

# Meccano frames

<https://github.com/heptagons/meccano/frames>

## Abstract

Meccano frames are groups of rigid meccano<sup>1</sup> strips. Can be used as internal diagonals of polygons to be rigid. The lengths of such diagonals are algebraic numbers of the form  $B + \frac{C\sqrt{D}}{A}$  or  $\frac{\sqrt{F+H\sqrt{G}}}{A}$ .

## 1 Triangular frame

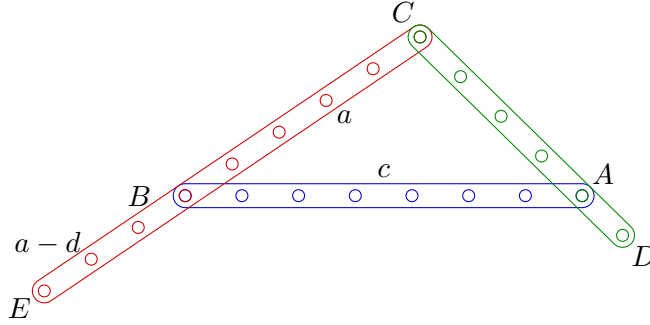


Figure 1: Triangular frame.

Figure 1 shows a triangular frame. With three strips we form the triangle  $\triangle ABC$ . At least we extend one of the two strips  $\overline{CB}$  and  $\overline{CA}$  to become  $\overline{CE}$  and  $\overline{CD}$ . The new vertices  $D$  and  $E$  distance is rigid of the form  $\frac{p\sqrt{s}}{q}$ , where  $p, q, s \in \mathbb{Z}^+$ .

First we identify five integer distances  $a, b, c, d, e$ :

$$a \equiv \overline{CB}, \quad b \equiv \overline{CA}, \quad c \equiv \overline{AB}, \quad c < a + b \quad (1)$$

$$d \equiv \overline{CB} + \overline{BE} = \overline{CE} \geq a \quad (2)$$

$$e \equiv \overline{CA} + \overline{AD} = \overline{CD} \geq b \quad (3)$$

We calculate the cosine of  $\angle BCA$ :

$$\theta \equiv \angle BCA \quad (4)$$

$$\cos \theta = \frac{a^2 + b^2 - c^2}{2ab} \quad (5)$$

Then we apply the cosine to the triangle  $\triangle CED$  to get the extensions distance  $\overline{DE}$ :

$$\begin{aligned} \overline{DE}^2 &= \overline{CD}^2 + \overline{CE}^2 - 2\overline{CD} \times \overline{CE} \cos \theta \\ &= d^2 + e^2 - 2de \cos \theta \\ &= d^2 + e^2 - de \left( \frac{a^2 + b^2 - c^2}{ab} \right) \end{aligned} \quad (6)$$

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<sup>1</sup> Meccano mathematics by 't Hooft

We extract the square root:

$$\begin{aligned}
\overline{DE} &= \sqrt{d^2 + e^2 - de \left( \frac{a^2 + b^2 - c^2}{ab} \right)} \\
&= \frac{\sqrt{a^2 b^2 (d^2 + e^2) - abde(a^2 + b^2 - c^2)}}{ab} \\
&= \frac{\sqrt{ab((ad - be)(bd - ae) + c^2 de)}}{ab}
\end{aligned} \tag{7}$$

## 1.1 Software

We write a software to report all the triangle frames with specific surd  $\sqrt{s}$  for a given maximum strips length. We can reject cases  $q \neq 1$  and  $s$  not square-free. Next list show all the triangles with  $q = 1$  and  $s = \sqrt{7}$  where  $c < a + b$ ,  $a \leq d \leq \max$ ,  $b \leq e \leq \max$ ,  $c \leq \max$ :

```

1  === RUN    TestFramesTriangleSurds
2  NewFrames().TriangleSurds surd=7 max=15
3      1) a=1 e=1+2 c=1 cos=1/2
4      2) d=1+1 e=1+2 c=1 cos=1/2
5      3) d=1+2 b=1 c=1 cos=1/2
6      4) d=1+2 e=1+1 c=1 cos=1/2
7      5) a=2 e=2+1 c=2 cos=1/2
8      6) d=2+1 b=2 c=2 cos=1/2
9      7) a=3 e=2+2 c=2 cos=3/4 CED=pi/2
10     8) d=3+1 e=2+1 c=2 cos=3/4 CDE=pi/2
11     9) d=4+2 e=4+4 c=1 cos=31/32
12    10) d=4+4 e=4+2 c=1 cos=31/32
13    11) a=7 e=5+1 c=3 cos=13/14
14    12) a=7 e=5+2 c=3 cos=13/14

