Steve Depp
413 – 55

Assignment 7
12 August 2019

## 1.1 Use EDA to justify a transformation of the simple returns to log returns.

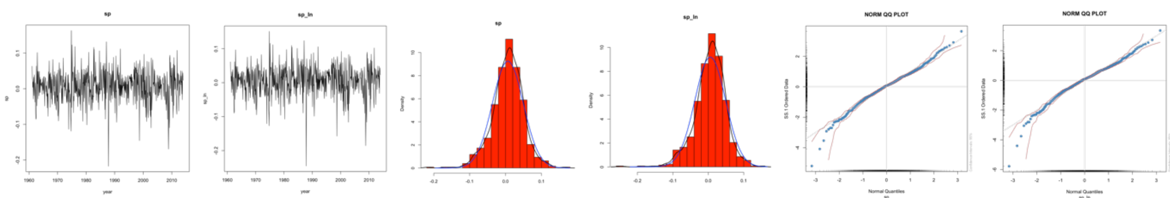
The following presents sample size, mean, standard deviation, median, 20% trimmed mean, min, max, range, skewness, excess kurtosis for S&P simple and log returns over 636 trading days. EDA first confirms there are no gaps in the 636 monthly observations and converts dates from YYYYMMDD format to monthly percent of year format.

| | n | mean | sd | median | trimmed | min | max | range | skew | kurtosis |
|---|---|---|---|---|---|---|---|---|---|---|
| **sp** | 636 | 0.006391434 | 0.04309955 | 0.009218500 | 0.007729418 | -0.2176300 | 0.1630470 | 0.3806770 | -0.4342593 | 1.772202 |
| **sp_ln** | 636 | 0.005439821 | 0.04339813 | 0.009176264 | 0.007377353 | -0.2454275 | 0.1510433 | 0.3964708 | -0.6829334 | 2.490092 |

```
sp  H1: mean=zeroNULL

        One Sample t-test

data:  obj
t = 3.7398, df = 635, p-value = 0.0002008
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 0.003035443 0.009747425
sample estimates:
  mean of x
0.006391434
```

```
sp_ln  H1: mean=zeroNULL

        One Sample t-test

data:  obj
t = 3.1611, df = 635, p-value = 0.001646
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 0.002060580 0.008819062
sample estimates:
  mean of x
0.005439821
```

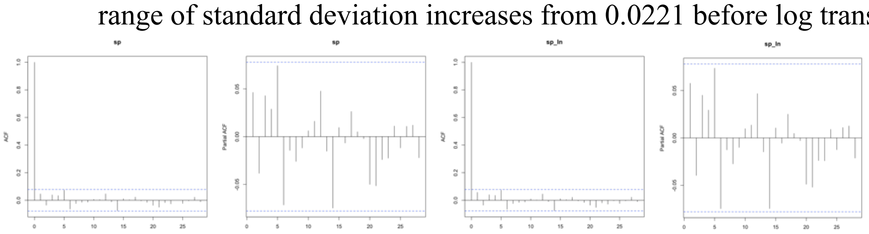
The **above** univariates describe distributions with non-zero mean departing from normal via leptokurtosis and negative skew. Log transformation shifts the distribution leftward, widens the range and increases peakedness as evidenced by more negative skew and more positive kurtosis. The leftward shift is greater for trimmed than for mean which may indicate the center shifts more than wings. Sample means, skew and kurtosis in both series allow us to reject the hypothesis of zero value with 99.9% confidence with the exception of untransformed mean where confidence is 99% as shown **above**.



Plots **above** demonstrate the leftward shifts via the depths of the sharp moves lower in time series, the peakedness via a higher mode in the histogram, and skew of outliers from 5 SDs in the QQ plot of simple returns versus 6 SD for log returns.

| | n | mean | sd | median | trimmed | min | max | range | skew | kurtosis |
|---|---|---|---|---|---|---|---|---|---|---|
| **sd_sp** | 10 | 0.04220037 | 0.007222820 | 0.04199668 | 0.04229961 | 0.03072901 | 0.05287779 | 0.02214877 | -0.1562605 | -1.390068 |
| **sd_sp_ln** | 10 | 0.04246217 | 0.007433149 | 0.04157324 | 0.04258650 | 0.03043878 | 0.05349088 | 0.02305211 | -0.1250292 | -1.319590 |

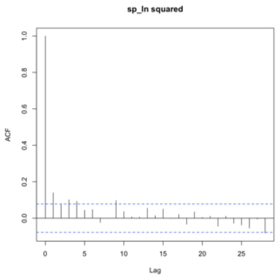
With log transformation, our time series does not approach closer to normality. To check whether log transformation stabilizes variance, both series were divided into 10 consecutive time segments and measured for standard deviation. Show in the **first table above** table, log transformation increases the standard deviation of returns (from 0.0431 to 0.0433) and increases the standard deviation of standard deviation from 0.0072 to 0.0074 in the **table just above**. The range of standard deviation increases from 0.0221 before log transformation to 0.0231 after.

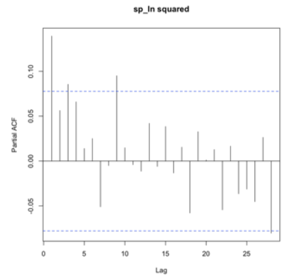


```
SD of sp_lnSD of sp_ln

        Box-Ljung test

data:  obj
X-squared = 11.768, df = 10, p-value = 0.3009
```

A review of auto correlation and partial autocorrelation for simple returns (left 2 plots **above**) and log returns (right 2 plots **above**) reveals similarity and no significant lines. Added benefit is that GARCH modeling can assume mean equation: r_t = mu_t + a_t if any mu_t without any autoregressive or moving average modeling of log returns. Box-Ljung portmanteau test Q statistic = 11.768 with 10 df gives a p-value 0.3009 which does not allow us to reject the null hypothesis of white noise.

