Meccano frames

https://github.com/heptagons/meccano/frames

Abstract

Meccano frames are groups of rigid meccano [1] strips. Can be used as internal diagonals of polygons to be rigid. The lengths of such diagonals are algebraic numbers of the form $\frac{B+C\sqrt{D}}{A}$ or $\frac{E\sqrt{F+H\sqrt{G}}}{A}$.

1 Triangular frame

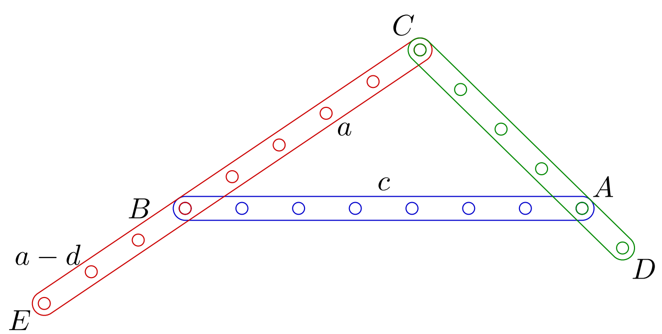

Figure 1: Triangular frame.

Figure 1 shows a triangular frame. With three strips we form the triangle $\triangle ABC$. At least we extend one of the two strips $\overline{CB}$ and $\overline{CA}$ to become $\overline{CE}$ and $\overline{CD}$. The new vertices D and E distance is rigid of the form $\dfrac{p\sqrt{s}}{q}$, where $p, q, s \in \mathbb{Z}^+$.

First we identify five integer distances a, b, c, d, e:

$$a \equiv \overline{CB}, \quad b \equiv \overline{CA}, \quad c \equiv \overline{AB}, \quad c < a + b \tag{1}$$
$$d \equiv \overline{CB} + \overline{BE} = \overline{CE} \geq a \tag{2}$$
$$e \equiv \overline{CA} + \overline{AD} = \overline{CD} \geq b \tag{3}$$

We calculate the cosine of $\angle BCA$:

$$\theta \equiv \angle BCA \tag{4}$$
$$\cos\theta = \frac{a^2 + b^2 - c^2}{2ab} \tag{5}$$

Then we apply the cosine to the triangle $\triangle CED$ to get the extensions distance $\overline{DE}$:

$$\begin{aligned} \overline{DE}^2 &= \overline{CD}^2 + \overline{CE}^2 - 2\overline{CD} \times \overline{CE}\cos\theta \\ &= d^2 + e^2 - 2de\cos\theta \\ &= d^2 + e^2 - de\left(\frac{a^2 + b^2 - c^2}{ab}\right) \end{aligned} \tag{6}$$

[1] Meccano mathematics by ‘t Hooft

We extract the square root:

$$
\begin{aligned}
\overline{DE} &= \sqrt{d^2 + e^2 - de\left(\frac{a^2 + b^2 - c^2}{ab}\right)} \\
&= \frac{\sqrt{a^2b^2(d^2 + e^2) - abde(a^2 + b^2 - c^2)}}{ab} \\
&= \frac{\sqrt{ab((ad - be)(bd - ae) + c^2de)}}{ab}
\end{aligned} \tag{7}
$$

1.1 Software

We write a software to report all the triangle frames with specific surd $\sqrt{s}$ for a given maximum strips length. We can reject cases $q \neq 1$ and s not square-free. Next list show all the triangles with $q = 1$ and $s = \sqrt{7}$ where $c < a + b$, $a \leq d \leq max$, $b \leq e \leq max$, $c \leq max$:

```
=== RUN   TestFramesTriangleSurds
NewFrames().TriangleSurds surd=7 max=15
  1) a=1 e=1+2 c=1 cos=1/2
  2) d=1+1 e=1+2 c=1 cos=1/2
  3) d=1+2 b=1 c=1 cos=1/2
  4) d=1+2 e=1+1 c=1 cos=1/2
  5) a=2 e=2+1 c=2 cos=1/2
  6) d=2+1 b=2 c=2 cos=1/2
  7) a=3 e=2+2 c=2 cos=3/4 CED=pi/2
  8) d=3+1 e=2+1 c=2 cos=3/4 CDE=pi/2
  9) d=4+2 e=4+4 c=1 cos=31/32
 10) d=4+4 e=4+2 c=1 cos=31/32
 11) a=7 e=5+1 c=3 cos=13/14
 12) a=7 e=5+2 c=3 cos=13/14
```

