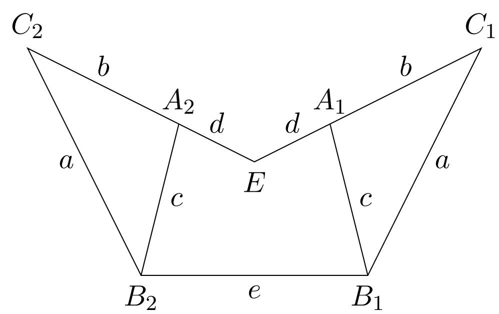
Horns unit

https://github.com/heptagons/meccano/units/horns

Abstract

Horns unit is a group of seven meccano [1] strips intended to build polygons. We found the formula to calculate the internal angles then look for polygons and found hexagons, octagons and dodecagons. We found no pentagons.



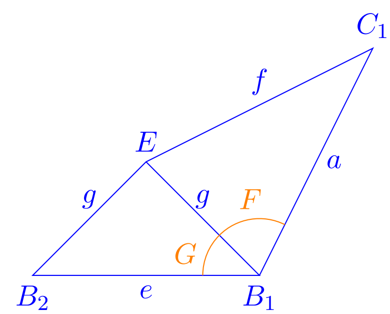


Figure 1: The **horn unit** has seven strips: Two of length a, two of length $b+d$, two of length c and one of length e. We expect to build polygons with internal angle $C_1B_1B_2$ and perimeter including segments a, e, a.

1 Algebra

From figure 1 we start with triangle $\triangle A_1B_1C1$. At vertex A_1 we have angle A and the supplement A':

$$A \equiv \angle B_1A_1C_1 \tag{1}$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc} \qquad \text{if and only if } a < b+c \tag{2}$$

$$A' \equiv \angle EA_1B_1 = \pi - A \tag{3}$$

$$\cos A' = \cos(\pi - A) = -\cos A = \frac{-b^2-c^2+a^2}{2bc} \tag{4}$$

We define $f \equiv b+d$ and $g \equiv \overline{EB_1}$. With the law of cosines we have:

$$f \equiv \boxed{b+d} \qquad \in \mathbb{N} \tag{5}$$

$$g^2 = c^2+d^2-2cd\cos A' \tag{6}$$

$$\begin{aligned} &= c^2+d^2-(2cd)\frac{-b^2-c^2+a^2}{2bc} \\ &= \frac{bc^2+bd^2+b^2d+c^2d-a^2d}{b} \\ &= \frac{(b+d)(bd+c^2)-a^2d}{b} \end{aligned} \tag{7}$$

[1] Meccano mathematics by 't Hooft

Define a new variable $h = (b+d)(bd+c^2) - a^2d$:

$$h \equiv \boxed{(bd+c^2)f - a^2d} \qquad \in \mathbb{Z} \tag{8}$$

$$g^2 = \boxed{\frac{h}{b}} \qquad \text{if and only if } 0 < h < b \tag{9}$$

We calculate angles $F \equiv \angle C_1B1E$ and $G \equiv \angle B_2B_1E$. We replace g^2 by h/b:

$$\cos F = \frac{a^2+g^2-f^2}{2ag} = \frac{a^2b - bf^2 + h}{2abg} \tag{10}$$

$$\cos G = \boxed{\frac{e}{2g}} \tag{11}$$

Define new variable $j = a^2b - bf^2 + h$ so:

$$j \equiv \boxed{a^2b - bf^2 + h} \qquad \in \mathbb{Z} \tag{12}$$

$$\cos F = \frac{a^2b - bf^2 + h}{2abg} = \boxed{\frac{j}{2abg}} \tag{13}$$

We calculate cosines squares and products. Again we replace g^2 by h/b:

$$\cos F \cos G = \frac{ej}{4abg^2} = \frac{bej}{4abh} = \boxed{\frac{ej}{4ah}} \qquad \in \mathbb{Q} \tag{14}$$

$$\cos^2 F = \frac{j^2}{4a^2b^2g^2} = \frac{bj^2}{4a^2b^2h} = \boxed{\frac{j^2}{4a^2bh}} \qquad \in \mathbb{Q} \tag{15}$$

$$\cos^2 G = \frac{e^2}{4g^2} = \boxed{\frac{be^2}{4h}} \qquad \in \mathbb{Q} \tag{16}$$