```

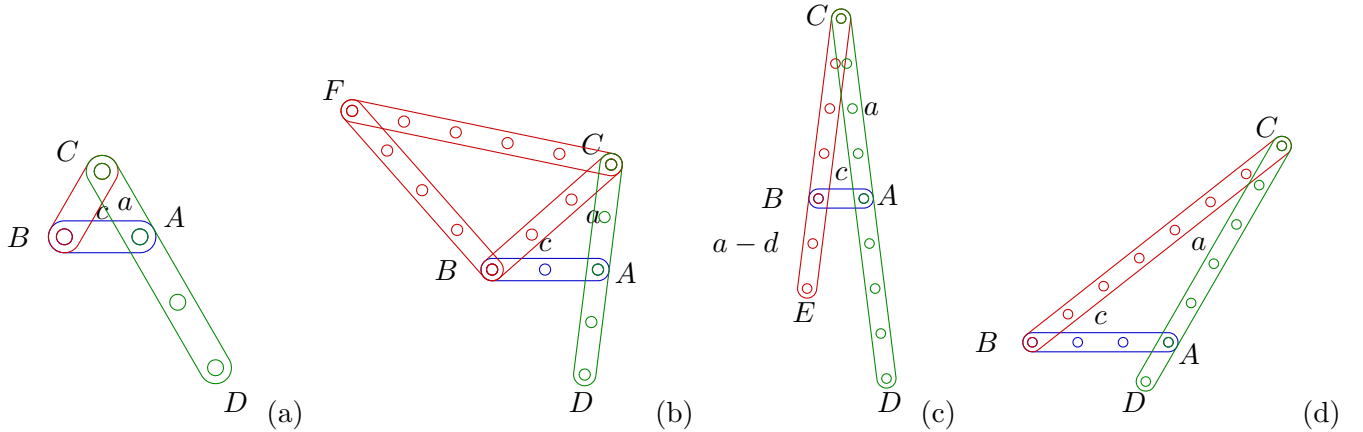


Figure 2: Some triangular frames with rigid distance  $\overline{DE} = \sqrt{7}$  found by the software.

Figure 2 show four cases of this list. The code is in the folder [github.com/heptagons/meccano/frames](https://github.com/heptagons/meccano/frames).

## 1.2 Triangular distance of the form $\sqrt{s} + f$

In the figure 2, the particular case (b), was reported with the angle  $CED = \pi/2$  which means we can append two extra strips to make a pythagorean triangle  $\triangle CEF$  where angle  $CEF = \pi/2$ , which makes the three vertices  $D, E, F$  collinear, so the rigid distance  $\overline{DF} = \sqrt{7} + 4$  is an algebraic number.

### 1.3 Another rigid distances $\sqrt{s} + h$

We explore a more complicated frame to get additional cases of distances  $\sqrt{s} + h$  without relying in an explicit pythagorean triangle as we saw in case (b) of figure 2.

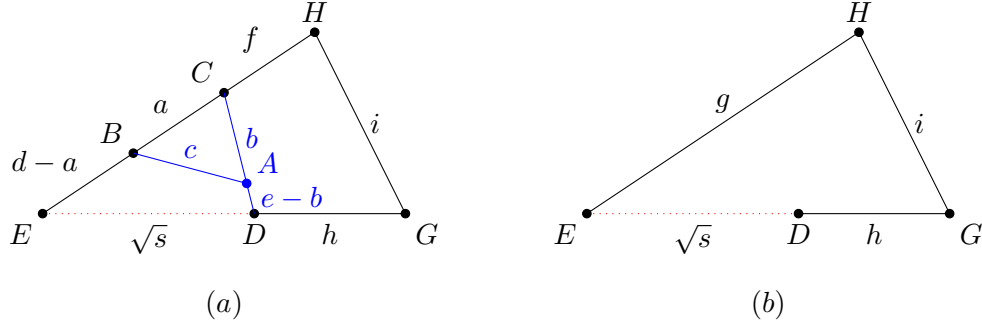


Figure 3: The five strips intended to form an algebraic distance  $\overline{EG} = \sqrt{s} + h$ .