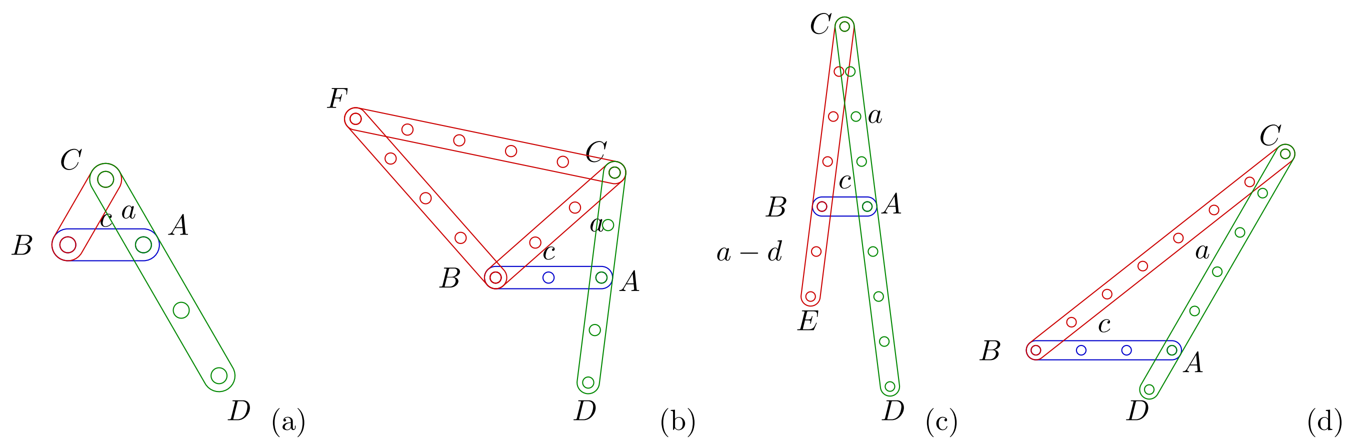

Figure 2: Some triangular frames with rigid distance $\overline{DE} = \sqrt{7}$ found by the software.

Figure 2 show four cases of this list. The code is in the folder `github.com/heptagons/meccano/frames`.

1.2 Triangular distance of the form $\sqrt{s} + f$

In the figure 2, the particular case (b), was reported with the angle $CED = \pi/2$ which means we can append two extra strips to make a pythagorean triangle $\triangle CEF$ where angle $CEF = \pi/2$, which makes the three vertices D, E, F collinear, so the rigid distance $\overline{DF} = \sqrt{7} + 4$ is an algebraic number.

1.3 Another rigid distances $\sqrt{s}+h$

We explore a more complicated frame to get additional cases of distances $\sqrt{s}+h$ without relying in an explicit pythagorean triangle as we saw in case (b) of figure 2.

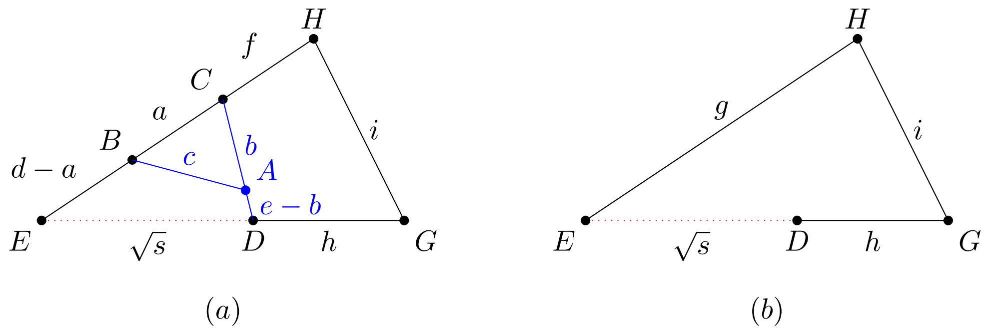

Figure 3: The five strips intented to form an algebraic distance $\overline{EG} = \sqrt{s}+h$.

