Meccano nonagon

https://github.com/heptagons/meccano/nona

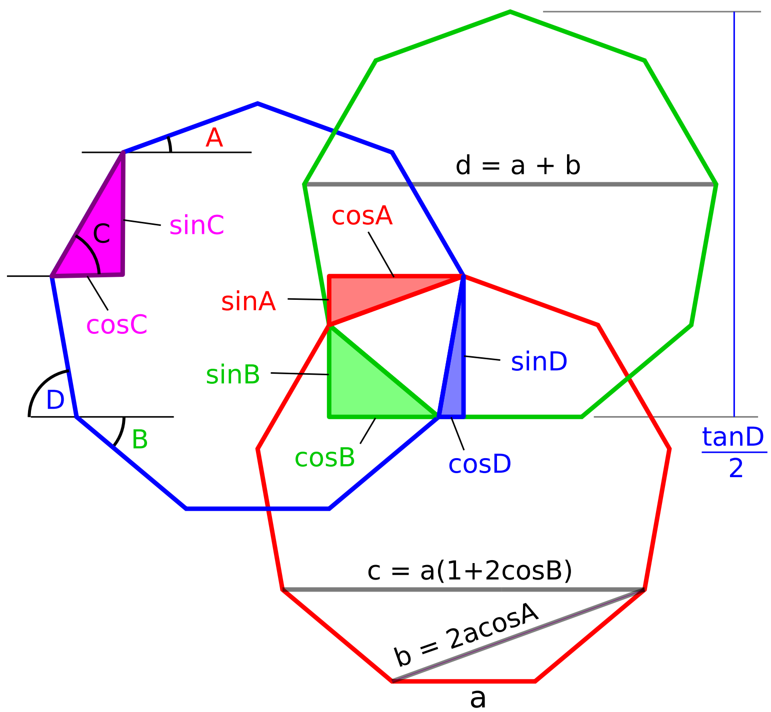

Figure 1: Three regular nonagons connected by an equilateral triangle. We note four angles in the figure A, B, C and D.

1 Algebra

Figure 1 shows three regular nonagons connected by an equilateral triangle. Four angles appear orthogonally in any regular nonagon:

$$A = \pi/9 = 20^\circ \tag{1}$$
$$B = 2\pi/9 = 40^\circ \tag{2}$$
$$C = 3\pi/9 = 60^\circ \tag{3}$$
$$D = 4\pi/9 = 80^\circ \tag{4}$$
$$A + B = D - A = C \tag{5}$$

The relations of angle C are those of equilateral triangle:

$$\cos C = -\frac{1}{2} \tag{6}$$
$$\sin C = \frac{\sqrt{3}}{2} \tag{7}$$

From the figure 1, cosines of angles A, B and D are related as:

$$\begin{aligned}
\cos A &= \cos B + \cos D && (8)\\
&= \cos(2A) + \cos(4A)\\
&= (2\cos^2 A - 1) + (1 - 8\cos^2 A + 8\cos^4 A)\\
&= 8\cos^4 A - 6\cos^2 A\\
1 &= 8\cos^3 A - 6\cos A && (9)
\end{aligned}$$

Previous cosines equation solves this cubic equation:

$$x^3 - \frac{3}{4}x - \frac{1}{8} = 0 \tag{10}$$
$$x_1 = +\cos A \approx +0.939692$$
$$x_2 = -\cos B \approx -0.766044$$
$$x_3 = -\cos D \approx -0.173648$$

From equation 10 we know the product of roots squares is $\frac{3}{2}$:

$$\begin{aligned}
\cos^2 A + \cos^2 B + \cos^2 D &= \frac{3}{2} && (11)\\
1 - \sin^2 A + 1 - \sin^2 B + 1 - \sin^2 D &= \frac{3}{2}\\
\sin^2 A + \sin^2 B + \sin^2 D &= \frac{3}{2} && (12)
\end{aligned}$$

From equation 10 we know the product of roots is $\frac{1}{8}$ matching the "Morrie's law":

$$\begin{aligned}
\cos A\cos B\cos D &= \frac{1}{8} && (13)\\
(1-\sin^2 A)(1-\sin^2 B)(1-\sin^2 D) &= \frac{1}{64}\\
(\sin A\sin B)^2 + (\sin A\sin D)^2 + (\sin C\sin D)^2 &= \frac{1}{64} - 1 + \sin^2 A + \sin^2 B + \sin^2 D + (\sin A\sin B\sin D)^2\\
&= \frac{1}{64} - 1 + \frac{3}{2} + \left(\frac{\sqrt{3}}{8}\right)^2 = \left(\frac{9}{4}\right)^2 && (14)
\end{aligned}$$

From the figure **??**, sines of angles A, B and D are related as:

$$\begin{aligned}
\sin A + \sin B &= \sin D && (15)\\
&= \sin(2A + B) && (16)\\
&= \sin(2A)\cos B + \cos(2A)\sin B && (17)\\
&= 2\sin A\cos A\cos B + (\cos^2 A - \sin^2 A)\sin B && (18)
\end{aligned}$$

Last equation solves this cubic equation:

$$y^3 - \frac{3y}{4} - \frac{3}{8} = 0$$
$$y_1 = -\sin A \approx -0.342020$$
$$y_2 = -\sin B \approx -0.642787$$
$$y_3 = +\sin C \approx +0.984807$$

More sines relations of angles A, B and C are:

$$\sin A \sin B \sin C = \frac{\sqrt{3}}{8}$$
$$\sin^2 A + \sin^2 B + \sin^2 C = \frac{3}{2}$$

Cosines and sines relations are:

$$\cos A \cos B - \sin A \sin B = \frac{1}{2}$$
$$\frac{1}{\cos C} - \frac{\sqrt{3}}{\sin C} = 4$$
$$\tan C - 4 \sin C = \sqrt{3}$$