Meccano triangles

https://github.com/heptagons/meccano/nest

Abstract

We construct meccano triangles. Basic triangles has the three sides as integers and calculate the internal diagonal distances. Such diagonals then are used as the new side of more complicated triangles and then again we calculate new distances formed and so on. Eventually we expect to find certain angles joining the triangles which can be used to construct regular polygons or more figures.

1 Triangles (a, b, c)

Triangles (a, b, c) have three sides a, b and c where $a, b, c \in \mathbb{N}$. To avoid repetitions and get only valid triangles, we consider only the cases:

$$a \geq b \geq c \tag{1}$$

$$a < b + c \tag{2}$$

The cosines of the three angles of triangle(a, b, c) are rationals $\cos A, \cos B, \cos C \in \mathbb{Q}$:

$$\cos \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} \dfrac{b^2 + c^2 - a^2}{2bc} \\ \dfrac{c^2 + a^2 - b^2}{2ca} \\ \dfrac{a^2 + b^2 - c^2}{2ab} \end{pmatrix} \equiv \begin{pmatrix} \frac{a_n}{a_d} \\ \frac{b_n}{b_d} \\ \frac{c_n}{c_d} \end{pmatrix} \in \mathbb{Q} \tag{3}$$

Where numerators are integers $a_n, b_n, c_n \in \mathbb{Z}$ and denominators are naturals $a_d, b_d, c_d \in \mathbb{N}$ and always $x_n <= x_d$ for $x = \{a, b, c\}$.

The sines of the three angles are algebraic $\sin A, \sin B, \sin C \in \mathbb{A}$:

$$\sin \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} \sqrt{1 - \cos^2 A} \\ \sqrt{1 - \cos^2 B} \\ \sqrt{1 - \cos^2 C} \end{pmatrix} = \begin{pmatrix} \dfrac{\sqrt{a_d^2 - a_n^2}}{a_d} \\ \dfrac{\sqrt{b_d^2 - b_n^2}}{b_d} \\ \dfrac{\sqrt{c_d^2 - c_n^2}}{c_d} \end{pmatrix} \equiv \begin{pmatrix} \frac{\sqrt{a_s}}{a_d} \\ \frac{\sqrt{b_s}}{b_d} \\ \frac{\sqrt{c_s}}{c_d} \end{pmatrix} \in \mathbb{A} \tag{4}$$

Where numbers inside square roots are naturals $a_s, b_s, c_s \in \mathbb{N}$ and $x_s = x_d^2 - x_n^2 \geq 0$ for $x = \{a, b, c\}$.

Table 1: Small triangles (a,b,c) vertices A,B,C cosines and sines.

	(a,b,c)	$\cos A$	$\cos B$	$\cos C$	$\sin A$	$\sin B$	$\sin C$
1	(1,1,1)	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
2	(2,2,1)	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{7}{8}$	$\frac{\sqrt{15}}{4}$	$\frac{\sqrt{15}}{4}$	$\frac{\sqrt{15}}{8}$
3	(3,2,2)	$-\frac{1}{8}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3\sqrt{7}}{8}$	$\frac{\sqrt{7}}{4}$	$\frac{\sqrt{7}}{4}$
4	(3,3,1)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{17}{18}$	$\frac{\sqrt{35}}{6}$	$\frac{\sqrt{35}}{6}$	$\frac{\sqrt{35}}{18}$
5	(3,3,2)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{7}{9}$	$\frac{2\sqrt{2}}{3}$	$\frac{2\sqrt{2}}{3}$	$\frac{4\sqrt{2}}{9}$
6	(4,3,2)	$-\frac{1}{4}$	$\frac{11}{16}$	$\frac{7}{8}$	$\frac{\sqrt{15}}{4}$	$\frac{3\sqrt{15}}{16}$	$\frac{\sqrt{15}}{8}$
7	(4,3,3)	$\frac{1}{9}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{4\sqrt{5}}{9}$	$\frac{\sqrt{5}}{3}$	$\frac{\sqrt{5}}{3}$
8	(4,4,1)	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{31}{32}$	$\frac{3\sqrt{7}}{8}$	$\frac{3\sqrt{7}}{8}$	$\frac{3\sqrt{7}}{32}$
9	(4,4,3)	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{23}{32}$	$\frac{\sqrt{55}}{8}$	$\frac{\sqrt{55}}{8}$	$\frac{3\sqrt{55}}{32}$
10	(5,3,3)	$-\frac{7}{18}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5\sqrt{11}}{18}$	$\frac{\sqrt{11}}{6}$	$\frac{\sqrt{11}}{6}$
11	(5,4,2)	$-\frac{5}{16}$	$\frac{13}{20}$	$\frac{37}{40}$	$\frac{\sqrt{231}}{16}$	$\frac{\sqrt{231}}{20}$	$\frac{\sqrt{231}}{40}$
12	(5,4,3)	0	$\frac{3}{5}$	$\frac{4}{5}$	1	$\frac{4}{5}$	$\frac{3}{5}$
13	(5,4,4)	$\frac{7}{32}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{5\sqrt{39}}{32}$	$\frac{\sqrt{39}}{8}$	$\frac{\sqrt{39}}{8}$
14	(5,5,1)	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{49}{50}$	$\frac{3\sqrt{11}}{10}$	$\frac{3\sqrt{11}}{10}$	$\frac{3\sqrt{11}}{50}$
15	(5,5,2)	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{23}{25}$	$\frac{2\sqrt{6}}{5}$	$\frac{2\sqrt{6}}{5}$	$\frac{4\sqrt{6}}{25}$
16	(5,5,3)	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{41}{50}$	$\frac{\sqrt{91}}{10}$	$\frac{\sqrt{91}}{10}$	$\frac{3\sqrt{91}}{50}$
17	(5,5,4)	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{17}{25}$	$\frac{\sqrt{21}}{5}$	$\frac{\sqrt{21}}{5}$	$\frac{4\sqrt{21}}{25}$
18	(7,6,5)	$\frac{1}{5}$	$\frac{19}{35}$	$\frac{5}{7}$	$\frac{2\sqrt{6}}{5}$	$\frac{12\sqrt{6}}{35}$	$\frac{2\sqrt{6}}{7}$

Data from `github.com/heptagons/meccano/nest/t_test.go TestTCosSin`

1.1 Triangle (a,b,c) diagonals

Within the triangle (a,b,c) we can form diagonals as lines joining integer points of a given side a,b,c with others points of another side. To calculate the diagonals we use the law of cosines. Using equation 3 we can calculate diagonals $\overline{b_i c_j}$:

$$\begin{aligned}\overline{b_i c_j} &= \sqrt{i^2+j^2-2ij\cos A} = \sqrt{i^2+j^2-2ij\frac{a_n}{a_d}} \\ &= \frac{\sqrt{a_d^2(i^2+j^2)-2a_n ij}}{a_d}\end{aligned} \tag{5}$$

where $i,j \in \mathbb{N}$ are sides points positions starting with 1 (don't confuse i with $\sqrt{-1}$) and$1 \le i \le b$, $1 \le j \le c$ and $i \ge j$. For the whole triangle we have:

$$\begin{pmatrix} \overline{b_i c_j} \\ \overline{a_i c_j} \\ \overline{a_i b_j} \end{pmatrix} = \begin{pmatrix} \dfrac{\sqrt{a_d^2(i^2+j^2)-2a_n ij}}{a_d} \\ \dfrac{\sqrt{b_d^2(i^2+j^2)-2b_n ij}}{b_d} \\ \dfrac{\sqrt{c_d^2(i^2+j^2)-2c_n ij}}{c_d} \end{pmatrix} \in \mathbb{A} \tag{6}$$

