

# Meccano frames

<https://github.com/heptagons/meccano/frames>

## Abstract

Meccano frames are groups of meccano <sup>1</sup> strips intended to be a base to build diverse meccano larger objects.

## 1 Triangular frame

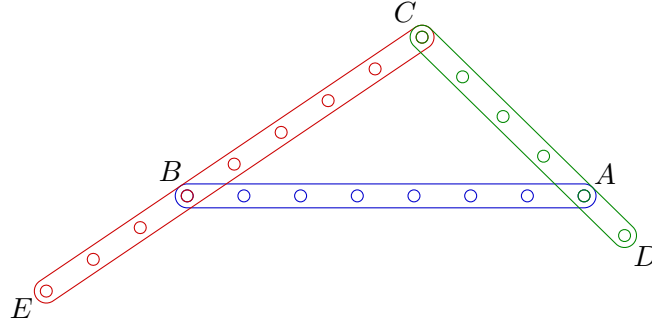


Figure 1: Triangular frame. With three strips we form the triangle  $\triangle ABC$ . At least we extend one of two strips  $\overline{CB}$  and  $\overline{CA}$  to become  $\overline{CE}$  and  $\overline{CD}$ . The new vertices  $D$  and  $E$  are rigid as the triangle and we'll calculate the distance between them which most of the time is not longer an integer.

Figure 1 shows the three strips triangular frame with extensions. First we identify five integer distances  $a, b, c, d, e$ :

$$a \equiv \overline{CB}, \quad b \equiv \overline{CA}, \quad c \equiv \overline{AB}, \quad c < a + b \quad (1)$$

$$d \equiv \overline{CB} + \overline{BE} = \overline{CE} \geq a \quad (2)$$

$$e \equiv \overline{CA} + \overline{AD} = \overline{CD} \geq b \quad (3)$$

We calculate the cosine of  $\angle BCA$ :

$$\theta \equiv \angle BCA \quad (4)$$

$$\cos \theta = \frac{a^2 + b^2 - c^2}{2ab} \quad (5)$$

Then we apply the cosine to the triangle  $\triangle CED$  to get the extensions distance  $\overline{DE}$ :

$$\begin{aligned} \overline{DE}^2 &= \overline{CD}^2 + \overline{CE}^2 - 2\overline{CD} \times \overline{CE} \cos \theta \\ &= d^2 + e^2 - 2de \cos \theta \\ &= d^2 + e^2 - de \left( \frac{a^2 + b^2 - c^2}{ab} \right) \end{aligned} \quad (6)$$

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<sup>1</sup> Meccano mathematics by 't Hooft

We expect at most a value of the form  $\sqrt{s}/t$  where  $s, t \in \mathbb{Z}^+$  so we define the surd as:

$$\begin{aligned}\overline{DE} &= \sqrt{d^2 + e^2 - de \left( \frac{a^2 + b^2 - c^2}{ab} \right)} \\ &= \frac{\sqrt{a^2 b^2 (d^2 + e^2) - abde(a^2 + b^2 - c^2)}}{ab} \\ &= \frac{\sqrt{ab((ad - be)(bd - ae) + c^2 de)}}{ab} \\ &= \frac{\sqrt{s}}{t}\end{aligned}\tag{7}$$

