Meccano frames

https://github.com/heptagons/meccano/frames

Abstract

Meccano frames are groups of rigid meccano [1] strips. Can be used as internal diagonals of polygons to be rigid. The lengths of such diagonals are algebraic numbers of the form $B + \frac{C\sqrt{D}}{A}$ or $\frac{\sqrt{F+H\sqrt{G}}}{A}$.

1 Triangular frame

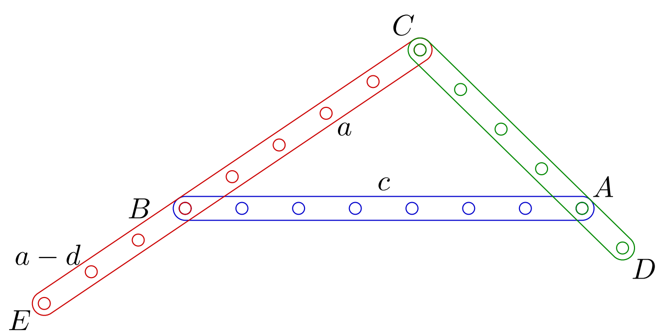

Figure 1: Triangular frame.

Figure 1 shows a triangular frame. With three strips we form the triangle $\triangle ABC$. At least we extend one of the two strips $\overline{CB}$ and $\overline{CA}$ to become $\overline{CE}$ and $\overline{CD}$. The new vertices D and E distance is rigid of the form $\dfrac{p\sqrt{s}}{q}$, where $p, q, s \in \mathbb{Z}^+$.

First we identify five integer distances a, b, c, d, e:

$$a \equiv \overline{CB}, \quad b \equiv \overline{CA}, \quad c \equiv \overline{AB}, \quad c < a + b \tag{1}$$

$$d \equiv \overline{CB} + \overline{BE} = \overline{CE} \geq a \tag{2}$$

$$e \equiv \overline{CA} + \overline{AD} = \overline{CD} \geq b \tag{3}$$

We calculate the cosine of $\angle BCA$:

$$\theta \equiv \angle BCA \tag{4}$$

$$\cos\theta = \frac{a^2 + b^2 - c^2}{2ab} \tag{5}$$

Then we apply the cosine to the triangle $\triangle CED$ to get the extensions distance $\overline{DE}$:

$$\begin{aligned}\overline{DE}^2 &= \overline{CD}^2 + \overline{CE}^2 - 2\overline{CD} \times \overline{CE}\cos\theta \\ &= d^2 + e^2 - 2de\cos\theta \\ &= d^2 + e^2 - de\left(\frac{a^2 + b^2 - c^2}{ab}\right)\end{aligned} \tag{6}$$

[1] Meccano mathematics by ‘t Hooft

We extract the square root:

$$\begin{aligned}
\overline{DE} &= \sqrt{d^2 + e^2 - de\left(\frac{a^2 + b^2 - c^2}{ab}\right)} \\
&= \frac{\sqrt{a^2b^2(d^2+e^2) - abde(a^2+b^2-c^2)}}{ab} \\
&= \frac{\sqrt{ab((ad-be)(bd-ae)+c^2de)}}{ab}
\end{aligned} \tag{7}$$

1.1 Software

We write a software to report all the triangle frames with specific surd $\sqrt{s}$ for a given maximum strips length. We can reject cases $q \neq 1$ and s not square-free. Next list show all the triangles with $q = 1$ and $s = \sqrt{7}$ where $c < a + b$, $a \leq d \leq max$, $b \leq e \leq max$, $c \leq max$:

```
=== RUN   TestFramesTriangleSurds
NewFrames().TriangleSurds surd=7 max=15
  1) a=1 e=1+2 c=1 cos=1/2
  2) d=1+1 e=1+2 c=1 cos=1/2
  3) d=1+2 b=1 c=1 cos=1/2
  4) d=1+2 e=1+1 c=1 cos=1/2
  5) a=2 e=2+1 c=2 cos=1/2
  6) d=2+1 b=2 c=2 cos=1/2
  7) a=3 e=2+2 c=2 cos=3/4 CED=pi/2
  8) d=3+1 e=2+1 c=2 cos=3/4 CDE=pi/2
  9) d=4+2 e=4+4 c=1 cos=31/32
 10) d=4+4 e=4+2 c=1 cos=31/32
 11) a=7 e=5+1 c=3 cos=13/14
 12) a=7 e=5+2 c=3 cos=13/14
```

