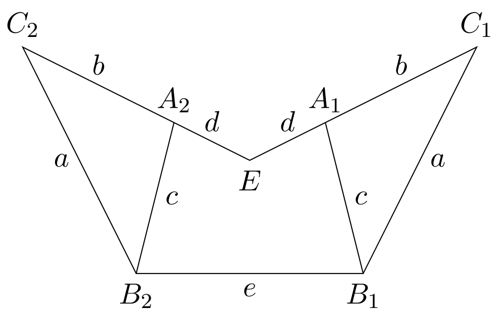
Horns unit

https://github.com/heptagons/meccano/units/horns

Abstract

Horns unit is a group of seven meccano [1] strips intended to build polygons.



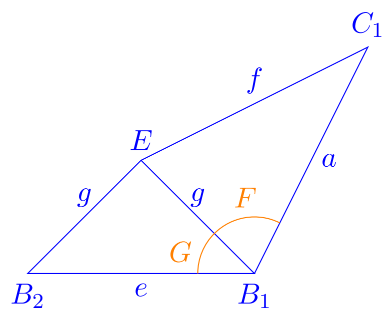


Figure 1: The **horn unit** has seven strips: Two of length a, two of length $b+d$, two of length c and one of length e. We expect to build polygons with internal angle $C_1B_1B_2$ and perimeter including segments a, e, a.

1 Algebra

From figure 1 we start with triangle $\triangle A_1B_1C1$. At vertex A_1 we have angle A and the supplement A':

$$A \equiv \angle B_1A_1C_1 \tag{1}$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc} \qquad \text{if and only if } a < b+c \tag{2}$$

$$A' \equiv \angle EA_1B_1 = \pi - A \tag{3}$$

$$\cos A' = \cos(\pi - A) = -\cos A = \frac{-b^2-c^2+a^2}{2bc} \tag{4}$$

We define $f \equiv b+d$ and $g \equiv \overline{EB_1}$. With the law of cosines we have:

$$f \equiv \boxed{b+d} \qquad \in \mathbb{N} \tag{5}$$

$$g^2 = c^2 + d^2 - 2cd\cos A' \tag{6}$$

$$= c^2 + d^2 - (2cd)\frac{-b^2-c^2+a^2}{2bc}$$

$$= \frac{bc^2+bd^2+b^2d+c^2d-a^2d}{b}$$

$$= \frac{(b+d)(bd+c^2)-a^2d}{b} \tag{7}$$

[1] Meccano mathematics by ‘t Hooft

Define a new variable $h = (b+d)(bd+c^2) - a^2 d$:

$$h \equiv \boxed{(bd+c^2)f - a^2 d} \qquad \in \mathbb{Z} \tag{8}$$

$$g^2 = \boxed{\frac{h}{b}} \qquad \text{if and only if } 0 < h < b \tag{9}$$

We calculate angles $F \equiv \angle C_1 B1E$ and $G \equiv \angle B_2 B_1 E$. We replace g^2 by h/b:

$$\cos F = \frac{a^2+g^2-f^2}{2ag} = \frac{a^2 b - bf^2 + h}{2abg} \tag{10}$$

$$\cos G = \boxed{\frac{e}{2g}} \tag{11}$$

Define new variable $j = a^2 b - bf^2 + h$ so:

$$j \equiv \boxed{a^2 b - bf^2 + h} \qquad \in \mathbb{Z} \tag{12}$$

$$\cos F = \frac{a^2 b - bf^2 + h}{2abg} = \boxed{\frac{j}{2abg}} \tag{13}$$

We calculate cosines squares and products. Again we replace g^2 by h/b:

$$\cos F \cos G = \frac{ej}{4abg^2} = \frac{bej}{4abh} = \boxed{\frac{ej}{4ah}} \qquad \in \mathbb{Q} \tag{14}$$

$$\cos^2 F = \frac{j^2}{4a^2b^2g^2} = \frac{bj^2}{4a^2b^2h} = \boxed{\frac{j^2}{4a^2bh}} \qquad \in \mathbb{Q} \tag{15}$$

$$\cos^2 G = \frac{e^2}{4g^2} = \boxed{\frac{be^2}{4h}} \qquad \in \mathbb{Q} \tag{16}$$

$$\cos^2 F \cos^2 G = \frac{be^2j^2}{16a^2bh^2} = \boxed{\frac{e^2j^2}{16a^2h^2}} \qquad \in \mathbb{Q} \tag{17}$$

$$\tag{18}$$

We calculate the sines part squared and set a common denominator as square $16a^2b^2h^2$:

$$(\sin F \sin G)^2 = (1 - cos^2 F)(1 - cos^2 G) \tag{19}$$

$$\begin{aligned}
&= 1 - cos^2 F - cos^2 G + cos^2 F \cos^2 G \\
&= 1 - \frac{j^2}{4a^2bh} - \frac{be^2}{4h} + \frac{e^2j^2}{16a^2h^2} \\
&= 1 - \frac{j^2}{4a^2bh} - \frac{be^2}{4h} + \frac{e^2j^2}{16a^2h^2} \\
&= \frac{16a^2b^2h^2 - (4bh)j^2 - (4a^2b^2h)be^2 + (b^2)e^2j^2}{16a^2b^2h^2} \\
&= \frac{16a^2b^2h^2 - 4bhj^2 - 4a^2b^3e^2h + b^2e^2j^2}{16a^2b^2h^2} \\
&= \frac{b(be^2-4h)(j^2-4a^2bh)}{16a^2b^2h^2}
\end{aligned} \tag{20}$$

Extract square root to get $\sin F \sin G = \sqrt{D}/A$ where $D, A \in \mathbb{Z}$:

$$\sin F \sin G = \boxed{\frac{\sqrt{b(be^2-4h)(j^2-4a^2bh)}}{4abh}} \qquad \in \mathbb{A} \tag{21}$$

We sum the angles F and G to get:

$$\begin{align}
F + G &\equiv \angle B_2 B_1 C_1 \tag{22}\\
\cos(F+G) &= \cos F \cos G - \sin F \sin G \tag{23}\\
&= \frac{ej}{4ah} - \frac{\sqrt{b(be^2-4h)(j^2-4a^2bh)}}{4abh} \tag{24}\\
&= \frac{bej - \sqrt{b(be^2-4h)(j^2-4a^2bh)}}{4abh} && \in \mathbb{A} \tag{25}
\end{align}$$