Meccano frames

https://github.com/heptagons/meccano/frames

Abstract

Meccano frames are groups of rigid meccano [1] strips. Can be used as internal diagonals of polygons we want to be rigid. The lengths of such diagonals are algebraic numbers of the form $\frac{z_2\sqrt{z_3+z_4\sqrt{z_5}}}{z_1}$ which in some cases can be denested as $\frac{z_2+z_3\sqrt{z_4}}{z_1}$ where $z_i \in \mathbb{Z}$.

1 Triangular frame

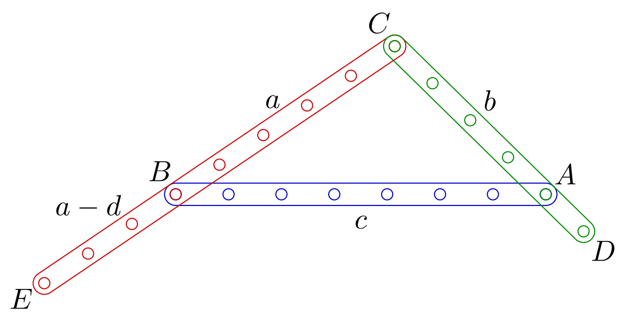

Figure 1: Triangular frame.

Figure 1 shows a triangular frame. With three strips we form the triangle $\triangle ABC$. At least we extend one of the two strips $\overline{CB}$ and $\overline{CA}$ to become $\overline{CE}$ and $\overline{CD}$. The new vertices D and E distance is rigid as the triangle with the form $\frac{z_2\sqrt{z_3}}{z_1}$, where $z_i \in \mathbb{Z}^+$.

First we identify five integer distances a, b, c, d, e:

$$a \equiv \overline{CB}, \quad b \equiv \overline{CA}, \quad c \equiv \overline{AB}, \quad c < a + b \tag{1}$$
$$d \equiv \overline{CB} + \overline{BE} = \overline{CE} \geq a \tag{2}$$
$$e \equiv \overline{CA} + \overline{AD} = \overline{CD} \geq b \tag{3}$$

We calculate the cosine of $\angle BCA$:

$$\theta \equiv \angle BCA \tag{4}$$
$$\cos\theta = \frac{a^2+b^2-c^2}{2ab} \tag{5}$$

We define $g \equiv \overline{DE}$ the triangular frame main diagonal. We calculate the diagonal with the law of

[1] Meccano mathematics by ‘t Hooft

cosines:

$$
\begin{aligned}
g^2 &= \overline{DE}^2 \\
&= \overline{CD}^2 + \overline{CE}^2 - 2\overline{CD} \times \overline{CE} \cos\theta \\
&= d^2 + e^2 - 2de\cos\theta \\
&= d^2 + e^2 - de\left(\frac{a^2+b^2-c^2}{ab}\right) \\
g &= \sqrt{d^2 + e^2 - de\left(\frac{a^2+b^2-c^2}{ab}\right)} \\
&= \frac{\sqrt{a^2b^2(d^2+e^2) - abde(a^2+b^2-c^2)}}{ab} \\
&= \frac{\sqrt{ab((ad-be)(bd-ae)+c^2de)}}{ab} \qquad (6)
\end{aligned}
$$

1.1 Triangular frame software

From the last equation of diagonal g we identify two `input` integers i_1, i_2 which are used to get $g(i)$. Then the nested radicals software will return square-free `output` integers z_1, z_2, z_3 as $g(z)$:

$$i_1 = ab \qquad (7)$$

$$i_2 = ab((ad-be)(bd-ae)+c^2de) \qquad (8)$$

$$g(i) = \frac{\sqrt{i_2}}{i_1} \qquad (9)$$

$$g(z) = \frac{z_2\sqrt{z_3}}{z_1} \qquad (10)$$

We request a software report for all the triangle frames with specific distance $\sqrt{z_3}$ for a given maximum strips length. This report reject the triangles where $z_1, z_2 \neq 1$. Next example report list all the triangles $z_3 = 7, max = 15$ so the software filters are $c < a + b$, $a \leq d \leq max$, $b \leq e \leq max$, $c \leq max$:

```
=== RUN   TestFramesTriangleSurds
NewFrames().TriangleSurds surd=7 max=15
  1) a=1 e=1+2 c=1 cos=1/2
  2) d=1+1 e=1+2 c=1 cos=1/2
  3) d=1+2 b=1 c=1 cos=1/2
  4) d=1+2 e=1+1 c=1 cos=1/2
  5) a=2 e=2+1 c=2 cos=1/2
  6) d=2+1 b=2 c=2 cos=1/2
  7) a=3 e=2+2 c=2 cos=3/4 CED=pi/2
  8) d=3+1 e=2+1 c=2 cos=3/4 CDE=pi/2
  9) d=4+2 e=4+4 c=1 cos=31/32
 10) d=4+4 e=4+2 c=1 cos=31/32
 11) a=7 e=5+1 c=3 cos=13/14
 12) a=7 e=5+2 c=3 cos=13/14
```

The code is in the folder `github.com/heptagons/meccano/frames`.

