Meccano pentagons gallery

https://github.com/heptagons/meccano/penta/gallery

Abstract

We show constructions of meccano rigid regular pentagons from side 12 to 3. We restrict all internal strips, we call diagonals, to remain inside the pentagon's perimeter and that don't overlap. Several programs found the solutions and we show some alternatives and prove the claimed values are exact.

1 Pentagons of size 12

A program found that side 12 is the smallest pentagon that can be made rigid with a rhoumbus and two strips as diagonals so need only 4 strips as diagonals. We show two cases.

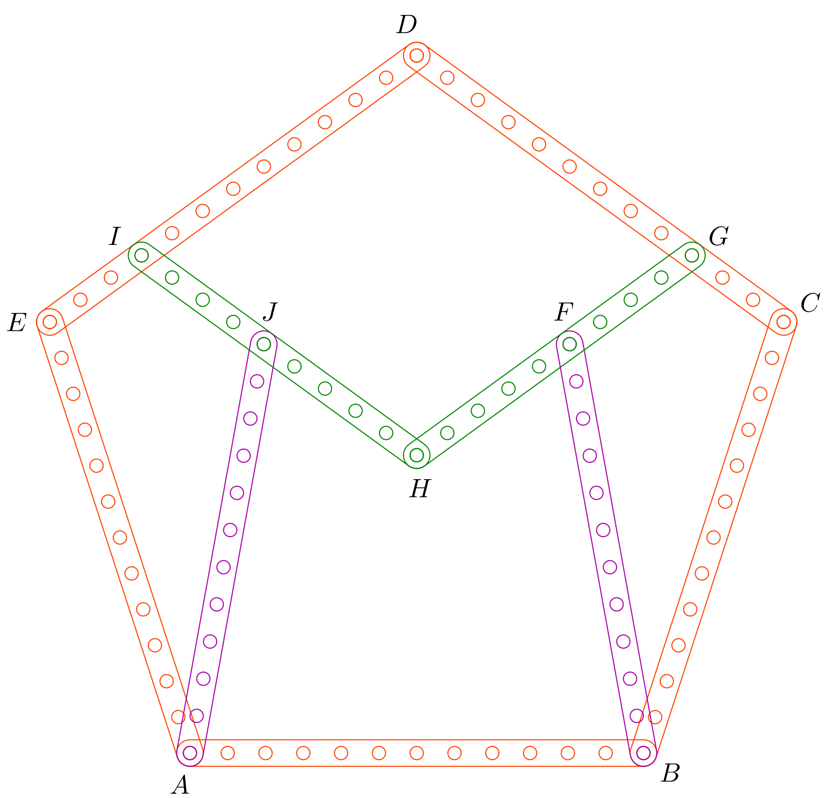

Figure 1: Regular pentagon size 12 (case a) made rigid with only four internal strips.

Figure 1 show a regular pentagon A, B, C, D, E of side 12 with a rhombus D, I, H, G of side 9. We prove strips AJ, BF are correct. First we calculate the abscissas going through vertices A, E, I, J substracting when we move to the left and adding when we move to the right:

$$
\begin{aligned}
AJ_x &= AE_x + EI_x + IJ_x && (1)\\
&= -\overline{AE}\cos\left(\frac{2\pi}{5}\right) + \overline{EI}\cos\left(\frac{\pi}{5}\right) + \overline{IJ}\cos\left(\frac{\pi}{5}\right)\\
&= -12\left(\frac{\sqrt{5}-1}{4}\right) + 3\left(\frac{1+\sqrt{5}}{4}\right) + 4\left(\frac{1+\sqrt{5}}{4}\right) = \frac{19-5\sqrt{5}}{4} && (2)
\end{aligned}
$$

Then we calculate the ordinates going to the same order of vertices adding when we go up and substracting when we go down:

$$
\begin{aligned}
AJ_y &= -AE_y + EI_y + IJ_y && (3)\\
&= \overline{AE}\sin\left(\frac{2\pi}{5}\right) + \overline{EI}\sin\left(\frac{\pi}{5}\right) - \overline{IJ}\sin\left(\frac{\pi}{5}\right)\\
&= 12\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right) + 3\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right) - 4\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right)\\
&= \frac{12\sqrt{10+2\sqrt{5}} - \sqrt{10-2\sqrt{5}}}{4} = \frac{\sqrt{1450+190\sqrt{5}}}{4} && (4)
\end{aligned}
$$

