Middle East Technical University
Department of Electrical and Electronics Engineering
EE301: Signals and Systems I

Solutions for Homework 0
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1. (a)

$$\underbrace{3\exp\left(j\frac{\pi}{2}\right)}_{3j}+\underbrace{3\exp\left(-j\frac{\pi}{2}\right)}_{-3j}+\underbrace{\exp\left(j\pi\right)}_{-1}-\underbrace{4\exp\left(-j\frac{\pi}{2}\right)}_{-4j}+4j+\underbrace{2\exp\left(j\frac{\pi}{6}\right)}_{\sqrt{3}+j}=(\sqrt{3}-1)+9j$$

(b)

$$\begin{aligned}\frac{(1-j)\,(5-5j)\,(\sqrt{3}-j)}{10j\,(5-j5\sqrt{3})}&=\frac{\sqrt{2}\exp\left(-j\frac{\pi}{4}\right)5\sqrt{2}\exp\left(-j\frac{\pi}{4}\right)2\exp\left(-j\frac{\pi}{6}\right)}{10\exp\left(j\frac{\pi}{2}\right)10\exp\left(-j\frac{\pi}{3}\right)}\\&=\frac{1}{5}\exp\left(-j\tfrac{5\pi}{6}\right)\\&=\frac{1}{10}\left(-\sqrt{3}-j\right)\end{aligned}$$

(c)

$$\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}j\right)^5=\left(\exp\left(-j\frac{\pi}{4}\right)\right)^5=\exp\left(-j\frac{5\pi}{4}\right)=\frac{-1}{\sqrt{2}}+\frac{1}{\sqrt{2}}j$$

(d)

$$\left(\cos\frac{\pi}{6}-j\sin\frac{\pi}{6}\right)^6=\left(\exp\left(-j\frac{\pi}{6}\right)\right)^6=\exp\left(-j\pi\right)=-1$$

2. We assume that

$$\exp(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!}$$

for any $z\in\mathbb{C}$. Let $z=j\theta$,

$$\exp\left(j\theta\right)=\sum_{n=0}^{\infty}\frac{(j\theta)^n}{n!}=1+j\theta+\frac{(j\theta)^2}{2!}+\frac{(j\theta)^3}{3!}+\frac{(j\theta)^4}{4!}+\cdots=1+j\theta-\frac{\theta^2}{2!}-j\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+\dots$$

It is clear that even and odd n values correspond to real and imaginary parts, respectively.

$$\exp(j\theta)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\theta^{2n}+j\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\theta^{2n+1}$$

The Maclaurin series of cosine and sine functions have explicit forms as follows:

$$\cos\theta=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\theta^{2n}\qquad\sin\theta=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\theta^{2n+1}$$

Substituting them, we obtain Euler's formula.

$$\exp\left(j\theta\right)=\cos\theta+j\sin\theta$$

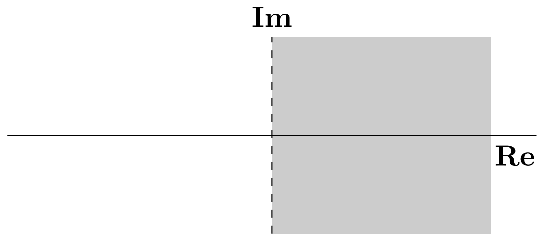
3. For $z = x + jy$,

$$\exp(z) = \exp(x + jy) = \exp(x)\exp(jy),$$

and hence, we can show that the given equality holds by using Euler's formula.

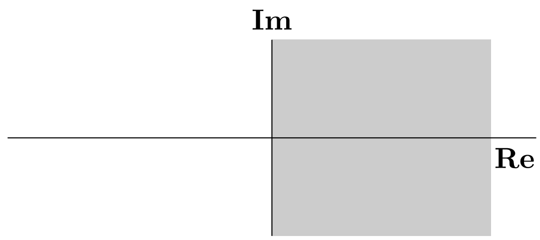
$$\exp(z) = \exp(x)[\cos(y) + j\sin(y)]$$

4. (a) On the complex plane, **Re** is the real axis and **Im** is the imaginary axis.



$$S_1 = \left\{ z \middle| \mathbf{Re}(z) > 0 \right\}$$

(b)

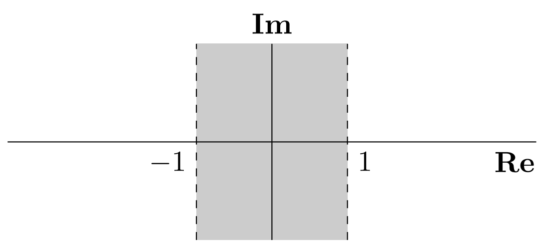


$$S_2 = \left\{ z \middle| \mathbf{Re}(z) \geqslant 0 \right\}$$

i. $S_1 \subset S_2$, and S_2 includes the imaginary axis, $\mathbf{Re}(z) = 0$.

