

Triple unit

<https://github.com/heptagons/meccano/units/triple>

Abstract

A **Triple unit** is a group of **five** meccano ¹ strips a, b, c, d, e forming **three equal angles** θ intended to build three consecutive perimeter sides of some regular polygons. We look for integer values of strip e in function of integer values of sides a, b, c, d and a particular angle θ . We confirm a generic equation found matches the one used to build pentagons of type 2 ². Here we found a lot of hexagons and filter some not trivial solutions. We look for octagons, decagons and dodecagons.

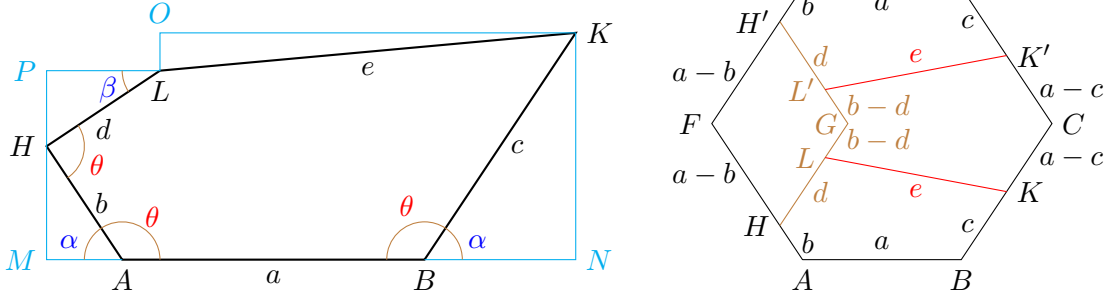


Figure 1: At the left we have the Triple unit (three angles θ) with the strips a, b, c, d, e . At the right we use two units to build a regular polygon of side a extending strips b, c, d to fix everthing.

1 Algebra

From nodes A and B of fig 1 we get α from θ ($\pi = 180^\circ$):

$$\begin{aligned}\theta &= \pi - \alpha \\ \alpha &= \pi - \theta\end{aligned}\tag{1}$$

And from node H we get β from θ :

$$\begin{aligned}\theta &= \alpha + \beta \\ \beta &= \theta - \alpha = \theta - (\pi - \theta) = 2\theta - \pi\end{aligned}\tag{2}$$

We calculate horizontal segment \overline{OK} :

$$\begin{aligned}\overline{OK} &= \overline{MA} + a + \overline{BN} - \overline{PL} \\ &= b \cos \alpha + a + c \cos \alpha - d \cos \beta \\ &= a + (b + c) \cos \alpha - d \cos \beta \\ &= a + (b + c) \cos (\pi - \theta) - d \cos (2\theta - \pi) \\ &= a - (b + c) \cos \theta + d \cos (2\theta)\end{aligned}\tag{3}$$

¹ Meccano mathematics by 't Hooft

² Meccano pentagons

And vertical segment \overline{OL} :

$$\begin{aligned}
\overline{OL} &= \overline{KN} - \overline{PH} - \overline{HM} \\
&= c \sin \alpha - d \sin \beta - b \sin \alpha \\
&= (c - b) \sin \alpha - d \sin \beta \\
&= (c - b) \sin (\pi - \theta) - d \sin (2\theta - \pi) \\
&= (c - b) \sin \theta + d \sin (2\theta)
\end{aligned} \tag{4}$$

So we can express e in function of a, b, c, d and angle θ :

$$\begin{aligned}
e^2 &= (\overline{OK})^2 + (\overline{OL})^2 \\
&= (a - (b + c) \cos \theta + d \cos(2\theta))^2 + ((c - b) \sin \theta + d \sin(2\theta))^2 \\
&= a^2 + (b^2 + 2bc + c^2) \cos^2 \theta + d^2 \cos^2(2\theta) + (c^2 - 2cb + b^2) \sin^2 \theta + d^2 \sin^2(2\theta) \\
&\quad - 2a(b + c) \cos \theta + 2ad \cos(2\theta) - 2(b + c)d \cos \theta \cos(2\theta) \\
&\quad + 2(c - b)d \sin \theta \sin(2\theta) \\
&= a^2 + b^2 + c^2 + d^2 + 2bc \cos^2 \theta - 2bc \sin^2 \theta \\
&\quad - 2a(b + c) \cos \theta + 2ad \cos(2\theta) \\
&\quad - 2d((b + c) \cos \theta \cos(2\theta) + (b - c) \sin \theta \sin(2\theta))
\end{aligned} \tag{5}$$

