

Meccano nonagon

<https://github.com/heptagons/meccano/nona>

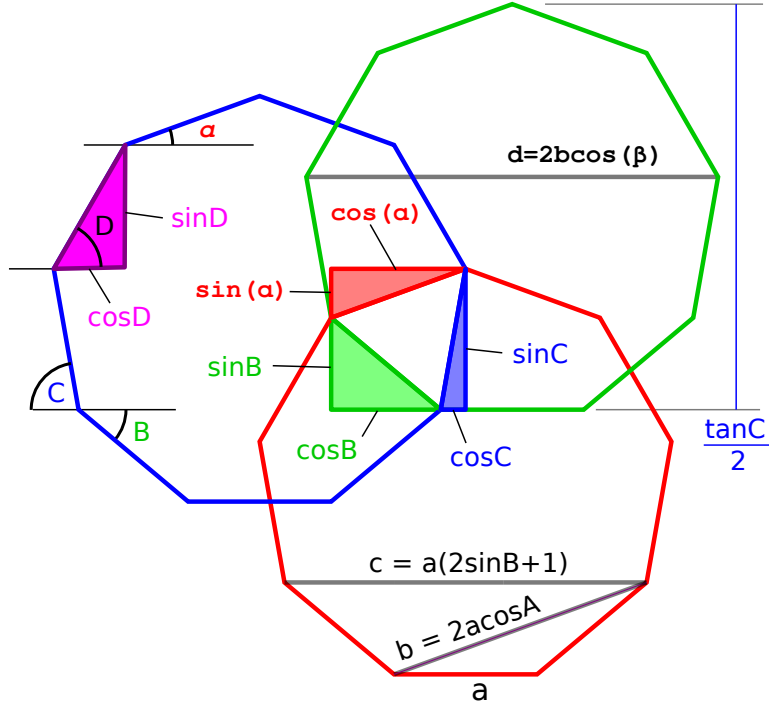


Figure 1: Three regular nonagons connected by an equilateral triangle. We note four angles in the figure A , B , C and D .

1 Algebra

Figure 1 shows three regular nonagons connected by an equilateral triangle. Four angles appear orthogonally in any regular nonagon:

$$\alpha = \pi/9 = 20^\circ \quad (1)$$

$$\beta = 2\pi/9 = 40^\circ \quad (2)$$

$$\gamma = 3\pi/9 = 60^\circ \quad (3)$$

$$\delta = 4\pi/9 = 80^\circ \quad (4)$$

$$\alpha + \beta = \delta - \alpha = \gamma \quad (5)$$

The relations of angle C are those of equilateral triangle:

$$\cos \gamma = -\frac{1}{2} \quad (6)$$

$$\sin \gamma = \frac{\sqrt{3}}{2} \quad (7)$$

From the figure 1, cosines of angles α, β, δ are related as:

$$\cos \alpha = \cos \beta + \cos \delta \quad (8)$$

$$= \cos(2\alpha) + \cos(4\alpha)$$

$$= (2 \cos^2 \alpha - 1) + (1 - 8 \cos^2 \alpha + 8 \cos^4 \alpha)$$

$$= 8 \cos^4 \alpha - 6 \cos^2 \alpha$$

$$1 = 8 \cos^3 \alpha - 6 \cos \alpha \quad (9)$$

The last equation is a depressed cubic equation with a negative discriminant:

$$t^3 + pt + q = 0 \quad (10)$$

$$p = -\frac{3}{4} \quad (11)$$

$$q = -\frac{1}{8} \quad (12)$$

$$\Delta = \frac{q^2}{4} + \frac{p^3}{27} = -\frac{3}{64}$$

The negative discriminant means we have three real roots which can be found by trigonometry:

$$t_k = 2\sqrt{-\frac{p}{3}} \cos \left(\frac{1}{3} \arccos \left(\frac{3q}{2p} \sqrt{\frac{-3}{p}} \right) - k \frac{2\pi}{3} \right) \quad \text{for } k = 0, 1, 2.$$