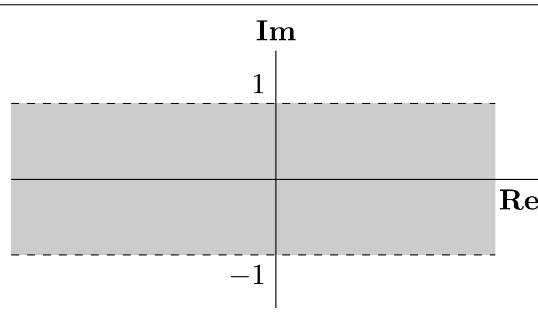
ii. $S_{im} = \{z = x + jy | x = 0\}$ is the set of complex numbers on the imaginary axis, and $S_2 = S_1 \cup S_{im}$. We can also relate them as $S_1 = S_2 \setminus S_{im}$ and $S_{im} = S_2 \setminus S_1$.

(c)



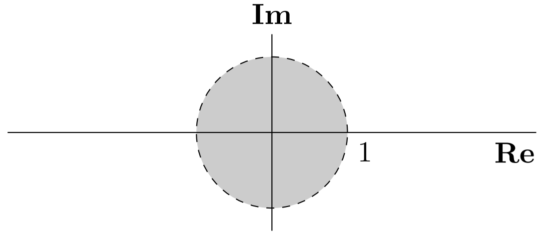
$$S_3 = \left\{ z \middle| -1 < \mathbf{Re}(z) < 1 \right\}$$

(d)



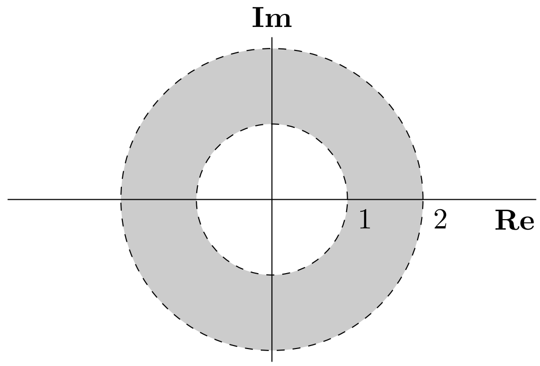
$$S_4 = \left\{ z \middle| -1 < \mathbf{Im}(z) < 1 \right\}$$

(e)



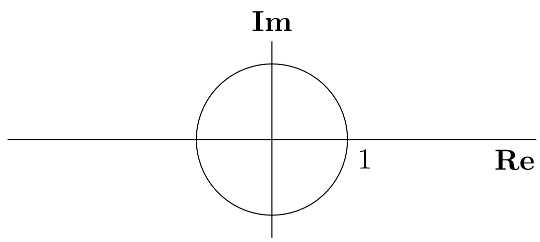
$$S_5 = \left\{ z \middle| \, |z| < 1 \right\}$$

(f)



$$S_6 = \left\{ z \middle| 1 < |z| < 2 \right\}$$

(g)



$$S_7 = \left\{ z \middle| \, |z| = 1 \right\}$$

The set S_7 is called the unit circle since it includes the circle of unit radius, i.e., of radius 1.

i. $S_{ud} = \left\{ z \middle| \, |z| \leqslant 1 \right\}$ is the set of all complex numbers z with modulus less than or equal to 1, the closed unit disk.

ii. $S_8 = \left\{ z \middle| 0 < |z| \leqslant 1 \right\}$ is the set of all complex numbers z with modulus less than or equal to 1 except $|z| = 0$. The set S_8 is called the punctured (closed) unit disk since it includes the unit circle (disk boundary) and a disk with radius 1 except the center of disk (the hole at the center).

