

Meccano hexagons gallery

<https://github.com/heptagons/meccano/hexa/gallery>

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Abstract

We build meccano ¹ rigid regular hexagons from sides 4 to 24. Hexagon perimeter has 6 equal strips connected with 6 bolts. To make it rigid we want to add a maximum of 3 internal strips connected with at most 4 extra bolts. The extra strips also must remain totally inside the perimeter, don't overlap with any other and must not be parallel to any external strip. With algebra and then software, we produce **hexagonal triangles** which have an internal angle of 120° and we use their sides as the internal strips.

1 Algebra and software

1.1 Hexagon angles

We look strips that can make rigid two consecutive internal sides the regular hexagon. Figure 1 show the first four cases found. From any figure we have the internal hexagon angle is $\theta \equiv \angle GBC = 2\pi/3$. First we define the hexagon side as $p \equiv \overline{BC}$. From the triangle $\triangle GBC$ we define the other two sides as $b \equiv \overline{GB}$ and $c \equiv \overline{GC}$. By the law of cosines we know that:

$$\begin{aligned} c &= \sqrt{b^2 + p^2 - 2bp \cos \theta} \\ &= \sqrt{b^2 + p^2 - 2bp \left(-\frac{1}{2}\right)} = \sqrt{b^2 + p^2 + bp} \end{aligned} \quad (1)$$

Then we define $a \equiv p + b$ to get:

$$c = \sqrt{a^2 + b^2 - ab} \quad \text{where } a > b \quad (2)$$

a	c	p	b
8	7	5	3
15	13	8	7
21	19	16	5
35	31	24	11
40	37	33	7
48	43	35	13

Table 1: **Hexagonal triangles** with sides $c > p > b$ with an internal angle $2\pi/3$.

Our software iterates first $0 < a < \max$ and then $1 < b < a$ and record all c that is an integer. The first cases of such triangles with sides $c > p > b$ are shown table 1 and we call them **Hexagonal triangles**.

Figure 1 shows hexagons of sizes $p = \{5, 8, 16, 24\}$ with perimeter strips in orange made rigid adding three internal green strips of length $c = \{7, 13, 19, 31\}$.

¹ Meccano mathematics by 't Hooft

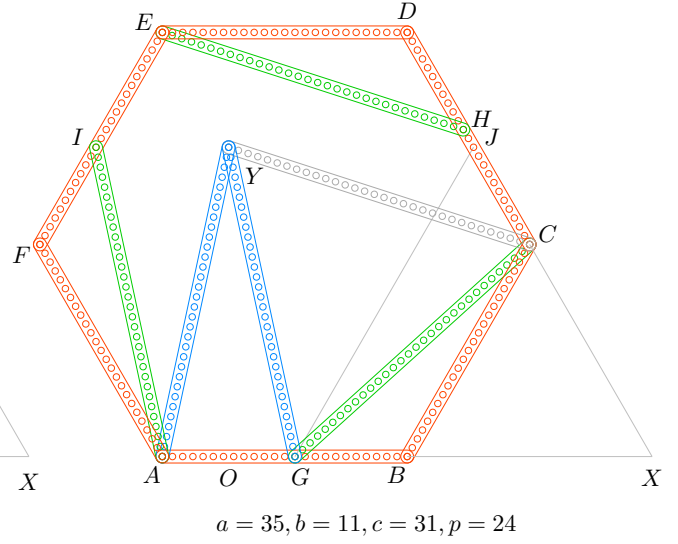
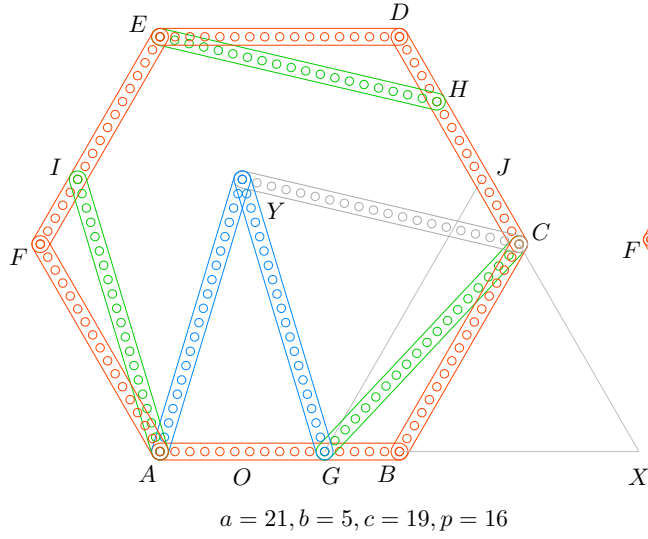
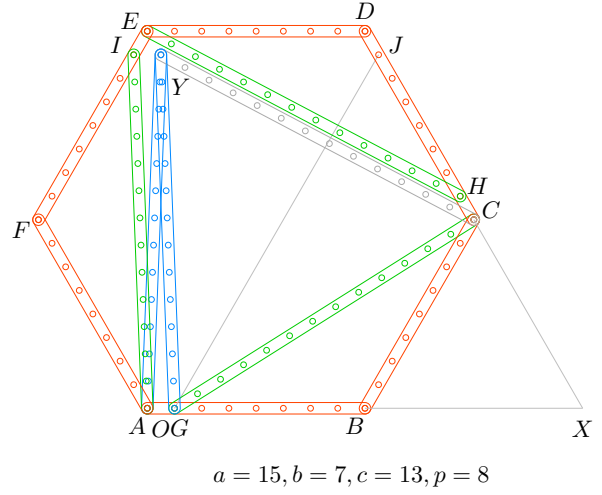
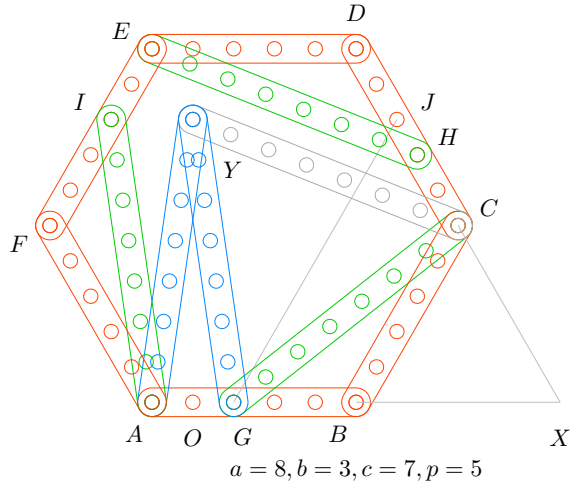