$$t = ab \tag{8}$$

$$s = t((ad - be)(bd - ae) + c^2 de) \tag{9}$$

### 1.1 Rigid distances $\sqrt{s}$ and $\sqrt{s} + f$

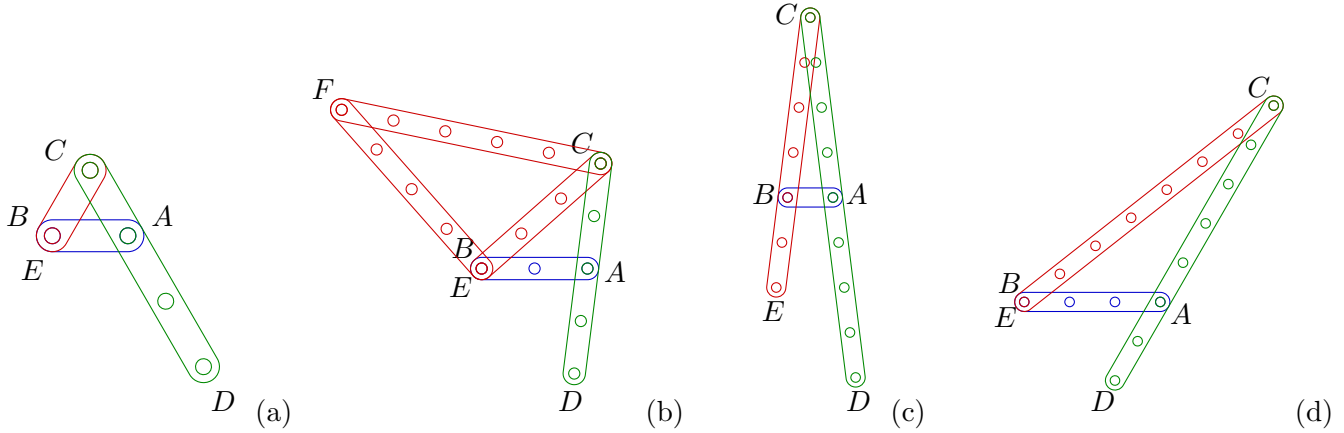


Figure 2: Four frames with rigid distance  $\overline{DE} = \sqrt{7}$  reported by the factory software mentioned in this section. The particular case (b), was reported with the angle  $CED = \pi/2$  which means we can append two extra strips to make a pythagorean triangle  $\triangle CEF$  where angle  $CEF = \pi/2$ , which makes the three vertices  $D, E, F$  collinear, so the rigid distance  $\overline{DF} = \sqrt{7} + 4$  is an algebraic number.

We write a factory to report all the triangles with specific surd  $\sqrt{s}$  for a given maximum strips length. We reject  $t \neq 1$  and  $s$  as not square-free, which includes pythagorean triangles. Next list show all the triangles with  $s = \sqrt{7}, t = 1$  where  $c < a + b$ ,  $a \leq d \leq \max$ ,  $b \leq e \leq \max$ ,  $c \leq \max$ :

```
1 === RUN    TestFramesTriangleSurd
2 NewFrames().TriangleSurd surd=7 max=15
3 1) a=1 e=1+2 c=1 cos=1/2
4 2) d=1+1 e=1+2 c=1 cos=1/2
5 3) d=1+2 b=1 c=1 cos=1/2
6 4) d=1+2 e=1+1 c=1 cos=1/2
7 5) a=2 e=2+1 c=2 cos=1/2
8 6) d=2+1 b=2 c=2 cos=1/2
9 7) a=3 e=2+2 c=2 cos=3/4 CED=pi/2
10 8) d=3+1 e=2+1 c=2 cos=3/4 CDE=pi/2
11 9) d=4+2 e=4+4 c=1 cos=31/32
12 10) d=4+4 e=4+2 c=1 cos=31/32
13 11) a=7 e=5+1 c=3 cos=13/14
14 12) a=7 e=5+2 c=3 cos=13/14
```

Figure 2 show four cases of this list. The code is in the folder [github.com/heptagons/meccano/frames](https://github.com/heptagons/meccano/frames).

## 2 More rigid distances $\sqrt{s} + h$

We explore a more complicated frame to get additional cases of distances  $\sqrt{s} + h$  without relying in an explicit pythagorean triangle as we saw in case (b) of figure 2.

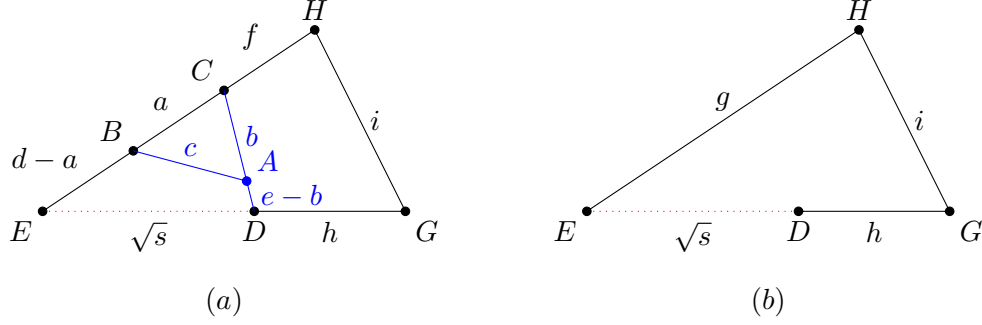


Figure 3: The five strips intended to form an algebraic distance  $\overline{EG} = \sqrt{s} + h$ .