From figure 3 (a) we know  $\sqrt{s}$  distance between nodes  $E$  and  $D$  is produced by the three strips frame  $a + d$ ,  $b + e$  and  $c$ . Using the law of cosines we calculate the angle  $\theta = \angle CED$  in terms of  $\sqrt{s}$ :

$$\begin{aligned} \cos \theta &= \frac{d^2 + (\sqrt{s})^2 - e^2}{2d\sqrt{s}} \\ &= \frac{(d^2 + s - e^2)\sqrt{s}}{2ds} \end{aligned} \tag{8}$$

$$= \frac{m\sqrt{s}}{n} \tag{9}$$

$$m = d^2 + s - e^2 \tag{10}$$

$$n = 2ds \tag{11}$$

From figure 3 (a) we notice two sets of points are collinear:  $\{E, B, C, H\}$  and  $\{E, D, G\}$ . Using the law of cosines we calculate the angle  $\theta = \angle HEG$  in terms of distances  $g$ ,  $\sqrt{s} + h$ ,  $i$ :

$$\begin{aligned} \cos \theta &= \frac{g^2 + (\sqrt{s} + h)^2 - i^2}{2g(\sqrt{s} + h)} \\ &= \frac{g^2 + s + 2\sqrt{s}h + h^2 - i^2}{2g(\sqrt{s} + h)} \\ &= \frac{g^2 + s + h^2 - i^2 + 2\sqrt{s}h}{2g(\sqrt{s} + h)} \end{aligned} \tag{12}$$

We multiply both numerator and denominator by  $\sqrt{s} - h$  to eliminate the surd from denominator:

$$\begin{aligned}
\cos \theta &= \frac{(s + g^2 + h^2 - i^2)(\sqrt{s} - h) + 2\sqrt{s}h(\sqrt{s} - h)}{2g(\sqrt{s} + h)(\sqrt{s} - h)} \\
&= \frac{(s + g^2 + h^2 - i^2)(\sqrt{s} - h) + 2sh - 2\sqrt{s}h^2}{2g(s - h^2)} \\
&= \frac{-h(s + g^2 + h^2 - i^2 - 2s) + (s + g^2 + h^2 - i^2 - 2h^2)\sqrt{s}}{2g(s - h^2)} \\
&= \frac{h(s - g^2 - h^2 + i^2) + (s + g^2 - h^2 - i^2)\sqrt{s}}{2g(s - h^2)} \\
&= \frac{o + p\sqrt{s}}{q}
\end{aligned} \tag{13}$$

$$o = h(s - g^2 - h^2 + i^2) \tag{14}$$

$$p = s + g^2 - h^2 - i^2 \tag{15}$$

$$q = 2g(s - h^2) \tag{16}$$

We compare both cosines equations 9 and 13:

$$\frac{m\sqrt{s}}{n} = \frac{o + p\sqrt{s}}{q} \tag{17}$$

Since all variables are integers we need two conditions. First  $o$  should be zero. And second  $\frac{m}{n} = \frac{p}{q}$ .

For condition 1, we force  $o$  to be zero:

$$\begin{aligned}
o &= 0 \\
h(s - g^2 - h^2 + i^2) &= 0 \\
s &= g^2 + h^2 - i^2
\end{aligned} \tag{18}$$

For condition2, we force  $m, n, p, q$  as:

$$\begin{aligned}
\frac{m}{n} &= \frac{p}{q} \\
\frac{d^2 + s - e^2}{2ds} &= \frac{s + g^2 - h^2 - i^2}{2g(s - h^2)}
\end{aligned} \tag{19}$$

We replace the value of  $s$  of last equation RHS with the value of equation 18 of condition 1:

$$\begin{aligned}
\frac{d^2 - e^2 + s}{ds} &= \frac{s + g^2 - h^2 - i^2}{g(s - h^2)} \\
&= \frac{g^2 + h^2 - i^2 + g^2 - h^2 - i^2}{g(g^2 + h^2 - i^2 - h^2)} \\
&= \frac{2(g^2 - i^2)}{g(g^2 - i^2)} \\
&= \frac{2}{g} \\
(d^2 - e^2 + s)g &= 2ds
\end{aligned} \tag{20}$$

