Meccano triangles

https://github.com/heptagons/meccano/nest

Abstract

We construct meccano triangles. Basic triangles has the three sides as integers and calculate the internal diagonal distances. Such diagonals then are used as the new side of more complicated triangles and then again we calculate new distances formed and so on. Eventually we expect to find certain angles joining the triangles which can be used to construct regular polygons or more figures.

1 Triangle (a, b, c)

A triangle (a, b, c) has the tree sides a, b and c where $a, b, c \in \mathbb{N}$. To avoid repetitions we consider only the cases:

$$a \geq b \geq c \tag{1}$$

$$a < b + c \tag{2}$$

1.1 Triangle (a, b, c) diagonals

To calculate the diagonals from side a to side b we start calculating $\cos C$ where C is the opposite angle of side c:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \tag{3}$$

Then with the $\cos C$ we can calculate every diagonal $\overline{a_x b_y}$ with the law of cosines:

$$\overline{a_x b_y} = \sqrt{x^2 + y^2 - 2xy \cos C} \tag{4}$$

$$= \sqrt{x^2 + y^2 - 2xy \frac{a^2 + b^2 - c^2}{2ab}} \tag{5}$$

$$= \frac{\sqrt{a^2 b^2 (x^2 + y^2) - abxy(a^2 + b^2 - c^2)}}{ab} \tag{6}$$

where $1 \leq x \leq a$, $1 \leq y \leq b$ and $x - y \geq 0$.

By inspection we deduce that for basic meccano triangles:

$$a, b, c \in \mathbb{N} \tag{7}$$

$$\cos A, \cos B, \cos C \in \mathbb{Q} \tag{8}$$

$$\overline{a_x b_y}, \overline{b_y c_z}, \overline{a_x c_z} \in \mathbb{A} \tag{9}$$

1.2 Example triangle [7,6,5]

Figure 1 show triangle $[7, 6, 5]$ diagonals $b_y c_z$.

Figure 2 show triangle $[7, 6, 5]$ diagonals $a_x c_z$. The diagonals join points from side a nodes described as a_x to side c nodes described as c_z in all combinations.

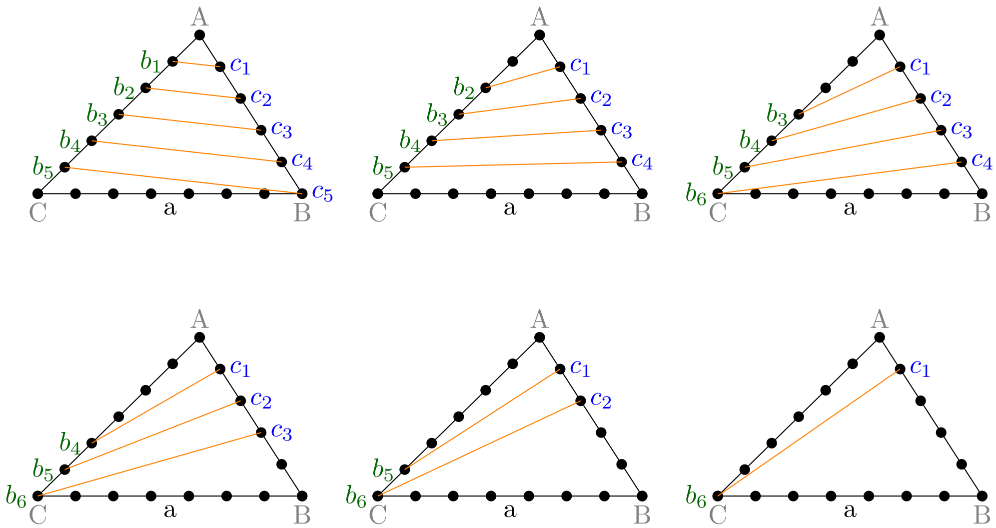

Figure 1: Triangle $[7, 6, 5]$, bc diagonals.