Finally we calculate the distance $\overline{AJ}$ which coincides with strip size 11:

$$
\begin{aligned}
\overline{AJ} &= \sqrt{(AJ_x)^2 + (AJ_y)^2} && (5)\\
&= \sqrt{\left(\frac{19-5\sqrt{5}}{4}\right)^2 + \frac{1450+190\sqrt{5}}{16}}\\
&= \sqrt{\frac{486-190\sqrt{5}}{16} + \frac{1450+190\sqrt{5}}{16}} = \sqrt{121} = 11 && (6)
\end{aligned}
$$

Figure 2 show a regular pentagon A, B, C, D, E of size 12 with a rhombus D, I, H, G of size 12. We prove strips GH, IJ are correct. First we calculate the abscissas going through vertices G, A, E, H substracting when we move to the left and adding when we move to the right:

$$
\begin{aligned}
GH_x &= -GA_x - AE_x + EH_x && (7)\\
&= -\overline{GA} - \overline{AE}\cos\left(\frac{2\pi}{5}\right) + \overline{EH}\cos\left(\frac{\pi}{5}\right)\\
&= -4 - 12\left(\frac{\sqrt{5}-1}{4}\right) + 3\left(\frac{1+\sqrt{5}}{4}\right) = \frac{-1-9\sqrt{5}}{4} && (8)
\end{aligned}
$$

Then we calculate the ordinates going to the same order of vertices adding when we go up and substracting when we go down:

$$
\begin{aligned}
GH_y &= AG_y + AE_y - EH_y && (9)\\
&= 0 + \overline{AE}\sin\left(\frac{2\pi}{5}\right) - \overline{EH}\sin\left(\frac{\pi}{5}\right)\\
&= 12\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right) - 3\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right)\\
&= \frac{12\sqrt{10+2\sqrt{5}} - 3\sqrt{10-2\sqrt{5}}}{4} = \frac{\sqrt{1530-18\sqrt{5}}}{4} && (10)
\end{aligned}
$$

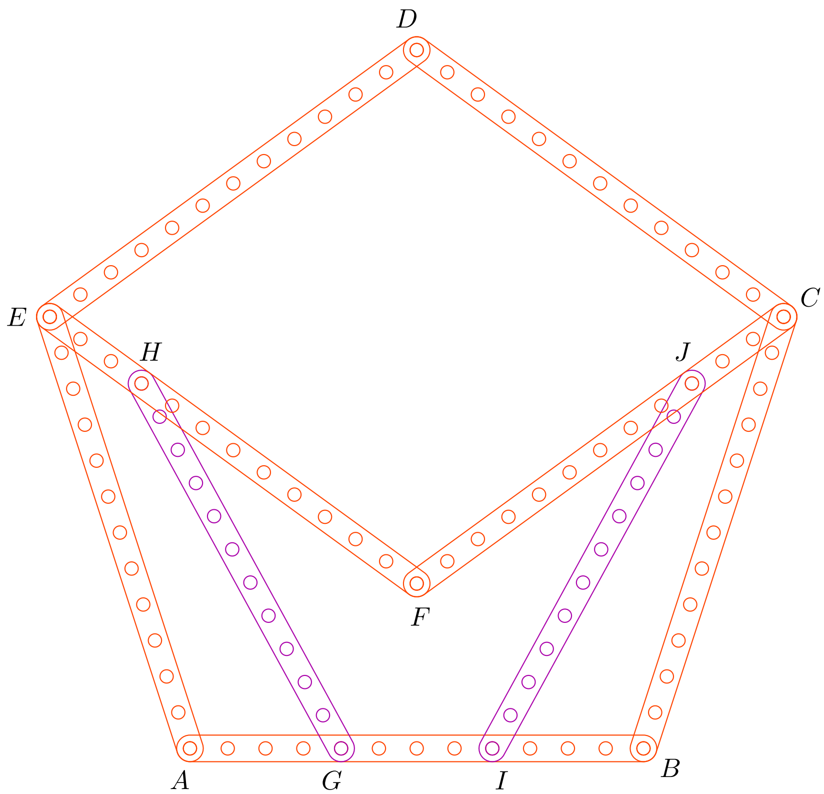

Figure 2: Regular pentagon size 12 case (b) made rigid with only four internal strips.