$$\cos^2 F \cos^2 G = \frac{be^2j^2}{16a^2bh^2} = \boxed{\frac{e^2j^2}{16a^2h^2}} \qquad \in \mathbb{Q} \tag{17}$$

$$\tag{18}$$

We calculate the sines part squared and set a common denominator as square $16a^2b^2h^2$:

$$(\sin F \sin G)^2 = (1 - cos^2F)(1 - cos^2G) \tag{19}$$

$$\begin{aligned}
&= 1 - cos^2F - cos^2G + cos^2F\cos^2 G \\
&= 1 - \frac{j^2}{4a^2bh} - \frac{be^2}{4h} + \frac{e^2j^2}{16a^2h^2} \\
&= 1 - \frac{j^2}{4a^2bh} - \frac{be^2}{4h} + \frac{e^2j^2}{16a^2h^2} \\
&= \frac{16a^2b^2h^2 - (4bh)j^2 - (4a^2b^2h)be^2 + (b^2)e^2j^2}{16a^2b^2h^2} \\
&= \frac{16a^2b^2h^2 - 4bhj^2 - 4a^2b^3e^2h + b^2e^2j^2}{16a^2b^2h^2} \\
&= \frac{b(be^2-4h)(j^2-4a^2bh)}{16a^2b^2h^2}
\end{aligned} \tag{20}$$

Extract square root to get $\sin F \sin G = \sqrt{D}/A$ where $D, A \in \mathbb{Z}$:

$$\sin F \sin G = \boxed{\frac{\sqrt{b(be^2-4h)(j^2-4a^2bh)}}{4abh}} \qquad \in \mathbb{A} \tag{21}$$

We sum the angles F and G to get:

$$\begin{aligned}
F + G &\equiv \angle B_2 B_1 C_1 && (22) \\
\cos(F+G) &= \cos F \cos G - \sin F \sin G && (23) \\
&= \frac{ej}{4ah} - \frac{\sqrt{b(be^2-4h)(j^2-4a^2bh)}}{4abh} \\
&= \frac{bej - \sqrt{b(be^2-4h)(j^2-4a^2bh)}}{4abh} \qquad \in \mathbb{A} && (24)
\end{aligned}$$

2 Software

3 Examples

3.1 Hexagons examples

```
segments min=0 max=20 a,b,c,d,e:
  1) 3,7,5,0,10
  2) 3,8,7,0,13
  3) 10,13,7,0,13
  4) 8,13,7,0,14
  5) 3,8,7,1,15
  6) 5,8,7,2,15
  7) 11,7,7,7,15
  8) 7,13,8,0,16
  9) 3,8,7,2,17
 10) 5,8,7,3,17
 11) 16,14,6,7,17
 12) 8,7,5,7,18
 13) 3,8,7,3,19
 14) 5,8,7,4,19
 15) 7,3,5,11,19
 16) 7,8,5,6,19
--- PASS: TestHornsEHexagons (28.73s)
```

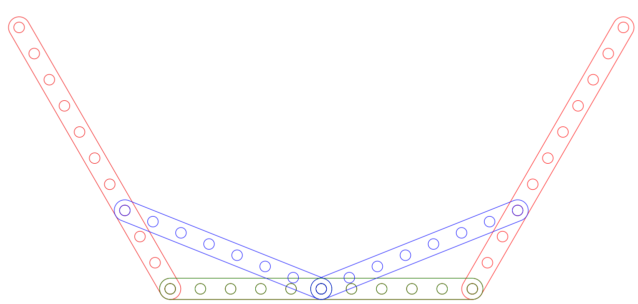


Figure 2: Hexagon size 10.

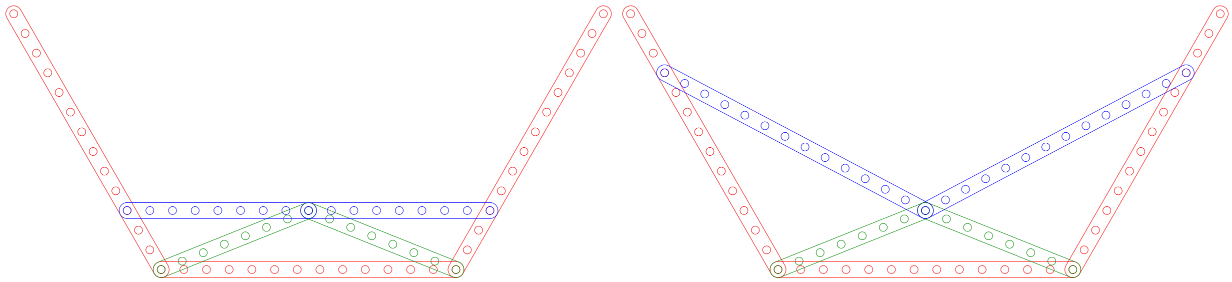


Figure 3: Hexagons size 13.

