Meccano pentagons

https://github.com/heptagons/meccano/penta

1 Meccano pentagons

To identify a pentagon we use two angles A and B. Some identities are solved for $a + b\sqrt{5}$ values to be used later.

$$
\begin{aligned}
5A &= 2\pi \\
5B &= \pi \\
4\cos(A) &= -1 + \sqrt{5} \\
4\cos(B) &= 1 + \sqrt{5} \\
8\cos^2(A) &= 3 - \sqrt{5} \\
8\cos^2(B) &= 3 + \sqrt{5} \\
4\cos(A)\cos(B) &= 1 \\
8\sin^2(A) &= 5 + \sqrt{5} \\
8\sin^2(B) &= 5 - \sqrt{5} \\
4\sin(A)\sin(B) &= \sqrt{5}
\end{aligned}
$$

1.1 Pentagons of type 1

A pentagon of type 1 is shown in figure 1. We note three rods (or sections of rods) a, b, and c at fixed angles and with integer sizes as it should be for any meccano figure. We want to find the fourth rod d which also needs to be of integer size to make the pentagon.

We start by looking the rods' related formulas:

$$
\begin{aligned}
d_x^2 &= ((a+c)\cos(A) + b)^2 \\
&= (a+c)^2\cos^2(A) + 2(a+c)b\cos(A) + b^2 \\
d_y^2 &= ((a-c)\sin(A))^2 \\
&= (a-c)^2\sin^2(A) \\
d^2 &= d_x^2 + d_y^2 \\
&= (a+c)^2\cos^2(A) + (a-c)^2\sin^2(A) + 2(a+c)b\cos(A) + b^2 \\
&= (a+c)^2(3-\sqrt{5})/8 \\
&\qquad + (a-c)(5+\sqrt{5})/8 \\
&\qquad + 2(a+c)b(-1+\sqrt{5})/4 \\
&\qquad + b^2 \\
&= m\sqrt{5} + n
\end{aligned}
$$

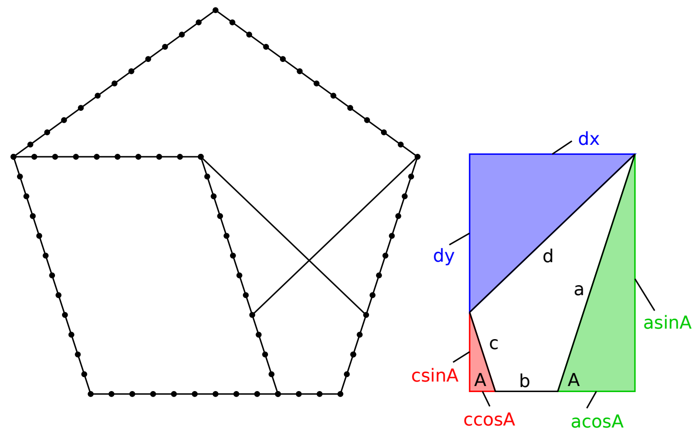

Figure 1: Meccano pentagon of type 1. From rods a, b and c with integer lengths we expect to find the rod d length as integer too. Actually, the pentagon shown is the unique solved so far for small values of rods, $a = 12$.

We define two variables m and n. m is the sum of all the terms multiplied by $\sqrt{5}$ while n is the sum of all the terms not multipled by $\sqrt{5}$.

$$\begin{aligned} 8m &= -(a+c)^2 + (a-c)^2 + 4(a+c)b \\ &= 4(a+c)b - 4ac \\ 8n &= 3(a+c)^2 + 5(a-c)^2 - 4(a+c)b + 8b^2 \end{aligned}$$

Simplifying we get a value for the rod d^2 in function of the rest of rods.

$$\begin{aligned} m &= \frac{ab - ac + bc}{2} \\ n &= a^2 + b^2 + c^2 - \frac{ab + ac + bc}{2} \\ &= a^2 + b^2 + c^2 - ac - m \\ d^2 &= m\sqrt{5} + a^2 + b^2 + c^2 - ac - m \end{aligned}$$

Now, we want rod d to be as simple as possible so is good idea to make $m = 0$ wich requires $ac = (a+c)b$. This way the rod d is a simpler function of a, b and c.

$$\begin{aligned} ac &= (a+c)b \\ d &= \sqrt{a^2 + b^2 + c^2 - ac} \end{aligned}$$

1.1.1 Pentagon type 1 search

With last equations, a program can iterate over the integer values of the rods a, b and c to discover the rod d to be integer too. Next javascript program was run and found a single solution $a = 12, b = 3, c = 4, d = 11$ after 5000 iterations. Scaled solutions are discarded as repetitions.

