Real Business Cycle

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1 Hodrick-Prescott Filter

In analyzing real world datas, it is often useful to separate the trend data from business cycle variation. In essence, we want

$$\ln(Y_t) = \ln(Y_t^{\text{trend}}) + \ln(Y_t^{\text{cycle}})$$

To simplify the expression, we can rewrite this as

$$y_t = y_t^{\text{trend}} + y_t^{\text{cycle}}$$

Now, the HP-filter makes the decomposition by solving the minimization problem of

$$\min_{y_t^{\text{trend}}} \sum_{t=1}^{T} (y_t - y_t^{\text{trend}})^2 + \lambda \sum_{t=1}^{T} \left[(y_{t+1}^{\text{trend}} - y_t^{\text{trend}}) - (y_t^{\text{trend}} - y_{t-1}^{\text{trend}}) \right]^2$$

subject to

$$y_t = y_t^{\text{trend}} + y_t^{\text{cycle}}$$

The first term measures the difference between trend and the actual data, while the second term measures the difference between changes in trend.

Thus, the parameter λ measures the trade off between fit of the data and movement in trends. The higher the λ, the more weight we put on the smoothness of the trend.

If $\lambda = 0$ then all the weight are put on the first term, so to minimize the difference we would just set trend equals to y_t

2 The Model

2.1 Households

Household solves the utility maximization problem of

$$\max_{c_t} u(c_t) = \sum_{t=0}^{T} \beta^t U(c_t)$$

subject to

$$\begin{aligned} c_t + a_{t+1} &= w_t + (1 + r_t)a_t \\ c_t &\geq 0 \\ a_{T+1} &= 0 \end{aligned}$$

The usual Inada conditions and restriction applies here.

Set up the Lagrangian gives

$$L = \sum_{t=0}^{T} \beta^t U(c_t) + \sum_{t=0}^{T} \lambda_t \big(w_t + (1+r_t)a_t - c_t - a_{t+1} \big)$$

Solving for FOCs give

$$\begin{aligned} \beta^t U'(c_t) &= \lambda_t \\ \beta^{t+1} U'(c_{t+1}) &= \lambda_{t+1} \\ \lambda_t &= \lambda_{t+1}(1 + r_{t+1}) \\ U'(c_t) &= \beta(1 + r_{t+1})U'(c_{t+1}) \end{aligned}$$

where the last equation is our familiar Euler equation.

In steady state, we have

$$1 = \beta(1+r)$$

Using the budget constraint

$$c_t = w_t + (1+r_t)a_t - a_{t+1}$$
$$c_{t+1} = w_{t+1} + (1+r_{t+1})a_{t+1} - a_{t+2}$$

We can rewrite the Euler equation as

$$U'(w_t + (1+r_t)a_t - a_{t+1})$$
$$= \beta(1+r_{t+1})U'(w_{t+1} + (1+r_{t+1})a_{t+1} - a_{t+2})$$

2.2 Firms

The production technology is standard neoclassical Cobb-Douglas production function

$$y_t = A_t k_t^\alpha n_t^{\alpha-1}$$

For now, we assume $A_t = A$

$$y_t = A k_t^\alpha n_t^{\alpha-1}$$

We denote the rental price per unit of capital as μ_t, so the return rate is

$$r_t = \mu_t - \delta$$

Then the firms solves the profit maximization problem

$$\max_{n_t, k_t}(y_t - w_t n_t - \mu_t k_t)$$

subject to

$$y_t = A k_t^\alpha n_t^{1-\alpha}$$
$$k_t, n_t \geq 0$$

The FOC yields

$$w_t = (1-\alpha)A\left(\frac{k_t}{n_t}\right)^\alpha$$

$$\mu_t = \alpha A\left(\frac{k_t}{n_t}\right)^{\alpha-1}$$

So the profit function becomes

$$\pi_t = A k_t^\alpha n_t^{1-\alpha} - w_t n_t - \mu_t k_t$$
$$= A k_t^\alpha n_t^{1-\alpha} - (1-\alpha)A\left(\frac{k_t}{n_t}\right)^\alpha n_t$$
$$- \alpha A\left(\frac{k_t}{n_t}\right)^{\alpha-1} k_t = 0$$

2.3 Competitive Equilibrium

Since the economy resource constraint is

$$y_t = c_t + I_t$$
$$I_t = k_{t+1} - (1-\delta)k_t$$
$$c_t + k_{t+1} - (1-\delta)k_t = A k_t^\alpha n_t^{1-\alpha}$$

The last equation is then the exact market clearing condition. We would also have $n_t = 1, a_t = k_t$.

