Triple unit

https://github.com/heptagons/meccano/units/triple

Abstract

Triple unit is a group of five meccano [1] strips a, b, c, d, e intended to build regular polygons three consecutive perimeter sides. This unit has three angles equal to the polygon internal angle θ. Triple unis has been using to build the pentagon type 2 mentioned in pentagons paper[2].

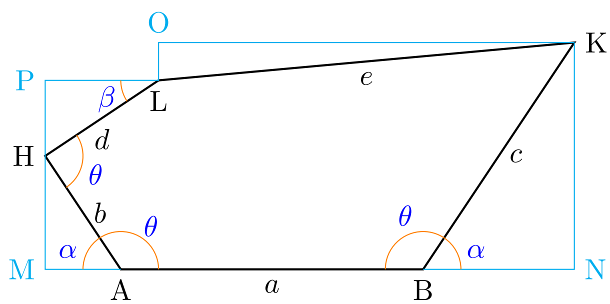

Figure 1: Triple unit has five strips a, b, c, d, e

From nodes A and B of fig 1 we get α from θ ($\pi = 180°$):

$$\begin{aligned} \theta &= \pi - \alpha \\ \alpha &= \pi - \theta \end{aligned} \tag{1}$$

And from node H we get β from θ:

$$\begin{aligned} \theta &= \alpha + \beta \\ \beta &= \theta - \alpha = \theta - (\pi - \theta) = 2\theta - \pi \end{aligned} \tag{2}$$

We calculate horizontal segment $\overline{OK}$:

$$\begin{aligned} \overline{OK} &= \overline{MA} + a + \overline{BN} - \overline{PL} \\ &= b\cos\alpha + a + c\cos\alpha - d\cos\beta \\ &= a + (b+c)\cos\alpha - d\cos\beta \\ &= a + (b+c)\cos(\pi - \theta) - d\cos(2\theta - \pi) \\ &= a - (b+c)\cos\theta + d\cos(2\theta) \end{aligned} \tag{3}$$

And vertical segment $\overline{OL}$:

$$\begin{aligned} \overline{OL} &= \overline{KN} - \overline{PH} - \overline{HM} \\ &= c\sin\alpha - d\sin\beta - b\sin\alpha \\ &= (c-b)\sin\alpha - d\sin\beta \\ &= (c-b)\sin(\pi - \theta) - d\sin(2\theta - \pi) \\ &= (c-b)\sin\theta - d\sin(2\theta) \end{aligned} \tag{4}$$

[1] Meccano mathematics by ‘t Hooft
[2] Meccano pentagons

So we can express e in function of a, b, c, d and angles α, β:

$$
\begin{aligned}
e^2 &= (\overline{OK})^2 + (\overline{OL})^2 \\
&= (a - (b+c)\cos\theta + d\cos(2\theta))^2 + ((c-b)\sin\theta - d\sin(2\theta))^2 \\
&= a^2 + (b^2 + 2bc + c^2)\cos^2\theta + d^2\cos^2(2\theta) - 2a(b+c)\cos\theta + 2ad\cos(2\theta) - 2(b+c)d\cos\theta\cos(2\theta) \\
&\quad (c^2 - 2bc + b^2)\sin^2\theta - 2(c-b)d\sin\theta\sin(2\theta) + d^2\sin^2(2\theta) \\
&= a^2 + (b^2 + c^2)(cos^2\theta + sin^2\theta) + d^2(cos^2(2\theta) + sin^2(2\theta)) \\
&\quad + 2bc\cos^2\theta - 2a(b+c)\cos\theta + 2ad\cos(2\theta) - 2(b+c)d\cos\theta\cos(2\theta) - 2bc\sin^2\theta - 2(c-b)d\sin\theta\sin(2\theta) \\
&= a^2 + b^2 + c^2 + d^2 + 2bc(\cos^2\theta - \sin^2\theta) - 2a(b+c)\cos\theta + 2ad\cos(2\theta) \\
&\quad - 2(b+c)d\cos\theta\cos(2\theta) - 2(c-b)d\sin\theta\sin(2\theta) \\
&= a^2 + b^2 + c^2 + d^2 + 2bc\cos(2\theta) - 2a(b+c)\cos\theta + 2ad\cos(2\theta) \\
&\quad - 2bd(\cos\theta\cos(2\theta) - sin\theta\sin(2\theta)) - 2cd(\cos\theta\cos(2\theta) + sin\theta\sin(2\theta)) \\
&= a^2 + b^2 + c^2 + d^2 + 2(bc + ad)\cos(2\theta) - 2a(b+c)\cos\theta \\
&\quad - 2bd\cos(\theta + 2\theta) - 2cd\cos(\theta - 2\theta) \\
&= a^2 + b^2 + c^2 + d^2 + 2(bc + ad)\cos(2\theta) - 2(ab + ac)\cos\theta - 2bd\cos(3\theta) - 2cd\cos(-\theta) \\
&= a^2 + b^2 + c^2 + d^2 - 2(ab + ac + cd)\cos\theta + 2(bc + ad)\cos(2\theta) - 2bd\cos(3\theta) \qquad (5)
\end{aligned}
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