Middle East Technical University
Department of Electrical and Electronics Engineering
EE301: Signals and Systems I

Solutions for Homework 5
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1. (a) The linearity and differentiation in the time domain properties of the Laplace transform can be stated as below.

$$\begin{aligned}
x(t) &\overset{\mathcal{L}}{\longleftrightarrow} X(s), && \text{ROC} : R_X \\
ax(t) &\overset{\mathcal{L}}{\longleftrightarrow} aX(s), && \text{ROC} : R_X && \text{(Linearity)} \\
\frac{d\,x(t)}{dt} &\overset{\mathcal{L}}{\longleftrightarrow} sX(s), && \text{ROC} : \text{At least } R_X && \text{(Differentiation in the time domain)}
\end{aligned}$$

By using these properties, we apply the Laplace transform to both sides of the given differential equation

$$\frac{d^2\,y(t)}{dt^2} + 2\frac{d\,y(t)}{dt} - 3y(t) = \frac{d\,x(t)}{dt} - 2x(t),$$

and obtain

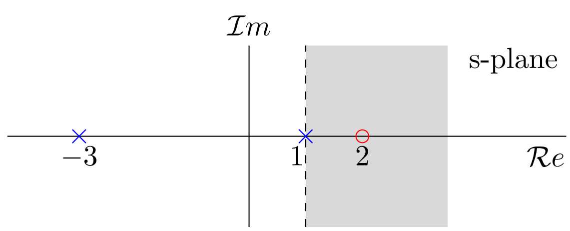
$$s^2Y(s) + 2sY(s) - 3Y(s) = sX(s) - 2X(s).$$

Then, we find the system function $H(s)$.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s-2}{s^2+2s-3} = \frac{s-2}{(s+3)(s-1)}$$

(b) The system function $H(s)$ has one zero at $s = 2$, and two poles, one at $s = -3$, the other at $s = 1$. Since the order of the denominator exceeds the order of the numerator by 1, $H(s)$ has one zero at infinity.

Owing to causality of the system, ROC is the right-half plane to the right of the rightmost pole, 1, namely the ROC of $H(s)$ is $\mathcal{R}e\,\{s\} > 1$.



ROC for $H(s)$: $\mathcal{R}e\{s\} > 1$

(c) We can use the eigenfunction property of complex exponential signals to find an input signal $x(t)$ resulting in an output that is identically zero.

Let $x(t) = e^{s_0 t}$, where $s_0 \in \mathbb{C}$. Then, the output signal is

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)\,h(\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} e^{s_0(t-\tau)}\,h(\tau)\,\mathrm{d}\tau = e^{s_0 t}\underbrace{\int_{-\infty}^{\infty} h(\tau)\,e^{-s_0\tau}\,\mathrm{d}\tau}_{H(s_0)}$$

and it is identically zero when $H(s_0) = 0$. From part (b), we know that $H(s)$ has a zero at $s = 2$. Therefore, $x(t) = e^{2t}$ will result an output that is identically zero, $y(t) = 0$.

(d) Let $x_1(t) = e^{2t}\,u(t)$. Then, we can express the given input signal $x(t)$ as

$$x(t) = e^{2t}\,u(t) - e^{2(t-1)}\,u(t-1) = x_1(t) - x_1(t-1).$$

Since the given system is LTI, we have the following statements.

Time invariance: $\quad x_1(t-1) \to y_1(t-1) \Longleftarrow x_1(t) \to y_1(t)$

Linearity: $\quad x_1(t) - x_1(t-1) \to y_1(t) - y_1(t-1) \Longleftarrow x_1(t) \to y_1(t)$ and $x_1(t-1) \to y_1(t-1)$

$$x_1(t) = e^{2t}\,u(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad X_1(s) = \frac{1}{s-2}, \qquad \text{ROC} : \mathcal{R}e\,\{s\} > 2$$

The Laplace transform of the system response to $x_1(t)$ is

$$\mathcal{L}\,\{y_1(t)\} = Y_1(s) = H(s)\,X_1(s) = \frac{s-2}{(s+3)(s-1)} \times \frac{1}{s-2} = \frac{1}{(s+3)(s-1)}$$

The possible ROCs that can be associated with the pole-zero pattern of $Y_1(s)$ are $\mathcal{R}e\{s\} < -3$, $-3 < \mathcal{R}e\{s\} < 1$ and $\mathcal{R}e\{s\} > 1$. We know that the ROC of $Y_1(s)$ must contain the intersection of the ROCs of $H(s)$ and $X_1(s)$, $\mathcal{R}e\,\{s\} > 2$. Hence, the ROC of $Y_1(s)$ is $\mathcal{R}e\{s\} > 1$.

