Decidability of Presburger Arithmetic

How far does decidability go?

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March 13, 2025

What's being discussed?

- Today I'll talk about *Presburger Arithmetic* (PrA), what it is, and how far it can be extended whilst still being decidable. The theory is named in honour of Mojżesz Presburger, who published the original proof of its decidability in 1929.
- Here, when I say that a theory is *decidable*, I mean that there exists a computable algorithm which, in finite time, tells us whether any given sentence of a theory is true or not.
- And by *sentence of a theory*, I mean a sentence of PrA — we'll go into this more formally soon, but for the time being, think of statements of PrA as mathematical statements over the integers which only mention addition, but not multiplication.

Mojżesz Presburger

Why care?

- Why care about the decidability of arithmetic theories? We know from Gödel that all the interesting ones are undecidable. But we do have a few reasons to care:
  - There are algorithms which only need to make use of first-order logic with a few additional mathematical constructs, and so it'd be nice to know in advance if these algorithms are workable.
  - Even if a theory is decidable, there are still questions about the computational complexity required to decide that theory (and in particular you can get to some silly large complexity classes in this field). Asking these sorts of questions, then, gives us an idea of how hard and complex certain theories are in comparison with others.
  - It's fun to think about logic, really.
    (this is the reason that I care about it)
- Despite the first paper on Presburger arithmetic being published in 1929, it's still actively developed and studied — indeed, I'm doing a master's project on it next year.

A Definition of FOL

- First off is first-order logic (FOL) over the integers, with addition and order. For a brief refresher, we'll be building up our formulae as Presburger [5] does by starting with terms:
  - Constants 0 and 1,
  - Integer variables $x, y, z, \ldots$, and
  - Sums of terms, using $+$.

  We then construct atomic formulae:
  - Equalities $a = b$, and
  - Inequalities $a < b$.

  Finally, we build up meaningful (well-formed) formulae, using:
  - Negation: If ϕ is a formula, then $\neg\phi$ is a formula,
  - Implication: If ϕ and ψ are formulae, then $\phi \to \psi$ is a formula, and
  - Existential quantification: If ϕ is a formula and x a variable, then $\exists x\ \phi$ is a formula.

  What about conjunction, disjunction, and universal quantification? We just take these as abbreviations of $\neg(\phi \to \neg\psi), \neg\phi \to \psi$ and $\neg\exists x\ \neg\phi$, and get the semantics that we want.

We'll skip the semantics of FOL, but it's what we'd expect.

A definition of PrA

We have our logic and the symbols of it; let's consider the proof system. Again we follow Presburger [5] closely. So let's consider a set of formulae A, of which the following (instantiations of the schemata) are members:

- $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$
- $(\neg\phi \to \phi) \to \phi$
- $\phi \to (\neg\phi \to \psi)$

then the following sentences of equality:

- $a = a$
- $(a = b) \to ((a = c) \to (b = c))$

and the following sentences defining the semantics of $+$:

- $(a = b) \to (a + c = b + c)$
- $(a + c = b + c) \to (a = b)$
- $a + b = b + a$
- $a + (b + c) + (a + b) + c$
- $a + 0 = a$
- $\exists b(a + b = c)$

A definition of PrA

Then, we add recursive schemata to the set of formulae A:

- For each $n \in \mathbb{N}$, a formula $(a + \ldots_n + a = b + \ldots_n + b) \to (a = b)$,
- For each $n \in \mathbb{N}$, a formula $\exists x\ (x + \ldots_n + x = a) \vee (x + \ldots_n + x = a + 1) \vee \cdots \vee (x + \ldots_n + x = a + 1 + \ldots_{n-1} + 1)$, and
- For each $n \in \mathbb{N}$, a formula $\neg(a + \ldots_n + a + 1 = 0)$.

To finish off the theory, we close our set of formulae A under the following operations:

- If $\phi \in A$, then $\phi[x/y] \in A$ also. (i.e. every substitution of variables in ϕ is in A as well)
- If $\phi \in A$ and $\phi \to \psi \in A$, then $\psi \in A$ also.
- If $\exists x \phi \to \psi \in A$, then $\phi \to \psi \in A$ also.
- If $\phi \to \psi \in A$, and x is (a) free in q and (b) doesn't appear in ψ, then $\exists x \phi \to \psi \to \psi \in A$ also.

