Meccano triangles

https://github.com/heptagons/meccano/nest

Abstract

We construct meccano triangles. Basic triangles has the three sides as integers and calculate the internal diagonal distances. Such diagonals then are used as the new side of more complicated triangles and then again we calculate new distances formed and so on. Eventually we expect to find certain angles joining the triangles which can be used to construct regular polygons or more figures.

1 Triangle (a, b, c)

A triangle (a, b, c) has the tree integer sides a, b and c. To avoid repetitions we consider only the cases:

$$a \geq b \geq c \qquad \in \mathbb{N} \tag{1}$$

$$a < b + c \tag{2}$$

We calculate the three angles:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \in \mathbb{Q} \tag{3}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} \qquad \in \mathbb{Q} \tag{4}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \qquad \in \mathbb{Q} \tag{5}$$

1.1 Triangle (a, b, c) diagonals

To calculate the diagonals we use the law of cosines. With the $\cos A$ we can calculate every diagonal $\overline{b_m c_n}$ with:

$$\overline{b_m c_n} = \sqrt{m^2 + n^2 - 2mn \cos A} \tag{6}$$

$$= \sqrt{m^2 + n^2 - 2mn \frac{b^2 + c^2 - a^2}{2bc}} \tag{7}$$

$$= \frac{\sqrt{b^2c^2(m^2 + n^2) - bcmn(b^2 + c^2 - a^2)}}{bc} \qquad \in \mathbb{A} \tag{8}$$

where $1 \leq m \leq b$, $1 \leq n \leq c$ and $m - n \geq 0$. Similarly:

$$\overline{c_m a_n} = \frac{\sqrt{c^2a^2(m^2 + n^2) - camn(c^2 + a^2 - b^2)}}{ac} \qquad \in \mathbb{A} \tag{9}$$

$$\overline{a_m b_n} = \frac{\sqrt{a^2b^2(m^2 + n^2) - abmn(a^2 + b^2 - c^2)}}{ab} \qquad \in \mathbb{A} \tag{10}$$

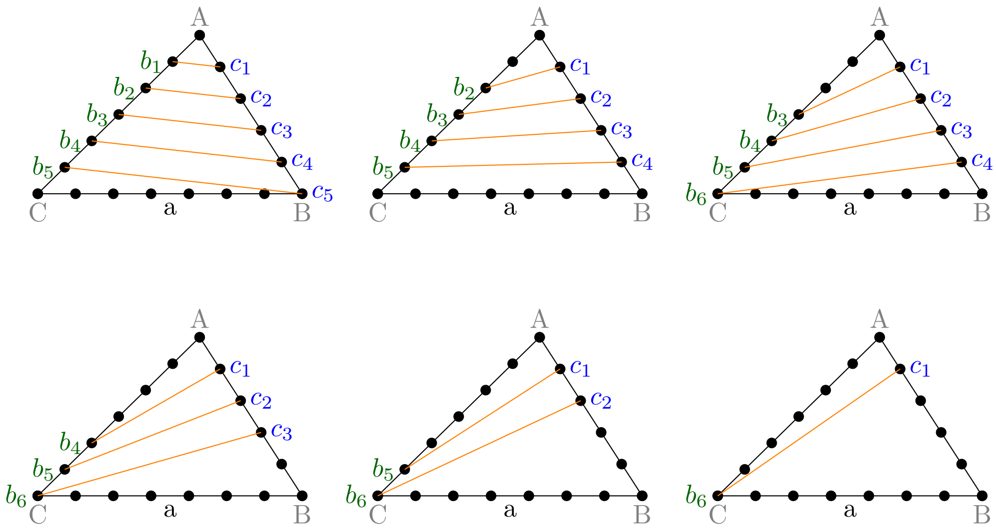

Figure 1: Triangle $(7, 6, 5)$ $b_m c_n$ diagonals $(m \geq n)$. For top to bottom and left to right we have six groups of diagonals. Each group is defined by m and n indices difference: $m - n = 0$, $m - n = 1$, ... $m - n = 5$.