To find the inverse Laplace transform, $y_1(t)$, by using the Laplace transform pairs, we first obtain the partial-fraction expansion for $Y_1(s)$.

$$Y_1(s) = \frac{1}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1},$$

where A and B are found as below.

$$A = \big[(s+3)\,Y_1(s)\big]\Big|_{s=-3} = \left(\frac{1}{s-1}\right)\Bigg|_{s=-3} = \frac{-1}{4}$$

$$B = \big[(s-1)\,Y_1(s)\big]\Big|_{s=1} = \left(\frac{1}{s+3}\right)\Bigg|_{s=1} = \frac{1}{4}$$

$$e^{-\alpha t}\,u(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad \frac{1}{s+\alpha}, \qquad \text{ROC} : \mathcal{R}e\,\{s\} > -\alpha$$

$$y_1(t) = \frac{1}{4}\,e^{t}\,u(t) - \frac{1}{4}\,e^{-3t}\,u(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad Y_1(s) = \frac{1}{(s+3)(s-1)}, \qquad \text{ROC} : \mathcal{R}e\,\{s\} > 1$$

$$x(t) = x_1(t) - x_1(t-1) \to y(t) = y_1(t) - y_1(t-1)$$

$$y(t) = \frac{1}{4}\big(e^{t} - e^{-3t}\big)u(t) - \frac{1}{4}\big(e^{(t-1)} - e^{-3(t-1)}\big)u(t-1)$$

2.

$$X(s) = \frac{s^2}{(s^2-1)(s^2-4s+5)(s^2+4s+5)}$$

We know that a second-order polynomial equation in a single variable s given by

$$as^2 + bs + c = 0,$$

with $a \neq 0$, has the roots

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

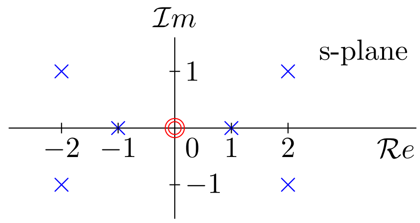
By using this formula, the poles of $X(s)$ are found as follows:

$$s^2 - 1 = 0 \implies s_1 = -1, \quad s_2 = 1$$

$$s^2 - 4s + 5 = 0 \implies s_3 = 2 - j, \quad s_4 = 2 + j$$

$$s^2 + 4s + 5 = 0 \implies s_5 = -2 - j, \quad s_6 = -2 + j$$

The pole-zero plot of $X(s)$ is shown below.



$$X(s) = \frac{A}{(s+1)} + \frac{B}{(s-1)} + \frac{C}{(s-2+j)} + \frac{D}{(s-2-j)} + \frac{E}{(s+2+j)} + \frac{F}{(s+2-j)}$$

$$A = \left[(s+1)\,X(s)\right]\Big|_{s=-1} = \left(\frac{s^2}{(s-1)(s^2-4s+5)(s^2+4s+5)}\right)\Bigg|_{s=-1} = \frac{-1}{40}$$

$$B = \left[(s-1)\,X(s)\right]\Big|_{s=1} = \left(\frac{s^2}{(s+1)(s^2-4s+5)(s^2+4s+5)}\right)\Bigg|_{s=1} = \frac{1}{40}$$

$$C = \left[(s-2+j)\,X(s)\right]\Big|_{s=2-j} = \left(\frac{s^2}{(s^2-1)(s-2-j)(s^2+4s+5)}\right)\Bigg|_{s=2-j} = \frac{1}{160}(-3+j4)$$

We have $D = C^* = (-3-j4)/160$ since s_3 and s_4 are complex conjugate.

$$E = \left[(s+2+j)\,X(s)\right]\Big|_{s=-2-j} = \left(\frac{s^2}{(s^2-1)(s^2-4s+5)(s+2-j)}\right)\Bigg|_{s=-2-j} = \frac{1}{160}(3+j4)$$

Similarly, we have $F = E^* = (3-j4)/160$ since s_5 and s_6 are complex conjugate.