Finally we calculate the distance $\overline{GH}$ which coincides with strip size 11:

$$\overline{GH} = \sqrt{(GH_x)^2 + (GH_y)^2} \tag{11}$$

$$= \sqrt{\left(\frac{-1-9\sqrt{5}}{4}\right)^2 + \frac{1530-18\sqrt{5}}{16}}$$

$$= \sqrt{\frac{406+18\sqrt{5}}{16} + \frac{1530-18\sqrt{5}}{16}} = \sqrt{121} = 11 \tag{12}$$

2 Pentagon of size 11

Figure 3 show a rigid regular pentagon A, B, C, D, E of size 11. A program found this is the smallest pentagon having a consecutive sides diagonal of the form $\frac{z_2 + z_3\sqrt{5}}{z_1}$ instead of the nested form $\frac{z_2\sqrt{z_3 + z_4\sqrt{5}}}{z_1}$ where z_i are integers. The mentioned diagonal is the distance $\overline{CF}$ in the figure which can be calculated with the law of cosines knowing angle $\angle CBF = \frac{3\pi}{5}$ and denesting the result. We calculate the angle $\angle CFB$

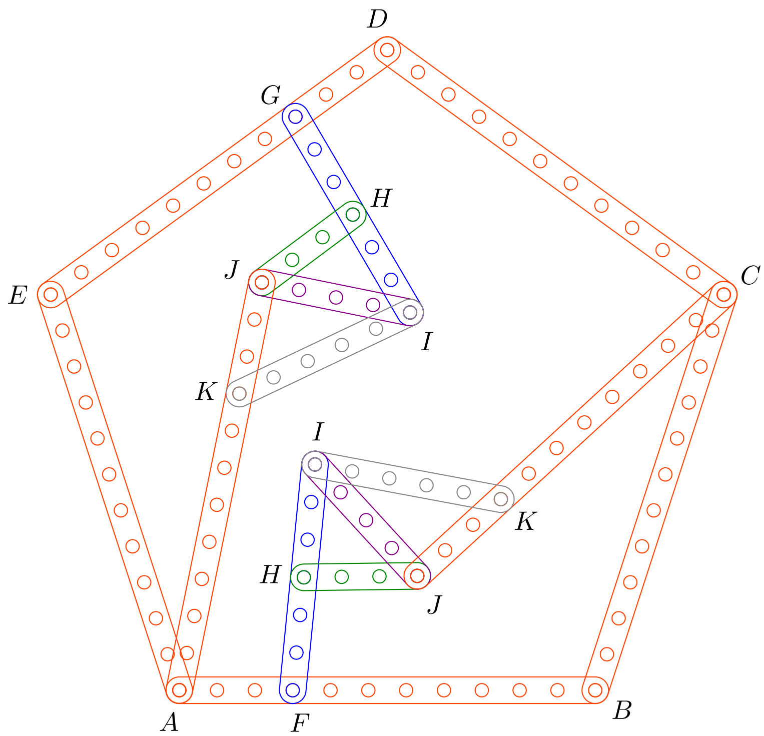

Figure 3: Regular pentagon size 11 made rigid with two internal clusters of five strips each.

for the drawing:

$$\overline{CF}^2 = \overline{BC}^2 + \overline{BF}^2 - 2(\overline{BC})(\overline{BF})\cos\left(\frac{3\pi}{5}\right) \tag{13}$$
$$= 11^2 + 8^2 - 2(11)(8)\left(\frac{1-\sqrt{5}}{4}\right) = 141 + 44\sqrt{5}$$
$$\overline{CF} = \sqrt{141 + 44\sqrt{5}} = 11 + 2\sqrt{5} \tag{14}$$
$$\cos(\angle CFB) = \frac{\overline{CF}^2 + \overline{BF}^2 - \overline{BC}^2}{2(\overline{CF})(\overline{BF})}$$
$$= \frac{141 + 44\sqrt{5} + 8^2 - 11^2}{2(11+2\sqrt{5})(8)} = \frac{21 + 11\sqrt{5}}{44 + 8\sqrt{5}} = \frac{121 + 79\sqrt{5}}{404} \tag{15}$$