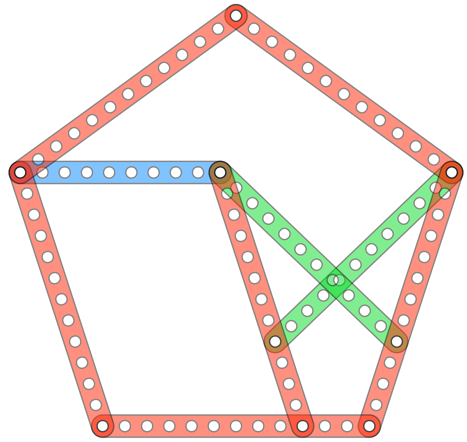

Figure 2: The smallest and maybe unique of pentagons of type 1. Is composed of 6 rods of length 12 in color red, 2 rods of length 11 in green and 1 rod of size 9 in blue.

```
function meccano_pentagons_1(sols)
{
  this.find = (max)=> {
    for (let a=1; a < max; a++)
      for (let b=1; b <= max; b++)
        for (let c=0; c <= a; c++)
          if (a*c == (a + c)*b)
            mZero(a, b, c)
  }

  const mZero = (a, b, c)=> {
    const d = Math.sqrt(a*a + b*b + c*c - a*c)
    if (d > 0 && d % 1 === 0)
      dInteger(a, b, c, d)
  }

  const dInteger = (a, b, c, d) => {
    for (let i=0; i < sols.length; i++) {
      const s = sols[i]
      if (a % s.a == 0) {
        const f = a / s.a
        const bS = (b % s.b == 0) && b / s.b == f
        const cS = (c % s.c == 0) && c / s.c == f
        const dS = (d % s.d == 0) && d / s.d == f
        if (bS && cS && dS)
          return // scaled solution already
      }
    }
    sols.push({ a:a, b:b, c:c, d:d }) // solution!
  }
}
```

1.1.2 Pentagons of type 1 results

Figure 2 shows the first pentagon of type 1 found.

1.2 Pentagons of type 2

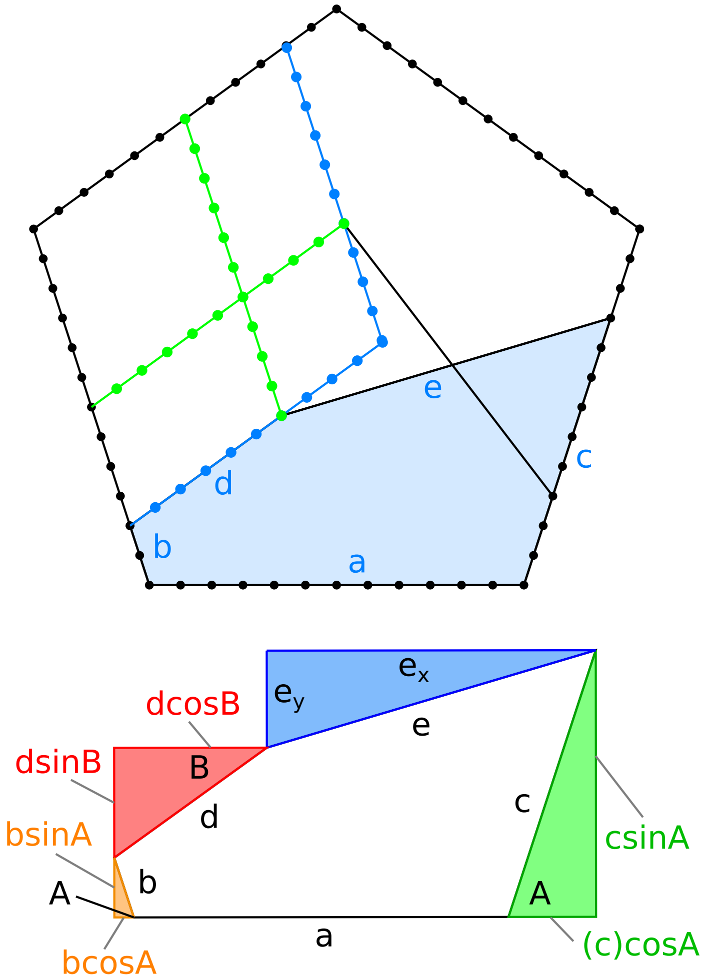

Figure 3: Meccano pentagon of type 2. For rods a, b, c and d with integer lengths we expect to find the rod e with integer length too. Actually, the example shown is the smallest found $a = 12, b = 2, c = 9, d = 6, e = 11$. For each solution there are two versions whether the green rods are used or the blue ones.

A pentagon of type 2 is shown in figure 3. We identify in this type of pentagon four rods a, b, c and d at fixed angles. We want to find a fifth rod e with integer length to make the pentagon.

We start with the rods relation formulas

$$
\begin{aligned}
e_x &= b\cos(A) + a + c\cos(A) - d\cos(B) \\
&= a + (b+c)\cos(A) - d\cos(B) \\
e_y &= c\sin(A) - b\sin(A) - d\sin(B) \\
&= (c-b)\sin(A) - d\sin(B) \\
e^2 &= e_x^2 + e_y^2 \\
&= a^2 + (b+c)^2\cos^2(A) + d^2\cos^2(B) \\
&\quad + 2a(b+c)\cos(A) - 2ad\cos(B) \\
&\quad - 2(b+c)d\cos(A)cos(B) \\
&\quad + (c-b)^2\sin^2(A) + d^2\sin^2(B) \\
&\quad - 2(c-b)d\sin(A)\sin(B) \\
&= a^2 - 2(b+c)d/4 \\
&\quad + (b+c)^2(3-\sqrt{5})/8 \\
&\quad + d^2(3+\sqrt{5})/8 \\
&\quad + 2a(b+c)(-1+\sqrt{5})/4 \\
&\quad - 2ad(1+\sqrt{5})/4 \\
&\quad + (c-b)^2(5+\sqrt{5})/8 \\
&\quad + d^2(5-\sqrt{5})/8 \\
&\quad - 2(c-b)d(\sqrt{5})/4 \\
&= m\sqrt{5} + n
\end{aligned}
$$

As we did with the pentagon type 1, we define variables m and n:

$$
\begin{aligned}
8m &= -(b+c)^2 + d^2 + 4a(b+c) - 4ad + (c-b)^2 - d^2 - 4(c-b)d \\
8n &= 8a^2 + 3(b+c)^2 + 3d^2 - 4a(b+c) - 4ad - 4(b+c)d + 5(c-b)^2 + 5d^2
\end{aligned}
$$

Simplifying, we get a value for rod e in function of the rest of rods:

$$
\begin{aligned}
m &= \frac{(a-b)(c-d) + ab - cd}{2} \\
n &= a^2 + b^2 + c^2 + d^2 - \frac{(a+b)(c+d) + ab + cd}{2} \\
&= a^2 + b^2 + c^2 + d^2 - ad - bc - cd - m \\
e^2 &= m\sqrt{5} + a^2 + b^2 + c^2 + d^2 - ad - bc - cd - m
\end{aligned}
$$

Again we decide to make $m = 0$ which now requires $cd = (a-b)(c-d) + ab$. This way the rod e is a simple function of rods a, b, c and d:

$$
\begin{aligned}
cd &= (a-b)(c-d) + ab \\
e &= \sqrt{a^2 + b^2 + c^2 + d^2 - ad - bc - cd}
\end{aligned}
$$