$$X(s) = \frac{-1}{40(s+1)} + \frac{1}{40(s-1)} + \frac{-3+j4}{160(s-2+j)} + \frac{-3-j4}{160(s-2-j)} + \frac{3+j4}{160(s+2+j)} + \frac{3-j4}{160(s+2-j)}$$

Combining the complex conjugate terms, we obtain an alternative expression for $X(s)$.

$$\frac{-3+j4}{160(s-2+j)} + \frac{-3-j4}{160(s-2-j)} = \frac{2\mathcal{R}e\left\{(-3+j4)(s-2-j)\right\}}{160(s^2-4s+5)} = \frac{-3s+10}{80(s^2-4s+5)}$$

$$\frac{3+j4}{160(s+2+j)} + \frac{3-j4}{160(s+2-j)} = \frac{2\mathcal{R}e\left\{(3+j4)(s+2-j)\right\}}{160(s^2+4s+5)} = \frac{3s+10}{80(s^2+4s+5)}$$

$$\begin{aligned} X(s) &= \frac{-1}{40(s+1)} + \frac{1}{40(s-1)} + \frac{-3s+10}{80(s^2-4s+5)} + \frac{3s+10}{80(s^2+4s+5)} \\ &= \frac{1}{80}\left(\frac{-2}{s+1} + \frac{2}{s-1} - \frac{3(s-2)}{(s-2)^2+1} + \frac{4}{(s-2)^2+1} + \frac{3(s+2)}{(s+2)^2+1} + \frac{4}{(s+2)^2+1}\right) \end{aligned}$$

The Laplace transform pairs used to find the time-domain signal having the Laplace transform $X(s)$ are given as follows:

$$e^{-\alpha t}\,u(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad \frac{1}{s+\alpha}, \qquad \text{ROC} : \mathcal{R}e\{s\} > -\alpha$$

$$-e^{-\alpha t}\,u(-t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad \frac{1}{s+\alpha}, \qquad \text{ROC} : \mathcal{R}e\{s\} < -\alpha$$

$$\left[e^{-\alpha t}\cos(\omega_0 t)\right]\,u(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}, \qquad \text{ROC} : \mathcal{R}e\{s\} > -\alpha$$

$$-\left[e^{-\alpha t}\cos(\omega_0 t)\right]u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}, \qquad \text{ROC}: \mathcal{R}e\{s\} < -\alpha$$

$$\left[e^{-\alpha t}\sin(\omega_0 t)\right]u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega_0}{(s+\alpha)^2+\omega_0^2}, \qquad \text{ROC}: \mathcal{R}e\{s\} > -\alpha$$

$$-\left[e^{-\alpha t}\sin(\omega_0 t)\right]u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega_0}{(s+\alpha)^2+\omega_0^2}, \qquad \text{ROC}: \mathcal{R}e\{s\} < -\alpha$$

(a) For $R_X : \mathcal{R}e\{s\} < -2$, the ROC is to the left of all poles and each term of $X(s)$ corresponds to the left-sided signal.

$$x(t) = \frac{1}{80}\left[2e^{-t} - 2e^{t} + e^{2t}\big(3\cos(t) - 4\sin(t)\big) - e^{-2t}\big(3\cos(t) + 4\sin(t)\big)\right]u(-t)$$

Remark: If we use $X(s)$ without combining the terms corresponding to the complex conjugate pairs such that

$$X(s) = \frac{-1}{40(s+1)} + \frac{1}{40(s-1)} + \frac{-3+j4}{160(s-2+j)} + \frac{-3-j4}{160(s-2-j)} + \frac{3+j4}{160(s+2+j)} + \frac{3-j4}{160(s+2-j)},$$

then we find an equivalent expression for $x(t)$ as follows:

$$\begin{aligned}
x(t) &= \frac{1}{160}\Big[4e^{-t} - 4e^{t} + \underbrace{(3-j4)}_{r\,e^{-j\theta}}e^{(2-j)t} + \underbrace{(3+j4)}_{r\,e^{j\theta}}e^{(2+j)t} - \underbrace{(3+j4)}_{r\,e^{j\theta}}e^{-(2+j)t} - \underbrace{(3-j4)}_{r\,e^{-j\theta}}e^{-(2-j)t}\Big]u(-t) \\
&= \frac{1}{160}\left[4e^{-t} - 4e^{t} + e^{2t}\big(r\,e^{-j(t+\theta)} + r\,e^{j(t+\theta)}\big) - e^{-2t}\big(r\,e^{-j(t-\theta)} + r\,e^{j(t-\theta)}\big)\right]u(-t) \\
&= \frac{1}{160}\left[4e^{-t} - 4e^{t} + 2r\,e^{2t}\cos(t+\theta) - 2r\,e^{-2t}\cos(t-\theta)\right]u(-t)
\end{aligned}$$

