Introduction to synthetic homotopy theory

Quick intro to fundamental math

- We are interested in objects + morphisms (aka category)

→ Holy grail Classification.

List of non-isomorphic objects s.t. any object is isomorphic to one in the list

Strategy Have invariant, i.e "things" unchanged by isomorphisms (aka functors)

Example the dimension of vector space.

→ Usually this is not possible

2 solutions

→ restricts to 'well-behaved' objects. (aka sub-category)

→ consider a weaker notion of 'being then same' than iso (aka localisation of a category)

Main example

- Topological spaces cannot be classified
- Topological spaces up to weak equivalences can be partially classified.

<u>Drawing</u>

$$\mathrm{Top} \xrightarrow{\pi_n} \mathrm{Groups}^{\mathbb{N}}$$

you localize at this invariant,

Def Spaces up to weak equivalences are called homotopy types

Theorem Homotopy types are equivalently:

- Spaces up to weak equivalences — Historical
- Simplicial sets up to weak equivalences. — Usual definition
- Kan simplicial sets up to equivalences
- ∞-groupoids up to equivalences — (relevant this class.

Theorem (Kapulkin, Lumsdaine, Voevodsky)

<u>Types</u> from type theory can be interpreted as homotopy <u>types</u>.

<u>Rk</u> Lucky coincidence that types occur twice

Homotopy Type Theory (Steve Awodey)

HoTT

Synthetic homotopy theory

we prove theorems about types

+ we conclude it is true for homotopy types.

References

- $\boxed{\text{HoTT book}}$
- Egbert Rijke Lecture notes
- Lecture notes for this class.

EPIT spring school this week

Friday 14h → Lecture be Egbert on synthetic homotopy theory.

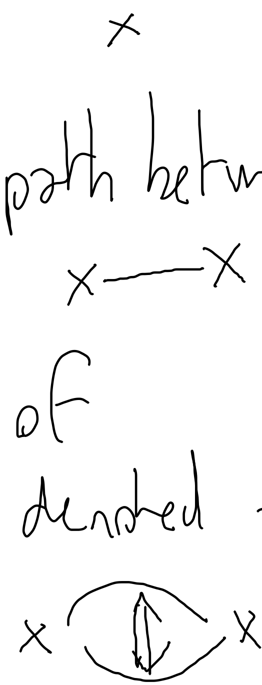
∞-groupoids (classical point of view)

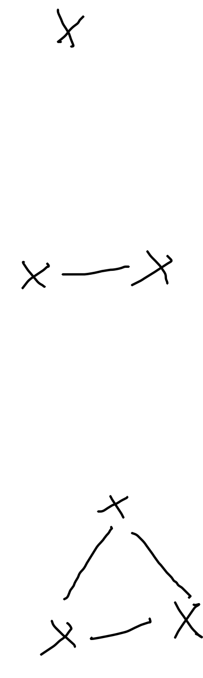
→ The model for homotopy types closest to type theory.

'Definition'

