$$|a_k(x+t) - a_k(x)| \leqslant 2^{-k-1}. \tag{5}$$

It follows from (1) and (5) that for any $|t| \leqslant 1$,

$$\sum_{k=n}^{\infty} |a_k(x+t) - a_k(x)| \leqslant 2^{-n} \leqslant 2|t| \leqslant 2|t|^s. \tag{6}$$

Summing inequalities (3) and (6), we obtain

$$\sum_{k=0}^{\infty} |a_k(x+t) - a_k(x)| \leqslant \{2 + n2^{-n(1-s)}\}|t|^s. \tag{7}$$

It is clear that there is a constant M_s such that

$$2 + n2^{-n(1-s)} \leqslant M_s \tag{8}$$

for every integer $n \geqslant 0$. Now (7) and (8) imply that

$$|f(x+t) - f(x)| \leqslant M_s|t|^s$$

whenever $0 < |t| \leqslant 1$.

Now we consider the case $|t| > 1$. It follows from (4) that

$$0 \leqslant f(x) \leqslant 1,$$

and for any $|t| > 1$, this yields

$$|f(x+t) - f(x)| \leqslant 1 \leqslant |t|^s$$

for any two real numbers x and t. Hence, choosing the same M_s as above ensures that

$$|f(x+t) - f(x)| \leqslant M_s|t|^s$$

for any t, and the proof is complete.

References

1. Patrick Billingsley, Van der Waerden's continuous nowhere differentiable function, this MONTHLY, 89 (1982) 691.

2. F. S. Cater, On Van der Waerden's nowhere differentiable function, this MONTHLY, 91 (1984) 307–308.

A NOTE ON EASY PROOFS OF STIRLING'S THEOREM*

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Introduction. For Stirling's Theorem

$$\lim_{n\to\infty} \frac{\sqrt{2\pi}\, e^{-n} n^{n-.5}}{\Gamma(n)} = 1 \tag{1}$$

or, equivalently, multiplying numerator and denominator each by n,

$$\lim_{n\to\infty} \frac{\sqrt{2\pi}\, e^{-n} n^{n+.5}}{n!} = 1.$$

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Khan [2] and Wong [5] give easy proofs valid for integers n only, by applying the Central Limit Theorem to Gamma and Poisson random variables respectively.

In Sections 1 and 2 these proofs are shortened; also they are made more elementary and Khan's extended to non-integers n by noticing that, not the Central Limit Theorem, but only the limit theorem for moment generating functions (or characteristic functions), is needed.

Section 3 gives an even shorter and more elementary proof by applying the inversion theorem for characteristic functions to a Gamma random variable. This method also has the advantage that it can be used (but the only proofs we have are too long and complicated to be satisfactory) to prove the extended Stirling Theorem

$$\frac{\Gamma(n)}{\sqrt{2\pi}\, n^{n-.5}e^{-n}} = 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + O\left(\frac{1}{n^4}\right). \tag{2}$$

1. Khan's Proof. Because it uses the Central Limit Theorem, Khan's [2] proof is valid for integers n only. By using instead the limit theorem for moment generating functions, his proof can be made valid for non-integers as well. This proof begins by calculating the average deviation of a Gamma(n) random variable:

LEMMA 1. *If the random variable X has Gamma(n) probability density, with $n > 0$ not necessarily an integer, given by*

$$\frac{1}{\Gamma(n)}x^{n-1}e^{-x} \quad \text{for } x > 0,$$

then for $Z = (X-n)/\sqrt{n}$ we have

$$E|Z| = \frac{2e^{-n}n^{n-.5}}{\Gamma(n)}. \tag{3}$$

Proof. (A shorter proof of (3) than Khan's).

$$\begin{aligned} E|X-n| &= \frac{1}{\Gamma(n)}\int_0^\infty |x-n|x^{n-1}e^{-x}\,dx \\ &= -\frac{1}{\Gamma(n)}\int_0^n (x-n)x^{n-1}e^{-x}\,dx + \frac{1}{\Gamma(n)}\int_n^\infty (x-n)x^{n-1}e^{-x}\,dx \\ &= -\frac{-2}{\Gamma(n)}\int_0^n (x-n)x^{n-1}e^{-x}\,dx + \frac{1}{\Gamma(n)}\int_0^\infty (x-n)x^{n-1}e^{-x}\,dx. \end{aligned}$$

This last integral is $E(X-n) = EX - n = 0$, giving

$$E|X-n| = -\frac{2}{\Gamma(n)}\int_0^n x^n e^{-x}\,dx + \frac{2}{\Gamma(n)}\int_0^n nx^{n-1}e^{-x}\,dx.$$

Integrating the first of these integrals by parts gives

$$E|X-n| = \frac{2}{\Gamma(n)}e^{-n}n^n$$

and division by $\sqrt{n}$ gives (3), completing the proof of the lemma.

