Meccano four frame

https://github.com/heptagons/meccano/frames/four

Abstract

Four frame is a group of four rigid meccano [1] strips.
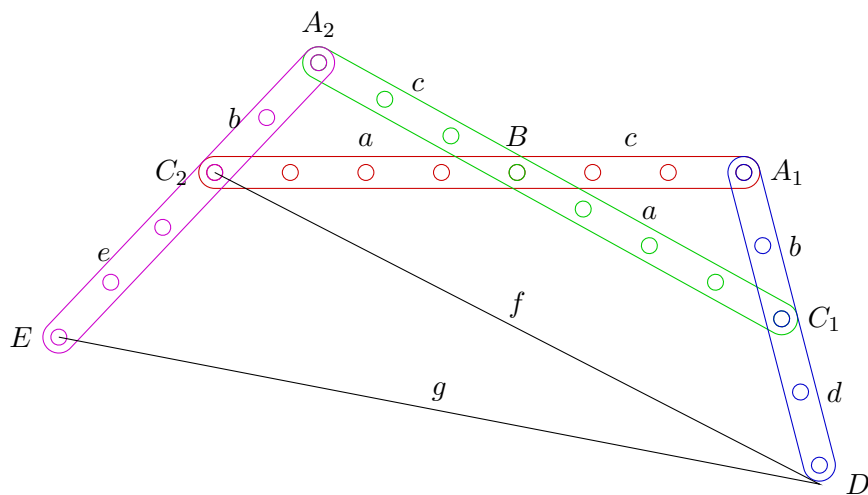
1 Four frame

Figure 1: Antisymmetric four frame.

Figure 1 show the antisymmetric four-strips frame. From the figure we define $\alpha \equiv \angle BA_1C_1$ and define integers $m = b^2 + c^2 - a^2$ and $n = 2bc$ using the law of cosines, then we calculate $\cos\alpha$ and $\sin\alpha$:

$$(\alpha, m, n) \equiv (\angle BA_1C_1, b^2 + c^2 - a^2, 2bc) \tag{1}$$

$$\cos\alpha = \frac{m}{n} \tag{2}$$

$$\sin\alpha = \frac{\sqrt{n^2 - m^2}}{n} \tag{3}$$

From the figure 1 we define $\gamma \equiv \angle BA_2C_2$ and define integers $s = a^2 + b^2 - c^2$ and $t = 2ab$ and calculate $\cos\gamma$ and $\sin\gamma$:

$$(\gamma, s, t) \equiv (\angle BA_2C_2, a^2 + b^2 - c^2, 2ab) \tag{4}$$

$$\cos\gamma = \frac{s}{t} \tag{5}$$

$$\sin\gamma = \frac{\sqrt{t^2 - s^2}}{t} \tag{6}$$

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[1] Meccano mathematics by 't Hooft

We calculate the distance $f = \overline{C_2D}$ with the law of cosines using angle α and defining integers $x = a+c$ and $y = b+d$:

$$(x, y) \equiv (a + c, b + d) \tag{7}$$

$$f^2 = (a + c)^2 + (b + d)^2 - 2(a + c)(b + d)\cos\alpha \tag{8}$$

$$= x^2 + y^2 - \frac{2mxy}{n} \tag{9}$$

$$f = \frac{\sqrt{n^2(x^2 + y^2) - 2mnxy}}{n} \tag{10}$$

We define a new integer $z \equiv n^2(x^2 + y^2) - 2mnxy$ so we have:

$$z \equiv n^2(x^2 + y^2) - 2mnxy \tag{11}$$

$$f = \frac{\sqrt{z}}{n} \tag{12}$$

We define angle $\theta \equiv \angle A_1 C_2 D$ and calculate $\cos\theta$ and $\sin\theta$:

$$\theta \equiv \angle A_1 C2D \tag{13}$$

$$\cos\theta = \frac{(a + c)^2 + f^2 - (b + d)^2}{2(a + c)f}$$

$$= \frac{x^2 + f^2 - y^2}{2xf}$$

$$= \frac{x^2 + x^2 + y^2 - \dfrac{2mxy}{n} - y^2}{2x\dfrac{\sqrt{z}}{n}}$$

$$= \frac{nx - my}{\sqrt{z}} \tag{14}$$

$$\sin^2\theta = 1 - \cos^2\theta = \frac{z - (nx - my)^2}{z}$$

$$= \frac{n^2(x^2 + y^2) - 2mnxy - (nx - my)^2}{z}$$

$$= \frac{n^2(x^2 + y^2) - 2mnxy - n^2x^2 + 2nxmy - m^2y^2}{z}$$

$$\sin\theta = \frac{y\sqrt{n^2 - m^2}}{\sqrt{z}} \tag{15}$$

We define angle $\phi \equiv \angle A_2 C_2 D$ and we note is the sum of angles $\theta + \gamma$ and we calculate $\cos\phi$:

$$\phi \equiv \angle A_2 C_2 D \tag{16}$$

$$= \theta + \gamma \tag{17}$$

$$\cos\phi = \cos(\theta + \gamma) \tag{18}$$

$$= \cos\theta\cos\gamma - \sin\theta\sin\gamma$$

$$= \frac{(nx - my)s}{\sqrt{z}t} - \frac{(y\sqrt{n^2 - m^2})\sqrt{t^2 - s^2}}{\sqrt{z}t}$$