Figure 1: First four cases where internal strip $c = \overline{GC}$ is an integer and makes rigid two consecutive regular hexagon sides $p = \overline{AB} = \overline{BC}$. Our software inspect two integers $a > b$ and looks for $c = \sqrt{a^2 + b^2 - ab}$ to be an integer. In the figures $b = \overline{GB}$ and $a = \overline{GX} = \overline{XJ} = \overline{JG} = p + b$.

1.1.1 Hexagon height

In figure 1 we have also an equilateral triangle $\triangle GCY$ and an isoscelles triangle $\triangle AGY$. The base of the isoscelles triangle is $x \equiv \overline{AG} = \overline{AB} - \overline{GB} = p - b$ and the equals sides are $\overline{AY} = \overline{GY} = c$. So we can calculate the height $y \equiv \overline{OY}$ substituting c using equation 1:

$$\begin{aligned}
 y &= \sqrt{(\overline{GY})^2 - (\overline{AO})^2} \\
 &= \sqrt{c^2 - \left(\frac{p-b}{2}\right)^2} \\
 &= \sqrt{b^2 + p^2 + bp - \left(\frac{p-b}{2}\right)^2} = \frac{(p+b)\sqrt{3}}{2} = \frac{a\sqrt{3}}{2}
 \end{aligned} \tag{3}$$

