Meccano nonagon

https://github.com/heptagons/meccano/nona

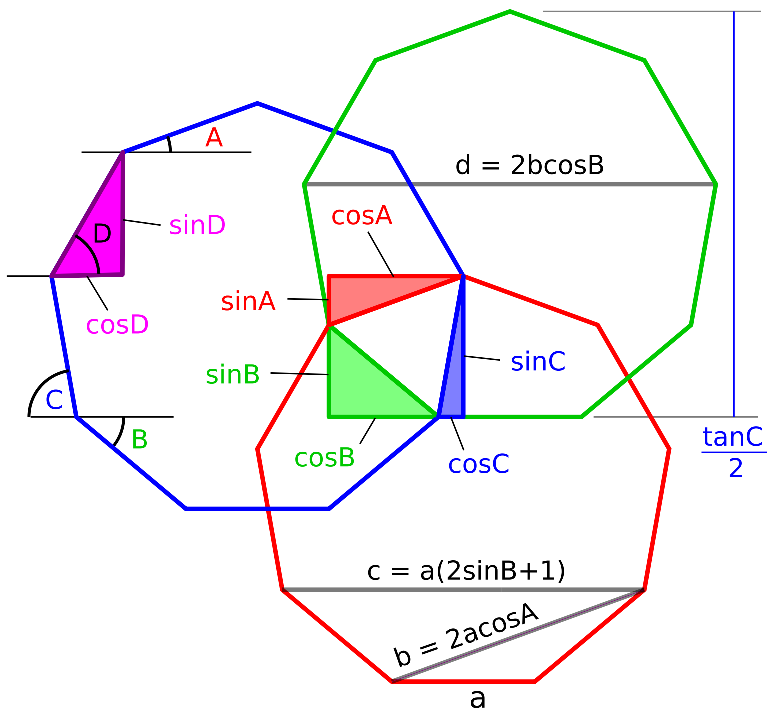

Figure 1: Three regular nonagons connected by an equilateral triangle. We note four angles in the figure A, B, C and D.

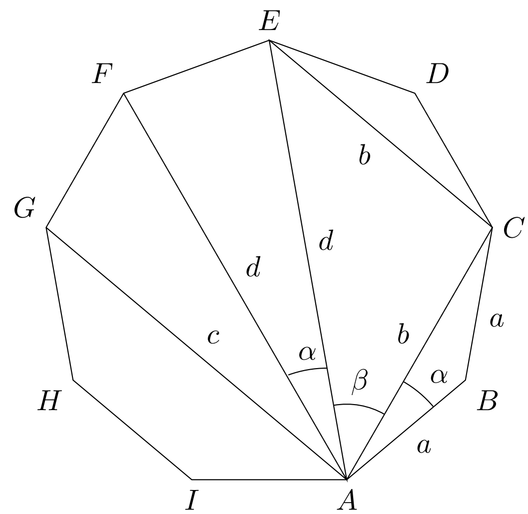

Figure 2: Nonagon

1 Algebra

Figure 1 shows three regular nonagons connected by an equilateral triangle. Four angles appear orthogonally in any regular nonagon:

$$\alpha = \pi/9 = 20^\circ \tag{1}$$
$$\beta = 2\pi/9 = 40^\circ \tag{2}$$
$$\gamma = 3\pi/9 = 60^\circ \tag{3}$$
$$\delta = 4\pi/9 = 80^\circ \tag{4}$$
$$\alpha + \beta = \delta - \alpha = \gamma \tag{5}$$

The relations of angle C are those of equilateral triangle:

$$\cos C = -\frac{1}{2} \tag{6}$$
$$\sin C = \frac{\sqrt{3}}{2} \tag{7}$$

From the figure 1, cosines of angles A, B and D are related as:

$$\begin{aligned}
\cos A &= \cos B + \cos D && (8)\\
&= \cos(2A) + \cos(4A)\\
&= (2\cos^2 A - 1) + (1 - 8\cos^2 A + 8\cos^4 A)\\
&= 8\cos^4 A - 6\cos^2 A\\
1 &= 8\cos^3 A - 6\cos A && (9)
\end{aligned}$$