1.2 Example triangle (7,6,5)

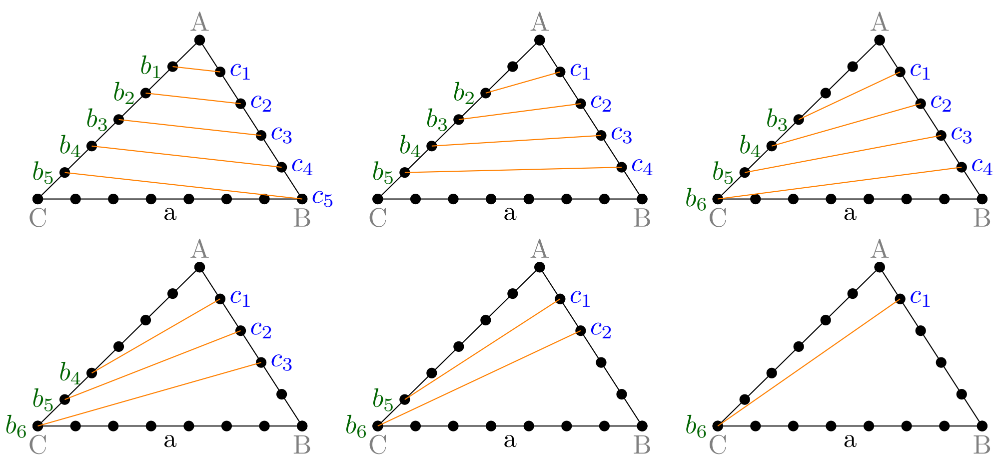

Figure 1: Triangle $(7,6,5)$ $b_i c_j$ diagonals $(i \geq j)$.

Figure 1 show triangle $(7,6,5)$ diagonals $b_m c_n$ for vertex A. The diagonals values are set in a first matrix with columns $b_1, ..., b_6$ and rows $c_1, ..., c_5$. Empty cells are repetitions.

$$b_{1:6}c_{1:5} = \begin{pmatrix} \frac{2\sqrt{10}}{5} & \frac{\sqrt{105}}{5} & \frac{2\sqrt{55}}{5} & \frac{\sqrt{385}}{5} & 2\sqrt{6} & \frac{\sqrt{865}}{5} \\ & \frac{4\sqrt{10}}{5} & \frac{\sqrt{265}}{5} & \frac{2\sqrt{105}}{5} & 5 & \frac{4\sqrt{55}}{5} \\ & & \frac{6\sqrt{10}}{5} & \frac{\sqrt{505}}{5} & 2\sqrt{7} & \frac{3\sqrt{105}}{5} \\ & & & \frac{8\sqrt{10}}{5} & \sqrt{33} & \frac{2\sqrt{265}}{5} \\ & & & & 2\sqrt{10} & \end{pmatrix} \tag{7}$$

Figure 2 show triangle $(7,6,5)$ diagonals $a_i c_j$ for vertex B. The diagonals are set in a second matrix with columns $a_1, ..., a_7$ and rows $c_1, ..., c_5$. Emtpy cells are repetitions. Values at column 7 are repeated and already accounted in previous matrix.

$$a_{1:7}c_{1:5} \begin{pmatrix} \frac{4\sqrt{70}}{35} & \frac{3\sqrt{385}}{35} & \frac{2\sqrt{2065}}{35} & \frac{\sqrt{15505}}{35} & \frac{12\sqrt{7}}{7} & \frac{\sqrt{37345}}{35} & \frac{2\sqrt{265}}{5} \\ & \frac{8\sqrt{70}}{35} & \frac{\sqrt{7945}}{35} & \frac{6\sqrt{385}}{35} & \frac{\sqrt{889}}{7} & \frac{4\sqrt{2065}}{35} & \frac{3\sqrt{105}}{5} \\ & & \frac{12\sqrt{70}}{35} & \frac{\sqrt{14665}}{35} & \frac{2\sqrt{217}}{7} & \frac{9\sqrt{385}}{35} & \frac{4\sqrt{55}}{5} \\ & & & \frac{16\sqrt{70}}{35} & \frac{3\sqrt{105}}{7} & \frac{2\sqrt{7945}}{35} & \frac{\sqrt{865}}{5} \\ & & & & \frac{4\sqrt{70}}{7} & \frac{\sqrt{1393}}{7} & \end{pmatrix} \tag{8}$$

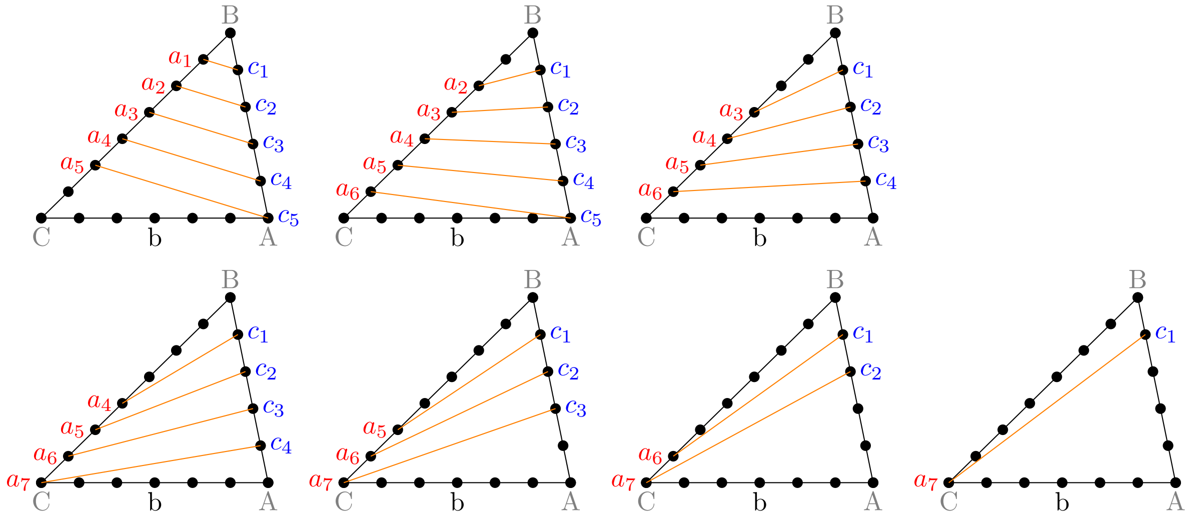

Figure 2: Triangle $(7, 6, 5)$, $a_i c_j$ diagonals $(i \geq j)$.

