

1 Polygons algebraic integers

We develop code to operate complicated numbers appearing in polygons constructions, like the number:

$$\cos \frac{2\pi}{17} = \frac{-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + 2\sqrt{17 + 3\sqrt{17} - \sqrt{170 + 38\sqrt{17}}}}{16} \quad (1.1)$$

Define A_0 , A_1 , A_2 and A_3 algebraic integers in the numerator of equation (1.1):

$$A_0 = \pm b \quad (1.2)$$

$$A_1 = \pm c\sqrt{\pm d} \quad (1.3)$$

$$A_2 = \pm e\sqrt{f \pm g\sqrt{\pm h}} \quad (1.4)$$

$$A_3 = \pm i\sqrt{j \pm k\sqrt{\pm l} \pm m\sqrt{\pm n \pm o\sqrt{\pm p}}} \quad (1.5)$$

Define a function F and apply it to the variables above b to p :

$$F = F(x, F, F, \dots, F) \quad (1.6)$$

$$= x\sqrt{F + F + \dots + F} \quad (1.7)$$

$$F_0(b) = F(b) \quad (1.8)$$

$$F_1(c, d) = F(c, F_0(d)) \quad (1.9)$$

$$F_2(e, f, g, h) = F(e, F_0(f), F_1(g, h)) \quad (1.10)$$

$$F_3(i, j, k, l, m, n, o, p) = F(i, F_0(j), F_1(k, l), F_2(m, n, o, p)) \quad (1.11)$$

So first equation can be expressed as:

$$\cos \frac{2\pi}{17} = \frac{F_0(-1) + F_1(1, 17) + F_2(1, 34, -2, 17) + F_3(2, 17, 3, 17, -1, 170, 38, 17)}{16} \quad (1.12)$$

1.1 32 bits limits

Algebraic integers **parts** are expressed as *AZ32* golang objects. The variable o (see line 2) is the number's part outside square root. The array i (see line 3) is the number's parts sum inside square root. Symbolically:

$$a = F(o, F(o, \dots), \dots) \quad (1.13)$$

$$n = \text{len}(a.i) \quad (1.14)$$

$$\text{value} = \begin{cases} a.o & n = 0 \\ a.o\sqrt{\sum_{j=0}^n a.i[j]} & n > 0 \end{cases} \quad (1.15)$$

```

1 type AZ32 struct {
2     o int32
3     i []*AZ32
4 }
5
6 type AZ32s struct { // factory
7 }
8
9
10 func (a *AZ32s) F0(b int32) *AZ32 {
11     return &AZ32{
12         o: b,
13     }
14 }
15
16 func (a *AZ32s) F1(c, d int32) *AZ32 {
17     return &AZ32 {

```

```

18     o: c,
19     i: []*AZ32{
20         a.F0(d),
21     },
22 }
23 }
24
25 func (a *AZ32s) F2(e, f, g, h int32) *AZ32 {
26     return &AZ32 {
27         o: e,
28         i: []*AZ32{
29             a.F0(f),
30             a.F1(g, h),
31         },
32     }
33 }
34
35 func (a *AZ32s) F3(i, j, k, l, m, n, o, p int32) *AZ32 {
36     return &AZ32 {
37         o: i,
38         i: []*AZ32{
39             a.F0(j),
40             a.F1(k, l),
41             a.F2(m, n, o, p),
42         },
43     }
44 }

```

1.2 N32, I32, AI32, AQ32

```

1 type N32 uint32 // range 0 - 0xffffffff
2
3 type I32 struct {
4     s bool // sign: true means negative
5     n N32  // positive value
6 }
7
8 type AI32 struct {
9     o *I32 // outside radical
10    i *I32 // inside radical square-free
11    e *AI32 // inside radical extension
12 }
13
14 type AQ32 struct {
15     nums []*AI32 // numerator sum
16     den  N32     // denominator
17 }

```

In this list we define four 32 bit numbers in Golang

1.3 Reductions R32

Reductions factory *R32* produce irreducibles *AI32* and *AQ32* using a precomputed fixed list of 32-bit primes. The reduction simplify and standarize the algebraic numbers for further operations like cloning, addition, multiplication, inversion and square root extractions with results also reduced.

```

1 type Red32 struct {
2     primes []N32
3 }
4

```

code.

