

Meccano nonagon

<https://github.com/heptagons/meccano/nona>

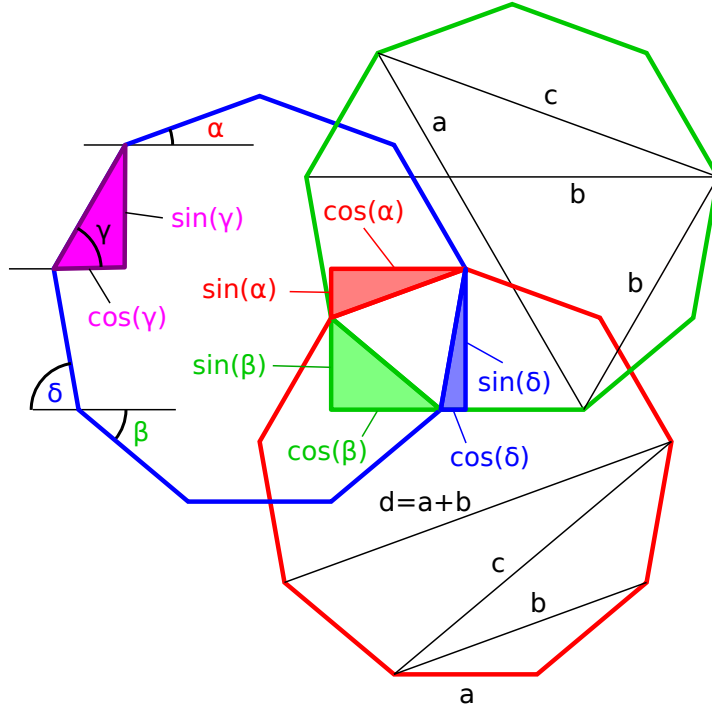


Figure 1: Three regular nonagons connected by an equilateral triangle to prove two relations: $\cos \alpha = \cos \beta + \cos \delta$ and $\sin \alpha + \sin \beta = \sin \delta$.

1 Algebra

Figure 1 shows three regular nonagons connected by an equilateral triangle. Four angles appear orthogonally in any regular nonagon:

$$\alpha = \frac{\pi}{9}, \quad \beta = \frac{2\pi}{9}, \quad \gamma = \frac{3\pi}{9}, \quad \delta = \frac{4\pi}{9} \quad (1)$$

$$\alpha + \beta = \delta - \alpha = \gamma \quad (2)$$

The relations of angle γ are those of equilateral triangle:

$$\cos \gamma = -\frac{1}{2}, \quad \sin \gamma = \frac{\sqrt{3}}{2} \quad (3)$$

From the figure 1, cosines of angles α, β, δ are related as:

$$\cos \alpha = \cos \beta + \cos \delta \quad (4)$$

$$= \cos(2\alpha) + \cos(4\alpha)$$

$$= (2 \cos^2 \alpha - 1) + (1 - 8 \cos^2 \alpha + 8 \cos^4 \alpha)$$

$$= 8 \cos^4 \alpha - 6 \cos^2 \alpha$$

$$1 = 8 \cos^3 \alpha - 6 \cos \alpha \quad (5)$$

The last equation is a depressed cubic equation with a negative discriminant:

$$t^3 + pt + q = 0 \quad \text{where } p = -\frac{3}{4}, q = -\frac{1}{8} \quad (6)$$

$$\Delta = \frac{q^2}{4} + \frac{p^3}{27} = -\frac{3}{64}$$

The negative discriminant means we have three real roots which can be found by trigonometry:

$$t_k = 2\sqrt{-\frac{p}{3}} \cos \left(\frac{1}{3} \arccos \left(\frac{3q}{2p} \sqrt{\frac{-3}{p}} \right) - k \frac{2\pi}{3} \right) \quad \text{for } k = 0, 1, 2.$$

$$= \cos \left(\frac{1}{3} \arccos \left(\frac{1}{2} \right) - k \frac{2\pi}{3} \right) \quad \text{for } k = 0, 1, 2.$$

$$= \cos \left(\frac{1}{3} \left(\frac{\pi}{3} \right) - k \frac{2\pi}{3} \right) \quad \text{for } k = 0, 1, 2.$$

$$t_0 = \cos \left(\frac{\pi}{9} \right) = \cos \alpha \approx +0.939692620785908 \quad (7)$$

$$t_1 = \cos \left(-\frac{2\pi}{9} \right) = -\cos \beta \approx -0.766044443118978 \quad (8)$$

$$t_2 = \cos \left(-\frac{4\pi}{9} \right) = -\cos \delta \approx -0.173648177666930 \quad (9)$$