From figure 3 (a) we know  $\sqrt{s}$  distance between nodes  $E$  and  $D$  is produced by the three strips frame  $a + d$ ,  $b + e$  and  $c$ . Using the law of cosines we calculate the angle  $\theta = \angle CED$  in terms of  $\sqrt{s}$ :

$$\begin{aligned} \cos \theta &= \frac{d^2 + (\sqrt{s})^2 - e^2}{2d\sqrt{s}} \\ &= \frac{(d^2 + s - e^2)\sqrt{s}}{2ds} \end{aligned} \tag{10}$$

$$= \frac{m\sqrt{s}}{n} \tag{11}$$

$$m = d^2 + s - e^2 \tag{12}$$

$$n = 2ds \tag{13}$$

From figure 3 (a) we notice two sets of points are collinear:  $\{E, B, C, H\}$  and  $\{E, D, G\}$ . Using the law of cosines we calculate the angle  $\theta = \angle HEG$  in terms of distances  $g, \sqrt{s} + h, i$ :

$$\begin{aligned} \cos \theta &= \frac{g^2 + (\sqrt{s} + h)^2 - i^2}{2g(\sqrt{s} + h)} \\ &= \frac{g^2 + s + 2\sqrt{s}h + h^2 - i^2}{2g(\sqrt{s} + h)} \\ &= \frac{g^2 + s + h^2 - i^2 + 2\sqrt{s}h}{2g(\sqrt{s} + h)} \end{aligned} \tag{14}$$

We multiply both numerator and denominator by  $\sqrt{s} - h$  to eliminate the surd from denominator:

$$\begin{aligned}
\cos \theta &= \frac{(s + g^2 + h^2 - i^2)(\sqrt{s} - h) + 2\sqrt{s}h(\sqrt{s} - h)}{2g(\sqrt{s} + h)(\sqrt{s} - h)} \\
&= \frac{(s + g^2 + h^2 - i^2)(\sqrt{s} - h) + 2sh - 2\sqrt{s}h^2}{2g(s - h^2)} \\
&= \frac{-h(s + g^2 + h^2 - i^2 - 2s) + (s + g^2 + h^2 - i^2 - 2h^2)\sqrt{s}}{2g(s - h^2)} \\
&= \frac{h(s - g^2 - h^2 + i^2) + (s + g^2 - h^2 - i^2)\sqrt{s}}{2g(s - h^2)} \\
&= \frac{o + p\sqrt{s}}{q}
\end{aligned} \tag{15}$$

$$o = h(s - g^2 - h^2 + i^2) \tag{16}$$

$$p = s + g^2 - h^2 - i^2 \tag{17}$$

$$q = 2g(s - h^2) \tag{18}$$

We compare both cosines equations 11 and 15:

$$\frac{m\sqrt{s}}{n} = \frac{o + p\sqrt{s}}{q} \tag{19}$$

Since all variables are integers we need two conditions. First  $o$  should be zero. And second  $\frac{m}{n} = \frac{p}{q}$ .

For condition 1, we force  $o$  to be zero:

$$\begin{aligned}
o &= 0 \\
h(s - g^2 - h^2 + i^2) &= 0 \\
s &= g^2 + h^2 - i^2
\end{aligned} \tag{20}$$

For condition2, we force  $m, n, p, q$  as:

$$\begin{aligned}
\frac{m}{n} &= \frac{p}{q} \\
\frac{d^2 + s - e^2}{2ds} &= \frac{s + g^2 - h^2 - i^2}{2g(s - h^2)}
\end{aligned} \tag{21}$$

We replace the value of  $s$  of last equation RHS with the value of equation 20 of condition 1:

$$\begin{aligned}
\frac{d^2 - e^2 + s}{ds} &= \frac{s + g^2 - h^2 - i^2}{g(s - h^2)} \\
&= \frac{g^2 + h^2 - i^2 + g^2 - h^2 - i^2}{g(g^2 + h^2 - i^2 - h^2)} \\
&= \frac{2(g^2 - i^2)}{g(g^2 - i^2)} \\
&= \frac{2}{g} \\
(d^2 - e^2 + s)g &= 2ds
\end{aligned} \tag{22}$$