Thus, I cannot justify via EDA, the log transformation of S&P returns. There are other advantages, time additivty and numerical stability, that would argue for log transformation. We will continue with the log transformed series.




```
Box-Ljung test

data:  obj^2
X-squared = 38.052, df = 10, p-value = 3.717e-05
```

**<u>Use your analysis to select an appropriate model.</u>**

S&P log returns squared autocorrelations in ACF plot and a Box-Ljung test with p-value = 3.717e-05 allow us to reject the null hypothesis of white noise and infer the presence of GARCH effects in the S&P log return series. ACF and PACF show significance 1, 3, 4 and 9 lag. PACF 1 lag significance might suggest investigation into unit roots in the conditional variance series.

```
Title:
 Augmented Dickey-Fuller Test

Test Results:
  PARAMETER:
    Lag Order: 5
  STATISTIC:
    Dickey-Fuller: -9.8801
  P VALUE:
    0.01
```

```
Title:
 Augmented Dickey-Fuller Test

Test Results:
  PARAMETER:
    Lag Order: 5
  STATISTIC:
    Dickey-Fuller: -5.8812
  P VALUE:
    0.01
```

```
#######################
# KPSS Unit Root Test #
#######################

Test is of type: mu with 6 lags.

Value of test-statistic is: 0.0827

Critical value for a significance level of:
                10pct  5pct 2.5pct  1pct
critical values 0.347 0.463  0.574 0.739
```

```
#######################
# KPSS Unit Root Test #
#######################

Test is of type: mu with 6 lags.

Value of test-statistic is: 0.1162

Critical value for a significance level of:
                10pct  5pct 2.5pct  1pct
critical values 0.347 0.463  0.574 0.739
```

An Augmented Dickey-Fuller test p-value near zero and KPSS Unit Root test suggest an absence of unit roots for the S&P log return series (**left above**) and its squared series (**right above)**.

We will examine GARCH models employing innovations distributed Gaussian, Student-t, skewed Student-t, an IGARCH model and an APARCH model. In each case, we will examine the residuals fromm these models for stationarity and consistency, specifically looking at mean, skew, kurtosis, clustered variance, outliers, serial correlation / GARCH effects, and unit roots. Equations will be written for each model where $r_t$ be the log of S&P returns. AIC and BIC will be examined for comparative purposes.

<u>Gaussian</u>

Model to be fitted:

$$r_t = \mu_t + a_t$$
$$a_t = \sigma_t\varepsilon_t$$
$$\sigma^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1\sigma_{t-1}^2$$

Fitted model:

$$r_t = 6.1617e10\text{-}3 + a_t$$
$$a_t = \sigma_t\varepsilon_t$$
$$\sigma_t^2 = 9.0757e\text{-}5 + 0.12931a^2_{t-1} + 0.83185\sigma_{t-1}^2$$

<u>I-GARCH</u>

Model to be fitted:

$$r_t = \mu_t + a_t$$
$$a_t = \sigma_t\varepsilon_t$$
$$\sigma_t^2 = \alpha_0 + \beta_1\sigma_{t-1}^2 + (1-\beta_1)a_{t-1}^2$$

Fitted model:

$$r_t = 0.00540272 + at$$
$$\sigma_t^2 = 0.88699174\,\sigma_{t-1}^2 + (1-0.88691174)a_{t-1}^2.$$

Students-t

Model to be fitted:

$r_t = \mu_t + a_t$

$a_t = \sigma_t\varepsilon_t$

$\sigma^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1\sigma_{t-1}^2$

Fitted model:

$r_t = 0.00790252 + a_t$

$a_t = \sigma_t\varepsilon_t$

$\sigma_t = 0.00012287 + 0.13623688a^2_{t-1} + 0.80613861\sigma_{t-1}^2$

Skewed Student-t

Model to be fitted:

$r_t = \mu_t + a_t$

$a_t = \sigma_t\varepsilon_t$

$\sigma^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1\sigma_{t-1}^2$

Fitted model:

$r_t = 6.1527e\text{-}3 + a_t$

$a_t = \sigma_t\varepsilon_t$

$\sigma_t = 9.6189e\text{-}5 + 0.13375a^2_{t-1} + 0.82181\sigma_{t-1}^2$

APARCH

Model to be fitted:

$r_t = \mu_t + a_t$

$a_t = \sigma_t\varepsilon_t$

$\sigma_t = \omega + \alpha ( |a_{t-1}| - \gamma_1 a_{t-1}) + \beta\sigma_{t-1}$

Fitted model:

$r_t = 0.00526 + at$

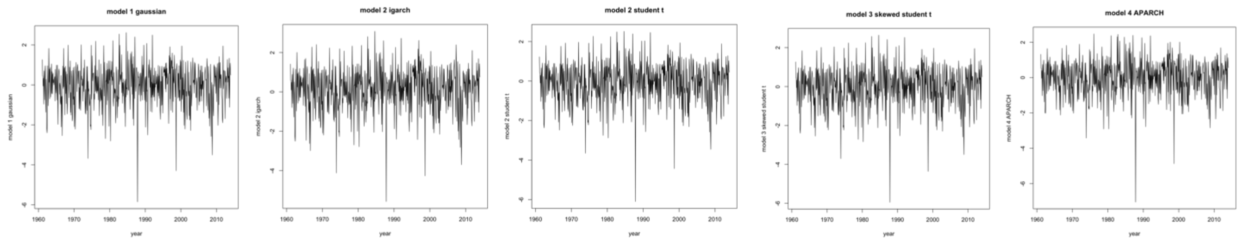
$a_t = \sigma_t\varepsilon_t \qquad \varepsilon_t \sim t^*_{5.71}$

$\sigma^2 = 1.626e10\text{-}4 + 0.07896(|a_{t-1}| - 0.6939a_{t-1})^2 + 0.7956\ \sigma_{t-1}^2$