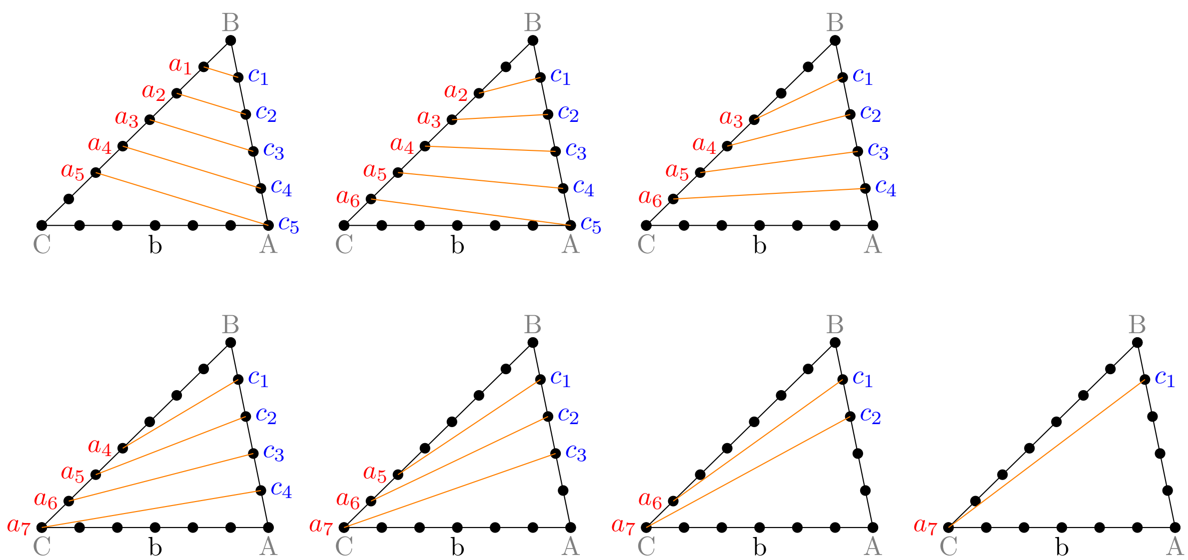

Figure 2: Triangle $(7, 6, 5)$, $a_m c_n$ diagonals $(m \geq n)$. We have also six group. But here we found diagonals repetead already found in previous figure. Diagonals repeated are all including a_7 points.

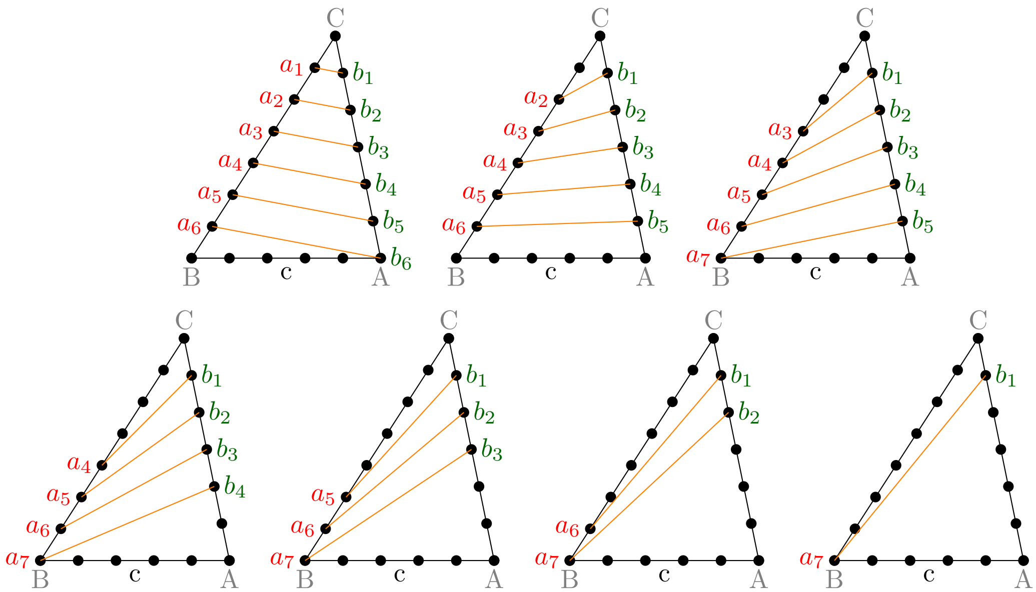

Figure 3: Triangle $(7,6,5)$, $a_m b_n$ diagonals $(m \geq n)$. Here we have diagonals repeteated already in previous figure. Using matrices we can reject such diagonals rejecting matrices columns.

