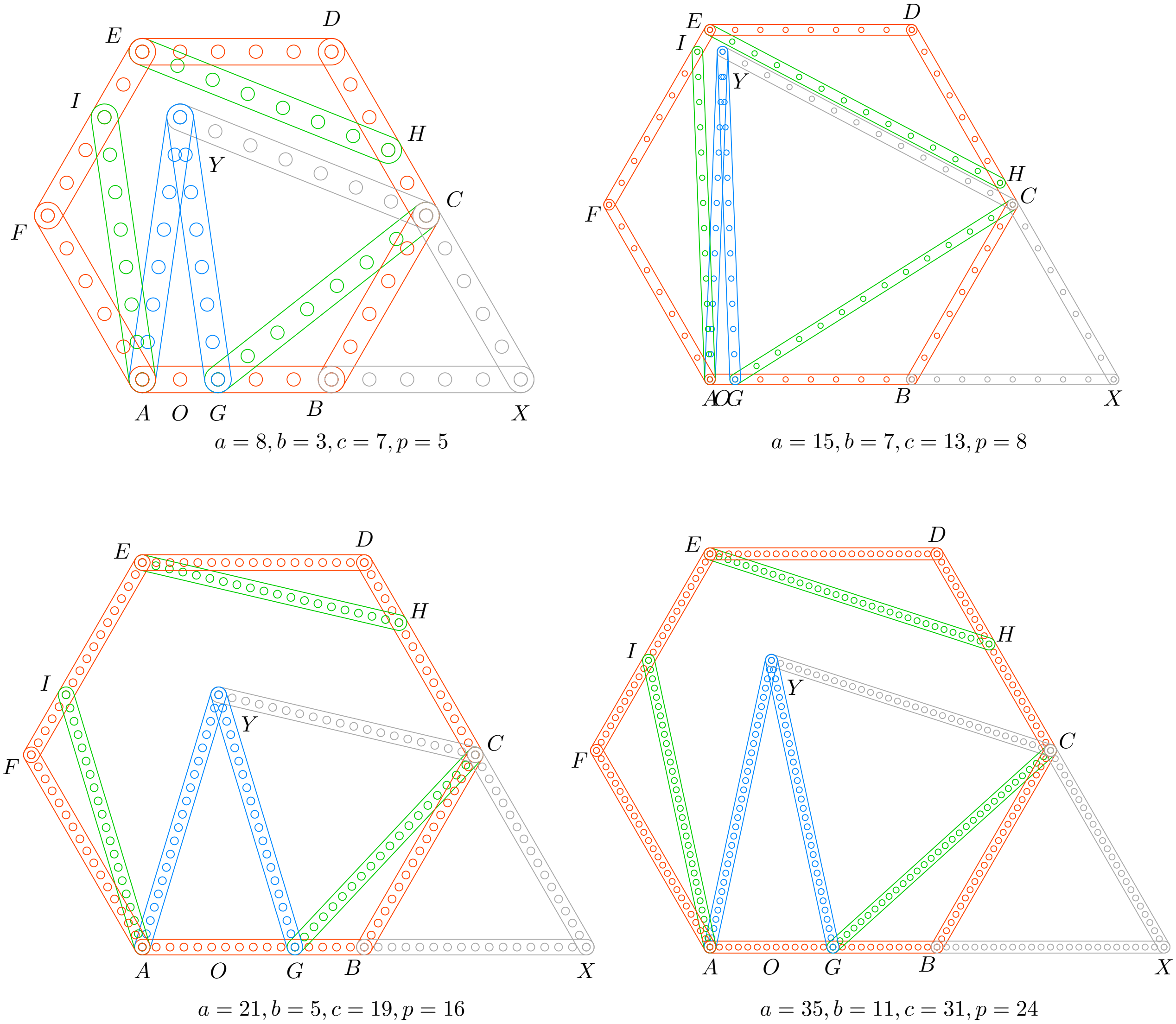