Now, to prove Stirling's Theorem, consider the moment generating function of Z which is given, for $t/\sqrt{n} < 1$, by

$$Ee^{t(X-n)/\sqrt{n}} = e^{-t\sqrt{n}}\frac{1}{\Gamma(n)}\int_0^\infty e^{tx/\sqrt{n}}e^{-x}x^{n-1}\,dx$$

$$= e^{-t\sqrt{n}}\left(1 - t/\sqrt{n}\right)^{-n} = e^{-t\sqrt{n} - n\log(1 - t/\sqrt{n})}$$
$$= \exp\left\{\frac{t^2}{2} + O\left(1/\sqrt{n}\right)\right\}.$$

As $n \to \infty$, this converges to $\exp(t^2/2)$ for every t, giving the moment generating function of the Standard Normal distribution, showing that the distribution function of Z converges to the Standard Normal. Moreover $EZ^2 = 1$ for every n, so the moment convergence theorem (Loève [**3**], p. 184) shows that $E|Z|$ must converge to the first absolute moment $\sqrt{(2/\pi)}$ of the Standard Normal. From (3),

$$\lim_{n\to\infty} \frac{2e^{-n}n^{n-.5}}{\Gamma(n)} = \sqrt{\frac{2}{\pi}}.$$

Division by $\sqrt{(2/\pi)}$ now gives Stirling's theorem (1).

NOTE. Having proved Stirling's theorem over integer and non-integer values n, we can now easily conclude for fixed a that

$$\lim_{n\to\infty} \frac{\Gamma(n+a)}{n^a\Gamma(n)} = 1, \tag{4}$$

instead of proving (4) and using it to derive Stirling's Theorem as Titchmarsh [**4**], page 58, does.

2. Wong's Proof. Wong's [**5**] proof of Stirling's theorem is valid for integers n only and can not be extended to non-integers. His proof can be made slightly more elementary by noticing that, not the Central Limit Theorem, but only the limit theorem for moment generating functions is needed; it can be made easier by using the more familiar $E|X - n|$ instead of $E(X - n)^-$ which eliminates the need to appeal to the Tucker theorem. This proof begins by calculating the average deviation of a Poisson (n) random variable.

LEMMA 2. *If the random variable X has Poisson (n) distribution with $n > 0$ an integer, then for $Z = (X - n)/\sqrt{n}$ we have*

$$E|Z| = \frac{2e^{-n}n^{n+.5}}{n!}. \tag{5}$$

Proof. This proof of (5) is a discrete version of the proof of (3):

$$E|X - n| = \sum_{x=0}^{\infty} |x - n| e^{-n}\frac{n^x}{x!}$$
$$= -e^{-n}\sum_{x=0}^{n-1}(x - n)\frac{n^x}{x!} + e^{-n}\sum_{x=n}^{\infty}(x - n)\frac{n^x}{x!}$$
$$= -2e^{-n}\sum_{x=0}^{n-1}(x - n)\frac{n^x}{x!} + e^{-n}\sum_{x=0}^{\infty}(x - n)\frac{n^x}{x!}.$$

This last sum is $E(X - n) = EX - n = 0$. The sum preceding it telescopes to give

$$E|X - n| = \frac{2e^{-n}n^n}{(n-1)!}$$

and division by $\sqrt{n}$ gives (5), completing the proof of the lemma.

Now, to prove Stirling's Theorem, consider the moment generating function of Z, which is given by

$$Ee^{t(X-n)/\sqrt{n}} = e^{-t\sqrt{n}} \sum_{x=0}^{\infty} e^{tx/\sqrt{n}} n^x \frac{e^{-n}}{x!}$$

$$= e^{-t/\sqrt{n}} \exp\{n(e^{t/\sqrt{n}} - 1)\} = \exp\left\{\frac{t^2}{2} + O(1/\sqrt{n})\right\},$$

and Stirling's theorem (1) now follows in exactly the same way as in Section 1.

NOTE. If this proof is tried with n not an integer, it gives the same result but with n replaced by $[n]$, the greatest integer in n.

3. A New Proof. This proof is shorter than the preceding two, is slightly more elementary, and is valid for arbitrary n:

If Z has continuous probability density function $f(\ )$ and integrable characteristic function $\phi(\ )$, then the inversion theorem for characteristic functions, Chung [**1**], p. 143, gives

$$f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t)\, dt. \tag{6}$$

For the Z of Section 1 we have

$$f(0) = \frac{n^{n-.5}e^{-n}}{\Gamma(n)}$$

and, just as done in Section 1 for the moment generating function,

$$\phi(t) = e^{-it\sqrt{n}}(1 - it/\sqrt{n})^{-n} = e^{-t^2/2 + O(1/\sqrt{n})}.$$

Moreover, $\phi(t)$ is easily seen to be dominated by $1/(1 + t^2/2)$ for $n \geq 2$, so the dominated convergence theorem gives

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t^2/2}\, dt = 1/\sqrt{2\pi}.$$

Taking limits of the two sides of (6) for this Z therefore gives

$$\lim_{n\to\infty} \frac{n^{n-.5}e^{-n}}{\Gamma(n)} = 1/\sqrt{2\pi},$$

and multiplication by $\sqrt{2\pi}$ now gives Stirling's Theorem (1).

NOTE. It would be nice to have a proof of the extended Stirling Theorem (2) that is comparable in ease to these proofs of (1). The proof of this section, by using an expansion of $\phi(t)$ in (6), can be used to get a proof of (2), but the proof as we have it is too long and complicated to be satisfactory.

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ANSWER TO PHOTO ON PAGE 373

Richard Courant.