This set A is then just the set of formulae of PrA, just as defined by Presburger [5].

Some Examples

- The above is a little bit confusing at first, but it helps to know that most sensible statements of arithmetic that only mention addition are statements of PrA. For instance, we can express:
  - That all integers are positive, negative or zero:

    $$\forall x\ (x < 0 \lor x = 0 \lor 0 < x)$$

  - Constant multiplication:

    $$e.g.\ a + a + a + a = b + b + b \text{ expresses that } 4a = 3b$$

  - Modular congruences:

    $$a \equiv_n b \text{ can be expressed by } \exists x(nx + a = b)$$

  - Discrete differentiation: given a function f, we can write

    $$\Delta_a f(x) \stackrel{\text{def}}{=} f(x + a) - f(x)$$

    which gives us the *discrete derivative* of that function, letting us state some facts about calculus in PrA.
- It turns out that PrA is quite a brave little theory, then — so it's good news that it's decidable!

A Decidability Proof for PrA

Let's now prove that PrA is decidable, closely following Presburger [5]. We first define *ground statements*:

Definition

Ground statements are those of the form $a = b$ or $a \equiv_n b$.

It can be seen right away that all ground statements are equivalent either to $c = 0$ or $c \equiv_n 0$, where c is some natural number. And from this we can see that such statements are decidable:

- If we have $c = 0$, then it's immediate; we just check if c is zero.
- If we have $c \equiv_n 0$, we just check if c is a multiple of n. In particular, we can check whether $c = n$, then whether $c = n + n$, and so on. Since this process is bounded by c, it's decidable.

Ground statements have particularly nice properties about decidability, then.

A Decidability Proof for PrA

Indeed, we get the following lemma:

Lemma

Every formula ϕ of PrA can be transformed into an equivalent formula ϕ', where ϕ':

- *is in disjunctive normal form, and*
- *where the members of each disjunct are either ground statements, or negations of ground statements.*

We can then see that every such formula ϕ' can easily be decided; we just have to go over each of the disjuncts, and check that at least one of them is true. If so, then we have a true formula. Otherwise, we have a false formula. This lemma, then, is key to Presburger's proof! So let's prove it.

A Decidability Proof for PrA

Let's work through Presburger's proof, and whilst at it try to prove our own example; let's show commutativity of addition:

$$\forall x \forall y \ (x + y = y + x).$$

We can then get rid of the universal quantifiers to write this as:

$$\neg \exists x \exists y \ \neg (x + y = y + x)$$

On with the proof, then.

Proof.

- It's fairly easy to show that, for any formula ϕ, we can transform it into a formula ψ where ψ is in *disjunctive prenex form* — that just means that
  - the formula is of the form $Q_1 Q_2 \ldots Q_n \chi$, where each Q_i is either $\exists x_i$ or $\neg \exists x_i$, and
  - the quantifier-free part of the formula, ϕ, is in disjunctive normal form (DNF) (so a disjunction of clauses).

  We'll skip over the proof of that here. Conveniently, our example is already in DNF, and so we don't need to do anything at this point.

A Decidability Proof for PrA

Proof. (Cont.)

- Given a formula ψ in disjunctive prenex form, we can then distribute the innermost existential quantifier over each of the disjuncts, to obtain an equivalent formula. Formally, we'd transform

$$Q_1 Q_2 \dots Q_n \, (\chi_1 \vee \chi_2 \vee \dots \vee \chi_m)$$

  into

$$Q_1 Q_2 \dots Q_{n-1}(Q_n\chi_1 \vee Q_n\chi_2 \vee \dots \vee Q_n\chi_m).$$

  Again, conveniently, our example is in the right form, and so we stick with

$$\neg\exists x \exists y \; \neg(x + y = y + x).$$

- At this point, we have to deal with arbitrary disjuncts of the form $Q\chi_i = Q(r_1 \wedge r_2 \wedge \dots \wedge r_p)$, where each r_i is a ground statement. Let's follow Presburger, and only consider the case where each disjunct has just two conjuncts — where each disjunct is of the form $Q(r_1 \wedge r_2)$.
- Each ground term has one of the following forms:
  - An equation, $a = b$,
  - A congruence, $a \equiv_n b$,
  - A negated equation, $\neg(a = b)$, and
  - A negated congruence, $\neg(a \equiv_n b)$.

