Back to univalence

Intuition The ∞-groupoid of ∞-groupoids has:

- ∞-groupoids as points.
- equivalence of ∞-groupoids as paths.

Definition For $A : \mathcal{U}$ we have
$\mathrm{id}_A : A \simeq A$

$f : A \to B$
$f : A \simeq B$
I mean f is an equivalence

Univalence Reformulated

For all $A, B : \mathcal{U}$, I have

$$\mathrm{univ} : (A =_{\mathcal{U}} B) \simeq (A \simeq B)$$

and $\mathrm{univ}(\mathrm{refl}_A) =_{A \simeq A} \mathrm{id}_A$

Analyse identity types for type formers

Axiom $$(A =_{\mathcal{U}} B) \simeq (A \simeq B)$$ (univalence)

Consequence of univalence

$$(f =_{A \to B} g) \simeq (a : A) \to f a =_B g a$$

(function extensionality)

$$(f =_{(a:A) \to B(a)} g) \simeq (a : A) \to f a =_{B(a)} g a$$

true in plain type theory

$$(a,b) =_{A \times B} (a',b') \simeq (a =_A a') \times (b =_B b')$$

$$(a,b) =_{(x:A) \times B(x)} (a',b') \simeq (p : a =_A a') \times (\mathrm{transport}^B_p(b) =_{B(a')} b')$$

Summary of Part 1 (Type Formers)

- We have identity types and path induction giving all the ∞-groupoids rules we want.

- Σ, Π and $\mathcal{U}$ with formulas for their identity types.

$f =_{A \to \mathcal{U}} g$

is

$(a : A) \to fa =_{\mathcal{U}} ga$

is

$(a : A) \to fa \simeq ga$

→ This all can be interpreted in homotopy types.

Part 2 : Grothendieck correspondence

Definition For B a type, I define

$$\mathcal{U}_{/B} :\equiv (A : \mathcal{U}) \times A \to B$$

"types over B"

"$\mathcal{U}$ over B"

Lemma $f : A \to B$ and $g : A' \to B$

$$(f =_{\mathcal{U}/B} g) \simeq (\varepsilon : A \simeq A') \times g \circ \varepsilon =_{A \to B} f$$

$$\begin{tikzcd} A \arrow[rr, dashed, "\simeq"] \arrow[dr, "f"'] & \arrow[d, phantom, "="] & A' \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

Theorem (Grothendieck correspondence)

For B a type we have

$$\mathcal{U}/B \simeq B \to \mathcal{U}$$

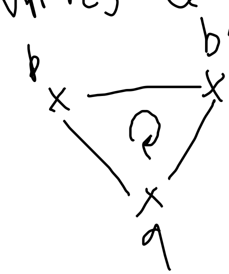
Rq This can have many interpretations

- In ~~∞-groupoids~~ The ∞-groupoid of ∞-groupoids over B is equivalent to the ∞-groupoid of morphisms for B to the ∞-groupoid of ∞ groupoids
- In topos theory, it means '$\mathcal{U}$ classifies all morphisms'
- From homotopy theory, it means all map is homotopic to a fibration.

Def A type A is <u>contractible</u> if we have $a : A$ (called the <u>center of contraction</u>) such that $(b : A) \to a = b$

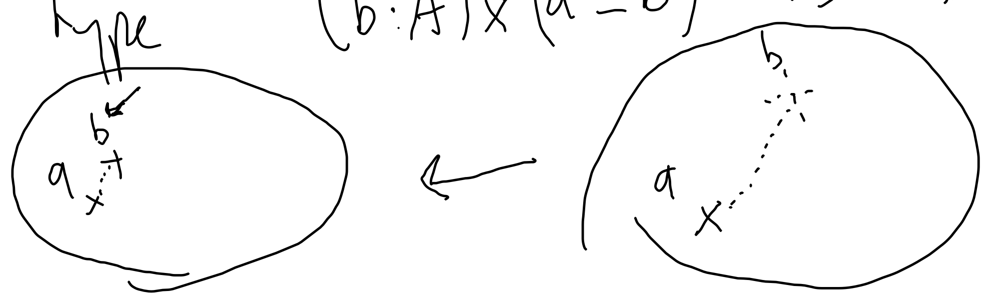
Rq This is a lot stronger than being path connected. Because the path in $a = b$ varies 'continuously' in b.