From figure 3 (a) we know $\sqrt{s}$ distance between nodes E and D is produced by the three strips frame $a+d$, $b+e$ and c. Using the law of cosines we calculate the angle $\theta = \angle CED$ in terms of $\sqrt{s}$:

$$\begin{aligned}
\cos\theta &= \frac{d^2 + (\sqrt{s})^2 - e^2}{2d\sqrt{s}} \\
&= \frac{(d^2 + s - e^2)\sqrt{s}}{2ds} \qquad (8)\\
&= \frac{m\sqrt{s}}{n} \qquad (9)\\
m &= d^2 + s - e^2 \qquad (10)\\
n &= 2ds \qquad (11)
\end{aligned}$$

From figure 3 (a) we notice two sets of points are collinear: $\{E, B, C, H\}$ and $\{E, D, G\}$. Using the law of cosines we calculate the angle $\theta = \angle HEG$ in terms of distances $g, \sqrt{s}+h, i$:

$$\begin{aligned}
\cos\theta &= \frac{g^2 + (\sqrt{s}+h)^2 - i^2}{2g(\sqrt{s}+h)} \\
&= \frac{g^2 + s + 2\sqrt{s}h + h^2 - i^2}{2g(\sqrt{s}+h)} \\
&= \frac{g^2 + s + h^2 - i^2 + 2\sqrt{s}h}{2g(\sqrt{s}+h)} \qquad (12)
\end{aligned}$$

We multiply both numerator and denominator by $\sqrt{s} - h$ to eliminate the surd from denominator:

$$\begin{aligned}
\cos\theta &= \frac{(s+g^2+h^2-i^2)(\sqrt{s}-h)+2\sqrt{s}h(\sqrt{s}-h)}{2g(\sqrt{s}+h)(\sqrt{s}-h)} \\
&= \frac{(s+g^2+h^2-i^2)(\sqrt{s}-h)+2sh-2\sqrt{s}h^2}{2g(s-h^2)} \\
&= \frac{-h(s+g^2+h^2-i^2-2s)+(s+g^2+h^2-i^2-2h^2)\sqrt{s}}{2g(s-h^2)} \\
&= \frac{h(s-g^2-h^2+i^2)+(s+g^2-h^2-i^2)\sqrt{s}}{2g(s-h^2)} \\
&= \frac{o+p\sqrt{s}}{q} && (13) \\
o &= h(s-g^2-h^2+i^2) && (14) \\
p &= s+g^2-h^2-i^2 && (15) \\
q &= 2g(s-h^2) && (16)
\end{aligned}$$

We compare both cosines equations 9 and 13:

$$\frac{m\sqrt{s}}{n} = \frac{o+p\sqrt{s}}{q} \tag{17}$$

Since all variables are integers we need two conditions. First o should be zero. And second $\frac{m}{n} = \frac{p}{q}$.

For condition 1, we force o to be zero:

$$\begin{aligned}
o &= 0 \\
h(s-g^2-h^2+i^2) &= 0 \\
s &= g^2+h^2-i^2 && (18)
\end{aligned}$$

For condition2, we force m, n, p, q as:

$$\begin{aligned}
\frac{m}{n} &= \frac{p}{q} \\
\frac{d^2+s-e^2}{2ds} &= \frac{s+g^2-h^2-i^2}{2g(s-h^2)}
\end{aligned} \tag{19}$$

We replace the value of s of last equation RHS with the value of equation 18 of condition 1:

$$\begin{aligned}
\frac{d^2-e^2+s}{ds} &= \frac{s+g^2-h^2-i^2}{g(s-h^2)} \\
&= \frac{g^2+h^2-i^2+g^2-h^2-i^2}{g(g^2+h^2-i^2-h^2)} \\
&= \frac{2(g^2-i^2)}{g(g^2-i^2)} \\
&= \frac{2}{g} \\
(d^2-e^2+s)g &= 2ds && (20)
\end{aligned}$$

