Meccano polygon diagonals

https://github.com/heptagons/meccano/penta

Abstract

We construct meccano [1] polygon internal diagonals.

1 Polygon diagonals

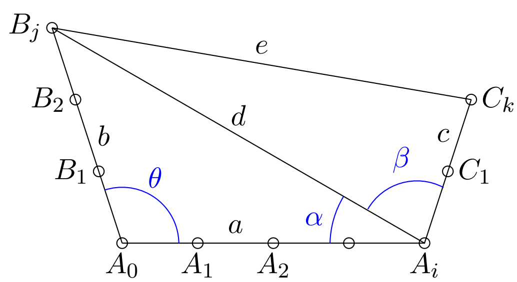

Figure 1: Meccano polygon three consecutive sides segments $a \geq\ b \geq c$ can form two diagonals d and e.

2 Regular polygon diagonals

In the regular polygon all internal angles are equal to θ. From figure 1 the polygon is regular if $\alpha + \beta = \theta$ so we have:

$$\alpha = \angle A_0 A_i B_j \tag{1}$$
$$\beta = \angle B_j A_i C_k \tag{2}$$
$$\theta = \angle B_j A_0 A_i = \angle A_0 A_i C_k \tag{3}$$
$$\alpha + \beta = \theta \tag{4}$$

We use the cosines sum identity to express $\cos\beta$ in function of the rest of variables. We define $u = \cos\theta$:

$$u \equiv \cos\theta \tag{5}$$
$$= \cos(\alpha+\beta) \tag{6}$$
$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{7}$$
$$\sin\beta = \frac{\cos\alpha\cos\beta - u}{\sin\alpha} \tag{8}$$
$$\sin^2\beta = \frac{(\cos\alpha\cos\beta - u)^2}{\sin^2\alpha} \tag{9}$$
$$1 - \cos^2\beta = \frac{\cos^2\alpha\cos^2\beta - 2u\cos\alpha\cos\beta + u^2}{sin^2\alpha} \tag{10}$$

[1] Meccano mathematics by ‘t Hooft

We set $X = \cos\beta$ and rearrange the last equation to get:

$$X^2 - 2u\cos\alpha X + u^2 - \sin^2\alpha = 0 \tag{11}$$

And solve the quadratic equation $AX^2 + BX + C = 0$ to get $\cos\beta$ in function of u and α:

$$\begin{aligned}\cos\beta &= \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \\ &= \frac{2u\cos\alpha \pm \sqrt{4u^2\cos^2\alpha - 4(u^2 - \sin^2\alpha)}}{2} \\ &= u\cos\alpha \pm \sqrt{u^2\cos^2\alpha - u^2 + \sin^2\alpha}\end{aligned} \tag{12}$$

Now, we need to find the values of $\cos\alpha$, $\sin\alpha$ and $\cos\beta$ which in turn need the value of d, all in terms of a, b, c the segments of the polygon perimeter.

For the value of d we use the law of cosines:

$$\begin{aligned}d &= \sqrt{a^2 + b^2 - 2ab\cos\theta} \\ &= \sqrt{a^2 + b^2 - 2abu}\end{aligned} \tag{13}$$

Using the law of cosines we calculate the angles $\alpha = \angle A_0 A_i B_j$ and $\beta = \angle B_j A_i C_k$:

$$\begin{aligned}\cos\alpha &= \frac{a^2 + d^2 - b^2}{2ad} \\ &= \frac{a^2 + (a^2 + b^2 - 2abu) - b^2}{2ad} \\ &= \frac{a - bu}{d}\end{aligned} \tag{14}$$

$$\begin{aligned}\cos\beta &= \frac{c^2 + d^2 - e^2}{2cd} \\ &= \frac{c^2 + (a^2 + b^2 - 2abu) - e^2}{2cd} \\ &= \frac{a^2 + b^2 + c^2 - e^2 - 2abu}{2cd}\end{aligned} \tag{15}$$

We define new variable f to simplify $\cos\beta$ to obtain:

$$f \equiv \frac{a^2 + b^2 + c^2 - e^2}{2} \tag{16}$$

$$\cos\beta = \frac{f - abu}{cd} \tag{17}$$

We calculate $\sin^2\alpha = 1 - \cos^2\alpha$:

$$\begin{aligned}\sin^2\alpha &= 1 - \frac{(a - bu)^2}{d^2} \\ &= \frac{d^2 - a^2 + 2abu - b^2u^2}{d^2} \\ &= \frac{(a^2 + b^2 - 2abu) - a^2 + 2abu - b^2u^2}{d^2} \\ &= \frac{b^2(1 - u^2)}{d^2}\end{aligned} \tag{18}$$