In line 1 we define the natural number *N32* with a range of $0 < n \leq 2^{32} - 1$.

In line 3 we define the integer number *I32*, the number sign is negative if *s* is true and the number value always is a positive. If *I32* is nil, then we assume the number is zero.

In line 8 we define the algebraic integer number *AI32*. The number is recursive with a value of

$$\pm o \sqrt{\pm i \pm e.o \sqrt{\pm e.i \pm e.e.o \dots}} \quad (1.16)$$

where each sign \pm corresponds to its integer sign *s* of the values of integers *o* and *i*; a *AI32* with value nil corresponds to a zero. In line 14 we define the algebraic rational number *AQ32*. In the numerator has a sum of several integer numbers *AI32* and in the denominator a natural number *N32* other than zero.

```

5 func NewRed32() *Red32 {
6     value := 0xffff
7     f := make([]bool, value)
8     for i := 2; i <= int(math.Sqrt(float64(value))); i++ {
9         if f[i] == false {
10             for j := i * i; j < value; j += i {
11                 f[j] = true
12             }
13         }
14     }
15     primes := make([]N32, 0)
16     for i := N32(2); i < N32(value); i++ {
17         if f[i] == false {
18             primes = append(primes, i)
19         }
20     }
21     return &Red32{
22         primes: primes,
23     }
24 }

```

Numbers of the form $x\sqrt{y}$ are reduced with a function call named $roi(x, y)$. Numbers of the form $x\sqrt{y + z\sqrt{\dots}}$ are reduced with function named $roie(x, y, z)$. The factory produces *AI32* numbers calling both functions as necessary to return irreducible algebraic integers. As an example, this is the process to reduce number A_3 :

$$A_3 = \pm i \sqrt{\pm j \pm k \sqrt{\pm l \pm m \sqrt{\pm n}}} \quad (1.17)$$

$$\boxed{m_1, n_1 = roi(m, n)} \quad (1.18)$$

$$= \pm i \sqrt{\pm j \pm k \sqrt{\pm l \pm m_1 \sqrt{\pm n_1}}} \quad (1.19)$$

$$\boxed{k_1, l_1, m_2 = roie(k, l, m_1)} \quad (1.20)$$

$$= \pm i \sqrt{\pm j \pm k_1 \sqrt{\pm l_1 \pm m_2 \sqrt{\pm n_1}}} \quad (1.21)$$

$$\boxed{i_1, j_1, k_2 = roie(i, j, k_1)} \quad (1.22)$$

$$= \pm i_2 \sqrt{\pm j_2 \pm k_3 \sqrt{\pm l_1 \pm m_2 \sqrt{\pm n_1}}} \quad (1.23)$$

1.4 Reduction roi

This reduction is done for *AI32* numbers parts **without** extension e . This is the case of whole part $\pm c\sqrt{\pm d}$ of A_1 , part $\pm g\sqrt{\pm h}$ of A_2 and part $\pm m\sqrt{\pm n}$ of A_3 . Example of reducing $A_1 = \pm c\sqrt{\pm d}$. From d find two numbers p and d_1 , so p is the product of some primes even repeated and d_1 is square-free or 1:

$$A_1 = \pm c\sqrt{\pm d} \quad (1.24)$$

$$\boxed{d = p^2 d_1} \quad (1.25)$$

$$A_1 = \begin{cases} 0 & \text{case 1: if } c = 0 \text{ or } d = 0 \\ \pm cp\sqrt{\pm 1} & \text{case 2: if } d_1 = +1 \\ \pm c\sqrt{\pm d} & \text{case 3: if } p = 1 \\ \pm cp\sqrt{\pm d/p} & \text{case 4: otherwise} \end{cases} \quad (1.26)$$

For case 1 and case 2 we got A_1 degenerated into A_0 . For case 3 the number remains the same because was irreducible. For case 4 the reduction updates the values where $p > 1$ and $c_1 = cp$ and $d_1 = d/p$.

```

1 func (r *Red32) roi(out, inA Z) (ai *AI32, overflow bool) {
2     if out == 0 || inA == 0 { // case 1