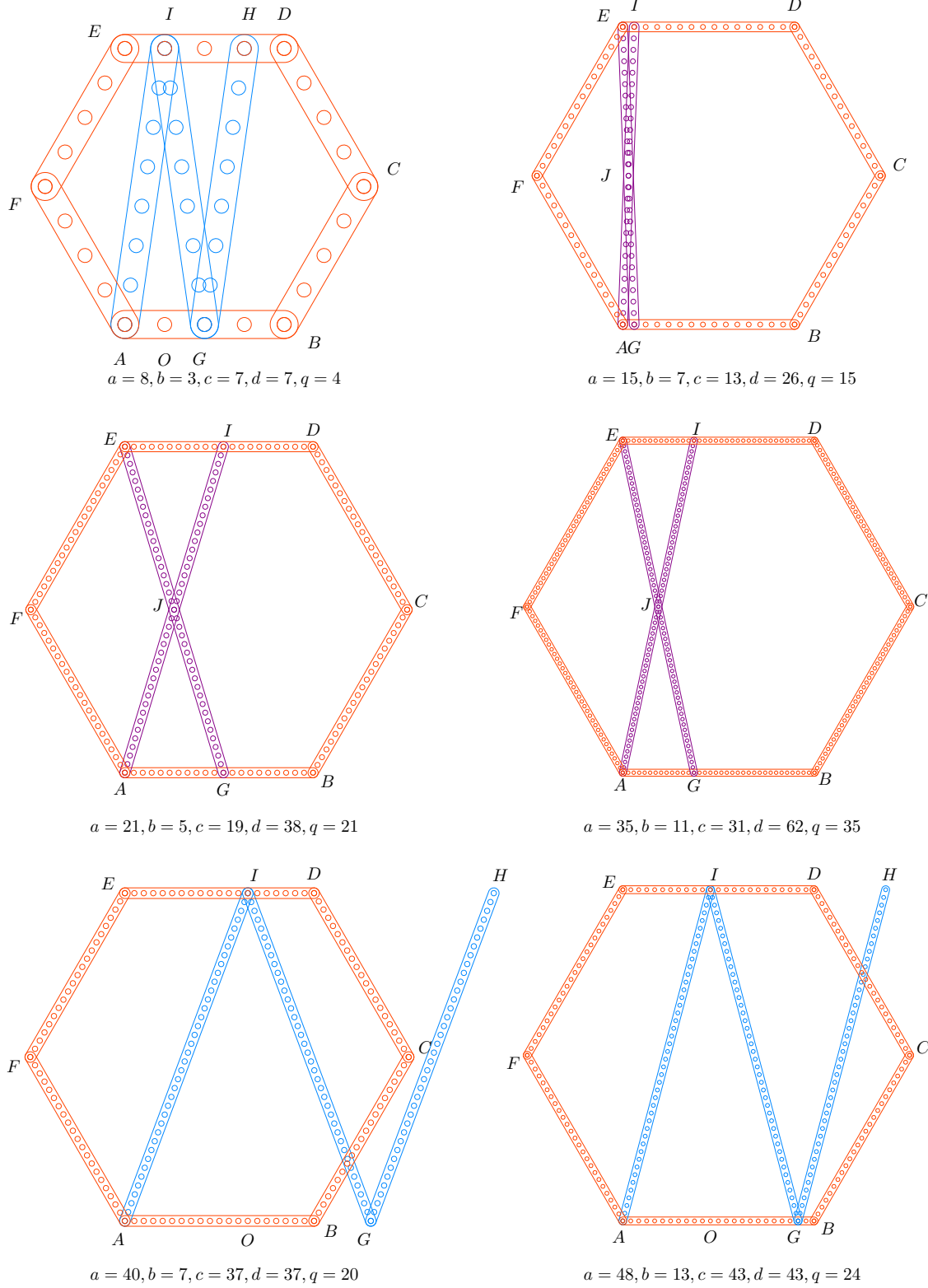


Figure 2: First six cases of integral distances c . When the distance $p - b = \overline{AG}$ is even, we use the strips $d = c$ (in blue) to join opposites sides of hexagons of side $q = a/2$. When is odd, we use the strips $d = 2c$ (in purple) to join opposites sides of hexagons of side $q = a$.

We know $\frac{a\sqrt{3}}{2}$ is the height of the regular hexagon of side $\frac{a}{2}$ so we can use the blue strips to connect

opposite sides. Figure 2 show the smaller hexagons that have integer strips connecting opposites sides.

1.1.2 Hexagon rigidity

Through the gallery we will use the green, blue and purple strips and their copies multiplied by integers as internal diagonals to make rigid regular hexagons from size 4 to 24. We prioritize minimum number of internal strips (2 or 3) and extra bolts (2 or 3 or 4) and the largest strips sizes as possible. We discard the hexagons which are scaled copies of smaller ones.

2 Hexagons of size $s < 10$

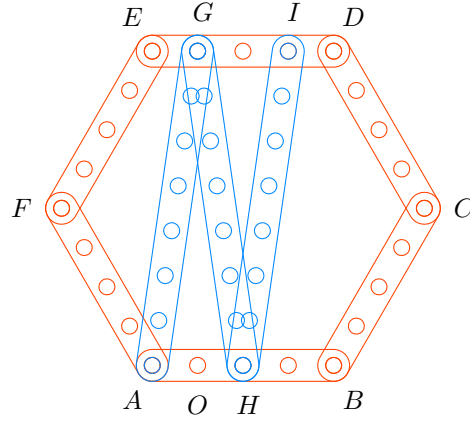


Figure 3: Hexagon of size 4 with three equal diagonals $\overline{GH} = \overline{HI} = \overline{IG} = 7$ and three extra bolts at G, H, I .

Figure 3 show a regular hexagon $ABCDEF$ of size 4 with three diagonals of size 7. We confirm the height of the hexagon with the distance $\overline{OG} = \sqrt{(AG)^2 - (AO)^2} = \sqrt{7^2 - 1^2} = 4\sqrt{3}$.

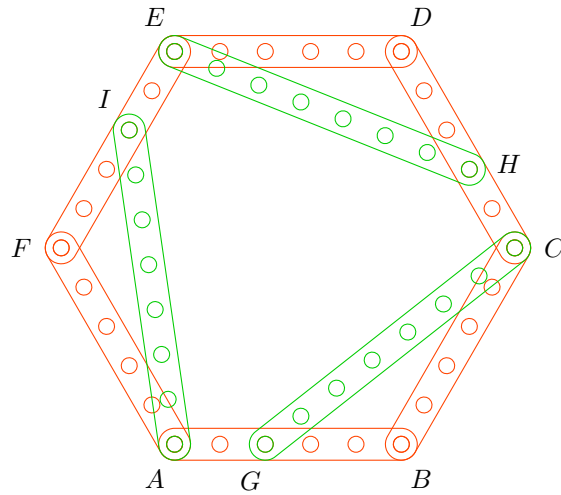


Figure 4: Hexagon of size 5 with three equal diagonals $\overline{GC} = \overline{HE} = \overline{IA} = 7$ and three extra bolts at G, H, I .

Figure 4 show a regular hexagon $ABCDEF$ of size 5 with three diagonals of size 7. We confirm the

angle of $\alpha \equiv \angle GBC = 120^\circ$ with the law of cosines.

$$\cos \alpha = \frac{(\overline{GB})^2 + (\overline{BC})^2 - (\overline{GC})^2}{2(\overline{GB})(\overline{BC})} = \frac{3^2 + 5^2 - 7^2}{2(3)(5)} = -\frac{1}{2}$$

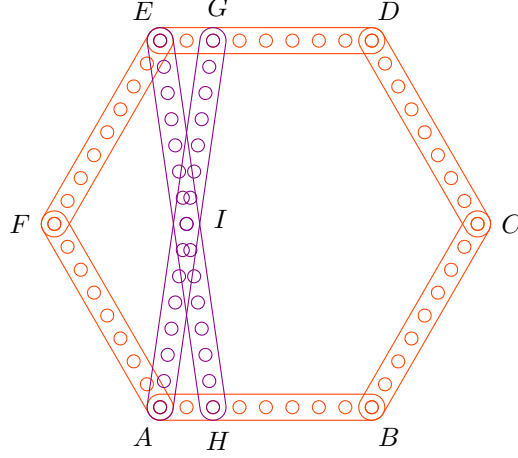


Figure 5: Hexagon of size 8 with **only** two diagonals $\overline{AH} = \overline{EI} = 14$ and three extra bolts at G, H, I .