$$= \cos \left(\frac{1}{3} \arccos \left(\frac{1}{2} \right) - k \frac{2\pi}{3} \right) \quad \text{for } k = 0, 1, 2.$$

$$= \cos \left(\frac{1}{3} \left(\frac{\pi}{3} \right) - k \frac{2\pi}{3} \right) \quad \text{for } k = 0, 1, 2.$$

$$t_0 = \cos \left(\frac{\pi}{9} \right) = \cos \alpha \approx +0.939692620785908 \quad (13)$$

$$t_1 = \cos \left(-\frac{2\pi}{9} \right) = -\cos \beta \approx -0.766044443118978 \quad (14)$$

$$t_2 = \cos \left(-\frac{4\pi}{9} \right) = -\cos \delta \approx -0.173648177666930 \quad (15)$$

From equation 10 we know the product of roots squares is $-2p = \frac{3}{2}$:

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \delta = \frac{3}{2} \quad (16)$$

$$1 - \sin^2 \alpha + 1 - \sin^2 \beta + 1 - \sin^2 \delta = \frac{3}{2}$$

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \delta = \frac{3}{2} \quad (17)$$

From equation 10 we know the product of roots is $-q = \frac{1}{8}$ matching the “Morrie’s law”:

$$\cos \alpha \cos \beta \cos \delta = \frac{1}{8} \quad (18)$$

$$\begin{aligned} (1 - \sin^2 \alpha)(1 - \sin^2 \beta)(1 - \sin^2 \delta) &= \frac{1}{64} \\ (\sin \alpha \sin \beta)^2 + (\sin \alpha \sin \delta)^2 + (\sin \beta \sin \delta)^2 &= \frac{1}{64} - 1 + \sin^2 \alpha + \sin^2 \beta + \sin^2 \delta + (\sin \alpha \sin \beta \sin \delta)^2 \\ &= \frac{1}{64} - 1 + \frac{3}{2} + \left(\frac{\sqrt{3}}{8}\right)^2 = \left(\frac{9}{4}\right)^2 \end{aligned} \quad (19)$$

From the figure 1, sines of angles α, β, δ are related as:

$$\sin \alpha + \sin \beta = \sin \delta \quad (20)$$

$$\begin{aligned} &= \sin(2\alpha + \beta) \\ &= \sin(2\alpha) \cos \beta + \cos(2\alpha) \sin \beta \\ &= (2 \sin \alpha \cos \alpha) \cos \beta + (1 - 2 \sin^2 \alpha) \sin \beta \\ \sin \alpha &= \sin \alpha (2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta) \\ 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta &= 1 \\ 2 \cos(\alpha + \beta) &= 2 \cos \gamma = 1 \end{aligned} \quad (21)$$

Product of sines of angles α, β, δ is using equations 20 and 21:

$$\begin{aligned} \sin \alpha \sin \beta \sin \delta &= \frac{1}{2}(2 \cos \alpha \cos \beta - 1)(\sin \alpha + \sin \beta) \\ &= \sin(2\alpha) \sin \beta + \sin \alpha \sin(2\beta) \\ &= \frac{\sqrt{3}}{8} \end{aligned} \quad (22)$$

Last equation solves this cubic equation:

$$\begin{aligned} y^3 - \frac{3y}{4} - \frac{3}{8} &= 0 \\ y_1 &= -\sin A \approx -0.342020 \\ y_2 &= -\sin B \approx -0.642787 \\ y_3 &= +\sin C \approx +0.984807 \end{aligned}$$

Cosines and sines relations are:

$$\begin{aligned} \cos A \cos B - \sin A \sin B &= \frac{1}{2} \\ \frac{1}{\cos C} - \frac{\sqrt{3}}{\sin C} &= 4 \\ \tan C - 4 \sin C &= \sqrt{3} \end{aligned}$$