$\boxed{TODO: Examples!!!}$

2 Triangle pair frame

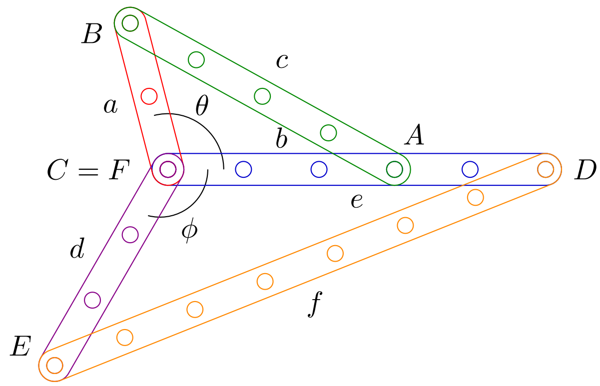

Figure 4: Triangle pair frame.

Figure 4 shows a triangle pair frame. The pair joins triangles $\triangle ABC$ and $\triangle DEF$ in such a way vertices C and F coincide and vertices A, C, D, F be collinear. With only five strips this frame is small and useful to make up the rigid polygons diagonals of the form $\overline{BE} = \dfrac{z_2\sqrt{z_3 + z_4\sqrt{z_5}}}{z_1}, z_i \in \mathbb{Z}$.

2.1 Triangle pair algebra

Using the law of cosines we calculate the angle $\theta = \angle ACB$ with defined variables m, n and the angle $\phi = \angle DFE$ with defined variables o, p:

$$(m, n) \equiv (a^2 + b^2 - c^2, 2ab), \quad |m| \leq n, \quad m, n \in \mathbb{Z} \tag{21}$$

$$\cos\theta = \frac{m}{n} \tag{22}$$

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \frac{\sqrt{n^2 - m^2}}{n} \tag{23}$$

$$(o, p) \equiv (d^2 + e^2 - f^2, 2de), \quad |o| \leq p, \quad o, p \in \mathbb{Z} \tag{24}$$

$$\cos\phi = \frac{o}{p} \tag{25}$$

$$\sin\phi = \sqrt{1 - \cos^2\phi} = \frac{\sqrt{p^2 - o^2}}{p} \tag{26}$$

Then, we use the cosines sum identity:

$$\begin{aligned}
\cos(\theta + \phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi \\
&= \left(\frac{m}{n}\right)\left(\frac{o}{p}\right) - \left(\frac{\sqrt{n^2 - m^2}}{n}\right)\left(\frac{\sqrt{p^2 - o^2}}{p}\right) \\
&= \frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{np} \qquad (27)
\end{aligned}$$

Finally we can calculate the distance $g \equiv \overline{BE}$ using the law of cosines:

$$
\begin{aligned}
g &\equiv \overline{BE} \\
&= \sqrt{a^2 + d^2 - 2ad\cos(\theta + \phi)} \\
&= \sqrt{a^2 + d^2 - 2ad\left(\frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{np}\right)} \\
&= \sqrt{a^2 + d^2 - 2ad\left(\frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{4abde}\right)} \\
&= \sqrt{a^2 + d^2 - \frac{mo - \sqrt{(n^2 - m^2)(p^2 - o^2)}}{2be}} \\
&= \frac{\sqrt{4b^2e^2(a^2 + d^2) - 2bemo + 2be\sqrt{(n^2 - m^2)(p^2 - o^2)}}}{2be} \qquad (28)
\end{aligned}
$$

2.2 Triangle pairs software

From the last equation of g we identify three `input` integers i_1, i_2, i_3 which are used to get $g(i)$. Then the nested radicals sofware returns `output` integers z_1, z_2, z_3, z_4, z_5 as $g(z)$:

$$i_1 \equiv 2be \qquad (29)$$

$$i_2 \equiv i_1^2(a^2 + d^2) - i_1 mo \qquad (30)$$

$$i_3 \equiv (n^2 - m^2)(p^2 - o^2) \qquad (31)$$

$$g(i) = \frac{\sqrt{i_2 + i_1\sqrt{i_3}}}{i_1} \qquad (32)$$

$$g(z) = \frac{z_2\sqrt{z_3 + z_4\sqrt{z_5}}}{z_1} \qquad (33)$$

We run a program to print a list of triangle pairs with sides $1 < a, b, c, d, e, f \leq max$ having a given distance $\overline{BE} = g$ or particular $z3, z4, z5$. Next example request a pairs list with $g = z_2\sqrt{46 + 18\sqrt{5}}/z_1$ up to strip length 10 so we set as limits $max = 10, z_3 = 46, z_4 = 18, z_5 = 5$ to get next report (text in blue):

Folder : `github.com/heptagons/meccano/frames`
Call : `NewFrames().TrianglePairsTex(10, [46 18 5])`

$(a,b,c) \oplus (d,e,f) \mapsto g$
$(2,1,2) \oplus (3,3,3) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(2,1,2) \oplus (3,8,7) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(2,2,2) \oplus (3,6,6) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(2,3,4) \oplus (3,5,7) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(2,4,4) \oplus (3,8,7) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(3,3,3) \oplus (2,4,4) \mapsto \dfrac{\sqrt{46+18\sqrt{5}}}{2}$
$(4,2,4) \oplus (6,6,6) \mapsto \sqrt{46+18\sqrt{5}}$
$(4,4,4) \oplus (6,7,8) \mapsto \sqrt{46+18\sqrt{5}}$
$(6,3,6) \oplus (4,4,4) \mapsto \sqrt{46+18\sqrt{5}}$
$(6,3,6) \oplus (9,9,9) \mapsto \dfrac{3\sqrt{46+18\sqrt{5}}}{2}$
$(6,6,6) \oplus (4,8,8) \mapsto \sqrt{46+18\sqrt{5}}$
$(6,7,8) \oplus (9,9,9) \mapsto \dfrac{3\sqrt{46+18\sqrt{5}}}{2}$

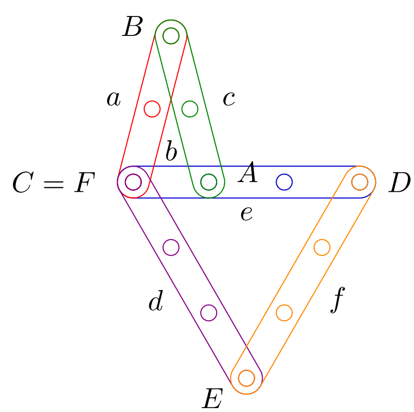

Figure 5: Triangle pair frame $(2,1,2) \oplus (3,3,3)$ makes $\overline{BE} = \dfrac{\sqrt{46+18\sqrt{5}}}{2}$.

In figure 5 we build a triangular pair following one of the last report results, when $abc = (2,1,2)$ and $def = (3,3,3)$.