*TODO : Examples!!!*

## 2 Triangle pair frame

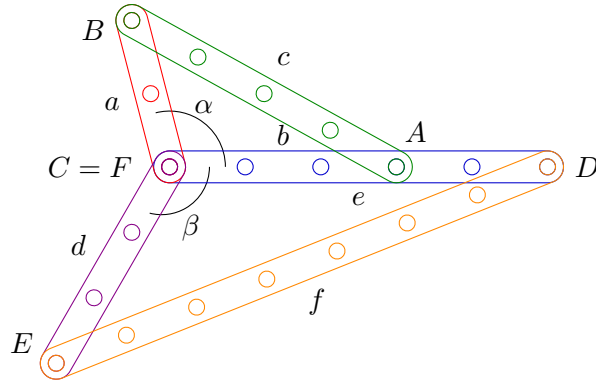


Figure 4: Triangle pair frame. We join triangles  $\triangle ABC$  and  $\triangle DEF$  in such a way vertices  $C$  and  $F$  coincide and vertices  $A, C, D, E$  be collinear. The result is a five strips frame. We are interested in the distance  $\overline{BE}$ .

Figure 4 shows a triangle pair frame. The triangles share a strip which contains four of the vertices. The remaining two vertices are separated by distances of the form  $\frac{\sqrt{F+G\sqrt{H}}}{A}$ . With only five strips this frame is small and useful to make up the diagonals inside polygons we want to be rigid.

### 2.1 Triangle pair algebra

First we calculate the cosines:

$$\cos \alpha = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos \beta = \frac{d^2 + e^2 - f^2}{2de}$$

We define integers  $m, n, o, p$  to simplify cosines and get sines:

$$(m, n) \equiv (a^2 + b^2 - c^2, 2ab) \quad |m| \leq n \quad (21)$$

$$(o, p) \equiv (d^2 + e^2 - f^2, 2de) \quad |o| \leq p \quad (22)$$

$$\cos \alpha = \frac{m}{n} \quad (23)$$

$$\cos \beta = \frac{o}{p} \quad (24)$$

$$\sin \alpha = \sqrt{1 - \cos^2 \alpha} = \frac{\sqrt{n^2 - m^2}}{n} \quad (25)$$

$$\sin \beta = \sqrt{1 - \cos^2 \beta} = \frac{\sqrt{p^2 - o^2}}{p} \quad (26)$$

Then, we use the cosines sum identity:

$$\begin{aligned} \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ &= \left(\frac{m}{n}\right) \left(\frac{o}{p}\right) - \left(\frac{\sqrt{n^2 - m^2}}{n}\right) \left(\frac{\sqrt{p^2 - o^2}}{p}\right) \\ &= \frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{np} \end{aligned} \quad (27)$$

Finally we can calculate the distance  $g \equiv \overline{BE}$  using the law of cosines:

$$\begin{aligned}
g &\equiv \overline{BE} \\
&= \sqrt{a^2 + d^2 - 2ad \cos(\alpha + \beta)} \\
&= \sqrt{a^2 + d^2 - 2ad \left( \frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{np} \right)} \\
&= \sqrt{a^2 + d^2 - 2ad \left( \frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{4abde} \right)} \\
&= \sqrt{a^2 + d^2 - \frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{2be}} \\
&= \frac{\sqrt{4b^2e^2(a^2 + d^2) - 2bem o + 2be\sqrt{(n^2 - m^2)(p^2 - o^2)}}}{2be}
\end{aligned} \tag{28}$$

For the software we can define integers  $A, F, G, H$  to calculate and reduce  $g$ :

$$A \equiv 2be \tag{29}$$

$$F \equiv A^2(a^2 + d^2) - Amo \tag{30}$$

$$G \equiv A \tag{31}$$

$$H \equiv (n^2 - m^2)(p^2 - o^2) \tag{32}$$

$$g = \frac{\sqrt{F + G\sqrt{H}}}{A} \tag{33}$$

## 2.2 Triangle pairs software

We run a program to inspect triangle pairs having a given distance  $g$ . The software iterates over the two triangles sides  $(a, b, c)$  and  $(d, e, f)$  up to a maximum strip length.