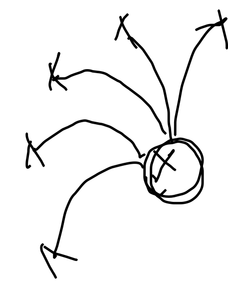
$p : b = b'$



E.g. $\mathbb{1}$ is contractible

Lemma (contractibility of singleton)
For all $a : A$ the type $(b : A) \times (a = b)$ is contractible





Proof $(a, \mathrm{refl}_a) : (b : A) \times (a = b)$

Now for all $(b : A)$ and $(p : a = b)$ I want

$$(a, \mathrm{refl}_a) =_{(b:A) \times a = b} (b, p)$$

$$\simeq (q : a = b) \times \mathrm{transport}_q^{\lambda x . a = x} (\mathrm{refl}_a) =_{a=b} p$$

$$\simeq (q : a = b) \times q =_{a=b} p \qquad \left(\text{as } \mathrm{tr}_q^{\lambda x . a = x} (\mathrm{refl}_a) = q\right)$$

inhabited (p, refl_p)

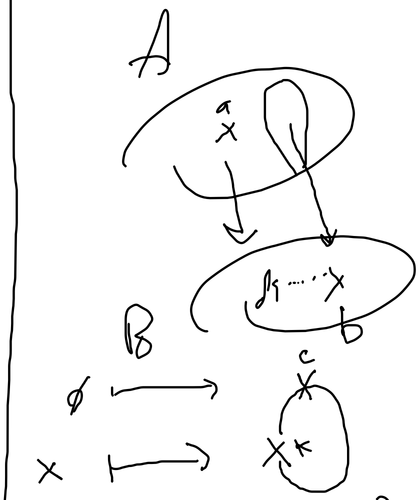
Rq Any contractible type is equivalent to $\mathbb{1}$.

Proof Grothendieck correspondence

For $A : \mathcal{U}$ and $f : A \to B$, I define

$$\mathrm{fib}_f : B \to \mathcal{U}$$

$$\mathrm{fib}_f(b) :\equiv (a : A) \times (f a =_B b)$$



$\mathrm{fib}_c = (* = c)$

Given $P : B \to \mathcal{U}$ I define

$$\begin{tikzcd} (x:B) \times P(x) \arrow[d, "\pi_B"] \\ B \end{tikzcd} \quad : \mathcal{U}/B$$

$$\begin{tikzcd} \mathcal{U}/B \arrow[r, bend left, "\mathrm{fib}"] & (B \to \mathcal{U}) \arrow[l, bend left, "\pi_B"] \end{tikzcd}$$

Induction for products

to define $f : A \times B \to C$

it is enough to define

$$f(x, y) :\equiv \; ? \; : C$$

for all $x : A$ and $y : B$

$\pi_B(x, y) :\equiv x$

From $A : \mathcal{U}$ and $f : A \to B$, I need to show:

$$A \overset{?}{\simeq} (b:B) \times \mathrm{fib}_f(b)$$

\begin{tikzcd}
A \arrow[dr, "f"'] & & (b:B) \times \mathrm{fib}_f(b) \arrow[dl, "\pi_B"] \\
& B &
\end{tikzcd}

$(fa, (-)) \quad (fa, a, \mathrm{refl}_{fa}) \mapsfrom (a, fa, \mathrm{refl}_{fa})$

$$\underbrace{(b:B) \times \mathrm{fib}_f(b) \simeq (b:B) \times (a:A) \times (fa = b) \simeq (a:A) \times \underbrace{(b:B) \times (fa = b)}_{\text{contractible}}}$$

$$\mathrm{IS}$$

\begin{tikzcd}
(b:B) \times \mathrm{fib}_f(b) \arrow[d, "\pi_B"] & & & \\
B & A \arrow[l, "f"'] & A \arrow[l, equal] \simeq A \times \mathbb{1} &
\end{tikzcd}