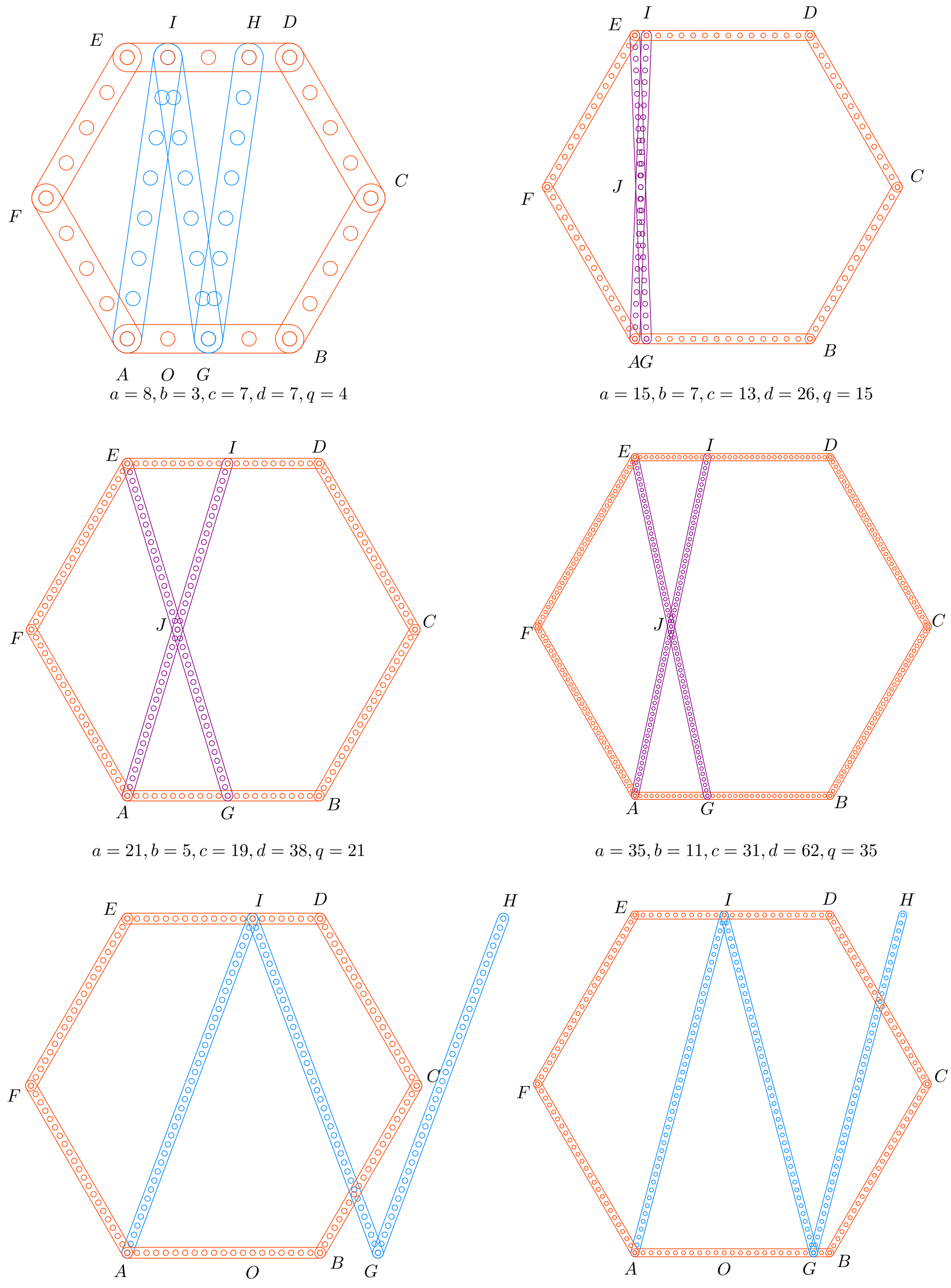


