Meccano polygon diagonals

https://github.com/heptagons/meccano/penta

Abstract

We construct meccano [1] polygon internal diagonals.

1 Polygon diagonals

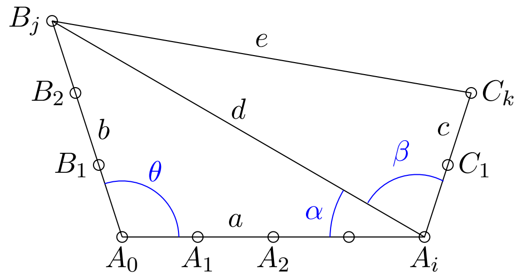

Figure 1: Meccano polygon three consecutive sides segments $a \geq b \geq c$ can form two diagonals d and e.

2 Regular polygon diagonals

In the regular polygon all internal angles are equal to θ. From figure 1 the polygon is regular if $\alpha + \beta = \theta$ so we have:

$$\alpha = \angle A_0 A_i B_j \tag{1}$$
$$\beta = \angle B_j A_i C_k \tag{2}$$
$$\theta = \angle B_j A_0 A_i = \angle A_0 A_i C_k \tag{3}$$
$$\alpha + \beta = \theta \tag{4}$$

We use the cosines sum identity to express $\cos\beta$ in function of the rest of variables. We define $u = \cos\theta$:

$$\begin{aligned} u &\equiv \cos\theta && (5) \\ &= \cos(\alpha+\beta) && (6) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta && (7) \\ \sin\beta &= \frac{\cos\alpha\cos\beta - u}{\sin\alpha} && (8) \\ \sin^2\beta &= \frac{(\cos\alpha\cos\beta - u)^2}{\sin^2\alpha} && (9) \\ 1-\cos^2\beta &= \frac{\cos^2\alpha\cos^2\beta - 2u\cos\alpha\cos\beta + u^2}{sin^2\alpha} && (10) \end{aligned}$$

[1] Meccano mathematics by ‘t Hooft

We set $X = \cos\beta$ and rearrange the last equation to get:

$$X^2 - 2u\cos\alpha X + u^2 - \sin^2\alpha = 0 \tag{11}$$

And solve the quadratic equation $AX^2 + BX + C = 0$ to get $\cos\beta$ in function of u and α:

$$\begin{aligned}\cos\beta &= \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\\ &= \frac{2u\cos\alpha \pm \sqrt{4u^2\cos^2\alpha - 4(u^2 - \sin^2\alpha)}}{2}\\ &= u\cos\alpha \pm \sqrt{u^2\cos^2\alpha - u^2 + \sin^2\alpha}\end{aligned} \tag{12}$$

Now, we need to find the values of $\cos\alpha$, $\sin\alpha$ and $\cos\beta$ which in turn need the value of d, all in terms of a, b, c the segments of the polygon perimeter.

For the value of d we use the law of cosines:

$$\begin{aligned}d &= \sqrt{a^2 + b^2 - 2ab\cos\theta}\\ &= \sqrt{a^2 + b^2 - 2abu}\end{aligned} \tag{13}$$

Using the law of cosines we calculate the angles $\alpha = \angle A_0A_iB_j$ and $\beta = \angle B_jA_iC_k$:

$$\begin{aligned}\cos\alpha &= \frac{a^2 + d^2 - b^2}{2ad}\\ &= \frac{a^2 + (a^2 + b^2 - 2abu) - b^2}{2ad}\\ &= \frac{a - bu}{d}\end{aligned} \tag{14}$$

$$\begin{aligned}\cos\beta &= \frac{c^2 + d^2 - e^2}{2cd}\\ &= \frac{c^2 + (a^2 + b^2 - 2abu) - e^2}{2cd}\\ &= \frac{a^2 + b^2 + c^2 - e^2 - 2abu}{2cd}\end{aligned} \tag{15}$$

We define new variable f to simplify $\cos\beta$ to obtain:

$$f \equiv \frac{a^2 + b^2 + c^2 - e^2}{2} \tag{16}$$

$$\cos\beta = \frac{f - abu}{cd} \tag{17}$$

We calculate $\sin^2\alpha = 1 - \cos^2\alpha$:

$$\begin{aligned}\sin^2\alpha &= 1 - \frac{(a - bu)^2}{d^2}\\ &= \frac{d^2 - a^2 + 2abu - b^2u^2}{d^2}\\ &= \frac{(a^2 + b^2 - 2abu) - a^2 + 2abu - b^2u^2}{d^2}\\ &= \frac{b^2(1 - u^2)}{d^2}\end{aligned} \tag{18}$$

We plug the values of $\cos\alpha, \cos\beta, \sin^2\alpha$ in equation 12 to get:

$$\begin{aligned}
\frac{f-abu}{cd} &= \left(\frac{a-bu}{d}\right)u \pm \sqrt{\left(\frac{a-bu}{d}\right)^2 u^2 - u^2 + \frac{b^2(1-u^2)}{d^2}} \\
\frac{f-abu}{c} &= (a-bu)u \pm \sqrt{(a-bu)^2u^2 - d^2u^2 + b^2(1-u^2)} \\
f &= (ab+ac-bcu)u \pm c\sqrt{(a-bu)^2u^2 - d^2u^2 + b^2 - b^2u^2)} \\
&= abu + acu - bcu^2 \pm c\sqrt{a^2u^2 - 2abu^3 + b^2u^4 - d^2u^2 + b^2 - b^2u^2} \qquad (19)
\end{aligned}$$