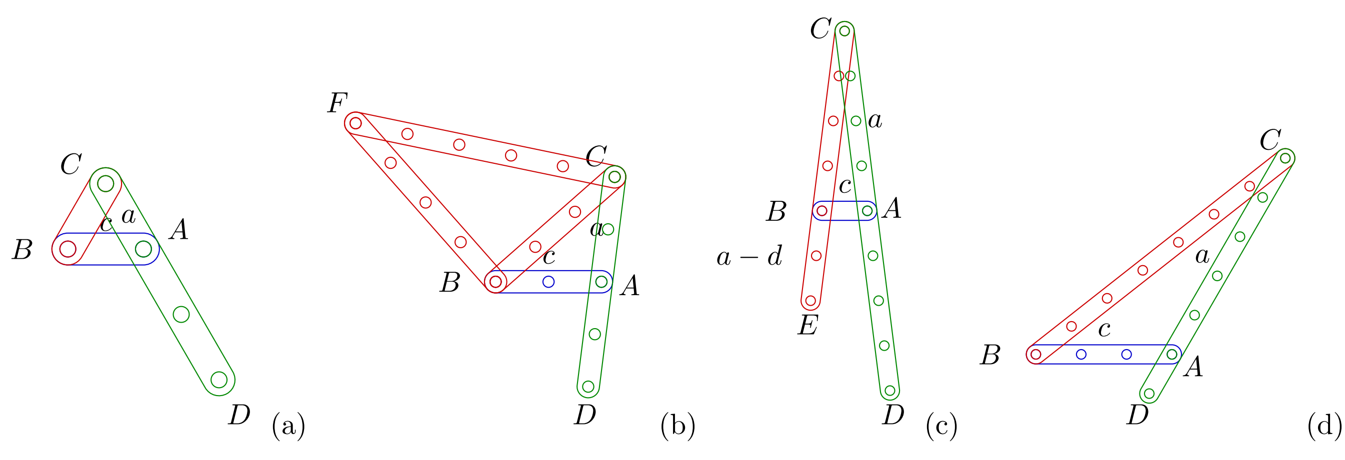

Figure 2: Some triangular frames with rigid distance $\overline{DE} = \sqrt{7}$ found by the software.

Figure 2 show four cases of this list. The code is in the folder `github.com/heptagons/meccano/frames`.

1.2 Triangular distance of the form $\sqrt{s} + f$

In the figure 2, the particular case (b), was reported with the angle $CED = \pi/2$ which means we can append two extra strips to make a pythagorean triangle $\triangle CEF$ where angle $CEF = \pi/2$, which makes the three vertices D, E, F collinear, so the rigid distance $\overline{DF} = \sqrt{7} + 4$ is an algebraic number.

1.3 Another rigid distances $\sqrt{s}+h$

We explore a more complicated frame to get additional cases of distances $\sqrt{s}+h$ without relying in an explicit pythagorean triangle as we saw in case (b) of figure 2.

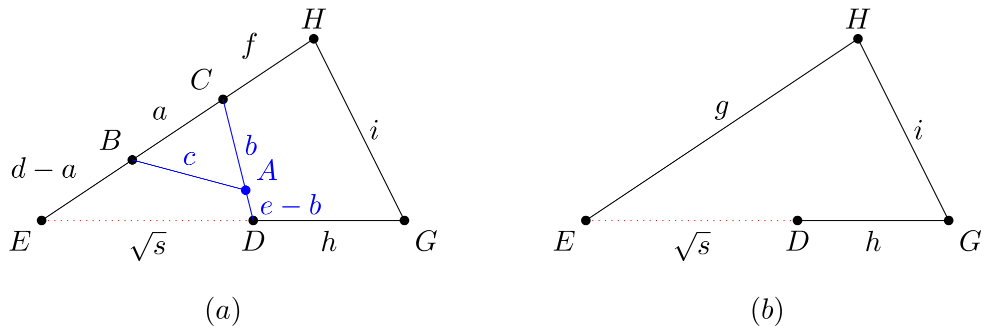

Figure 3: The five strips intented to form an algebraic distance $\overline{EG}=\sqrt{s}+h$.