From equation 6 we know the sum of roots squared is $-2p = \frac{3}{2}$:

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \delta = \frac{3}{2} \quad (10)$$

$$1 - \sin^2 \alpha + 1 - \sin^2 \beta + 1 - \sin^2 \delta = \frac{3}{2}$$

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \delta = \frac{3}{2} \quad (11)$$

From equation 6 we know the product of roots is $-q = \frac{1}{8}$ matching the “Morrie’s law”:

$$\cos \alpha \cos \beta \cos \delta = \frac{1}{8} \quad (12)$$

$$(1 - \sin^2 \alpha)(1 - \sin^2 \beta)(1 - \sin^2 \delta) = \frac{1}{64}$$

$$\begin{aligned} (\sin \alpha \sin \beta)^2 + (\sin \alpha \sin \delta)^2 + (\sin \beta \sin \delta)^2 &= \frac{1}{64} - 1 + \sin^2 \alpha + \sin^2 \beta + \sin^2 \delta + (\sin \alpha \sin \beta \sin \delta)^2 \\ &= \frac{1}{64} - 1 + \frac{3}{2} + \left(\frac{\sqrt{3}}{8} \right)^2 = \left(\frac{9}{4} \right)^2 \end{aligned} \quad (13)$$

From the figure 1, sines of angles α, β, δ are related as:

$$\sin \alpha + \sin \beta = \sin \delta \quad (14)$$

$$= \sin(2\alpha + \beta)$$

$$= \sin(2\alpha) \cos \beta + \cos(2\alpha) \sin \beta$$

$$= (2 \sin \alpha \cos \alpha) \cos \beta + (1 - 2 \sin^2 \alpha) \sin \beta$$

$$\sin \alpha = \sin \alpha (2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta)$$

$$2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta = 1 \quad (15)$$

$$2 \cos(\alpha + \beta) = 1 \implies 2 \cos \gamma = 1$$

Product of sines of angles α, β, δ is using equations 20 and 21:

$$\begin{aligned} \sin \alpha \sin \beta \sin \delta &= \frac{1}{2} (2 \cos \alpha \cos \beta - 1) (\sin \alpha + \sin \beta) \\ &= \sin(2\alpha) \sin \beta + \sin \alpha \sin(2\beta) \\ &= \frac{\sqrt{3}}{8} \end{aligned} \quad (16)$$

Last equation solves this cubic equation:

$$y^3 - \frac{3y}{4} - \frac{3}{8} = 0$$

$$y_1 = -\sin A \approx -0.342020$$

$$y_2 = -\sin B \approx -0.642787$$

$$y_3 = +\sin C \approx +0.984807$$

1.1 Segments a, b, c, d

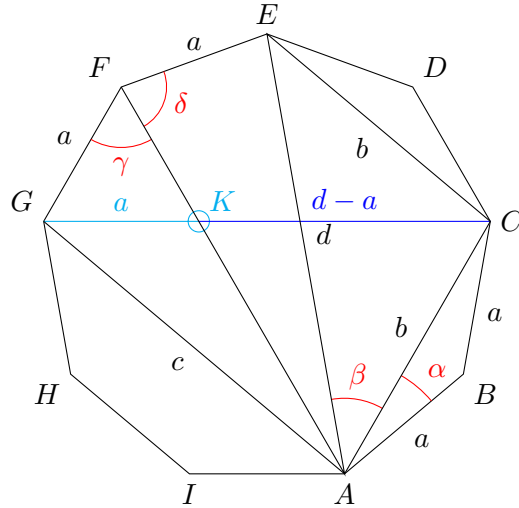


Figure 2: The nonagon perimeter side a and the three internal diagonals b, c, d . Also shown the base angle α and three more $\beta = 2\alpha$, $\gamma = 3\alpha$ and $\delta = 4\alpha$.