2.1 Construction of distance $11 + 2\sqrt{5}$

A five strips cluster can create a rigid distance like $11+2\sqrt{5}$. In the figure, three strips $\overline{FI} = 2\overline{HJ}, \overline{FI} > \overline{IJ}$ builds a right angle $\angle FJI = \pi$, since triangle $\triangle IJH$ is isosceles ($\overline{FH} = \overline{HI} = \overline{JH}$). These three strips also build a distance $\overline{FJ} = \sqrt{\overline{FI}^2 - \overline{IJ}^2} = \sqrt{6^2 - 4^2} = 2\sqrt{5}$. Now we attach strip $\overline{CJ}$ making a second right triangle $\angle CJI = \pi$ using strip $\overline{IK} = 5$ as pythagorean diagonal ($\overline{JK} = 3, \overline{IJ} = 4$). We have two right triangles at vertice J so vertices F, J, C are collinear, so we can calculate the distance $\overline{FC} = \overline{CJ} + \overline{JF} = 11 + 2\sqrt{5}$. We repeat the five-strips cluster between vertices A, G preventing overlaps of any strips. Since the clusters are rigid we formed two rigid triangles $\triangle ABC, \triangle DEA$ so the pentagon is rigid.

The program found the next pentagon of this type is a lot bigger: $\overline{BC} = 246, \overline{BF} = 70, \overline{CF} = 41 + 105\sqrt{5}$.

3 Pentagon of size 10

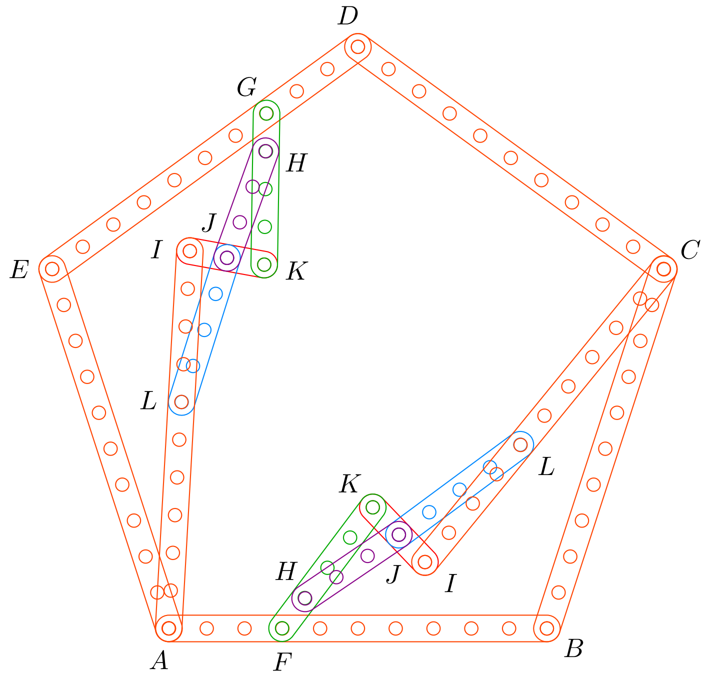

Figure 4: Regular pentagon size 10 made rigid with two internal clusters of five strips each.

Figure 4 show a rigid regular pentagon A, B, C, D, E of size 10. We calculate a diagonal joining two consecutive sides relative primes to have something exclusive to the size 10, we choose $\overline{BF} : \overline{BC} = 7 : 10$. With the law of cosines we calculate $\overline{CF}$. We calculate the angle $\angle CFB$ for the drawing:

$$\begin{aligned} \overline{CF}^2 &= \overline{BC}^2 + \overline{BF}^2 - 2(\overline{BC})(\overline{BF})\cos\left(\frac{3\pi}{5}\right) \\ &= 10^2 + 7^2 - 2(10)(7)\left(\frac{1-\sqrt{5}}{4}\right) = 114 + 35\sqrt{5} \end{aligned} \tag{16}$$

$$\overline{CF} = \sqrt{114 + 35\sqrt{5}} \tag{17}$$

$$\begin{aligned} \cos(\angle CFB) &= \frac{\overline{CF}^2 + \overline{BF}^2 - \overline{BC}^2}{2(\overline{CF})(\overline{BF})} \\ &= \frac{114 + 35\sqrt{5} + 7^2 - 10^2}{2(\sqrt{114+35\sqrt{5}})(7)} = \frac{9 + 5\sqrt{5}}{2\sqrt{114 + 35\sqrt{5}}} \end{aligned} \tag{18}$$

3.1 Construction of distance $\sqrt{114 + 35\sqrt{5}}$

Number $\sqrt{114 + 35\sqrt{5}}$ cannot be denested so we need to solve this with a cluster of strips. A program found a lot of solutions for this distance using five strips, so we choose one narrow enough to fit inside the decagon.