From figure 3 (a) we know $\sqrt{s}$ distance between nodes E and D is produced by the three strips frame $a+d$, $b+e$ and c. Using the law of cosines we calculate the angle $\theta=\angle CED$ in terms of $\sqrt{s}$:

$$
\begin{aligned}
\cos\theta &= \frac{d^2+(\sqrt{s})^2-e^2}{2d\sqrt{s}} \\
&= \frac{(d^2+s-e^2)\sqrt{s}}{2ds} && (8)\\
&= \frac{m\sqrt{s}}{n} && (9)\\
m &= d^2+s-e^2 && (10)\\
n &= 2ds && (11)
\end{aligned}
$$

From figure 3 (a) we notice two sets of points are collinear: $\{E,B,C,H\}$ and $\{E,D,G\}$. Using the law of cosines we calculate the angle $\theta=\angle HEG$ in terms of distances $g, \sqrt{s}+h, i$:

$$
\begin{aligned}
\cos\theta &= \frac{g^2+(\sqrt{s}+h)^2-i^2}{2g(\sqrt{s}+h)} \\
&= \frac{g^2+s+2\sqrt{s}h+h^2-i^2}{2g(\sqrt{s}+h)} \\
&= \frac{g^2+s+h^2-i^2+2\sqrt{s}h}{2g(\sqrt{s}+h)} && (12)
\end{aligned}
$$

We multiply both numerator and denominator by $\sqrt{s} - h$ to eliminate the surd from denominator:

$$
\begin{aligned}
\cos\theta &= \frac{(s + g^2 + h^2 - i^2)(\sqrt{s} - h) + 2\sqrt{s}h(\sqrt{s} - h)}{2g(\sqrt{s} + h)(\sqrt{s} - h)} \\
&= \frac{(s + g^2 + h^2 - i^2)(\sqrt{s} - h) + 2sh - 2\sqrt{s}h^2}{2g(s - h^2)} \\
&= \frac{-h(s + g^2 + h^2 - i^2 - 2s) + (s + g^2 + h^2 - i^2 - 2h^2)\sqrt{s}}{2g(s - h^2)} \\
&= \frac{h(s - g^2 - h^2 + i^2) + (s + g^2 - h^2 - i^2)\sqrt{s}}{2g(s - h^2)} \\
&= \frac{o + p\sqrt{s}}{q} \qquad (13) \\
o &= h(s - g^2 - h^2 + i^2) \qquad (14) \\
p &= s + g^2 - h^2 - i^2 \qquad (15) \\
q &= 2g(s - h^2) \qquad (16)
\end{aligned}
$$

We compare both cosines equations 9 and 13:

$$\frac{m\sqrt{s}}{n} = \frac{o + p\sqrt{s}}{q} \qquad (17)$$

Since all variables are integers we need two conditions. First o should be zero. And second $\frac{m}{n} = \frac{p}{q}$.

For condition 1, we force o to be zero:

$$
\begin{aligned}
o &= 0 \\
h(s - g^2 - h^2 + i^2) &= 0 \\
s &= g^2 + h^2 - i^2 \qquad (18)
\end{aligned}
$$

For condition2, we force m, n, p, q as:

$$
\begin{aligned}
\frac{m}{n} &= \frac{p}{q} \\
\frac{d^2 + s - e^2}{2ds} &= \frac{s + g^2 - h^2 - i^2}{2g(s - h^2)}
\end{aligned}
\qquad (19)
$$

We replace the value of s of last equation RHS with the value of equation 18 of condition 1:

$$
\begin{aligned}
\frac{d^2 - e^2 + s}{ds} &= \frac{s + g^2 - h^2 - i^2}{g(s - h^2)} \\
&= \frac{g^2 + h^2 - i^2 + g^2 - h^2 - i^2}{g(g^2 + h^2 - i^2 - h^2)} \\
&= \frac{2(g^2 - i^2)}{g(g^2 - i^2)} \\
&= \frac{2}{g} \\
(d^2 - e^2 + s)g &= 2ds \qquad (20)
\end{aligned}
$$