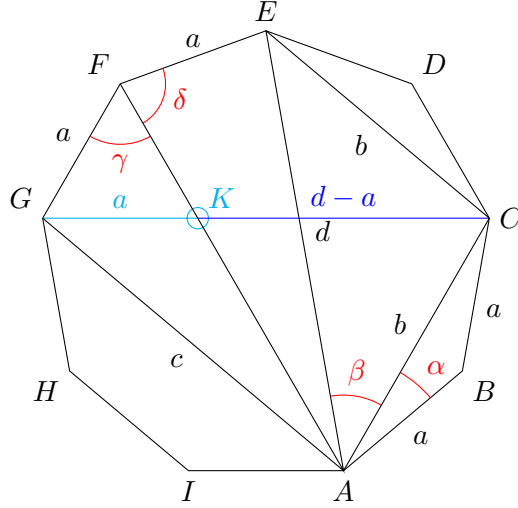


Figure 2: The nonagon perimeter side a and the three internal diagonals b, c, d . Also shown the base angle α and three more $\beta = 2\alpha$, $\gamma = 3\alpha$ and $\delta = 4\alpha$.

1.1 Segments a, b, c, d

From figure 2 we can calculate $\cos \alpha$ inspecting isosceles $\triangle ABC$ ($\overline{AB} = \overline{AC} = a$):

$$\begin{aligned} a^2 &= a^2 + b^2 - 2ab \cos \alpha \\ b^2 &= 2ab \cos \alpha \implies \boxed{b = 2a \cos \alpha} \end{aligned} \quad (23)$$

We calculate $\cos \beta$ inspecting isosceles $\triangle ACE$ ($\overline{AC} = \overline{AE} = b$):

$$\begin{aligned} b^2 &= b^2 + d^2 - 2bd \cos \beta \\ d^2 &= 2bd \cos \beta \implies \boxed{d = 2b \cos \beta} \end{aligned} \quad (24)$$

We calculate $\cos \delta$ inspecting isosceles $\triangle AEF$ ($\overline{AE} = \overline{AF} = d$):

$$\begin{aligned} d^2 &= a^2 + d^2 - 2ad \cos \delta \\ a^2 &= 2ad \cos \delta \implies \boxed{a = 2d \cos \delta} \end{aligned} \quad (25)$$

From equations 23, 24 and 25 we have:

$$\cos \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} = \begin{pmatrix} \frac{b}{2a} \\ \frac{d}{2b} \\ \frac{a}{2d} \end{pmatrix} \quad (26)$$

From equation 8 $\cos \alpha = \cos \beta + \cos \delta$:

$$\frac{b}{2a} = \frac{d}{2b} + \frac{a}{2d} \implies \boxed{\frac{1}{a/b} = \frac{1}{b/d} + \frac{1}{d/a}} \quad (27)$$

From figure 2 we inspect equilateral triangles $\triangle ACK$ and $\triangle FGK$ and found that:

$$\overline{FG} = \overline{GK} = \overline{KF} = a \quad (28)$$

$$\overline{AC} = \overline{CK} = \overline{KA} = b \quad (29)$$

$$\overline{CK} = \overline{CG} - \overline{KG}$$

$$b = d - a \implies \boxed{d = a + b} \quad (30)$$

Replacing d in optic equation 27 we get:

$$\frac{b}{a} = \frac{a+b}{b} + \frac{a}{a+b} \quad (31)$$

Making $e = b/a$ we update last equation as:

$$e \equiv \frac{b}{a}$$

$$e = \frac{1+e}{e} + \frac{1}{1+e} \implies \boxed{e^3 - 3e - 1 = 0} \quad (32)$$

$$e_0 \approx 1.87938524157182$$

$$e_1 \approx -1.53208888623796$$

$$e_2 \approx -0.347296355333861$$

$$e_0 + e_1 + e_2 = 0 \quad (33)$$

$$e_0 e_1 e_2 = 1 \quad (34)$$

$$= +2 \cos \alpha = b/a$$

$$= -2 \cos \beta = (a+b)/b$$

$$= -2 \cos \delta = a/(a+b)$$