```

```

3   return nil, false // zero
4   }
5   io := out; if out < 0 { io = -out }
6   ia := inA; if inA < 0 { ia = -inA }
7   if no, na, overflow := r.roiN(N(io), N(ia)); overflow {
8       return nil, true
9   } else { // cases 2,3,4
10      return &AI32{
11          o: &I32{ n:no, s:out < 0 },
12          i: &I32{ n:na, s:inA < 0 },
13      }, false
14  }
15  }
16
17 func (r *Red32) roiN(out, in N) (o N32, i N32, overflow bool) {
18     if out == 0 || in == 0 {
19         return 0, 0, false // zero
20     }
21     for _, prime := range r.primes {
22         p := N(prime)
23         if pp := p*p; in >= pp {
24             for {
25                 if in % pp == 0 { // reduce ok
26                     out *= p
27                     in /= pp
28                     continue // look for repeated squares in reduced in
29                 }
30                 break // check next prime
31             }
32         } else { // in has no more factors to check
33             break
34         }
35     }
36     if out > N32_MAX || in > N32_MAX {
37         return 0, 0, true // overflow
38     }
39     return N32(out), N32(in), false
40 }

```

1.5 Reduction *roi*

This reduction is done for *AI32* numbers **with** extension e . This is the case of part $\pm e\sqrt{\pm f + g\sqrt{\dots}}$ of A_2 , part $\pm i\sqrt{\pm j + k\sqrt{\dots}}$ of A_3 and part $\pm k\sqrt{\pm l + m\sqrt{\dots}}$ of A_3 . Example of reducing A_2 . First we reduce part $\pm g\sqrt{\pm h}$ with $roi(g, h)$ and apply the four cases to A_2 :

$$A_1 = \pm g\sqrt{\pm h} \tag{1.27}$$

$$\boxed{h = p^2 h_1} \tag{1.28}$$

$$A_2 = \pm e\sqrt{f \pm gp\sqrt{\pm h_1}} \tag{1.29}$$

$$A_2 = \begin{cases} 0 & \text{case5 : if } e = 0 \\ \pm e\sqrt{\pm f} & \text{case6 : if } g = 0 \text{ or } h = 0 \\ \pm e\sqrt{\pm f \pm gp} & \text{case7 : if } h_1 = +1 \\ \pm e\sqrt{\pm f \pm gp\sqrt{\pm h_1}} & \text{case8 : if } p \geq 1 \end{cases} \tag{1.30}$$

For case 5 we have that A_2 is zero. For case 6 we reduce A_2 with $roi(\pm e, \pm f)$. For case 7 we reduce A_2 with $roi(\pm e, \pm f \pm gp)$.

For case 8 we reduce A_2 with $roia(\pm e, \pm f, \pm gp)$ where h_1 is irreducible. Reduction $roie$ look for another product of primes q such that at the same time $f = q^2 f_1$ and $gp = q^2 g_1$:

$$A_2 = \pm e \sqrt{\pm f \pm gp \sqrt{\pm h_1}} \quad (1.31)$$

$$\boxed{f = q^2 f_1} \quad (1.32)$$

$$\boxed{gp = q^2 g_1} \quad (1.33)$$

$$A_2 = \pm e_1 \sqrt{\pm f_1 \pm g_1 \sqrt{\pm h_1}} \quad (1.34)$$

Where $e_1 = eq$, $f_1 = f/q^2$, $g_1 = gp/q^2$.

```

1 func (r *Red32) roie(out, inA, inB Z) (o, i, j Z, overflow bool) {
2   if out == 0 { // case 1
3     return 0, 0, 0, false // zero
4   }
5   io := out; if out < 0 { io = -out }
6   ia := inA; if inA < 0 { ia = -inA }
7   ib := inB; if inB < 0 { ib = -inB }
8   if no, na, nb, overflow := r.roieN(N(io), N(ia), N(ib)); overflow {
9     return 0, 0, 0, true // overflow
10  } else {
11    zo := Z(no); if out < 0 { zo = -zo }
12    za := Z(na); if inA < 0 { za = -za }
13    zb := Z(nb); if inB < 0 { zb = -zb }
14    return zo, za, zb, false
15  }
16 }
17
18 func (r *Red32) roieN(out, inA, inB N) (o, i, j N32, overflow bool) {
19   if inA > 1 && inB > 1 {
20     for _, prime := range r.primes {
21       p := N(prime)
22       pp := p*p
23       for {
24         if inA % pp == 0 && inB % pp == 0 {
25           out *= p // multiply by p
26           inA /= pp // divide by p squared
27           inB /= pp // divide by p squared
28           continue
29         } else {
30           break
31         }
32       }
33     }
34   }
35   if out > N32_MAX || inA > N32_MAX || inB > N32_MAX {
36     return 0, 0, 0, true // overflow
37   }
38   return N32(out), N32(inA), N32(inB), false
39 }