Figure 5 show hexagon $ABCDEF$ of size 8. We confirm the height of the hexagon $\overline{AE} = \sqrt{(\overline{AG})^2 - (\overline{AH})^2} = \sqrt{14^2 - 2^2} = 8\sqrt{3}$.

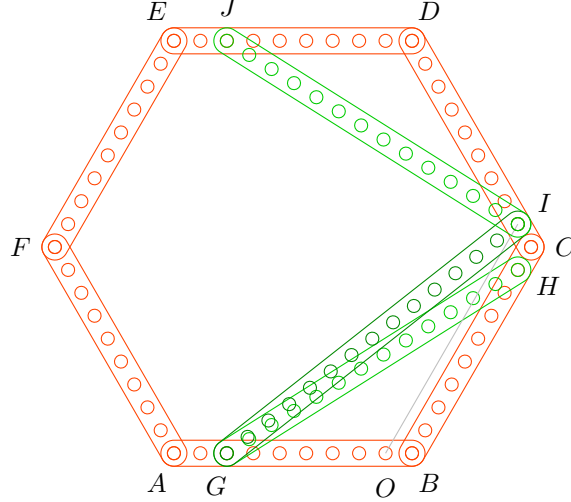


Figure 6: Hexagon of size 9 with three diagonals $\overline{GH} = \overline{IJ} = 13, \overline{GI} = 14$ and four extra bolts at G, H, I, J .

Figure 6 show hexagon $ABCDEF$ of size 9. Triangles $\triangle HBG$ and $\triangle JID$ have sides $(13, 8, 7)$ which are hexagonal triangles. Triangle $\triangle GIO$ has sides $(14, 10, 6)$ which is hexagonal triangle $(7, 5, 3)$ multiplied by 2.

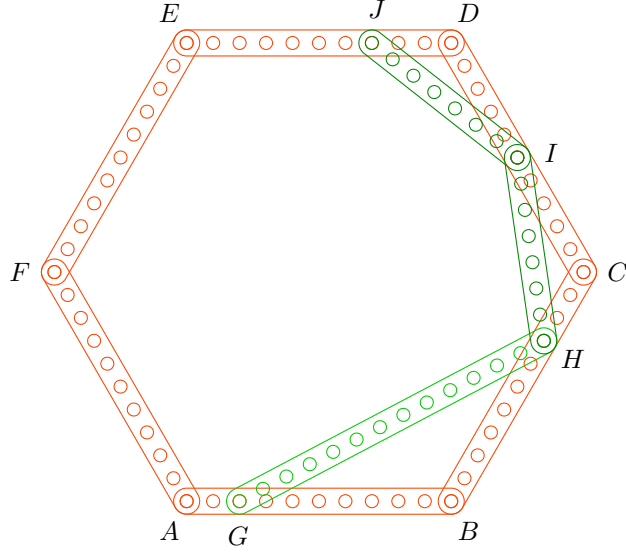


Figure 7: Hexagon of size 10 with three diagonals $\overline{HI} = \overline{IJ} = 7$, $\overline{GH} = 13$ and four extra bolts at G, H, I, J .

Figure 7 show hexagon $ABCDEF$ of size 10. Triangles $\triangle HIC$ and $\triangle JID$ have sides $(7, 5, 3)$ which are hexagonal triangles. Triangle $\triangle HGB$ has size $(13, 8, 7)$ which is an hexagonal triangle.

3 Hexagons of size $10 < s < 20$

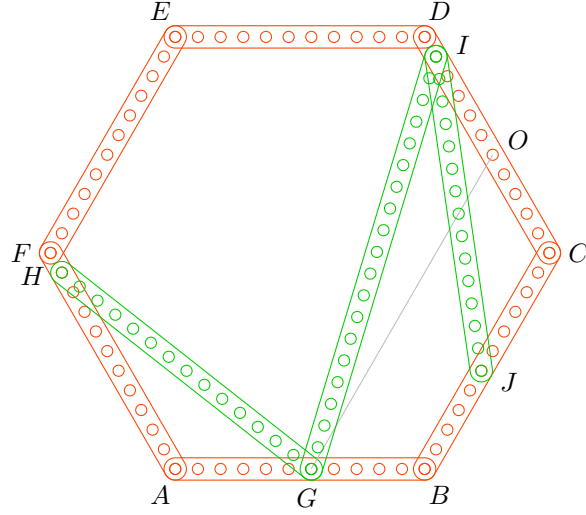


Figure 8: Hexagon of size 11 with three diagonals $\overline{GH} = \overline{IJ} = 14$ and $\overline{IG} = 19$ and four extra bolts at G, H, I, J .

Figure 8 show hexagon $ABCDEF$ of size 11. Triangles $\triangle GHA$ and $\triangle JIC$ have sides $(14, 10, 6)$ which are the hexagonal triangle $(7, 5, 3)$ multiplied by 2. Triangle $\triangle IGO$ has sides $(19, 16, 5)$ which is an hexagonal triangle.