$$r\,e^{j\theta} = 3 + j4 \implies r = 5, \quad \theta = 0.9273$$

$$x(t) = \frac{1}{80}\left[2e^{-t} - 2e^{t} + 5e^{2t}\cos(t+0.9273) - 5e^{-2t}\cos(t-0.9273)\right]u(-t)$$

(b) For $R_X : -2 < \mathcal{R}e\{s\} < -1$, the ROC is to the right of the poles at $s = -2-j$ and $s = -2+j$ so that each of these terms corresponds to the right-sided signal, and each of the other terms of $X(s)$ corresponds to the left-sided signal.

$$x(t) = \frac{1}{80}\left[2e^{-t} - 2e^{t} + e^{2t}\big(3\cos(t) - 4\sin(t)\big)\right]u(-t) + \frac{1}{80}\left[e^{-2t}\big(3\cos(t) + 4\sin(t)\big)\right]u(t)$$

$$x(t) = \frac{1}{80}\left[2e^{-t} - 2e^{t} + 5e^{2t}\cos(t+0.9273)\right]u(-t) + \frac{1}{80}\left[5e^{-2t}\cos(t-0.9273)\right]u(t)$$

(c) For $R_X : -1 < \mathcal{R}e\{s\} < 1$, the ROC is to the right of the poles at $s = -2-j$, $s = -2+j$ and $s = -1$ so that each of these terms corresponds to the right-sided signal, and each of the other terms of $X(s)$ corresponds to the left-sided signal.

$$x(t) = \frac{1}{80}\left[-2e^{t} + e^{2t}\big(3\cos(t) - 4\sin(t)\big)\right]u(-t) + \frac{1}{80}\left[-2e^{-t} + e^{-2t}\big(3\cos(t) + 4\sin(t)\big)\right]u(t)$$

$$x(t) = \frac{1}{80}\left[-2e^{t} + 5e^{2t}\cos(t+0.9273)\right]u(-t) + \frac{1}{80}\left[-2e^{-t} + 5e^{-2t}\cos(t-0.9273)\right]u(t)$$

(d) For $R_X : 1 < \mathcal{R}e\{s\} < 2$, the ROC is to the left of the poles at $s = 2-j$ and $s = 2+j$ so that each of these terms corresponds to the left-sided signal, and each of the other terms of $X(s)$ corresponds to the right-sided signal.

$$x(t) = \frac{1}{80}\left[e^{2t}\big(3\cos(t) - 4\sin(t)\big)\right]u(-t) + \frac{1}{80}\left[2e^{t} - 2e^{-t} + e^{-2t}\big(3\cos(t) + 4\sin(t)\big)\right]u(t)$$

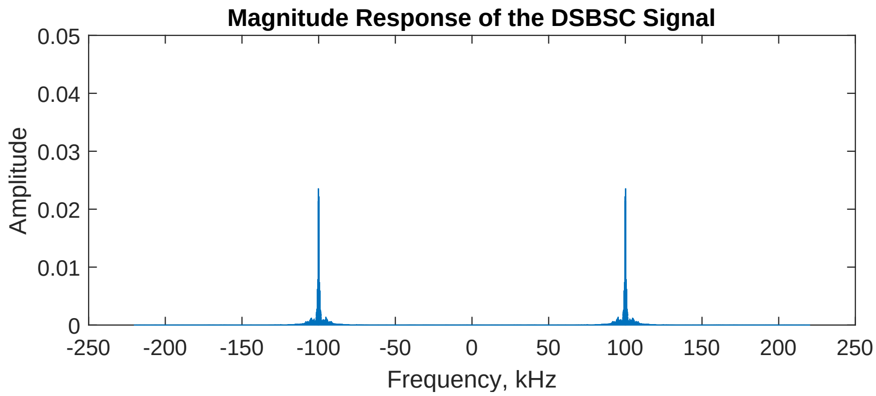
$$x(t) = \frac{1}{80}\left[5e^{2t}\cos(t+0.9273)\right]u(-t) + \frac{1}{80}\left[2e^{t} - 2e^{-t} + 5e^{-2t}\cos(t-0.9273)\right]u(t)$$