Figure 5 shows how to prove the cluster selected is correct. In the figure we have two isoscelles triangles $\triangle IKL_1$ and $\triangle JKH$. The sides IL_1 and KH are extended to double the original size to the vertices L_2 and H_1 building two right angles $\angle IKL_2$ and $\angle KJH_1$. The right triangles permit the calculation of the abscissas and ordinates of vertices C and F to calculate their distance.

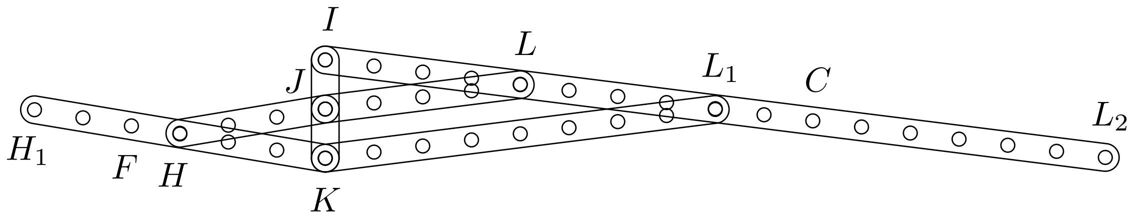

Figure 5: Construction of distance $\overline{CF} = \sqrt{114 + 35\sqrt{5}}$

From the figure we calculate $\overline{KL_2}$ and $\overline{JH_1}$ from their respective right triangles:

$$\overline{KL_2} = \sqrt{(\overline{IL_2})^2 - (IK)^2} = \sqrt{16^2 - 2^2} = 6\sqrt{7} \tag{19}$$

$$\overline{JH_1} = \sqrt{(\overline{KH_1})^2 - (KJ)^2} = \sqrt{6^2 - 1^2} = \sqrt{35} \tag{20}$$

Assuming vertice K is at the origin we can calculate the abscissas C_x, F_x and ordinates C_y, F_y of vertices C and F using as factors $c = \dfrac{\overline{IC}}{\overline{IL_2}} = \dfrac{10}{16} = \dfrac{5}{8}$ and $f = \dfrac{\overline{KF}}{\overline{KH1}} = \dfrac{4}{6} = \dfrac{2}{3}$:

$$C_x = +c(\overline{KL_2}) = \frac{5}{8}(6\sqrt{7}) = \frac{15}{4}\sqrt{7} \tag{21}$$

$$F_x = -f(\overline{JH_1}) = -\frac{2}{3}\sqrt{35} \tag{22}$$

$$C_y = +(\overline{KI}) - c(\overline{KI}) = 2 - \frac{5}{8}(2) = \frac{3}{4} \tag{23}$$

$$F_y = +f(\overline{KJ}) = \frac{2}{3}(1) = \frac{2}{3} \tag{24}$$

Finally we calculate the distance $\overline{CF}$:

$$\begin{aligned}
\overline{CF} &= \sqrt{(C_x - F_x)^2 + (C_y - F_y)^2} \\
&= \sqrt{\left(\frac{15}{4}\sqrt{7} + \frac{2}{3}\sqrt{35}\right)^2 + \left(\frac{3}{4} - \frac{2}{3}\right)^2} \\
&= \sqrt{\frac{1575}{16} + 35\sqrt{5} + \frac{140}{9} + \frac{1}{144}} = \sqrt{114 + 35\sqrt{5}}
\end{aligned} \tag{25}$$

A minimal part with five strips of the construction of figure 5 including only vertices F, H, I, J, K, L, C is used twice to make rigid the pentagon of side 10 as show in figure 4.