*TODO : Examples!!!*

### 3 Five strips frame

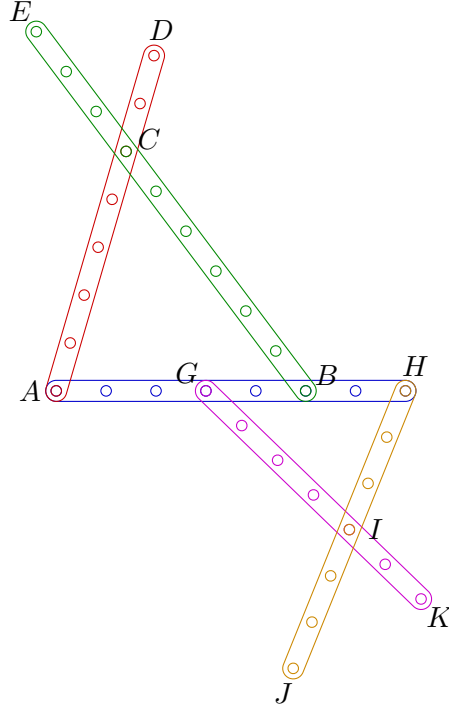


Figure 4: Five strips frame. We construct two triangles  $\triangle ABC$  and  $\triangle GHI$ . Extending the strips we get four vertices  $E, D, J, K$  which can form four rigid distances of surd type:  $\overline{DJ}, \overline{DK}, \overline{EJ}, \overline{EK}$ .

Figure 4 shows a frame with five strips. The frame has eleven variables:

$$a = \overline{BC}, \quad b = \overline{AC}, \quad c = \overline{AB} \quad (23)$$

$$d = \overline{AD}, \quad e = \overline{AE} \quad (24)$$

$$f = \overline{AG} \quad (25)$$

$$g = \overline{HI}, \quad h = \overline{GI}, \quad i = \overline{GH} \quad (26)$$

$$j = \overline{HJ}, \quad k = \overline{HK} \quad (27)$$

Assume vertex A is at the origin. Let  $\alpha = \angle BAC$ , and  $D_x, D_y$  the abscissa and ordinate of vertex D so we have:

$$t \equiv b^2 + c^2 - a^2 \quad (28)$$

$$x \equiv 4b^2c^2 - t^2 \quad (29)$$

$$\cos \alpha = \frac{t}{2bc} \quad (30)$$

$$\sin \alpha = \frac{\sqrt{x}}{2bc} \quad (31)$$

$$D_x = d \sin \alpha = \frac{d\sqrt{x}}{2bc} \quad (32)$$

$$D_y = d \cos \alpha = \frac{dt}{2bc} \quad (33)$$

$$D_x^2 + D_y^2 = d^2 \quad (34)$$

Let  $\delta = \angle HGI$  and  $K_x, K_y$  the abscissa and ordinate of vertex  $K$  so we have:

$$v \equiv h^2 + i^2 - g^2 \quad (35)$$

$$y \equiv 4h^2i^2 - v^2 \quad (36)$$

$$\cos \delta = \frac{v}{2hi} \quad (37)$$

$$\sin \delta = \frac{\sqrt{y}}{2hi} \quad (38)$$

$$K_x = f + k \sin \delta = f + \frac{k\sqrt{y}}{2hi} \quad (39)$$

$$K_y = -k \cos \delta = -\frac{kv}{2hi} \quad (40)$$

$$K_x^2 + K_y^2 = f^2 + 2fk \sin \delta + k^2 \quad (41)$$

$$= f^2 + k^2 + \frac{fk\sqrt{y}}{hi} \quad (42)$$

We calculate the distance  $\overline{DK}$ :

$$\begin{aligned} \overline{DK}^2 &= (D_x + K_x)^2 + (D_y + K_y)^2 \\ &= D_x^2 + 2D_xK_x + K_x^2 + D_y^2 + 2D_yK_y + K_y^2 \\ &= (D_x^2 + D_y^2) + (K_x^2 + K_y^2) + 2D_xK_x + 2D_yK_y \\ &= d^2 + f^2 + k^2 + \frac{fk\sqrt{y}}{hi} + 2 \left( \frac{d\sqrt{x}}{2bc} \right) \left( f + \frac{k\sqrt{y}}{2hi} \right) + 2 \left( \frac{dt}{2bc} \right) \left( -\frac{kv}{2hi} \right) \\ &= d^2 + f^2 + k^2 - \frac{dtkv}{2bchi} + \frac{fk\sqrt{y}}{hi} + \frac{df\sqrt{x}}{bc} + \frac{dk\sqrt{xy}}{2bchi} \end{aligned} \quad (43)$$