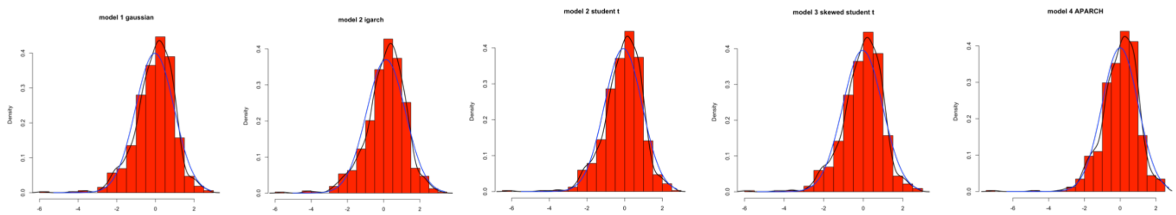
Examining the standardized residuals from each:

| | n | mean | sd | median | trimmed | min | max | range | skew | kurtosis |
|---|---|---|---|---|---|---|---|---|---|---|
| **Gauss** | 636 | -0.031802101 | 0.9987887 | 0.07446634 | 0.02217629 | -5.850012 | 2.617485 | 8.467496 | -0.7549695 | 2.261937 |
| **igarch** | 636 | 0.127798791 | 1.0770816 | 0.24908368 | 0.18689387 | -5.571858 | 3.075483 | 8.647341 | -0.7269258 | 1.831151 |
| **Student-t** | 636 | -0.073637219 | 1.0015256 | 0.03151231 | -0.02102811 | -6.098559 | 2.514928 | 8.613487 | -0.7707017 | 2.471563 |
| **skewed-Student** | 636 | -0.031883957 | 1.0089733 | 0.07547517 | 0.02243288 | -5.952538 | 2.640600 | 8.593137 | -0.7580896 | 2.307599 |
| **APARCH** | 636 | -0.005574764 | 1.0098444 | 0.10527400 | 0.04400036 | -7.043798 | 2.471629 | 9.515427 | -0.9468771 | 4.125376 |

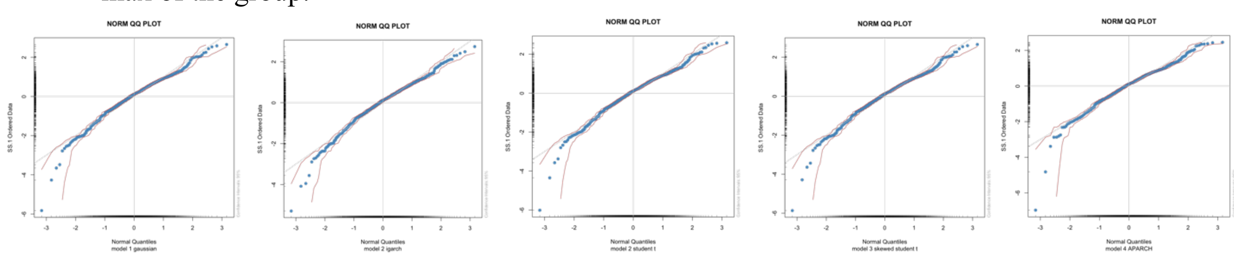
From univariates **above**, the shapes of these standardized residual distributions are similarly departed from normal with (a) excess kurtosis, greatest for APARCH and least for IGARCH, (b) similar levels of negative skew except APARCH, (c) similar min, max, range excepting APARCH with the lowest min and IGARCH with the highest min (d) the shift rightward in the distribution of IGARCH residuals is greatest at the min, then in the max and next in the center.




The time series plots of standardized residuals **above** represent white noise with varying degrees of negative skew, mostly from the APARCH model followed by student-t, then skewed student-t, gaussian and IGARCH.



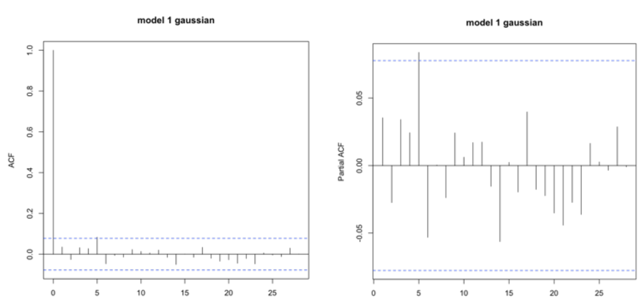

The above histograms seem to demonstrate that the skew in these distributions arise from the leftward outliers.  APARCH has the tallest mode and 2 tall bars in the center; those together with its most leftward outlier of the group, drive its which drive its skew and kurtosis measures to the max of the group.




Here again, the RHS of the distribution adheres to normality while the leftward departs mostly for APARCH, Student-t, Gaussian, skewed Student-t and lastly IGARCH.  The departures and their relative magnitude result from 4 outlier residuals.

$a_t = \sigma_t \varepsilon_t$

Since these are the residuals from modeling conditional variance and the differences between these models is the distribution of epsilon, it would appear that the distribution of epsilon is absorbing the negative skew mostly in APARCH and skewed Student-t, leaving GARCH to model other elements of conditional variance.




The standardized residual ACF and PACF plots for all 5 models are the same with one line significant in ACF and PACF at lag 5. There were no significant AC or PAC in the ACF and PACF plots of squared residuals. Not shown, but a Box-Ljung portmanteau test for all models demonstrated no residual GARCH effects with p-values > 0.55 for all models (greatest for APARCH).

Based upon the above, all models are adequate. The parameters are significant at the 99.9% level, except omegas with p-values ranging 0.02 to 0.03 and alpha and gamma for APARCH with p-values 0.24 and 0.33 respectively. Without considering APARCH yet, I would select the model employing skewed Student's for innovation distribution based upon the AIC and BIC scores below. The skew and shape parameters contribute significantly to the model and improve its performance as measured by AIC and BIC which penalize for additional parameters.

| | AIC | BIC |
|---|---|---|
| GARCH | -3.50 | -3.47 |
| Student's-t | -3.55 | -3.51 |
| Skew Student's-t | -3.57 | -3.53 |
| APARCH | -3.59 | -3.55 |

1.2 Model adequacy checking for the skewed Student's t model is shown above as is the equation for the fitted model.

| meanForecast | meanError | standardDeviation |
|---|---|---|
| 0.006152749 | 0.03322843 | 0.03322843 |
| 0.006152749 | 0.03393009 | 0.03393009 |
| 0.006152749 | 0.03458728 | 0.03458728 |
| 0.006152749 | 0.03520380 | 0.03520380 |
| 0.006152749 | 0.03578300 | 0.03578300 |

1.3 1-step and 5-step ahead predictions from the skewed Student's t model are shown **above.**

1.4 A fitted APARCH model is shown above together with adequacy checking. The model is adequate, but the parameter coefficients are not significant. We cannot reject the hypothesis that the ARCH and leverage parameters, (alpha and omega), are zero based upon p-values 0.24 and 0.33 respectively. It is possible that the distribution does not adhere sufficiently to the heavy tail that APARCH assumes; possibly the outliers bias the measurement of skew and kurtosis in log S&P returns and the actual distribution is closer to normal, thus rendering the omega parameter not significant.

