Synthetic Homotopy Theory TD3: Homotopy groups

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Exercise 1 Eckmann-Hilton argument

In this exercise we want to show that $|\Omega^2 X|_0$ has a structure of **abelian** group for any pointed type X.

Question 1 Assume given a set X with two binary operations $\_\cdot\_$ and $\_\otimes\_$ with the same unit e, meaning that for all $x : X$ we have:

$$x \cdot e = e \cdot x = x$$

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Moreover assume that:

$$(a \otimes b) \cdot (c \otimes d) = (a \cdot c) \otimes (b \cdot d)$$

Prove that $\_\cdot\_$ and $\_\otimes\_$ coincide and commute.

Question 2 Assume given $A : \mathscr{U}$ with $p, q : x =_A y$ and $p', q' : y =_A z$. Moreover assume given $h : p = q$ and $h' : p' = q'$. Define:

$$h \otimes h' : p \cdot q = p' \cdot q'$$

Question 3 Assume given a type X with $x : X$. Prove that $\_\cdot\_$ and $\_\otimes\_$ induces operations on $\mathrm{refl}_x =_{x =_X x} \mathrm{refl}_x$ obeying the hypothesis from question 1.

Question 4 Conclude that $|\Omega^2 X|_0$ has a structure of abelian group.

Exercise 2 Loop and suspension

Recall that we defined $\Sigma : \mathscr{U}_* \to \mathscr{U}_*$ and $\Omega : \mathscr{U}_* \to \mathscr{U}_*$ by:

$$\Sigma X :\equiv (\Sigma X, \mathbf{N})$$

$$\Omega X :\equiv (* =_X *, \mathrm{refl}_*)$$

In this exercise we want to show that for any $X, Y : \mathscr{U}_*$ we have:

$$(\Sigma X \to_* Y) \simeq (X \to_* \Omega Y)$$

Question 1 Assume given $f : \Sigma X \to_* Y$ (recall this means we have $*_f : f(\mathbf{N}) = *$). We define $\psi(f) : X \to \Omega Y$ by:

$$\psi(f, x) :\equiv *_f^{-1} \cdot \mathrm{ap}_f(\mathbf{merid}_x \cdot \mathbf{merid}_*^{-1}) \cdot *_f$$

Show this actually defines a map:

$$\psi : (\Sigma X \to_* Y) \to (X \to_* \Omega Y)$$

Question 2 Assume given $g : X \to_* \Omega Y$, we define $\phi(g) : \Sigma X \to Y$ by:

$$\phi(g, \mathbf{N}) :\equiv *$$

$$\phi(g, \mathbf{S}) :\equiv *$$

$$\mathrm{ap}_{\phi(g)} \mathbf{merid}_x :\equiv f(x)$$

Show this actually defines a map:

$$\phi : (X \to_* \Omega Y) \to (\Sigma X \to_* Y)$$

Question 3 Show that given $f, g : \Sigma X \to Y$, in order to prove $f = g$ it is enough to give:

$$p : f(\mathbf{N}) = g(\mathbf{N})$$

$$q : f(\mathbf{S}) = g(\mathbf{S})$$

$$h : (x : X) \to \mathrm{ap}_f(\mathbf{merid}_x) \cdot q = p \cdot \mathrm{ap}_g(\mathbf{merid}_x)$$

Using this show that for all $f : \Sigma X \to_* Y$ we have:

$$\phi(\psi(f)) =_{\Sigma X \to Y} f$$

Question 4 (Optional) Can you prove that:

$$\phi(\psi(f)) =_{\Sigma X \to_* Y} f$$

(Can you understand what you need to prove? Hint: that paths between basepoints agree).

Question 5 Show that:

$$\psi(\phi(g)) =_{X \to \Omega Y} g$$

for all $g : X \to_* \Omega Y$.

Question 6 (Optional) Try to prove:

$$\psi(\phi(g)) =_{X \to_* \Omega Y} g$$

(Can you understand what you need to prove? Hint: that paths between basepoints agree).

Question 7 Conclude from the previous questions that:

$$(\Sigma X \to_* Y) \simeq (X \to_* \Omega Y)$$

* Exercise 3 Canonical fiber sequence

Question 1 Prove that any fiber sequence is equal to a canonical one.

Question 2 Show that giving $Y : \mathcal{U}_*$ and a map in $Y \to_* Z$ is the same as giving $P : X \to \mathcal{U}$ and $*_P : P(*_Z)$.

Question 3 Using previous questions, prove that any fiber sequence is of the form:

$$P(*_Z) \xrightarrow{\mathbf{inc}}_* (x : Z) \times P(x) \xrightarrow{p_Z}_* Z$$

where $P(_Z*)$ is pointed by $*_P$ and $(z : Z) \times P(x)$ is pointed by $(*_Z, *_P)$, with p_Z the projection and $\mathbf{inc}(q) :\equiv (*_Z, q)$.

Exercise 4 Representable invariants

A representable invariant (implicitly: on based types) is a map $F : \mathcal{U}_* \to \mathrm{Set}_*$ of the form:

$$\lambda(X : \mathcal{U}_*).|A \to_* X|_0$$

for some fixed A, with $|A \to_* X|_0$ pointed by the constant map.

Question 1 Show that homotopy groups are representable invariants.

We define the product of two pointed types $X, Y : \mathcal{U}_*$ as $X \times Y$ pointed by $(*_X, *_Y)$.

Question 2 Show that:

$$F(X \times Y) = F(X) \times F(Y)$$

for any representable invariant F.

A family of pointed types is a map $X : I \to \mathcal{U}_*$ for I a set. We denote such a family by $(X_i)_{i:I}$, with X_i denoting the element $X(i)$. The product $\Pi_{i:I} X_i$ of a family of type is defined as the type $(i : I) \to X(i)$ pointed by $\lambda(i : I).*_{X(i)}$.

Question 3 Using an appropriate version of the axiom of choice, prove that:

$$F(\Pi_{i:I} X_i) = (i : I) \to F(X_i)$$

Question 4 (Optional) Let $X \to_* Y \to_* Z$ be a fiber sequence, prove that we have an exact sequence of pointed set:

$$F(X) \to_* F(Y) \to_* F(Z)$$

for any representable invariant F. (Hint: use the "canonical fiber sequence" exercise)

Exercise 5 The long fiber sequence of a map

We want to prove that given a fiber sequence:

$$X \xrightarrow{f}_* Y \xrightarrow{g}_* Z$$

we have a long fiber sequence:

$$\cdots \to_* \Omega X \to_* \Omega Y \to_* \Omega Z \to_* X \to_* Y \to_* Z$$

Question 1 Show that given a fiber sequence:

$$X \to_* Y \to_* Z$$

we have a fiber sequence:

$$\Omega Z \to_* X \to_* Y$$

(Hint: use the "canonical fiber sequence" exercise. The necessary map from ΩZ to $P(*_Z)$ sends r to $\mathbf{tr}_r^P(*_P)$).

Question 2 Conclude by iterating the previous question.

Question 3 (Optional) Can you build a long fiber sequence:

$$\cdots \to_* \Omega^2 X \xrightarrow{\Omega^2 f}_* \Omega^2 Y \xrightarrow{\Omega^2 g}_* \Omega^2 Z \xrightarrow{\Omega\delta}_* \Omega X \xrightarrow{\Omega f}_* \Omega Y \xrightarrow{\Omega g}_* \Omega Z \xrightarrow{\delta}_* X \xrightarrow{f}_* Y \xrightarrow{g}_* Z$$

(you need to be careful about maps).