A Decidability Proof for PrA

Proof. (Cont.)

Conveniently, negated congruences can be reduced to congruences in the following way:

$$\neg(a \equiv_n b) \equiv (a + 1 \equiv_n b) \vee (a + 2 \equiv_n b) \vee \cdots \vee (a + n - 1 \equiv_n b)$$

and so we'll ignore that case here.
Given this, there are six possibilities for what $\exists x(r_1 \wedge r_2)$ is:

1. An equation and an equation,
2. An equation and a negated equation,
3. An equation and a congruence,
4. A negated equation and a negated equation,
5. A negated equation and a congruence, and
6. A congruence and a congruence.

Let's consider how to decide each of these in turn.

A Decidability Proof for PrA

Proof. (Cont.)

1. If we have two equations, then we can transform them into the form

$$\exists x(c_1 x + a_1 = b_1 \wedge c_2 x + a_2 = b_2)$$

i.e. two linear equations in x. Say that n is equal to $c_1 c_2$; then the above is equivalent to

$$\exists x(nx + c_2 a_1 = c_2 b_1 \wedge nx + c_1 a_2 = c_1 b_2)$$

which we may then rewrite, substituting on the right-hand side, as

$$\exists x(nx + c_2 a_1 = c_2 b_1 \wedge c_2 b_1 + c_1 a_2 = c_1 b_2 + c_2 a_1).$$

We can then bring the existential quantifier in:

$$(\exists x \ nx + c_2 a_1 = c_2 b_1) \wedge c_2 b_1 + c_1 a_2 = c_1 b_2 + c_2 a_1$$

and finally use some modular magic to obtain:

$$(c_2 a_1 \equiv_n c_2 b_1) \wedge (c_2 b_1 + c_1 a_2 = c_1 b_2 + c_2 a_1).$$

We know this last formula to be decidable; it's the conjunction of two decidable statements.

A Decidability Proof for PrA

- A brief note: it looks like on the last slide, we used multiplication to express e.g. $c_2 a_1$! Wasn't that banned — aren't we supposed to just limit ourselves to addition?
- Here it's worthwhile to make a distinction between the object language and the metalanguage. We, from our armchairs, are allowed to talk about multiplication as much as we want. From inside the theory, however, talk of multiplication (by a variable) is banned.
- In the above, expressions such as $c_2 a_1$ are just convenient metalinguistic names for terms of the object language. Inside the theory, no multiplication has gone on; these are just constants.
- If this is still unpalatable, we can rewrite our final expression from before as just

  $$(c \equiv_n d) \wedge (d + c' = d' + c)$$

  which doesn't mention multiplication at all — it's just using constant symbols, which we know to be legitimate abbreviations for $1 + 1 + \ldots_c + 1$.

A Decidability Proof for PrA

Let's continue with the proof:

Proof. (Cont.)

1. Suppose now that we have $\exists x\ (r_1 \wedge r_2)$, with r_1 an equation and r_2 a negated equation. We know that all the same transformations as in the above case hold, and so it's easy here to run the same proof to get

$$(c_2 a_1 \equiv_n c_2 b_1) \wedge \neg(c_2 b_1 + c_1 a_2 = c_1 b_2 + c_2 a_1)$$

which we know is all decidable.

2. If we have an equation and a congruence, then we can apply the fact that $a \equiv_n b$ holds *iff* $ma \equiv_{mn} mb$ holds, to transform our conjunction into

$$(c_2 a_1 \equiv_n c_2 b_1) \wedge (c_2 b_1 + c_1 a_2 \equiv_m c_1 b_2 + c_2 a_1)$$

which, again, is decidable.

We'll skip the rest of the proofs to save time; they all follow this general pattern, though, of eliminating the innermost existential quantifier.

A Decidability Proof for PrA

Proof. (Cont.)