```

1.6 B, D, H, N

We define four numbers of increasing complexity:

$$B \equiv \frac{A_0}{a} \quad (1.35)$$

$$D \equiv \frac{A_0 + A_1}{a} \quad (1.36)$$

$$H \equiv \frac{A_0 + A_1 + A_2}{a} \quad (1.37)$$

$$N \equiv \frac{A_0 + A_1 + A_2 + A_3}{a} \quad (1.38)$$

2 functions

Each of the radicals r_0, \dots, r_3 has a function to read their corresponding signs and integers variables:

$$f_0 \equiv f(\pm b) \quad (2.1)$$

$$f_1 \equiv f(\pm c, d) \quad (2.2)$$

$$f_2 \equiv f(\pm e, f, \pm g, h) \quad (2.3)$$

$$f_3 \equiv f(\pm i, j, \pm k, l, \pm m, n) \quad (2.4)$$

Each f_0, \dots, f_4 reduces the values with gcd and root simplifications.

Each of the algebraic numbers B, D, H and N has a function to read their radicals functions as inputs:

$$f_B \equiv f(f_0(\dots), a) \quad (2.5)$$

$$f_D \equiv f(f_0(\dots), f_1(\dots), a) \quad (2.6)$$

$$f_H \equiv f(f_0(\dots), f_1(\dots), f_2(\dots), a) \quad (2.7)$$

$$f_N \equiv f(f_0(\dots), f_1(\dots), f_2(\dots), f_3(\dots), a) \quad (2.8)$$

Each f_B, \dots, f_N adds the radicals reducing once more the variables with gcd root simplifications and now considering the denominator a .

3 Examples

3.1 f_B examples

$$\cos 0 = 1 \implies f_B(f_0(1), 1) \quad (3.1)$$

$$\sin \frac{\pi}{6} = \frac{1}{2} \implies f_B(f_0(1), 2) \quad (3.2)$$

3.2 f_D examples

$$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \implies f_D(\emptyset, f_1(1, 2), 2) \quad (3.3)$$

$$\sin \frac{\pi}{10} = \frac{-1 + \sqrt{5}}{4} \implies f_D(f_0(-1), f_1(1, 5), 4) \quad (3.4)$$

3.3 f_H examples

$$\sin \frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4} \implies f_H(\emptyset, \emptyset, f_2(1, 10, -2, 5), 4) \quad (3.5)$$

$$\sin \frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4} \implies f_H(\emptyset, f_1(1, 6), f_2(1, 2, 0, 0), 4)* \quad (3.6)$$

$$\sin \frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2} \implies f_H(\emptyset, \emptyset, f_2(1, 2, 1, 3), 2) \quad (3.7)$$

$$\cos \frac{\pi}{15} = \frac{1 + \sqrt{5} + \sqrt{30-6\sqrt{5}}}{8} \implies f_E(f_0(1), f_1(1, 5), f_2(1, 30, -6, 5), 8) \quad (3.8)$$

3.4 f_N examples

$$\begin{aligned} \cos \frac{\pi}{16} &= \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \\ &\implies f_N(\emptyset, \emptyset, \emptyset, f_3(1, 2, 1, 2, 1, 2), 2) \end{aligned} \quad (3.9)$$

$$\begin{aligned} \cos \frac{\pi}{24} &= \frac{\sqrt{2 + \sqrt{2 + \sqrt{3}}}}{2} \\ &\implies f_N(\emptyset, \emptyset, \emptyset, f_3(1, 2, 1, 2, 1, 3), 2) \end{aligned} \quad (3.10)$$

$$\begin{aligned} \cos \frac{2\pi}{17} &= \frac{-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + 2\sqrt{17 + 3\sqrt{17} - \sqrt{170 + 38\sqrt{17}}}}{16} \\ &\implies f_N(f_0(-1), f_1(1, 17), f_2(1, 34, -2, 17), f_3(2, 17, 3, 17, -1, 170, +38, 17), 16) \end{aligned} \quad (3.11)$$

