Synthetic Homotopy Theory TD2: Higher inductive types, truncations and logic

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Exercise 1 Higher inductive types

Question 1 Let A be a type, show that:

$$(S^1 \to A) \simeq (x : A) \times (x = x)$$

Question 2 Prove that $\Sigma \mathbf{2} \simeq S^1$.

Question 3 (Optional) Prove that $S^1 \times S^1 \simeq T$. Prove that $\Sigma S^1 \simeq S^2$.

Exercise 2

In this exercise we use results from TD1 to prove truncation results for types.

Question 1 Show that $\mathbf{1}$ is contractible.

Question 2 Show that $\mathbf{0}$ is a proposition.

Question 3 Show that $\mathbf{2}$ and $\mathbb{N}$ are sets. Show that they are not propositions.

Question 4 Let A be a proposition. Show that $A + \neg A$ is a proposition. Find $A, B : \mathrm{Prop}$ such that $A + B$ is not a proposition (Hint: recall that $\mathbf{1} + \mathbf{1} = \mathbf{2}$).

Question 5 Show that the type of sets is a groupoid.

Question 6 (Optional) Show that the type of contractible types is contractible.

Exercise 3 Identity types in set-truncations

Question 1 Let A be a type, using the encode-decode method show that:

$$[x] =_{|A|_0} [y] \simeq |x =_A y|$$

Exercise 4 Decidable types and propositions

Definition 1. *A proposition A is called decidable if we can prove $A + \neg A$. A type X is said to have decidable equality if for all $x, y : X$ the type $(x = y) + \neg(x = y)$ is provable.*

Question 1 Let A be a decidable proposition, prove that:

$$(A = \mathbf{0}) + (A = \mathbf{1})$$

Question 2 Show that types with decidable equality are sets (Hint: use the encode-decode method to compute the identity types of a type with decidable equality).

Remark 1. *It is quite natural to ask whether all propositions are decidable. This cannot be proved or disproved in HoTT.*

Exercise 5 The law of the excluded middle

In this exercise we consider the law of the excluded middle in type theory. The naive law of the excluded middle is an inhabitant of the type $(A : \mathcal{U}) \to A + \neg A$.

Question 1 Show that univalence contradicts the naive law of the excluded middle.

Definition 2. *The law of excluded middle is the following type:*

$$(A : \mathrm{Prop}) \to A + \neg A$$

Equivalently it says that all propositions are decidable.

The law of excluded cannot be proved nor disproved in HoTT. Its interpretation in homotopy types is true.

Question 2 Assuming the law of the excluded middle, show that $\mathrm{Prop} =_{\mathcal{U}} \mathbf{2}$.

Exercise 6 Propositional truncation

Question 1 Let A be a type. Show that the map $[\_] : A \to |A|$ induces equivalences:

$$\lambda f, x. f([x]) : (|A| \to B) \to (A \to B)$$

for all proposition B.

Question 2 Let A be a type. Assume given a type C with a map $t : A \to C$ such that we have equivalences:

$$\lambda f. f \circ t : (C \to B) \to (A \to B)$$

for all proposition B. Show that $C \simeq |A|$.

Question 3 Assuming the law of excluded middle, prove that $\neg\neg A$ is equivalent to $|A|$.

Exercise 7 Connected types

Recall that a type X is said connected if $(x, y : X) \to |x = y|$.

Question 1 Let X be a connected type with $x : X$. Show that $|X|_0 = \mathbf{1}$.

Question 2 Let X be a type such that $|X|_0 = \mathbf{1}$, show that X is connected.

Question 3 Show that the circle is connected.

Question 4 Assume given $X : \mathcal{U}$ with $x : X$. Show that ΣX is connected.

Exercise 8 The axiom of choice

Definition 3. *The naive axiom of choice says that for any types $A, B : \mathcal{U}$ and $P : A \to B \to \mathcal{U}$ we have an inhabitant of:*

$$\big((x : A) \to (y : B) \times P(x, y)\big) \to (f : A \to B) \times \big((x : A) \to P(x, f(x))\big)$$

Question 1 Prove the naive axiom of choice.

Definition 4. *The axiom of choice says that for all* **set** *A with $P : A \to \mathcal{U}$ we have:*

$$\big((x : A) \to |P(x)|\big) \to |(x : A) \to P(x)|$$

Remark 2. *There exists a lot of variants of this axiom in HoTT, for example P can be a family of sets, or we can use set-truncation rather than propositional truncation, etc.*

Question 2 (Optional) Prove that the same axiom for all $A : \mathcal{U}$ contradicts univalence.

Exercise 9 Diaconescu's theorem

In this exercise we show that the axiom of choice imply the law of the excluded middle. Assume given $A : \mathrm{Prop}$.

Question 1 Using the encode-decode method, prove that ΣA is a set and that:

$$(\mathbf{N} =_{\Sigma A} \mathbf{S}) \simeq A$$

Question 2 We define $f : \mathbf{2} \to \Sigma A$ by $f(0) :\equiv \mathbf{N}$ and $f(1) :\equiv \mathbf{S}$. Show that:

$$(x : \Sigma A) \to |\mathbf{fib}_f(x)|$$

Question 3 Using the axiom of choice, conclude that:

$$|(g : \Sigma A \to \mathbf{2}) \times (f \circ g \sim \mathbf{id})|$$

Question 4 Assume given $g : \Sigma A \to \mathbf{2}$ such that $f \circ g \sim \mathbf{id}$. Show that:

$$(g(\mathbf{N}) =_{\mathbf{2}} g(\mathbf{S})) + (g(\mathbf{N}) \neq_{\mathbf{2}} g(\mathbf{S}))$$

Show that $g(\mathbf{N}) = g(\mathbf{S}) \to \mathbf{N} = \mathbf{S}$ and $g(\mathbf{N}) \neq g(\mathbf{S}) \to \mathbf{N} \neq \mathbf{S}$.

Question 5 From the previous questions, conclude that the axiom of choice implies the law of the excluded-middle.