1.2.1 Pentagon type 2 search

With last equations, another program, for the pentagon type 2, can iterate over the integer values of rods a, b, c and d to discover a rod e with integer length too. Next javascript program was run and found 40 different pentagons with rods length <= 183.

```
func pentagons_type_2(max int) {

  sols := make([][]int, 0)

  add := func(a, b, c, d, e int) {
    for _, s := range sols {
      if a % s[0] != 0 { continue }
      // new a is a factor of previous a
      f := a / s[0]
      if t := b % s[1] == 0 && b / s[1] == f; !t { continue }
      if t := c % s[2] == 0 && c / s[2] == f; !t { continue }
      if t := d % s[3] == 0 && d / s[3] == f; !t { continue }
      if t := e % s[4] == 0 && e / s[4] == f; !t { continue }
      return // scaled solution already found (reject)
    }
    // solution!
    sols = append(sols, []int{ a, b, c, d, e })
    fmt.Printf("%3d a=%3d b=%3d c=%3d d=%3d e=%3d\n", len(sols), a, b, c, d, e)
  }

  check := func(a, b, c, d int) {
    f := float64(a*a + b*b + c*c + d*d - a*d - b*c - c*d)
      if f < 0 {
        return
      }
    if e := int(math.Sqrt(f)); math.Pow(float64(e), 2) == f {
      add(a, b, c, d, e)
    }
  }

    for a := 1 ; a < max; a++ {
      for b := 1; b < a; b++ {
          for c := 1; c < a; c++ {
              for d := 1; d < a; d++ {
                if ((a - b)*(c - d) + a*b == c*d) {
                    check(a, b, c, d)
                  }
                }
            }
        }
    }
}
```

1.3 Type 2 results

The program found as much as 124 pentagons of type 2 for a <= 488.

```
  1 a= 12 b=  2 c=  9 d=  6 e= 11
  2 a= 12 b=  6 c=  3 d= 10 e= 11
  3 a= 31 b=  4 c= 28 d= 16 e= 31
  4 a= 31 b= 15 c=  3 d= 27 e= 31
```