We plug the values of $\cos\alpha, \cos\beta, \sin^2\alpha$ in equation 12 to get:

$$
\begin{aligned}
\frac{f-abu}{cd} &= \left(\frac{a-bu}{d}\right)u \pm \sqrt{\left(\frac{a-bu}{d}\right)^2 u^2 - u^2 - \frac{b^2(1-u^2)}{d^2}} \\
\frac{f-abu}{c} &= (a-bu)u \pm \sqrt{(a-bu)^2u^2 - d^2u^2 - b^2(1-u^2)} \\
f &= (ab+ac-bcu)u \pm c\sqrt{(a-bu)^2u^2 - d^2u^2 - b^2 + b^2u^2)} \\
&= abu + acu - bcu^2 \pm c\sqrt{a^2u^2 - 2abu^3 + b^2u^4 - d^2u^2 - b^2 + b^2u^2} \qquad (19)
\end{aligned}
$$

2.1 Regular polygon diagonal e

We define variables m, n to simplify f, so we have:

$$m = abu + acu - bcu^2 \qquad (20)$$

$$n = a^2u^2 - 2abu^3 + b^2u^4 - d^2u^2 - b^2 + b^2u^2 \qquad (21)$$

$$f = m + c\sqrt{n} \qquad (22)$$

$$\frac{a^2+b^2+c^2-e^2}{2} = m + c\sqrt{n} \qquad (23)$$

$$e^2 = a^2 + b^2 + c^2 - 2m - 2c\sqrt{n} \qquad (24)$$

3 Regular pentagon diagonals

For the regular pentagon we have $u = \cos\theta = \cos(3\pi/5)$:

$$u = \frac{1-\sqrt{5}}{4} \qquad (25)$$

$$u^2 = \frac{3-\sqrt{5}}{8} \qquad (26)$$

$$u^3 = \frac{2-\sqrt{5}}{8} \qquad (27)$$

$$u^4 = \frac{7-3\sqrt{5}}{32} \qquad (28)$$

We plug the value of pentagon's u in equation 13 to get d^2 for the pentagon:

$$
\begin{aligned}
d^2 &= a^2 + b^2 - 2ab\left(\frac{1-\sqrt{5}}{4}\right) \\
&= \frac{4a^2 + 4b^2 - 2ab + 2ab\sqrt{5}}{4} \qquad (29)
\end{aligned}
$$

We define variables d_1, d_2 to simplify the previous equation:

$$d_1 = 4a^2 + 4b^2 - 2ab \qquad (30)$$

$$d_2 = 2ab \qquad (31)$$

$$d^2 = \frac{d_1 + d_2\sqrt{5}}{4} \qquad (32)$$

We plug the values of pentagon's u, u^2, u^3, u^4 in equations 20 and 21 to get pentagon's m, n:

$$
\begin{aligned}
m &= ab\left(\frac{1-\sqrt{5}}{4}\right) + ac\left(\frac{1-\sqrt{5}}{4}\right) - bc\left(\frac{3-\sqrt{5}}{8}\right) \\
&= \frac{2ab - 2ab\sqrt{5} + 2ac - 2ac\sqrt{5} - 3bc + bc\sqrt{5}}{8} \\
&= \frac{2ab + 2ac - 3bc + (bc - 2ab - 2ac)\sqrt{5}}{8} \qquad (33) \\
n &= a^2\left(\frac{3-\sqrt{5}}{8}\right) - 2ab\left(\frac{2-\sqrt{5}}{8}\right) + b^2\left(\frac{7-3\sqrt{5}}{3}2\right) - d^2\left(\frac{3-\sqrt{5}}{8}\right) - b^2 + b^2\left(\frac{3-\sqrt{5}}{8}\right) \\
&= \frac{12a^2 - 4a^2\sqrt{5} - 16ab + 8\sqrt{5} + 7b^2 - 3b^2\sqrt{5} - 12d^2 + 4d^2\sqrt{5} - 32b^2 + 12b^2 - 4b^2\sqrt{5}}{32} \\
&= \frac{12a^2 - 16ab - 13b^2 - 12d^2 + (-4a^2 + 8 - 3b^2 + 4d^2 - 4b^2)\sqrt{5}}{32} \qquad (34)
\end{aligned}
$$