Figure 1: First four cases where internal strip $c = \overline{GC}$ is an integer and makes rigid two consecutive regular hexagon sides $p = \overline{AB} = \overline{BC}$. Our software inspect two integers $a > b$ and looks for $c = \sqrt{a^2 + b^2 - ab}$ to be an integer. In the figures $b = \overline{GB}$ and $a = \overline{GX} = p + b$.

1.1.1 Hexagon height

In figure 1 we have also an equilateral triangle $\triangle GCY$ and an isoscelles triangle $\triangle AGY$. The base of the isoscelles triangle is $x \equiv \overline{AG} = \overline{AB} - \overline{GB} = p - b$ and the equals sides are $\overline{AY} = \overline{GY} = c$. So we can calculate the height $y \equiv \overline{OY}$ substituting c using equation 1:

$$\begin{aligned} y &= \sqrt{(\overline{GY})^2 - (\overline{AO})^2} \\ &= \sqrt{c^2 - \left(\frac{p-b}{2}\right)^2} \\ &= \sqrt{b^2 + p^2 + bp - \left(\frac{p-b}{2}\right)^2} = \frac{(p+b)\sqrt{3}}{2} = \frac{a\sqrt{3}}{2} \end{aligned} \tag{3}$$



$a = 8, b = 3, c = 7, d = 7, q = 4$

$a = 15, b = 7, c = 13, d = 26, q = 15$

$a = 21, b = 5, c = 19, d = 38, q = 21$

$a = 35, b = 11, c = 31, d = 62, q = 35$

$a = 40, b = 7, c = 37, d = 37, q = 20$

$a = 48, b = 13, c = 43, d = 43, q = 24$

Figure 2: First six cases of integral distances c. When the distance $p - b = \overline{AG}$ is even, we use the strips $d = c$ (in blue) to join opposites sides of hexagons of side $q = a/2$. When is odd, we use the strips $d = 2c$ (in purple) to join opposites sides of hexagons of side $q = a$.

We know $\dfrac{a\sqrt{3}}{2}$ is the height of the regular hexagon of side $\dfrac{a}{2}$ so we can use the blue strips to connect opposite sides. Figure 2 show the smaller hexagons that have integer strips connecting opposites sides.

1.1.2 Hexagon rigidity

Through the gallery we will use the green, blue and purple strips and their scaled copies as internal diagonals to make rigid regular hexagons from size 4 to 24. We prioritize minimum number of internal strips (2 or 3) and extra bolts (2 or 3 or 4) and the largest strips sizes as possible. We discard the hexagons which are scaled copies of smaller ones.

2 Hexagons of size $s < 10$

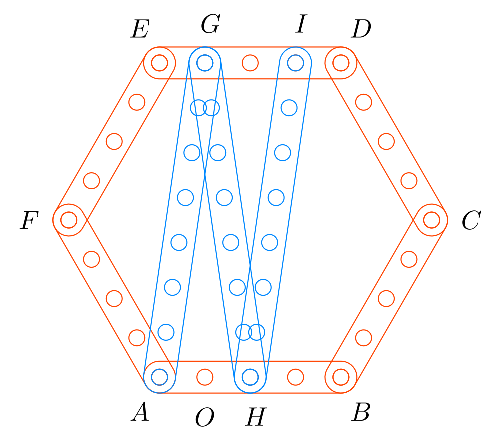


Figure 3: Hexagon of size $s = 4$ with three diagonals $c = \overline{GH} = \overline{HI} = \overline{IG} = 7$.

Figure 3 show a regular hexagon $ABCDEF$ of size 4 with three diagonals of size 7. We confirm the height of the hexagon since the distance $\overline{OG} = \sqrt{(\overline{AG})^2 - (\overline{AO})^2} = \sqrt{7^2 - 1^2} = 4\sqrt{3}$.

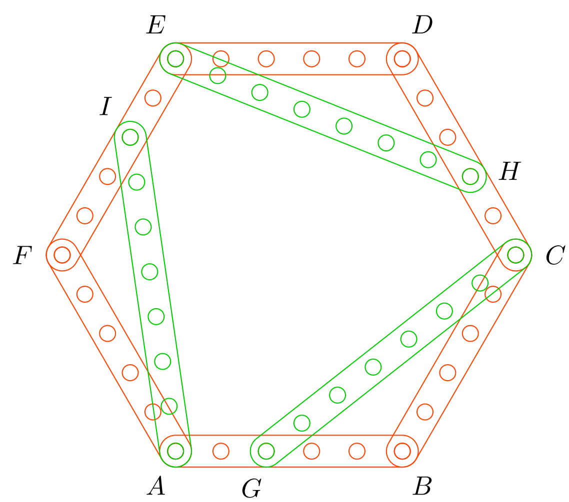


Figure 4: Hexagon of size $s = 5$ with three diagonals $c = \overline{GC} = \overline{HE} = \overline{IA} = 7$.