2.1 **Use EDA to show these data are stationary.**

The following presents sample size, mean, standard deviation, median, 20% trimmed mean, min, max, range, skewness, excess kurtosis for USDJPY daily log returns over 2210 trading days. There are no dates associated with this data; so, we will assume they are contiguous days.

| n | mean | sd | median | trimmed | min | max | range | skew | kurtosis |
|---|---|---|---|---|---|---|---|---|---|
| 2210 | -3.883753e-05 | 0.006809707 | 0 | 3.459545e-05 | -0.05215648 | 0.03342812 | 0.0855846 | -0.3511912 | 5.005496 |

```
        One Sample t-test

data:  obj
t = -0.26811, df = 2209, p-value = 0.7886
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 -0.0003229030  0.0002452279
sample estimates:
    mean of x
-3.883753e-05
```

A 1-sampled t-test cannot reject the null hypothesis that the mean = 0 as shown on the right **above**. Otherwise the distribution departs from normal via kurtosis and moderate skew, both of which are non-zero significant at the 99.9% confidence level.

```
Title:
 Augmented Dickey-Fuller Test

Test Results:
  PARAMETER:
    Lag Order: 5
  STATISTIC:
    Dickey-Fuller: -20.4787
  P VALUE:
    0.01
```

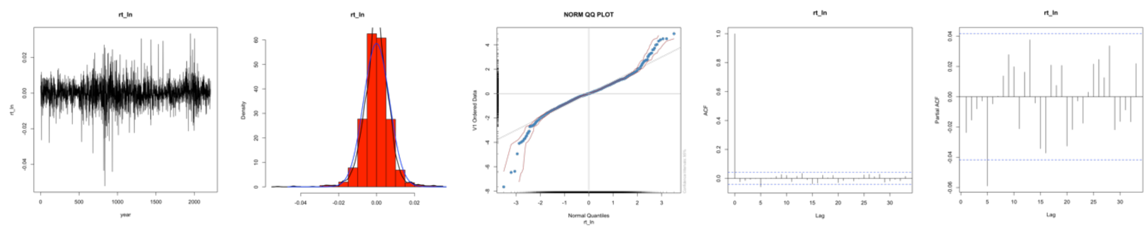
```
Title:
 Augmented Dickey-Fuller Test

Test Results:
  PARAMETER:
    Lag Order: 5
  STATISTIC:
    Dickey-Fuller: -20.4787
  P VALUE:
    0.01
```

```
#######################
# KPSS Unit Root Test #
#######################

Test is of type: mu with 8 lags.

Value of test-statistic is: 0.2915

Critical value for a significance level of:
                10pct  5pct 2.5pct  1pct
critical values 0.347 0.463  0.574 0.739
```

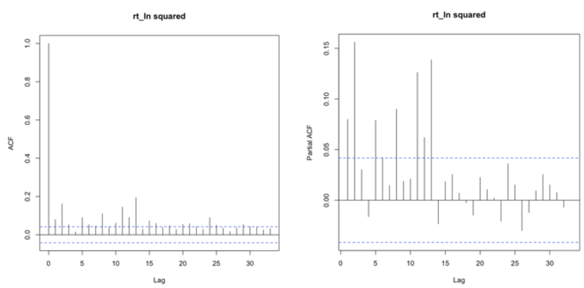
```
#######################
# KPSS Unit Root Test #
#######################

Test is of type: mu with 8 lags.

Value of test-statistic is: 0.7868

Critical value for a significance level of:
                10pct  5pct 2.5pct  1pct
critical values 0.347 0.463  0.574 0.739
```

```
        Box-Ljung test

data:  obj
X-squared = 12.659, df = 10, p-value = 0.2434

        Box-Ljung test

data:  obj^2
X-squared = 145.86, df = 10, p-value < 2.2e-16
```

An Augmented Dickey Fuller unit root test of log returns (on the **left above**) and their squares (on the **right above**) allows us to reject the hypothesis of unit roots with 99.9% confidence level. The log return series is white noise. A KPSS test for unit roots confirms this result for log returns (on the **left above**) but contradicts Dickey Fuller for the square of the log returns (on the **right above**) with a test statistic 0.7868 and p-value < 1%. A repeated KPSS test across different lags via the diff() function confirms that there is a unit root in the squared log returns. A Box-Ljung portmanteau test with p-value < 2.2e-16, shown **above right**, allow us to reject the null hypothesis of white noise in the squared log returns and infer the presence of GARCH effects, confirming to some extent structure found by KPSS. The Box-Ljung test of unsquared log returns would not allow us to reject the null hypothesis of white noise found by Dickey Fuller and KPSS.




The **above** plots are consistent with a normal, but fat-tailed distribution, stationary in that it exhibits constant variation around a zero mean with no autocorrelation in the ACF or PACF plots except possibly by chance at lag 5 in the PACF plot.




Consistent with KPSS tests and Box-Ljung tests of squared log returns, the **above** ACF and PACF plots of squared log returns exhibit auto correlation at mutliple lags: 1,2,5,7,11,12 and 13 in the PACF on the **right above** and all but 8 or 9 different lags in the ACF plot on the **left above**.

**Use your analysis to select an appropriate model. Write the equation for this model.**

For problem 2.1 and 2.2, we examine 4 models: Three are GARCH models employing innovations distributed Gaussian, Student-t, skewed Student-t. The fourth is an IGARCH model.

For problem 2.3 and 2.4, we examine 6 models: Three are TGARCH models with Gaussian, Student-t, skewed Student-t innovations. There is a single APARCH model and a GARCH model with leverage and Student t innovations with and without a mean.

In all 10 cases, we examine the standardized model residuals for stationarity and consistency, specifically looking at mean, skew, kurtosis, clustered variance, outliers, left over serial correlation / GARCH effects, and unit roots.

After a brief examination of univariates, all 10 equations are written for each model where $r_t$ be the log of USDJPY returns for the first four and $r_t$ will be 100*the log of USDJPY returns for the final 6. For problems 2.1 and 2.2 four models adequacy is checked. For problems 2.3 and 2.4, 6 models will be examined for adequacy and significance. AIC and BIC will be examined across all models for comparative purposes

| | n | mean | sd | median | trimmed | min | max | range | skew | kurtosis |
|---|---|---|---|---|---|---|---|---|---|---|
| **Gauss 1.** | 2210 | -0.015174760 | 1.0006824 | -0.013206604 | -0.005349414 | -5.798802 | 5.794046 | 11.59285 | -0.1155665 | 3.386022 |
| **Student-t 2.** | 2210 | -0.003917028 | 1.0056565 | 0.000000000 | 0.005873351 | -5.823417 | 5.844873 | 11.66829 | -0.1137961 | 3.396542 |
| **skewed-Student 3.** | 2210 | -0.003968491 | 1.0060079 | 0.000000000 | 0.005860549 | -5.823866 | 5.846509 | 11.67038 | -0.1137504 | 3.396548 |
| **igarch 4.** | 2210 | -0.001994991 | 1.0227776 | 0.000000000 | 0.006384979 | -5.824695 | 6.139601 | 11.96430 | -0.1034341 | 3.478785 |
| **TGARCH Gauss 5.** | 2210 | -0.005391986 | 0.6809707 | -0.001508232 | 0.001951313 | -5.217156 | 3.341304 | 8.55846 | -0.3511912 | 5.005496 |
| **TGARCH Student 6.** | 2210 | -0.008850133 | 0.6809707 | -0.004966379 | -0.001506834 | -5.220614 | 3.337846 | 8.55846 | -0.3511912 | 5.005496 |
| **TGARCH skewed Student 7.** | 2210 | -0.005391986 | 0.6809707 | -0.001508232 | 0.001951313 | -5.217156 | 3.341304 | 8.55846 | -0.3511912 | 5.005496 |
| **Student Lev w mean 8.** | 2210 | -0.012482829 | 1.0052105 | -0.009368999 | -0.002476944 | -5.867094 | 5.684900 | 11.55199 | -0.1510000 | 3.277361 |
| **Student Lev wo mean 9.** | 2210 | -0.004518780 | 1.0053297 | 0.000000000 | 0.005478704 | -5.864773 | 5.696823 | 11.56160 | -0.1507279 | 3.281175 |
| **APARCH Student wo mean 10.** | 2210 | -0.004518780 | 1.0053297 | 0.000000000 | 0.005478704 | -5.864773 | 5.696823 | 11.56160 | -0.1507279 | 3.281175 |

The **above** univariates compare distributions of standardized residuals from 10 models that depart from normality via 2 general levels of leptokurtosis and slight negative skew. For models 1, 2, 3, 4, 8, 9 and 10 which are GARCH variations, IGARCH, and APARCH, leptokurtois is roughtly ~3.3-3.5 and skew is ~-0.10-0.15. For models 5, 6, 7 that are TGARCH, leptokurtosis is 5.01 and skew is -0.35. For all, a single sample t-test confirms mean = zero and kurtosis is non-zero, both with 99.9% confidence. For all models, except 2, 3, 4, the hypothesis of a symmetric distribution can be rejected with 99% or greater confidence. For models 2 and 3 confidence is 95% (p-value 0.03) and for model 4, confidence is 95% (p-value 0.047). For all, the trimmed > median indicating some right skew in the center of the distritbuions. The range is smalest for TGARCH models while other models have ranges 35% broader.

As with problem 1, I will write the formula here for all models for comparision purposes and make reference back to this point when a question asks for specific fitted models.

1. <u>Gaussian</u>

Model to be fitted:

$$r_t = \mu_t + a_t$$
$$a_t = \sigma_t\varepsilon_t$$
$$\sigma^2 = \alpha_0 + \alpha_1 a_{t-1}{}^2 + \beta_1\sigma_{t-1}{}^2$$

Fitted model:

$$r_t = 7.0196e\text{-}03 + a_t$$
$$a_t = \sigma_t\varepsilon_t$$
$$\sigma_t^2 = 4.2428e\text{-}07 + 3.3612e\text{-}02a^2_{t-1} + 9.5703e\text{-}01\sigma_{t-1}{}^2$$

2. <u>Students-t (no mean)</u>

Model to be fitted:

$$r_t = a_t$$
$$a_t = \sigma_t\varepsilon_t$$
$$\sigma^2 = \alpha_0 + \alpha_1 a_{t-1}{}^2 + \beta_1\sigma_{t-1}{}^2$$

Fitted model:

$$r_t = a_t$$
$$a_t = \sigma_t\varepsilon_t$$
$$\sigma_t = 3.9977e\text{-}07 + 3.2543e\text{-}02a^2_{t-1} + 9.5824e\text{-}01\sigma_{t-1}{}^2$$

3. <u>Skewed Student-t (no mean)</u>

Model to be fitted:

$$r_t = a_t$$
$$a_t = \sigma_t\varepsilon_t$$
$$\sigma^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1\sigma_{t-1}^2$$

Fitted model:

$$r_t = a_t$$
$$a_t = \sigma_t\varepsilon_t$$
$$\sigma_t = 4.1298e\text{-}07 + 3.2995e\text{-}02 a^2_{t-1} + 9.5746e\text{-}01\sigma_{t-1}^2$$

4. I-GARCH (no mean)

Model to be fitted:

$$r_t = a_t$$
$$a_t = \sigma_t\varepsilon_t$$
$$\sigma_t^2 = \alpha_0 + \beta_1\sigma_{t-1}^2 + (1-\beta_1)a_{t-1}^2$$

Fitted model:

$$r_t = at$$
$$\sigma_t^2 = 0.97372893\sigma_{t-1}^2 + (1 - 0.97372893)a_{t-1}^2.$$

5. TGARCH Gauss

Model to be fitted:

$$r_t = \mu_t + a_t$$
$$a_t = \sigma_t\varepsilon_t$$
$$\sigma_t^2 = \alpha_0 + \beta_1\sigma_{t-1}^2 + \alpha_1 a_{t-1}^2 + \gamma_1\, a^2_{t-1} * N_{t-1}$$

Fitted model:

$$r_t = 0.00150823 + a_t$$
$$\sigma_t^2 = 0.00598220 + 0.94852658\sigma_{t-1}^2 + 0.02061994a^2_{t-1} + 0.03431583a^2_{t-1} * N_{t-1}$$

where

$$N_{t-i} = \begin{cases} 1 & \text{if } a_{t-i} < 0, \\ 0 & \text{if } a_{t-i} \geq 0, \end{cases}$$

6. TGARCH Student's-t

Model to be fitted:

$$r_t = \mu_t + a_t$$
$$a_t = \sigma_t\varepsilon_t$$
$$\sigma_t^2 = \alpha_0 + \beta_1\sigma_{t-1}^2 + \alpha_1 a_{t-1}^2 + \gamma_1\, a^2_{t-1} * N_{t-1}$$

Fitted model:

$$r_t = 0.00496638 + a_t$$
$$\sigma_t^2 = 0.00514514 + 0.95356064\sigma_{t-1}^2 + 0.02043414a^2_{t-1} + 0.02720600a^2_{t-1} * N_{t-1}$$

where

$$N_{t-i} = \begin{cases} 1 & \text{if } a_{t-i} < 0, \\ 0 & \text{if } a_{t-i} \geq 0, \end{cases}$$

7. TGARCH Skewed Student's-t

Model to be fitted:

$$r_t = \mu_t + a_t$$
$$a_t = \sigma_t\varepsilon_t$$
$$\sigma_t^2 = \alpha_0 + \beta_1\sigma_{t-1}^2 + \alpha_1 a_{t-1}^2 + \gamma_1\, a^2_{t-1} * N_{t-1}$$

Fitted model:

$$r_t = 0.00150823 + a_t$$

$\sigma_t^2$ = $0.00598220 + 0.94852658\sigma_{t-1}^2$
$+ 0.02061004a^2_{t-1} + 0.03431583a^2_{t-1} * N_{t-1}$
where
$$N_{t-i} = \begin{cases} 1 & \text{if } a_{t-i} < 0, \\ 0 & \text{if } a_{t-i} \geq 0, \end{cases}$$

8. Student's-t Levered with mean

Model to be fitted:

$r_t = a_t + \mu_t$

$a_t = \sigma_t\varepsilon_t$

$\sigma_t = \omega + \alpha\,|a_{t-1}| + \gamma_1\,|a_{t-1}|*N + \beta\sigma_{t-1}$

where $N_{t-i} = \begin{cases} 1 & \text{if } a_{t-i} < 0, \\ 0 & \text{if } a_{t-i} \geq 0, \end{cases}$

Fitted model:

$r_t$ = at + 0.004962

$a_t = \sigma_t\varepsilon_t$

$\sigma^2 = 0.005151 + 0.032648\,(|a_{t-1}| - 0.208510a_{t-1})^2 + 0.953515\sigma_{t-1}^2$

where $N_{t-i} = \begin{cases} 1 & \text{if } a_{t-i} < 0, \\ 0 & \text{if } a_{t-i} \geq 0, \end{cases}$

9. Student's-t Levered (no mean)

Model to be fitted:

$r_t = a_t$

$a_t = \sigma_t\varepsilon_t$

$\sigma_t = \omega + \alpha\,|a_{t-1}| + \gamma_1\,|a_{t-1}|*N + \beta\sigma_{t-1}$

where $N_{t-i} = \begin{cases} 1 & \text{if } a_{t-i} < 0, \\ 0 & \text{if } a_{t-i} \geq 0, \end{cases}$

Fitted model:

$r_t$ = at

$a_t = \sigma_t\varepsilon_t$

$\sigma^2 = 0.005198 + 0.032667\,(|a_{t-1}| - 0.211693a_{t-1})^2 + 0.953479\,\sigma_{t-1}^2$

where $N_{t-i} = \begin{cases} 1 & \text{if } a_{t-i} < 0, \\ 0 & \text{if } a_{t-i} \geq 0, \end{cases}$

( 8 and 9 formulae from http://hedibert.org/wp-content/uploads/2018/05/garchmodeling.pdf )

10. APARCH Student's t (no mean)

Model to be fitted:

$r_t = a_t$

$a_t = \sigma_t\varepsilon_t$

$\sigma_t = \omega + \alpha\,(\,|a_{t-1}| - \gamma_1\,a_{t-1}) + \beta\sigma_{t-1}$

Fitted model:

$r_t$ = at

$a_t = \sigma_t\varepsilon_t$

$\sigma^2 = 0.005198 + 0.032667\,(|a_{t-1}| - 0.211693a_{t-1})^2 + 0.953479\,\sigma_{t-1}^2$

For 3 of the first 4 models, here are AIC and BIC.

| | AIC | BIC |
|---|---|---|
| Gaussian (no mean | -7.25 | -7.24 |
| Student's-t (no mean) | -7.33 | -7.32 |

| | | |
|---|---|---|
| Skew Student's-t (no mean) | -7.33 | -7.32 |
| IGARCH | NA | NA |

On the basis of the above figures, the GARCH model employing skewed Student's t distributed innovations would be the choice. All four models have parameters significant at the 99.9% level except omega which is significant at the 95% level (p-values ~ 0.02).