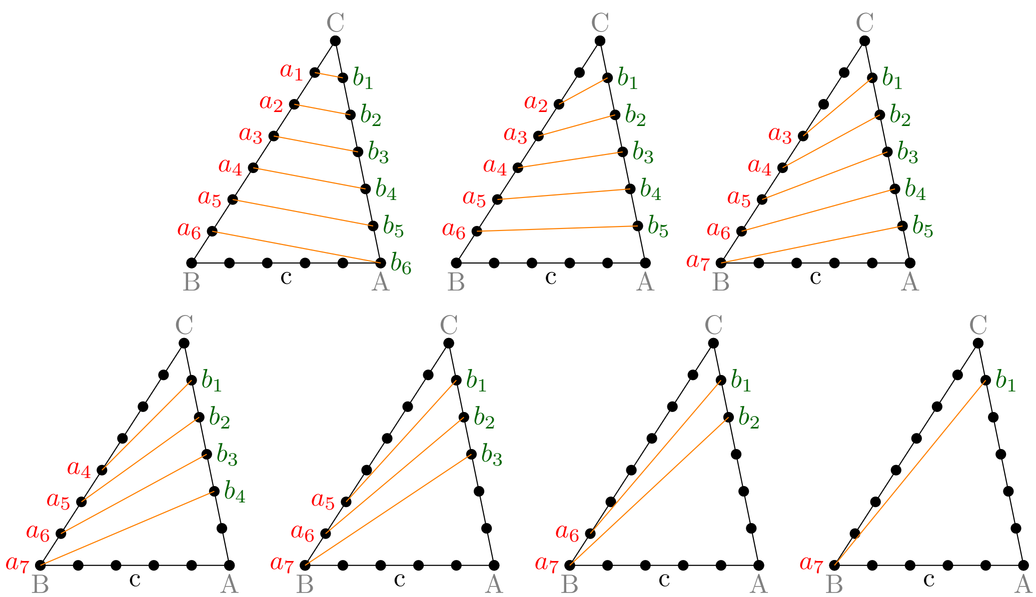

Figure 3: Triangle $(7, 6, 5)$, $a_i b_j$ diagonals $(i \geq j)$

Figure 3 show triangle $(7, 6, 5)$ diagonals $a_i b_j$ for vertex C. The diagonals are set in a third matrix with columns $a_1, ..., a_7$ and rows $b_1, ..., b_6$. Empty cells are repetitions. Values at columns 6 and 7 are repeated and already in previous matrices.

$$a_{1:7}b_{1:6} = \begin{pmatrix} \frac{2\sqrt{7}}{7} & \frac{\sqrt{105}}{7} & \frac{2\sqrt{70}}{7} & \frac{\sqrt{553}}{7} & \frac{2\sqrt{231}}{7} & \frac{\sqrt{1393}}{7} & 2\sqrt{10} \\ & \frac{4\sqrt{7}}{7} & \frac{\sqrt{217}}{7} & \frac{2\sqrt{105}}{7} & \frac{\sqrt{721}}{7} & \frac{4\sqrt{70}}{7} & \sqrt{33} \\ & & \frac{6\sqrt{7}}{7} & \frac{\sqrt{385}}{7} & \frac{2\sqrt{154}}{7} & \frac{3\sqrt{105}}{7} & 2\sqrt{7} \\ & & & \frac{8\sqrt{7}}{7} & \frac{\sqrt{609}}{7} & \frac{2\sqrt{217}}{7} & 5 \\ & & & & \frac{10\sqrt{7}}{7} & \frac{\sqrt{889}}{7} & 2\sqrt{6} \\ & & & & & \frac{12\sqrt{7}}{7} & \end{pmatrix} \tag{9}$$

The values of the matrices are calculated with the code: `github.com/heptagons/meccano/nest/t_test.go TestT765diags`.

2 Triangles$(\sqrt{\alpha}, b, c)$

Triangles$(\sqrt{\alpha}, b, c)$ have three sides with lengths $a = \sqrt{\alpha}$, b and c where $\alpha, b, c \in \mathbb{N}$ and α is square-free. We have:

$$\sqrt{\alpha} > b \geq c \qquad \Longrightarrow \alpha > b^2 \geq c^2 \tag{10}$$
$$\sqrt{\alpha} < b + c \qquad \Longrightarrow \alpha < (b+c)^2 \tag{11}$$

We calculate the triangle cosines. $\cos A_\alpha$ is rational and $\cos B_\alpha$ and $\cos C_\alpha$ are algebraic:

$$\cos A_\alpha = \frac{b^2 + c^2 - (\sqrt{\alpha})^2}{2bc} = \frac{b^2 + c^2 - \alpha}{2bc} \qquad \equiv \frac{\alpha_n}{\alpha_d} \in \mathbb{Q} \tag{12}$$
$$\cos B_\alpha = \frac{(\sqrt{\alpha})^2 + c^2 - b^2}{2\sqrt{\alpha}c} = \frac{(\alpha + c^2 - b^2)\sqrt{\alpha}}{2\alpha c} \qquad \in \mathbb{A} \tag{13}$$
$$\cos C_\alpha = \frac{(\sqrt{\alpha})^2 + b^2 - c^2}{2\sqrt{\alpha}b} = \frac{(\alpha + b^2 - c^2)\sqrt{\alpha}}{2\alpha b} \qquad \in \mathbb{A} \tag{14}$$

2.1 Triangle $(\sqrt{\alpha}, b, c)$ diagonals

The only possible diagonals are for sides with integers points, that is segments $\overline{b_i c_j}$. Using the law of cosines:

$$\begin{aligned} \overline{b_i c_j} &= \sqrt{i^2 + j^2 - 2ij \cos A_\alpha} \\ &= \sqrt{i^2 + j^2 - 2ij\frac{\alpha_n}{\alpha_d}} \\ &= \frac{\sqrt{\alpha_d^2(i^2 + j^2) - 2\alpha_n ij}}{\alpha_d} \qquad \in \mathbb{A} \end{aligned} \tag{15}$$

where $1 \leq i \leq b$, $1 \leq j \leq c$ and $i \geq j$.

2.2 Example triangles$(2\sqrt{6}, b, c)$

In this case $\sqrt{\alpha} = 2\sqrt{6}$ so $\alpha = 24$. Then sets $i = j = \{1, 2, 3, 4\}$ because sets $b^2 = c^2 = \{1, 4, 9, 16\} < 24$. We form a matrix with the values $(b + c)^2$ and satisfying $b \geq c$:

$$(b_i + c_j)^2 = \begin{array}{c} \\ c=1 \\ c=2 \\ c=3 \\ c=4 \end{array} \begin{array}{c} \begin{array}{cccc} b=1 & b=2 & b=3 & b=4 \end{array} \\ \begin{pmatrix} 2 & 9 & 16 & 25 \\ \times & 16 & 25 & 36 \\ \times & \times & 36 & 49 \\ \times & \times & \times & 64 \end{pmatrix} \end{array} \tag{16}$$

Then we remove the cells that don't satisfy the condition $\alpha < (b + c)^2$:

$$(b_m + c_n)^2 = \begin{array}{c} \\ c=1 \\ c=2 \\ c=3 \\ c=4 \end{array} \begin{array}{c} \begin{array}{cccc} b=1 & b=2 & b=3 & b=4 \end{array} \\ \begin{pmatrix} \times & \times & \times & 25 \\ & \times & 25 & 36 \\ & & 36 & 49 \\ & & & 64 \end{pmatrix} \end{array} \tag{17}$$

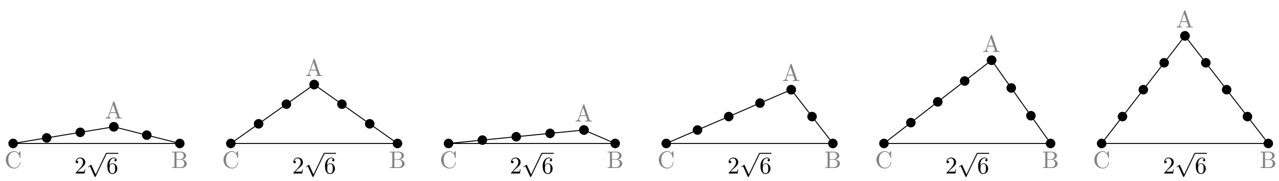

Figure 4: All triangles with sides $a = 2\sqrt{6} > b \geq c$

Each remaining cell in the matrix corresponds to a particular triangle:

$$(2\sqrt{6}, b, c) = \begin{array}{c} \\ (2\sqrt{6}, 3, 2) \\ (2\sqrt{6}, 3, 3) \\ (2\sqrt{6}, 4, 1) \\ (2\sqrt{6}, 4, 2) \\ (2\sqrt{6}, 4, 3) \\ (2\sqrt{6}, 4, 4) \end{array} \begin{array}{c} \begin{array}{ccc} \cos A_\alpha & \cos B_\alpha & \cos C_\alpha \end{array} \\ \begin{pmatrix} -\frac{11}{12} & \frac{19\sqrt{6}}{48} & \frac{29\sqrt{6}}{72} \\ -\frac{1}{3} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{3} \\ -\frac{7}{8} & \frac{3\sqrt{6}}{8} & \frac{13\sqrt{6}}{32} \\ -\frac{1}{4} & \frac{\sqrt{6}}{4} & \frac{3\sqrt{6}}{8} \\ \frac{1}{24} & \frac{17\sqrt{6}}{72} & \frac{31\sqrt{6}}{96} \\ \frac{1}{4} & \frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} \end{pmatrix} \end{array} \tag{18}$$