2.1 Regular polygon diagonal e

We define variables m, n to simplify f. For n we substiute d^2 from equation 13 so we have:

$$\begin{aligned}
m &\equiv abu + acu - bcu^2 \\
&= a(b+c)u - bcu^2 \qquad (20) \\
n &\equiv a^2u^2 - 2abu^3 + b^2u^4 - d^2u^2 + b^2 - b^2u^2 \\
&= b^2 + (a^2 - d^2 - b^2)u^2 - 2abu^3 + b^2u^4 \\
&= b^2 + (a^2 - (a^2 + b^2 - 2abu) - b^2)u^2 - 2abu^3 + b^2u^4 \\
&= b^2(1-u^2)^2 \qquad (21)
\end{aligned}$$

We substitue m, n in f where we choose the positive sign and take the absolute value of $\sqrt{n} = |b(1-u^2)|$:

$$\begin{aligned}
f &= m + c\sqrt{n} \\
&= a(b+c)u - bcu^2 + bc(|1-u^2|) \qquad (22)
\end{aligned}$$

3 Regular pentagon diagonal e

For the regular pentagon we have $u = \cos\theta = \cos(3\pi/5)$:

$$u = \frac{1-\sqrt{5}}{4} \qquad (23)$$

$$u^2 = \frac{3-\sqrt{5}}{8} \qquad (24)$$

$$|1-u^2| = \frac{5+\sqrt{5}}{8} \qquad (25)$$

We plug the value of pentagon's u in equation 13 to get pentagon's d_5:

$$\begin{aligned}
d_5 &= \sqrt{a^2 + b^2 - 2ab\left(\frac{1-\sqrt{5}}{4}\right)} \\
&= \frac{\sqrt{4a^2 + 4b^2 - 2ab - 2ab\sqrt{5}}}{2} \qquad (26)
\end{aligned}$$

We plug the values of pentagon's u, u^2 and $|1-u^2|$ in equation 22 to get pentagon's f_5:

$$\begin{aligned}
f_5 &= a\,(b+c)\left(\frac{1-\sqrt{5}}{4}\right) - bc\left(\frac{3-\sqrt{5}}{8}\right) + bc\left(\frac{5+\sqrt{5}}{8}\right) \\
&= \frac{2a(b+c) - 3bc + 5bc + (-2a(b+c) + bc + bc)\sqrt{5}}{8} \\
&= \frac{bc + a(b+c) + (bc - a(b+c))\sqrt{5}}{4} \qquad (27)
\end{aligned}$$

Finally we get the generic pentagon diagonal e_5 in function of only a, b, c: From the definition of f in equation 16 we have:

$$a^2 + b^2 + c^2 - e_5^2 = 2f_5$$
$$e_5 = \sqrt{a^2 + b^2 + c^2 - \frac{bc + a(b+c) + (bc - a(b+c))\sqrt{5}}{2}} \tag{28}$$

3.1 Regular pentagon diagonal d

From the figure so far we know that when $c = 0$, e becomes d so we can confirm this:

$$\begin{aligned} d &= e \qquad \texttt{when } c = 0 \\ &= \sqrt{a^2 + b^2 - \frac{ab - ab\sqrt{5}}{2}} \end{aligned} \tag{29}$$

which coincides with d_5 in equation 26 $\quad\square$.

3.2 Regular pentagon width W

The regular pentagon width W is defined as the distance between two farthest separated points, which equals the diagonal length D which is given by:

$$W = D = \frac{1+\sqrt{5}}{2}a \tag{30}$$

In our case the width is the diagonal d when $a = b$ or also de e when $a = b, c = 0$.

$$\begin{aligned} W_5 &= d \qquad \texttt{when } a = b \\ &= \frac{\sqrt{4a^2 + 4a^2 + 2a^2 + 2a^2\sqrt{5}}}{2} \\ &= \frac{\sqrt{6 - 2\sqrt{5}}}{2}a \\ &= \frac{\sqrt{5}+1}{2}a \end{aligned} \tag{31}$$

which coincides with above equation of pentagon's $W \quad \square$.

3.3 Regular pentagon height H

In the regular pentagon the height H is the distance from one side of length a to the opposite vertex:

$$H = \frac{\sqrt{5 + 2\sqrt{5}}}{2}a \tag{32}$$

For the height to occur we apply $b = a$ and $c = a/2$ in e_5 equation 28:

$$\begin{aligned} H_5 &= e \qquad \texttt{when } b = a \texttt{ and } c = a/2 \\ &= \sqrt{a^2 + a^2 + \frac{a^2}{4} - \frac{\frac{a^2}{2} + a\left(a + \frac{a}{2}\right) + \left(\frac{a^2}{2} - a\left(a + \frac{a}{2}\right)\right)\sqrt{5}}{2}} \\ &= a\sqrt{\frac{9}{4} - \frac{2-\sqrt{5}}{2}} \\ &= a\sqrt{\frac{5 + 2\sqrt{5}}{4}} \end{aligned} \tag{33}$$

which coincides with regular pentagon height H $\square$.