Meccano fox-surd frame

https://github.com/heptagons/meccano/frames/fox-surd

Abstract

Meccano [1] fox-surd frame is a generalization of fox-frame[2] where two of the original five strips are no longer integers but surds which must be solved using several more strips.

1 Pentagons fox-surd

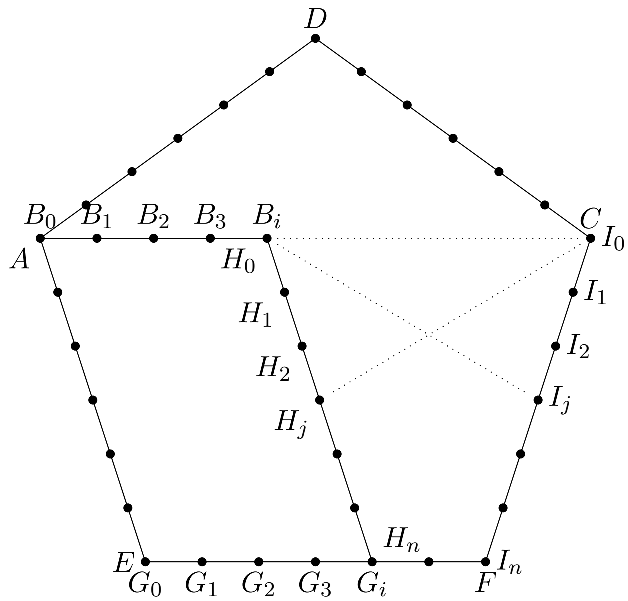

Figure 1: Pentagon of size n where each segment separated by circles represents a unit. We have a surd frame formed by the six points: B_i, I_0, H_j, I_j, H_n and I_n. By iterating the values $i, j = 0, ..., n$ we'll get diverse frames.

From figure 1 the fox-surd frame has three real strips of integer size:

- $\overline{B_iG_i}$ of size n.
- $\overline{G_iI_n}$ of size $n - i$, where $i = 0, ..., n$.
- $\overline{I_nI_0}$ of size n.

The other two strips are generic in the sense the sizes can be surds:

- $\overline{B_iI_j}$ of size $f(n, i, j)$, where $i, j = 0, ..., n$.
- $\overline{I_0H_j}$ of equal size of $\overline{B_iI_j}$.

[1] Meccano mathematics by 't Hooft
[2] Meccano fox frame

From the regular pentagon we know the main diagonal $\overline{AC}$ equals $\frac{1+\sqrt{5}}{4}n$ where n is the pentagon side size. We can calculate different segments of the main diagonal iterating $i = 0, ..., n$:

$$\begin{aligned} B_0 &\equiv A \\ \overline{B_0C} &= \frac{1+\sqrt{5}}{4}n && (1) \\ \overline{B_iC} &= \frac{1+\sqrt{5}}{4}n - i \\ &= \frac{n-4i}{4} + \frac{\sqrt{5}}{4}, \quad i = 0, ..., n && (2) \end{aligned}$$

From the regular pentagon we know the angle CB_iH_i equals $2\pi/5$ so we have:

$$\begin{aligned} \theta &\equiv \angle CB_iH_i && (3) \\ \cos\theta &= \cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4} && (4) \end{aligned}$$

1.1 Surds strips

Using the law of cosines we can calculate one of the frame surds $s_ij \equiv \overline{B_iH_j}$. We notice the value of $\overline{B_iH_i}$ equals i, and we'll use the values of $\overline{B_iC}$ from equation 2, and the cosine value from equation 4 to get:

$$\begin{aligned} s_i^2 &\equiv \overline{CH_i}^2 && (5) \\ &= \overline{B_iH_i}^2 + \overline{B_iC}^2 - 2\overline{B_iH_i} \times \overline{B_iC}\cos\theta \\ &= i^2 + \left(\frac{n-4i}{4} + \frac{\sqrt{5}}{4}\right)^2 - 2i\left(\frac{n-4i}{4} + \frac{\sqrt{5}}{4}\right)\left(\frac{-1+\sqrt{5}}{4}\right) && (6) \\ &= i^2 + \frac{1}{16}\left(n-4i+\sqrt{5}\right)^2 - \frac{2i}{16}\left(n-4i+\sqrt{5}\right)\left(-1+\sqrt{5}\right) && (7) \end{aligned}$$

We multiply both sides by 16 and substitute $x = n - 4i$:

$$\begin{aligned} (4s_i)^2 &= 16i^2 + x^2 + 2x\sqrt{5} + 5 - 2i\left(x+\sqrt{5}\right)\left(-1+\sqrt{5}\right) && (8) \\ &= 16i^2 + x^2 + 2x\sqrt{5} + 5 - 2i(-x + 5 + (x-1)\sqrt{5}) && (9) \\ &= 16i^2 + x^2 + 5 + 2ix - 10i + (2x - 2ix + 2i)\sqrt{5} && (10) \end{aligned}$$

We define two variables u and v in order to have $(4s_i)^2 = u + v\sqrt{5}$ and reduce x, so:

$$\begin{aligned} u &\equiv 16i^2 + x^2 + 5 + 2ix - 10i \\ &= 16i^2 + (n-4i)^2 + 5 + 2i(n-4i) - 10i \\ &= 16i^2 + n^2 - 8in + 16i^2 + 5 - 2in - 8i^2 \\ &= n^2 - 10in + 25i^2 - i^2 + 5 \\ &= (n-5i)^2 + 5 - i^2 && (11) \\ v &\equiv 2x - 2ix + 2i \\ &= 2x(1-i) + 2i \\ &= 2(n-4i)(1-i) + 2i && (12) \end{aligned}$$

Finally we have s_ij in function of n the side: