

Meccano four frame

<https://github.com/heptagons/meccano/frames/four>

Abstract

Four frame is a group of four rigid meccano ¹ strips.

1 Four frame

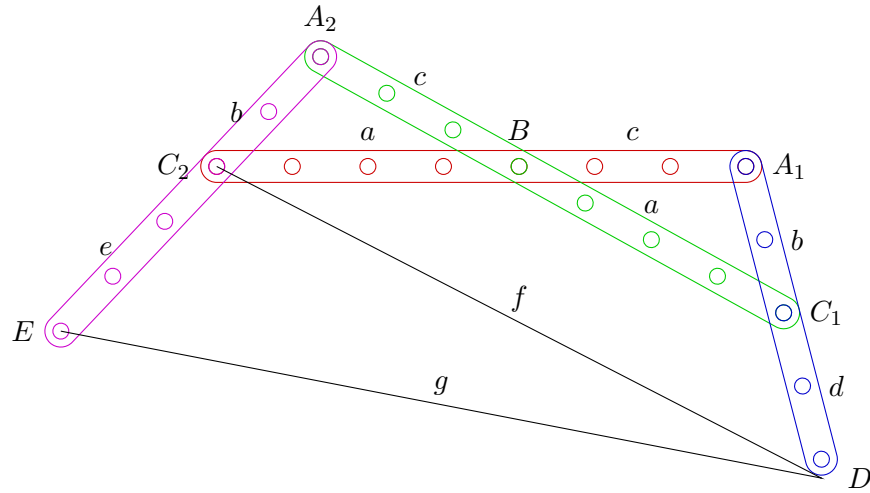


Figure 1: Antisymmetric four frame.

Figure 1 show the antisymmetric four-strips frame. From the figure we define $\alpha \equiv \angle BA_1C_1$ and define integers $m = b^2 + c^2 - a^2$ and $n = 2bc$ using the law of cosines, then we calculate $\cos \alpha$ and $\sin \alpha$:

$$(\alpha, m, n) \equiv (\angle BA_1C_1, b^2 + c^2 - a^2, 2bc) \quad (1)$$

$$\cos \alpha = \frac{m}{n} \quad (2)$$

$$\sin \alpha = \frac{\sqrt{n^2 - m^2}}{n} \quad (3)$$

From the figure 1 we define $\gamma \equiv \angle BA_2C_2$ and define integers $s = a^2 + b^2 - c^2$ and $t = 2ab$ and calculate $\cos \gamma$ and $\sin \gamma$:

$$(\gamma, s, t) \equiv (\angle BA_2C_2, a^2 + b^2 - c^2, 2ab) \quad (4)$$

$$\cos \gamma = \frac{s}{t} \quad (5)$$

$$\sin \gamma = \frac{\sqrt{t^2 - s^2}}{t} \quad (6)$$

¹ Meccano mathematics by 't Hooft

We calculate the distance $f = \overline{C_2D}$ with the law of cosines using angle α and defining integers $x = a + c$ and $y = b + d$:

$$(x, y) \equiv (a + c, b + d) \quad (7)$$

$$f^2 = (a + c)^2 + (b + d)^2 - 2(a + c)(b + d) \cos \alpha \quad (8)$$

$$= x^2 + y^2 - \frac{2mxy}{n} \quad (9)$$

$$f = \frac{\sqrt{n^2(x^2 + y^2) - 2mnxy}}{n} \quad (10)$$

We define a new integer $z \equiv n^2(x^2 + y^2) - 2mnxy$ so we have:

$$z \equiv n^2(x^2 + y^2) - 2mnxy \quad (11)$$

$$f = \frac{\sqrt{z}}{n} \quad (12)$$

We define angle $\theta \equiv \angle A_1C_2D$ and calculate $\cos \theta$ and $\sin \theta$:

$$\theta \equiv \angle A_1C_2D \quad (13)$$

$$\begin{aligned} \cos \theta &= \frac{(a + c)^2 + f^2 - (b + d)^2}{2(a + c)f} \\ &= \frac{x^2 + f^2 - y^2}{2xf} \\ &= \frac{x^2 + x^2 + y^2 - \frac{2mxy}{n} - y^2}{2x \frac{\sqrt{z}}{n}} \\ &= \frac{nx - my}{\sqrt{z}} \end{aligned} \quad (14)$$

$$\begin{aligned} \sin^2 \theta &= 1 - \cos^2 \theta = \frac{z - (nx - my)^2}{z} \\ &= \frac{n^2(x^2 + y^2) - 2mnxy - (nx - my)^2}{z} \\ &= \frac{n^2(x^2 + y^2) - 2mnxy - n^2x^2 + 2nxmy - m^2y^2}{z} \\ \sin \theta &= \frac{y\sqrt{n^2 - m^2}}{\sqrt{z}} \end{aligned} \quad (15)$$