$$\begin{aligned}
&= a^2 + b^2 + c^2 + d^2 + 2bc(\cos^2 \theta - \sin^2 \theta) - 2a(b + c) \cos \theta + 2ad \cos(2\theta) \\
&\quad - 2d(b(\cos \theta \cos(2\theta) + \sin \theta \sin(2\theta)) + c(\cos \theta \cos(2\theta) - \sin \theta \sin(2\theta))) \\
&= a^2 + b^2 + c^2 + d^2 + 2bc \cos(2\theta) - 2a(b + c) \cos \theta + 2ad \cos(2\theta) \\
&\quad - 2d(b \cos(\theta - 2\theta) + c \cos(\theta + 2\theta)) \\
&= a^2 + b^2 + c^2 + d^2 + 2(bc + ad) \cos(2\theta) - 2a(b + c) \cos \theta - 2d(b \cos \theta + c \cos(3\theta)) \\
&= a^2 + b^2 + c^2 + d^2 + 2(bc + ad) \cos(2\theta) - 2(ab + ac) \cos \theta - 2(bd \cos \theta + cd \cos(3\theta)) \\
&= \boxed{a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd) \cos \theta + 2(bc + ad) \cos(2\theta) - 2cd \cos(3\theta)}
\end{aligned} \tag{6}$$

2 Regular polygons

Polygon	θ	$\cos \theta$	$\cos(2\theta)$	$\cos(3\theta)$
Pentagon	$\frac{3\pi}{5}$	$\frac{1 - \sqrt{5}}{4}$	$\frac{-1 - \sqrt{5}}{4}$	$\frac{1 + \sqrt{5}}{4}$
Hexagon	$\frac{2\pi}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1
Heptagon	$\frac{5\pi}{7}$			
Octagon	$\frac{3\pi}{4}$			
Decagon	$\frac{4\pi}{5}$			
Dodecagon	$\frac{5\pi}{6}$			

Table 1: Regular polygons internal angles and cosines.

2.1 Equilateral pentagon

We replace the cosines for pentagon in table 1 in e^2 equation:

$$\begin{aligned}
e^2 &= a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd) \cos \theta + 2(bc + ad) \cos(2\theta) - 2cd \cos(3\theta) \\
&= a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd) \left(\frac{1 - \sqrt{5}}{4} \right) + 2(bc + ad) \left(\frac{-1 - \sqrt{5}}{4} \right) - 2cd \left(\frac{1 + \sqrt{5}}{4} \right) \\
&= a^2 + b^2 + c^2 + d^2 - \frac{ab + ac + bd + bc + ad + cd}{2} + \frac{ab + ac + bd - bc - ad - cd}{2} \sqrt{5}
\end{aligned} \tag{7}$$

e cannot to be and integer if the factor of $\sqrt{5}$ is not zero so we force this factor to be zero:

$$\begin{aligned}
ab + ac + bd - bc - ad - cd &= 0 \\
ab + ac + bd &= bc + ad + cd
\end{aligned} \tag{8}$$

We replace $ab + ac + bd$ by $bc + ad + cd$ in the e^2 equation to get:

$$\begin{aligned}
e^2 &= a^2 + b^2 + c^2 + d^2 - \frac{(bc + ad + cd) + bc + ad + cd}{2} + \frac{0}{2} \sqrt{5} \\
&= a^2 + b^2 + c^2 + d^2 - bc - ad - cd \\
e &= \sqrt{a^2 + b^2 + c^2 + d^2 - bc - ad - cd}
\end{aligned} \tag{9}$$

The last formula matches the formula used in the paper Meccano pentagons which finds several pentagons of type 2. Only when we get e integer we have a solution.

2.2 Equilateral hexagon

We replace the cosines for hexagon in table 1 in e^2 equation:

$$\begin{aligned}
e^2 &= a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd) \cos \theta + 2(bc + ad) \cos(2\theta) - 2cd \cos(3\theta) \\
&= a^2 + b^2 + c^2 + d^2 - 2(ab + ac + bd) \left(-\frac{1}{2} \right) + 2(bc + ad) \left(-\frac{1}{2} \right) - 2cd(1) \\
&= a^2 + b^2 + c^2 + d^2 + ab + ac + bd - bc - ad - 2cd \\
&= (a + b)^2 + (c - d)^2 - ab + ac + bd - bc - ad \\
&= (a + b)^2 + (c - d)^2 + (c - d)(a - b) - ab \\
&= (a + b)^2 + (c - d)(a - b + c - d) - ab \\
e &= \sqrt{(a + b)^2 + (c - d)(a - b + c - d) - ab}
\end{aligned} \tag{10}$$

We wrote software code to look for hexagons using the formula for e and set several filters to prevent trivial solutions. We say an hexagon is nice when $e \leq a$. Next is a partial list of nice hexagons:

1	1	a=	7	b=	1	c=	2	d=	6	e=	7
2	2	a=	7	b=	1	c=	4	d=	6	e=	7
3	3	a=	13	b=	2	c=	5	d=	11	e=	13
4	4	a=	13	b=	2	c=	6	d=	11	e=	13
5	5	a=	14	b=	1	c=	6	d=	13	e=	13
6	6	a=	14	b=	1	c=	7	d=	13	e=	13
7	7	a=	15	b=	1	c=	5	d=	14	e=	14
8	8	a=	15	b=	1	c=	9	d=	14	e=	14
9	9	a=	19	b=	2	c=	3	d=	17	e=	19
10	10	a=	19	b=	2	c=	14	d=	17	e=	19
11	11	a=	20	b=	1	c=	4	d=	19	e=	19

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12 12 a= 20 b= 1 c= 15 d= 19 e= 19
13 ...
14 105 a= 58 b= 5 c= 10 d= 53 e= 57
15 106 a= 58 b= 5 c= 43 d= 53 e= 57
16 107 a= 59 b= 1 c= 27 d= 58 e= 52
17 108 a= 59 b= 1 c= 31 d= 58 e= 52
18 109 a= 59 b= 4 c= 11 d= 55 e= 57
19 110 a= 59 b= 4 c= 44 d= 55 e= 57
20 111 a= 59 b= 5 c= 19 d= 54 e= 56
21 112 a= 59 b= 5 c= 35 d= 54 e= 56
22 --- PASS: TestHexagonsNice (0.01s)

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Results from github.com/heptagons/meccano/units/triple/triple_test.go TestHexagonsNice
The results has related pairs and there are several ways to build an hexagon from each pair.

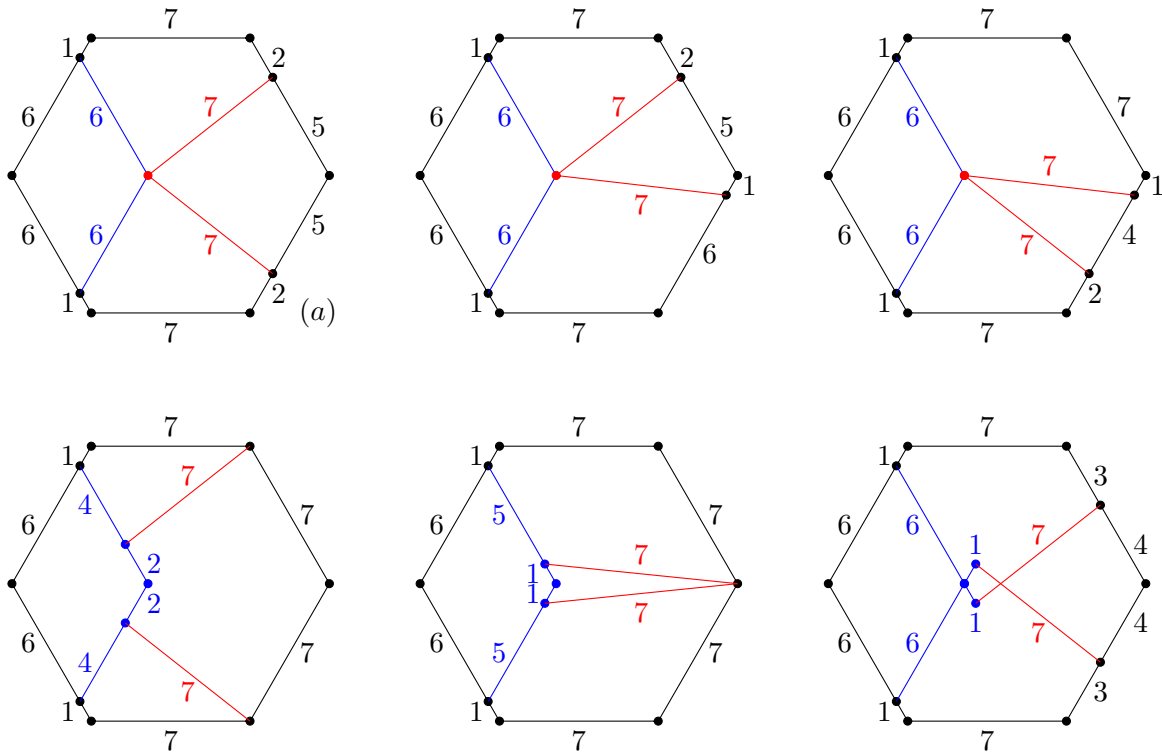


Figure 2: Constructions options of the nice hexagon side $a = 7, b = 1, e = 7$.