- Given a method, for each type of formula $\exists x\ (r_1 \wedge r_2)$, we can generalise this to arbitrary long series of conjunctions.
- So a method of proof has been given: for any formula ϕ of PrA, we:
  - Convert it to disjunctive prenex form,
  - For each of n quantifiers:
    - Distribute the innermost quantifier over the disjunctions, and
    - Eliminate the quantifier by liberal application of modular congruences over ground terms.
- This gets us a disjunction over clauses whose members are all ground terms. We know that we can decide all of these. So, for any formula ϕ of PrA, it's equivalent to a formula ϕ' which can be decided.

$\square$

Great! That's just the proof that we wanted.

Our Example

- Not all formulae need congruences; the case of two negations holds more easily.
- So let's take our example from earlier, on the commutativity of addition:

  $$\neg\exists x \exists y\ \neg(x + y = y + x)$$

  and, to fit this into our proof, transform it to the equivalent conjunction

  $$\neg\exists x \exists y\ (\neg(x + y = y + x) \wedge \neg(x + y = y + x))$$

  which we then balance the equations to get

  $$\neg\exists x(\neg(x = x) \wedge \neg(x = x))$$

  which immediately looks suspect, and so we balance again to get

  $$\neg(\neg(0 = 0) \wedge \neg(0 = 0))$$

  which we can decide (as true!) right away.

Multiplication

- There's an interesting remark at the end of Presburger's paper [5]. He says "Should we want to introduce the multiplication symbol to our system, we would encounter unsolved problems in the proof of decidability". We do.
- "[In] order to prove the decidability of the expended system we would have to be able to decide each special case of Fermat's last theorem."
- Now, Fermat's last theorem hadn't been proven at the time (this was 1929). But Presburger was right that there's an unsolved problem in the proof of decidability.
- Unfortunately, Gödel showed that, if we add multiplication, we get an undecidable theory.
- We'll skip the proof here (there's a whole eight-week Part C course in the maths department on this, if you're interested), but know that this is an unfortunate result.

Kurt Gödel

Decidable Extensions

- We know that PrA, extended with multiplication, is undecidable. So is addition the limit, or can we add other interesting predicates and still get a decidable theory out the other end?
- In 1960, Büchi [1] showed that PrA can be extended with a predicate V_k, where $V_k(x, y)$ holds if x is the largest power of k which divides y. So, $V_2(4, 12)$ holds, as does $V_3(9, 36)$. Büchi showed that this theory was decidable.
- Out of this, one can define a predicate which holds just of the powers of two. Counter-intuitively, this means that PrA extended with an exponentiation function is decidable, whereas extension by multiplication is not.
- In 1980, Semënov [4] showed that PrA extended with a predicate that identifies Fibonacci numbers is decidable, as is PrA extended with a predicate that identifies values of the factorial function.
- There's no clear boundary to what is and isn't decidable; but decidability in PrA gives us a good way of judging how simple or complex a predicate is.

Undecidable Extensions

- We know that adding multiplication to PrA gets us an undecidable theory. This gets us a pretty effective method of showing that an extension of PrA is undecidable; all we need to do is show that, in that theory, we can define multiplication.
- Here's a quick example:

Theorem

$Th(\mathbb{Z}; +, <, \cdot^2)$, where $\cdot^2$ is the squaring function, is undecidable.

Proof.

Consider, where a and b are integers, the term. $(a+b)^2 - a^2 - b^2$. We know, from outside the theory, that this is equal to
$(a^2 + 2ab + b^2) - a^2 - b^2$.
So, we can write $c + c = (a+b)^2 - a^2 - b^2$, which holds *iff* $c = ab$. All the symbols there are expressible in our theory. So it follows that multiplication is definable in $Th(\mathbb{Z}; +, <, \cdot^2)$; so the theory is undecidable. $\square$

Undecidable Extensions

- This generalises to higher-degree polynomials.

Theorem

$Th(\mathbb{Z}; +, <, \cdot^3)$, where $\cdot^3$ is the cubing function, is undecidable.

Proof.