Figure 2 show four triangles from the mentioned report.

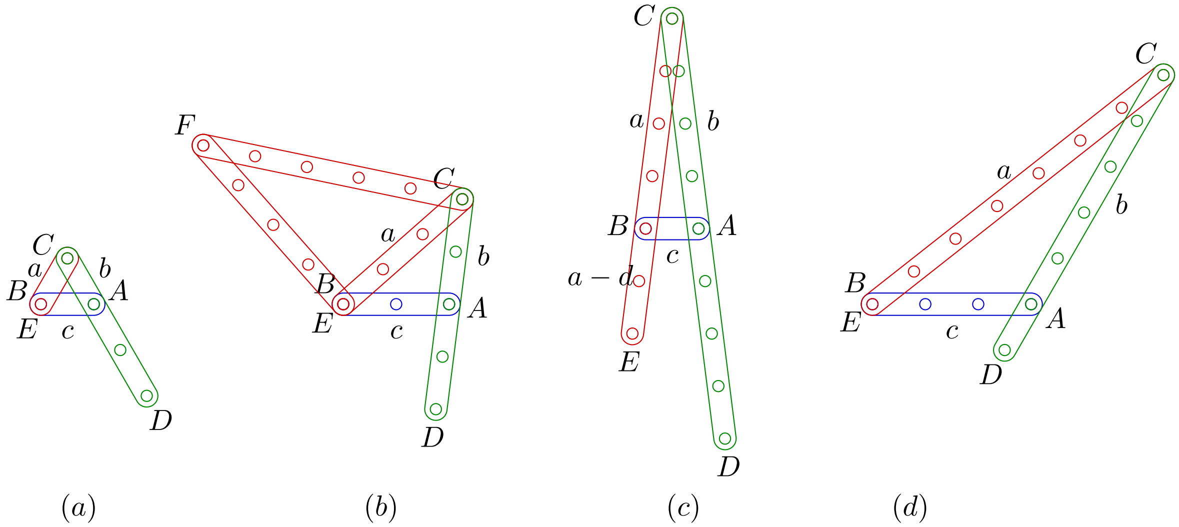

Figure 2: Some triangular frames with distances $g = \overline{DE} = \sqrt{7}$ found by the software.

1.2 Triangular frame distance of the form $\sqrt{z_3} + z_4$

In the figure 2, the particular triangle at (b) was reported with the angle $\angle CED = \pi/2$. For such triangle, if we add a triangle $\triangle CBF$ where angle $\angle CBF = \pi/2$ also, then we'll have vertices D, E, F collinear. With two extra strips we can form a pythagorean triangle sharing the strip a. The figure (b) shows the pythagorean triangle with sides $3, 4, 5$. This five-strips frame has a new distance:

$$\begin{aligned} h &= \overline{DF} \\ &= \overline{DB} + \overline{BF} \\ &= \sqrt{7} + 4. \end{aligned}$$

1.3 Another rigid distances $\sqrt{z_3} + z_4$

We explore a more complicated frame to get additional cases of distances $\sqrt{s} + h$ without relying in an explicit pythagorean triangle as we saw in case (b) of figure 2.

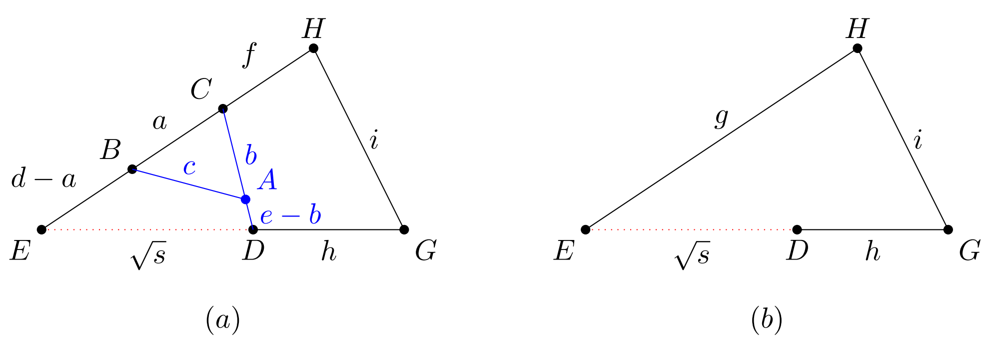

Figure 3: The five strips intented to form an algebraic distance $\overline{EG} = \sqrt{s} + h$.