An ∞-groupoid consists of

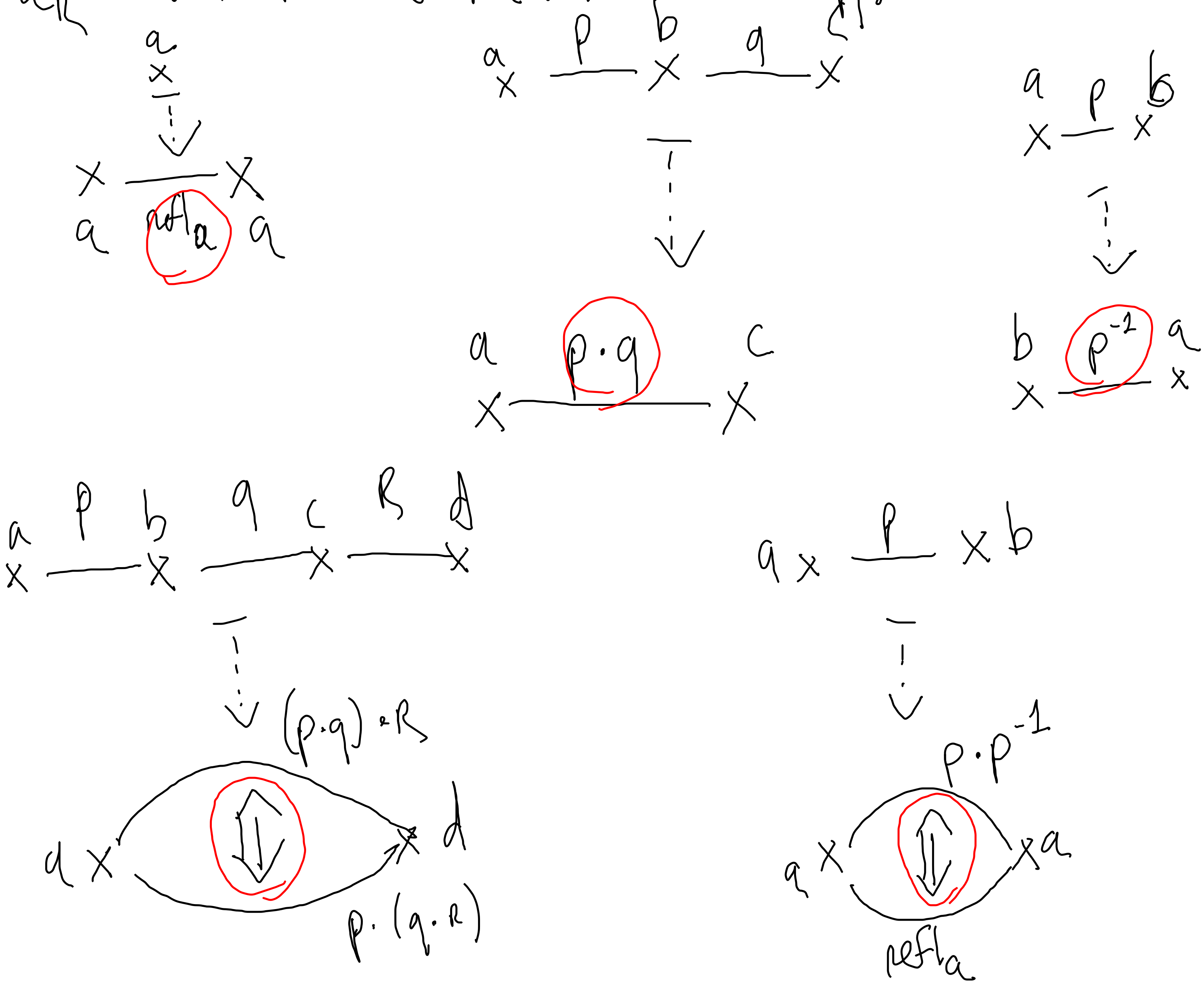
- A set of points A_0
- For $x, y \in A_0$, a set of path between them $A_1(x,y)$
- For $p, q \in A_1(x,y)$ a set of deformation from p to q denoted $A_2(p,q)$
- ...





Together with the relevant compositions.

E.g



PART I : Type Formers

① Identity types

$$\frac{A \text{ type} \quad a, b : A}{a =_A b \text{ type}} \qquad \frac{A \text{ type} \quad a : A}{\mathrm{refl}_a : a =_A a}$$

Intuition $a =_A b$ is ∞-groupoid of paths from a to b in A.

Path induction

To prove/define $t(x,y,p) : B(x,y,p)$

<u>for all</u> $x,y : A$ and $p : x = y$

It is enough to define

$$t(x,x,\mathrm{refl}_x) : B(x,x,\mathrm{refl}_x)$$

<u>Idea</u> This implies all ∞-groupoid compositions for a type A.

Example

- For all $a, b, c : A$ and $p : a = b$ and $q : b = c$
  we want to define $p \cdot q : a = c$

  It is enough to define $\boxed{\mathrm{refl}_a \cdot q :\equiv q}$ (Coq/HoTT)

- For all $a, b, c, d : A$ and $p : a = b$, $q : b = c$, $r : c = d$

  $$\mathrm{assoc}(p, q, r) : (p \cdot q) \cdot r = p \cdot (q \cdot r)$$

  It is enough to define $\mathrm{assoc}(\mathrm{refl}_a, q, r)$

  $$:\equiv \mathrm{refl}_{q \cdot r} \quad : \underbrace{(\mathrm{refl}_a \cdot q) \cdot r}_{\equiv\ q \cdot r} = \underbrace{\mathrm{refl}_a \cdot (q \cdot r)}_{\equiv\ q \cdot r}$$

⚠ very important

We $\boxed{\text{cannot}}$ use path induction to prove

for all $x : A$ and $p : x = x$ we have $P(x, p)$

Dependent functions (sections, Π-types, dependent products)

$$\frac{\begin{array}{l} A \text{ type} \\ B(x) \text{ type depending on } x : A \end{array}}{(x:A) \to B(x)}$$