Figure 4 show the triangles $(2\sqrt{6}, b, c)$. The cosines are calculated by code at:

`github.com/heptagons/meccano/nest/t_test.go TestTslursA`

3 Triangles$(a, \sqrt{\beta}, c)$

Triangles$(a, \sqrt{\beta}, c)$ have the three sides a, $b = \sqrt{\beta}$ and c where $a, \beta, c \in \mathbb{N}$ and β is square-free. We have:

$$a > \sqrt{\beta} > c \qquad \Longrightarrow \; a^2 > \beta > c^2 \tag{19}$$

$$a < \sqrt{\beta} + c \qquad \Longrightarrow \; (a - c)^2 < \beta \tag{20}$$

We calculate the triangle cosines:

$$\cos A_\beta = \frac{(\sqrt{\beta})^2 + c^2 - a^2}{2\sqrt{\beta}c} = \frac{(\beta + c^2 - a^2)\sqrt{\beta}}{2\beta c} \qquad \in \mathbb{A} \tag{21}$$

$$\cos B_\beta = \frac{a^2 + c^2 - (\sqrt{\beta})^2}{2ac} = \frac{a^2 + c^2 - \beta}{2ac} \qquad \equiv \frac{\beta_n}{\beta_d} \in \mathbb{Q} \tag{22}$$

$$\cos C_\beta = \frac{a^2 + (\sqrt{\beta})^2 - c^2}{2a\sqrt{\beta}} = \frac{(a^2 + \beta - c^2)\sqrt{\beta}}{2a\beta} \qquad \in \mathbb{A} \tag{23}$$

3.1 Triangle $(a, \sqrt{\beta}, c)$ diagonals

The only possible diagonals are for sides with integers points, that is segments $\overline{a_i c_j}$. Using the law of cosines:

$$\overline{a_i c_j} = \sqrt{i^2 + j^2 - 2ij \cos B_\beta} \tag{24}$$

$$= \sqrt{i^2 + j^2 - 2ij \frac{\beta_n}{\beta_d}} \tag{25}$$

$$= \frac{\sqrt{\beta_d^2 (i^2 + j^2) - 2\beta_n ij}}{\beta_d} \qquad \in \mathbb{A} \tag{26}$$

where $1 \leq i \leq a$, $1 \leq j \leq c$ and $i \geq j$.

3.2 Example triangles$(a, 2\sqrt{6}, c)$

In this case $\sqrt{\beta} = 2\sqrt{6}$ so $\beta = 24$. Then $i = \{5, 6, 7, ...\}$ because $a^2 = \{25, 36, 49, ...\} > 24$ and $j = \{1, 2, 3, 4\}$ because $c^2 = \{1, 4, 9, 16\} < 24$. We form a matrix with the values $(a - c)^2$:

$$(a_i - c_j)^2 = \begin{array}{c} \\ c=1 \\ c=2 \\ c=3 \\ c=4 \end{array} \begin{array}{c} \begin{array}{cccccc} a=5 & a=6 & a=7 & a=8 & a=9 & ... \end{array} \\ \begin{pmatrix} 16 & 25 & 36 & 49 & 64 & ... \\ 9 & 16 & 25 & 36 & 49 & ... \\ 4 & 9 & 16 & 25 & 36 & ... \\ 1 & 4 & 9 & 16 & 25 & ... \end{pmatrix} \end{array} \tag{27}$$

We remove cells which don't satisfy the condition $(a - c)^2 < \beta = 24$:

$$(a_i - c_j)^2 = \begin{array}{c} \\ c=1 \\ c=2 \\ c=3 \\ c=4 \end{array} \begin{array}{c} \begin{array}{cccccc} a=5 & a=6 & a=7 & a=8 & a=9 & ... \end{array} \\ \begin{pmatrix} 16 & \times & \times & \times & \times & ... \\ 9 & 16 & \times & \times & \times & ... \\ 4 & 9 & 16 & \times & \times & ... \\ 1 & 4 & 9 & 16 & \times & ... \end{pmatrix} \end{array} \tag{28}$$

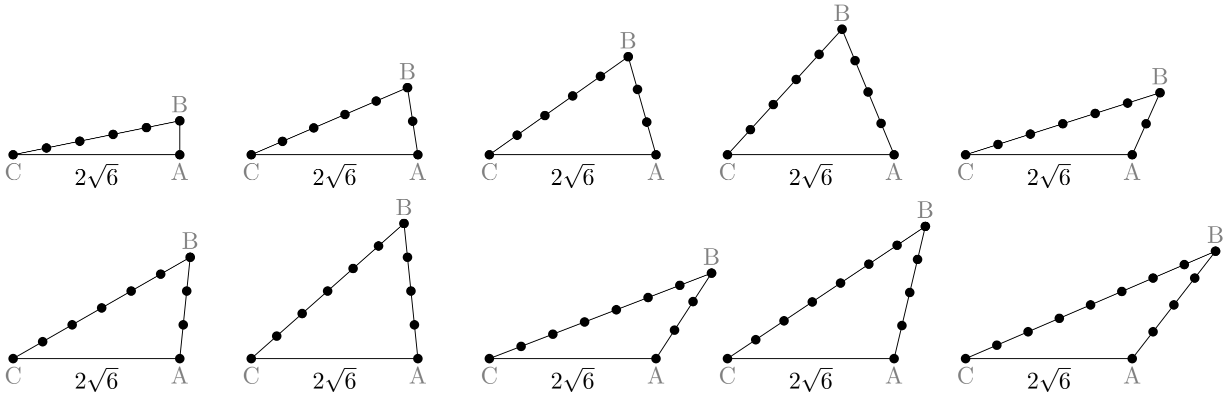

Figure 5: All triangles with sides $a > b = 2\sqrt{6} > c$

So we have ten triangles are valid:

$$(a, 2\sqrt{6}, c) = \begin{array}{c} \\ (5,2\sqrt{6},1) \\ (5,2\sqrt{6},2) \\ (5,2\sqrt{6},3) \\ (5,2\sqrt{6},4) \\ (6,2\sqrt{6},2) \\ (6,2\sqrt{6},3) \\ (6,2\sqrt{6},4) \\ (7,2\sqrt{6},3) \\ (7,2\sqrt{6},4) \\ (8,2\sqrt{6},4) \end{array} \begin{array}{c} \begin{array}{ccc} \cos A_\beta & \cos B_\beta & \cos C_\beta \end{array} \\ \begin{pmatrix} 0 & \frac{1}{5} & \frac{2\sqrt{6}}{5} \\ \frac{\sqrt{6}}{16} & \frac{1}{4} & \frac{3\sqrt{6}}{8} \\ \frac{\sqrt{6}}{9} & \frac{1}{3} & \frac{\sqrt{6}}{3} \\ \frac{5\sqrt{6}}{32} & \frac{17}{40} & \frac{11\sqrt{6}}{40} \\ \frac{\sqrt{6}}{6} & \frac{2}{3} & \frac{7\sqrt{6}}{18} \\ \frac{\sqrt{6}}{24} & \frac{7}{12} & \frac{17\sqrt{6}}{48} \\ \frac{\sqrt{6}}{24} & \frac{7}{12} & \frac{11\sqrt{6}}{36} \\ -\frac{2\sqrt{6}}{9} & \frac{17}{21} & \frac{8\sqrt{6}}{21} \\ -\frac{3\sqrt{6}}{32} & \frac{41}{56} & \frac{19\sqrt{6}}{56} \\ \frac{\sqrt{6}}{4} & \frac{7}{8} & \frac{3\sqrt{6}}{8} \end{pmatrix} \end{array} \tag{29}$$

