

# Horns unit

<https://github.com/heptagons/meccano/units/horns>

## Abstract

Horns unit is a group of seven meccano <sup>1</sup> strips intended to build polygons.

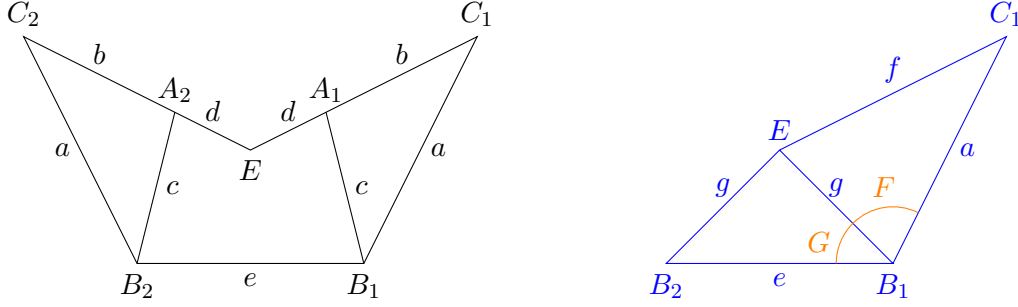


Figure 1: The **horn unit** has seven strips: Two of length  $a$ , two of length  $b + d$ , two of length  $c$  and one of length  $e$ . We expect to build polygons with internal angle  $C_1B_1B_2$  and perimeter including segments  $a, e, a$ .

## 1 Algebra

From figure 1 we start with triangle  $\triangle A_1B_1C_1$ . At vertex  $A_1$  we have angle  $A$  and the supplement  $A'$ :

$$A \equiv \angle B_1A_1C_1 \tag{1}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{if and only if } a < b + c \tag{2}$$

$$A' \equiv \angle EA_1B_1 = \pi - A \tag{3}$$

$$\cos A' = \cos(\pi - A) = -\cos A = \frac{-b^2 - c^2 + a^2}{2bc} \tag{4}$$

We define  $f \equiv b + d$  and  $g \equiv \overline{EB_1}$ . With the law of cosines we have:

$$f \equiv \boxed{b + d} \in \mathbb{N} \tag{5}$$

$$g^2 = c^2 + d^2 - 2cd \cos A' \tag{6}$$

$$\begin{aligned} &= c^2 + d^2 - (2cd) \frac{-b^2 - c^2 + a^2}{2bc} \\ &= \frac{bc^2 + bd^2 + b^2d + c^2d - a^2d}{b} \\ &= \frac{(b + d)(bd + c^2) - a^2d}{b} \end{aligned} \tag{7}$$

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<sup>1</sup> Meccano mathematics by 't Hooft

Define a new variable  $h = (b + d)(bd + c^2) - a^2d$ :

$$h \equiv \boxed{(bd + c^2)f - a^2d} \in \mathbb{Z} \quad (8)$$

$$g^2 = \boxed{\frac{h}{b}} \quad \text{if and only if } 0 < h < b \quad (9)$$

We calculate angles  $F \equiv \angle C_1B_1E$  and  $G \equiv \angle B_2B_1E$ . We replace  $g^2$  by  $h/b$ :

$$\cos F = \frac{a^2 + g^2 - f^2}{2ag} = \frac{a^2b - bf^2 + h}{2abg} \quad (10)$$

$$\cos G = \boxed{\frac{e}{2g}} \quad (11)$$

Define new variable  $j = a^2b - bf^2 + h$  so:

$$j \equiv \boxed{a^2b - bf^2 + h} \in \mathbb{Z} \quad (12)$$

$$\cos F = \frac{a^2b - bf^2 + h}{2abg} = \boxed{\frac{j}{2abg}} \quad (13)$$

We calculate cosines squares and products. Again we replace  $g^2$  by  $h/b$ :

$$\cos F \cos G = \frac{ej}{4abg^2} = \frac{bej}{4abh} = \boxed{\frac{ej}{4ah}} \in \mathbb{Q} \quad (14)$$

$$\cos^2 F = \frac{j^2}{4a^2b^2g^2} = \frac{bj^2}{4a^2b^2h} = \boxed{\frac{j^2}{4a^2bh}} \in \mathbb{Q} \quad (15)$$

$$\cos^2 G = \frac{e^2}{4g^2} = \boxed{\frac{be^2}{4h}} \in \mathbb{Q} \quad (16)$$

$$\cos^2 F \cos^2 G = \frac{be^2j^2}{16a^2bh^2} = \boxed{\frac{e^2j^2}{16a^2h^2}} \in \mathbb{Q} \quad (17)$$

$$(18)$$

We calculate the sines part squared and set a common denominator as square  $16a^2b^2h^2$ :

$$(\sin F \sin G)^2 = (1 - \cos^2 F)(1 - \cos^2 G) \quad (19)$$

$$= 1 - \cos^2 F - \cos^2 G + \cos^2 F \cos^2 G$$

$$= 1 - \frac{j^2}{4a^2bh} - \frac{be^2}{4h} + \frac{e^2j^2}{16a^2h^2}$$

$$= 1 - \frac{j^2}{4a^2bh} - \frac{be^2}{4h} + \frac{e^2j^2}{16a^2h^2}$$

$$= \frac{16a^2b^2h^2 - (4bh)j^2 - (4a^2b^2h)be^2 + (b^2)e^2j^2}{16a^2b^2h^2}$$

$$= \frac{16a^2b^2h^2 - 4bhj^2 - 4a^2b^3e^2h + b^2e^2j^2}{16a^2b^2h^2}$$

$$= \frac{b(be^2 - 4h)(j^2 - 4a^2bh)}{16a^2b^2h^2} \quad (20)$$

Extract square root to get  $\sin F \sin G = \sqrt{D}/A$  where  $D, A \in \mathbb{Z}$ :

$$\sin F \sin G = \boxed{\frac{\sqrt{b(be^2 - 4h)(j^2 - 4a^2bh)}}{4abh}} \in \mathbb{A} \quad (21)$$

We sum the angles  $F$  and  $G$  to get:

$$F + G \equiv \angle B_2 B_1 C_1 \quad (22)$$

$$\cos(F + G) = \cos F \cos G - \sin F \sin G \quad (23)$$

$$\begin{aligned} &= \frac{ej}{4ah} - \frac{\sqrt{b(be^2 - 4h)(j^2 - 4a^2bh)}}{4abh} \\ &= \frac{bej - \sqrt{b(be^2 - 4h)(j^2 - 4a^2bh)}}{4abh} \in \mathbb{A} \end{aligned} \quad (24)$$

## 2 Software