From figure 3 (a) we know $\sqrt{s}$ distance between nodes E and D is produced by the three strips frame $a + d$, $b + e$ and c. Using the law of cosines we calculate the angle $\theta = \angle CED$ in terms of $\sqrt{s}$:

$$
\begin{aligned}
\cos\theta &= \frac{d^2 + (\sqrt{s})^2 - e^2}{2d\sqrt{s}} \\
&= \frac{(d^2 + s - e^2)\sqrt{s}}{2ds} \qquad (11) \\
&= \frac{m\sqrt{s}}{n} \qquad (12) \\
m &= d^2 + s - e^2 \qquad (13) \\
n &= 2ds \qquad (14)
\end{aligned}
$$

From figure 3 (a) we notice two sets of points are collinear: $\{E, B, C, H\}$ and $\{E, D, G\}$. Using the law of cosines we calculate the angle $\theta = \angle HEG$ in terms of distances $g, \sqrt{s} + h, i$:

$$
\begin{aligned}
\cos\theta &= \frac{g^2 + (\sqrt{s} + h)^2 - i^2}{2g(\sqrt{s} + h)} \\
&= \frac{g^2 + s + 2\sqrt{s}h + h^2 - i^2}{2g(\sqrt{s} + h)} \\
&= \frac{g^2 + s + h^2 - i^2 + 2\sqrt{s}h}{2g(\sqrt{s} + h)} \qquad (15)
\end{aligned}
$$

We multiply both numerator and denominator by $\sqrt{s} - h$ to eliminate the surd from denominator:

$$
\begin{aligned}
\cos\theta &= \frac{(s + g^2 + h^2 - i^2)(\sqrt{s} - h) + 2\sqrt{s}h(\sqrt{s} - h)}{2g(\sqrt{s} + h)(\sqrt{s} - h)} \\
&= \frac{(s + g^2 + h^2 - i^2)(\sqrt{s} - h) + 2sh - 2\sqrt{s}h^2}{2g(s - h^2)} \\
&= \frac{-h(s + g^2 + h^2 - i^2 - 2s) + (s + g^2 + h^2 - i^2 - 2h^2)\sqrt{s}}{2g(s - h^2)} \\
&= \frac{h(s - g^2 - h^2 + i^2) + (s + g^2 - h^2 - i^2)\sqrt{s}}{2g(s - h^2)} \\
&= \frac{o + p\sqrt{s}}{q} \qquad (16) \\
o &= h(s - g^2 - h^2 + i^2) \qquad (17) \\
p &= s + g^2 - h^2 - i^2 \qquad (18) \\
q &= 2g(s - h^2) \qquad (19)
\end{aligned}
$$

We compare both cosines equations 12 and 16:

$$
\frac{m\sqrt{s}}{n} = \frac{o + p\sqrt{s}}{q} \qquad (20)
$$

Since all variables are integers we need two conditions. First o should be zero. And second $\frac{m}{n} = \frac{p}{q}$.

For condition 1, we force o to be zero:

$$
\begin{aligned}
o &= 0 \\
h(s - g^2 - h^2 + i^2) &= 0 \\
s &= g^2 + h^2 - i^2 \qquad (21)
\end{aligned}
$$

For condition2, we force m, n, p, q as:

$$\begin{aligned}
\frac{m}{n} &= \frac{p}{q} \\
\frac{d^2 + s - e^2}{2ds} &= \frac{s + g^2 - h^2 - i^2}{2g(s - h^2)}
\end{aligned} \tag{22}$$

We replace the value of s of last equation RHS with the value of equation 21 of condition 1:

$$\begin{aligned}
\frac{d^2 - e^2 + s}{ds} &= \frac{s + g^2 - h^2 - i^2}{g(s - h^2)} \\
&= \frac{g^2 + h^2 - i^2 + g^2 - h^2 - i^2}{g(g^2 + h^2 - i^2 - h^2)} \\
&= \frac{2(g^2 - i^2)}{g(g^2 - i^2)} \\
&= \frac{2}{g} \\
(d^2 - e^2 + s)g &= 2ds
\end{aligned} \tag{23}$$

$\boxed{TODO: Examples!!!}$

2 Triangle pair frame

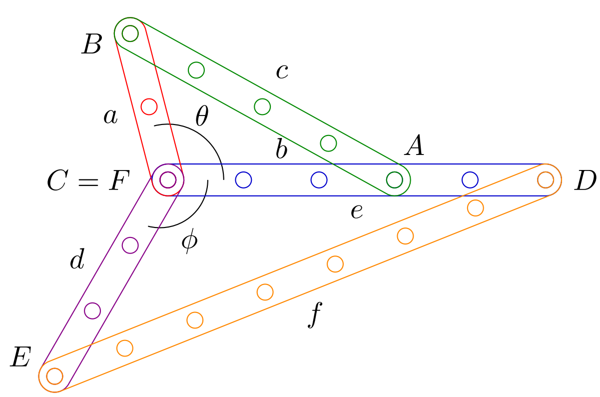

Figure 4: Triangle pair frame.