3 Triangle pair extended frame

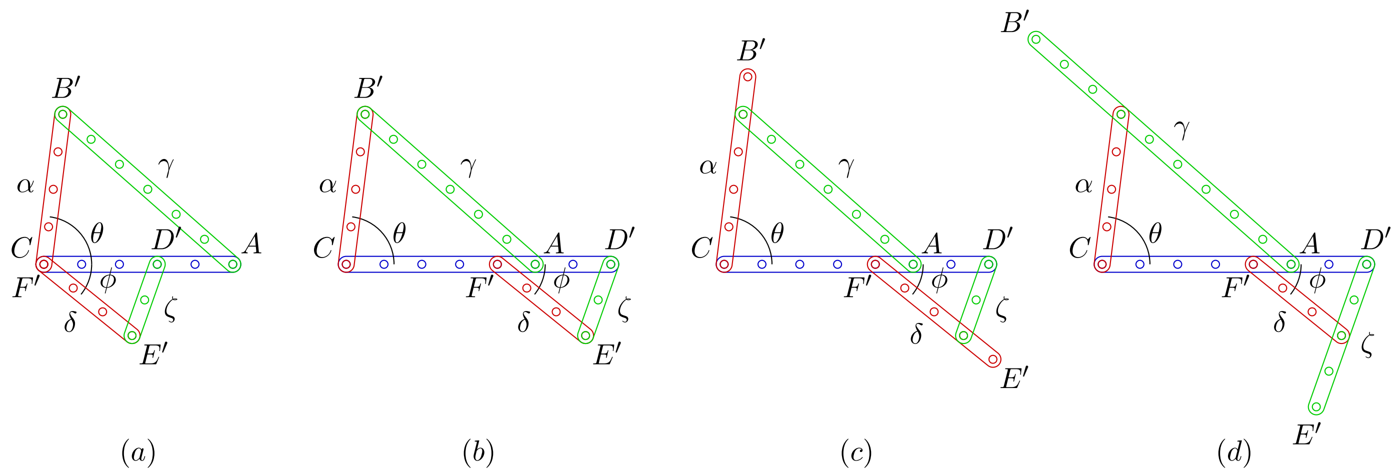

Figure 6: Triangle pair extended frame. Is like previous triangle pair frame except we can extend strips α or γ, δ or ζ, and we can separate vertices C and F'. Vertices A, C, D', F' remain collinear and we are interested in the distance $\overline{B'E'}$. We show four examples: (a) is the original triangle pair, (b) has moved the $\triangle D'E'F'$ to the right, (c) also extends strips α and δ and (d) extends strips γ and ζ.

Some triangle pair extended frames are shown in figure 6. As with figure 4 we also have two triangles with five strips, but we can perform one, two or three transformations on the frame:

1. Separate nodes C and F which moves $\triangle D'E'F'$.
2. Extends strip $a \to \alpha$ or strip $c \to \gamma$ but not both.
3. Extend strip $d \to \delta$ or strip $f \to \zeta$ but not both.

For each transformation we define three integers x, y_1, y_2:

$$x = \begin{cases} 0 & C, F \text{ vertices remain joined} \\ \geq 0 & \triangle DEF \text{ is moved to the right a distance equal to } x \end{cases} \tag{34}$$

$$y_1 = \begin{cases} 0 & \alpha = a, \quad \gamma = c \\ > 0 & \alpha = a + y_1, \quad \gamma = c \\ < 0 & \alpha = a, \quad \gamma = c + |y_1| \end{cases} \tag{35}$$

$$y_2 = \begin{cases} 0 & \delta = d, \quad \zeta = f \\ > 0 & \delta = d + y_2, \quad \zeta = f \\ < 0 & \delta = d, \quad \zeta = f + |y_2| \end{cases} \tag{36}$$

Let define $M(a, b, c)$ the triangle above, $N(d, e, f)$ the triangle below and $T(x, y_1, y_2)$ the transformations. Then we can describe the cases $(a) - (d)$ of figure 6 as operations:

$$\begin{aligned} (a) &: M(4,5,6) \oplus N(4,4,2) \oplus T(0,0,0) \\ (b) &: M(4,5,6) \oplus N(4,4,2) \oplus T(4,0,0) \\ (c) &: M(4,5,6) \oplus N(4,4,2) \oplus T(4,+2,+1) \\ (d) &: M(4,5,6) \oplus N(4,4,2) \oplus T(4,-3,-2) \end{aligned}$$

3.1 Triangle pair extended frame algebra

We are going to calculate the distance $\overline{B'E'}$ of the triangle pair extended using the M, N, T values. We start setting the vertex C at the origin of the standard two-dimensional graph and defining (B_x, By) the abscissa and ordinate of vertex B' and (E_x, E_y) the abscissa and ordinate of vertex E'.

For the triangle above we have two cases: $y_1 \geq 0$ and $y_1 < 0$. In the not-extended triangle pair we already calculated $\theta = \angle ACB, \cos\theta, \sin\theta$ based in m, n of equation 21. For the case $y_1 < 0$ here we calculate also $\omega = \angle BAC, \cos\omega, \sin\omega$ using two variables p, q. For both cases we define $u = |y_1|$ and finally we get (B_x, B_y):

$$(\omega, p, q) \equiv (\angle BAC, b^2 + c^2 - a^2, 2bc), \quad |p| \leq q, \quad p, q \in \mathbb{Z} \tag{37}$$

$$\cos\omega = \frac{p}{q} \tag{38}$$

$$\sin\omega = \sqrt{1 - \cos^2\omega} = \frac{\sqrt{q^2 - p^2}}{q} \tag{39}$$

$$\alpha = a + u \tag{40}$$

$$\gamma = c + u \tag{41}$$

$$B_x = \begin{cases} y_1 \geq 0 & \alpha\cos\theta \\ y_1 < 0 & b - \gamma\cos\omega \end{cases} \tag{42}$$

$$B_y = \begin{cases} y_1 \geq 0 & \alpha\sin\theta \\ y_1 < 0 & \gamma\sin\omega \end{cases} \tag{43}$$