Figure 4 show a regular hexagon $ABCDEF$ of size 5 with three diagonals of size 7. We confirm the angle of $\alpha \equiv \angle GBC = 120°$ with the law of cosines.

$$\cos\alpha = \frac{(\overline{GB})^2 + (\overline{BC})^2 - (\overline{GC})^2}{2(\overline{GB})(\overline{BC})} = \frac{3^2+5^2-7^2}{2(3)(5)} = -\frac{1}{2}$$

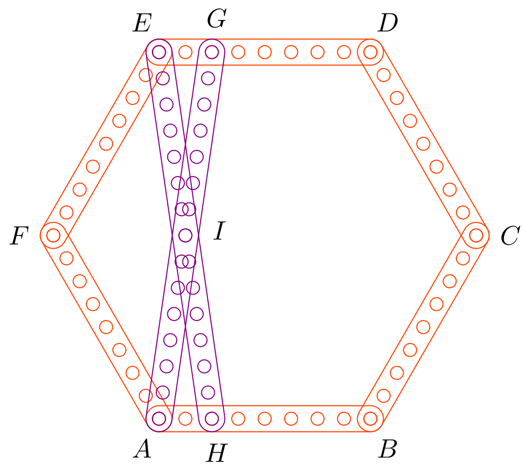


Figure 5: Hexagon of size 8 with **only** two diagonals $\overline{AH} = \overline{EI} = 14$ and extra bolts at G, H, I.

Figure 5 show hexagon $ABCDEF$ of size 8. We confirm the height of the hexagon $\overline{AE} = \sqrt{(\overline{AG})^2 - (\overline{AH})^2} = \sqrt{14^2 - 2^2} = 8\sqrt{3}$.

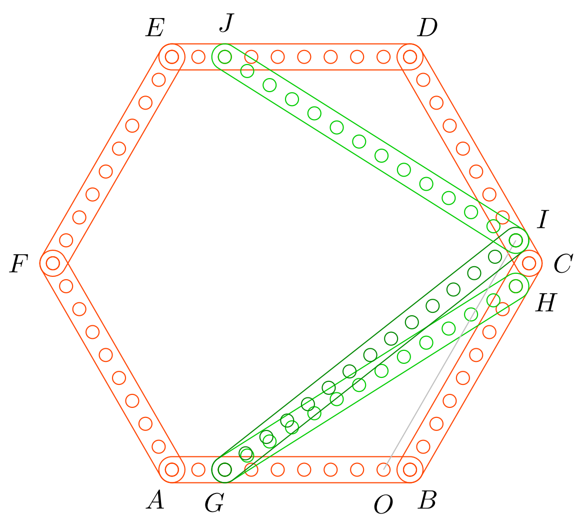


Figure 6: Hexagon of size 9 with three diagonals $\overline{GH} = \overline{IJ} = 13, \overline{GI} = 14$ and extra bolts at G, H, I, J.

Figure 6 show hexagon $ABCDEF$ of size 9. Triangles $\triangle HBG$ and $\triangle JID$ have sides $\{13, 8, 7\}$ which are hexagonal triangles. Triangle $\triangle GIO$ has sides $\{14, 10, 6\}$ which is hexagonal triangle $\{7, 5, 3\}$ multipled by 2.

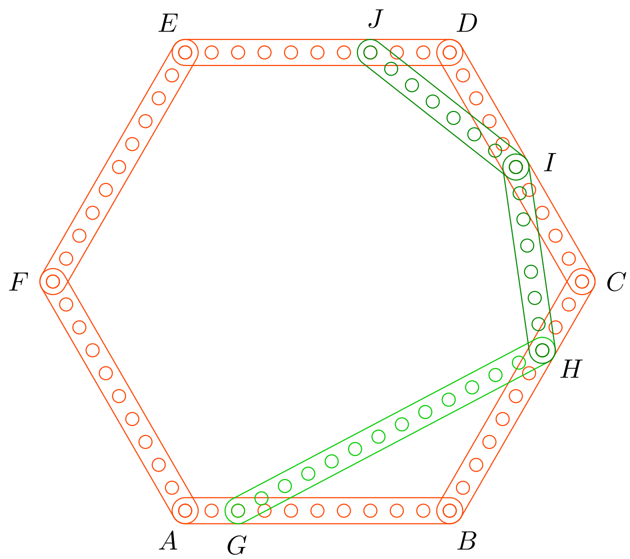


Figure 7: Hexagon of size 10 with three diagonals $\overline{HI} = \overline{IJ} = 7, \overline{GH} = 13$ and extra bolts at G, H, I, J.