$$(3.12)$$

4 Operations with result B

4.1 NewB $B = B_1$

$$B_1 = \frac{\pm b_1}{a_1} \quad (4.1)$$

$$\mathbf{Reduce} \{a, b\} = \{a_1/G, b_1/G\} \iff G = \gcd\{a_1, b_1\} > 1$$

$$= \frac{\pm b}{a} \quad (4.2)$$

4.2 AddBB $B = B_2 + B_3$

$$B_2 + B_3 = \frac{\pm b_2}{a_2} + \frac{\pm b_3}{a_3} \quad (4.3)$$

$$= \frac{\pm b_2 a_3 \pm b_3 a_2}{a_2 a_3} = \frac{q}{p} \quad (4.4)$$

$$\mathbf{Reduce} \{a_1, b_1\} = \{p/G, q/G\} \iff G = \gcd\{p, q\} > 1$$

$$= \frac{\pm b_1}{a_1} \mathbf{Solve as NewB} \quad (4.5)$$

4.3 MulBB $B = B_2 \times B_3$

$$B_2 \times B_3 = \frac{\pm b_2}{a_2} \times \frac{\pm b_3}{a_3} \quad (4.6)$$

$$= \frac{\pm b_2 b_3}{a_2 a_3} = \frac{q}{p} \quad (4.7)$$

$$\begin{aligned} & \textbf{Reduce } \{a_1, b_1\} = \{p/G, q/G\} \iff G = \gcd\{p, q\} > 1 \\ & = \frac{\pm b_1}{a_1} \textbf{ Solve as NewB} \end{aligned} \quad (4.8)$$

4.4 InvB $B = 1/B_2$

$$\frac{1}{B_2} = \frac{1}{\pm b_2/a_2} \quad (4.9)$$

$$= \frac{\pm a_2}{b_2} = \frac{q}{p} \quad (4.10)$$

$$\begin{aligned} & \textbf{Reduce } \{a_1, b_1\} = \{p/G, q/G\} \iff G = \gcd\{p, q\} > 1 \\ & = \frac{\pm b_1}{a_1} \textbf{ Solve as NewB} \end{aligned} \quad (4.11)$$

5 Operations with result D

5.1 NewD $D = D_1$

$$D_1 = \frac{\pm b_1 \pm c_1 \sqrt{d_1}}{a_1} \quad (5.1)$$

$$\begin{aligned} & \textbf{Reduce } \{p, q, r\} = \{a_1/G, b_1/G, c_1/G\} \iff G = \gcd\{a_1, b_1, c_1\} > 1 \\ & = \frac{\pm q \pm r \sqrt{d_1}}{p} \end{aligned} \quad (5.2)$$

$$\begin{aligned} & \textbf{Reduce } \{d\} = s^2 d_1 \iff s > 1 \\ & = \frac{\pm q \pm rs \sqrt{d}}{p} \end{aligned} \quad (5.3)$$

$$\begin{aligned} & \textbf{Reduce } \{a, b, c\} = \{p/G, q/G, rs/G\} \iff G = \gcd\{p, q, rs\} \\ & = \frac{\pm b \pm c \sqrt{d}}{a} \end{aligned} \quad (5.4)$$

5.2 SqrtB $D = \sqrt{B_2}$

$$\sqrt{B_2} = \sqrt{\frac{\pm b_2}{a_2}} \quad (5.5)$$

$$= \frac{\sqrt{a_2 b_2}}{a_2} \quad (5.6)$$

$$\begin{aligned} & \textbf{Set } \{a_1, b_1, c_1, d_1\} = \{a_2, 0, 1, a_2 b_2\} \\ & = \frac{\pm b_1 \pm c_1 \sqrt{d_1}}{a_1} \textbf{ Solve as NewD} \end{aligned} \quad (5.7)$$

5.3 InvD $D = 1/D_2$

$$\begin{aligned}
1/D_2 &= \frac{a_2}{\pm b_2 \pm c_2 \sqrt{d_2}} \\
&= \frac{\pm a_2 b_2 \mp a_2 c_2 \sqrt{d_2}}{b_2^2 - c_2^2 d_2} \\
&\quad \text{Set } \{a_1, b_1, c_1, d_1\} = \{b_2^2 - c_2^2 d_2, \pm a_2 b_2, \mp a_2 c_2, d_2\} \\
&= \frac{\pm b_1 \pm c_1 \sqrt{d_1}}{a_1} \text{ Solve as NewD}
\end{aligned}$$