**Unit Roots / GARCH effects in model standardized residuals**

Inasmuch as there was a unit root suggested by KPSS tests of the square of log returns at the 99% confidence level, a look at the IGARCH model is warranted. Additionally, $\alpha_1 + \beta_1 = 0.990642$, 0.990783 and 0.990455 for the first 3 models, and so IGARCH would be a logical choice as alpha and beta together are capturing nearly 100% of lingering effects from past observations.

```
#######################
# KPSS Unit Root Test #
#######################

Test is of type: mu with 8 lags.

Value of test-statistic is: 0.7868

Critical value for a significance level of:
                10pct  5pct 2.5pct  1pct
critical values 0.347 0.463  0.574 0.739
```

```
#######################
# KPSS Unit Root Test #
#######################

Test is of type: mu with 8 lags.

Value of test-statistic is: 0.7873

Critical value for a significance level of:
                10pct  5pct 2.5pct  1pct
critical values 0.347 0.463  0.574 0.739
```

For GARCH model 1, and TGARCH model 5, the **KPSS test** suggests unit roots in the **residuals squared** at the 99% confidence level as shown **above**. **KPSS residuals** test statistics at 8 lags allow rejection of the null hypothesis of no unit roots at the 90% confidence level for 6 of the 10 models except the TGARCH and APARCH models (5,6,7,10). These test results are not shown.

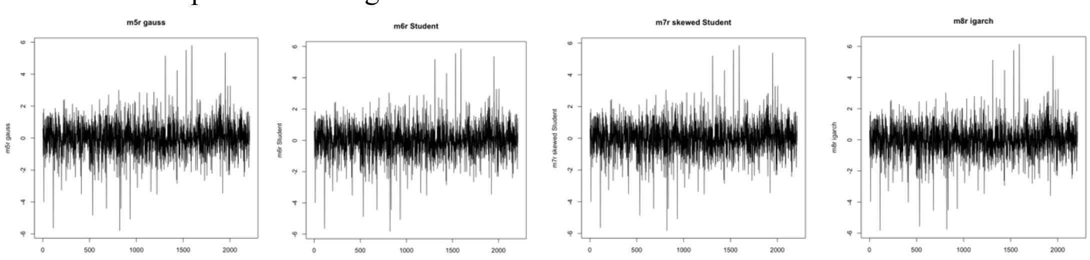
```
        Box-Ljung test

data:  obj^2
X-squared = 145.79, df = 10, p-value < 2.2e-16
```

```
        Box-Ljung test

data:  obj^2
X-squared = 145.63, df = 10, p-value < 2.2e-16
```

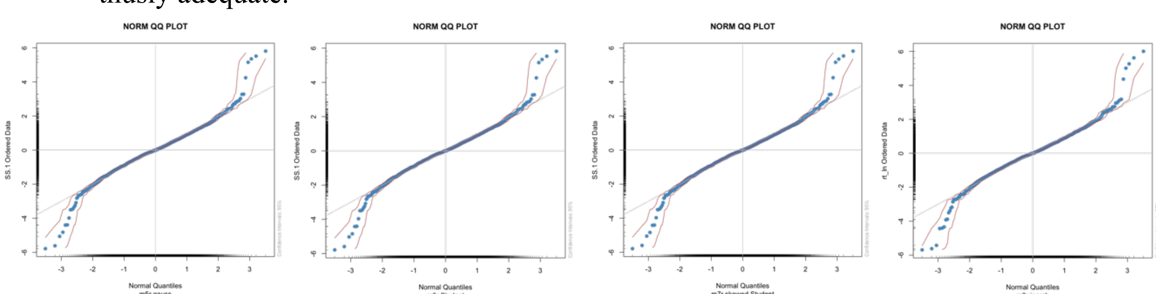
```
        Box-Ljung test

data:  obj^2
X-squared = 145.79, df = 10, p-value < 2.2e-16
```

For 7 of the 10 models, **Box Ljung** tests of residuals and residuals squared indicate white noise. Exceptions are shown above for the 3 TGARCH models **residuals squared** indicate the presence of **GARCH effects** after the model is run. **Dickey Fuller tests** of residuals and residuals squared do not suggest unit roots in any of the 10 models residuals or residuals squared.

2.2 On this basis, I would continue to prefer the GARCH model employing a skewed Student's -t innovation distribution on the basis of its AIC and BIC score. IGARCH warrants investigation based upon the residual unit roots and GARCH effects found in the above models and unit roots in the squares of the log returns.




On the basis of these plots, any one of the first 4 models time series appears stationary and is thusly adequate.




There appears to be no difference between the models and aside from fat tails in each one, they appear normal.

The ACF and PACF plots show no more than chance frequency of signifcant autocorrelation or partial autocorrelation for all of these 4 models.

Skewed Student-t (no mean)

Model to be fitted:

$$r_t = a_t$$
$$a_t = \sigma_t\varepsilon_t$$
$$\sigma^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1\sigma_{t-1}^2$$

Fitted model:

$$r_t = a_t$$
$$a_t = \sigma_t\varepsilon_t$$
$$\sigma_t = 4.1298e{-}07 + 3.2995e{-}02a^2_{t-1} + 9.5746e{-}01\sigma_{t-1}^2$$

On these bases, the models are deemed adequate and the model 3 with skewed Standardized innovations is selected. It's equation to be fitted and fitted coefficents are shown **above**.