4 Pentagons of size 9

Figure 6 show a rigid regular pentagon A, B, C, D, E of size 9. The regular pentagon distance $\overline{CE}$ is called width and equals $\dfrac{1+\sqrt{5}}{2}\overline{AB}$. Is easy to note the distance $\overline{FG}$ equals the width of smaller pentagons size $9 - 1 = 8$ plus 1. So we have:

$$\begin{aligned}
\overline{FG} &= \frac{1+\sqrt{5}}{2}(\overline{BC} - \overline{FC}) + \overline{FC} \\
&= \frac{1+\sqrt{5}}{2}(9-1) + 1 = 5 + 4\sqrt{5}
\end{aligned} \tag{26}$$

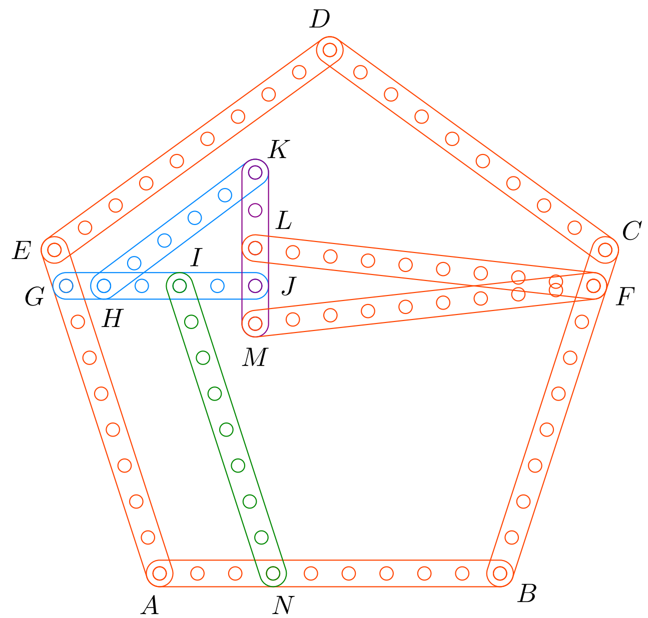

Figure 6: Regular pentagon size 9 (case a) made rigid with only six extra internal strips.

From the figure we see two right angles. Angle $\angle GJK = \pi$ because we have a Pythagorean triangle $\triangle HJK$. Angle $\angle FJM = \pi$ because we have an isosceles triangle $\triangle FLM$. The two right angles share vertice J so vertices G, J, F are collinear. First we calculate the distance $\overline{JF} = \sqrt{(LF)^2 - (LJ)^2} = \sqrt{9^2 - 1^2} = 4\sqrt{5}$ and finally the distance $\overline{GF} = \overline{GJ} + \overline{JF} = 5 + 4\sqrt{5}$ which matches the value in last equation above. To make rigid the pentagon we add strip $\overline{IN}$ parallel to side $\overline{GA}$.

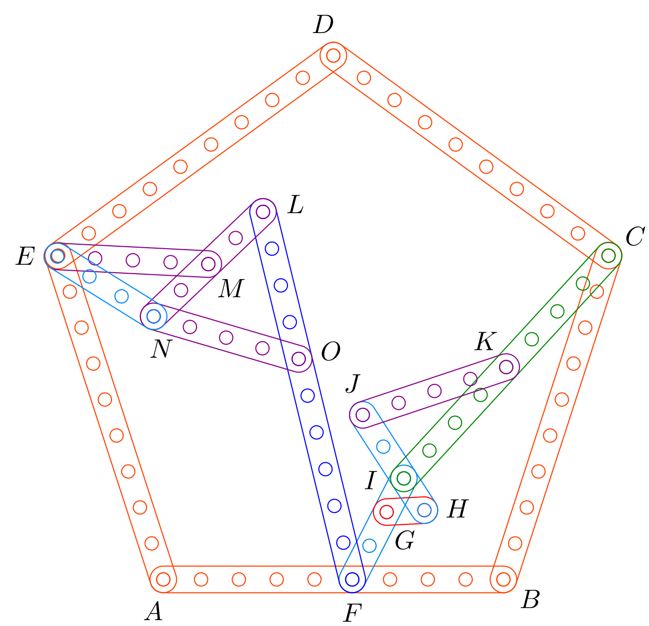

Figure 7: Regular pentagon size 9 (case b) made rigid with two internal clusters of five strips each.

