# Meccano fox-surd frame

https://github.com/heptagons/meccano/frames/fox-surd

**Abstract**

Meccano [1] fox-surd frame is a generalization of fox-frame[2] where at least one of the frame's strips size is no longer an integer but a surd.

## 1 Pentagons fox-surd

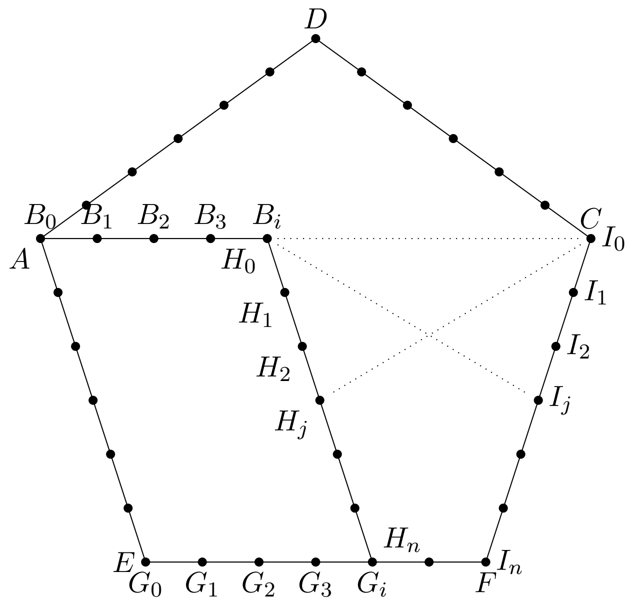


Figure 1: Pentagon of size $n$ where each segment separated by circles represents a unit. We have a surd frame formed by the six points: $B_i$, $I_0$, $H_j$, $I_j$, $H_n$ and $I_n$. By iterating the values $i, j = 0, ..., n$ we'll get diverse frames.

From figure 1 the fox-surd frame has three real strips of integer size:

- $\overline{B_iG_i}$ of size $n$.
- $\overline{G_iI_n}$ of size $n - i$, where $i = 0, ..., n$.
- $\overline{I_0I_n}$ of size $n$.

The other two strips are generic in the sense the sizes can be surds:

- $\overline{B_iI_j}$ of size to be determined $f(n, i, j)$, where $i, j = 0, ..., n$.
- $\overline{H_jI_0}$ of equal size of $\overline{B_iI_j}$.

[1] Meccano mathematics by 't Hooft
[2] Meccano fox frame

From the regular pentagon we know the main diagonal $\overline{AC}$ equals $\frac{1+\sqrt{5}}{2}n$ where $n$ is the pentagon side size. We can calculate different segments of the main diagonal iterating $i = 0, ..., n$:

$$\begin{aligned}
B_0 &\equiv A \\
\overline{B_0C} &= \frac{1+\sqrt{5}}{2}n \qquad (1) \\
\overline{B_iC} &= \frac{1+\sqrt{5}}{2}n - i \\
&= \frac{n-2i}{2} + \frac{n\sqrt{5}}{2}, \quad i = 0, ..., n \\
&= \frac{x_i}{2} + \frac{n\sqrt{5}}{2}, \quad x_i = n - 2i \qquad (2)
\end{aligned}$$

From the regular pentagon we know the angle $B_iCI_j$ equals $2\pi/5$ so we have:

$$\theta \equiv \angle B_iCI_j \qquad (3)$$

$$\cos\theta = \cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4} \qquad (4)$$

## 1.1 Pentagon surds sizes

Using the law of cosines we can calculate one of the frame surds $s_{ij} \equiv \overline{B_iI_j}$. We notice the value of $\overline{CI_j}$ equals $j$, and we'll use the values of $\overline{B_iC}$ from equation 2, and the cosine value from equation 4 to get:

$$\begin{aligned}
s_{ij}^2 &\equiv \overline{B_iI_j}^2 \qquad (5) \\
&= \overline{CI_j}^2 + \overline{B_iC}^2 - 2\overline{CI_j} \times \overline{B_iC}\cos\theta \\
&= j^2 + \left(\frac{x_i}{2} + \frac{n\sqrt{5}}{2}\right)^2 - 2j\left(\frac{x_i}{2} + \frac{n\sqrt{5}}{2}\right)\left(\frac{\sqrt{5}-1}{4}\right) \qquad (6) \\
&= j^2 + \frac{1}{4}\left(x_i + n\sqrt{5}\right)^2 - \frac{j}{4}\left(x_i + n\sqrt{5}\right)\left(\sqrt{5}-1\right) \qquad (7)
\end{aligned}$$

We multiply both sides by 4 and simplify:

$$\begin{aligned}
(2s_{ij})^2 &= 4j^2 + (x_i + n\sqrt{5})^2 - j(x_i + n\sqrt{5})(\sqrt{5}-1) \qquad (8) \\
&= 4j^2 + x_i^2 + 2x_in\sqrt{5} + 5n^2 - j(x_i\sqrt{5} - x_i + 5n - n\sqrt{5}) \qquad (9) \\
&= 4j^2 + x_i^2 + 5n^2 + x_ij - 5nj + (2nx_i - x_ij + nj)\sqrt{5} \qquad (10)
\end{aligned}$$

In order to have a simpler $(2s_i)^2 = u + v\sqrt{5}$ we define two variables $u$ and $v$. We replace again $x_i = n - 4i$ defined in equation 2:

$$\begin{aligned}
u &\equiv 4j^2 + x_i^2 + 5n^2 + x_ij - 5nj \\
&= 4j^2 + (n-2i)^2 + 5n^2 + (n-2i)j - 5nj \\
&= 4j^2 + n^2 - 4ni + 4i^2 + 5n^2 + nj - 2ij - 5nj \\
&= 6n^2 + 4i^2 + 4j^2 - 4ni - 4nj - 2ij \qquad (11) \\
v &\equiv 2nx_i - x_ij + nj \\
&= 2n(n-2i) - (n-2i)j + nj \\
&= 2n^2 - 4ni - nj + 2ij + nj \\
&= 2n^2 - 4ni + 2ij \qquad (12) \\
& \qquad (13)
\end{aligned}$$

Finally we have $s_{ij}$ in function of $n, i, j$ the side:

$$\begin{aligned} s_{ij} &= \frac{\sqrt{u + v\sqrt{5}}}{2} \\ &= \frac{\sqrt{6n^2 + 4i^2 + 4j^2 - 4ni - 4nj - 2ij + (2n^2 - 4ni + 2ij)\sqrt{5}}}{2} \end{aligned} \tag{14}$$

### 1.2 Pentagons surds simplification

If value $v$ from equation 12 is zero $s_{ij}$ simplifies to $\sqrt{u}/2$:

$$\begin{aligned} v &= 0 \\ 2n^2 - 4ni + 2ij &= 0 \\ n^2 - 2ni + ij &= 0 \end{aligned} \tag{15}$$