Figure 4 shows a triangle pair frame. The pair joins triangles $\triangle ABC$ and $\triangle DEF$ in such a way vertices C and F coincide and vertices A, C, D, F be collinear. With only five strips this frame is small and useful to make up the rigid polygons diagonals of the form $g = \overline{BE} = \frac{z_2\sqrt{z_3 + z_4\sqrt{z_5}}}{z_1}, z_i \in \mathbb{Z}$. In some cases the diagonal can be denested to the form $g = \frac{z_2 + z_3\sqrt{z_4}}{z_1}$.

2.1 Triangle pair algebra

Using the law of cosines we calculate the angle $\theta = \angle ACB$ with defined variables m, n and the angle $\phi = \angle DFE$ with defined variables o, p:

$$(\theta, m, n) \equiv (\angle ACB, a^2 + b^2 - c^2, 2ab), \quad |m| \leq n, \quad m, n \in \mathbb{Z} \tag{24}$$

$$\cos\theta = \frac{m}{n} \tag{25}$$

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \frac{\sqrt{n^2 - m^2}}{n} \tag{26}$$

$$(\phi, o, p) \equiv (\angle DFE, d^2 + e^2 - f^2, 2de), \quad |o| \leq p, \quad o, p \in \mathbb{Z} \tag{27}$$

$$\cos\phi = \frac{o}{p} \tag{28}$$

$$\sin\phi = \sqrt{1 - \cos^2\phi} = \frac{\sqrt{p^2 - o^2}}{p} \tag{29}$$

Then, we use the cosines sum identity:

$$\begin{aligned} \cos(\theta + \phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi \\ &= \left(\frac{m}{n}\right)\left(\frac{o}{p}\right) - \left(\frac{\sqrt{n^2 - m^2}}{n}\right)\left(\frac{\sqrt{p^2 - o^2}}{p}\right) \\ &= \frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{np} \end{aligned} \tag{30}$$

Finally we can calculate the distance $g \equiv \overline{BE}$ using the law of cosines:

$$\begin{aligned} g &\equiv \overline{BE} \\ &= \sqrt{a^2 + d^2 - 2ad\cos(\theta + \phi)} \\ &= \sqrt{a^2 + d^2 - 2ad\left(\frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{np}\right)} \\ &= \sqrt{a^2 + d^2 - 2ad\left(\frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{4abde}\right)} \\ &= \sqrt{a^2 + d^2 - \frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{2be}} \\ &= \frac{\sqrt{4b^2e^2(a^2 + d^2) - 2bemo + 2be\sqrt{(n^2 - m^2)(p^2 - o^2)}}}{2be} \end{aligned} \tag{31}$$

2.2 Triangle pairs software

From the last equation of g we identify three `input` integers i_1, i_2, i_3 which are used to get $g(i)$. Then the nested radicals sofware will return square-free `output` integers z_1, z_2, z_3, z_4, z_5 as $g(z)$:

$$i_1 \equiv 2be \tag{32}$$

$$i_2 \equiv i_1^2(a^2 + d^2) - i_1 mo \tag{33}$$

$$i_3 \equiv (n^2 - m^2)(p^2 - o^2) \tag{34}$$

$$g(i) = \frac{\sqrt{i_2 + i_1\sqrt{i_3}}}{i_1} \tag{35}$$

$$g(z) = \frac{z_2\sqrt{z_3 + z_4\sqrt{z_5}}}{z_1} \tag{36}$$

We run a program to print a list of triangle pairs with sides $1 < a, b, c, d, e, f \leq max$ having a given distance $\overline{BE} = g$ or particular $z3, z4, z5$. Next example request a pairs list with $g = z_2\sqrt{46 + 18\sqrt{5}}/z_1$ up to strip length 10 so we set as limits $max = 10, z_3 = 46, z_4 = 18, z_5 = 5$ to get next report (text in blue):

Folder : `github.com/heptagons/meccano/frames`
Call : `NewFrames().TrianglePairsTex(10, [46 18 5])`

$(a, b, c) \oplus (d, e, f) \mapsto g$
$(2, 1, 2) \oplus (3, 3, 3) \mapsto \dfrac{\sqrt{46 + 18\sqrt{5}}}{2}$
$(2, 1, 2) \oplus (3, 8, 7) \mapsto \dfrac{\sqrt{46 + 18\sqrt{5}}}{2}$
$(2, 2, 2) \oplus (3, 6, 6) \mapsto \dfrac{\sqrt{46 + 18\sqrt{5}}}{2}$
$(2, 3, 4) \oplus (3, 5, 7) \mapsto \dfrac{\sqrt{46 + 18\sqrt{5}}}{2}$
$(2, 4, 4) \oplus (3, 8, 7) \mapsto \dfrac{\sqrt{46 + 18\sqrt{5}}}{2}$
$(3, 3, 3) \oplus (2, 4, 4) \mapsto \dfrac{\sqrt{46 + 18\sqrt{5}}}{2}$
$(4, 2, 4) \oplus (6, 6, 6) \mapsto \sqrt{46 + 18\sqrt{5}}$
$(4, 4, 4) \oplus (6, 7, 8) \mapsto \sqrt{46 + 18\sqrt{5}}$
$(6, 3, 6) \oplus (4, 4, 4) \mapsto \sqrt{46 + 18\sqrt{5}}$
$(6, 3, 6) \oplus (9, 9, 9) \mapsto \dfrac{3\sqrt{46 + 18\sqrt{5}}}{2}$
$(6, 6, 6) \oplus (4, 8, 8) \mapsto \sqrt{46 + 18\sqrt{5}}$
$(6, 7, 8) \oplus (9, 9, 9) \mapsto \dfrac{3\sqrt{46 + 18\sqrt{5}}}{2}$

2.3 Triangle pair example

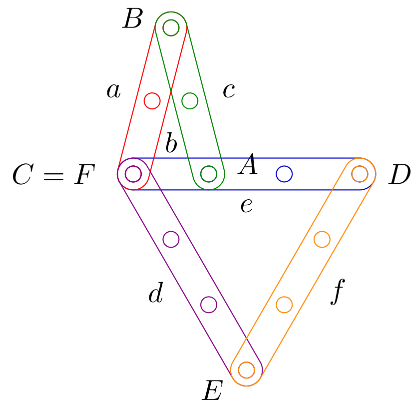

Figure 5: Triangle pair frame $(2, 1, 2) \oplus (3, 3, 3)$ makes $\overline{BE} = \dfrac{\sqrt{46 + 18\sqrt{5}}}{2}$.

In figure 5 we build a triangular pair following one of the last report results, when $abc = (2, 1, 2)$ and $def = (3, 3, 3)$.

3 Triangle pair extended frame

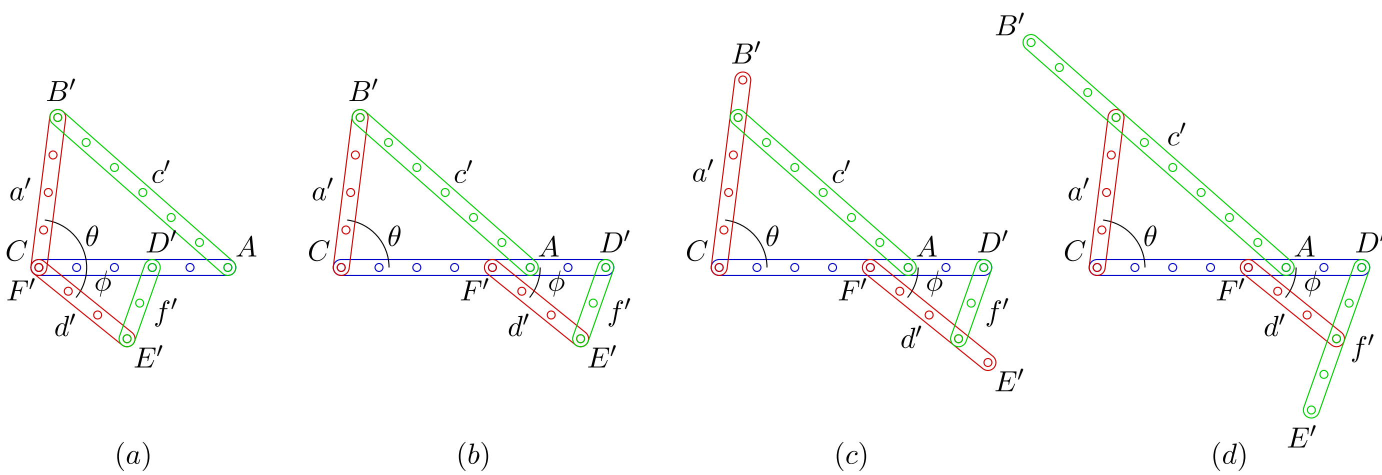

Figure 6: Triangle pair extended frame. Starts like previous triangle pair frame except we can extend strips a' or c', d' or f', and we can separate vertices C and F'. Vertices A, C, D', F' remain collinear and we are interested in the distance $g \equiv \overline{B'E'}$. We show four examples: (a) is the original triangle pair, (b) has moved the $\triangle D'E'F'$ to the right, (c) also extends strips a' and d' and (d) extends strips c' and f'.