6 Operations with result H

6.1 $D_1 + D_2 \mapsto H$::::i

$$D_1 + D_2 = \frac{\pm b_1 \pm c_1 \sqrt{d_1}}{a_1} + \frac{\pm b_2 \pm c_2 \sqrt{d_2}}{a_2} \quad (6.1)$$

$$= \frac{(\pm a_2 b_1 \pm a_1 b_2) \pm a_2 c_1 \sqrt{d_1} \pm a_1 c_2 \sqrt{d_2}}{a_1 a_2} \quad (6.2)$$

$$= \frac{\pm q \pm r \sqrt{d_1} \pm s \sqrt{d_2}}{p} \quad (6.3)$$

$$\text{where } \{p, q, r, s\} = \gcd\{a_1 a_2, (\pm a_2 b_1 \pm a_1 b_2), \pm a_2 c_1, \pm a_1 c_2\}$$

$$= \frac{\pm q \pm \sqrt{r^2 d_1 + s^2 d_2 \pm 2rs \sqrt{d_1 d_2}}}{p} \quad (6.4)$$

$$= \frac{\pm q \pm \sqrt{t \pm 2rsu \sqrt{h}}}{p} \quad (6.5)$$

$$\text{where } \{t\} = r^2 d_1 + s^2 d_2 \text{ and } \{u^2 h\} = d_1 d_2$$

$$= \frac{\pm q \pm v \sqrt{f \pm g \sqrt{h}}}{p} \quad (6.6)$$

$$\text{where } \{v^2 f\} = t \text{ and } \{v^2 g\} = 2rsu$$

$$= \frac{\pm d \pm e \sqrt{f \pm g \sqrt{h}}}{a} \quad (6.7)$$

$$\text{where } \{a, d, e\} = \gcd\{p, \pm q, \pm qv\}$$

$$(6.8)$$

6.2 $\sqrt{C_1} = F_2$

$$\begin{aligned}
\sqrt{C_1} &= \sqrt{\frac{a_1 \sqrt{c_1}}{b_1}} \\
&= \frac{\sqrt{a_1 b_1 \sqrt{c_1}}}{b_1} \\
&= \frac{m \sqrt{e_2 \sqrt{c_1}}}{b_1} & a_1 b_1 = m^2 e_2 \\
&= \frac{a_2 \sqrt{e_2 \sqrt{c_1}}}{b_2} & (a_2, b_2) = \gcd(m, b_1)
\end{aligned}$$

6.3 $C_1 + D_2 = F_3$

$$\begin{aligned}
C_1 + D_2 &= \frac{\pm a_1 \sqrt{c_1}}{b_1} + \frac{\pm a_2 \sqrt{c_2} \pm d_2}{b_2} \\
&= \frac{\pm a_1 b_2 \sqrt{c_1} \pm a_2 b_1 \sqrt{c_2} \pm d_2 b_1}{b_1 b_2} \\
&= \frac{\pm m \sqrt{c_1} \pm n \sqrt{c_2} \pm p}{o} \\
&= \frac{\sqrt{m^2 c_1 + n^2 c_2 \pm 2mn \sqrt{c_1 c_2}} \pm p}{o} \\
&= \frac{\sqrt{q \pm 2mnr \sqrt{f_3}} \pm p}{o} \\
&= \frac{s \sqrt{c_3} \pm e_3 \sqrt{f_3} \pm p}{o} \\
&= \frac{a_3 \sqrt{c_3} \pm e_3 \sqrt{f_3} \pm d_3}{b_3}
\end{aligned}$$

$$(\pm m, \pm n, \pm p, o) = \gcd(\pm a_1 b_2, \pm a_2 b_1, \pm d_2 b_1, b_1 b_2)$$

$$q = m^2 c_1 + n^2 c_2, c_1 c_2 = r^2 f_3$$

$$q = s^2 c_3, 2mnr = s^2 e_3$$

$$(a_3, b_3, \pm d_3) = \gcd(s, \pm p, o)$$

6.4 $1/D_1 = D_2$

$$\begin{aligned}
1/D_1 &= \frac{b_1}{\pm a_1 \sqrt{c_1} \pm d_1} \\
&= \frac{\pm a_1 b_1 \sqrt{c_1} \mp b_1 d_1}{a_1^2 c_1 - d_1^2} \\
&= \frac{a_2 \sqrt{c_1} \pm d_2}{b_2}
\end{aligned}$$

$$(a_2, b_2, d_2) = \gcd(\pm a_1 b_1, \mp b_1 d_1, a_1^2 c_1 - d_1^2)$$

6.5 $\sqrt{D_1} = F_2$ editing...