Figure 5 show the triangles $(a, 2\sqrt{6}, c)$. The cosines are calculated by code at:

`github.com/heptagons/meccano/nest/t_test.go TestTslursB`

4 Triangles$(a, b, \sqrt{\gamma})$

Triangles$(a, b, \sqrt{\gamma})$ have three sides a, b, $\sqrt{\gamma}$ where $a, b, \gamma \in \mathbb{N}$ and γ is square-free. We have:

$$a \geq b > \sqrt{\gamma} \qquad \Longrightarrow \quad a^2 \geq b^2 > \gamma \tag{30}$$

$$a < b + \sqrt{\gamma} \qquad \Longrightarrow \quad (a-b)^2 < \gamma \tag{31}$$

We calculate the triangle cosines:

$$\cos A_\gamma = \frac{b^2+\gamma-a^2}{2b\sqrt{\gamma}} = \frac{(b^2+\gamma-a^2)\sqrt{\gamma}}{2b\gamma} \qquad \in \mathbb{A} \tag{32}$$

$$\cos B_\gamma = \frac{a^2+\gamma-b^2}{2a\sqrt{\gamma}} = \frac{(a^2+\gamma-b^2)\sqrt{\gamma}}{2a\gamma} \qquad \in \mathbb{A} \tag{33}$$

$$\cos C_\gamma = \frac{a^2+b^2-(\sqrt{\gamma})^2}{2ab} = \frac{a^2+b^2-\gamma}{2ab} \qquad \equiv \frac{\gamma_n}{\gamma_d} \in \mathbb{Q} \tag{34}$$

4.1 Triangle $(a, b, \sqrt{\gamma})$ diagonals

The only possible diagonals are for sides with integers, that is $\overline{a_m b_n}$. Using the law of cosines:

$$\overline{a_m b_n} = \sqrt{m^2+n^2-2mn\cos C} \tag{35}$$

$$= \sqrt{m^2+n^2-2mn\frac{a^2+b^2-\gamma}{2ab}} \tag{36}$$

$$= \frac{\sqrt{a^2b^2(m^2+n^2)-acmn(a^2+b^2-\gamma)}}{ab} \qquad \in \mathbb{A} \tag{37}$$

where $1 \le m < a$, $1 \le n < b$ and $m \ge n$.

4.2 Example triangles$(a, b, 2\sqrt{6})$

In this case $\sqrt{\gamma} = 2\sqrt{6}$ so $\gamma = 24$. We form a matrix with with the values $(a-b)^2$ satisfying the condition $a^2 \ge b^2 > \gamma$:

$$(a_m - b_n)^2 = \begin{array}{c} \\ b=5 \\ b=6 \\ b=7 \\ b=8 \\ b=9 \\ b=10 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccccccc} a=5 & a=6 & a=7 & a=8 & a=9 & a=10 & a=11 & a=12 & \dots \end{array} \\ \begin{pmatrix} 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & \dots \\ & 0 & 1 & 4 & 9 & 16 & 25 & 36 & \dots \\ & & 0 & 1 & 4 & 9 & 16 & 25 & \dots \\ & & & 0 & 1 & 4 & 9 & 16 & \dots \\ & & & & 0 & 1 & 4 & 9 & \dots \\ & & & & & 0 & 1 & 4 & \dots \\ & & & & & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array} \tag{38}$$

We remove cells except those satisfying the condition $(a-b)^2 < \gamma$:

$$(a_m - b_n)^2 = \begin{array}{c} \\ b=5 \\ b=6 \\ b=7 \\ b=8 \\ b=9 \\ b=10 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccccccc} a=5 & a=6 & a=7 & a=8 & a=9 & a=10 & a=11 & a=12 & \dots \end{array} \\ \begin{pmatrix} 0 & 1 & 4 & 9 & 16 & \times & \times & \times & \dots \\ & 0 & 1 & 4 & 9 & 16 & \times & \times & \dots \\ & & 0 & 1 & 4 & 9 & 16 & \times & \dots \\ & & & 0 & 1 & 4 & 9 & 16 & \dots \\ & & & & 0 & 1 & 4 & 9 & \dots \\ & & & & & 0 & 1 & 4 & \dots \\ & & & & & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array} \tag{39}$$

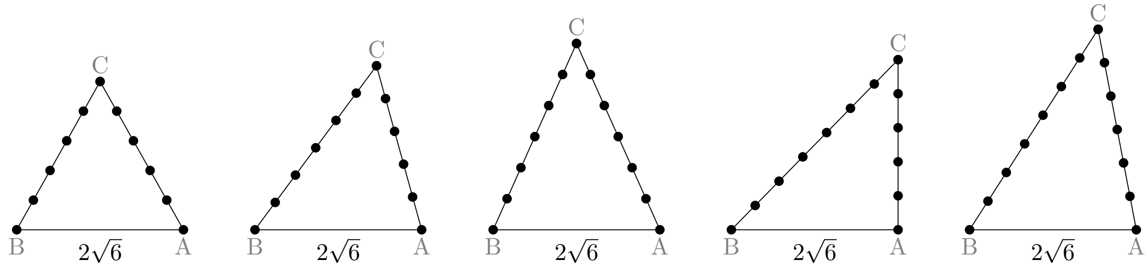

Figure 6: Some triangles with sides $a \geq b > c = 2\sqrt{6}$

So we found that infinite triangles are valid, the smaller ones are:

$$(a, b, 2\sqrt{6}) = \begin{array}{c} \\ (5,5,2\sqrt{6}) \\ (6,5,2\sqrt{6}) \\ (6,6,2\sqrt{6}) \\ (7,5,2\sqrt{6}) \\ (7,6,2\sqrt{6}) \\ (7,7,2\sqrt{6}) \\ \dots \end{array} \begin{array}{c} \begin{array}{ccc} \cos A & \cos B & \cos C \end{array} \\ \begin{pmatrix} \frac{\sqrt{6}}{5} & \frac{\sqrt{6}}{5} & \frac{13}{25} \\ \frac{13\sqrt{6}}{120} & \frac{35\sqrt{6}}{144} & \frac{37}{60} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{2}{3} \\ 0 & \frac{2\sqrt{6}}{7} & \frac{5}{7} \\ \frac{11\sqrt{6}}{144} & \frac{37\sqrt{6}}{168} & \frac{61}{84} \\ \frac{\sqrt{6}}{7} & \frac{\sqrt{6}}{7} & \frac{37}{49} \\ \dots & \dots & \dots \end{pmatrix} \end{array} \tag{40}$$

Figure 6 show some triangles $(a, b, 2\sqrt{6})$. The cosines are calculated by code at:

`github.com/heptagons/meccano/nest/t_test.go TestTslursC`

5 Triangles pairs

We can attach two triangles to share a common side and vertex to get more angles and diagonals.