1.2 Example triangle (7,6,5)

Figure **??** show triangle $(7,6,5)$ diagonals $b_m c_n$ for vertex A. For this figure we calculate the values and form a matrix. Empty cells are reflections:

$$\begin{pmatrix} \frac{2\sqrt{10}}{5} & \frac{\sqrt{105}}{5} & \frac{2\sqrt{55}}{5} & \frac{\sqrt{385}}{5} & 2\sqrt{6} & \frac{\sqrt{865}}{5} \\ & \frac{4\sqrt{10}}{5} & \frac{\sqrt{265}}{5} & \frac{2\sqrt{105}}{5} & \boxed{5} & \frac{4\sqrt{55}}{5} \\ & & \frac{6\sqrt{10}}{5} & \frac{\sqrt{505}}{5} & 2\sqrt{7} & \frac{3\sqrt{105}}{5} \\ & & & \frac{8\sqrt{10}}{5} & \sqrt{33} & \frac{2\sqrt{265}}{5} \\ & & & & 2\sqrt{10} & \boxed{7} \end{pmatrix} \tag{11}$$

Figure **??** show triangle $(7,6,5)$ diagonals $a_m c_n$ for vertex B. For this figure we calculate the values for a second matrix. Values at column 7 (at the right of separator $|$) are repeated and already in previous matrix. Empty cells are reflections.

$$\begin{pmatrix} \frac{4\sqrt{70}}{35} & \frac{3\sqrt{385}}{35} & \frac{2\sqrt{2065}}{35} & \frac{\sqrt{15505}}{35} & \frac{12\sqrt{7}}{7} & \frac{\sqrt{37345}}{35} & | & \frac{2\sqrt{265}}{5} \\ & \frac{8\sqrt{70}}{35} & \frac{\sqrt{7945}}{35} & \frac{6\sqrt{385}}{35} & \frac{\sqrt{889}}{7} & \frac{4\sqrt{2065}}{35} & | & \frac{3\sqrt{105}}{5} \\ & & \frac{12\sqrt{70}}{35} & \frac{\sqrt{14665}}{35} & \frac{2\sqrt{217}}{7} & \frac{9\sqrt{385}}{35} & | & \frac{4\sqrt{55}}{5} \\ & & & \frac{16\sqrt{70}}{35} & \frac{3\sqrt{105}}{7} & \frac{2\sqrt{7945}}{35} & | & \frac{\sqrt{865}}{5} \\ & & & & \frac{4\sqrt{70}}{7} & \frac{\sqrt{1393}}{7} & | & \boxed{6} \end{pmatrix} \tag{12}$$

Figure **??** show triangle $(7,6,5)$ diagonals $a_m b_n$ for vertex C. For this figure we calculate the values for a third matrix. Values at columns 6 and 7 (at the right of separator $|$) are repeated and already in previous matrices.

$$\left(\begin{array}{cccccccc} \frac{2\sqrt{7}}{7} & \frac{\sqrt{105}}{7} & \frac{2\sqrt{70}}{7} & \frac{\sqrt{553}}{7} & \frac{2\sqrt{231}}{7} & | & \frac{\sqrt{1393}}{7} & 2\sqrt{10} \\ & \frac{4\sqrt{7}}{7} & \frac{\sqrt{217}}{7} & \frac{2\sqrt{105}}{7} & \frac{\sqrt{721}}{7} & | & \frac{4\sqrt{70}}{7} & \sqrt{33} \\ & & \frac{6\sqrt{7}}{7} & \frac{\sqrt{385}}{7} & \frac{2\sqrt{154}}{7} & | & \frac{3\sqrt{105}}{7} & 2\sqrt{7} \\ & & & \frac{8\sqrt{7}}{7} & \frac{\sqrt{609}}{7} & | & \frac{2\sqrt{217}}{7} & 5 \\ & & & & \frac{10\sqrt{7}}{7} & | & \frac{\sqrt{889}}{7} & 2\sqrt{6} \\ & & & & & | & \frac{12\sqrt{7}}{7} & \boxed{5} \end{array}\right) \tag{13}$$