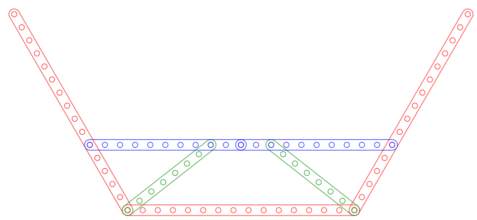


Figure 4: Horns() 60°.

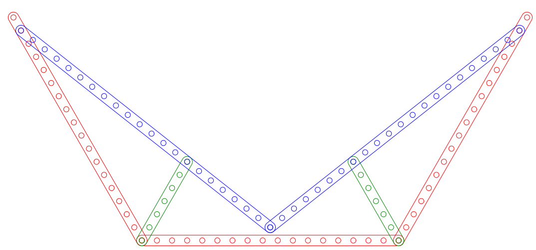


Figure 5: Horns() 60°.

3.2 Octagons examples

```
segments min=1 max=40 a,b,c,d,e:
  1) 2,3,3,3,8
  2) 3,2,3,7,12
  3) 14,9,9,9,16
  4) 14,11,11,11,24
  5) 21,21,14,6,24
  6) 7,18,17,3,28
  7) 9,10,11,17,36
  8) 9,20,19,7,36
  9) 10,9,11,21,40
 10) 35,27,22,18,40
--- PASS: TestHornsEOctagons (566.86s)
```

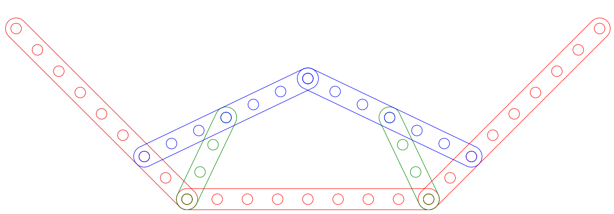


Figure 6: Horns(2,3,3,3,8) 45°.

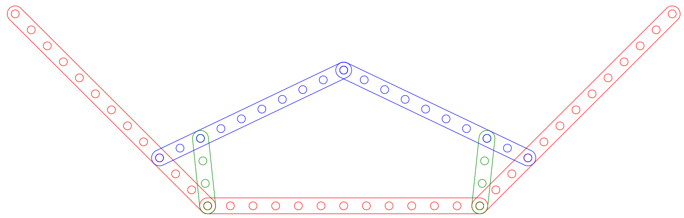


Figure 7: Horns(3,2,3,7,12) 45°.

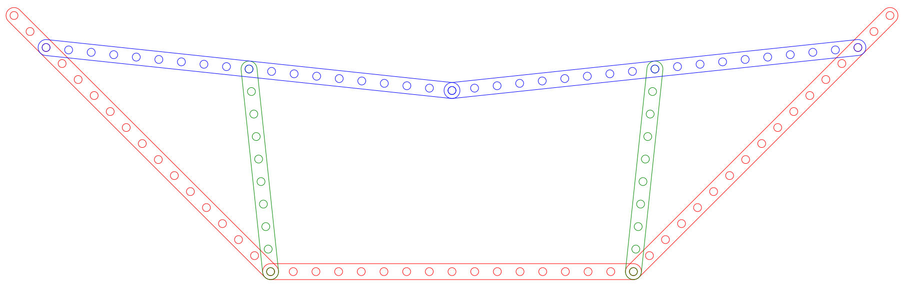


Figure 8: Horns(14,9,9,9,16) 45°.

