Meccano heptagons

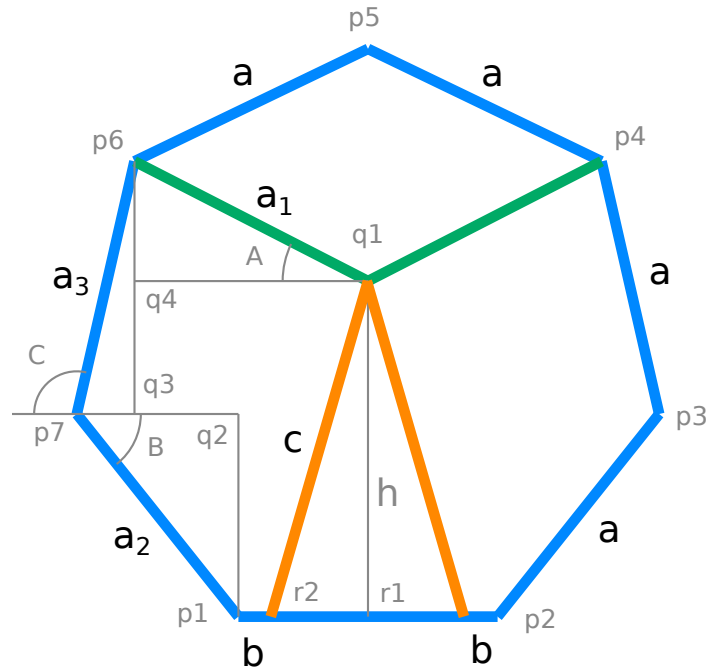https://github.com/heptagons/meccano/hepta

1 Meccano heptagons

Figure 1: A meccano regular heptagon layout. First we define two integers a and b where $a > 2b$. We look for a third integer c to make the heptagon.

Consider the regular heptagon in figure 1. By inspection we identify three angles A, B and C:

$$A = \frac{\pi}{7}$$
$$B = \frac{2\pi}{7}$$
$$C = \frac{4\pi}{7}$$

Then we find the sines of the angles, noticing that the regular heptagon side is $a = a_1 = a_2 = a_3$:

$$sinA = \frac{\overline{p_6 q_4}}{a_1}$$

$$sinB = \frac{\overline{p_1 q_2}}{a_2}$$

$$sinC = \frac{\overline{p_6 q_3}}{a_3}$$

From the figure the height h corresponds to:

$$\begin{aligned}
h &= \overline{p_1 q_2} + \overline{p_6 q_3} - \overline{p_6 q_4} \\
&= a_2 sinB + a_3 sinC - a_1 sinA \\
&= a(-sinA + sinB + sinC)
\end{aligned}$$

According to *heptagonal triangles*[1]

$$\begin{aligned}
sinA - sinB - sinC &= -\frac{\sqrt{7}}{2} \\
\frac{h}{a} &= \frac{\sqrt{7}}{2} \\
h &= \frac{\sqrt{7}a}{2}
\end{aligned}$$

Finally we get the c length as a function of lengths a and b:

$$\begin{aligned}
c^2 &= \overline{r_1 r_2}^2 + h^2 \\
&= \frac{(a-b)^2}{4} + \frac{7a^2}{4} \\
&= \frac{8a^2 - 2ab + b^2}{4}
\end{aligned}$$

1.1 Heptagons search

A valid meccano heptagon needs to have the three lenghts a, b and c as integers. With a software routine we look for c to be integer by incrementing the values of $a > b$.

1.1.1 Code

Following javascript code running inside a web page is used to find several heptagons:

```
1  <script type="text/javascript">
2  const gcd = (a, b)=> { return !b ? a : gcd(b, a % b) }
3
4  let i = 1
5  for (let a=2; a <= 100; a++) {
```

---

[1]https://en.wikipedia.org/wiki/Heptagonal_triangle

```
 6      for (let e=1; e < a; e++) {
 7        const c = Math.sqrt(7*a*a + e*e)/2;
 8        if ((c - parseInt(c)) == 0) {
 9          if (gcd(c, gcd(a, e)) == 1) {
10            console.log(`N=${i}: a=${a} b=${(a-e)/2} c=${c}`);
11            i++;
12          }
13        }
14      }
15  }
16  </script>
```