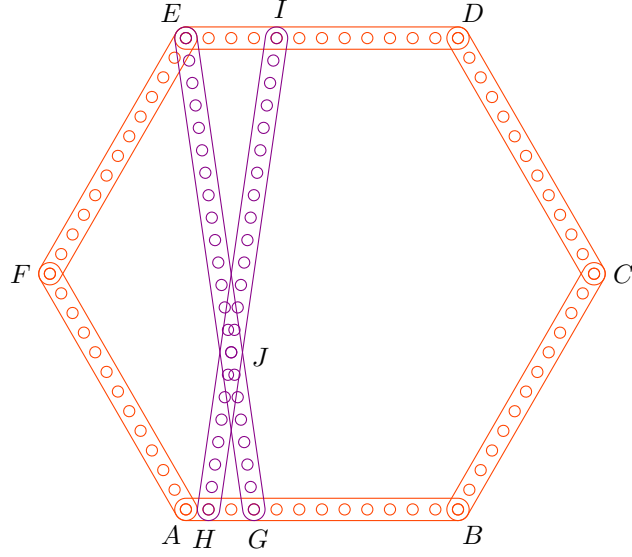


Figure 9: Hexagon of size 12 with **only** two diagonals $\overline{EG} = \overline{HI} = 21$ and four extra bolts at G, H, I, J .

Figure 9 show hexagon $ABCDEF$ of size 12. We confirm the hexagon height: $\overline{AE} = \sqrt{(\overline{EG})^2 - (\overline{AG})^2} = \sqrt{21^2 - 3^2} = 12\sqrt{3}$.

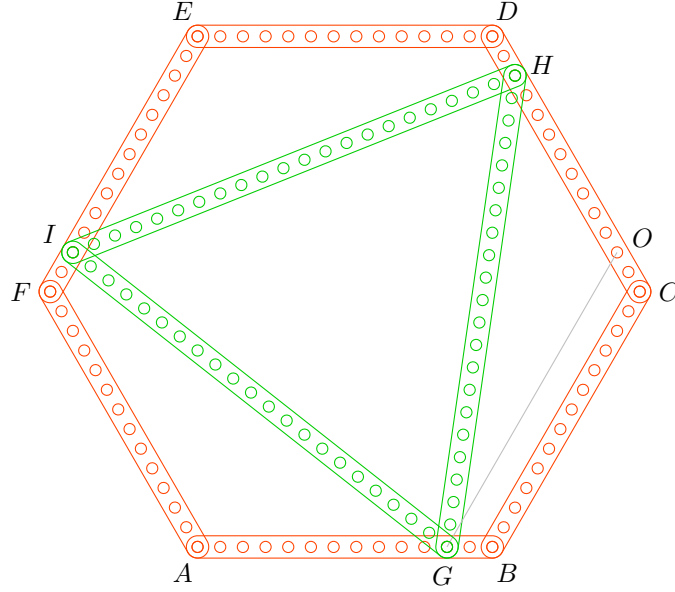


Figure 10: Hexagon of size 13 with three equal diagonals $\overline{GH} = \overline{HI} = \overline{IG} = 21$ and three bolts at vertices G, H, I .

Figure 10 show hexagon $ABCDEF$ of size $s = 13$. First we note the offset $o \equiv \overline{CO}$ which we use to calculate the sides $\{c, p, b\}$ of triangle $\triangle GOH$:

$$\begin{aligned}
 o &\equiv \overline{DH} = \overline{CO} = 2 \\
 c &\equiv \overline{GH} = 21 \\
 p &\equiv \overline{GO} = \overline{BC} + o = 13 + 2 = 15 \\
 b &\equiv \overline{OH} = \overline{CD} - 2o = 13 - 2(2) = 9
 \end{aligned} \tag{4}$$

The sides c, p, b are $(21, 15, 9)$ which is hexagonal triangle $(7, 5, 3)$ multiplied by 3. For this case having an equilateral triangle inside the hexagon we can calculate c in function of s, o using the equation 1:

$$\begin{aligned}
b &= s - 2o \\
p &= s + o \\
c &= \sqrt{b^2 + p^2 - bp} \\
&= \sqrt{(s - 2o)^2 + (s + o)^2 + (s - 2o)(s + o)} \\
&= \sqrt{3s^2 - 3so + 3o^2}
\end{aligned} \tag{5}$$

Using the formula our software found more cases shown in the table 2

s	o	c	p	b
13	2	21	15	9
23	1	39	24	21
37	11	57	48	15
59	13	93	72	33

Table 2: Hexagons of size s with inside equilateral triangles of side c .

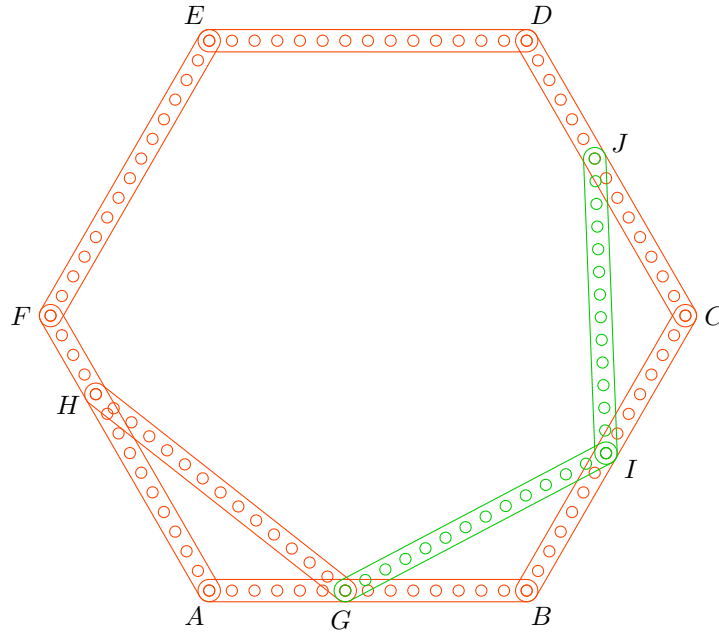


Figure 11: Hexagon of size 14 with three diagonals $\overline{GH} = 14, \overline{GI} = \overline{IJ} = 13$ and four extra bolts at G, H, I, J .