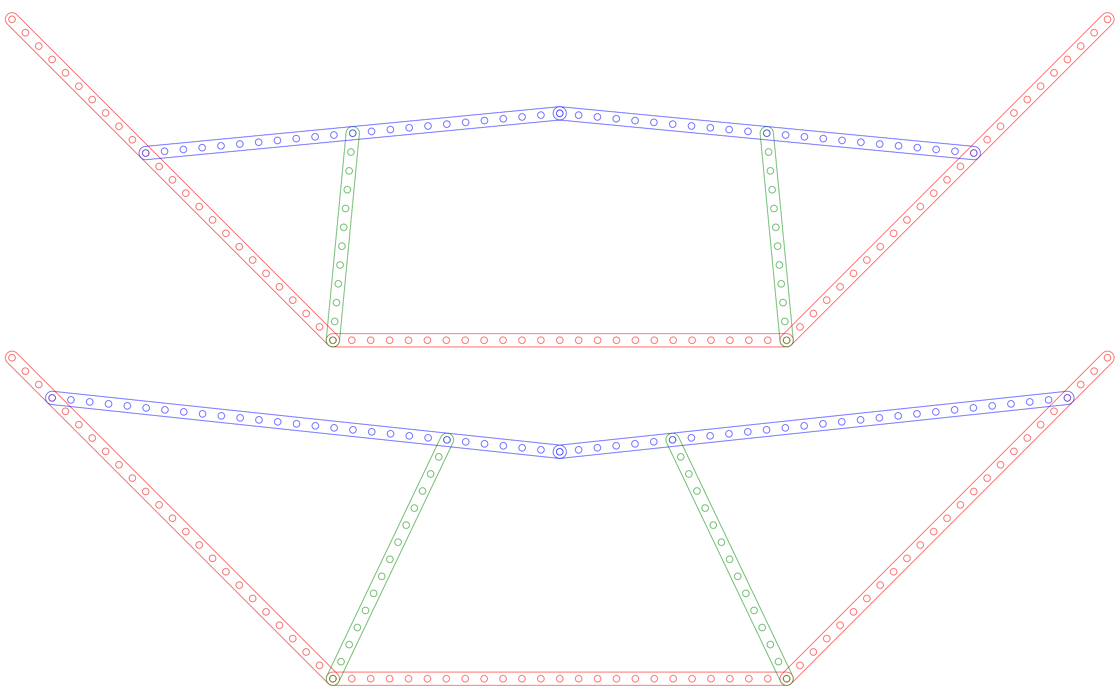


Figure 9: Octagons. Horns(14,11,11,11,24) and (21,21,14,6,24).

3.3 Dodecagons examples

```
segments min=1 max=40 a,b,c,d,e:
  1) 6,5,5,5,8
  2) 15,4,13,21,20
  3) 15,9,12,16,20
  4) 15,14,13,11,20
  5) 15,18,15,7,20
```

```
  6) 10,13,13,13,24
  7) 21,10,17,25,28
  8) 16,17,17,17,30
  9) 30,11,25,39,40
--- PASS: TestHornsEDodecagons (566.19s)
```

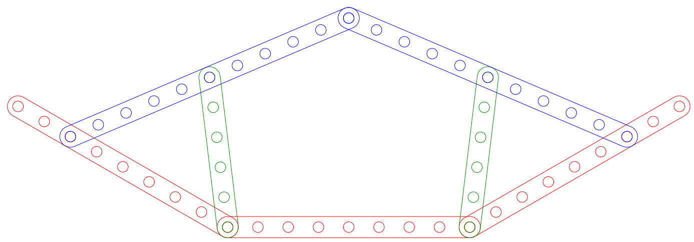


Figure 10: Dodecagon: Horns(6,5,5,5,8).

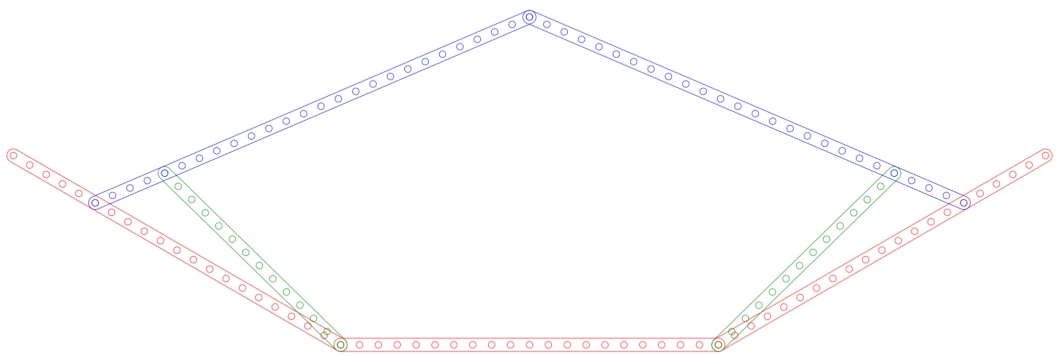


Figure 11: Dodecagons. Horns (15,4,13,21,20). Special case $d > e$.

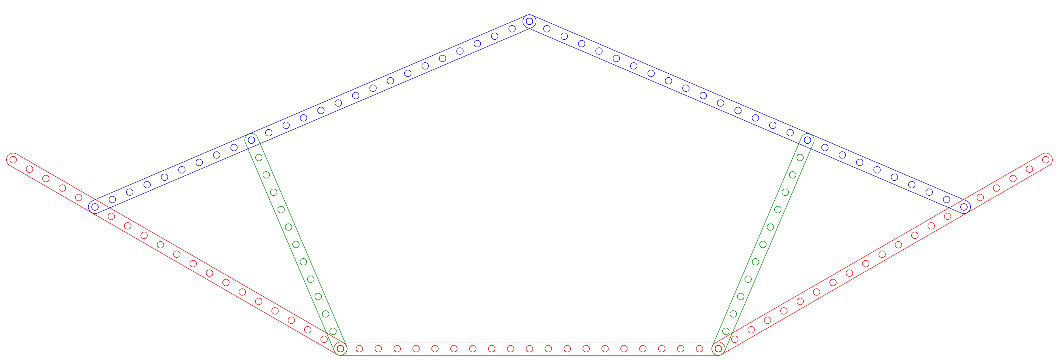


Figure 12: Dodecagons. Horns (15,9,12,16,20).

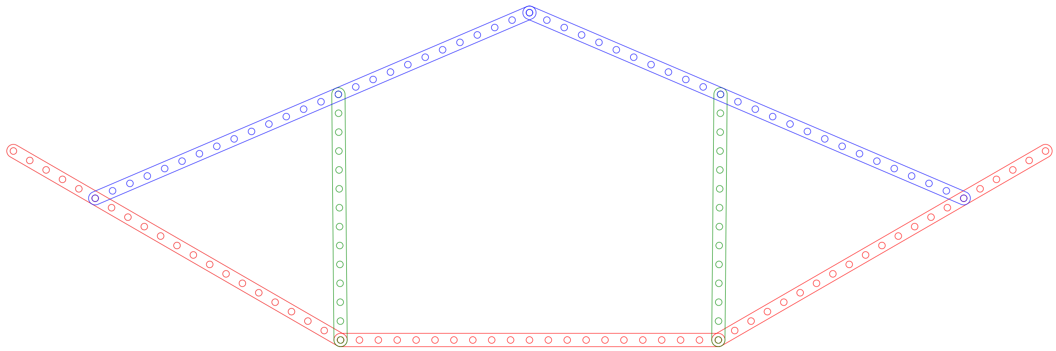


Figure 13: Dodecagons. Horns (15,14,13,11,20).

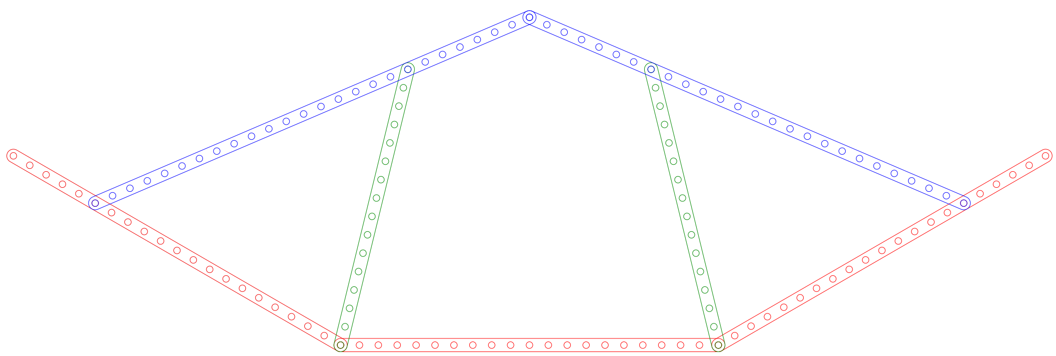


Figure 14: Dodecagons. Horns (15,18,15,7,20).