5.1 Triangles pairs rational cosines angles

When we sum two angles $Z = X + Y$, where $\cos X \equiv x_N/x_D$ and $\cos Y \equiv y_N/y_D$ we have:

$$\cos Z = \cos X \cos Y - \sin X \sin Y \tag{41}$$

$$= \frac{x_N}{x_D} \times \frac{y_N}{y_D} - \frac{\sqrt{x_S}}{x_D} \times \frac{\sqrt{y_S}}{y_D} \tag{42}$$

$$= \frac{x_N y_N - \sqrt{x_S y_S}}{x_D y_D} \tag{43}$$

5.1.1 Triangles pairs angles

Adding the angles of Triangles from $(1, 1, 1)$ to $(3, 3, 2)$ we get some sums. The values are calculated with code at:

`github.com/heptagons/meccano/nest/t_test.go TestTCosAplusB`

Pair	$vertex_X$	$vertex_Y$	$\cos(X+Y)$
1	(1,1,1)[A]	(1,1,1)[A]	$-\frac{1}{2}$
2	(2,2,1)[A]	(1,1,1)[A]	$\frac{1-3\sqrt{5}}{8}$
3	(2,2,1)[C]	(1,1,1)[A]	$\frac{7-3\sqrt{5}}{16}$
4	(2,2,1)[A]	(2,2,1)[A]	$-\frac{7}{8}$
5	(2,2,1)[A]	(2,2,1)[C]	$-\frac{1}{4}$
6	(2,2,1)[C]	(2,2,1)[C]	$\frac{17}{32}$
7	(3,2,2)[A]	(1,1,1)[A]	$-\frac{1+3\sqrt{21}}{16}$
8	(3,2,2)[B]	(1,1,1)[A]	$\frac{3-\sqrt{21}}{8}$
9	(3,2,2)[A]	(2,2,1)[A]	$-\frac{1+3\sqrt{105}}{32}$
10	(3,2,2)[A]	(2,2,1)[C]	$-\frac{7+3\sqrt{105}}{64}$
11	(3,2,2)[B]	(2,2,1)[A]	$\frac{3-\sqrt{105}}{16}$
12	(3,2,2)[B]	(2,2,1)[C]	$\frac{21-\sqrt{105}}{32}$
13	(3,2,2)[A]	(3,2,2)[A]	$-\frac{31}{32}$
14	(3,2,2)[A]	(3,2,2)[B]	$-\frac{3}{4}$
15	(3,2,2)[B]	(3,2,2)[B]	$\frac{1}{8}$
16	(3,3,1)[A]	(1,1,1)[A]	$\frac{1-\sqrt{105}}{12}$
17	(3,3,1)[C]	(1,1,1)[A]	$\frac{17-\sqrt{105}}{36}$
18	(3,3,1)[A]	(2,2,1)[A]	$\frac{1-5\sqrt{21}}{24}$
19	(3,3,1)[A]	(2,2,1)[C]	$\frac{7-5\sqrt{21}}{48}$
20	(3,3,1)[C]	(2,2,1)[A]	$\frac{17-5\sqrt{21}}{72}$
21	(3,3,1)[C]	(2,2,1)[C]	$\frac{119-5\sqrt{21}}{144}$
22	(3,3,1)[A]	(3,2,2)[A]	$-\frac{1+21\sqrt{5}}{48}$
23	(3,3,1)[A]	(3,2,2)[B]	$\frac{3-7\sqrt{5}}{24}$
24	(3,3,1)[C]	(3,2,2)[A]	$-\frac{17+21\sqrt{5}}{144}$
25	(3,3,1)[C]	(3,2,2)[B]	$\frac{51-7\sqrt{5}}{72}$
26	(3,3,1)[A]	(3,3,1)[A]	$-\frac{17}{18}$
27	(3,3,1)[A]	(3,3,1)[C]	$-\frac{1}{6}$
28	(3,3,1)[C]	(3,3,1)[C]	$\frac{127}{162}$
29	(3,3,2)[A]	(1,1,1)[A]	$\frac{1-2\sqrt{6}}{6}$
30	(3,3,2)[C]	(1,1,1)[A]	$\frac{7-4\sqrt{6}}{18}$
31	(3,3,2)[A]	(2,2,1)[A]	$\frac{1-2\sqrt{30}}{12}$
32	(3,3,2)[A]	(2,2,1)[C]	$\frac{7-2\sqrt{30}}{24}$
33	(3,3,2)[C]	(2,2,1)[A]	$\frac{7-4\sqrt{30}}{36}$
34	(3,3,2)[C]	(2,2,1)[C]	$\frac{49-4\sqrt{30}}{72}$
35	(3,3,2)[A]	(3,2,2)[A]	$-\frac{1+6\sqrt{14}}{24}$
36	(3,3,2)[A]	(3,2,2)[B]	$\frac{3-2\sqrt{14}}{12}$
Pair	$vertex_X$	$vertex_Y$	$\cos(X+Y)$

Pair	$vertex_X$	$vertex_Y$	$\cos(X+Y)$
37	(3,3,2)[C]	(3,2,2)[A]	$-\frac{7+12\sqrt{14}}{72}$
38	(3,3,2)[C]	(3,2,2)[B]	$\frac{21-4\sqrt{14}}{36}$
39	(3,3,2)[A]	(3,3,1)[A]	$\frac{1-2\sqrt{70}}{18}$
40	(3,3,2)[A]	(3,3,1)[C]	$\frac{17-2\sqrt{70}}{54}$
41	(3,3,2)[C]	(3,3,1)[A]	$\frac{7-4\sqrt{70}}{54}$
42	(3,3,2)[C]	(3,3,1)[C]	$\frac{119-4\sqrt{70}}{162}$
43	(3,3,2)[A]	(3,3,2)[A]	$-\frac{7}{9}$
44	(3,3,2)[A]	(3,3,2)[C]	$-\frac{1}{3}$
45	(3,3,2)[C]	(3,3,2)[C]	$\frac{17}{81}$
Pair	$vertex_X$	$vertex_Y$	$\cos(X+Y)$

We can also test more triangles and filter, for example, the cosines to contain term $\sqrt{5}$:

Pair	$vertex_X$	$vertex_Y$	$\cos(X+Y)$
1	(2,2,1)[A]	(1,1,1)[A]	$\frac{1-3\sqrt{5}}{8}$
2	(2,2,1)[C]	(1,1,1)[A]	$\frac{7-3\sqrt{5}}{16}$
3	(3,3,1)[A]	(3,2,2)[A]	$-\frac{1+21\sqrt{5}}{48}$
4	(3,3,1)[A]	(3,2,2)[B]	$\frac{3-7\sqrt{5}}{24}$
5	(3,3,1)[C]	(3,2,2)[A]	$-\frac{17+21\sqrt{5}}{144}$
6	(3,3,1)[C]	(3,2,2)[B]	$\frac{51-7\sqrt{5}}{72}$
7	(4,3,2)[A]	(1,1,1)[A]	$-\frac{1+3\sqrt{5}}{8}$
8	(4,3,2)[B]	(1,1,1)[A]	$\frac{11-9\sqrt{5}}{32}$
9	(4,4,1)[A]	(3,3,1)[A]	$\frac{1-21\sqrt{5}}{48}$
10	(4,4,1)[A]	(3,3,1)[C]	$\frac{17-21\sqrt{5}}{144}$
11	(4,4,1)[C]	(3,3,1)[A]	$\frac{31-21\sqrt{5}}{192}$
12	(4,4,1)[C]	(3,3,1)[C]	$\frac{527-21\sqrt{5}}{576}$
13	(5,3,3)[A]	(4,4,3)[A]	$-\frac{21+55\sqrt{5}}{144}$
14	(5,3,3)[A]	(4,4,3)[C]	$-\frac{161+165\sqrt{5}}{576}$
15	(5,3,3)[B]	(4,4,3)[A]	$\frac{15-11\sqrt{5}}{48}$
16	(5,3,3)[B]	(4,4,3)[C]	$\frac{115-33\sqrt{5}}{192}$
17	(5,4,3)[A]	(4,3,3)[A]	$\frac{-4\sqrt{5}}{9}$
18	(5,4,3)[A]	(4,3,3)[B]	$\frac{-\sqrt{5}}{3}$
19	(5,4,3)[B]	(4,3,3)[A]	$\frac{3-16\sqrt{5}}{45}$
20	(5,4,3)[B]	(4,3,3)[B]	$\frac{6-4\sqrt{5}}{15}$
21	(5,4,3)[C]	(4,3,3)[A]	$\frac{4-12\sqrt{5}}{45}$
22	(5,4,3)[C]	(4,3,3)[B]	$\frac{8-3\sqrt{5}}{15}$
23	(5,5,1)[A]	(4,4,3)[A]	$\frac{3-33\sqrt{5}}{80}$
24	(5,5,1)[A]	(4,4,3)[C]	$\frac{23-99\sqrt{5}}{320}$
Pair	$vertex_X$	$vertex_Y$	$\cos(X+Y)$

Pair	$vertex_X$	$vertex_Y$	$\cos(X+Y)$
25	(5,5,1)[C]	(4,4,3)[A]	$\frac{147-33\sqrt{5}}{400}$
26	(5,5,1)[C]	(4,4,3)[C]	$\frac{1127-99\sqrt{5}}{1600}$
Pair	$vertex_X$	$vertex_Y$	$\cos(X+Y)$

5.2 Triangles pairs diagonals

So we can calculate new diagonals from one triangle side to another triangle side:

$$\delta = \sqrt{m^2+n^2-2mn\cos Z} \tag{44}$$

$$= \sqrt{m^2+n^2-2mn\frac{b_1+c_1\sqrt{d_1}}{a_1}} \tag{45}$$

$$= \frac{\sqrt{a_1^2(m^2+n^2)-2mn(b_1+c_1\sqrt{d_1})}}{a_1} \tag{46}$$

$$= \frac{\sqrt{a_1^2(m^2+n^2)-2b_1mn-2c_1mn\sqrt{d_1}}}{a_1} \equiv \frac{b_2+c_2\sqrt{d_2+e_2\sqrt{f_2}}}{a_2} \tag{47}$$

5.3 Triangles pairs surds angles

When we sum two algebraic angles $W = U + V$ when $\cos U = \sqrt{u_n}/u_d$ and $\cos V = \sqrt{v_n}/v_d$ we have:

$$\cos W = \cos U \cos V - \sin U \sin V \tag{48}$$

$$= \cos U \cos V - \sqrt{1-\cos^2 U}\sqrt{1-\cos^2 V} \tag{49}$$

$$= \frac{\sqrt{u_n v_n}}{u_d v_d} - \sqrt{\frac{u_d^2-u_n}{u_d^2}}\sqrt{\frac{v_d^2-v_n}{v_d^2}} \tag{50}$$

$$= \frac{\sqrt{u_n v_n} - \sqrt{(u_d^2-u_n)(v_d^2-v_n)}}{u_d v_d} \tag{51}$$

6 Triangle triplets

We can attach three triangles to share a common vertex and two sides.

6.1 Triple angles $2A + D$ and $A + 2D$

Lets add three angles, two angles A from a pair of triangles (a, b, c) and one angle D from single triangle (d, e, f). For triangle (a, b, c) and using $\cos A$ and $\sin A$ from equations 3 and 4 we have:

$$\cos(2A) = 2\cos^2 A - 1 = 2\frac{a_N^2}{a_D^2} - 1 = \frac{2a_N^2 - a_D^2}{a_D^2} \tag{52}$$

$$\sin(2A) = 2\sin A \cos A = \frac{2a_N\sqrt{a_D^2-a_N^2}}{a_D^2} \tag{53}$$

As with equations 3 and 4 lets define cosines and sines of triangle (d, e, f):

$$\begin{pmatrix} \cos & \sin \end{pmatrix} \begin{pmatrix} D \\ E \\ F \end{pmatrix} = \begin{pmatrix} \frac{d_N}{d_D} & \frac{\sqrt{d_D^2 - d_N^2}}{d_D} \\ \frac{e_N}{e_D} & \frac{\sqrt{e_D^2 - e_N^2}}{e_D} \\ \frac{f_N}{f_D} & \frac{\sqrt{f_D^2 - f_N^2}}{f_D} \end{pmatrix} \tag{54}$$

Using $\cos(2A)$, $\sin(2A)$, $\cos D$ and $\sin D$ we have:

$$\cos(2A + D) = \cos(2A)\cos D - \sin(2A)\sin D \tag{55}$$

$$= \frac{(2a_N^2 - a_D^2)d_N - 2a_N\sqrt{(a_D^2 - a_N^2)(d_D^2 - d_N^2)}}{a_D^2 d_D} \tag{56}$$

Table 4: Pair of triangles (X_{ABC}) and (Y_{ABC}) adding three angles in two different forms $(2X + Y)$ and $(X + 2Y)$.

Pair	$X = A \vee B \vee C$	$Y = A \vee B \vee C$	$\cos(2X+Y)$	$\cos(X+2Y)$
1	(1,1,1)[A]	(1,1,1)[A]	$\frac{1}{2}$	$\frac{1}{2}$
2	(2,2,1)[A]	(1,1,1)[A]	$-\frac{7-3\sqrt{5}}{16}$	$-\frac{1-3\sqrt{5}}{8}$
3	(2,2,1)[C]	(1,1,1)[A]	$\frac{17+21\sqrt{5}}{64}$	$-\frac{7-3\sqrt{5}}{16}$
4	(2,2,1)[A]	(2,2,1)[A]	$\frac{1}{4}$	$\frac{1}{4}$
5	(2,2,1)[A]	(2,2,1)[C]	$-\frac{17}{32}$	$\frac{61}{64}$
6	(2,2,1)[C]	(2,2,1)[A]	$\frac{61}{64}$	$-\frac{17}{32}$
7	(2,2,1)[C]	(2,2,1)[C]	$\frac{7}{8}$	$\frac{7}{8}$
8	(3,2,2)[A]	(1,1,1)[A]	$-\frac{31+3\sqrt{21}}{64}$	$\frac{1+3\sqrt{21}}{16}$
9	(3,2,2)[B]	(1,1,1)[A]	$\frac{1+3\sqrt{21}}{16}$	$-\frac{3-\sqrt{21}}{8}$
10	(3,2,2)[A]	(2,2,1)[A]	$-\frac{31+3\sqrt{105}}{128}$	$\frac{7+3\sqrt{105}}{64}$
11	(3,2,2)[A]	(2,2,1)[C]	$-\frac{217+3\sqrt{105}}{256}$	$-\frac{17-21\sqrt{105}}{256}$
12	(3,2,2)[B]	(2,2,1)[A]	$\frac{1+3\sqrt{105}}{32}$	$-\frac{21-\sqrt{105}}{32}$
13	(3,2,2)[B]	(2,2,1)[C]	$\frac{7+3\sqrt{105}}{64}$	$\frac{51+7\sqrt{105}}{128}$
14	(3,2,2)[A]	(3,2,2)[A]	$-\frac{1}{8}$	$-\frac{1}{8}$
15	(3,2,2)[A]	(3,2,2)[B]	$-\frac{57}{64}$	$\frac{31}{32}$
16	(3,2,2)[B]	(3,2,2)[A]	$\frac{31}{32}$	$-\frac{57}{64}$
17	(3,2,2)[B]	(3,2,2)[B]	$\frac{3}{4}$	$\frac{3}{4}$
18	(3,3,1)[A]	(1,1,1)[A]	$-\frac{17-\sqrt{105}}{36}$	$-\frac{1-\sqrt{105}}{12}$
19	(3,3,1)[C]	(1,1,1)[A]	$\frac{127+17\sqrt{105}}{324}$	$-\frac{17-\sqrt{105}}{36}$
20	(3,3,1)[A]	(2,2,1)[A]	$-\frac{17-5\sqrt{21}}{72}$	$-\frac{7-5\sqrt{21}}{48}$
21	(3,3,1)[A]	(2,2,1)[C]	$-\frac{119-5\sqrt{21}}{144}$	$\frac{17+35\sqrt{21}}{192}$
22	(3,3,1)[C]	(2,2,1)[A]	$\frac{127+85\sqrt{21}}{648}$	$-\frac{119-5\sqrt{21}}{144}$