5. (a) Using Euler's formula, we obtain the Cartesian form as below.

$$(r \exp(j\theta))^n = (r \cos\theta + jr \sin\theta)^n$$

Then, using de Moivre's formula,

$$(r \cos\theta + jr \sin\theta)^n = r^n(\cos n\theta + j \sin n\theta),$$

we can show that

$$(r \exp(j\theta))^n = r^n(\cos n\theta + j \sin n\theta) = r^n \exp(jn\theta)$$

for any $n \in \mathbb{Z}$.

(b) We can write

$$(\exp(j\theta))^3 = \exp(j3\theta) = \cos 3\theta + j\sin 3\theta$$

by using the expression given in part (a) for $r = 1$. We can also express $(\exp(j\theta))^3$ as

$$\begin{aligned}(\exp(j\theta))^3 &= (\cos\theta + j\sin\theta)^3 \\ &= \cos^3\theta + j3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - j\sin^3\theta \\ &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + j(-\sin^3\theta + 3\cos^2\theta\sin\theta)\end{aligned}$$

by using Euler's formula and binomial series. Then, we obtain the trigonometric identities

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ \sin 3\theta &= -\sin^3\theta + 3\cos^2\theta\sin\theta\end{aligned}$$

by equating real and imaginary parts of $\exp(j3\theta)$ and $(\cos\theta + j\sin\theta)^3$.

(c) *i.* We have $r^n\exp(jn\theta) = \exp(j2\pi k) = \cos(2\pi k) + j\sin(2\pi k) = 1$ for $k = 0, \pm1, \pm2, \ldots$ and $n \in \mathbb{Z}^+$.

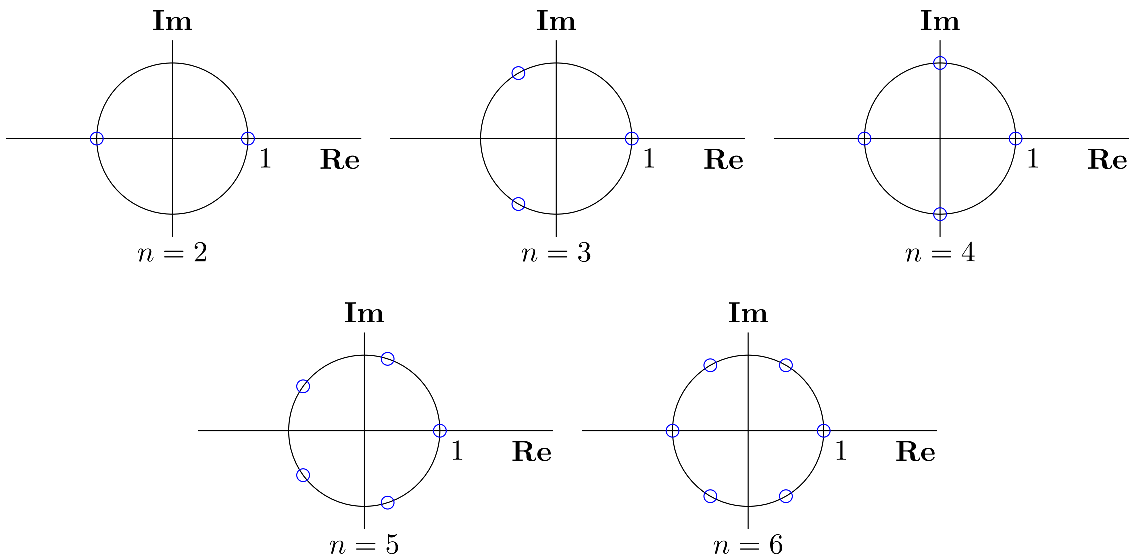
Hence, r^n must have a stable value with nth power changing, and $r = 1$ (remember that $1^a = 1^b$ for any $a, b \in \mathbb{Z}$).

$$r = 1 \implies \exp(jn\theta) = \exp(j2\pi k) \implies \theta = \frac{2\pi k}{n}$$

We know that the complex exponential $\exp(jn\theta)$ is identical to $\exp(jn(\theta + 2\pi))$ for integer n. Hence, we can write the interval of k:

$$\theta = \frac{2\pi k}{n} \text{ for } n \in \mathbb{Z}^+ \text{ and } k = 0, 1, \ldots, n-1$$

ii.



iii. The n-th roots of an arbitrary complex number $w = \rho\exp(j\phi)$ are $\rho^{1/n}\exp\left(j\dfrac{\phi + 2\pi k}{n}\right)$ for $k = 0, 1, \ldots, n-1$.

