Endogenous Growth

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So far our discussion on output assumes technology is an exogenous process, i.e. $Y = \tilde{F}(A, K, L) = F(K, AL)$ where F is CRS in K and L.

If A increases endogenously, it is unrealistic to assume F is CRS in K, L and A.

In perfectly competitive models, all factors (A, K, L) paid their marginal products, which not enough income is generated to do that. To see this, consider the **Euler's Theorem**: If $F(A, K, L)$ is a homogeneous function of degree $\lambda > 1$, then

$$\tilde{F}_A A + \tilde{F}_K K + \tilde{F}_L L = \lambda \tilde{F} > \tilde{F}$$

Four models and two solutions have been proposed to resolve this issue:

1. *Romer (1986)* — A grows externally (not exogenously) to economic activity
2. *Romer (1990)* — A grows purposefully and markets are not competitive.

1 AK Model — *Romer (1986)*

The *Romer (1986)* model is so called "AK Model" because the aggregate technology is effectively linear in capital. This model introduces learning-by-doing (LBD) externality, which can be concluded in two points:

- production process generates knowledge externalities as a by-product
- since knowledge creation is accidental, no one needs to be compensated for it.

1.1 Assumptions

There are two basic assumptions in this model:

1. **Learning-by-doing** works through each firm's investment. An increase in a firm's capital stock leads to an increase in its stock of knowledge A_i. The idea is supported more broadly by evidence that patents — a proxy for learning — closely follow investment in physical capital.
2. **Knowledge is a public good** that any other firm can access at zero cost. This assumption implies that the change in each firm's technology A_i corresponds to the economy's overall learning and is therefore proportional to the change in economy's average capital intensity.

1.2 Setup

The economy has a large number of small identical firms.

- Firm j's production function is CRS w.r.t its choice of capital and labor.

$$Y_t^j = (K_t^j)^\alpha (A_t L_t^j)^{1-\alpha} \implies y_t^j = A_t^{1-\alpha}(k_t^j)^\alpha$$

where $y^j \equiv Y^j/L^j, k^j \equiv K^j/L^j$

- At the aggregate level:

$$A_t = \hat{A}k_t$$

which embodies LBD externalities

- Each j's private marginal product:

$$\frac{\partial y_t^j}{\partial k_t^j} = \alpha A_t^{1-\alpha}(k_t^j)^{\alpha-1} = \alpha(\hat{A}k_t)^{1-\alpha}(k_t^j)^{\alpha-1}$$

$$= \alpha A \left(\frac{k_t}{k_t^j}\right)^{1-\alpha}$$

where $A \equiv \hat{A}^{1-\alpha}$

1.3 Equilibrium

In a symmetric equilibrium $k_t^j = k_t \forall j$, then the private marginal product becomes

$$\frac{\partial y_t^j}{\partial k_t^j} = \alpha A$$

The symmetric equilibrium makes sense as all firms are small, identical firms, which should result in homogeneous solutions. If the solution is homogeneous, then individual solution is the same as the average.

Social production function is

$$y_t = (\hat{A}k_t)^{1-\alpha}k_t^\alpha = Ak_t$$

Social marginal product of capital

$$\frac{\partial y_t}{\partial k_t} = A$$

The return to capital

$$r_t = \frac{\partial y_t^j}{\partial k_t^j}\Big|_{k_t^j = k_t} - \delta = \alpha A - \delta$$

Embed this production side in a Ramsey economy with CES utility function yields:

$$\left(\frac{C_{t+1}}{C_t}\right)^\sigma = \beta(1 + r_{t+1}) = \beta(1 + \alpha A - \delta)$$

$$\underbrace{\frac{c_{t+1}}{c_t}}_{1+g} = [\beta(1 + \alpha A - \delta)]^{1/\sigma}$$

Now, as long as $\beta(1 + \alpha A - \delta) > 1 \implies g > 0$, the economy grows forever. This requires β and A to be sufficiently large. Also, note that C_{t+1}, C_t are both consumption per worker and $1 + g$ is now endogenous.

Contrast this to Neo-classical model, where

$$1 + g_t \equiv \frac{C_{t+1}}{C_t} = [\beta(1 + f'(k_{t+1}) - \delta)]^{1/\sigma}$$

$g_t > 0$ is not feasible since $\lim_{k\to\infty} f'(k) = 0$ and $\beta(1 - \delta) < 1$.

This means if we have a BGP with endogenous/unbounded growth, then k_t needs to grow unboundedly, i.e. $\lim_{t\to\infty} k_t = \infty$, which contradicts with the initial assumption.

1.4 Takeaways

1. The key to producing endogenous growth is the absence of diminishing return with respect to reproducible inputs, which is capital here (see next model for a more subtle take).
2. The possibility of endogenous growth opens the door for policy to have a permanent effect on the growth rate
3. Competitive equilibrium in this economy is not Pareto-optimal.
   - Each j ignores the effect of their choice of k^j on k.
   - Subsidy to increase private return on investment form αA to A.

To see the last point, suppose for every unit of y^j, the government provides a subsidy ($\frac{1-\alpha}{\alpha}$), which increases the net revenue to

$$\left(1 + \frac{1-\alpha}{\alpha}\right) y^j = \frac{1}{\alpha} y^j$$

These result r = social return in competitive equilibrium. This subsidy can be funded through lump sum taxes on households, which meant Euler equation is unaffected but consumption growth increases to the optimal level. Thus, policy has a permanent effect on the growth rate.

2 Jones-Mannelli Model — *Jones (1995)*

Jones model is also known as the semi-endogenous growth model.

Suppose we break some of the Inada conditions in the production function:

$$Y = F(K, L) = BK^{\alpha}L^{1-\alpha} + bK, \text{ where } b > \delta > 0$$

This implies

$$F(0, L) = 0, \text{ but } F(K, 0) > 0$$
$$\lim_{K\to\infty} F_K = \lim_{K\to\infty} (\alpha BK^{\alpha-1}L^{1-\alpha} + b) = b > 0$$

The underlining idea is that capital is vital for production and more productive than labor in a very specific way.

Output per labor now becomes

$$y \equiv \frac{Y}{L} = f(k) = Bk^{\alpha} + bK$$

Embed this into the Ramsey economy with CES utility function yields

$$r_t = f'(k_t) - \delta = \alpha Bk_t^{\alpha-1} + b - \delta$$

So the Euler equation becomes

$$\frac{C_{t+1}}{C_t} = [\beta(1 + \alpha Bk_t^{\alpha-1} + b - \delta - \delta)]^{1/\sigma}$$

If there is endogenous growth

$$\frac{C_{t+1}}{C_t} \to [\beta(1 + \alpha B + b - \delta - \delta)]^{1/\sigma} > 1$$

as long as $\beta(1 + \beta - \delta) > 1$, which happens with high β and/or high b.

SIDENOTE

Since endogenous growth is usually about the production side, the consumption/preference side is often not too relevant.

3 Endogenous Growth with Human Capital — *Lucas (1988)*

The basic idea behind *Lucas (1988)* model is that if the Y is CRS in 2 reproducible inputs like physical and human capital, then the production function behaves like an AK technology.

For example, if h is humand capital per worker such that the production function becomes

$$Y_t = BK_t^{\alpha}(h_tL_t)^{1-\alpha} \implies y_t \equiv \frac{Y_t}{L_t} = Bk_t^{\alpha}h_t^{1-\alpha}$$

where h grows via investment (net of depreciation)

Embed this into a Solow economy

$$K_{t+1} = s_KY_t + (1 - \delta_k)K_t$$
$$(1 + n)k_{t+1} = s_Ky_t + (1 - \delta_k)k_t$$

Each worker augments their humand capital recording to

$$h_{t+1} = s_Hy_t + (1 - \delta_H)h_t$$

where the given initial conditions are $k_0 > 0, h_0 > 0$

The two difference equations that determine the dynamic of the system are

$$(1 + n)k_{t+1} = s_KBk_t^{\alpha}h_t^{1-\alpha} + (1 - \delta)k_t \quad (3.1)$$
$$h_{t+1} = s_HBk_t^{\alpha}h_t^{1-\alpha} + (1 - \delta)h_t \quad (3.2)$$

In the steady state (BGP) k and h are not constant, in fact, both grow at the same rate g, but $\frac{k}{h}$ is.

Let $x_t \equiv \frac{k_t}{h_t}$, now if we divide (3.1) by h_t gives

$$(1 + n)\frac{k_{t+1}}{h_t}\frac{h_{t+1}}{h_{t+1}} = s_KB\left(\frac{k_t}{h_t}\right)^{\alpha} + (1 - \delta)\left(\frac{k_t}{h_t}\right)$$
$$(1 + n)x_{t+1}\frac{h_{t+1}}{h_t} = s_KBx_t^{\alpha} + (1 - \delta)x_t \quad (3.3)$$

Repeat the same procedure for (3.2)

$$\frac{h_{t+1}}{h_t} = s_HBx_t^{\alpha} + (1 - \delta) \quad (3.4)$$

Suppose $\frac{h_{t+1}}{h_t} = 1 + g = \frac{k_{t+1}}{k_t}, x_t = x \forall t$ in the BGP. Then from (3.3) we have

$$(1 + n)(1 + g)x = s_KBx_{\alpha+(1-\delta)x}$$
$$(1 + n)(1 + g) = s_KBx_{\alpha-1+(1-\delta)} \quad (3.3a)$$

From (3.4) we have

$$(1 + g) = s_KBx^{\alpha} + (1 - \delta)x \quad (3.4a)$$

Now, we have two equations and two unknowns in (3.3a) and (3.4a). Suppose we have $\delta = 1$, then

$$(1 + n)(1 + g) = s_KBx_{\alpha-1}$$
$$(1 + g) = s_HBx^{\alpha}$$

Divide first by the second

$$1 + n = \frac{s_K}{s_H}\frac{1}{x} \implies x = \frac{s_K}{s_H}\frac{1}{1 + n}$$

Substitute this into the second equation

$$1 + g = \frac{B}{(1+n)^{\alpha}} s_K^{\alpha} s_H^{1-\alpha} > 1$$

Thus, this exhibits infinte growth under suitable restrictions, for example, large enough B .

Since in BGP, we have

$$x = \frac{s_K}{s_H} \frac{1}{1+n} \implies h_t = \frac{s_H}{s_K}(1+n)k_t$$

Substitute this into the production function

$$\begin{aligned} y_t &= Bk_t^{\alpha} \left[\frac{s_H}{s_K}(1+n)k_t \right]^{1-\alpha} \\ &= B \left(\frac{s_H}{s_K} \right)^{1-\alpha} (1+n)^{1-\alpha} k_t \end{aligned}$$

4 Comparasion

To give a clear comparison between the first two models we discussed and the Ramsey, here we drop down the Euler equation for the three models.

Ramsey Model($\sigma = 1$)

$$\frac{c_{t+1}}{c_t} = \beta(1 + \alpha B k_{t+1}^{\alpha-1} - \delta) \tag{a}$$

AK Model

$$\frac{c_{t+1}}{c_t} = \beta(1 + \alpha A - \delta) \tag{b}$$

Jones-Mannelli Model

$$\frac{c_{t+1}}{c_t} = \beta(1 + \alpha B k_{t+1}^{\alpha-1} + b - \delta) \tag{c}$$

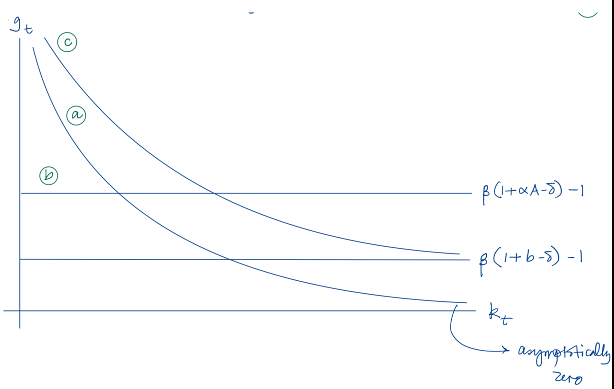

Figure 1: Comparasion Chart

5 Endogenous Growth through innovation

There are two approches to model technological progress:

1. Horizontal innovation — *Romer (1990)*: continuous expansion of input varieties used to manufacture the consumption good.
2. Vertical innovation — *Aghion & Howitt (1992)*: progressive imporvement in the quality of a limited number of intermediate goods.

Here, we are going to focus on the *Romer (1990)* model.

5.1 Setup

There are four sectors in *Romer (1990)* model, but the household sector is trivial as it follows the setup of Ramsey, so the discussion below focuses on the rest three production sectors.

5.1.1 Final goods sector

The production function of the final good section is

$$Y_t = L_{yt}^{1-\alpha} \left(\sum_{j=1}^{A_t} K_{jt}^{\alpha} \right) \tag{5.1}$$

where A_t is the number of intermediate input variesties $\{K_{jt}\}$ available at time t. L_y is labor used in final goods production. The production function exhibits CRS and the final goods sector is perfectly competitive.

Note K_j are not perfect substitutes. Elasticity of substitution between any pair of K_j is $\frac{1}{1-\alpha}$. K_i dos not affect the marginal product of $K_j, j \neq i$, i.e.

$$\frac{\partial Y_t}{\partial K_{jt}} = \alpha L_{yt}^{1-\alpha} K_{jt}^{\alpha-1}$$

Also, there is diminishing marginal product for any K_j that has already been invented.

5.1.2 R&D Sector

The R&D sector invents blueprints ("ideas") for the production of new types of intermediate inputs.

$$A_{t+1} - A_t = \theta A_t L_{At}, \theta > 0, L_{At} = L_t - L_{yt} \tag{5.2}$$

$A_{t+1} - A_t$ is the flow of new blueprints, A_t is the dynamc externality, i.e. accumulated knowledge makes it easier to invet new ideas.

A new blueprint can be put into production with 1-period lag, i.e. ideas inveted in t will lead to new k varieties from $t+1$.

R&D firms sell blueprints to intermediate goods/input producers.

5.1.3 Intermediate Goods

A blueprint produced at t leads to production of that K_j from $t+1$ onwards. The sale of blueprints give the producer perpetual monopoly rights to produce that K_j. The market power comes from imperfect substitutability.

Each K_j produced 1 for 1 using the final and depreciates fully upon use (5.3)

5.2 Solving the Model

To solve the model, we assume Ramsey with CES utility, which gives the household side as

$$\frac{c_{t+1}}{c_t} = (\beta R_{t+1})^{1/\sigma} \tag{5.4}$$

Now final goods producers produce Y_t using L_{yt} (hired at W_t) and $\{K_{jt}\}$ (produced at $\{P_{jt}\}$). Solving final goods producers profit maximization problem yields

$$\max_{L_{yt},\{K_{jt}\}} \Pi_t = L_{yt}^{1-\alpha}(\sum_{j=1}^{A_t} K_{jt}^{\alpha}) - W_t L_{yt}$$

$$-\sum_{j=1}^{A_t} P_{jt} K_{jt}$$

$$\implies (1-\alpha)L_{yt}^{-\alpha}(\sum_{j=1}^{A_t} K_{jt}^{\alpha}) = W_t \tag{5.5a}$$

$$\implies \alpha L_{yt}^{1-\alpha} K_{jt}^{\alpha-1} = P_{jt}, \forall j \in \{1,\dots,A_t\} \tag{5.5b}$$

The producer of K_{jt} is a monopolist who understand the demand for their product is given by (5.5b). This gives the profit maximization problem below

$$\max_{K_{jt}} \pi_{jt} = \underbrace{P(K_{jt})K_{jt}}_{(5.5b)} - \underbrace{K_{jt}}_{(5.3)}, \delta = 1$$

$$= \alpha L_{yt}^{1-\alpha} K_{jt}^{\alpha} - K_{jt}$$

FOC gives

$$\alpha^2 L_{yt}^{1-\alpha} K_{jt}^{\alpha-1} = 1 \implies K_{jt}^* = \alpha^{\frac{2}{1-\alpha}} L_{yt}, \forall j \tag{5.6}$$

Substitute this into (5.5b) gives

$$P_{jt}^* = \frac{1}{\alpha} \tag{5.7}$$

Since $\alpha < 1$, this means $P_{jt} > 1$, but the cost of producing K_{tj} is 1, meaning there is a constant markup, so the maximized profit for intermediate firms are

$$\begin{aligned}\pi_{jt}^* &= P_{jt}^* K_{jt}^* - K_{jt}^* \\ &= \frac{1-\alpha}{\alpha}\alpha^{2/(1-\alpha)} L_{yt}\end{aligned} \tag{5.8}$$

So there is a perpetual monopolist profit flow of $\{\pi_{jt}^*, \pi_{jt+1}^*, \pi_{jt+2}^*, \dots\}$

SIDENOTE

To simplify the assumption (and to get analytical solutions), suppose $L_t = L, \forall t$. We conjecture that in equilibrium

$$L_{yt} = L_y, L_{At} = L_A = L - L_y, r_t = r, \forall t$$

Now the steam of profit becomes $\{\pi_j^*, \pi_j^*, \pi_j^*, \dots\}$.

However, recall the timing assumption, to earn this profit flow, i.e. monoply rents, the producer would have to pruchase the blueprint fro K_j one peirod in advance.

Let the price of a new blueprint at t be P_{At}. Anyone can bid for such a blueprint. If there is free entry, then P_{At} is the maximum bid someone makes.

Note, the maximum bid equals to the PV of monopoly profits $t+1$ onwards, i.e.

$$\begin{aligned}P_{At}^j = P_A^j &= \frac{\pi_j^*}{1+r}\left[1 + \frac{1}{1+r} + \frac{1}{(1+r)^2}\right] \\ &= \frac{\pi_j^*}{1+r}\frac{1}{1-\frac{1}{1+r}} = \frac{\pi_j^*}{r}\end{aligned}$$

From (5.8), we can see that we have a symmetric equilibrium, i.e. π_j^* is the same $\forall j = \pi^*$.

$$\begin{aligned}\implies P_A^j = P_A &= \frac{\pi^*}{r}\forall j \\ &= \frac{1}{r}\left(\frac{1-\alpha}{\alpha}\right)\alpha^{2/(1-\alpha)} L_y\end{aligned} \tag{5.9}$$

SIDENOTE

Even though k producers earn monopoly profits, the PV of the net profit equals to 0 because of 9. This is because of the competitive aspect of the monopolistically competitive intermediate inputs sector, which also means zero equilibrium profits. Thus, the household budget constraint is unaffected by profits earned in this sector

We need to determine L_y and L_A. Note first, labor is perfectly mobile across the two sectors, which implies equilibrium wages have to be the same. Both sectors hire in perfectly competitive labor markets

$$\begin{aligned}
&VMP_L^Y = VMP_L^A \\
&\frac{\partial Y_t}{\partial L_t^Y} = P_{At}\frac{\partial(A_{t+1}-A_t)}{\partial L_t^A} \\
&(1-\alpha)L_{yt}^{-\alpha}\underbrace{(\sum_{j=1}^{A_t}K_{jt}^{\alpha})}_{A_t\bar{K}_t^{\alpha}} = P_{At}\theta A_t \\
&(1-\alpha)L_{yt}^{-\alpha}\bar{K}_t^{\alpha} = \frac{1}{r}\left(\frac{1-\alpha}{\alpha}\right)\alpha^{2/(1-\alpha)}L_{yt}\theta \\
&L_y^{\alpha}\alpha^{\frac{2\alpha}{1-\alpha}}L_y^{\alpha} = \frac{\theta}{\alpha r}\alpha^{\frac{2}{1-\alpha}}L_y \\
&L_y = \frac{\alpha r}{\theta}\alpha^{-2} = \frac{r}{\alpha\theta} \qquad (5.10)
\end{aligned}$$

Now if r is constant, so is L_y as conjectured.

The last step is to determine r and the economy's growth rate. From (5.4), we have

$$1+g = \frac{c_{t+1}}{c_t} = (\beta R)^{1/\sigma} = [\beta(1+r)]^{1/\sigma} \quad (5.4')$$

From (5.2), we have

$$\begin{aligned}
\frac{A_{t+1}-A_t}{A_t} &= \theta L_{At} = \theta(L-Ly) \\
&= \theta(L-\frac{r}{\alpha\theta}) \\
\implies g &= \theta(L-\frac{r}{\alpha\theta}) \qquad (5.2')
\end{aligned}$$

Solve (5.2') and (5.4') for (r, g), neither equation depends on t, thus our conjecture of constant (r, g) and (L_y, L_A) verified.

5.3 Closed-Form Solution

Suppose $\sigma = 1$, i.e. log utility

$$r = \frac{\alpha(1+\theta L-\beta)}{1+\alpha\beta}$$

$$g = \frac{\alpha\beta\theta L-(1-\beta)}{1+\alpha\beta}$$

Note that $g > 0$ only if $\theta L > \frac{1-\beta}{\alpha\beta}$, which can be interpreted as market size effect. Also, $\frac{\partial g}{\partial L} > 0$, $\frac{\partial g}{\partial \beta} > 0$.

However, g in efficiently low because of the two market failures

1. imperfect competition
2. dynamic externality

5.4 Social Planner's Problem

To maximize the lifetime utility of the representative households efficiently, we try to solve the social planner's problem, subject to

$$A_{t+1}-A_t = \theta A_t L_{At}$$

$$Y_t = (L_t - L_{At})^{1-\alpha}(\sum_{j=1}^{A_t}K_{jt}^{\alpha})$$

$$Y_t = C_t + A_{t+1}K_{t+1}$$

The result symmetric equilibrium

$$g^P = \beta\theta L-(1-\beta) > \frac{\alpha\beta\theta L-(1-\beta)}{1+\alpha\beta}$$

To counter the market failures,

1. Fixing the static inefficiency (imperfect competition)
    - subsidize intermediate capital good production
    - K_j increases leads to Y increase, but does not affect growth rate
2. Fixing the dynamic inefficiency (externality in knowledge production)
    - tax labor in final goods production to allow more worker in R&D sector
    - resulting growth higher but less than g^P

We need the combination of both to achieve g^P