Table 4: Pair of triangles (X_{ABC}) and (Y_{ABC}) adding three angles in two different forms ($2X+Y$) and ($X+2Y$).

Pair	$X = A \vee B \vee C$	$Y = A \vee B \vee C$	$\cos(2X+Y)$	$\cos(X+2Y)$
23	(3,3,1)[C]	(2,2,1)[C]	$\frac{889+85\sqrt{21}}{1296}$	$\frac{289+35\sqrt{21}}{576}$
24	(3,3,1)[A]	(3,2,2)[A]	$\frac{17+21\sqrt{5}}{144}$	$-\frac{31+21\sqrt{5}}{192}$
25	(3,3,1)[A]	(3,2,2)[B]	$-\frac{51-7\sqrt{5}}{72}$	$\frac{1+21\sqrt{5}}{48}$
26	(3,3,1)[C]	(3,2,2)[A]	$-\frac{127-357\sqrt{5}}{1296}$	$-\frac{527+21\sqrt{5}}{576}$
27	(3,3,1)[C]	(3,2,2)[B]	$\frac{381+119\sqrt{5}}{648}$	$\frac{17+21\sqrt{5}}{144}$
28	(3,3,1)[A]	(3,3,1)[A]	$\frac{1}{6}$	$\frac{1}{6}$
29	(3,3,1)[A]	(3,3,1)[C]	$-\frac{127}{162}$	$\frac{361}{486}$
30	(3,3,1)[C]	(3,3,1)[A]	$\frac{361}{486}$	$-\frac{127}{162}$
31	(3,3,1)[C]	(3,3,1)[C]	$\frac{17}{18}$	$\frac{17}{18}$
32	(3,3,2)[A]	(1,1,1)[A]	$-\frac{7-4\sqrt{6}}{18}$	$-\frac{1-2\sqrt{6}}{6}$
33	(3,3,2)[C]	(1,1,1)[A]	$\frac{17+56\sqrt{6}}{162}$	$-\frac{7-4\sqrt{6}}{18}$
34	(3,3,2)[A]	(2,2,1)[A]	$-\frac{7-4\sqrt{30}}{36}$	$-\frac{7-2\sqrt{30}}{24}$
35	(3,3,2)[A]	(2,2,1)[C]	$-\frac{49-4\sqrt{30}}{72}$	$\frac{17+14\sqrt{30}}{96}$
36	(3,3,2)[C]	(2,2,1)[A]	$\frac{17+56\sqrt{30}}{324}$	$-\frac{49-4\sqrt{30}}{72}$
37	(3,3,2)[C]	(2,2,1)[C]	$\frac{119+56\sqrt{30}}{648}$	$\frac{119+28\sqrt{30}}{288}$
38	(3,3,2)[A]	(3,2,2)[A]	$\frac{7+12\sqrt{14}}{72}$	$-\frac{31+6\sqrt{14}}{96}$
39	(3,3,2)[A]	(3,2,2)[B]	$-\frac{21-4\sqrt{14}}{36}$	$\frac{1+6\sqrt{14}}{24}$
40	(3,3,2)[C]	(3,2,2)[A]	$-\frac{17-168\sqrt{14}}{648}$	$-\frac{217+12\sqrt{14}}{288}$
41	(3,3,2)[C]	(3,2,2)[B]	$\frac{51+56\sqrt{14}}{324}$	$\frac{7+12\sqrt{14}}{72}$
42	(3,3,2)[A]	(3,3,1)[A]	$-\frac{7-4\sqrt{70}}{54}$	$-\frac{17-2\sqrt{70}}{54}$
43	(3,3,2)[A]	(3,3,1)[C]	$-\frac{119-4\sqrt{70}}{162}$	$\frac{127+34\sqrt{70}}{486}$
44	(3,3,2)[C]	(3,3,1)[A]	$\frac{17+56\sqrt{70}}{486}$	$-\frac{119-4\sqrt{70}}{162}$
45	(3,3,2)[C]	(3,3,1)[C]	$\frac{289+56\sqrt{70}}{1458}$	$\frac{889+68\sqrt{70}}{1458}$
46	(3,3,2)[A]	(3,3,2)[A]	$\frac{1}{3}$	$\frac{1}{3}$
47	(3,3,2)[A]	(3,3,2)[C]	$-\frac{17}{81}$	$\frac{241}{243}$
48	(3,3,2)[C]	(3,3,2)[A]	$\frac{241}{243}$	$-\frac{17}{81}$
49	(3,3,2)[C]	(3,3,2)[C]	$\frac{7}{9}$	$\frac{7}{9}$

Data from:`github.com/heptagons/meccano/nest/t_test.go TestTCos2AplusB`

6.2 Triple angles $Z = A + D + G$

$$\begin{aligned}
\cos(A+D) &= \cos A \cos D - \sin A \sin D \\
&= \frac{a_n d_n - \sqrt{a_s d_s}}{a_d d_d} && (57) \\
\sin(A+D) &= \sin A \cos D + \cos A \sin D \\
&= \frac{d_n \sqrt{a_s} + a_n \sqrt{d_s}}{a_d d_d} && (58) \\
\cos(A+D+G) &= \cos(A+D)\cos G - \sin(A+D)\sin G \\
&= \frac{a_n d_n g_n - g_n \sqrt{a_s d_s}}{a_d d_d g_d} - \frac{d_n \sqrt{a_s g_s} + a_n \sqrt{d_s g_s}}{a_d d_d g_d} \\
&= \frac{a_n d_n g_n - g_n \sqrt{a_s d_s} - d_n \sqrt{a_s g_s} - a_n \sqrt{d_s g_s}}{a_d d_d g_d} && (59)
\end{aligned}$$

Table 5: Pair of triangles (X_{ABC}) and (Y_{ABC}) adding three angles in two different forms $(2X + Y)$ and $(X + 2Y)$.

Pair	$X = A \vee B \vee C$	$Y = A \vee B \vee C$	$Z = A \vee B \vee C$	$\cos(X + Y + Z)$

Data from:`github.com/heptagons/meccano/nest/t_test.go TestTCosAplusBplusC`