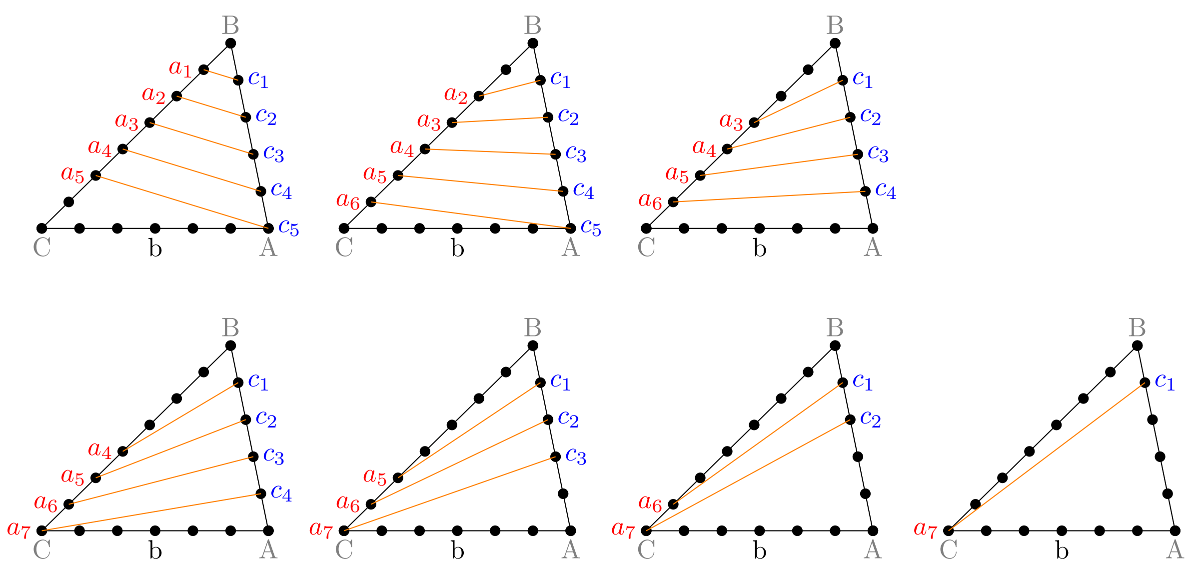

Figure 2: Triangle $[7, 6, 5]$, ac diagonals.

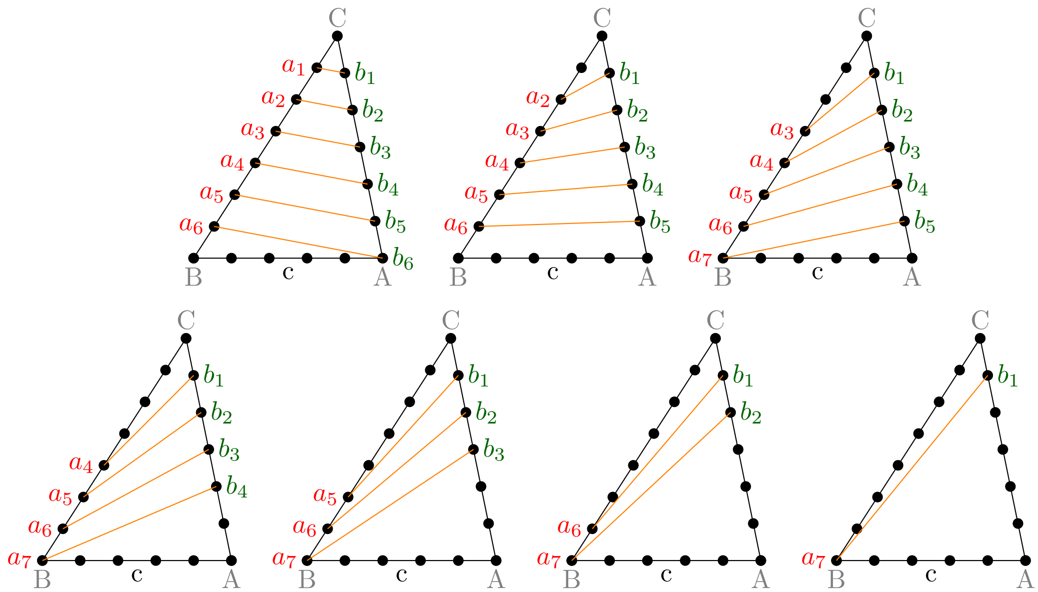

Figure 3: Triangle $[7, 6, 5]$, ab diagonals.

Figure 3 show triangle $[7, 6, 5]$ diagonals $a_x b_y$. The diagonals join points from side a nodes described as a_x to side b nodes described as b_y in all combinations.

Matrix for angle A diagonals joining sides b and c. Empty cells are repetitions:

$$\begin{pmatrix} \frac{2\sqrt{10}}{5} & \frac{\sqrt{105}}{5} & \frac{2\sqrt{55}}{5} & \frac{\sqrt{385}}{5} & 2\sqrt{6} & \frac{\sqrt{865}}{5} \\ & \frac{4\sqrt{10}}{5} & \frac{\sqrt{265}}{5} & \frac{2\sqrt{105}}{5} & \colorbox{green}{5} & \frac{4\sqrt{55}}{5} \\ & & \frac{6\sqrt{10}}{5} & \frac{\sqrt{505}}{5} & 2\sqrt{7} & \frac{3\sqrt{105}}{5} \\ & & & \frac{8\sqrt{10}}{5} & \sqrt{33} & \frac{2\sqrt{265}}{5} \\ & & & & 2\sqrt{10} & \boxed{7} \end{pmatrix} \tag{10}$$

Matrix for angle B diagonals joining sides a and c. Empty cells are repetitions. Values at column 7 (at the right of separator |) are repeated and already in previous matrix.