$$= \frac{s(nx - my) - y\sqrt{(n^2 - m^2)(t^2 - s^2)}}{\sqrt{z}t} \tag{19}$$

2

We simplify $\sqrt{(n^2 - m^2)(t^2 - s^2)}$:

$$
\begin{aligned}
(n^2 - m^2)(t^2 - s^2) &= (n - m)(n + m)(t - s)(t + s) \\
&= (2bc - (b^2 + c^2 - a^2))(2bc + b^2 + c^2 - a^2)(2ab - (a^2 + b^2 - c^2))(2ab + a^2 + b^2 - c^2) \\
&= (a^2 - (b - c)^2)((b + c)^2 - a^2)(c^2 - (a - b)^2)((a + b)^2 - c^2) \\
&= (a + b - c)(a - b + c)(b + c + a)(b + c - a)(c + a - b)(c - a + b)(a + b + c)(a + b - c) \\
&= (a + b + c)^2(a + b - c)^2(a - b + c)^2(-a + b + c)^2 \\
\sqrt{(n^2 - m^2)(t^2 - s^2)} &= (a + b + c)(a + b - c)(a - b + c)(-a + b + c) \\
&= ((a + b)^2 - c^2)(c^2 - (a - b)^2) \\
&= (s + t)(t - s) \\
&= (t^2 - s^2)
\end{aligned}
\tag{20}
$$

We substitute equation 20 into equation 19 and we get:

$$
\cos\phi = \frac{s(nx - my) - y(t^2 - s^2)}{\sqrt{z}t}
\tag{21}
$$

From the figure we define angle $\psi \equiv \angle DC_2E$ and we note equals angle $\pi - \phi$, so we have:

$$
\begin{aligned}
\psi &\equiv \angle DC_2E \tag{22} \\
&= \pi - \phi \tag{23} \\
\cos\psi &= \cos(\pi - \phi) \tag{24} \\
&= -\cos\phi \\
&= \frac{-s(nx - my) + y(t^2 - s^2)}{\sqrt{z}t} \tag{25}
\end{aligned}
$$

Finally with $\cos\psi$, e and f we can calculate distance $g = \overline{ED}$:

$$
\begin{aligned}
g^2 &= e^2 + f^2 - 2ef\cos\psi \tag{26} \\
&= e^2 + x^2 + y^2 - \frac{2mxy}{n} - 2e\left(\frac{\sqrt{z}}{n}\right)\left(\frac{-s(nx - my) + y(t^2 - s^2)}{\sqrt{z}t}\right) \tag{27} \\
&= e^2 + x^2 + y^2 - \frac{2mxy}{n} + 2e\frac{s(nx - my) - y(t^2 - s^2)}{nt} \tag{28} \\
&= e^2 + x^2 + y^2 - \frac{2mxy}{2bc} + 2e\frac{s(nx - my) - y(t^2 - s^2)}{4ab^2c} \tag{29} \\
&= \frac{4a^2b^2c^2(e^2 + x^2 + y^2) - (2a^2bc)(2mxy) + 2aces(nx - my) - 2acey(t^2 - s^2)}{4a^2b^2c^2} \\
&= \frac{2ac(2ab^2c(e^2 + x^2 + y^2) - 2abmxy + es(nx - my) - ey(t^2 - s^2))}{4a^2b^2c^2} \\
g &= \frac{\sqrt{2ac(2ab^2c(e^2 + x^2 + y^2) - 2abmxy + es(nx - my) - ey(t^2 - s^2))}}{2abc} \tag{30}
\end{aligned}
$$

1.1 Antisymmetric four frame software

From the last equation of g we identify two `input` integers i_1, i_2 which are used to get $g(i)$. Then the nested radicals software will return square-free `output` integers z_1, z_2, z_3 as $g(z)$:

$$i_1 = 2abc \tag{31}$$

$$i_2 = 2ac(2ab^2c(e^2 + x^2 + y^2) - 2abmxy + es(nx - my) - ey(t^2 - s^2)) \tag{32}$$

$$g(i) = \frac{\sqrt{i_2}}{i_1} \tag{33}$$

$$g(z) = \frac{z_2\sqrt{z_3}}{z_1} \tag{34}$$

where m, n are calculated with equations 1, x, y are calculated with equations 7 and s, t are calculated with equations 4.

2 Meccano triangles $abc = (2n + 5, 2n + 4, 2n + 3)$

The family of triangles $T_0 = \triangle ABC$ with sides $(2n+5, 2n+4, 2n+3)$ can be decomposed into two meccano subtriangles $T_1 = \triangle ABD$ with sides $(2n + 3, 2n, 2n + 3)$ and $T_2 = \triangle BCD$ with sides $(4, 2n + 3, 2n + 5)$. For the frames with four strips the important thing is that triangles T_0 and T_1 share a common angle α while triangles T_0 and T_2 share a common angle γ.

We calculate α_0 and α_1 with the law of cosines:

$$\begin{aligned}
\cos\alpha_0 &= \frac{(2n + 3)^2 + (2n + 4)^2 - (2n + 5)^2}{2(2n + 3)(2n + 4)} \\
&= \frac{4n^2 + 4n}{(2n + 3)(2n + 4)} \\
&= \frac{2n(2n + 2)}{(2n + 3)(2n + 4)} \\
\cos\alpha_1 &= \frac{(2n + 3)^2 + (2n)^2 - (2n + 3)^2}{2(2n + 3)(2n)} \\
&= \frac{4n^2}{4n(2n + 3)} \\
&= \frac{n}{2n + 3}
\end{aligned}$$

(35)

(36)