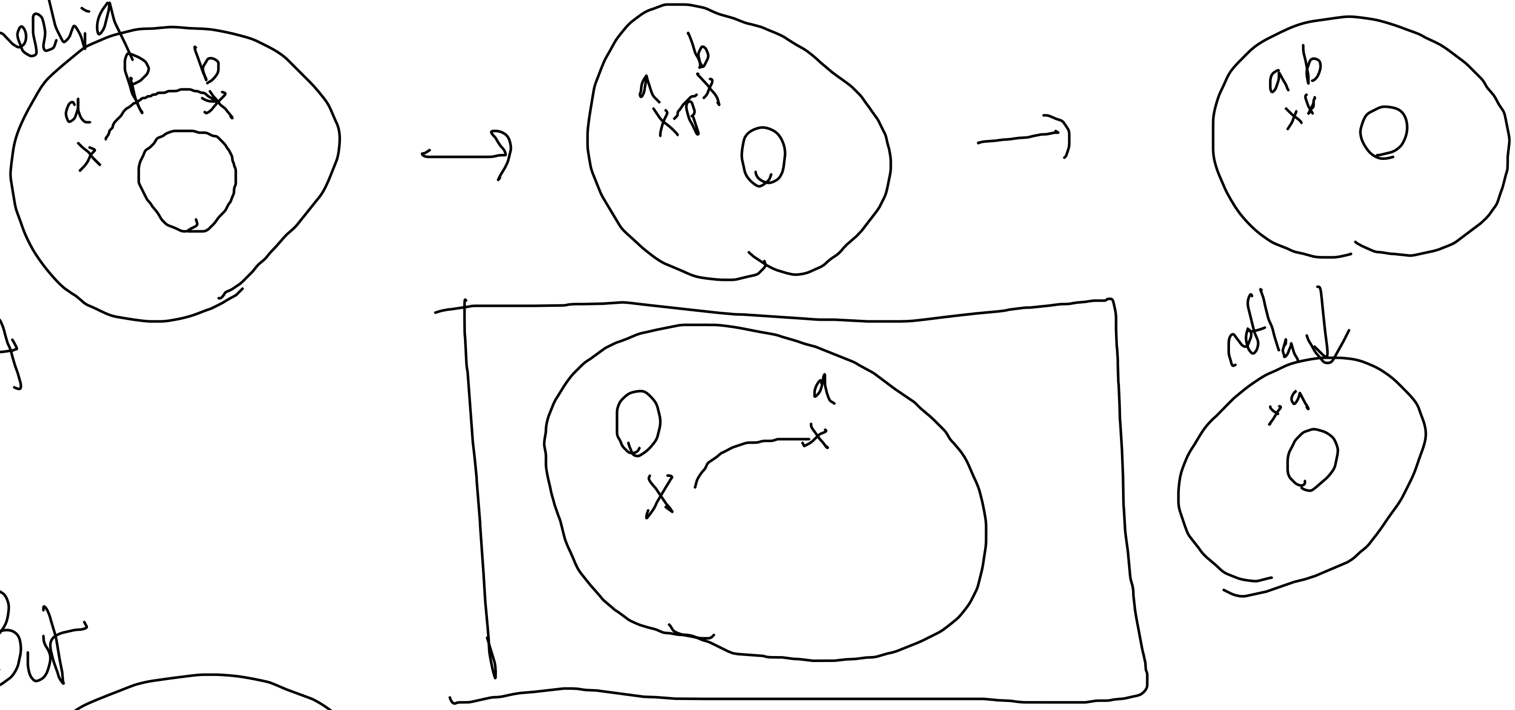
(Alt: $\Pi(x:A).B(x)$)

$$\frac{t(x) : B(x) \text{ depending on } x : A}{\lambda x. t(x) : (x:A) \to B(x)}$$

(Alt $x \mapsto t(x)$)

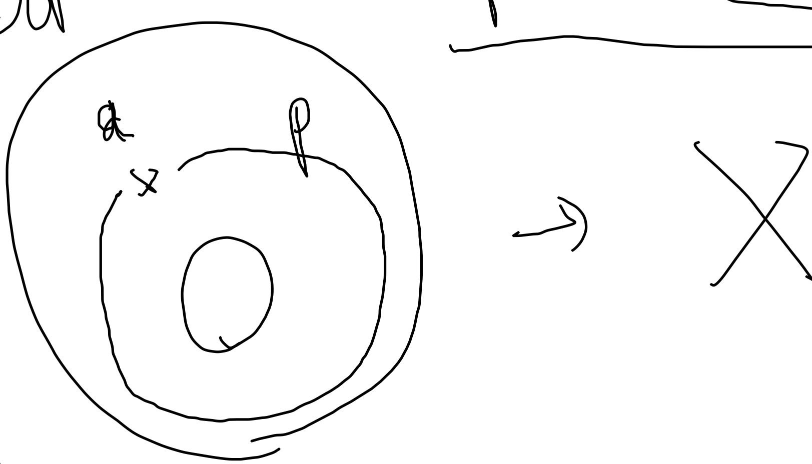
$$\frac{\begin{array}{l} f : (x:A) \to B(x) \\ a : A \end{array}}{f(a) : B(a)}$$