Next example request distances of the form  $\sqrt{46 + 18\sqrt{5}}$  up to strip length 10:

*Folder* : `github.com/heptagons/meccano/frames`

*Call* : `NewFrames().TrianglePairsTex(10, [46 18 5])`

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$$\begin{aligned}
(a, b, c) \oplus (d, e, f) &\mapsto g \\
(2, 1, 2) \oplus (3, 3, 3) &\mapsto \frac{\sqrt{46 + 18\sqrt{5}}}{2} \\
(2, 1, 2) \oplus (3, 8, 7) &\mapsto \frac{\sqrt{46 + 18\sqrt{5}}}{2} \\
(2, 2, 2) \oplus (3, 6, 6) &\mapsto \frac{\sqrt{46 + 18\sqrt{5}}}{2} \\
(2, 3, 4) \oplus (3, 5, 7) &\mapsto \frac{\sqrt{46 + 18\sqrt{5}}}{2} \\
(2, 4, 4) \oplus (3, 8, 7) &\mapsto \frac{\sqrt{46 + 18\sqrt{5}}}{2} \\
(3, 3, 3) \oplus (2, 4, 4) &\mapsto \frac{\sqrt{46 + 18\sqrt{5}}}{2} \\
(4, 2, 4) \oplus (6, 6, 6) &\mapsto \sqrt{46 + 18\sqrt{5}} \\
(4, 4, 4) \oplus (6, 7, 8) &\mapsto \sqrt{46 + 18\sqrt{5}} \\
(6, 3, 6) \oplus (4, 4, 4) &\mapsto \sqrt{46 + 18\sqrt{5}} \\
(6, 3, 6) \oplus (9, 9, 9) &\mapsto \frac{3\sqrt{46 + 18\sqrt{5}}}{2} \\
(6, 6, 6) \oplus (4, 8, 8) &\mapsto \sqrt{46 + 18\sqrt{5}} \\
(6, 7, 8) \oplus (9, 9, 9) &\mapsto \frac{3\sqrt{46 + 18\sqrt{5}}}{2}
\end{aligned}$$

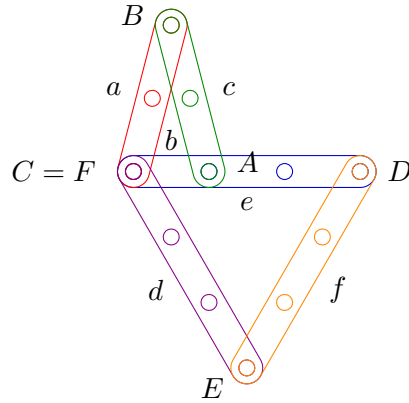


Figure 5: Triangle pair frame  $(2, 1, 2) \oplus (3, 3, 3)$  makes  $\overline{BE} = \frac{\sqrt{46 + 18\sqrt{5}}}{2}$ .

