Synthetic Homotopy Theory Homework: The Impredicative Circle

Definition 1. *A map $f : A \to B$ is called an immersion if for all $x, y : A$, the map:*

$$\mathbf{ap}_f : x =_A y \to f(x) =_B f(y)$$

is an equivalence.

Goal 1. *In this exercise we want to construct an immersion of S^1 into a type built without higher inductive type.*

Question 1 Let A be a pointed connected type, and let B be any type. Assume given $f : A \to B$ such that:

$$\mathbf{ap}_f : * = * \to f(*) = f(*)$$

is an equivalence (here $*$ is the basepoint of A). Prove that f is an immersion.

Question 2 Assume given B a type with $b : B$ such that we have an isomorphism of group:

$$\epsilon : \mathbb{Z} \to (b =_B b)$$

We define $f : S^1 \to B$ by:

$$f(\mathbf{base}) :\equiv b$$

$$\mathbf{ap}_f(\mathbf{loop}) :\equiv \epsilon(1)$$

Using question 1 show that f is an immersion.

Definition 2. *We denote by* Aut *the type $(X : \mathscr{U}) \times (X =_{\mathscr{U}} X)$.*

Question 3 Assume given (X, p) and (Y, q) in Aut. Prove that:

$$(X, p) =_{\text{Aut}} (Y, q) \simeq (\epsilon : X =_{\mathscr{U}} Y) \times (p \cdot \epsilon = \epsilon \cdot q)$$

Definition 3. *We denote by G the type of $\epsilon : \mathbb{Z} \simeq \mathbb{Z}$ such that we have the following property:*

$$(n : \mathbb{Z}) \to \epsilon(n+1) = \epsilon(n) + 1$$

Question 4 We denote by s the equivalence in $\mathbb{Z} \simeq \mathbb{Z}$ with underlying map $x \mapsto x + 1$. Using question 3 prove that:

$$\big((\mathbb{Z}, s) =_{\text{Aut}} (\mathbb{Z}, s)\big) =_{\mathscr{U}} G$$

Question 5 Show that G has a group structure where multiplication is the composition of equivalences. We admit that the equality from question 4 induces an isomorphism of group.

Question 6 Show that we have an isomorphism of group:

$$\epsilon \mapsto \epsilon(0) : G \to \mathbb{Z}$$

Question 7 Conclude that we have an immersion of S^1 into Aut.

Bonus Question Show that if A is a pointed-connected type and B is any type, an immersion $f : A \to B$ induces an equivalence:

$$A \simeq (b : B) \times |f(*) = b|$$

Remark 1. *We define:*

$$T :\equiv (X : \mathscr{U}) \times (p : X =_{\mathscr{U}} X) \times |(\mathbb{Z}, s) = (X, p)|$$

From question 7 and the bonus question, we can deduce that:

$$S^1 \simeq T$$

In fact we can define the circle as T. This means that the circle can be defined from propositional truncation, without using higher inductive type.

It should be noted that T is one universe higher than the circle.

Remark 2. *By our correspondance between groups and connected pointed groupoids, it is natural to see the type T from the previous remark as the type of $\mathbb{Z}$-torsors (see `https://ncatlab.org/nlab/show/torsor` for a definition of torsors, and `http://math.ucr.edu/home/baez/torsors.html` for an entertaining introduction).*

This can be generalized from $\mathbb{Z}$ to any group G, by giving a definition (without using higher inductive) of the type of G-torsors , and proving that it is equivalent to BG *(recall that $S^1 = \mathrm{B}\mathbb{Z}$). This gives an insight into the classical definition of G-torsors: they form a large type equivalent to* BG.