From figure 2 we can calculate $\cos \alpha$ inspecting isosceles $\triangle ABC$ ($\overline{AB} = \overline{AC} = a$):

$$a^2 = a^2 + b^2 - 2ab \cos \alpha$$

$$b^2 = 2ab \cos \alpha \implies \boxed{b = 2a \cos \alpha} \quad (17)$$

We calculate $\cos \beta$ inspecting isosceles $\triangle ACE$ ($\overline{AB} = \overline{AE} = b$):

$$\begin{aligned} b^2 &= b^2 + d^2 - 2bd \cos \beta \\ d^2 &= 2bd \cos \beta \implies \boxed{d = 2b \cos \beta} \end{aligned} \quad (18)$$

We calculate $\cos \delta$ inspecting isosceles $\triangle AEF$ ($\overline{AE} = \overline{AF} = d$):

$$\begin{aligned} d^2 &= a^2 + d^2 - 2ad \cos \delta \\ a^2 &= 2ad \cos \delta \implies \boxed{a = 2d \cos \delta} \end{aligned} \quad (19)$$

From equations 17, 18 and 19 we have:

$$\cos \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} = \begin{pmatrix} \frac{b}{2a} \\ \frac{d}{2b} \\ \frac{a}{2d} \end{pmatrix} \quad (20)$$

From equation 4 $\cos \alpha = \cos \beta + \cos \delta$:

$$\frac{b}{2a} = \frac{d}{2b} + \frac{a}{2d} \implies \boxed{\frac{1}{a/b} = \frac{1}{b/d} + \frac{1}{d/a}} \quad (21)$$

From figure 2 we inspect equilateral triangles $\triangle ACK$ and $\triangle FGK$ and found that:

$$\overline{FG} = \overline{GK} = \overline{KF} = a \quad (22)$$

$$\overline{AC} = \overline{CK} = \overline{KA} = b \quad (23)$$

$$\begin{aligned} \overline{CK} &= \overline{CG} - \overline{KG} \\ b &= d - a \implies \boxed{d = a + b} \end{aligned} \quad (24)$$

Replacing d in optic equation 21 we get:

$$\frac{b}{a} = \frac{a+b}{b} + \frac{a}{a+b} \quad (25)$$

Making $e = b/a$ we update last equation as:

$$\begin{aligned} e &\equiv \frac{b}{a} \\ e &= \frac{1+e}{e} + \frac{1}{1+e} \implies \boxed{e^3 - 3e - 1 = 0} \end{aligned} \quad (26)$$

$$\begin{aligned} e_0 &\approx 1.87938524157182 & &= +2 \cos \alpha = b/a \\ e_1 &\approx -1.53208888623796 & &= -2 \cos \beta = (a+b)/b \\ e_2 &\approx -0.347296355333861 & &= -2 \cos \delta = a/(a+b) \end{aligned}$$

$$e_0 + e_1 + e_2 = 0 \quad (27)$$

$$e_0 e_1 e_2 = 1 \quad (28)$$

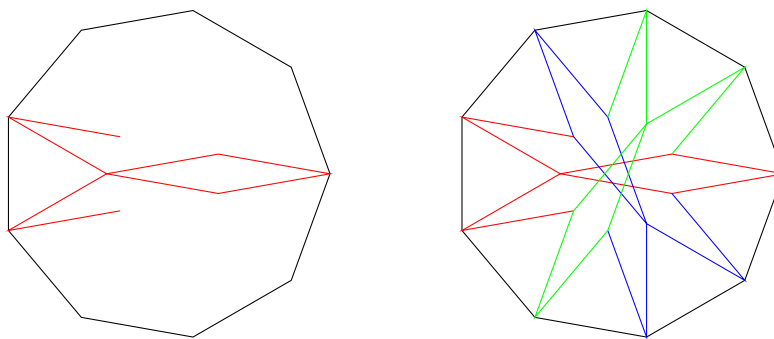


Figure 3: Rigid nonagon. 33 equal strips: Perimeter=9, diagonals= 3×8 .

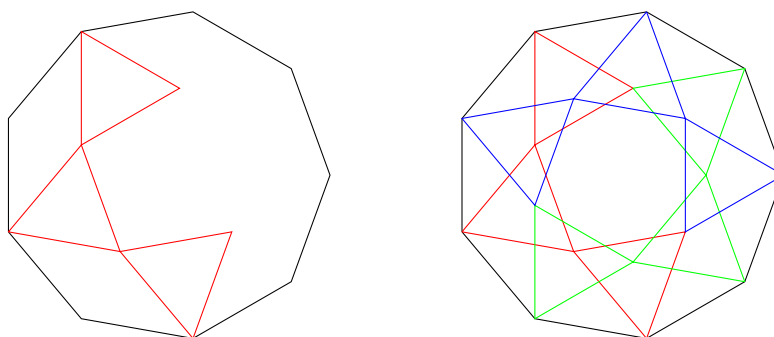


Figure 4: Rigid nonagon. 36 equal strips: Perimeter=9, diagonals= 3×9 .