Previous cosines equation is the depressed cubic equation with a negative discriminant:

$$t^3 + pt + q = 0 \tag{10}$$
$$p = -\frac{3}{4} \tag{11}$$
$$q = -\frac{1}{8} \tag{12}$$
$$\Delta = \frac{q^2}{4} + \frac{p^3}{27} = -\frac{3}{64}$$

The negative discriminant means we have three real roots which can be found by a geometric interpretation:

$$\begin{aligned}
t_k &= 2\sqrt{-\frac{p}{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right) - k\frac{2\pi}{3}\right) && \texttt{for } k = 0,1,2. \\
&= \cos\left(\frac{1}{3}\arccos\left(\frac{1}{2}\right) - k\frac{2\pi}{3}\right) && \texttt{for } k = 0,1,2. \\
&= \cos\left(\frac{1}{3}\left(\frac{\pi}{3}\right) - k\frac{2\pi}{3}\right) && \texttt{for } k = 0,1,2. \\
t_0 &= \cos\left(\frac{\pi}{9}\right) = \cos A \approx +0.939692 && (13) \\
t_1 &= \cos\left(-\frac{2\pi}{9}\right) = -\cos B \approx -0.766044 && (14) \\
t_2 &= \cos\left(-\frac{4\pi}{9}\right) = -\cos D \approx -0.173648 && (15)
\end{aligned}$$

From equation 10 we know the product of roots squares is $-2p = \frac{3}{2}$:

$$\begin{aligned}
\cos^2 A + \cos^2 B + \cos^2 D &= \frac{3}{2} && (16) \\
1 - \sin^2 A + 1 - \sin^2 B + 1 - \sin^2 D &= \frac{3}{2} \\
\sin^2 A + \sin^2 B + \sin^2 D &= \frac{3}{2} && (17)
\end{aligned}$$

From equation 10 we know the product of roots is $-q = \frac{1}{8}$ matching the "Morrie's law":

$$\begin{aligned}
\cos A \cos B \cos D &= \frac{1}{8} && (18) \\
(1 - \sin^2 A)(1 - \sin^2 B)(1 - \sin^2 D) &= \frac{1}{64} \\
(\sin A \sin B)^2 + (\sin A \sin D)^2 + (\sin C \sin D)^2 &= \frac{1}{64} - 1 + \sin^2 A + \sin^2 B + \sin^2 D + (\sin A \sin B \sin D)^2 \\
&= \frac{1}{64} - 1 + \frac{3}{2} + \left(\frac{\sqrt{3}}{8}\right)^2 = \left(\frac{9}{4}\right)^2 && (19)
\end{aligned}$$

From the figure 1, sines of angles A, B and D are related as:

$$\begin{aligned}
\sin A + \sin B &= \sin D && (20) \\
&= \sin(2A + B) \\
&= \sin(2A)\cos B + \cos(2A)\sin B \\
&= 2\sin A \cos A \cos B + (1 - 2\sin^2 A)\sin B \\
\sin A &= 2\sin A \cos A \cos B - 2\sin^2 A \sin B \\
1 &= 2\cos A \cos B - 2\sin A \sin B && (21) \\
&= 2\cos(A + B) = 2\cos C
\end{aligned}$$

Nonagon height is sum of all sines:

$$\sin A + \sin B + \sin C + \sin D = 2\sin D + \frac{\sqrt{3}}{2} \quad (22)$$

Last equation solves this cubic equation:

$$y^3 - \frac{3y}{4} - \frac{3}{8} = 0$$

$$y_1 = -\sin A \approx -0.342020$$

$$y_2 = -\sin B \approx -0.642787$$

$$y_3 = +\sin C \approx +0.984807$$

More sines relations of angles A, B and D are:

$$\sin A \sin B \sin D = \frac{\sqrt{3}}{8}$$

$$\sin^2 A + \sin^2 B + \sin^2 D = \frac{3}{2}$$

Cosines and sines relations are:

$$\cos A \cos B - \sin A \sin B = \frac{1}{2}$$

$$\frac{1}{\cos C} - \frac{\sqrt{3}}{\sin C} = 4$$

$$\tan C - 4 \sin C = \sqrt{3}$$