Figure 7 show hexagon $ABCDEF$ of size 10. Triangles $\triangle HIC$ and $\triangle JID$ have sides $\{7, 5, 3\}$ which are hexagonal triangles. Triangle $\triangle HGB$ has size $\{13, 8, 7\}$ which is an hexagonal triangle.

3 Hexagons of size $10 < s < 20$

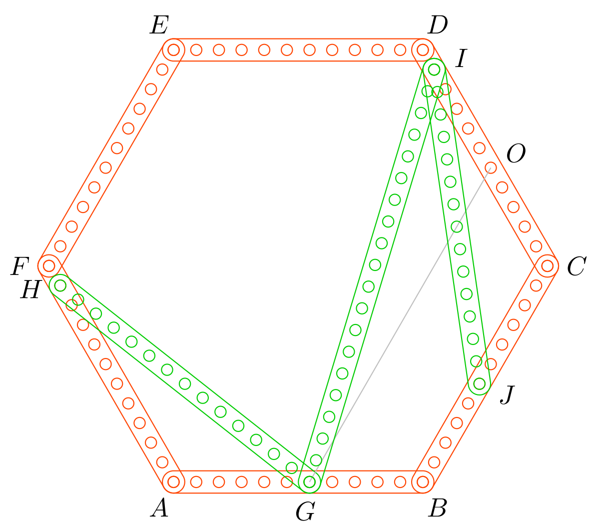


Figure 8: Hexagon of size 11 with three diagonals $\overline{GH} = \overline{IJ} = 14$ and $\overline{IG} = 19$ and four extra bolts in vertices G, H, I, J.

Figure 8 show hexagon $ABCDEF$ of size 11. Triangles $\triangle GHA$ and $\triangle JIC$ have sides $\{14, 10, 6\}$ which are the hexagonal triangle $\{7, 5, 3\}$ multipled by 2. Triangle $\triangle IGO$ has sides $\{19, 16, 5\}$ which is an hexagonal triangle.

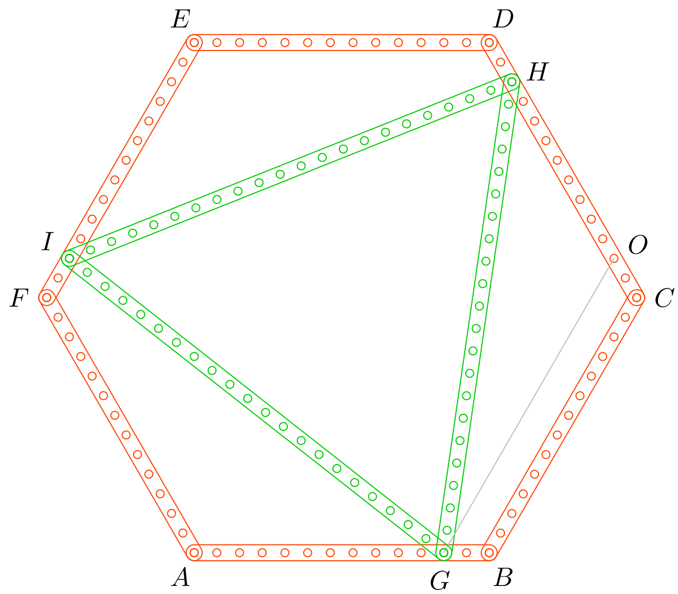


Figure 9: Hexagon of size 13 with three diagonals $\overline{GH} = \overline{HI} = \overline{IG} = 21$ and three bolts at vertices G, H, I.

Figure 9 show hexagon of size $s = 13$. First we note an offset $o \equiv \overline{CO}$ which we use to calculate the sides $\{p', b', c'\}$ of triangle $\triangle GJH$:

$$\begin{aligned} o &\equiv \overline{DH} = \overline{CO} = 2 \\ p' &\equiv \overline{GO} = \overline{BC} + o = 13 + 2 = 15 \\ b' &\equiv \overline{OH} = \overline{CD} - 2o = 13 - 2(2) = 9 \\ c' &\equiv \overline{GH} = 21 \end{aligned} \tag{4}$$

We confirm sides $\{c', p', b'\}$ corresponds to hexagonal triangle $\{7, 5, 3\}$ multipled by 3. This case has an equilateral triangle $\triangle GHI$ inside the hexagon because is a special of the general case when:

$$\begin{aligned} nb &= s - 2o \\ np &= s + o \\ nc &= \sqrt{(nb)^2 + (np)^2 - (nb)(np)} \\ &= \sqrt{(s-2o)^2 + (s+o)^2 + (s-2o)(s+o)} \\ &= \sqrt{3s^2 - 3so + 3o^2} \end{aligned} \tag{5}$$

First terms of this case is shown in the table 2

s	o	c	p	b
13	2	21	15	9
23	1	39	24	21
37	11	57	48	15
59	13	93	72	33

Table 2: Hexagons of size s with inside equilateral triangles of side c.