- Here we can deploy some machinery we saw near the start of the lecture — the *discrete derivative*, or Δ. Here's a refresher of the definition:

$$\Delta_n f(x) = f(x+n) - f(x)$$

- Consider the value of $\Delta_1(x^3)$. We can see that this is equal to $(x+1)^3 - x^3$, and this expands out to

$$(x^3 + 3x^2 + 3x + 1) - x^3$$

which gets us the equation

$$3x^2 + 3x + 1.$$

Undecidable Extensions

Proof (Cont.)

- Now, we know that we can express that $y = 3x^2 + 3x + 1$ in PrA with cubing. But we also know that we can do constant multiplication and addition in our theory. In particular, we can rearrange the above to get

$$y + y + y = (\Delta_1(x^3)) - 3x - 1$$

  which holds just when $y = x^2$.

- We already know that PrA extended with a squaring predicate is undecidable. So, since we can define the squaring predicate in PrA extended with cubing, it follows that $Th(\mathbb{Z}; +, <, \cdot^3)$ is undecidable.

$\square$

This reveals a strategy which can be used more generally to show that an extension of PrA is undecidable: if some polynomial can be defined in the theory, then discretely differentiate it until you reach x^2, which we know gives you an undecidable theory. This is what my research has been on!

Recent Work

- We've seen that, if multiplication can be defined inside an extension of Presburger arithmetic, then that extension is undecidable. So how far can we extend Presburger arithmetic?
- Work I've done with Jakub Konieczny (here in the department!) asks whether Presburger arithmetic extended with *sub-polynomial Hardy field functions* is decidable. The technicalities of Hardy field functions can be left aside for now — thankfully, all logarithmico-exponential functions are Hardy field functions, and so it helps to think of these in particular. The logarithmico-exponential functions are just those built up from real-valued constants, variables, addition, multiplication, exponentiation and the taking of logarithms.
- A sub-polynomial function is just a function that grows more slowly than some polynomial. More formally, f is sub-polynomial if

$$\lim \frac{f(x)}{x^n} > 0 \quad \text{and} \quad \lim \frac{f(x)}{x^{n+1}} = 0$$

  for some $n \in \mathbb{N}$.
- There's a snag in that Hardy field functions are, in general, not integer-valued. So, given a Hardy field function f, we consider the rounded version, $\lfloor f \rceil$, where $\lfloor f \rceil(x) = \lfloor f(x) \rceil$, the nearest integer to $f(x)$. More formally, $\lfloor x \rceil = \lfloor x + 0.5 \rfloor$.

A Proof Overview

- Intuitively, PrA extended by an arbitrary sub-polynomial Hardy field function shouldn't be decidable. Some examples of these arbitrary Hardy field functions include x^2 and x^3, which we know are immediately problematic. But does this hold, in general, for all Hardy field functions?
- Now, the research hasn't been finished, but Jakub and I have had the following proof idea here:
  - First show that all sub-polynomial Hardy field functions satisfy the P_d property, and
  - Then show that, in any extension of PrA by a function with the P_d property, multiplication is definable, and so the extension is undecidable.
- Let's cover these two sub-proofs in term. First we give the P_d property:

Definition

A function f satisfies the *P_d property* *iff* there exists a polynomial p, of degree exactly d, such that there exists some $L \in \mathbb{N}$ where, for all $0 \leq m \leq L$, we have that $|f(x+m) - p(x)| < \epsilon$ for some specified ϵ, subject to the condition that we don't have $\langle f(x) \rangle < 0.5$ and $\langle f(x+L) \rangle > 0.5$, or vice versa.
We write $\langle x \rangle$ to mean the decimal part of x.

A Proof Overview

- Let's now show that all sub-polynomial Hardy field functions have the P_d property. Let's prove the case where $d = 2$, and where our Hardy field function is just x^α where $\alpha \in (2, 3)$ for the time being; then we have a Hardy field function such that

$$\lim \frac{f(x)}{x^2} > 0 \quad \text{and} \quad \lim \frac{f(x)}{x^3} = 0.$$

- It happens that all Hardy field functions have well-defined Taylor expansions, and so we already have a candidate for what our polynomial p will be. But we need our polynomial to be of exactly degree 2 — it's not clear that a degree-2 Taylor expansion will be very good at approximating a function over long intervals.
- Counterintuitively, though, they can:

Proposition

Given a Hardy field function f such that

$$\lim \frac{f(x)}{x^2} > 0 \quad \textit{and} \quad \lim \frac{f(x)}{x^3} = 0,$$

there exists a degree-2 Taylor expansion of it that approximates it over an arbitrarily long interval.