Inheritig



A

But



A

Rq if B does depend on A, we write

$$A \to B.$$

<u>Intuition</u> $A \to B$ is ∞-groupoid of morphism of ∞-groupoid from A to B

As spaces such morphisms are <u>continuous</u>.

<u>Example</u> For $a, b : A$ and $p : a =_A b$ and $f : A \to B$ we have $ap_f(p) : f(a) =_B f(b)$.
defined by $\boxed{ap_f(refl_a) :\equiv refl_{f(a)}}$ ...

Dependent products (Σ-types, dependent sums)

$$\frac{A \text{ type} \qquad B(x) \text{ type on } x:A}{(x:A) \times B(x)} \qquad \frac{a : A \qquad b : B(a)}{(a,b) : (x:A) \times B(x)}$$

(Alt. $\Sigma(x:A).B(x)$)

<u>Rmk</u> B does not depend on A we write $A \times B$.

<u>Induction</u> We can assume any $z : (x:A) \times B(x)$ of the form (x,y) where $x : A$, $y : B(x)$

Universe

$$\frac{}{\mathcal{U}\ \text{type}}$$

inhabitants of $\mathcal{U}$ are types

$\boxed{\text{Intuition} \quad \mathcal{U} \text{ is the } \infty\text{-groupoid of } \infty\text{-groupoids}}$

$\boxed{\text{Rq} \quad \mathcal{U} : \mathcal{U} \text{ is inconsistent}}$

Example $B : A \to \mathcal{U}$ is a family of type indexed by A.

We can define $\text{transport}^B_p(y) : B(b)$

for $a, b : A$, $y : B(a)$, $p : a = b$

by $\boxed{\text{transport}^B_{\text{refl}_a}(y) :\equiv y \ : B(a)}$

$$x : A \vdash x =_A x \to \mathcal{U}$$

$$\cancel{\mathcal{U} : \mathcal{U}} \quad , \quad \cancel{x =_A x \to \mathcal{U} : \mathcal{U}}$$

$$\cancel{\lambda x . \, x =_A x \to \mathcal{U}} \qquad A \to \boxed{\mathcal{U}}$$

$\vdash$ $\mathcal{U}_1 : \mathcal{U}_\omega$, $\mathcal{U}_2, \ldots$ is consistent.

types $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \ldots$

$\mathcal{U}_n$ does not depend on $n : \mathbb{N}$ internally

but not $n : \mathbb{N} \cancel{\vdash \mathcal{U}_n} \text{ type}$

Univalence

Def A map $f : A \to B$ is an equivalence if:

- We have $g_1 : B \to A$ s.t. $g_1 \circ f = id$
- We have $g_2 : B \to A$ s.t. $f \circ g_2 = id$

Intuition An equivalence is an equivalence of ∞-groupoid.

Notation $A \simeq B$ is the type of equivalence from A to B.

Univalence

Univalence axiom

The map $A =_{\mathcal{U}} B \to A \simeq B$ is an equivalence.

Corollary

We have a map $A \simeq B \to A =_{\mathcal{U}} B$

Corollary We have $p : \mathrm{Bool} =_{\mathcal{U}} \mathrm{Bool}$ such $p \neq \mathrm{Refl}_{\mathrm{Bool}}$

Proof $(\mathrm{Bool} \simeq \mathrm{Bool}) \simeq \{ id, not \}$

Theorem Univalence axiom is validated by the interpretation of types as ∞-groupoids.

$\boxed{\text{From now on we assume univalence}}$

It is surprising, as it means that path induction is compatible with non-constant path.

$n \in \mathbb{N}$, we have $\mathcal{U}_n$ univalent.

Déf $\mathcal{U}$ is impredicative if

$$\frac{P(A) : \mathcal{U} \text{ depending on } A : \mathcal{U}}{(A : \mathcal{U}) \to P(A) : \mathcal{U}}$$

In this class this is never the case.