4 Hexagons of size ≥ 20

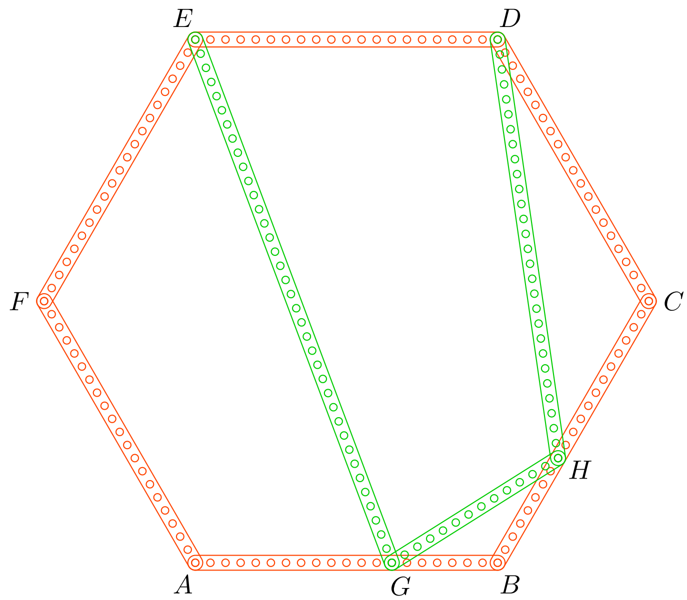


Figure 10: Hexagon of size 20 with three diagonals $\overline{GH} = 13$, $\overline{HD} = 28$ and $\overline{EG} = 37$ and **only** two extra bolts at vertices G and H.

Figure 10 show regular hexagon $ABCDEF$ of size 20. Triangle $\triangle GBH$ sides are $\{13, 8, 7\}$ which is an hexagonal triangle. Triangle $\triangle HCD$ sides are $\{28, 20, 12\}$ which is hexagonal triangle $\{7, 5, 3\}$ scaled by 4. Right triangle $\triangle EAG$ has side $\overline{AE} = \sqrt{(\overline{EG})^2 - (\overline{AG})^2} = \sqrt{37^2 - 13^2} = 20\sqrt{3}$ equals to hexagon height.

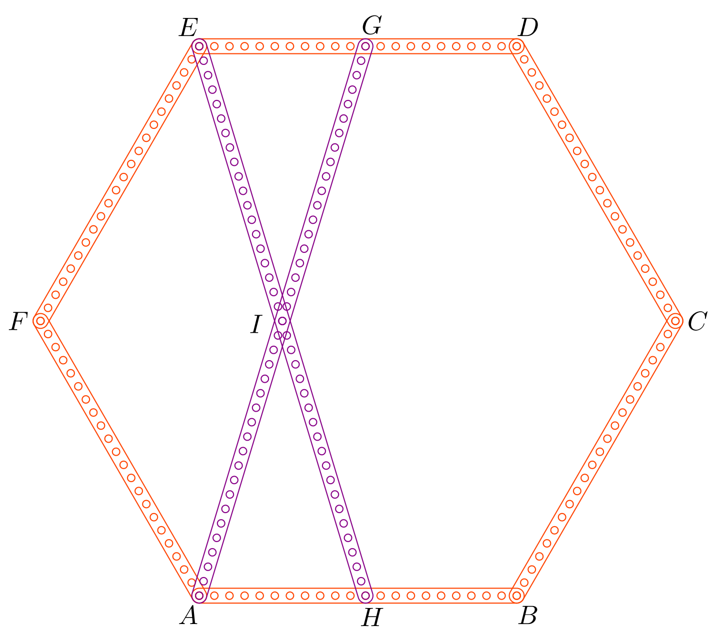


Figure 11: Hexagon of size 21 with **only** two diagonals $\overline{AG} = \overline{EH} = 38$. Segments $\overline{AH} = \overline{EG} = 11$, segment $\overline{AI} = 19$. Three extra bolts at H, G, I.