2.3 Let rt be the daily log return. For numeric stability, consider the percentage log return, i.e. yt = 100rt. Write the equation for the model to be fitted. The final 6 models above are considered and compared via AIC and BIC:

| | AIC | BIC |
|---|---|---|
| TGARCH Gauss | 1.952512* | 1.961450* |
| TGARCH Student's t | 1.952512* | 1.961450* |
| TGARCH Skew Student's-t | 1.875165* | 1.890643* |
| GARCH Student's t Levered (with mean) | 1.875165 | 1.890643 |
| GARCH Student's t Levered (no mean) | 1.874339 | 1.887237 |
| APARCH Student's t (no mean) | 1.874339 | 1.887237 |

* computed from Log Likelihood

```
Coefficient(s):
        Estimate  Std. Error  t value   Pr(>|t|)
mu    0.00150823  0.01292617  0.11668  0.9071133
omega 0.00598220  0.00192504  3.10758  0.0018863 **
alpha 0.02061004  0.00655225  3.14549  0.0016581 **
gam1  0.03431583  0.01241667  2.76369  0.0057152 **
beta  0.94852658  0.01010117 93.90261 < 2.22e-16 ***
```

TGARCH with Gauss

```
Coefficient(s):
        Estimate  Std. Error  t value Pr(>|t|)
mu    0.00496638  0.01187945  0.41806 0.675900
omega 0.00514514  0.00217856  2.36171 0.018191 *
alpha 0.02043414  0.00837567  2.43970 0.014699 *
gam1  0.02720600  0.01367878  1.98892 0.046710 *
beta  0.95356064  0.01182846 80.61579  < 2e-16 ***
shape 5.70653968  0.65103473  8.76534  < 2e-16 ***
```

TGARCH with Student's t

```
Coefficient(s):
        Estimate  Std. Error  t value   Pr(>|t|)
mu    0.00150823  0.01292617  0.11668  0.9071133
omega 0.00598220  0.00192504  3.10758  0.0018863 **
alpha 0.02061004  0.00655225  3.14549  0.0016581 **
gam1  0.03431583  0.01241667  2.76369  0.0057152 **
beta  0.94852658  0.01010117 93.90261 < 2.22e-16 ***
```

TGARCH with skewed Student's t

```
         Estimate  Std. Error  t value Pr(>|t|)
mu     click to scroll output; double click to hide   0.418   0.6761
omega    0.005151    0.002042    2.523   0.0116 *
alpha1   0.032648    0.007982    4.090 4.31e-05 ***
gamma1   0.208510    0.103211    2.020   0.0434 *
beta1    0.953515    0.010901   87.471  < 2e-16 ***
shape    5.706231    0.650726    8.769  < 2e-16 ***
```

GARCH with Student's t Levereed with mean

```
         Estimate  Std. Error  t value Pr(>|t|)
omega    0.005198    0.002052    2.533   0.0113 *
alpha1   0.032667    0.007994    4.087 4.38e-05 ***
gamma1   0.211693    0.102923    2.057   0.0397 *
beta1    0.953479    0.010927   87.262  < 2e-16 ***
shape    5.712793    0.651888    8.763  < 2e-16 ***
```

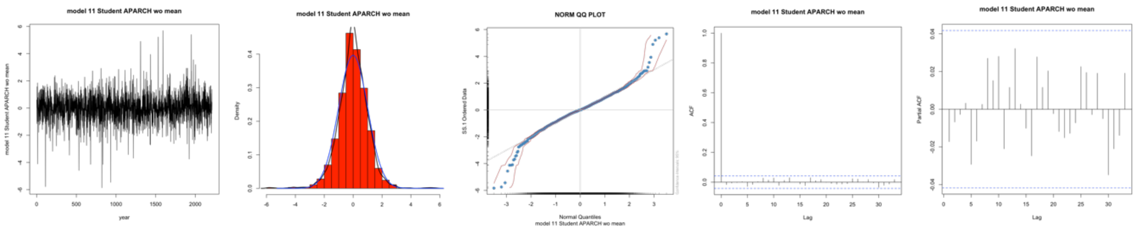
GARCH with Students t Levered without mean

```
         Estimate  Std. Error  t value Pr(>|t|)
omega    0.005198    0.002052    2.533   0.0113 *
alpha1   0.032667    0.007994    4.087 4.38e-05 ***
gamma1   0.211693    0.102923    2.057   0.0397 *
beta1    0.953479    0.010927   87.262  < 2e-16 ***
shape    5.712793    0.651888    8.763  < 2e-16 ***
```

APARCH with Students t without mean

2.4 As shown **above**, levereage or gamma is a significant parameter for the last 3 models (2 GARCH and 1 APARCH) at the 95% confidence level. P-values are higher for the 2 models without mean, and the mean for these last 2 models is one less parameter to calculate; hence the BIC measure of performance is slightly better for these 2 identical models. The mean is not significantly away from zero according to the model t-tests; so that rules out 4 of the 6 models on that basis (as well as the BIC measure). The leverage parameter is addressing the fat tailed nature of the distribtion.

Since the final 2 are algebraically identical, we can select either one and move on to adequacy.




Please see above plots. Times series plots represent stationarity. The histogram represents normality. The QQ shows fat tails but otherwise normality. Neither ACF nor PACF of residuals (or residuals squared, not shown) exhibit autocorrelation. Thus, the model is adequate.

<u>APARCH Student's t (no mean)</u>

Model to be fitted:

$$r_t = a_t$$
$$a_t = \sigma_t \varepsilon_t$$
$$\sigma_t = \omega + \alpha\,(\,|a_{t-1}| - \gamma_1\, a_{t-1}) + \beta\sigma_{t-1}$$

Fitted model:

$$r_t = at$$
$$a_t = \sigma_t \varepsilon_t$$
$$\sigma^2 = 0.005198 + 0.032667\,(|a_{t-1}| - 0.211693a_{t-1})^2 + 0.953479\,\sigma_{t-1}{}^2$$

Restating the equations for the model to be fitted and fitted **above**.

3.0 The work accumulated above presents evidence in the form of fitted models for a distribution of returns available in the USDJPY market that has fat tails and zero mean. This would present evidence that trading daily USDJPY may yield negative returns after bid ask is considered, but that buying out of the money options, even on a slightly higher volatility than at the money options, would not necessarily be a money losing venture. How much higher the volatility could be for out of the money strikes depends upon expectations for volatility movement. Based upon the limited information and modeling done, APARCH or GARCH models employing Student's-t distributed innovations would be a good starting point for comprehending the volatility market, better than models which estimate a mean.