A Proof Overview

- Let's show that our result holds.

Proof.

Say that L is the width of the interval that we want to approximate, within an error of ϵ. Say further that N will be the offset of the Taylor expansion. Suppose that we have a degree-2 Hardy field function f of degree 2. We know that, if $|p(x)| \leq M$, where $x \in (N - L, N + L)$, then the remainder term of the Taylor expansion is bounded above. That is,

$$|R_2(x)| \leq M\frac{L^3}{3!}$$

where $R_2(x)$ is the remainder term of the Taylor expansion.
Given that we want to have $|R_2(x)| \leq \epsilon$ for some $\epsilon > 0$, we set the bound above to be

$$M\frac{L^3}{3!} = \epsilon$$

and resolve for M:

$$M = \frac{\epsilon 3!}{L^3}.$$

Proof.

Given the last slide, we can therefore bound our Taylor expansion p by

$$|p(x)| \leq \frac{\epsilon 3!}{L^3}.$$

Now, we know that our Hardy field function f is of the form x^α, and so its 3^{rd} derivative is equal to $(\prod_{0 \leq i \leq 2}(\alpha - i))x^{\alpha-3}$. So, we know what our Taylor expansion will be, namely:

$$(\prod_{0 \leq i \leq 2}(\alpha - i))x^{\alpha-3} \leq \frac{\epsilon 3!}{L^3}.$$

Finally, we rearrange to get x out of the above, which gives us this horrible equation:

$$x \geq \left(\frac{\epsilon 3!}{L^3 \prod_{0 \leq i \leq 2}(\alpha - i)}\right)^{\frac{1}{\alpha-3}}$$

and then we can just take our N to be such that $N \geq x$.
So, for any given interval width L, we can find an N as per the above method, such that there is a Taylor expansion p of f with $|f(x-N) - p(x)| < \epsilon$ for all x in $0 \leq x < L$.
With a little bit of tweaking at the end, we get that all sub-polynomial f of degree α have the P_2 property, as we wanted. □

The result generalises to all d, and so we have what we want.

A Proof Overview

- So we've proved that some of the sub-polynomial Hardy field functions that we care about have the P_d property. Now we need to show that all functions with the P_d property lead to an undecidable theory.
- This is the bit we've got up to in the research! Right now, the idea is to take the polynomial, differentiate it, and then use the fact that the Hardy field function coincides with the differentiated polynomial to show that we can define multiplication.
- This is deploying the strategy we used before, where we differentiate a function and use that to define multiplication. But there are snags, particularly since we have to deal with the intricacies of rounding.
- If anyone has any ideas on these, do let me know!

Conclusion

- Today, we've learned about Presburger arithmetic, and seen how it can be decided via quantifier-elimination procedures.
- We've also seen some instances of decidable and undecidable extensions of it.
- Finally, we've looked at how we can use the tools of maths and analysis in undecidability proofs.

Further Reading

- *Mojżesz presburger: Life and work* by Jan Zygmunt [6] is a great short biography of Presburger's life; it's a good exposition to work in the Warsaw school as well, and the environment Presburger worked in.
- *A Survival Guide to Presburger Arithmetic* by Christoph Haase [2] gives a good exposition of the theory, and areas of interest in its study; in particular, the complexity of fragments of Presburger arithmetic are covered. There's a wealth of references in here as well.
- *Decidability of Extensions of Presburger Arithmetic by Generalised Polynomials* by Jakub Konieczny [3] is a proof that PrA, extended by a certain sort of generalised polynomial, is undecidable. The proof contains the brilliant trick of defining multiplication over a *multiplicative tuple*, rather than just directly — it's a great showcase of just how much PrA can do, even by itself.

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Goodbye!

Thanks for listening!

Any questions?