Figure 11 show hexagon $ABCDEF$ of size 14. Triangle $\triangle GHA$ has sides $(14, 10, 6)$ which is hexagonal triangle $(7, 5, 3)$ multiplied by 2. Triangles $\triangle IGB$ and $\triangle JCI$ have sides $\{13, 8, 7\}$ which are hexagonal triangles.

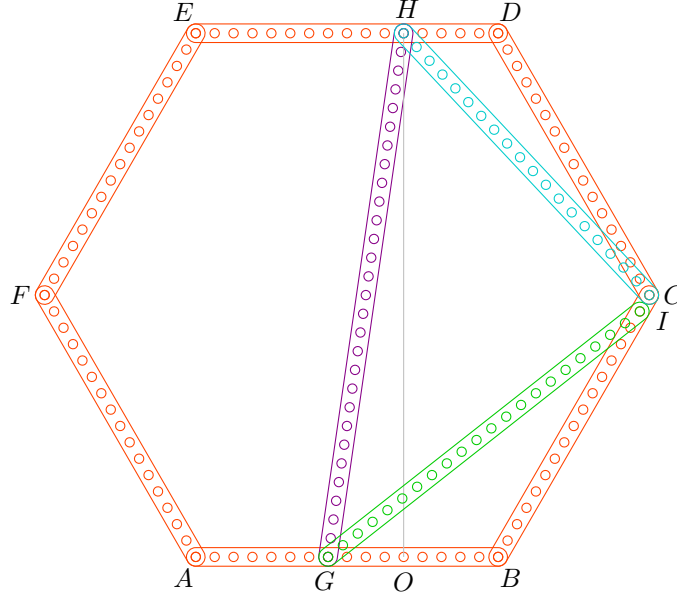


Figure 12: Hexagon of size 16 with three diagonals $\overline{GH} = 28, \overline{GI} = 21, \overline{CH} = 19$ and three extra bolts at G, H, I .

Figure 12 show hexagon $ABCDEF$ of size 16. Hexagon's height is $\overline{OH} = \sqrt{(\overline{GH})^2 - (\overline{GO})^2} = \sqrt{28^2 - 4^2} = 16\sqrt{3}$. Triangle $\triangle IBG$ sides are $(21, 15, 9)$ which is hexagonal triangle $(7, 5, 3)$ multiplied by 3. Triangle $\triangle HCD$ sides are $(19, 16, 5)$ which is an hexagonal triangle.

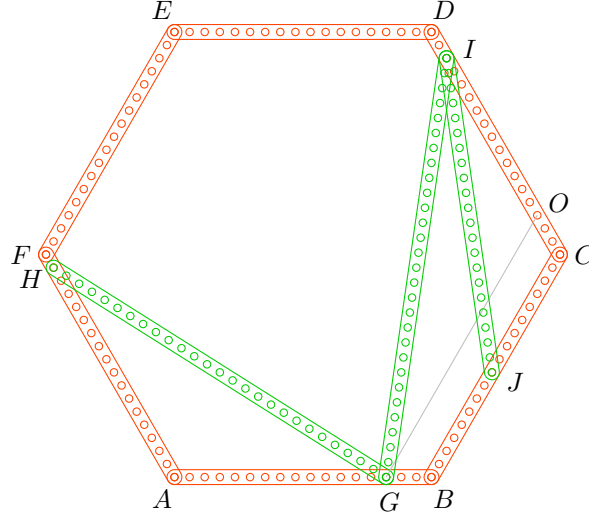


Figure 13: Hexagon of size 17 with three diagonals $\overline{GH} = 26, \overline{GI} = 28, \overline{IJ} = 21$ and four bolts at vertices G, H, I, J .

Figure 13 show hexagon $ABCDEF$ of size 17. Triangle $\triangle GHA$ sides are $(26, 16, 14)$ which is hexagonal triangle $(13, 8, 7)$ multiplied by 2. Triangle $\triangle IGO$ sides are $(28, 20, 12)$ which is hexagonal triangle $(7, 5, 3)$ multiplied by 4. Triangle $\triangle JIC$ has sides $(21, 15, 9)$ which is hexagonal triangle $(7, 5, 3)$ multiplied by 3.

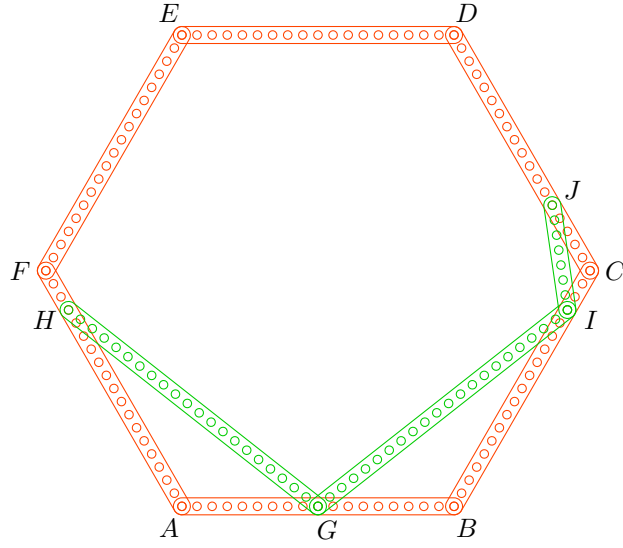


Figure 14: Hexagon of size 18 with three diagonals $\overline{GH} = \overline{GI} = 21, \overline{IJ} = 7$ and four bolts at vertices G, H, I, J .