The cube roots of $27j = 27\exp\left(j\frac{\pi}{2}\right)$ are $z_k = 3\exp\left(j\left(\frac{\pi}{6} + \frac{2\pi}{3}k\right)\right)$ for $k = 0, 1, 2$.

$$z_0 = 3\exp\left(j\frac{\pi}{6}\right) \qquad z_1 = 3\exp\left(j\frac{5\pi}{6}\right) \qquad z_2 = 3\exp\left(j\frac{3\pi}{2}\right)$$

6. (a) Let $S = 1 + z + z^2 + \cdots + z^n$. Then, we obtain the identity as below.

$$S - zS = 1 + z + z^2 + \cdots + z^n - z(1 + z + z^2 + \cdots + z^n) = 1 - z^{n+1}$$

$$S - zS = S(1-z) = 1 - z^{n+1} \implies S = \frac{1 - z^{n+1}}{1-z}$$

$$\therefore \quad 1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1-z} \text{ for any } z \in \mathbb{C}.$$

(b) Let $z = r\exp(j\theta)$, where $r = |z|$ and $\theta = \angle z$, then

$$\lim_{n\to\infty} z^{n+1} = \lim_{n\to\infty} |z|^{n+1}(\exp(j\theta))^{n+1}.$$

Provided that $|z| < 1$, $|z|^{n+1} \to 0$ as $n \to \infty$. Then, we take the limit of z^{n+1} and S as $n \to \infty$ by using the condition $|z| < 1$.

$$\lim_{n\to\infty} z^{n+1} = \lim_{n\to\infty} |z|^{n+1}(\exp(j\theta))^{n+1} = 0$$

$$\lim_{n\to\infty} S = \lim_{n\to\infty} \frac{1 - z^{n+1}}{1-z} = \lim_{n\to\infty} \frac{1}{1-z} - \underbrace{\lim_{n\to\infty} \frac{z^{n+1}}{1-z}}_{0} = \frac{1}{1-z}$$

(c) To derive Lagrange's trigonometric identity,

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left[\left(n + \frac{1}{2}\right)\theta\right]}{2\sin\frac{1}{2}\theta}, \tag{1}$$

let $z = \exp(j\theta)$, then use the identity given in part (a).

$$1 + \exp(j\theta) + (\exp(j\theta))^2 + \cdots + (\exp(j\theta))^n = \frac{1 - (\exp(j\theta))^{n+1}}{1 - \exp(j\theta)} \tag{2}$$

Using de Moivre's formula (for $r = 1$), we can express LHS of (2) in a different way as below.

$$1 + (\cos\theta + j\sin\theta) + (\cos 2\theta + j\sin 2\theta) + \cdots + (\cos n\theta + j\sin n\theta) = \frac{1 - (\exp(j\theta))^{n+1}}{1 - \exp(j\theta)} \tag{3}$$

If we compare (1) and (3), we observe that the LHS of (1) is identical to the real part of LHS of (3). To obtain the real part of RHS of (3), we perform some algebraic manipulations.

$$\frac{1 - (\exp(j\theta))^{n+1}}{1 - \exp(j\theta)} = \frac{1 - (\exp(j\theta))^{n+1}}{\exp\left(j\frac{\theta}{2}\right)\left[\exp\left(-j\frac{\theta}{2}\right) - \exp\left(j\frac{\theta}{2}\right)\right]} = \frac{-\exp\left(-j\frac{\theta}{2}\right) + (\exp(j\theta))^{n+\frac{1}{2}}}{2j\sin\frac{1}{2}\theta}$$

Then, the real part of RHS of (3) is

$$\mathbf{Re}\left\{\frac{1 - (\exp(j\theta))^{n+1}}{1 - \exp(j\theta)}\right\} = \frac{1}{2} + \frac{\sin\left[\left(n + \frac{1}{2}\right)\theta\right]}{2\sin\frac{1}{2}\theta},$$

which is identical to the RHS of (1). Notice that the sine terms in the numerator belong to real part owing to j term in denominator.

Thus, the real part of LHS and RHS of (3) give Lagrange's trigonometric identity (1).