Figure 7 show a regular pentagon A, B, C, D, E of size 9 made it rigid with the help of clusters fixing the distances $\overline{CF}$ and $\overline{EF}$. Pentagon size 9 is the smaller one with diagonals where consecutive side segments fractions are $\overline{BF}/\overline{BC} = \dfrac{4}{9}$ and $\overline{AF}/\overline{AE} = \dfrac{5}{9}$. We calculate the diagonals $\overline{CF}$, $\overline{EF}$ and the angles to side $\overline{AB}$ using the law of cosines and the internal pentagon angle $\theta = \angle FBC = \angle FAE = \frac{3\pi}{5}$ where $\cos\theta = \frac{1-\sqrt{5}}{4}$:

$$\overline{CF} = \sqrt{\overline{BC}^2 + \overline{BF}^2 - 2(\overline{BC})(\overline{BF})\cos\theta} = \sqrt{9^2 + 4^2 - 2(9)(4)\left(\frac{1-\sqrt{5}}{4}\right)} = \sqrt{79+18\sqrt{5}} \quad (27)$$

$$\cos(\angle CFB) = \frac{\overline{CF}^2 + \overline{BF}^2 - \overline{BC}^2}{2(\overline{CF})(\overline{BF})} = \frac{79+18\sqrt{5}+4^2-9^2}{2(\sqrt{79+18\sqrt{5}})(4)} = \frac{7+9\sqrt{5}}{4\sqrt{79+18\sqrt{5}}} \quad (28)$$

$$\overline{EF} = \sqrt{\overline{AE}^2 + \overline{AF}^2 - 2(\overline{AE})(\overline{AF})\cos\theta} = \sqrt{9^2 + 5^2 - 2(9)(5)\left(\frac{1-\sqrt{5}}{4}\right)} = \frac{\sqrt{334+90\sqrt{5}}}{2} \quad (29)$$

$$\cos(\angle EFA) = \frac{\overline{EF}^2 + \overline{AF}^2 - \overline{EA}^2}{2(\overline{EF})(\overline{AF})} = \frac{\frac{334+90\sqrt{5}}{4} + 5^2 - 9^2}{2\left(\frac{\sqrt{334+90\sqrt{5}}}{2}\right)(5)} = \frac{11+9\sqrt{5}}{2\sqrt{334+90\sqrt{5}}} \quad (30)$$

Our software found several options with five strips to build distances $\sqrt{79+18\sqrt{5}}$ and $\frac{\sqrt{334+90\sqrt{5}}}{2}$.

4.1 Construction of distance $\sqrt{79+18\sqrt{5}}$

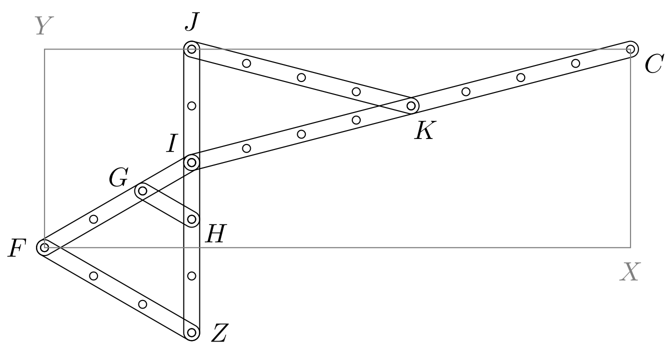

Figure 8: Construction of distance $\overline{OZ} = \sqrt{79+18\sqrt{5}}$

Figure 8 show one of several ways to build the distance $\sqrt{79+18\sqrt{5}}$. Equilateral triangle $\triangle OIH_2$ and isosceles $\triangle IJK$ share vertice I and the base $\overline{JH_2}$ which help to form rectangle $OXZY$ with base $\overline{OX}$ and height $\overline{OY}$ useful to calculate the diagonal $\overline{OZ}$:

$$\begin{aligned}
\overline{OX} &= \overline{YJ} + \overline{JZ} \\
&= \sqrt{\overline{OH_2}^2 - \left(\frac{\overline{IH_2}}{2}\right)^2} + \sqrt{\overline{IZ}^2 - \overline{IJ}^2} = \sqrt{3^2 - \left(\frac{3}{2}\right)^2} + \sqrt{8^2 - 2^2} = \frac{3\sqrt{3}}{2} + 2\sqrt{15} \\
\overline{OY} &= \overline{JI} + \frac{\overline{IH_2}}{2} = 2 + \frac{3}{2} = \frac{7}{2} \\
\overline{OZ} &= \sqrt{\overline{OX}^2 + \overline{OY}^2} = \sqrt{\left(\frac{3\sqrt{3}}{2} + 2\sqrt{15}\right)^2 + \left(\frac{7}{2}\right)^2} = \sqrt{79+18\sqrt{5}} \quad (31)
\end{aligned}$$

