Meccano four frame

https://github.com/heptagons/meccano/frames/four

Abstract

Four frame is a group of four rigid meccano [1] strips.

1 Four frame

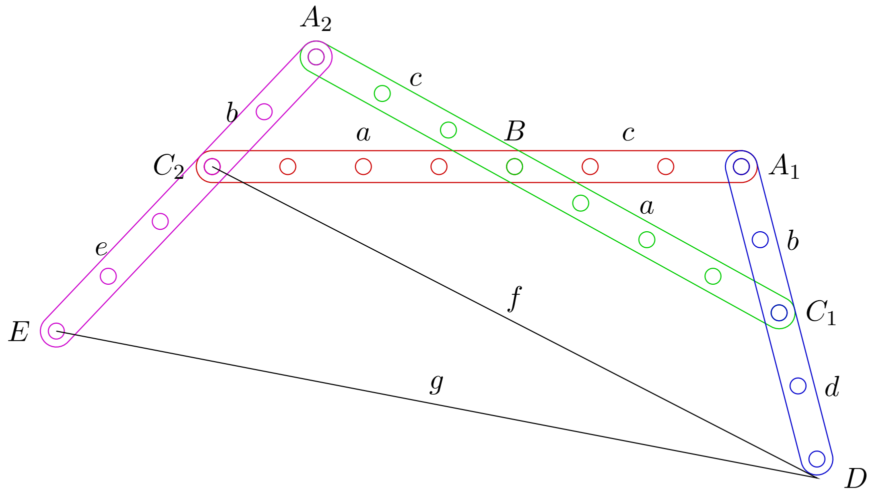

Figure 1: Antisymmetric four frame.

Figure 1 show the antisymmetric four-strips frame. From the figure we define $\alpha \equiv \angle BA_1C_1$ and define integers $m = b^2 + c^2 - a^2$ and $n = 2bc$ using the law of cosines, then we calculate $\cos\alpha$ and $\sin\alpha$:

$$(\alpha, m, n) \equiv (\angle BA_1C_1, b^2 + c^2 - a^2, 2bc) \tag{1}$$

$$\cos\alpha = \frac{m}{n} \tag{2}$$

$$\sin\alpha = \frac{\sqrt{n^2 - m^2}}{n} \tag{3}$$

We calculate the distance $f = \overline{C_2D}$ with the law of cosines using angle α and defining integers $x = a+c$ and $y = b + d$:

$$x \equiv a + c \tag{4}$$

$$y \equiv b + d \tag{5}$$

$$f^2 = (a + c)^2 + (b + d)^2 - 2(a + c)(b + d)\cos\alpha \tag{6}$$

$$= x^2 + y^2 - \frac{2mxy}{n} \tag{7}$$

$$f = \frac{\sqrt{n^2(x^2 + y^2) - 2mnxy}}{n} \tag{8}$$

[1] Meccano mathematics by ‘t Hooft

We define a new integer $z \equiv n^2(x^2+y^2) - 2mnxy$ so we have:

$$
\begin{aligned}
z &\equiv n^2(x^2+y^2) - 2mnxy && (9)\\
f &= \frac{\sqrt{z}}{n} && (10)
\end{aligned}
$$

We define angle $\theta \equiv \angle A_1C_2D$ and calculate $\cos\theta$ and $\sin\theta$:

$$
\begin{aligned}
\theta &\equiv \angle A_1C2D && (11)\\
\cos\theta &= \frac{(a+c)^2 + f^2 - (b+d)^2}{2(a+c)f}\\
&= \frac{x^2+f^2-y^2}{2xf}\\
&= \frac{x^2+x^2+y^2-\dfrac{2mxy}{n}-y^2}{2x\dfrac{\sqrt{z}}{n}}\\
&= \frac{nx-my}{\sqrt{z}} \equiv \frac{o}{p} && (12)\\
\sin\theta &= \frac{\sqrt{p^2-o^2}}{p}\\
&= \frac{\sqrt{n^2(x^2+y^2)-2mnxy-(nx-my)^2}}{p}\\
&= \frac{\sqrt{n^2(x^2+y^2)-2mnxy-n^2x^2+2nxmy-m^2y^2}}{p}\\
&= \frac{\sqrt{n^2y^2-m^2y^2}}{p} \equiv \frac{q}{p} && (13)
\end{aligned}
$$

From the figure 1 we define $\gamma \equiv \angle BA_2C_2$ and define integers $s = a^2+b^2-c^2$ and $t = 2ab$ and calculate $\cos\gamma$ and $\sin\gamma$:

$$
\begin{aligned}
(\gamma, s, t) &\equiv (\angle BA_2C_2, a^2+b^2-c^2, 2ab) && (14)\\
\cos\gamma &= \frac{s}{t} && (15)\\
\sin\gamma &= \frac{\sqrt{t^2-s^2}}{t} && (16)
\end{aligned}
$$

We define angle $\phi \equiv \angle A_2C_2D$ and we note is the sum of angles $\theta+\gamma$ and we calculate $\cos\phi$:

$$
\begin{aligned}
\phi &\equiv \angle A_2C_2D && (17)\\
&= \theta+\gamma && (18)\\
\cos\phi &= \cos(\theta+\gamma) && (19)\\
&= \cos\theta\cos\gamma - \sin\theta\sin\gamma\\
&= \frac{os}{pt} - \frac{q\sqrt{t^2-s^2}}{pt}\\
&= \frac{os - q\sqrt{t^2-s^2}}{pt} && (20)
\end{aligned}
$$

From the figure we define angle $\psi \equiv \angle DC_2E$ and we note equals angle $\pi - \phi$, so we have:

$$\psi \equiv \angle DC_2E \tag{21}$$

$$= \pi - \phi \tag{22}$$

$$\cos\psi = \cos(\pi - \phi) \tag{23}$$

$$= -\cos\phi$$

$$= \frac{q\sqrt{t^2 - s^2} - os}{pt} \tag{24}$$

Finally with $\cos\psi$, e and f we can calculate distance $g = \overline{ED}$:

$$g^2 = e^2 + f^2 - 2ef\cos\psi \tag{25}$$

$$= e^2 + x^2 + y^2 - \frac{2mxy}{n} - 2e\left(\frac{\sqrt{z}}{n}\right)\left(\frac{q\sqrt{t^2 - s^2} - os}{\sqrt{z}t}\right) \tag{26}$$

$$\tag{27}$$