$\boxed{TODO: Examples!!!}$

2 Triangle pair frame

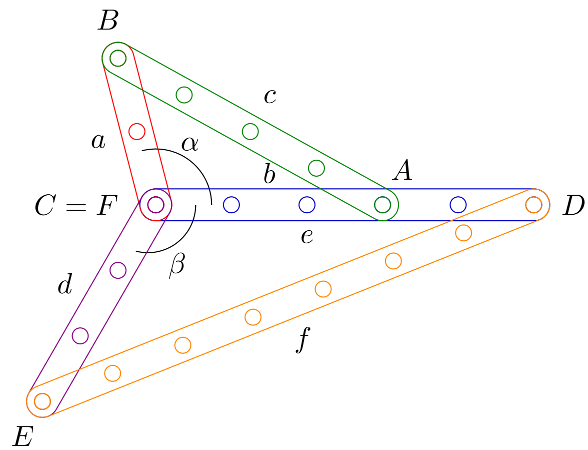

Figure 4: Triangle pair frame. We join triangles $\triangle ABC$ and $\triangle DEF$ in such a way vertices C and F coincide and vertices A, C, D, E be collinear. The result is a five strips frame. We are interested in the distance $\overline{BE}$.

Figure 4 shows a triangle pair frame. The triangles share a strip which contains four of the vertices. The remaining two vertices are separated by distances of the form $\dfrac{\sqrt{F+G\sqrt{H}}}{A}$. With only five strips this frame is small and useful to make up the diagonals inside polygons we want to be rigid.

2.1 Triangle pair algebra

First we calculate the cosines:

$$\cos\alpha = \frac{a^2+b^2-c^2}{2ab}$$
$$\cos\beta = \frac{d^2+e^2-f^2}{2de}$$

We define integers m, n, o, p to simplify cosines and get sines:

$$(m,n) \equiv (a^2+b^2-c^2, 2ab) \quad |m| \leq n \tag{21}$$
$$(o,p) \equiv (d^2+e^2-f^2, 2de) \quad |o| \leq p \tag{22}$$
$$\cos\alpha = \frac{m}{n} \tag{23}$$
$$\cos\beta = \frac{o}{p} \tag{24}$$
$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \frac{\sqrt{n^2-m^2}}{n} \tag{25}$$
$$\sin\beta = \sqrt{1-\cos^2\beta} = \frac{\sqrt{p^2-o^2}}{p} \tag{26}$$

Then, we use the cosines sum identity:

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
&= \left(\frac{m}{n}\right)\left(\frac{o}{p}\right) - \left(\frac{\sqrt{n^2-m^2}}{n}\right)\left(\frac{\sqrt{p^2-o^2}}{p}\right) \\
&= \frac{mo - \sqrt{(n^2-m^2)(p^2-o^2)}}{np} \qquad (27)
\end{aligned}$$

Finally we can calculate the distance $g \equiv \overline{BE}$ using the law of cosines:

$$\begin{aligned}
g &\equiv \overline{BE} \\
&= \sqrt{a^2+d^2-2ad\cos(\alpha+\beta)} \\
&= \sqrt{a^2+d^2-2ad\left(\frac{mo-\sqrt{(n^2-m^2)(p^2-o^2)}}{np}\right)} \\
&= \sqrt{a^2+d^2-2ad\left(\frac{mo-\sqrt{(n^2-m^2)(p^2-o^2)}}{4abde}\right)} \\
&= \sqrt{a^2+d^2-\frac{mo-\sqrt{(n^2-m^2)(p^2-o^2)}}{2be}} \\
&= \frac{\sqrt{4b^2e^2(a^2+d^2)-2bemo+2be\sqrt{(n^2-m^2)(p^2-o^2)}}}{2be} \qquad (28)
\end{aligned}$$

For the software we can define integers A, F, G, H to calculate and reduce g:

$$A \equiv 2be \qquad (29)$$

$$F \equiv A^2(a^2+d^2) - Amo \qquad (30)$$

$$G \equiv A \qquad (31)$$

$$H \equiv (n^2-m^2)(p^2-o^2) \qquad (32)$$

$$g = \frac{\sqrt{F+G\sqrt{H}}}{A} \qquad (33)$$

2.2 Triangle pairs software

We run a program to inspect triangle pairs having a given distance g. The software iterates over the two triangles sides (a, b, c) and (d, e, f) up to a maximum strip length.