We show some triangle pair extended frames in figure 6. As with not-extended triangle pair of figure 4 we also have two triangles with five strips, but we can perform one, two or three transformations on the frame:

1. Separate nodes C and F which moves $\triangle D'E'F'$.
2. Extends strip a' or strip c' but not both.

3. Extend strip d' or strip f' but not both.

For each transformation we define three integers x, y_1, y_2:

$$x = \begin{cases} 0 & C, F \text{ vertices remain joined} \\ \geq 0 & \triangle DEF \text{ is moved to the right a distance equal to } x \end{cases} \tag{37}$$

$$y_1 = \begin{cases} \geq 0 & a' = a + y_1 \quad c' = c \\ < 0 & a' = a \qquad\quad\; c' = c + |y_1| \end{cases} \tag{38}$$

$$y_2 = \begin{cases} \geq 0 & d' = d + y_2 \quad f' = f \\ < 0 & d' = d \qquad\quad\; f' = f + |y_2| \end{cases} \tag{39}$$

Let define $M(a, b, c)$ the triangle above, $N(d, e, f)$ the triangle below and $T(x, y_1, y_2)$ the transformations. Then we can describe the cases $(a) - (d)$ of figure 6 as operations:

$$(a) : M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(0, 0, 0)$$
$$(b) : M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(4, 0, 0)$$
$$(c) : M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(4, +2, +1)$$
$$(d) : M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(4, -3, -2)$$

3.1 Triangle pair extended frame algebra

We are going to calculate the diagonal $g \equiv \overline{B'E'}$ of the triangle pair extended using the M, N, T values. We start setting the vertex C at the origin of the standard two-dimensional graph and defining (B_x, By) the abscissa and ordinate of vertex B' and (E_x, E_y) the abscissa and ordinate of vertex E'.

For the triangle above $M(a, b, c)$ we have two cases: $y_1 \geq 0$ and $y_1 < 0$. In the not-extended triangle pair we already calculated $\theta = \angle ACB, \cos\theta, \sin\theta$ based in m, n of equation 24. For the case $y_1 < 0$ here, we calculate also $\omega = \angle BAC, \cos\omega, \sin\omega$ using two variables p, q and finally we get (B_x, B_y):

$$(\omega, p, q) \equiv (\angle BAC, b^2 + c^2 - a^2, 2bc), \quad |p| \leq q, \quad p, q \in \mathbb{Z} \tag{40}$$

$$\cos\omega = \frac{p}{q} \tag{41}$$

$$\sin\omega = \sqrt{1 - \cos^2\omega} = \frac{\sqrt{q^2 - p^2}}{q} \tag{42}$$

$$a' = a + |y_1| \tag{43}$$

$$c' = c + |y_1| \tag{44}$$

$$B_x = \begin{cases} y_1 \geq 0 & a' \cos\theta \\ y_1 < 0 & b - c' \cos\omega \end{cases} \tag{45}$$

$$B_y = \begin{cases} y_1 \geq 0 & a' \sin\theta \\ y_1 < 0 & c' \sin\omega \end{cases} \tag{46}$$

For the triangle below $N(d, e, f)$ we have two cases: $y_2 \geq 0$ and $y_2 < 0$. In both cases we will use $x \geq 0$ always for simplicity. In the not-extended triangle pair we already calculated $\phi = \angle DFE, \cos\phi, \sin\phi$ defining o, p in equation 27. For the case $y_2 < 0$ here we calculate also $\psi = \angle EDF, \cos\psi, \sin\psi$ using two variables r, s and finally we get (E_x, E_y):

$$(\psi, r, s) \equiv (\angle EDF, e^2 + f^2 - d^2, 2ef), \quad |r| \leq s, \quad r, s \in \mathbb{Z} \tag{47}$$

$$\cos\psi = \frac{r}{s} \tag{48}$$

$$\sin\psi = \sqrt{1 - \cos^2\psi} = \frac{\sqrt{s^2 - r^2}}{s} \tag{49}$$

$$d' = d + |y_2| \tag{50}$$

$$f' = f + |y_2| \tag{51}$$

$$E_x = \begin{cases} y_2 \geq 0 & x + d'\cos\phi \\ y_2 < 0 & x + e - f'\cos\psi \end{cases} \tag{52}$$

$$E_y = \begin{cases} y_2 \geq 0 & d'\sin\phi \\ y_2 < 0 & f'\sin\psi \end{cases} \tag{53}$$

With the four components B_x, B_y, E_x, E_y we can calculate $g = \overline{B'E'}$. For the abscissas we compute a difference $B_x - E_x$ since they both grow to the same direction: to the right. For the ordinates we compute an addition $B_y + E_y$ since they grow in opposite directions: above and below.

$$g = \sqrt{(B_x - E_x)^2 + (B_y + E_y)^2} \tag{54}$$

$$= \sqrt{(B_x^2 + B_y^2) + (E_x^2 + E_y^2) - 2B_xE_x + 2B_yE_y} \tag{55}$$

We need to calculate separated four types of diagonals $g^{++}, g^{+-}, g^{-+}, g^{--}$ according the signs of y_1 and y_2 as described in the next table:

g	y_1	y_2
g^{++}	≥ 0	≥ 0
g^{+-}	≥ 0	< 0
g^{-+}	< 0	≥ 0
g^{--}	< 0	< 0

3.2 Triangle pair extended g^{++} ($y_1 \geq 0$ and $y_2 \geq 0$)

For g^{++} we calculate sums and products of Bx, By, Ex, Ey when $y_1 \geq 0$ and $y_2 \geq 0$:

$$a' = a + u, \qquad d' = d + v \tag{56}$$

$$\begin{aligned}(B_x^2 + B_y^2)^{++} &= a'^2\cos^2\theta + a'^2\sin^2\theta \\ &= a'^2 \end{aligned} \tag{57}$$

$$\begin{aligned}(E_x^2 + E_y^2)^{++} &= (x + d'\cos\phi)^2 + (d'\sin\phi)^2 \\ &= x^2 + 2xd'\cos\phi + d'^2\cos^2\phi + d'^2\sin^2\phi \\ &= x^2 + 2xd'\cos\phi + d'^2 \\ &= x^2 + d'^2 + \frac{2xd'o}{p} \end{aligned} \tag{58}$$

$$\begin{aligned}(B_xE_x)^{++} &= (a'\cos\theta)(x + d'\cos\phi) \\ &= \frac{a'mx}{n} + \frac{a'md'o}{np} \end{aligned} \tag{59}$$

$$\begin{aligned}(B_yE_y)^{++} &= (a'\sin\theta)(d'\sin\phi) \\ &= \frac{a'd'\sqrt{(n^2 - m^2)(p^2 - o^2)}}{np} \end{aligned} \tag{60}$$

We subsitute the products in equation 55:

$$
\begin{aligned}
g^{++} &= \sqrt{(B_x^2+B_y^2)^{++}+(E_x^2+E_y^2)^{++}-2(B_xE_x)^{++}+2(B_yE_y)^{++}} \\
&= \sqrt{a'^2+\left(x^2+d'^2+\frac{2xd'o}{p}\right)-2\left(\frac{a'mx}{n}+\frac{a'md'o}{np}\right)+2\left(\frac{a'd'\sqrt{(n^2-m^2)(p^2-o^2)}}{np}\right)} \\
&= \sqrt{a'^2+x^2+d'^2+\frac{2xd'o}{p}-\frac{2a'mx}{n}-\frac{2a'md'o}{np}+\frac{2a'd'\sqrt{(n^2-m^2)(p^2-o^2)}}{np}} \\
&= \frac{\sqrt{n^2p^2(a'^2+x^2+d'^2)+2xd'on^2p-2a'mxnp^2-2a'md'onp+2a'd'np\sqrt{(n^2-m^2)(p^2-o^2)}}}{np} \\
&= \frac{\sqrt{n^2p^2(a'^2+x^2+d'^2)+2np(xd'on-a'mxp-a'md'o)+2npa'd'\sqrt{(n^2-m^2)(p^2-o^2)}}}{np} \qquad (61)
\end{aligned}
$$

3.3 Triangle pair extended g^{++} software

From the last equation of g^{++} we identify four `input` integer variables to calculate software $g^{++}(i)$ which will be reduced or even denested as $g^{++}(z)$:

$$
\begin{aligned}
i_1 &= np \qquad (62) \\
i_2 &= i_1^2(a'^2+x^2+d'^2)+2i_1(xd'on-a'mxp-a'md'o) \qquad (63) \\
i_3 &= 2i_1a'd' \\
i_4 &= (n^2-m^2)(p^2-o^2) \qquad (64) \\
g^{++}(i) &= \frac{\sqrt{i_2+i_3\sqrt{i_4}}}{i_1} \qquad (65) \\
g^{++}(z) &= \frac{z_2\sqrt{z_3+z_4\sqrt{z_5}}}{z_1} \quad \texttt{or} \quad \frac{z_2+\sqrt{z_3}}{z_1} \qquad (66)
\end{aligned}
$$

We run a program to print a list of triangle pairs extended with sides $1 < a', b, c, d', e, f \leq max$ having a given distance $g = \overline{B'E'}$ for particular z_3, z_4, z_5. Next example request a pairs list with $g = z_2\sqrt{46+18\sqrt{5}}/z_1$ up to strip length 3, but we show just a few (text in blue):