For the triangle below $N(d, e, f)$ we have two cases: $y_2 \geq 0$ and $y_2 < 0$. In both cases we will use $x \geq 0$ always for simplicity. In the not-extended triangle pair we already calculated $\phi = \angle DFE, \cos\phi, \sin\phi$ defining o, p in equation 24. For the case $y_2 < 0$ here we calculate also $D = \angle EDF, \cos D, \sin D$ using two variables r, s. For both cases we define $v = |y_2|$ and finally we get (E_x, E_y):

$$(r, s) \equiv (d^2 + f^2 - e^2, 2df), \quad |r| \leq s, \quad r, s \in \mathbb{Z} \tag{44}$$

$$\cos D = \frac{r}{s} \tag{45}$$

$$\sin D = \sqrt{1 - \cos^2 D} = \frac{\sqrt{s^2 - r^2}}{s} \tag{46}$$

$$\delta = d + v \tag{47}$$

$$\zeta = f + v \tag{48}$$

$$E_x = \begin{cases} y_2 \geq 0 & x + \delta\cos\phi \\ y_2 < 0 & x + e - \zeta\cos D \end{cases} \tag{49}$$

$$E_y = \begin{cases} y_2 \geq 0 & -\delta\sin\phi \\ y_2 < 0 & -\zeta\sin D \end{cases} \tag{50}$$

With the four components B_x, B_y, E_x, E_y we can calculate $g = \overline{B'E'}$:

$$g = \sqrt{(B_x + E_x)^2 + (B_y + E_y)^2} \tag{51}$$

$$= \sqrt{(B_x^2 + B_y^2) + (E_x^2 + E_y^2) + 2B_xE_x + 2B_yE_y} \tag{52}$$

For $y_1 \geq 0$ and $y_2 \geq 0$ we have $m = y_1, n = y_2$:

$$\alpha = a + m, \qquad \delta = d + n \tag{53}$$

$$\begin{aligned} B_x^2 + B_y^2 &= \alpha^2 \cos^2 C + \alpha^2 \sin^2 C \\ &= \alpha^2 \end{aligned} \tag{54}$$

$$\begin{aligned} E_x^2 + E_y^2 &= (x + \delta \cos F)^2 + (-\delta \sin F)^2 \\ &= x^2 + 2x\delta \cos F + \delta^2 \cos^2 F + \delta^2 \sin^2 F \\ &= x^2 + 2x\delta \cos F + \delta^2 \\ &= \frac{f_2 x^2 + 2x\delta f_1 + f_2 \delta^2}{f_2} \end{aligned} \tag{55}$$

$$\begin{aligned} B_x E_x &= (\alpha \cos C)(x + \delta \cos F) \\ &= \frac{\alpha c_1 (f_2 x + \delta f_1)}{c_2 f_2} \end{aligned} \tag{56}$$

$$\begin{aligned} B_y E_y &= (\alpha \sin C)(-\delta \sin F) \\ &= -\frac{\alpha\delta\sqrt{(c_2^2 - c_1^2)(f_2^2 - f_1^2)}}{c_2 f_2} \end{aligned} \tag{57}$$

$$\begin{aligned} g &= \overline{B'E'} \\ &= \sqrt{\alpha^2 + \frac{f_2 x^2 + 2x\delta f_1 + f_2 \delta^2}{f_2} + \frac{2\alpha c_1 (f_2 x + \delta f_1)}{c_2 f_2} - \frac{2\alpha\delta\sqrt{(c_2^2 - c_1^2)(f_2^2 - f_1^2)}}{c_2 f_2}} \\ &= \sqrt{\frac{c_2 f_2 \alpha^2 + c_2(f_2 x^2 + 2x\delta f_1 + f_2 \delta^2) + 2\alpha c_1 (f_2 x + \delta f_1) - 2\alpha\delta\sqrt{(c_2^2 - c_1^2)(f_2^2 - f_1^2)}}{c_2 f_2}} \end{aligned} \tag{58}$$