Since we know the wage and capital return from the firm's profit maximization problem, we can substitute those into the Euler's equation, which gives

$$U'\big((1-\alpha)Ak_t^\alpha + (1+\alpha A(k_t)^{\alpha-1} - \delta)k_t - k_{t+1}\big)$$
$$= \beta(1+\alpha A(k_{t+1})^{\alpha-1} - \delta)A(k_{t+1})^{\alpha-1} - \delta)k_{t+1}$$
$$- k_{t+2}\big)$$

$$U'(Ak_t^\alpha + (1-\delta)k_t - k_{t+1}) = \beta(1+\alpha A(k_{t+1})^{\alpha-1} - \delta)$$
$$\cdot U'(Ak_{t+1}^\alpha + (1-\delta)k_{t+1} - k_{t+2})$$

Given the boundary condition $k_0 = a_0$, and $k_{T+1} = a_{T+1} = 0$, we can solve for the problem.

2.4 Social Planner Problem

The social planner solves the problem of

$$\max_{\{c_t, k_{t+1}\}_{t=0}^T} \sum_{t=0}^T \beta^t U(c_t)$$
$$\text{s.t.} \quad c_t + k_{t+1} - (1-\delta)k_t = Ak_t^\alpha$$
$$c_t \geq 0, \quad k_0 > 0 \,\text{given}$$

Set up the Lagrangian and solve for the FOC gives

$$L = \sum_{t=0}^T \beta^t U(c_t) + \sum_{t=0}^T \lambda_t\big[Ak_t^\alpha - c_t - k_{t+1} + (1-\delta)k_t\big]$$
$$\frac{\partial L}{\partial c_t} = \beta^t U'(c_t) - \lambda_t = 0$$
$$\frac{\partial L}{\partial c_{t+1}} = \beta^{t+1} U'(c_{t+1}) - \lambda_{t+1} = 0$$
$$\frac{\partial L}{\partial k_{t+1}} = -\lambda_t + \lambda_{t+1}\big[\alpha A k_{t+1}^{\alpha-1} + (1-\delta)\big] = 0$$

Combining the FOCs give the Euler equation

$$\begin{aligned}\beta^t U'(c_t) = \lambda_t &= \lambda_{t+1}\big[\alpha A k_{t+1}^{\alpha-1} + (1-\delta)\big] \\ &= \beta^{t+1}U'(c_{t+1})\big[\alpha A k_{t+1}^{\alpha-1} + (1-\delta)\big]\end{aligned}$$

$$U'(c_t) = \beta U'(c_{t+1})\big[\alpha A k_{t+1}^{\alpha-1} + (1-\delta)\big]$$

Using the resource constraints

$$\begin{aligned}c_t &= Ak_t^\alpha - k_{t+1} + (1-\delta)k_t \\ c_{t+1} &= Ak_{t+1}^\alpha - k_{t+2} + (1-\delta)k_{t+1} \\ U'(Ak_t^\alpha - k_{t+1} + (1-\delta)k_t) &= \beta U'(Ak_{t+1}^\alpha - k_{t+2} + \\ &(1-\delta)k_{t+1}) \cdot \big[\alpha A k_{t+1}^{\alpha-1} + (1-\delta)\big]\end{aligned}$$

which is exactly the same as the Euler equation from the competitive equilibrium.

2.5 Steady State

In steady state, we set $c_t = c_{t+1} = c^*$ and $k_t = k_{t+1} = k^*$, which changes the Euler equation to

$$1 = \beta\big[\alpha A (k^*)^{\alpha-1} + (1-\delta)\big]$$

Solving for k^* gives

$$k^* = \left[\frac{\alpha A \beta}{1-(1-\delta)\beta}\right]^{\frac{1}{1-\alpha}}$$

and the steady state consumption is

$$c^* = A(k^*)^\alpha - \delta k^*$$

3 Dynamic Analysis

Since we have the Euler equation

$$\begin{aligned}&U'(Ak_t^\alpha + (1-\delta)k_t - k_{t+1}) = \beta(1+ \\ &\alpha A(k_{t+1})^{\alpha-1} - \delta)U'(Ak_{t+1}^\alpha + (1-\delta)k_{t+1} - k_{t+2})\end{aligned}$$