Fox-face unit

https://github.com/heptagons/meccano/fox-face

Abstract

Fox-face is a group of five meccano [1] strips not forming implicit triangles but a fox-faced figure used to build a regular pentagon. Here, we'll look for other angles but not only pentagon's $\cos 2\pi/5$.

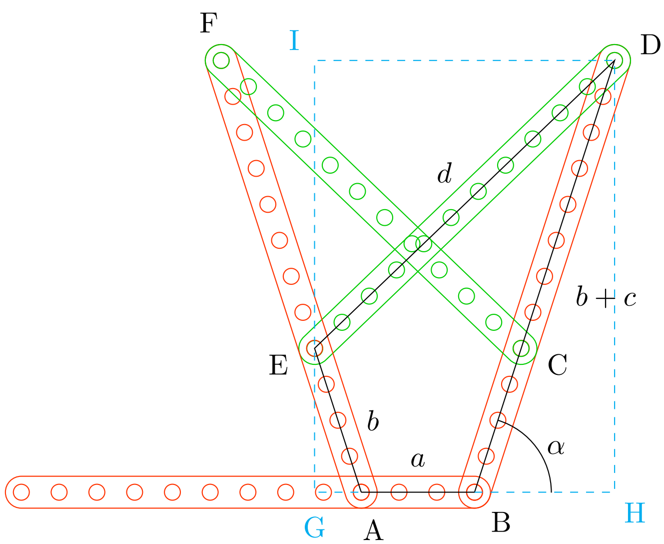

Figure 1: Fox-figure

Figure 1 show the so called fox-face unit. Has five strips of three types:

- Single $\overline{AB}$ of length a.
- Pair { $\overline{BD}$, $\overline{AF}$ } of length $b + c$.
- Pair { $\overline{DE}$, $\overline{CF}$ } of length d.

In other words we have four different distances:

- a distance of segment $\overline{AB}$.
- b distance of segments $\overline{BC}$ and $\overline{AE}$.
- c distance of segments $\overline{CD}$ and $\overline{EF}$.
- d distance of segments $\overline{DE}$ and $\overline{CF}$.

We are going to test several values of (a, b, c, d) and calculate the angle $\angle HBD$. First we'll calculate a formula and then we'll run a program iterating integer values.

[1] Meccano mathematics by 't Hooft

1 Algebra

From figure 1 we define $\theta = \angle HBD$ and figure out sines and cosines:

$$\theta \equiv \angle HBD = \angle GAE \tag{1}$$
$$\overline{BH} = (b+c)\cos\theta \tag{2}$$
$$\overline{DH} = (b+c)\sin\theta \tag{3}$$
$$\overline{AG} = b\cos\theta \tag{4}$$
$$\overline{EG} = b\sin\theta \tag{5}$$

We calculate d in function of (a,b,c):

$$\begin{aligned}
d^2 &= (\overline{DE})^2 \\
&= (\overline{DI})^2 + (\overline{EI})^2 \\
&= (\overline{AG} + \overline{AB} + \overline{BH})^2 + (\overline{DH} - \overline{EG})^2 && (6) \\
&= (b\cos\theta + a + (b+c)\cos\theta)^2 + ((b+c)\sin\theta - b\sin\theta)^2 && (7) \\
&= (a + (2b+c)\cos\theta)^2 + (c\sin\theta)^2) \\
&= a^2 + 2a(2b+c)\cos\theta + (2b+c)^2\cos^2\theta + c^2\sin^2\theta \\
&= a^2 + 2a(2b+c)\cos\theta + (4b^2+4bc+c^2)\cos^2\theta + c^2\sin^2\theta \\
&= a^2 + 2a(2b+c)\cos\theta + (4b^2+4bc)\cos^2\theta + c^2 \\
&= 4b(b+c)\cos^2\theta + 2a(2b+c)\cos\theta + a^2 + c^2 && (8)
\end{aligned}$$

Let do $X = \cos^2\theta$ so last equation can be written as:

$$4b(b+c)X^2 + 2a(2b+c)X + a^2 + c^2 - d^2 = 0 \tag{9}$$

So we can calculate $X = \cos^2\theta$ with the quadratic formula:

$$\begin{aligned}
\cos\theta &= \frac{-2a(2b+c) \pm \sqrt{4a^2(2b+c)^2 - 16b(b+c)(a^2+c^2-d^2)}}{8b(b+c)} \\
&= \frac{-a(2b+c) \pm \sqrt{a^2c^2 + 4b(b+c)(d^2-c^2)}}{4b(b+c)} && (10)
\end{aligned}$$

1.1 Test pentagon known case

Fox-face unit appears in the single solution found of the meccano regular pentagon type 1 construction [2].

