**Question 1** Assume given x, y : A, we want to prove that:

$$\mathbf{ap}: x =_A y \to f(x) =_B f(y)$$

is an equivalence. Since A is connected we know that |x = \*| and |y = \*|, but as being an equivalence is a proposition, we can eliminate propositional truncation and assume x = \* and y = \*. Then by path induction we need to prove that:

$$ap : * =_A * \to f(*) =_B f(*)$$

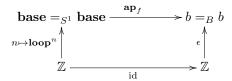
is an equivalence, and this is the hypothesis.

Question 2 From question 1 we just need to prove that:

$$\mathbf{ap}_f: \mathbf{base} =_{S^1} \mathbf{base} \to b =_B b$$

is an equivalence.

We have the following diagram of morphisms of group:



We want to show that it commutes. But morphisms of groups out of  $\mathbb{Z}$  are equal if and only if they have the same value on 1, so we just need to check that:

$$\mathbf{ap}_f(\mathbf{loop}) = \epsilon(1)$$

This is true by definition. Now we have a commuting square where three maps are equivalences, and therefore so is the fourth.

Question 3 By our analysis of identity types in product types, we know that:

$$(X,p) =_{\operatorname{Aut}} (Y,q) \ \simeq \ (\epsilon: X =_{\operatorname{\mathcal{U}}} Y) \times \mathbf{tr}_{\epsilon}^{\lambda X.X = X}(p) = q$$

So it is enough to show that for all  $X,Y:\mathcal{U}$  with  $\epsilon:X=Y,\,p:X=X$  and q:Y=Y we have:

$$(\mathbf{tr}_{\epsilon}^{\lambda X.X=X}(p) = q) \simeq (p \cdot \epsilon = \epsilon \cdot q)$$

By path induction we can assume that  $\epsilon$  is **refl**, and then we need to show that:

$$(p=q) \simeq (p \cdot \mathbf{refl} = \mathbf{refl} \cdot q)$$

but this is easy.

**Question 4** By question 3 we know that:

$$((\mathbb{Z}, s) =_{\mathrm{Aut}} (\mathbb{Z}, s)) =_{\mathcal{U}} (\epsilon : \mathbb{Z} = \mathbb{Z}) \times (s \cdot \epsilon =_{\mathbb{Z} = \mathbb{Z}} \epsilon \cdot s)$$

By univalence this is equivalent to:

$$(\epsilon : \mathbb{Z} \simeq \mathbb{Z}) \times (s \circ \epsilon =_{\mathbb{Z} \simeq \mathbb{Z}} \epsilon \circ s)$$

But by function extensionnality (and the analysis of equality between equivalences) this is equivalent to:

$$(\epsilon : \mathbb{Z} \simeq \mathbb{Z}) \times ((n : \mathbb{Z}) \to s(\epsilon(n)) =_{\mathbb{Z}} \epsilon(s(n)))$$

which is what we want.

**Question 5** It is known that  $\mathbb{Z} \simeq \mathbb{Z}$  has a group structure induced by composition. But G is the type of elements of  $\epsilon : \mathbb{Z} \simeq \mathbb{Z}$  verifying the proposition:

$$P(\epsilon) :\equiv (n : \mathbb{Z}) \to \epsilon(n) + 1 = \epsilon(n+1)$$

So it is enough to check that P is stable by composition and inverse in order to conclude.

For f, g satisgying P, so does  $g \circ f$ . Indeed for all  $n : \mathbb{Z}$  we have f(n+1) = f(n) + 1 and g(n+1) = g(n) + 1, we see that:

$$g(f(n+1)) = g(f(n)+1) = g(f(n))+1$$

so that  $g \circ f$  is in G.

If f satisfies P, so does  $f^{-1}$ , indeed:

$$f^{-1}(n) + 1 = f^{-1}(n+1)$$

is equivalent to:

$$f(f^{-1}(n) + 1) = f(f^{-1}(n+1))$$

because f is an equivalence, and then:

$$f(f^{-1}(n) + 1) = f(f^{-1}(n)) + 1 = n + 1 = f(f^{-1}(n+1))$$

**Question 6** As a preliminary result, we omit the proof by a straightforward induction on  $n : \mathbb{Z}$  that for f : G we have:

$$f(n) = f(0) + n$$

We call  $\psi$  the map  $f \mapsto f(0) : G \to \mathbb{Z}$ .

First we check that  $\psi$  is a morphism of group, meaning that for f,g:G, we have:

$$f(g(0)) = f(0) + g(0)$$

But f(n) = f(0) + n for all  $n : \mathbb{Z}$ , so we can conclude. (To be complete we need to check id(0) = 0 which is obvious).

Now we check that  $\psi$  is an equivalence. We define  $\phi: \mathbb{Z} \to G$  by:

$$\psi(n) :\equiv (m \mapsto n + m)$$

It is easy to check that  $\psi(n)$  is in G. Then:

• For f:G we have that  $\phi(\psi(f))\equiv\phi(f(0))\equiv(m\mapsto f(0)+m)$ , but we have seen that f(m)=f(0)+m so we can conclude that:

$$\phi(\psi(f)) = f$$

by function extensionnality.

• For  $m: \mathbb{Z}$  we see that  $\psi(\phi(m)) = 0 + m = m$  by definition.

So we indeed have an isomorphism of group  $\psi: G \to \mathbb{Z}$ .

Question 7 By question 5 and 6 we know that:

$$(\mathbb{Z}, s) =_{\operatorname{Aut}} (\mathbb{Z}, s)$$

is isomorphic as group to  $\mathbb{Z}$ . But from question 2 this gives an immersion for  $S^1$  of Aut, sending base to  $(\mathbb{Z}, s)$