	(e)	For $R_X : \mathcal{R}e\{s\} > 2$, the ROC is to the right of all poles and each term of $X(s)$ corresponds to the right-sided signal. $$x(t) = \frac{1}{80}\left[2e^{t} - 2e^{-t} - e^{2t}\big(3\cos(t) - 4\sin(t)\big) + e^{-2t}\big(3\cos(t) + 4\sin(t)\big)\right] u(t)$$ $$x(t) = \frac{1}{80}\left[2e^{t} - 2e^{-t} - 5e^{2t}\cos(t + 0.9273) + 5e^{-2t}\cos(t - 0.9273)\right] u(t)$$

3.	(a)	$$\alpha^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}, \qquad \text{ROC} : \lvert z\rvert > \lvert\alpha\rvert$$ $$\left(\frac{5}{6}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{5}{6}z^{-1}}, \qquad \text{ROC} : \lvert z\rvert > \frac{5}{6}$$ $$\left(\frac{5}{6}\right)^{n-6} u[n-6] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-6}}{1-\frac{5}{6}z^{-1}}, \qquad \text{ROC} : \lvert z\rvert > \frac{5}{6} \qquad \text{(Time shifting)}$$ $$\left(\frac{5}{6}\right)^{n} u[n-6] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{\left(\frac{5}{6}z^{-1}\right)^6}{1-\frac{5}{6}z^{-1}}, \qquad \text{ROC} : \lvert z\rvert > \frac{5}{6} \qquad \text{(Linearity)}$$
	(b)	$$u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}}, \qquad \text{ROC} : \lvert z\rvert > 1$$ $$n\,u[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{d}{dz}\left(\frac{1}{1-z^{-1}}\right) = \frac{z^{-1}}{(1-z^{-1})^2}, \qquad \text{ROC} : \lvert z\rvert > 1 \qquad \text{(Differentiation in the } z\text{-domain)}$$
	(c)	$$\left(\frac{2}{9}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{2}{9}z^{-1}}, \qquad \text{ROC} : \lvert z\rvert > \frac{2}{9}$$ $$n\left(\frac{2}{9}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{d}{dz}\left(\frac{1}{1-\frac{2}{9}z^{-1}}\right) = \frac{\frac{2}{9}z^{-1}}{\left(1-\frac{2}{9}z^{-1}\right)^2}, \qquad \text{ROC} : \lvert z\rvert > \frac{2}{9} \qquad \text{(Differentiation in the } z\text{-domain)}$$
	(d)	By using Euler's formula, $e^{j\theta} = \cos(\theta) + j\sin(\theta)$, we can express the given signal as follows: $$e^n \sin(n)\,u[n] = e^n\left(\frac{e^{jn} - e^{-jn}}{j2}\right) u[n] = \frac{e^{(1+j)n} - e^{(1-j)n}}{j2}\,u[n]$$ $$e^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-e\,z^{-1}}, \qquad \text{ROC} : \lvert z\rvert > e$$ $$e^{jn} e^n u[n] = e^{(1+j)n} u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-e\,(e^{-j}z)^{-1}}, \qquad \text{ROC} : \lvert z\rvert > e \qquad \text{(Scaling in the } z\text{-domain)}$$ $$e^{-jn} e^n u[n] = e^{(1-j)n} u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-e\,(e^{j}z)^{-1}}, \qquad \text{ROC} : \lvert z\rvert > e \qquad \text{(Scaling in the } z\text{-domain)}$$ Then, we obtain the z-transform of the given signal by using the linearity property of the z-transform. $$\begin{aligned} \mathcal{Z}\{e^n \sin(n)\,u[n]\} &= \frac{1}{j2}\left(\frac{1}{1-e\,(e^{-j}z)^{-1}} - \frac{1}{1-e\,(e^{j}z)^{-1}}\right) \\ &= \frac{\left(1-e\,(e^{j}z)^{-1}\right) - \left(1-e\,(e^{-j}z)^{-1}\right)}{j2\left(1-e\,(e^{-j}z)^{-1}\right)\left(1-e\,(e^{j}z)^{-1}\right)} \\ &= \frac{e\left(e^{j} - e^{-j}\right) z^{-1}}{j2\left(1 - e\left(e^{j}+e^{-j}\right) z^{-1} + e^2 z^{-2}\right)} \\ &= \frac{e\,\sin(1)\,z^{-1}}{1-2e\,\cos(1)\,z^{-1} + e^2 z^{-2}}, \qquad \text{ROC} : \lvert z\rvert > e \end{aligned}$$