In this case we have $a = 3$, $b = 4$, $c = 8$ and $d = 11$. Applying these values in the last equation we have:

$$\begin{aligned}
\cos\theta &= \frac{-48 \pm \sqrt{11520}}{192} \\
&= \frac{-1 \pm \sqrt{5}}{4} && (11)
\end{aligned}$$

Since $\cos 2\pi/5 = (\sqrt{5}-1)/4$ the equation passes one test. In the next section, a program will find more angles changing the distances a, b, c, d.

Polygon	$\angle ABC$	α	$\cos\alpha$
Pentagon	$72°$	$\frac{2\pi}{5}$	$\frac{\sqrt{5}-1}{4}$
Hexagon	$120°$	$\frac{\pi}{3}$	$\frac{1}{2}$
Octagon	$135°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
Dodecagon	$150°$	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{3}$

Table 1: Regular polygons with $\cos\alpha$ of the form $\frac{B+C\sqrt{D}}{A}$.

1.2 Angle for regular polygons

From figure 1 we notice angle $\angle ABC$ can be used as internal angle to construct a regular polygon. The internal angle is the complement of angle α. Since $\cos\alpha$ is an algebraic number of the form $\frac{B+C\sqrt{D}}{A}$ we can expect only a small group of regular polygons constructable. Table 1 list such polygons excluding triangles and squares.

2 Program

Next program using brute-force found a single pentagon, several hexagons and none of octagons or dodecagons. This paper conjectures that by using fox-face unit we can construct a single pentagon, infinite hexagons and none octagon and none dodecagon.

```
func FoxFace(max N32, found func(a, b, c, d N32, cos *A32)) {
  factory := NewA32s()
  n1 := N32(1)
  for a := n1; a <= max; a++ {
    for b := n1; b <= max; b++ {
      ab := NatGCD(a, b)
      for c := n1; c <= max; c++ {
        abc := NatGCD(ab, c)
        na := N32(4)*b*(b+c)         // 4b(b+c)
        zb := -Z(a)*(2*Z(b) + Z(c)) // -a(2b+c)
        zc := Z(1)                   // 1
        a2c2 := Z(a*a)*Z(c*c)        // a
        for d := c; d <= max; d++ { // d >= c always
          if g := NatGCD(abc, d); g > 1 {
            continue // skip scale repetitions, eg. [1,2,3,4] = [2,4,6,8]
          }
          if zd := a2c2 + 4*Z(b)*Z(b+c)*(Z(d*d) - Z(c*c)); zd < 0 {
            // skip imaginary numbers invalid fox-face, like d too short
          } else if cos, err := factory.ANew3(N(na), zb, zc, zd); err != nil {
            // silent overflow
          } else {
            found(a, b, c, d, cos)
          }
        }
      }
```

[2] Meccano pentagons

```
    }
  }
}
```

2.1 Hexagons

The more interesting costructions are the hexagons, since the algorithms found several. We run and filter the solutions where $\cos\alpha = 1/2$. In order to build efficient hexagons we impose another condition $a > b+c$. This way the hexagons size will be a.

```
func TestFoxFaceHexagons(t *testing.T) {
  max := N32(40)
  fmt.Printf("max-lenght=%d a,b,c,d efficient hexagons:\n", max)
  i := 0
  FoxFace(max, func(a, b, c, d N32, cos *A32) {
    if cos.Equals(2,1) { //  cos 60
      // Efficient hexagons are those when a > b+c
      if a >= b+c {
        i++
        fmt.Printf("% 3d %d,%d,%d,%d\n", i, a, b, c, d)
      }
    }
  })
}
```

We found 42 different hexagons when the maximum strip is of size 40 as shown in next table. The rows four numbers correspond to the lenghts of the strips a, b, c, d:

```
  1 4,1,3,7
  2 9,1,6,14
  3 11,5,5,19
  4 12,4,5,19
  5 13,2,9,21
  6 13,3,5,19
  7 14,1,9,21
  8 14,2,5,19
  9 15,1,5,19
 10 15,1,14,26
 11 17,3,12,28
 12 18,6,11,31
 13 19,1,12,28
 14 19,5,11,31
 15 20,4,11,31
 16 20,13,7,37
 17 21,3,11,31
 18 21,4,15,35
 19 21,11,10,38
 20 21,12,7,37
 21 22,2,11,31
```

```
 22 22,3,15,35
 23 22,11,7,37
 24 23,1,11,31
 25 23,1,21,39
 26 23,2,15,35
 27 23,9,10,38
 28 23,10,7,37
 29 24,1,15,35
 30 24,9,7,37
 31 25,7,10,38
 32 25,8,7,37
 33 26,7,7,37
 34 27,5,10,38
 35 27,6,7,37
 36 28,5,7,37
 37 29,3,10,38
 38 29,4,7,37
 39 30,3,7,37
 40 31,1,10,38
 41 31,2,7,37
 42 32,1,7,37
```

When we extend the maximum size to 100 we found 350 hexagons where the last one has strips $a = 84, b = 1, c = 11, d = 91$.