$fa = fa \mapsfrom a \mapsto$

this side is done

From $P : B \to \mathcal{U}$, I need

$$\mathrm{fib}_{\pi_B} =_{B \to \mathcal{U}} P \qquad \text{where } \pi_B : (b : B) \times Pb \to B$$

$$\simeq : (b : B) \to \mathrm{fib}_{\pi_B}(b) \simeq P(b)$$

but $\mathrm{fib}_{\pi_B}(b) :\equiv \Big(z : (x : B) \times Px\Big) \times \big(\pi_B(z) = b\big)$

$$\simeq (x : B) \times (y : Px) \times \pi_B(x, y) = b$$

$$\simeq (x : B) \times (x = b) \times Px$$

$$\simeq \mathbb{1} \times Pb$$

$$\simeq Pb$$

$\boxed{(x)x \text{ is } x(x)}$

Summary of part 2

We proved $\mathcal{U}/B \simeq B \to \mathcal{U}$

By ~~p~~we can conclude this is true for homotopy types

But we will give three variants of this theorem in HoTT, interpreted as three important theorems on homotopy types.

To do this we will define:

- propositions, sets and groupoids
- groups as groupoids

Part 3 : Propositions, sets and groupoids

Def (Proposition) a type A is a proposition if $(x, y : A) \to x = y$.

Intuition a proposition is a proof irrelevant type.

E.g.
- $\mathbb{1}$ and $\mathbb{0}$ are propositions
- $P : A \to \Omega$ a family of propositions (A is any Type) then $(x : A) \to P_x$ is a proposition.
- $$\frac{}{\text{then } (x : A) \times P_x \text{ is a proposition}} \quad (A \text{ is a prop})$$

More examples

- being an equivalence is a proposition.

$$\boxed{f : A \to B \quad \text{or} \quad f : A \simeq B}$$

not a proposition $\longrightarrow$ $\cancel{\left(\begin{array}{l} g : B \to A \\ g \circ f = id \\ f \circ g = id \end{array}\right)}$

$\boxed{\simeq (a : A) \to a = a}$

$\Updownarrow$

a proposition $\longrightarrow$ $\left(\begin{array}{l} g_1, g_2 : B \to A \\ g_1 \circ f = id \\ f \circ g_2 = id \end{array}\right)$

- 'being a proposition' is a proposition.

$(x, y : A) \to x = y$

$$\boxed{(\varepsilon, \varepsilon' : (x, y : A) \to x =_A y) \to \varepsilon = \varepsilon'}$$

Lemma If A is a proposition, $x, y : A$ then $x = y$ is contractible.

<u>theorem</u> (propositionnal extensionnality)

For A, B two propositions

$$(A =_{\mathcal{U}} B) \;\simeq\; A \leftrightarrow B$$

$$A \leftrightarrow B \equiv (A \to B) \times (B \to A)$$

is

$$A =_{(\cancel{X}:\mathcal{U}) \times isProp(X)} B$$

$\boxed{a : A}$

$$\begin{tikzcd} A \arrow[d, shift right] \\ \mathbb{1} \arrow[u, shift right, "a"'] \end{tikzcd}$$

$$\boxed{\mathcal{U}, =, \to \;\overset{\text{defines}}{\longmapsto}\; \text{Prop}}$$

Coq impredicative Prop | HoTT homotopy proposition

Prop | hProp

Sets

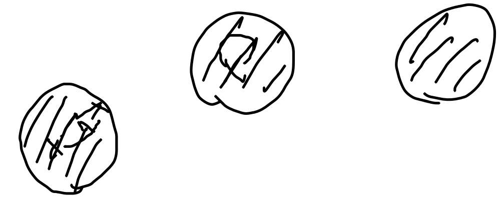
~~Set in Co~~

Def A type A is a set if for all $x, y : A$

$$x =_A y \text{ is a proposition.}$$

Intuition Sets are types with proof-irrelevant equalities.

$$\boxed{\text{Coq Set} \neq \text{HoTT homotopy set / hSet}}$$



Intuition Spaces with no nontrivial paths.

equivalently A is a set if for all $p : a = a$

$$p = \mathrm{refl}_a$$

Stability

$$\frac{A \text{ any type} \qquad B : A \to \mathcal{U} \text{ family of sets}}{(x:A) \to B(x) \text{ is a set}}$$

$$\frac{A \text{ a set} \qquad B : A \to \mathcal{U} \text{ family of sets}}{(x:A) \times B(x) \text{ is a set.}}$$

Examples

- Propositions are sets.
- Bool and $\mathbb{N}$ are sets
- $\mathrm{Prop} :\equiv (X:\mathcal{U}) \times (x,y:X) \to x = y$ is a set

Rq Being a set is a proposition.