$$\begin{aligned}
\sqrt{D_1} &= \sqrt{\frac{\pm a_1 \sqrt{c_1} \pm d_1}{b_1}} \\
&= \frac{\sqrt{\pm b_1 d_1 \pm a_1 b_1 \sqrt{f_2}}}{b_1} \\
&= \frac{m \sqrt{c_2} \pm e_2 \sqrt{f_2}}{b_1} \\
&= \frac{a_2 \sqrt{c_2} \pm e_2 \sqrt{f_2}}{b_2}
\end{aligned}$$

$$f_2 = c_1$$

$$\pm b_1 d_1 = m^2 c_2, \pm a_1 b_1 = m^2 e_2$$

$$(a_2, b_2) = \gcd(m, b_1)$$

6.6 $D_1 + D_2 = F_3$

$$\begin{aligned}
D_1 + D_2 &= \frac{\pm a_1 \sqrt{c_1} \pm d_1}{b_1} + \frac{\pm a_2 \sqrt{c_2} \pm d_2}{b_2} \\
&= \frac{\pm a_1 b_2 \sqrt{c_1} \pm a_2 b_1 \sqrt{c_2} \pm d_1 b_2 \pm d_2 b_1}{b_1 b_2} \\
&= \frac{\pm m \sqrt{c_1} \pm n \sqrt{c_2} \pm p}{o} & (\pm m, \pm n, \pm p, o) = \gcd(\pm a_1 b_2, \pm a_2 b_1, \pm d_1 b_2 \pm d_2 b_1, b_1 b_2) \\
&= \frac{\sqrt{m^2 c_1 + n^2 c_2 \pm 2mn \sqrt{c_1 c_2}} \pm p}{o} \\
&= \frac{\sqrt{q \pm 2mnr \sqrt{f_3}} \pm p}{o} & q = m^2 c_1 + n^2 c_2, c_1 c_2 = r^2 f_3 \\
&= \frac{s \sqrt{c_3 \pm e_3 \sqrt{f_3}} \pm p}{o} & q = s^2 c_3, 2mnr = s^2 e_3 \\
&= \frac{a_3 \sqrt{c_3 \pm e_3 \sqrt{f_3}} \pm d_3}{b_3} & (a_3, b_3, \pm d_3) = \gcd(s, \pm p, o)
\end{aligned}$$

6.7 $D_1 \times D_2 = F_3$

$$\begin{aligned}
D_1 \times D_2 &= \frac{\pm a_1 \sqrt{c_1} \pm d_1}{b_1} \times \frac{\pm a_2 \sqrt{c_2} \pm d_2}{b_2} \\
&= \frac{\pm a_1 a_2 \sqrt{c_1 c_2} \pm a_1 d_2 \sqrt{c_1} \pm a_2 d_1 \sqrt{c_2} \pm d_1 d_2}{b_1 b_2}
\end{aligned}$$

6.8 MulDD $D_1 \times D_2 \mapsto H$???

$$\begin{aligned}
D_1 \times D_2 &= \frac{\pm b_1 \pm c_1 \sqrt{d_1}}{a_1} \times \frac{\pm b_2 \pm c_2 \sqrt{d_2}}{a_2} \\
&= \frac{\pm b_1 b_2 \pm b_1 c_2 \sqrt{d_2} \pm b_2 c_1 \sqrt{d_1} \pm c_1 c_2 \sqrt{d_1 d_2}}{a_1 a_2} \\
&= \frac{\pm a_1 a_2 m \sqrt{c_3}}{b_1 b_2} & c_1 c_2 = m^2 c_3 \\
&= \frac{\pm a_3 \sqrt{c_3}}{b_3} & (\pm a_3, b_3) = \gcd(\pm a_1 a_2 m, b_1 b_2)
\end{aligned}$$