### 3 Triangle pair extended frame

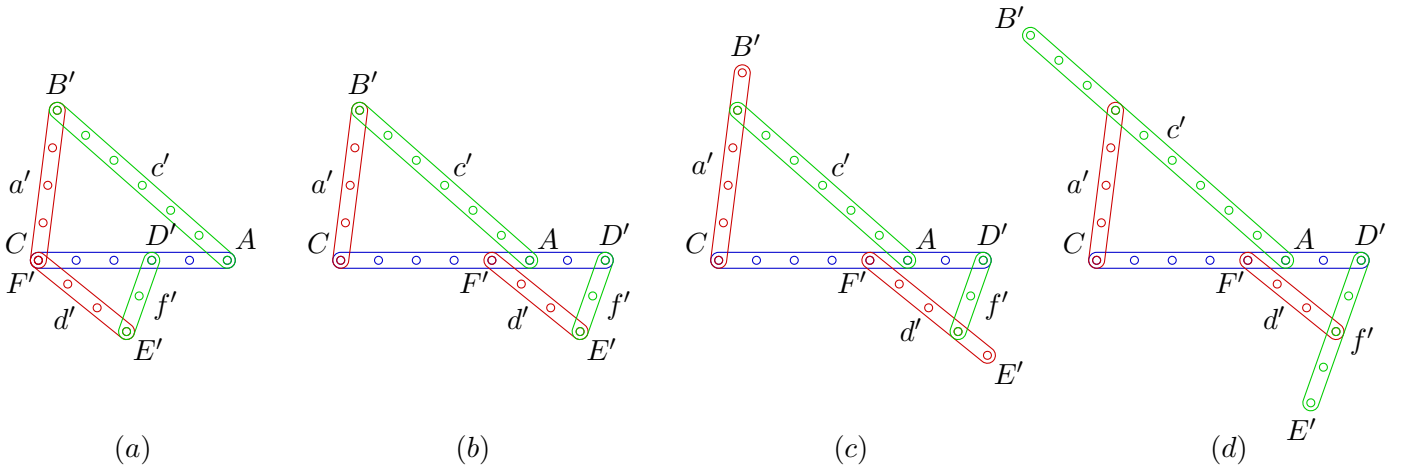


Figure 6: Triangle pair extended frame. Is like previous triangle pair frame except we can extend strips  $a$  or  $c$ ,  $d$  or  $f$ , and we can separate vertices  $C$  and  $F'$ . Vertices  $A, C, D', F'$  remain collinear and we are interested in the distance  $\overline{B'E'}$ . We show four examples: (a) is the original triangle pair, (b) has moved the  $\triangle D'E'F'$  to the right, (c) also extends strips  $a$  and  $d$  and (d) extends strips  $c$  and  $f$ .

The triangle pair extended frame is shown in figure 6. As with figure 4 we also have two triangles with five strips, but we can do up to three transformations: 1) Separate nodes  $C$  and  $F$  which moves  $\triangle D'E'F'$ , 2) extends strip  $a$  or strip  $c$  but not both and 3) extend strip  $d$  or strip  $f$  but not both. We define three integers  $x, y_1, y_2$  to do the transformations:

$$x = \begin{cases} 0 & C, F \text{ vertices remain joined} \\ \geq 0 & \triangle DEF \text{ is moved to the right a distance equal to } x \end{cases} \quad (34)$$

$$y_1 = \begin{cases} 0 & a' = a, \quad c' = c \\ > 0 & a' = a + y_1, \quad c' = c \\ < 0 & a' = a, \quad c' = c - y_1 \end{cases} \quad (35)$$

$$y_2 = \begin{cases} 0 & d' = d, \quad f' = f \\ > 0 & d' = d + y_2, \quad f' = f \\ < 0 & d' = d, \quad f' = f - y_2 \end{cases} \quad (36)$$

Let define  $M(a, b, c)$  the triangle above,  $N(d, e, f)$  the triangle below and  $T(x, y_1, y_2)$  the transformations. Then we can describe the cases (a) – (d) of figure 6 as:

$$\begin{aligned} (a) &: M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(0, 0, 0) \\ (b) &: M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(4, 0, 0) \\ (c) &: M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(4, +2, +1) \\ (d) &: M(4, 5, 6) \oplus N(4, 4, 2) \oplus T(4, -3, -2) \end{aligned}$$

#### 3.1 Triangle pair extended frame algebra

We are going to calculate the distance  $\overline{B'E'}$  of the triangle pair extended using the  $M, N, T$  values. We start setting the vertex  $C$  at the origin of the standard two-dimensional graph and defining  $(B_x, B_y)$  the abscissa and ordinate of vertex  $B'$  and  $(E_x, E_y)$  the abscissa and ordinate of vertex  $E'$ .



For the triangle above  $M(a, b, c)$  we have:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad (37)$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \quad (38)$$

$$B_x(a, b, c, y_1) = \begin{cases} y_1 \geq 0 & (a + y_1) \cos C \\ y_1 < 0 & b - (c - y_1) \cos A \end{cases} \quad (39)$$

$$B_y(a, b, c, y_1) = \begin{cases} y_1 \geq 0 & (a + y_1) \sin C \\ y_1 < 0 & (c - y_1) \sin A \end{cases} \quad (40)$$

For the triangle below  $N(d, e, f)$  we have:

$$\cos D = \frac{e^2 + f^2 - d^2}{2ef} \quad (41)$$

$$\cos F = \frac{d^2 + e^2 - f^2}{2de} \quad (42)$$

$$E_x(d, e, f, x, y_2) = \begin{cases} y_2 \geq 0 & x + (d + y_2) \cos F \\ y_2 < 0 & x + e - (f - y_2) \cos D \end{cases} \quad (43)$$

$$E_y(d, e, f, y_2) = \begin{cases} y_2 \geq 0 & -(d + y_2) \sin F \\ y_2 < 0 & -(f - y_2) \sin D \end{cases} \quad (44)$$

With the four components  $B_x, B_y, E_x, E_y$  we can have the distance:

$$\overline{B'E'} = \sqrt{(B_x + E_x)^2 + (B_y + E_y)^2} \quad (45)$$