We use a smaller part of this construction, the five strips $\overline{OI}, \overline{KJ}, \overline{JH}, \overline{HL}, \overline{IZ}$, as a cluster to made rigid the consecutive strips $\overline{AB}, \overline{BC}$ of the pentagon of side 9 of figure 7.

4.2 Construction of distance $\dfrac{\sqrt{334+90\sqrt{5}}}{2}$

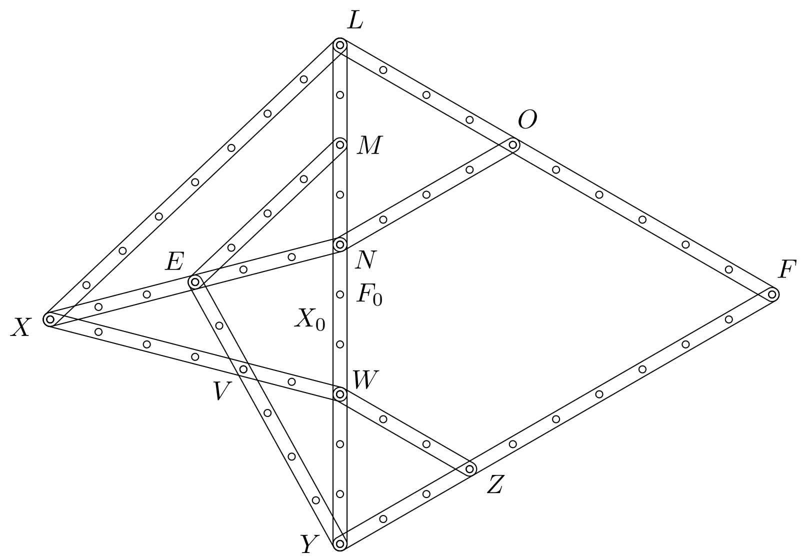

Figure 9: Construction of distance $\overline{EF} = \dfrac{\sqrt{334+90\sqrt{5}}}{2}$

Figure 9 show equilateral triangle $\triangle FLY$ and isoscelles triangle $\triangle NXW$ sharing strip $\overline{LY}$ which helps to calculate abscissas and ordinates of vertices E, F. Consider vertice Y located at the origin:

$$E_x = -\left(\frac{\overline{NE}}{\overline{NX}}\right)\overline{XX_0} = -\frac{3}{6}\sqrt{\overline{NX}^2 - \overline{NX_0}^2} = -\frac{1}{2}\sqrt{6^2 - \left(\frac{3}{2}\right)^2} = -\frac{3\sqrt{15}}{4} \tag{32}$$

$$E_y = \overline{YN} - \left(\frac{\overline{NE}}{\overline{NX}}\right)\overline{NX_0} = 6 - \left(\frac{3}{6}\right)\left(\frac{3}{2}\right) = \frac{21}{4} \tag{33}$$

$$F_x = \overline{F_0F} = \sqrt{\overline{YF}^2 - \overline{YF_0}^2} = \sqrt{10^2 - 5^2} = 5\sqrt{3} \tag{34}$$

$$F_y = \overline{YF_0} = 5 \tag{35}$$

$$\overline{EF} = \sqrt{(E_x - F_x)^2 + (E_y - F_y)^2} = \sqrt{\left(-\frac{3\sqrt{15}}{4} - 5\sqrt{3}\right)^2 + \left(\frac{21}{4} - 5\right)^2} = \frac{\sqrt{334+90\sqrt{5}}}{2} \tag{36}$$

We form a cluster from the last construction to be applied in the pentagon of side 9. We choose the five strips with vertices E, N, M, L, O, F. Is easy to prove strip $\overline{EM}$ is correct in the cluster comparing equal cosines at vertice Y for triangles $\triangle YVW, \triangle YEN, \triangle YEM$ using the law of cosines for each triangle.