4.	(a)	The given signals $x[n]$ and $y[n]$ are right-sided. We specify their ROCs according to the property that the ROC is the region in the z-plane outside the outermost pole of the z-transform of the right-sided signal. $$x[n] = \left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{4}\right)^{n-1} u[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) = \frac{1}{1-\frac{1}{3}z^{-1}} - \frac{4}{1-\frac{1}{4}z^{-1}}, \quad \text{ROC} : \lvert z \rvert > \frac{1}{3}$$ $$y[n] = \left(\frac{1}{4}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z) = \frac{1}{1-\frac{1}{4}z^{-1}}, \quad \text{ROC} : \lvert z \rvert > \frac{1}{4}$$ By using the convolution property of the z-transform, $$y[n] = h[n] * x[n] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z) = H(z)\,X(z), \qquad \text{ROC : At least } R_H \cap R_X$$ we can find the z-transfer function of the system, $H(z)$. $$H(z) = \frac{Y(z)}{X(z)} = \frac{\dfrac{1}{1-\frac{1}{4}z^{-1}}}{\dfrac{-3\left(1-\frac{13}{36}z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}} = \frac{\frac{1}{3}z^{-1}-1}{3\left(1-\frac{13}{36}z^{-1}\right)}, \quad \text{ROC} : \lvert z \rvert > \frac{13}{36}$$ To design a causal system, we choose the ROC of $H(z)$ as the exterior of a circle, including infinity, i.e., $z = \infty$.
	(b)	$$\left(\frac{13}{36}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{13}{36}z^{-1}}, \qquad \text{ROC} : \lvert z \rvert > \frac{13}{36}$$ $$\left(\frac{13}{36}\right)^{n-1} u[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{1-\frac{13}{36}z^{-1}}, \qquad \text{ROC} : \lvert z \rvert > \frac{13}{36} \quad \text{(Time shifting)}$$ By using these z-transform pairs and the linearity property of the z-transform, we find the impulse response of the system. $$h[n] = \frac{1}{9}\left(\frac{13}{36}\right)^{n-1} u[n-1] - \frac{1}{3}\left(\frac{13}{36}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} H(z) = \frac{\frac{1}{3}z^{-1}-1}{3\left(1-\frac{13}{36}z^{-1}\right)}, \quad \text{ROC} : \lvert z \rvert > \frac{13}{36}$$
	(c)	$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{1}{3}z^{-1}-1}{3\left(1-\frac{13}{36}z^{-1}\right)} \implies 3\left(1-\frac{13}{36}z^{-1}\right)Y(z) = X(z)\left(\frac{1}{3}z^{-1}-1\right)$$ $$3Y(z) - \frac{13}{12}z^{-1}Y(z) = \frac{1}{3}z^{-1}X(z) - X(z)$$ By using the linearity and time-shifting properties of the z-transform, we take the inverse z-transform of both sides of the this equation and obtain the difference equation together with the condition of initial rest for the system. $$3y[n] - \frac{13}{12}y[n-1] = \frac{1}{3}x[n-1] - x[n]$$

5.	1.	The DSBSC signal is given as $$s(t) = x_c(t)\,\cos(2\pi f_c t),$$ where $f_c = 100$ kHz. The plot of the magnitude response of $s[n] = s(n/F_s)$, where $F_s = 441$ kHz, is given below.



