Derivation of Normal Equation

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Define the cost function as:

$$\mathbf{J}(\mathbf{w}) = \frac{1}{2}\sum_{n=1}^{N}(Z_n^\top \mathbf{w} - t_n)^2$$

let $Z_n \in \mathbb{R}^m$ with samples $z_1, z_2, z_3, ..., z_m$.
let $\mathbf{Z}$ be a n by m matrix with each row vector as Z_i and let $\mathbf{t}$ be a column vector with n dimensions $t_1, t_2, ...t_n$:

$$\mathbf{Z} = \begin{bmatrix} Z_1 \\ Z_2 \\ ... \\ Z_n \end{bmatrix}$$

$$\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ ... \\ t_n \end{bmatrix}$$

since

$$\sum_{n=1}^{N} a_n^2 \iff ||\mathbf{a}||^2$$

We can rewrite the cost function as:

$$\mathbf{J}(\mathbf{w}) = \frac{1}{2}||\mathbf{Zw} - \mathbf{t}||^2, \text{ where } \mathbf{J}(\mathbf{w}) : \mathbb{R}^m \to \mathbb{R}$$

Using chain rule to differentiate the cost function against $\mathbf{w}$ with the variable declaration:

$$\mathbf{u} = \mathbf{Zw} - \mathbf{t}$$
$$\mathbf{J} = \frac{1}{2}||\mathbf{u}||^2$$

Then we have that:

$$D_{\mathbf{u}}(\mathbf{J}) = D_{\mathbf{u}}(\frac{1}{2}||\mathbf{u}||^2) = \mathbf{u}^\top$$
$$D_{\mathbf{w}}(\mathbf{u}) = D_{\mathbf{w}}(\mathbf{Zw} - \mathbf{t}) = \mathbf{z}$$
$$D_{\mathbf{w}}(\mathbf{J}(\mathbf{w})) = D_{\mathbf{u}}(\mathbf{J})D_{\mathbf{w}}(\mathbf{u}) = \mathbf{u}^\top\mathbf{Z} = (\mathbf{Zw} - \mathbf{t})^\top\mathbf{Z}$$

Set gradient to zero to find the point of minimal cost function (if possible, can someone knowledgeable tell why we can directly get minimal by setting gradient to zero?):

$$
\begin{aligned}
&D_{\mathbf{w}}\mathbf{J}(\mathbf{w}) = 0\\
&\implies 0 = (\mathbf{Z}\mathbf{w} - \mathbf{t})^{\top}\mathbf{Z}\\
&\implies 0 = (\mathbf{Z}\mathbf{w})^{\top}\mathbf{z} - \mathbf{t}^{\top}\mathbf{Z}\\
&\implies 0 = \mathbf{w}^{\top}\mathbf{Z}^{\top}\mathbf{Z} - \mathbf{t}^{\top}\mathbf{Z}\\
&\implies \mathbf{t}^{\top}\mathbf{Z} = \mathbf{w}^{\top}\mathbf{Z}^{\top}\mathbf{Z}\\
&\implies \mathbf{Z}^{\top}\mathbf{t} = \mathbf{Z}^{\top}\mathbf{Z}\mathbf{w}\\
&\implies \mathbf{w} = (\mathbf{Z}^{\top}\mathbf{Z})^{-1}\mathbf{Z}^{\top}\mathbf{t},\ \text{which is the normal equation}
\end{aligned}
$$