Folder : `github.com/heptagons/meccano/frames`

Call : `NewFrames().TrianglePairsExtPlusPlusTex(3, [46 18 5])`

$M(a,b,c) \oplus N(d,e,f) \oplus T(x,y1,y2) \mapsto g$
$(1,1,1) \oplus (1,2,2) \oplus (0,1,2) \mapsto \frac{\sqrt{46+18\sqrt{5}}}{2}$
$(1,1,1) \oplus (1,2,2) \oplus (0,2,1) \mapsto \frac{\sqrt{46+18\sqrt{5}}}{2}$
$(1,1,1) \oplus (2,1,2) \oplus (0,1,1) \mapsto \frac{\sqrt{46+18\sqrt{5}}}{2}$
$(1,1,1) \oplus (2,1,2) \oplus (0,2,0) \mapsto \frac{\sqrt{46+18\sqrt{5}}}{2}$
$(1,1,1) \oplus (2,1,2) \oplus (2,2,0) \mapsto \frac{\sqrt{46+18\sqrt{5}}}{2}$
$(1,2,2) \oplus (1,1,1) \oplus (0,1,2) \mapsto \frac{\sqrt{46+18\sqrt{5}}}{2}$
$(1,2,2) \oplus (1,1,1) \oplus (0,2,1) \mapsto \frac{\sqrt{46+18\sqrt{5}}}{2}$
$(1,2,2) \oplus (2,2,2) \oplus (0,1,1) \mapsto \frac{\sqrt{46+18\sqrt{5}}}{2}$
$(1,2,2) \oplus (2,2,2) \oplus (0,2,0) \mapsto \frac{\sqrt{46+18\sqrt{5}}}{2}$
$(1,2,2) \oplus (3,3,3) \oplus (0,1,0) \mapsto \frac{\sqrt{46+18\sqrt{5}}}{2}$

3.4 Triangle pair extended examples

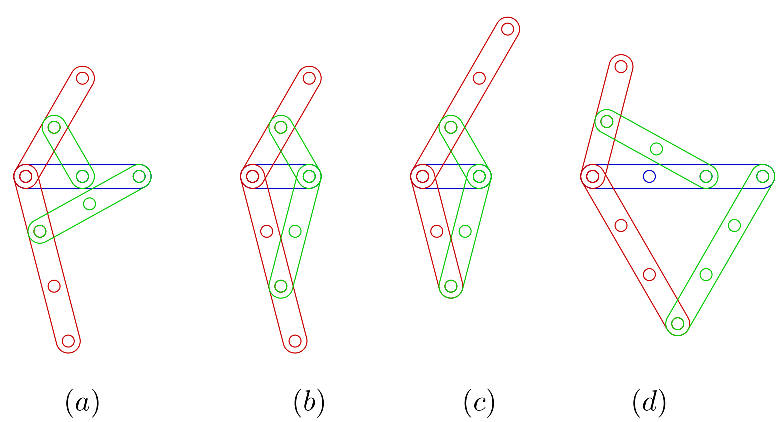

Figure 7: Four examples of $\overline{B'E'} = \frac{\sqrt{46+18\sqrt{5}}}{2}$ taken from the last report. B' is the top vertice of the red strip above and E' is the bottom vertice of the red strip below.

Figure 7 show some pairs from the last report.

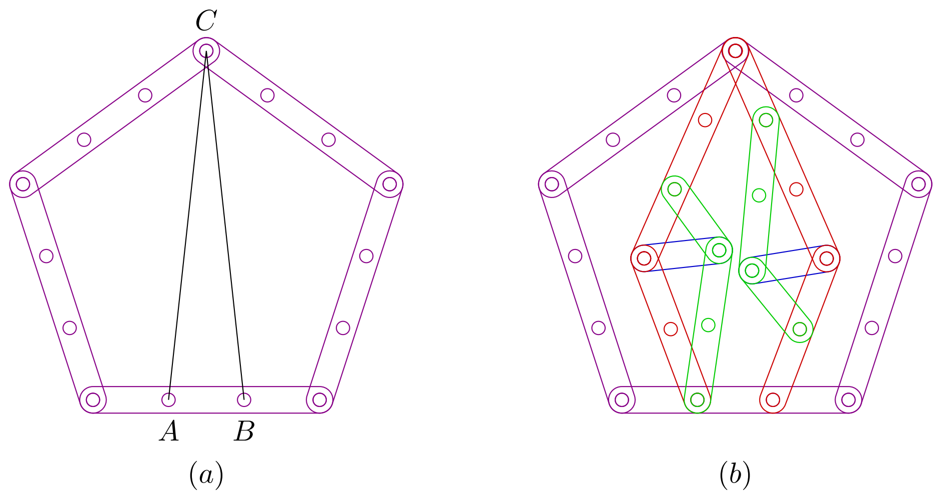

Figure 8: Regular pentagon made rigid with two triangle pairs extended frames as diagonals. In (a) we calculate the distances $\overline{AC} = \overline{BC} = \frac{\sqrt{46+18\sqrt{5}}}{2}$. In (b) we replace the distances with two frames shown in figure 7. The pentagon is rigid since triangle ABC is rigid.

In figure 8 we show a pentagon made rigid using the frames found so far. In (a) we show the pentagon of size 3 and two diagonals $\overline{AB} = \overline{BC}$. We know every regular pentagon height is $H = t\frac{\sqrt{5+2\sqrt{5}}}{2}$ where t is the pentagon's side, in our case $t = 3$. Then $\overline{AB} = \sqrt{H^2 + (\frac{1}{2})^2} = \frac{\sqrt{46+18\sqrt{5}}}{2}$. This is exactly the value of the diagonals we were looking for above. In (b) we place two diagonals formed with triangles pairs small and narrow enough to fit inside the pentagon preventing collapsing strips or bolts.