

Meccano polygon diagonals

<https://github.com/heptagons/meccano/penta>

Abstract

We construct meccano ¹ polygon internal diagonals.

1 Polygon diagonals

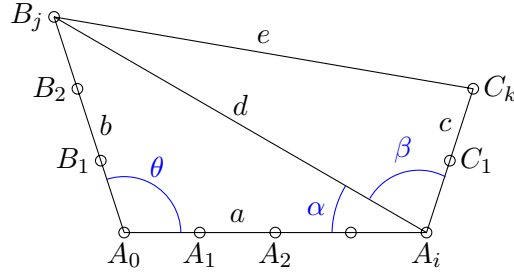


Figure 1: Meccano polygon three consecutive sides segments $a \geq b \geq c$ can form two diagonals d and e .

2 Regular polygon diagonals

In the regular polygon all internal angles are equal to θ . From figure 1 the polygon is regular if $\alpha + \beta = \theta$ so we have:

$$\alpha = \angle A_0 A_i B_j \quad (1)$$

$$\beta = \angle B_j A_i C_k \quad (2)$$

$$\theta = \angle B_j A_0 A_i = \angle A_0 A_i C_k \quad (3)$$

$$\alpha + \beta = \theta \quad (4)$$

We use the cosines sum identity to express $\cos \beta$ in function of the rest of variables. We define $u = \cos \theta$:

$$u \equiv \cos \theta \quad (5)$$

$$= \cos(\alpha + \beta) \quad (6)$$

$$= \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad (7)$$

$$\sin \beta = \frac{\cos \alpha \cos \beta - u}{\sin \alpha} \quad (8)$$

$$\sin^2 \beta = \frac{(\cos \alpha \cos \beta - u)^2}{\sin^2 \alpha} \quad (9)$$

$$1 - \cos^2 \beta = \frac{\cos^2 \alpha \cos^2 \beta - 2u \cos \alpha \cos \beta + u^2}{\sin^2 \alpha} \quad (10)$$

¹ Meccano mathematics by 't Hooft

We set $X = \cos \beta$ and rearrange the last equation to get:

$$X^2 - 2u \cos \alpha X + u^2 - \sin^2 \alpha = 0 \quad (11)$$

And solve the quadratic equation $AX^2 + BX + C = 0$ to get $\cos \beta$ in function of u and α :

$$\begin{aligned} \cos \beta &= \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \\ &= \frac{2u \cos \alpha \pm \sqrt{4u^2 \cos^2 \alpha - 4(u^2 - \sin^2 \alpha)}}{2} \\ &= u \cos \alpha \pm \sqrt{u^2 \cos^2 \alpha - u^2 + \sin^2 \alpha} \end{aligned} \quad (12)$$

Now, we need to find the values of $\cos \alpha$, $\sin \alpha$ and $\cos \beta$ which in turn need the value of d , all in terms of a, b, c the segments of the polygon perimeter.

For the value of d we use the law of cosines:

$$\begin{aligned} d &= \sqrt{a^2 + b^2 - 2ab \cos \theta} \\ &= \sqrt{a^2 + b^2 - 2abu} \end{aligned} \quad (13)$$

Using the law of cosines we calculate the angles $\alpha = \angle A_0 A_i B_j$ and $\beta = \angle B_j A_i C_k$:

$$\begin{aligned} \cos \alpha &= \frac{a^2 + d^2 - b^2}{2ad} \\ &= \frac{a^2 + (a^2 + b^2 - 2abu) - b^2}{2ad} \\ &= \frac{a - bu}{d} \end{aligned} \quad (14)$$

$$\begin{aligned} \cos \beta &= \frac{c^2 + d^2 - e^2}{2cd} \\ &= \frac{c^2 + (a^2 + b^2 - 2abu) - e^2}{2cd} \\ &= \frac{a^2 + b^2 + c^2 - e^2 - 2abu}{2cd} \end{aligned} \quad (15)$$