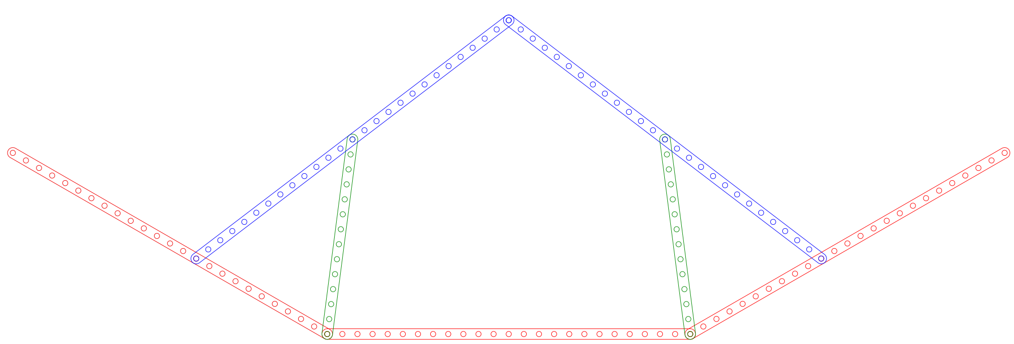


Figure 15: Dodecagon. Horns(10,13,13,13,24).

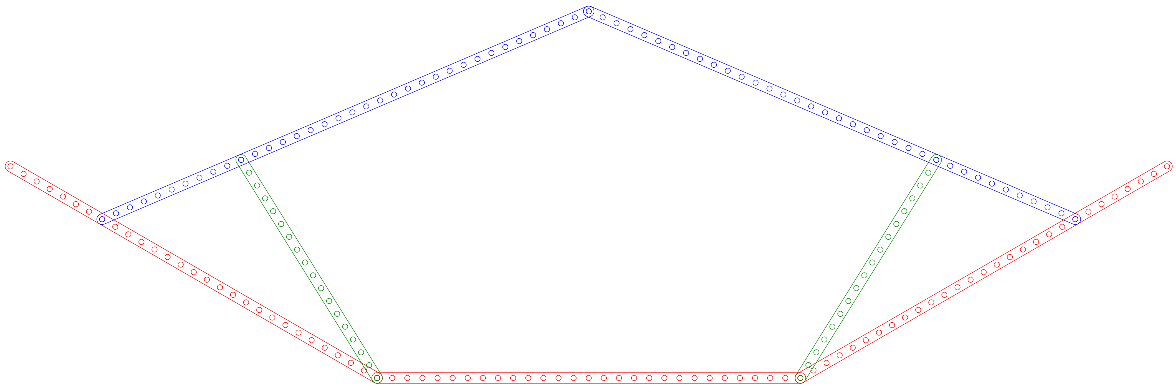


Figure 16: Dodecagon. Horns(21,10,17,25,28).

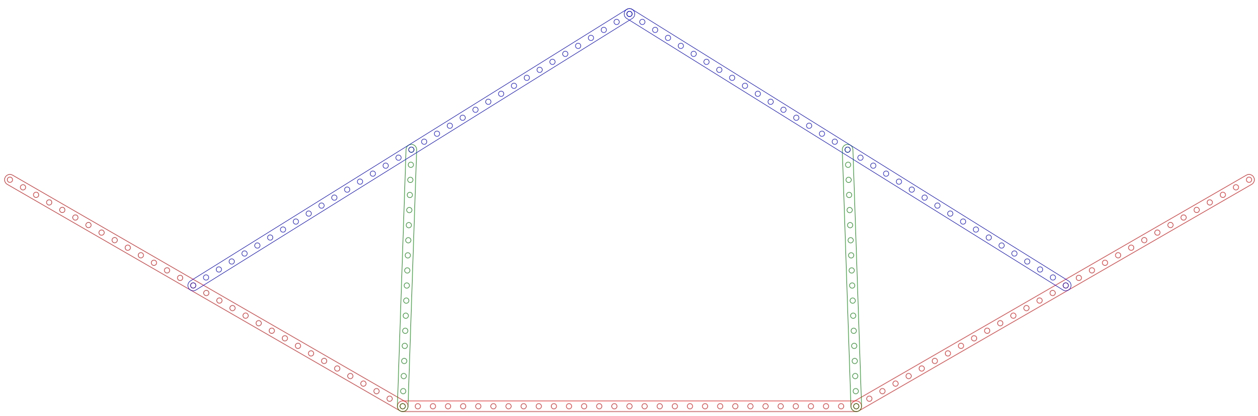


Figure 17: Dodecagon. Horns(16,17,17,17,30).