Figure 14 show hexagon $ABCDEF$ of size 18. Triangles $\triangle GHA$ and $\triangle GIB$ have sides $(21, 15, 9)$ which are hexagonal triangles $(7, 5, 3)$ multiplied by 3. Triangle $\triangle IJC$ sides are $(7, 5, 3)$ which is an hexagonal triangle.

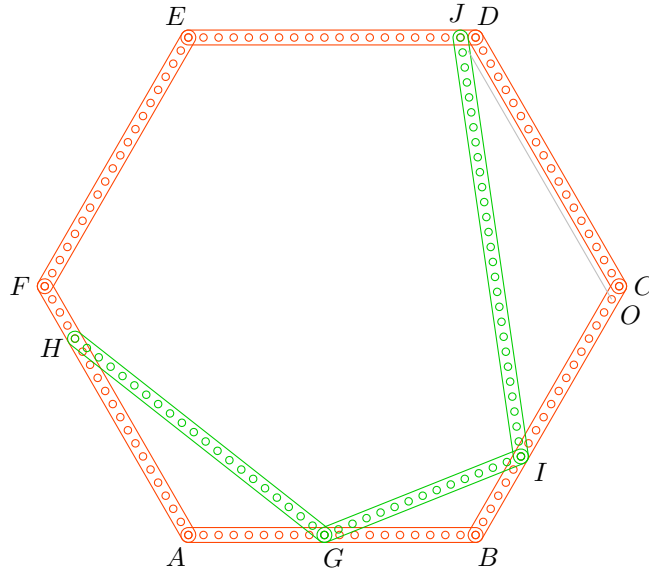


Figure 15: Hexagon of size 19 with three diagonals $\overline{GH} = 21, \overline{GI} = 14, \overline{IJ} = 28$ and four bolts at vertices G, H, I, J .

Figure 15 show hexagon $ABCDEF$ of size 19. Triangle $\triangle GHA$ has sides $(21, 15, 9)$ which correspond to hexagonal triangle $(7, 5, 3)$ multiplied by 3. Triangle $\triangle GIB$ has sides $(14, 10, 6)$ which correspond to hexagonal triangle $(7, 5, 3)$ multiplied by 2. Triangle $\triangle IJO$ has sides $(28, 20, 12)$ which correspond to hexagonal triangle $(7, 5, 3)$ multiplied by 4.

4 Hexagons of size ≥ 20

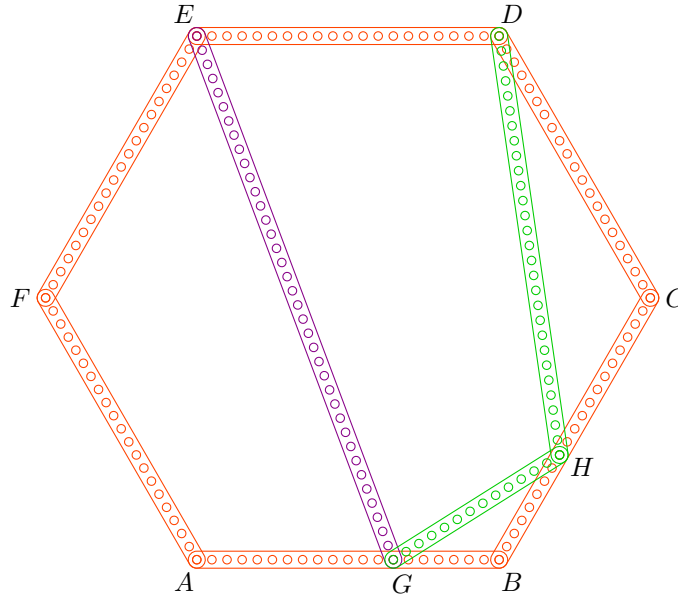


Figure 16: Hexagon of size 20 with three diagonals $\overline{GH} = 13$, $\overline{HD} = 28$ and $\overline{EG} = 37$ and **only** two extra bolts at vertices G and H .

Figure 16 show regular hexagon $ABCDEF$ of size 20. Triangle $\triangle GBH$ sides are $\{13, 8, 7\}$ which is an hexagonal triangle. Triangle $\triangle HCD$ sides are $\{28, 20, 12\}$ which is hexagonal triangle $\{7, 5, 3\}$ scaled by 4. Right triangle $\triangle EAG$ has side $\overline{AE} = \sqrt{(\overline{EG})^2 - (\overline{AG})^2} = \sqrt{37^2 - 13^2} = 20\sqrt{3}$ equals to hexagon height.