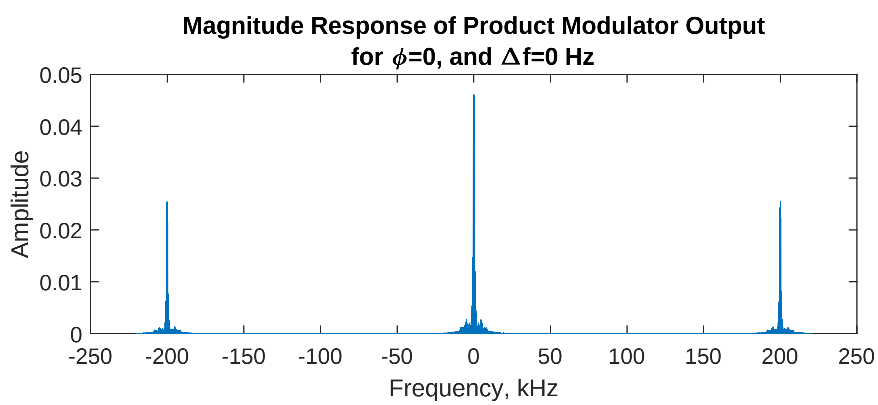
To plot the magnitude response of $s[n] = s(n/F_s)$, the MATLAB code is given below.

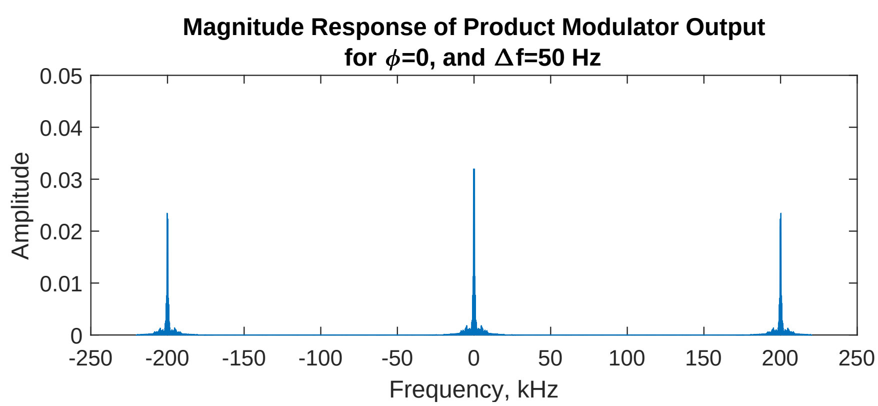
```
clc;close all;
fc = 100e3;
s = x.*cos(2*pi*fc*t');
w_axis = (-Fs/2:Fs/(upsmp_rate*voice_len-1):Fs/2)/1e3; % in kHz

% Observe magnitude response of the DSBSC signal
figure, plot(w_axis, fftshift(abs(fft(s)))/Fs);
title('Magnitude Response of the DSBSC Signal');
xlabel('Frequency, kHz'), ylabel('Amplitude');
set(gca,'ylim',[0, 0.05]);
```

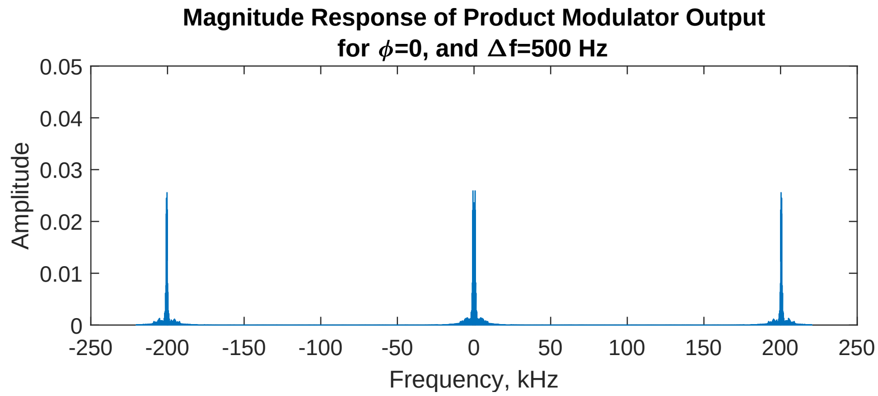
2. (a) For $\phi = 0$ and $\Delta f = 0, 50, 500, 5000$ Hz in