2 Triangles($\sqrt{a}, b, c$)

Triangles($\sqrt{a}, b, c$) have the tree sides $\sqrt{a}$, b and c so we have:

$$a^2, b, c \in \mathbb{N} \tag{14}$$
$$\sqrt{a} \geq b \geq c \tag{15}$$
$$\sqrt{a} < b + c \tag{16}$$

Raising to power 2 the inequalities we have:

$$a \geq b^2 \geq c^2 \tag{17}$$
$$a < (b+c)^2 \tag{18}$$

We calculate the triangle cosines:

$$\cos A = \frac{b^2 + c^2 - (\sqrt{a})^2}{2bc} = \frac{b^2 + c^2 - a}{2bc} \qquad \in \mathbb{Q} \tag{19}$$
$$\cos B = \frac{(\sqrt{a})^2 + c^2 - b^2}{2\sqrt{a}c} = \frac{(a + c^2 - b^2)\sqrt{a}}{2ac} \qquad \in \mathbb{A} \tag{20}$$
$$\cos C = \frac{(\sqrt{a})^2 + b^2 - c^2}{2\sqrt{a}b} = \frac{(a + b^2 - c^2)\sqrt{a}}{2ab} \qquad \in \mathbb{A} \tag{21}$$

2.1 Triangle ($\sqrt{a}, b, c$) diagonals

The only possible diagonals are for sides with integers, that is $\overline{b_m c_n}$. Using the law of cosines:

$$\overline{b_m c_n} = \sqrt{m^2 + n^2 - 2mn\cos A} \tag{22}$$
$$= \sqrt{m^2 + n^2 - 2mn\frac{b^2 + c^2 - a}{2bc}} \tag{23}$$
$$= \frac{\sqrt{b^2c^2(m^2+n^2) - bcmn(b^2 + c^2 - a)}}{bc} \qquad \in \mathbb{A} \tag{24}$$

where $1 \leq m \leq b$, $1 \leq n \leq c$ and $m - n \geq 0$.

2.2 Example triangles($2\sqrt{6}, b, c$)

In this case $\sqrt{a} = 2\sqrt{6}$ so $a = 24$. Then $m = n = \{1, 2, 3, 4\}$ because $b^2 = c^2 = \{1, 4, 9, 16\} < 24$. We form a matrix with with the values $(b+c)^2$:

$$(b_i + c_j)^2 = \begin{array}{c} \\ c=1 \\ c=2 \\ c=3 \\ c=4 \end{array} \begin{array}{c} \begin{array}{cccc} b=1 & b=2 & b=3 & b=4 \end{array} \\ \begin{pmatrix} 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \\ 25 & 36 & 49 & 64 \end{pmatrix} \end{array} \tag{25}$$

Then we remove cells which don't fulfil condition $b \geq c$:

$$(b_i + c_j)^2 = \begin{array}{cc} & \begin{array}{cccc} b=1 & b=2 & b=3 & b=4 \end{array} \\ \begin{array}{c} c=1 \\ c=2 \\ c=3 \\ c=4 \end{array} & \begin{pmatrix} 2 & 9 & 16 & 25 \\ \times & 16 & 25 & 36 \\ \times & \times & 36 & 49 \\ \times & \times & \times & 64 \end{pmatrix} \end{array} \tag{26}$$

Then we remove cells which don't fulfil condition $a < (b + c)^2$:

$$(b_i + c_j)^2 = \begin{array}{cc} & \begin{array}{cccc} b=1 & b=2 & b=3 & b=4 \end{array} \\ \begin{array}{c} c=1 \\ c=2 \\ c=3 \\ c=4 \end{array} & \begin{pmatrix} \times & \times & \times & 25 \\ & \times & 25 & 36 \\ & & 36 & 49 \\ & & & 64 \end{pmatrix} \end{array} \tag{27}$$

So only six triangles are valid:

$$(2\sqrt{6}, b, c) = \begin{array}{cc} & \begin{array}{ccc} \cos A & \cos B & \cos C \end{array} \\ \begin{array}{c} (2\sqrt{6}, 4, 1) \\ (2\sqrt{6}, 3, 2) \\ (2\sqrt{6}, 4, 2) \\ (2\sqrt{6}, 3, 3) \\ (2\sqrt{6}, 4, 3) \\ (2\sqrt{6}, 4, 4) \end{array} & \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \\ & & \\ & & \\ & & \end{pmatrix} \end{array} \tag{28}$$