We define angle $\phi \equiv \angle A_2C_2D$ and we note is the sum of angles $\theta + \gamma$ and we calculate $\cos \phi$:

$$\phi \equiv \angle A_2C_2D \quad (16)$$

$$= \theta + \gamma \quad (17)$$

$$\cos \phi = \cos(\theta + \gamma) \quad (18)$$

$$\begin{aligned} &= \cos \theta \cos \gamma - \sin \theta \sin \gamma \\ &= \frac{(nx - my)s}{\sqrt{z}t} - \frac{(y\sqrt{n^2 - m^2})\sqrt{t^2 - s^2}}{\sqrt{z}t} \\ &= \frac{s(nx - my) - y\sqrt{(n^2 - m^2)(t^2 - s^2)}}{\sqrt{z}t} \end{aligned} \quad (19)$$

We simplify $\sqrt{(n^2 - m^2)(t^2 - s^2)}$:

$$\begin{aligned}
(n^2 - m^2)(t^2 - s^2) &= (n - m)(n + m)(t - s)(t + s) \\
&= (2bc - (b^2 + c^2 - a^2))(2bc + b^2 + c^2 - a^2)(2ab - (a^2 + b^2 - c^2))(2ab + a^2 + b^2 - c^2) \\
&= (a^2 - (b - c)^2)((b + c)^2 - a^2)(c^2 - (a - b)^2)((a + b)^2 - c^2) \\
&= (a + b - c)(a - b + c)(b + c + a)(b + c - a)(c + a - b)(c - a + b)(a + b + c)(a + b - c) \\
&= (a + b + c)^2(a + b - c)^2(a - b + c)^2(-a + b + c)^2 \\
\sqrt{(n^2 - m^2)(t^2 - s^2)} &= (a + b + c)(a + b - c)(a - b + c)(-a + b + c) \\
&= ((a + b)^2 - c^2)(c^2 - (a - b)^2) \\
&= (s + t)(t - s) \\
&= (t^2 - s^2)
\end{aligned} \tag{20}$$

We substitute equation 20 into equation 19 and we get:

$$\cos \phi = \frac{s(nx - my) - y(t^2 - s^2)}{\sqrt{z}t} \tag{21}$$

From the figure we define angle $\psi \equiv \angle DC_2E$ and we note equals angle $\pi - \phi$, so we have:

$$\psi \equiv \angle DC_2E \tag{22}$$

$$= \pi - \phi \tag{23}$$

$$\cos \psi = \cos(\pi - \phi) \tag{24}$$

$$= -\cos \phi$$

$$= \frac{-s(nx - my) + y(t^2 - s^2)}{\sqrt{z}t} \tag{25}$$

Finally with $\cos \psi$, e and f we can calculate distance $g = \overline{ED}$:

$$g^2 = e^2 + f^2 - 2ef \cos \psi \tag{26}$$

$$= e^2 + x^2 + y^2 - \frac{2mxy}{n} - 2e \left(\frac{\sqrt{z}}{n} \right) \left(\frac{-s(nx - my) + y(t^2 - s^2)}{\sqrt{z}t} \right) \tag{27}$$

$$= e^2 + x^2 + y^2 - \frac{2mxy}{n} + 2e \frac{s(nx - my) - y(t^2 - s^2)}{nt} \tag{28}$$

$$= e^2 + x^2 + y^2 - \frac{2mxy}{2bc} + 2e \frac{s(nx - my) - y(t^2 - s^2)}{4ab^2c} \tag{29}$$

$$= \frac{4a^2b^2c^2(e^2 + x^2 + y^2) - (2a^2bc)(2mxy) + 2aces(nx - my) - 2acey(t^2 - s^2)}{4a^2b^2c^2}$$

$$= \frac{2ac(2ab^2c(e^2 + x^2 + y^2) - 2abmxy + es(nx - my) - ey(t^2 - s^2))}{4a^2b^2c^2}$$

$$g = \frac{\sqrt{2ac(2ab^2c(e^2 + x^2 + y^2) - 2abmxy + es(nx - my) - ey(t^2 - s^2))}}{2abc} \tag{30}$$

1.1 Antisymmetric four frame software

From the last equation of g we identify two **input** integers i_1, i_2 which are used to get $g(i)$. Then the nested radicals software will return square-free **output** integers z_1, z_2, z_3 as $g(z)$:

$$i_1 = 2abc \tag{31}$$

$$i_2 = 2ac(2ab^2c(e^2 + x^2 + y^2) - 2abmxy + es(nx - my) - ey(t^2 - s^2)) \tag{32}$$

$$g(i) = \frac{\sqrt{i_2}}{i_1} \tag{33}$$

$$g(z) = \frac{z_2\sqrt{z_3}}{z_1} \tag{34}$$

where m, n are calculated with equations 1, x, y are calculated with equations 7 and s, t are calculated with equations 4.