Next example request distances of the form $\sqrt{46+18\sqrt{5}}$ up to strip length 10:

Folder : `github.com/heptagons/meccano/frames`

Call : `NewFrames().TrianglePairsTex(10, [46 18 5])`

$(a,b,c) \oplus (d,e,f) \mapsto g$
$(2,1,2) \oplus (3,3,3) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(2,1,2) \oplus (3,8,7) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(2,2,2) \oplus (3,6,6) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(2,3,4) \oplus (3,5,7) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(2,4,4) \oplus (3,8,7) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(3,3,3) \oplus (2,4,4) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(4,2,4) \oplus (6,6,6) \mapsto \sqrt{46+18\sqrt{5}}$
$(4,4,4) \oplus (6,7,8) \mapsto \sqrt{46+18\sqrt{5}}$
$(6,3,6) \oplus (4,4,4) \mapsto \sqrt{46+18\sqrt{5}}$
$(6,3,6) \oplus (9,9,9) \mapsto \dfrac{3\sqrt{46+18\sqrt{5}}}{2}$
$(6,6,6) \oplus (4,8,8) \mapsto \sqrt{46+18\sqrt{5}}$
$(6,7,8) \oplus (9,9,9) \mapsto \dfrac{3\sqrt{46+18\sqrt{5}}}{2}$

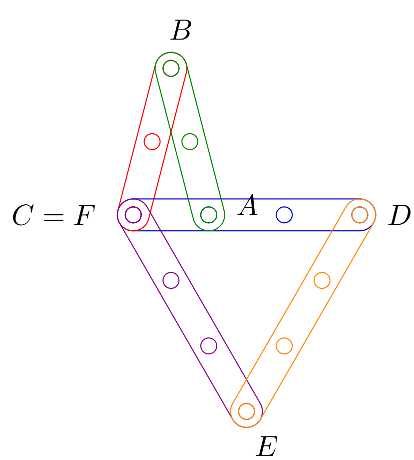

Figure 5: Triangle pair frame $(2,1,2) \oplus (3,3,3)$ makes $\overline{BE} = \dfrac{\sqrt{46+18\sqrt{5}}}{2}$.

3 Two triangles with offsets

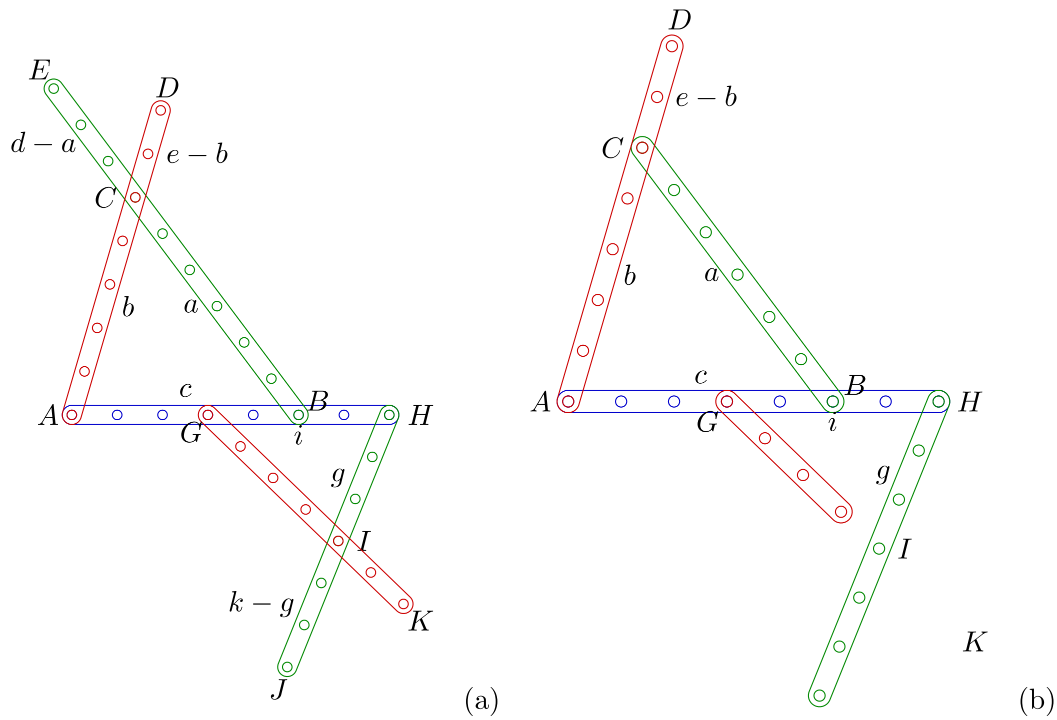

Figure 6: Frame of two triangles with offsets. We construct two triangles $\triangle ABC$ and $\triangle GHI$. Extending the strips we get four vertices E, D, J, K which can form four rigid distances of surd type: $\overline{DJ}, \overline{DK}, \overline{EJ}, \overline{EK}$.

Figure 6 shows a frame with five strips. The frame has eleven variables:

$$a = \overline{BC}, \quad b = \overline{AC}, \quad c = \overline{AB} \tag{34}$$

$$d = \overline{AE}, \quad e = \overline{AD} \tag{35}$$

$$f = \overline{AG} \tag{36}$$

$$g = \overline{HI}, \quad h = \overline{GI}, \quad i = \overline{GH} \tag{37}$$

$$j = \overline{HJ}, \quad k = \overline{HK} \tag{38}$$

Assume vertex A is at the origin. Let $\alpha = \angle BAC$, and D_x, D_y the abscissa and orditate of vertex D so we have:

$$t \equiv b^2 + c^2 - a^2 \tag{39}$$

$$x \equiv 4b^2c^2 - t^2 \tag{40}$$

$$\cos\alpha = \frac{t}{2bc} \tag{41}$$

$$\sin\alpha = \frac{\sqrt{x}}{2bc} \tag{42}$$

$$D_x = d\sin\alpha = \frac{d\sqrt{x}}{2bc} \tag{43}$$

$$D_y = d\cos\alpha = \frac{dt}{2bc} \tag{44}$$

$$D_x^2 + D_y^2 = d^2 \tag{45}$$

Let $\delta = \angle HGI$ and K_x, K_y the abscissa and ordinate of vertex K so we have:

$$
\begin{aligned}
v &\equiv h^2 + i^2 - g^2 && (46)\\
y &\equiv 4h^2 i^2 - v^2 && (47)\\
\cos\delta &= \frac{v}{2hi} && (48)\\
\sin\delta &= \frac{\sqrt{y}}{2hi} && (49)\\
K_x &= f + k\sin\delta = f + \frac{k\sqrt{y}}{2hi} && (50)\\
K_y &= -k\cos\delta = -\frac{kv}{2hi} && (51)\\
K_x^2 + K_y^2 &= f^2 + 2fk\sin\delta + k^2 && (52)\\
&= f^2 + k^2 + \frac{fk\sqrt{y}}{hi} && (53)
\end{aligned}
$$

We calculate the distance $\overline{DK}$:

$$
\begin{aligned}
\overline{DK}^2 &= (D_x + K_x)^2 + (D_y + K_y)^2\\
&= D_x^2 + 2D_xK_x + K_x^2 + D_y^2 + 2D_yK_y + K_y^2\\
&= (D_x^2 + D_y^2) + (K_x^2 + K_y^2) + 2D_xK_x + 2D_yK_y\\
&= d^2 + f^2 + k^2 + \frac{fk\sqrt{y}}{hi} + 2\left(\frac{d\sqrt{x}}{2bc}\right)\left(f + \frac{k\sqrt{y}}{2hi}\right) + 2\left(\frac{dt}{2bc}\right)\left(-\frac{kv}{2hi}\right)\\
&= d^2 + f^2 + k^2 - \frac{dtkv}{2bchi} + \frac{fk\sqrt{y}}{hi} + \frac{df\sqrt{x}}{bc} + \frac{dk\sqrt{xy}}{2bchi} && (54)
\end{aligned}
$$