Figure 11 show regular hexagon $ABCDEF$ of size 21. We confirm the height of the hexagon is

$\overline{AE} = \sqrt{(\overline{EH})^2 - (\overline{AH})^2} = \sqrt{38^2 - 11^2} = 21\sqrt{3}.$

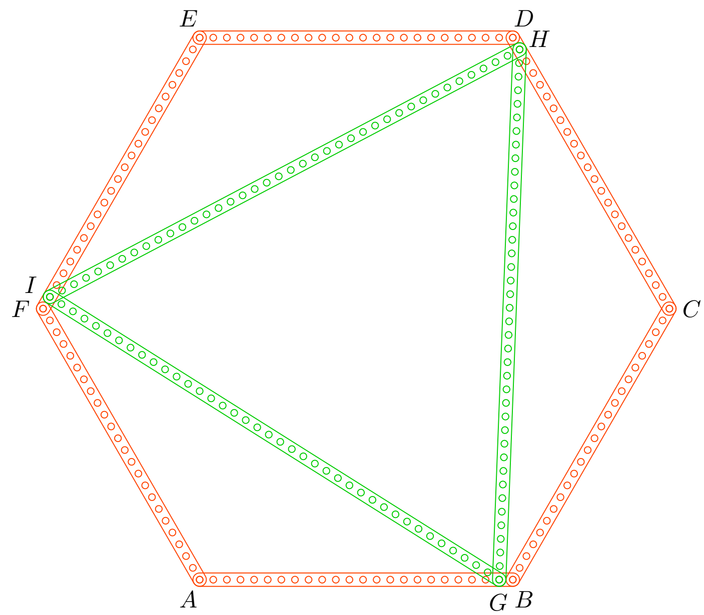


Figure 12: Hexagon of size $s = 23$ with three diagonals $c = \overline{GH} = \overline{HI} = \overline{IG} = 39$.

Figure 12 show a regular hexagon $ABCDEF$ of size $s = 23$. Is the second hexagon having an equilateral triangle inside, in this case of size $c = 39$ and described in table 2.

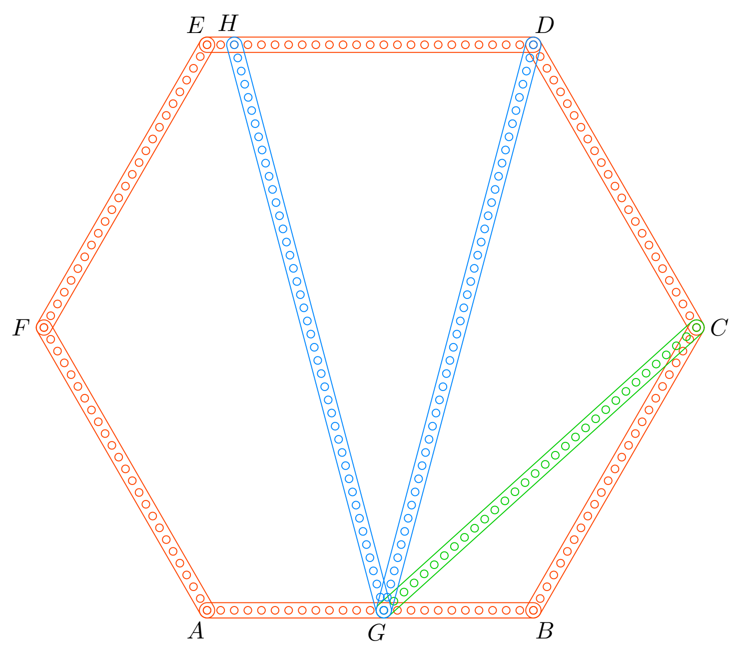


Figure 13: Hexagon of size $s = 24$ with diagonals $\overline{GC} = 31$ and $\overline{GD} = \overline{GH} = 43$ with **only** two extra bolts at vertices G and H. Segments $\overline{GB} = 11$ and $\overline{DH} = 22$.

Figure 13 show regular hexagon $ABCDEF$ of size 24. Triangle $\triangle GBC$ has sides $\{31, 24, 11\}$ which is an hexagonal triangle in table 1. Triangle $\triangle DHG$ is the isosceles triangle shown in figure 2 case $a = 48$.