Question 1 Assume given $x, y : A$, we want to prove that:

$$\mathbf{ap} : x =_A y \to f(x) =_B f(y)$$

is an equivalence. Since A is connected we know that $|x = *|$ and $|y = *|$, but as being an equivalence is a proposition, we can eliminate propositional truncation and assume $x = *$ and $y = *$. Then by path induction we need to prove that:

$$\mathbf{ap} : * =_A * \to f(*) =_B f(*)$$

is an equivalence, and this is the hypothesis.

Question 2 From question 1 we just need to prove that:

$$\mathbf{ap}_f : \mathbf{base} =_{S^1} \mathbf{base} \to b =_B b$$

is an equivalence.

We have the following diagram of morphisms of group:

$$\begin{tikzcd}
\mathbf{base} =_{S^1} \mathbf{base} \arrow[r, "\mathbf{ap}_f"] & b =_B b \\
\mathbb{Z} \arrow[u, "n \mapsto \mathbf{loop}^n"] \arrow[r, "\mathrm{id}"'] & \mathbb{Z} \arrow[u, "\epsilon"']
\end{tikzcd}$$

We want to show that it commutes. But morphisms of groups out of $\mathbb{Z}$ are equal if and only if they have the same value on 1, so we just need to check that:

$$\mathbf{ap}_f(\mathbf{loop}) = \epsilon(1)$$

This is true by definition. Now we have a commuting square where three maps are equivalences, and therefore so is the fourth.

Question 3 By our analysis of identity types in product types, we know that:

$$(X, p) =_{\mathrm{Aut}} (Y, q) \;\simeq\; (\epsilon : X =_{\mathcal{U}} Y) \times \mathbf{tr}_{\epsilon}^{\lambda X.X=X}(p) = q$$

So it is enough to show that for all $X, Y : \mathcal{U}$ with $\epsilon : X = Y$, $p : X = X$ and $q : Y = Y$ we have:

$$(\mathbf{tr}_{\epsilon}^{\lambda X.X=X}(p) = q) \;\simeq\; (p \cdot \epsilon = \epsilon \cdot q)$$

By path induction we can assume that ϵ is **refl**, and then we need to show that:

$$(p = q) \;\simeq\; (p \cdot \mathbf{refl} = \mathbf{refl} \cdot q)$$

but this is easy.

Question 4 By question 3 we know that:

$$\big((\mathbb{Z}, s) =_{\mathrm{Aut}} (\mathbb{Z}, s)\big) \ =_{\mathcal{U}} \ (\epsilon : \mathbb{Z} = \mathbb{Z}) \times (s \cdot \epsilon =_{\mathbb{Z}=\mathbb{Z}} \epsilon \cdot s)$$

By univalence this is equivalent to:

$$(\epsilon : \mathbb{Z} \simeq \mathbb{Z}) \times (s \circ \epsilon =_{\mathbb{Z}\simeq\mathbb{Z}} \epsilon \circ s)$$

But by function extensionnality (and the analysis of equality between equivalences) this is equivalent to:

$$(\epsilon : \mathbb{Z} \simeq \mathbb{Z}) \times \big((n : \mathbb{Z}) \to s(\epsilon(n)) =_{\mathbb{Z}} \epsilon(s(n))\big)$$

which is what we want.

Question 5 It is known that $\mathbb{Z} \simeq \mathbb{Z}$ has a group structure induced by composition. But G is the type of elements of $\epsilon : \mathbb{Z} \simeq \mathbb{Z}$ verifying the proposition:

$$P(\epsilon) \ :\equiv \ (n : \mathbb{Z}) \to \epsilon(n) + 1 = \epsilon(n+1)$$

So it is enough to check that P is stable by composition and inverse in order to conclude.

For f, g satisgying P, so does $g \circ f$. Indeed for all $n : \mathbb{Z}$ we have $f(n+1) = f(n) + 1$ and $g(n+1) = g(n) + 1$, we see that:

$$g(f(n+1)) = g(f(n) + 1) = g(f(n)) + 1$$

so that $g \circ f$ is in G.

If f satisfies P, so does f^{-1}, indeed:

$$f^{-1}(n) + 1 = f^{-1}(n+1)$$

is equivalent to:

$$f(f^{-1}(n) + 1) = f(f^{-1}(n+1))$$

because f is an equivalence, and then:

$$f(f^{-1}(n) + 1) = f(f^{-1}(n)) + 1 = n + 1 = f(f^{-1}(n+1))$$

Question 6 As a preliminary result, we omit the proof by a straightforward induction on $n : \mathbb{Z}$ that for $f : G$ we have:

$$f(n) = f(0) + n$$

We call ψ the map $f \mapsto f(0) : G \to \mathbb{Z}$.

First we check that ψ is a morphism of group, meaning that for $f, g : G$, we have:

$$f(g(0)) = f(0) + g(0)$$

But $f(n) = f(0) + n$ for all $n : \mathbb{Z}$, so we can conclude. (To be complete we need to check $\mathrm{id}(0) = 0$ which is obvious).

Now we check that ψ is an equivalence. We define $\phi : \mathbb{Z} \to G$ by:

$$\psi(n) \;:\equiv\; (m \mapsto n + m)$$

It is easy to check that $\psi(n)$ is in G. Then:

- For $f : G$ we have that $\phi(\psi(f)) \equiv \phi(f(0)) \equiv (m \mapsto f(0) + m)$, but we have seen that $f(m) = f(0) + m$ so we can conclude that:

  $$\phi(\psi(f)) = f$$

  by function extensionnality.

- For $m : \mathbb{Z}$ we see that $\psi(\phi(m)) = 0 + m = m$ by definition.

So we indeed have an isomorphism of group $\psi : G \to \mathbb{Z}$.

Question 7 By question 5 and 6 we know that:

$$(\mathbb{Z}, s) =_{\mathrm{Aut}} (\mathbb{Z}, s)$$

is isomorphic as group to $\mathbb{Z}$. But from question 2 this gives an immersion for S^1 of Aut, sending **base** to $(\mathbb{Z}, s)$