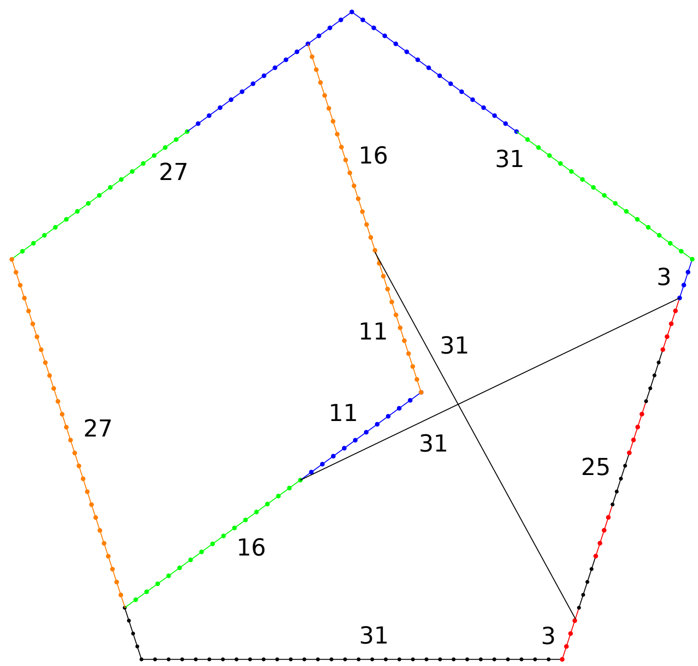

Figure 4: Pentagon of type 2 with $a = 31$. This construction requires 7 rods of length 31 and 2 rods of length 27.

```
  5  a= 38  b= 12  c= 18  d= 21  e= 31
  6  a= 38  b= 17  c= 20  d= 26  e= 31
  7  a= 48  b=  8  c= 24  d= 21  e= 41
  8  a= 48  b= 12  c=  9  d= 20  e= 41
  9  a= 48  b= 27  c= 24  d= 40  e= 41
 10  a= 48  b= 28  c= 39  d= 36  e= 41
 11  a= 72  b= 21  c= 48  d= 40  e= 61
 12  a= 72  b= 24  c= 16  d= 39  e= 61
 13  a= 72  b= 32  c= 24  d= 51  e= 61
 14  a= 72  b= 33  c= 56  d= 48  e= 61
 15  a= 78  b= 27  c=  4  d= 42  e= 71
 16  a= 78  b= 36  c= 74  d= 51  e= 71
 .  .  .
119  a=488  b= 72  c= 15  d= 96  e=451
120  a=488  b=132  c=423  d=276  e=451
121  a=488  b=152  c=269  d=272  e=401
122  a=488  b=212  c= 65  d=356  e=451
123  a=488  b=216  c=219  d=336  e=401
124  a=488  b=392  c=473  d=416  e=451
```

Figures 4, 5 and 6 show some of the pentagons of type 2 found.

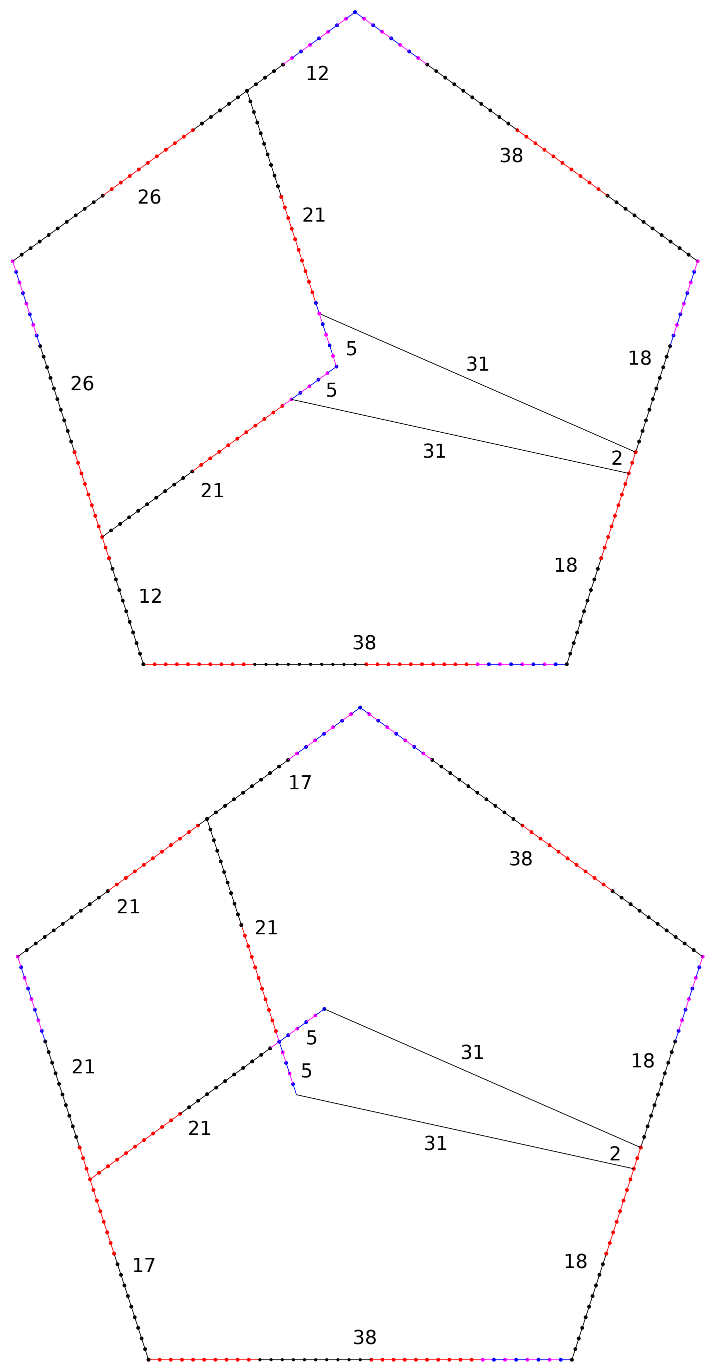

Figure 5: Pentagons of type 2 with $a = 38$. Each construction requires 5 rods of length 38, 2 rods of length 31 and 2 rods of length 26