We substitute d^2 of last equation with equation 32 value in terms of d_1, d_2 to isolate correctly $\sqrt{5}$ factors:

$$
\begin{aligned}
n &= \frac{12a^2 - 16ab - 13b^2 - 3(d_1 + d_2\sqrt{5}) + (-4a^2 + 8 - 3b^2 + (d_1 + d_2\sqrt{5}) - 4b^2)\sqrt{5}}{32} \\
&= \frac{12a^2 - 16ab - 13b^2 - 3d_1 + 5d_2 + (-4a^2 + 8 - 3b^2 + d_1 - 3d_2)\sqrt{5}}{32} \qquad (35)
\end{aligned}
$$

We define m_1, m_2, n_1, n_2 to simplify previous formulas of m, n and let them to be functions of only a, b, c, to obtain:

$$
\begin{aligned}
m_1 &= 2ab + 2ac - 3bc \qquad (36) \\
m_2 &= bc - 2ab - 2ac \qquad (37) \\
n_1 &= 12a^2 - 16ab - 13b^2 - 3d_1 + 5d_2 \\
&= 12a^2 - 16ab - 13b^2 - 3(4a^2 + 4b^2 - 2ab) + 5(2ab) \\
&= -10ab - 15b^2 \qquad (38) \\
n_2 &= -4a^2 + 8 - 3b^2 + (4a^2 + 4b^2 - 2ab) - 3(2ab) \\
&= b^2 - 8ab + 8 \qquad (39) \\
m &= \frac{m_1 + m_2\sqrt{5}}{8} \\
&= \frac{2ab + 2ac - 3bc + (bc - 2ab - 2ac)\sqrt{5}}{8} \qquad (40) \\
n &= \frac{n_1 + n_2\sqrt{5}}{32} \\
&= \frac{-20ab - 30b^2 + (2b^2 - 16ab + 16)\sqrt{5}}{64} \qquad (41)
\end{aligned}
$$

Finally we plug m_1, m_2, n_1, n_2 in equation 24 to get e in function of a, b, c:

$$\begin{aligned}
e^2 &= a^2 + b^2 + c^2 - 2m - 2c\sqrt{n} \\
&= a^2 + b^2 + c^2 - 2\left(\frac{2ab + 2ac - 3bc + (bc - 2ab - 2ac)\sqrt{5}}{8}\right) - 2c\sqrt{\frac{-20ab - 30b^2 + (2b^2 - 16ab + 16)\sqrt{5}}{64}} \\
&= a^2 + b^2 + c^2 - \frac{2ab + 2ac - 3bc + (bc - 2ab - 2ac)\sqrt{5}}{4} - \frac{c\sqrt{-20ab - 30b^2 + (2b^2 - 16ab + 16)\sqrt{5}}}{4}
\end{aligned} \tag{42}$$

From the figure we know that when $c = 0$ e^2 becomes d^2 so we can confirm this:

$$\begin{aligned}
d^2 &= e^2_{c=0} \\
&= a^2 + b^2 - \frac{2ab - 2ab\sqrt{5}}{4} \quad \square
\end{aligned} \tag{43}$$

3.1 Regular pentagon width W

The regular pentagon width W is defined as the distance between two farthest separated points, which equals the diagonal length D which is given by:

$$W = D = \frac{1 + \sqrt{5}}{2}a \tag{44}$$

In our case the width is the diagonal d when $a = b$ or also de e when $a = b, c = 0$.

$$\begin{aligned}
d_W &= \frac{\sqrt{4a^2 + 4a^2 + 2a^2 + 2a^2\sqrt{5}}}{2} \\
&= \frac{\sqrt{6 - 2\sqrt{5}}}{2}a \\
&= \frac{\sqrt{5} + 1}{2}a \quad \square
\end{aligned} \tag{45}$$

3.2 Regular pentagon height H

In the regular pentagon the height H is the distance from one side of length a to the opposite vertex:

$$H = \frac{\sqrt{5 + 2\sqrt{5}}}{2}a \tag{46}$$

For the height to occur (coincident with diagonal e) we need $a = b = c/2$. We plug $b = a$ and $c = a/2$ in equation 47:

$$\begin{aligned}
H^2 &= e^2 \\
&= a^2 + a^2 + \frac{a^2}{4} - \frac{2a^2 + a^2 - \frac{3a^2}{2} + \left(\frac{a^2}{2} - 2a^2 - a^2\right)\sqrt{5}}{4} - \frac{\frac{a}{2}\sqrt{-20a^2 - 30a^2 + (2a^2 - 16a^2 + 16)\sqrt{5}}}{4}
\end{aligned} \tag{47}$$