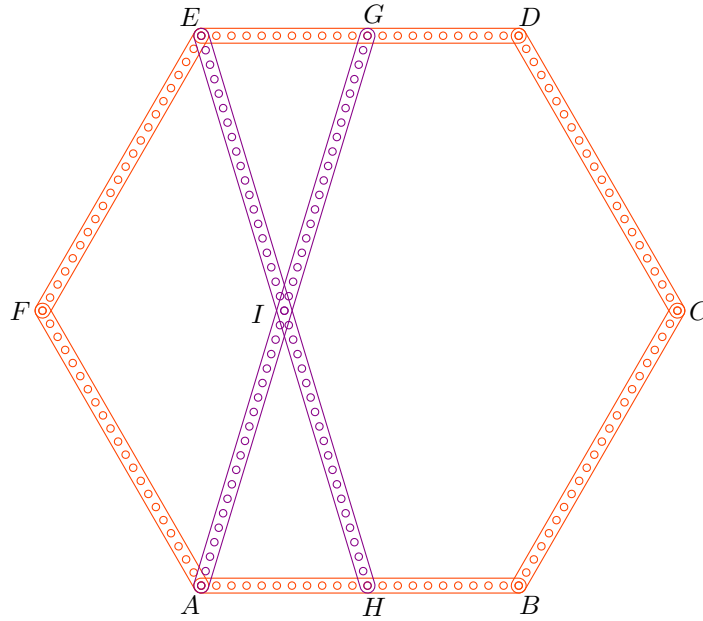


Figure 17: Hexagon of size 21 with **only** two diagonals $\overline{AG} = \overline{EH} = 38$. Segments $\overline{AH} = \overline{EG} = 11$, segment $\overline{AI} = 19$. Three extra bolts at H, G, I .

Figure 17 show regular hexagon $ABCDEF$ of size 21. We confirm the height of the hexagon is

$$\overline{AE} = \sqrt{(\overline{EH})^2 - (\overline{AH})^2} = \sqrt{38^2 - 11^2} = 21\sqrt{3}.$$

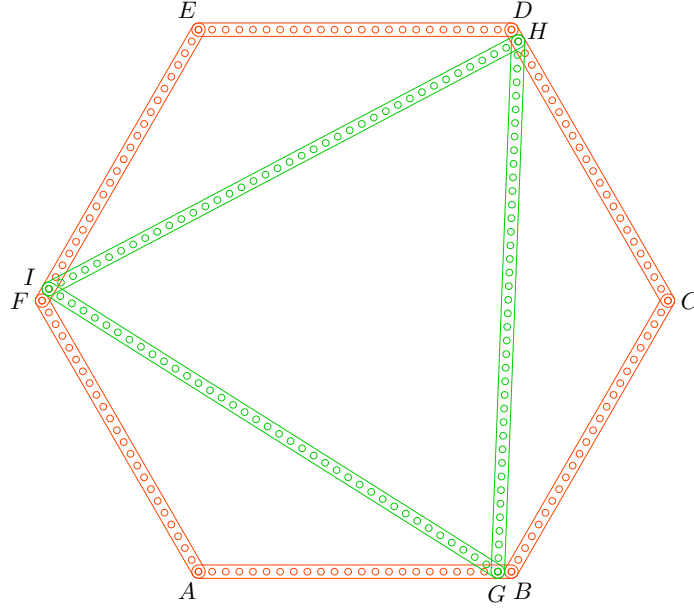


Figure 18: Hexagon of size $s = 23$ with three diagonals $c = \overline{GH} = \overline{HI} = \overline{IG} = 39$.

Figure 18 show a regular hexagon $ABCDEF$ of size $s = 23$. Is the second hexagon having an equilateral triangle inside, in this case of size $c = 39$ and described in table 2.

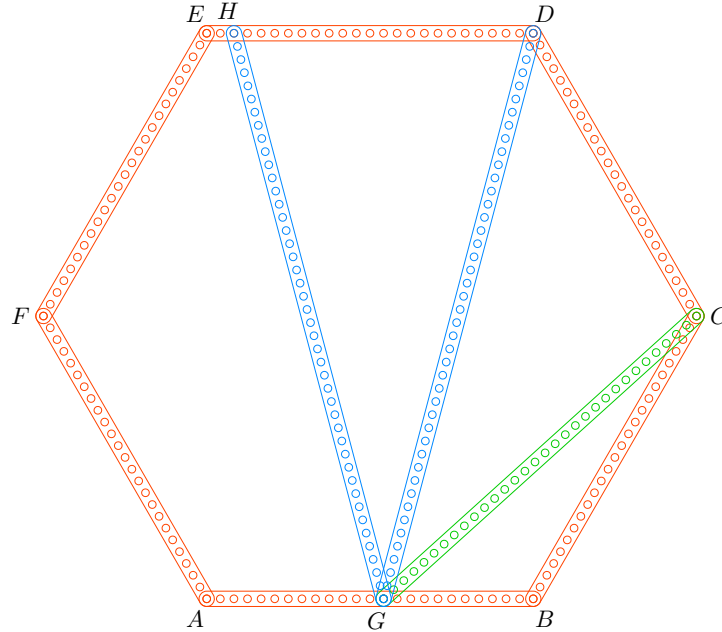


Figure 19: Hexagon of size $s = 24$ with diagonals $\overline{GC} = 31$ and $\overline{GD} = \overline{GH} = 43$ with **only** two extra bolts at vertices G and H . Segments $\overline{GB} = 11$ and $\overline{DH} = 22$.

Figure 19 show regular hexagon $ABCDEF$ of size 24. Triangle $\triangle GBC$ has sides $\{31, 24, 11\}$ which is an hexagonal triangle in table 1. Triangle $\triangle DHG$ is the isoscelles triangle shown in figure 2 case $a = 48$.