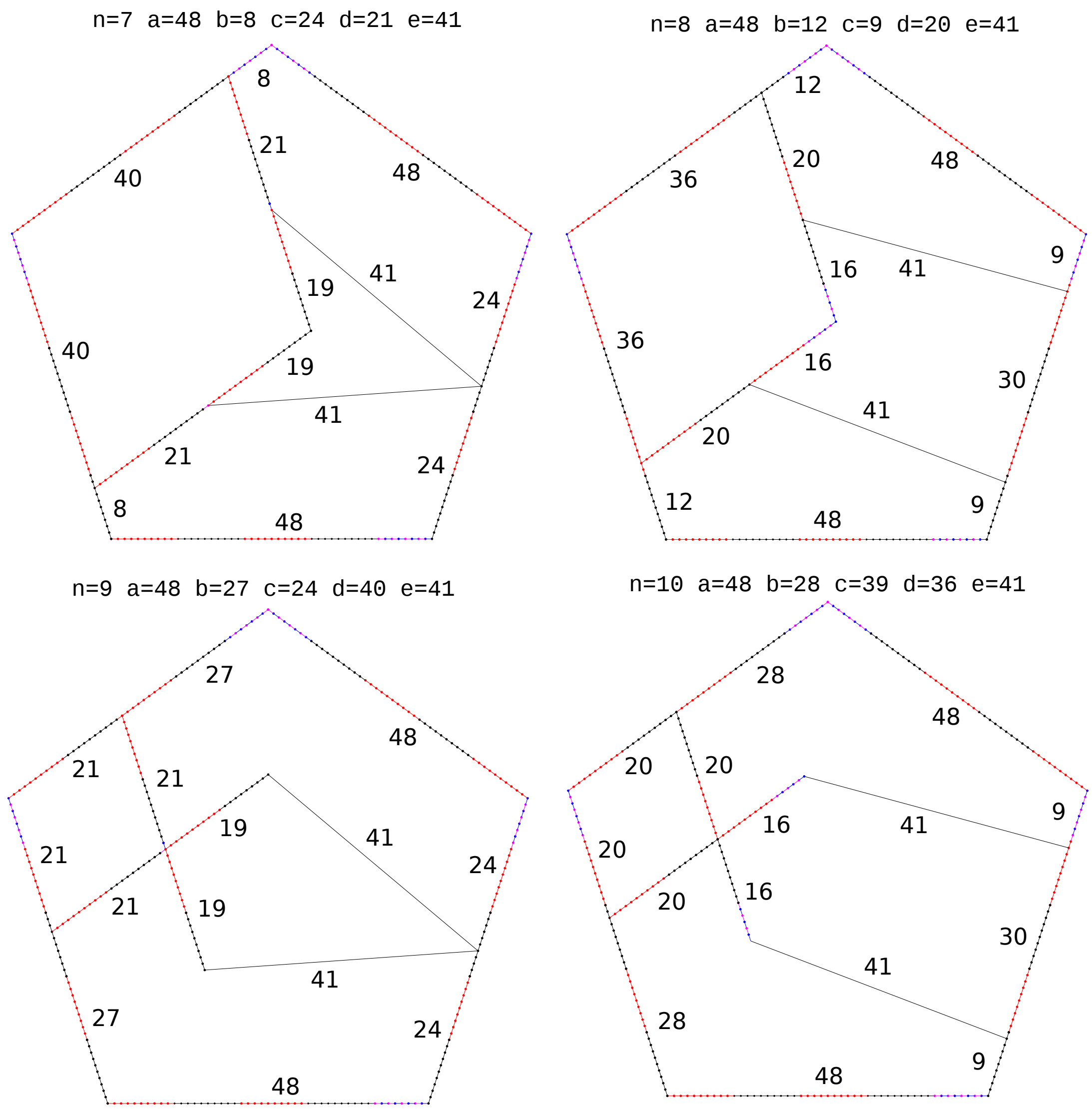

Figure 6: Pentagons of type 2 with $a = 48$