1.1.2 Results

Browser console first heptagons with $a < 100$:

```
 1  N= 1:  a= 3  b= 1  c=   4
 2  N= 2:  a= 8  b= 1  c=  11
 3  N= 3:  a=33  b= 2  c=  46
 4  N= 4:  a=40  b=17  c=  53
 5  N= 5:  a=55  b=14  c=  74
 6  N= 6:  a=65  b=31  c=  86
 7  N= 7:  a=85  b=14  c= 116
 8  N= 8:  a=91  b= 2  c= 128
 9  N= 9:  a=95  b= 1  c= 134
10  N=10:  a=96  b=47  c= 127
```

1.1.3 Smallest examples
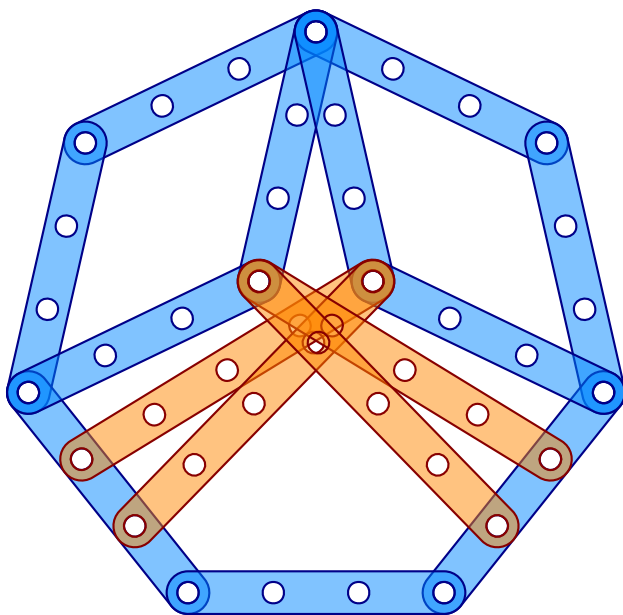
Figures 2 and 3 show the first two heptagons.

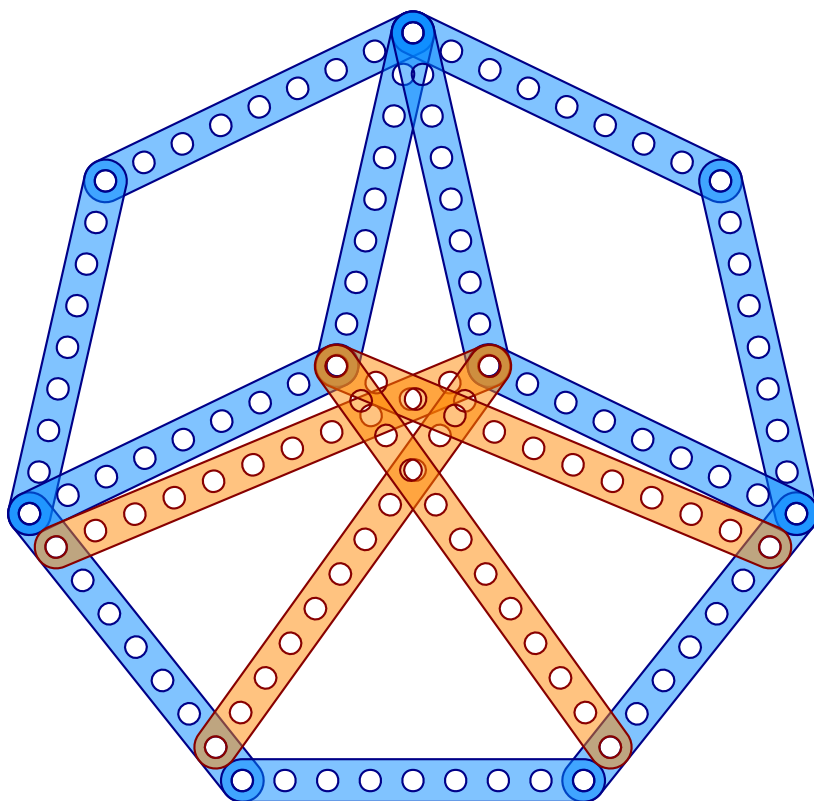Figure 2: The first meccano heptagon with values $a = 3$, $b = 1$ and $c = 4$.

Figure 3: The second meccano heptagon with values $a = 8$, $b = 1$ and $c = 11$.