$$\left(\begin{array}{cccccc|c} \frac{4\sqrt{70}}{35} & \frac{3\sqrt{385}}{35} & \frac{2\sqrt{2065}}{35} & \frac{\sqrt{15505}}{35} & \frac{12\sqrt{7}}{7} & \frac{\sqrt{37345}}{35} & \frac{2\sqrt{265}}{5} \\ & \frac{8\sqrt{70}}{35} & \frac{\sqrt{7945}}{35} & \frac{6\sqrt{385}}{35} & \frac{\sqrt{889}}{7} & \frac{4\sqrt{2065}}{35} & \frac{3\sqrt{105}}{5} \\ & & \frac{12\sqrt{70}}{35} & \frac{\sqrt{14665}}{35} & \frac{2\sqrt{217}}{7} & \frac{9\sqrt{385}}{35} & \frac{4\sqrt{55}}{5} \\ & & & \frac{16\sqrt{70}}{35} & \frac{3\sqrt{105}}{7} & \frac{2\sqrt{7945}}{35} & \frac{\sqrt{865}}{5} \\ & & & & \frac{4\sqrt{70}}{7} & \frac{\sqrt{1393}}{7} & \boxed{6} \end{array}\right) \tag{11}$$

Matrix for angle C diagonals joining sides a and b. Empty cells are repetitions. Values at columns 6 and 7 (at the right of separator |) are repeated and already in previous matrices.

$$\left(\begin{array}{ccccc|cc}
\frac{2\sqrt{7}}{7} & \frac{\sqrt{105}}{7} & \frac{2\sqrt{70}}{7} & \frac{\sqrt{553}}{7} & \frac{2\sqrt{231}}{7} & \frac{\sqrt{1393}}{7} & 2\sqrt{10} \\
 & \frac{4\sqrt{7}}{7} & \frac{\sqrt{217}}{7} & \frac{2\sqrt{105}}{7} & \frac{\sqrt{721}}{7} & \frac{4\sqrt{70}}{7} & \sqrt{33} \\
 & & \frac{6\sqrt{7}}{7} & \frac{\sqrt{385}}{7} & \frac{2\sqrt{154}}{7} & \frac{3\sqrt{105}}{7} & 2\sqrt{7} \\
 & & & \frac{8\sqrt{7}}{7} & \frac{\sqrt{609}}{7} & \frac{2\sqrt{217}}{7} & 5 \\
 & & & & \frac{10\sqrt{7}}{7} & \frac{\sqrt{889}}{7} & 2\sqrt{6} \\
 & & & & & \frac{12\sqrt{7}}{7} & \boxed{5}
\end{array}\right) \tag{12}$$

2 Triangles($\sqrt{a}, b, c$)

Triangles($\sqrt{a}, b, c$) have the tree sides $\sqrt{a}$, b and c where $a^2, b, c \in \mathbb{N}$. So we have:

$$\sqrt{a} \geq b \geq c \tag{13}$$

$$a \geq b^2 \geq c^2 \tag{14}$$

And:

$$\sqrt{a} < b + c \tag{15}$$

$$a < (b+c)^2 \tag{16}$$

2.1 Example triangles($2\sqrt{6}, b, c$)

In this case $\sqrt{a} = 2\sqrt{6}$ so $a = 24$. Then $b = \{1, 2, 3, 4\}$ because $b^2 = \{1, 4, 9, 16\} < 24$. Also $c = \{1, 2, 3, 4\}$ because $c^2 = \{1, 4, 9, 16\} < 24$. We form a matrix with with the values $(b+c)^2$:

$$(b_i + c_j)^2 = \begin{array}{c} \\ c=1 \\ c=2 \\ c=3 \\ c=4 \end{array} \begin{array}{c} \begin{array}{cccc} b=1 & b=2 & b=3 & b=4 \end{array} \\ \begin{pmatrix} 2 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \\ 25 & 36 & 49 & 64 \end{pmatrix} \end{array} \tag{17}$$

Then we remove cells which don't fulfil condition $b \geq c$:

$$(b_i + c_j)^2 = \begin{array}{c} \\ c=1 \\ c=2 \\ c=3 \\ c=4 \end{array} \begin{array}{c} \begin{array}{cccc} b=1 & b=2 & b=3 & b=4 \end{array} \\ \begin{pmatrix} 2 & 9 & 16 & 25 \\ & 16 & 25 & 36 \\ & & 36 & 49 \\ & & & 64 \end{pmatrix} \end{array} \tag{18}$$

Then we remove cells which don't fulfil condition $a < (b+c)^2$:

$$(b_i + c_j)^2 = \begin{array}{c} \\ c=1 \\ c=2 \\ c=3 \\ c=4 \end{array} \begin{array}{c} \begin{array}{cccc} b=1 & b=2 & b=3 & b=4 \end{array} \\ \begin{pmatrix} & & & 25 \\ & & 25 & 36 \\ & & 36 & 49 \\ & & & 64 \end{pmatrix} \end{array} \tag{19}$$

So only six triangles are valid: $(2\sqrt{6}, 4, 1)$, $[2\sqrt{6}, 3, 2]$, $[2\sqrt{6}, 4, 2]$, $[2\sqrt{6}, 3, 3]$, $[2\sqrt{6}, 4, 3]$ and $[2\sqrt{6}, 4, 4]$.