We define new variable f to simplify $\cos \beta$ to obtain:

$$f \equiv \frac{a^2 + b^2 + c^2 - e^2}{2} \quad (16)$$

$$\cos \beta = \frac{f - abu}{cd} \quad (17)$$

We calculate $\sin^2 \alpha = 1 - \cos^2 \alpha$:

$$\begin{aligned} \sin^2 \alpha &= 1 - \frac{(a - bu)^2}{d^2} \\ &= \frac{d^2 - a^2 + 2abu - b^2 u^2}{d^2} \\ &= \frac{(a^2 + b^2 - 2abu) - a^2 + 2abu - b^2 u^2}{d^2} \\ &= \frac{b^2(1 - u^2)}{d^2} \end{aligned} \quad (18)$$

We plug the values of $\cos \alpha, \cos \beta, \sin^2 \alpha$ in equation 12 to get:

$$\begin{aligned}
\frac{f - abu}{cd} &= \left(\frac{a - bu}{d} \right) u \pm \sqrt{\left(\frac{a - bu}{d} \right)^2 u^2 - u^2 + \frac{b^2(1 - u^2)}{d^2}} \\
\frac{f - abu}{c} &= (a - bu)u \pm \sqrt{(a - bu)^2 u^2 - d^2 u^2 + b^2(1 - u^2)} \\
f &= (ab + ac - bcu)u \pm c\sqrt{(a - bu)^2 u^2 - d^2 u^2 + b^2 - b^2 u^2} \\
&= abu + acu - bcu^2 \pm c\sqrt{a^2 u^2 - 2abu^3 + b^2 u^4 - d^2 u^2 + b^2 - b^2 u^2}
\end{aligned} \tag{19}$$

2.1 Regular polygon diagonal e

We define variables m, n to simplify f . For n we substitute d^2 from equation 13 so we have:

$$\begin{aligned}
m &\equiv abu + acu - bcu^2 \\
&= a(b + c)u - bcu^2
\end{aligned} \tag{20}$$

$$\begin{aligned}
n &\equiv a^2 u^2 - 2abu^3 + b^2 u^4 - d^2 u^2 + b^2 - b^2 u^2 \\
&= b^2 + (a^2 - d^2 - b^2)u^2 - 2abu^3 + b^2 u^4 \\
&= b^2 + (a^2 - (a^2 + b^2 - 2abu) - b^2)u^2 - 2abu^3 + b^2 u^4 \\
&= b^2(1 - u^2)^2
\end{aligned} \tag{21}$$

We substitute m, n in f where we choose the positive sign and take the absolute value of $\sqrt{n} = |b(1 - u^2)|$:

$$\begin{aligned}
f &= m + c\sqrt{n} \\
&= a(b + c)u - bcu^2 + bc(|1 - u^2|)
\end{aligned} \tag{22}$$

3 Regular pentagon diagonal e

For the regular pentagon we have $u = \cos \theta = \cos(3\pi/5)$:

$$u = \frac{1 - \sqrt{5}}{4} \tag{23}$$

$$u^2 = \frac{3 - \sqrt{5}}{8} \tag{24}$$

$$|1 - u^2| = \frac{5 + \sqrt{5}}{8} \tag{25}$$

We plug the value of pentagon's u in equation 13 to get pentagon's d_5 :

$$\begin{aligned}
d_5 &= \sqrt{a^2 + b^2 - 2ab \left(\frac{1 - \sqrt{5}}{4} \right)} \\
&= \frac{\sqrt{4a^2 + 4b^2 - 2ab - 2ab\sqrt{5}}}{2}
\end{aligned} \tag{26}$$

We plug the values of pentagon's u, u^2 and $|1 - u^2|$ in equation 22 to get pentagon's f_5 :

$$\begin{aligned}
f_5 &= a(b + c) \left(\frac{1 - \sqrt{5}}{4} \right) - bc \left(\frac{3 - \sqrt{5}}{8} \right) + bc \left(\frac{5 + \sqrt{5}}{8} \right) \\
&= \frac{2a(b + c) - 3bc + 5bc + (-2a(b + c) + bc + bc)\sqrt{5}}{8} \\
&= \frac{bc + a(b + c) + (bc - a(b + c))\sqrt{5}}{4}
\end{aligned} \tag{27}$$

Finally we get the generic pentagon diagonal e_5 in function of only a, b, c : From the definition of f in equation 16 we have:

$$a^2 + b^2 + c^2 - e_5^2 = 2f_5$$

$$e_5 = \sqrt{a^2 + b^2 + c^2 - \frac{bc + a(b+c) + (bc - a(b+c))\sqrt{5}}{2}} \quad (28)$$

We define integers p, q to simplify computer algebra software to obtain $e(a, b, c)$:

$$p \equiv (a-b)^2 + (a-c)^2 + (b-c)^2 + 2a^2 + 2b^2 + 2c^2 \quad (29)$$

$$q \equiv 2(ab + ac - bc) \quad (30)$$

$$e(a, b, c) = \frac{\sqrt{p + q\sqrt{5}}}{2} \quad (31)$$

3.1 Regular pentagon diagonal d

From the figure so far we know that when $c = 0$, e becomes d so we can confirm this:

$$d = e \quad \text{when } c = 0$$

$$= \sqrt{a^2 + b^2 - \frac{ab - ab\sqrt{5}}{2}} \quad (32)$$

which coincides with d_5 in equation 26 \square .

3.2 Regular pentagon width W

The regular pentagon width W is defined as the distance between two farthest separated points, which equals the diagonal length D which is given by:

$$W = D = \frac{1 + \sqrt{5}}{2}a \quad (33)$$

In our case the width is the diagonal d when $a = b$ or also de e when $a = b, c = 0$.

$$W_5 = d \quad \text{when } a = b$$

$$= \frac{\sqrt{4a^2 + 4a^2 + 2a^2 + 2a^2\sqrt{5}}}{2}$$

$$= \frac{\sqrt{6 - 2\sqrt{5}}}{2}a$$

$$= \frac{\sqrt{5} + 1}{2}a \quad (34)$$

which coincides with above equation of pentagon's W \square .

3.3 Regular pentagon height H

In the regular pentagon the height H is the distance from one side of length a to the opposite vertex:

$$H = \frac{\sqrt{5 + 2\sqrt{5}}}{2}a \quad (35)$$

For the height to occur we apply $b = a$ and $c = a/2$ in e_5 equation 28:

$$\begin{aligned}
H_5 &= e \quad \text{when } b = a \text{ and } c = a/2 \\
&= \sqrt{a^2 + a^2 + \frac{a^2}{4} - \frac{\frac{a^2}{2} + a\left(a + \frac{a}{2}\right) + \left(\frac{a^2}{2} - a\left(a + \frac{a}{2}\right)\right)\sqrt{5}}{2}} \\
&= a\sqrt{\frac{9}{4} - \frac{2 - \sqrt{5}}{2}} \\
&= a\sqrt{\frac{5 + 2\sqrt{5}}{4}}
\end{aligned} \tag{36}$$

which coincides with regular pentagon height H \square .

3.4 Regular pentagon side $a = 3$ diagonals e

Table 1: All regular pentagon diagonals e for $a = 3, a \geq b, b \geq c$

a	b	c	e
3	1	0	$\frac{\sqrt{34 + 6\sqrt{5}}}{2}$
3	1	1	$\frac{5 + \sqrt{5}}{2}$
3	2	0	$\sqrt{10 + 3\sqrt{5}}$
3	2	1	$\frac{\sqrt{34 + 15\sqrt{5}}}{2}$
3	2	2	$2 + \sqrt{5}$
3	3	0	$\frac{3 + 3\sqrt{5}}{2}$
3	3	1	$\frac{\sqrt{46 + 18\sqrt{5}}}{2}$
3	3	2	$\frac{\sqrt{46 + 18\sqrt{5}}}{2}$
3	3	3	$\frac{3 + 3\sqrt{5}}{2}$

In table 1 we list all the internal diagonals inside the regular pentagon of size $a = 3$. Row data was calculated with:

Folder : `github.com/heptagons/meccano/penta/`

Code : `NewDiagonals().Get(3, 3, func(a, b, c int, surd *nest.A32))`

In table ?? we draw the pentagon of size 3 perimeter and the two diagonals for segments $e(3, 3, 1)$ and $e(3, 3, 2)$ equal to $\frac{\sqrt{46 + 18\sqrt{5}}}{2}$.