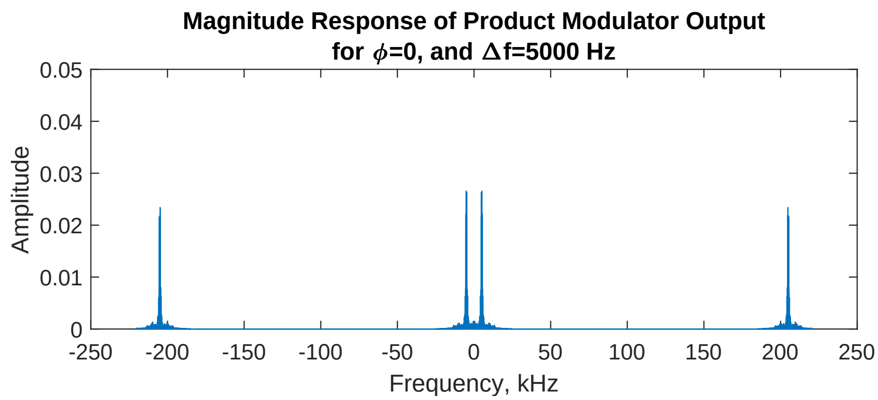
$$v(t) = 2\,s(t)\,\cos\left[2\pi(f_c + \Delta f)t + \phi\right],$$

plots of the magnitude response of $v[n] = v(n/F_s)$ are given below.









To obtain these plots, the MATLAB code is given below.

```
fDELTA = [0,50,500,5000];
phi = 0;
for i=1:numel(fDELTA)
    fdelta = fDELTA(i);
    v = 2*s.*cos(2*pi*(fc+fdelta)*t'+phi);
    w_axis = (-Fs/2:Fs/(upsmp_rate*voice_len-1):Fs/2)/1e3;% kHz

    % Observe magnitude response of product modulator output
    figure, plot(w_axis, fftshift(abs(fft(v)))/Fs);
    title(['Magnitude Response of Product Modulator Output' ...
      newline 'for\phi=' num2str(phi) ', and \Deltaf=' ...
      num2str(fdelta) ' Hz']);
    xlabel('Frequency, kHz'), ylabel('Amplitude');
    set(gca,'ylim',[0, 0.05]);
end
```

(b) For the DSBSC signal

$$s(t) = x_c(t)\,\cos(2\pi f_c t),$$

the output of product modulator can be expressed as follows:

$$\begin{aligned}
v(t) &= 2\,s(t)\,\cos\left[2\pi(f_c+\Delta f)t+\phi\right]\\
&= 2\,x_c(t)\,\cos(2\pi f_c t)\,\cos\left[2\pi(f_c+\Delta f)t+\phi\right]\\
&= 2\,x_c(t)\cdot\frac{e^{j2\pi f_c t}+e^{-j2\pi f_c t}}{2}\cdot\frac{e^{j(2\pi(f_c+\Delta f)t+\phi)}+e^{-j(2\pi(f_c+\Delta f)t+\phi)}}{2}\\
&= x_c(t)\cdot\frac{e^{j(2\pi(2f_c+\Delta f)t+\phi)}+e^{-j(2\pi(2f_c+\Delta f)t+\phi)}+e^{j(2\pi\Delta f t+\phi)}+e^{-j(2\pi\Delta f t+\phi)}}{2}\\
&= x_c(t)\left(\cos\left[2\pi(2f_c+\Delta f)t+\phi\right]+\cos\left(2\pi\Delta f t+\phi\right)\right)
\end{aligned}$$

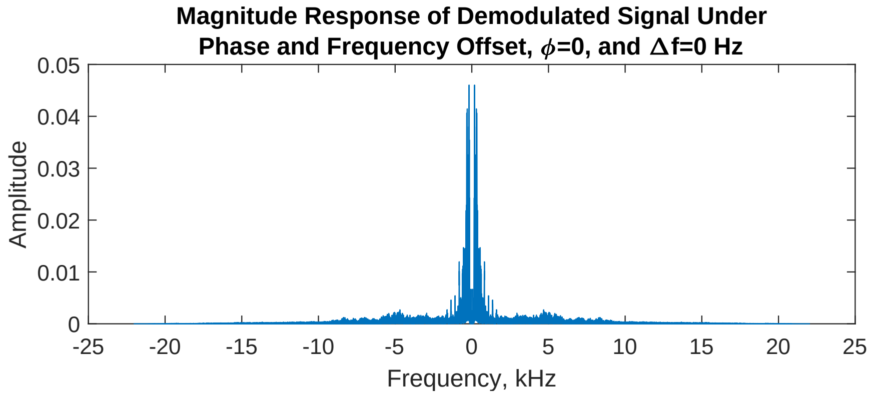
Assuming that $v(t)$ passes through an ideal low pass filter, whose cutoff frequency is below f_c, the demodulator output can be written as

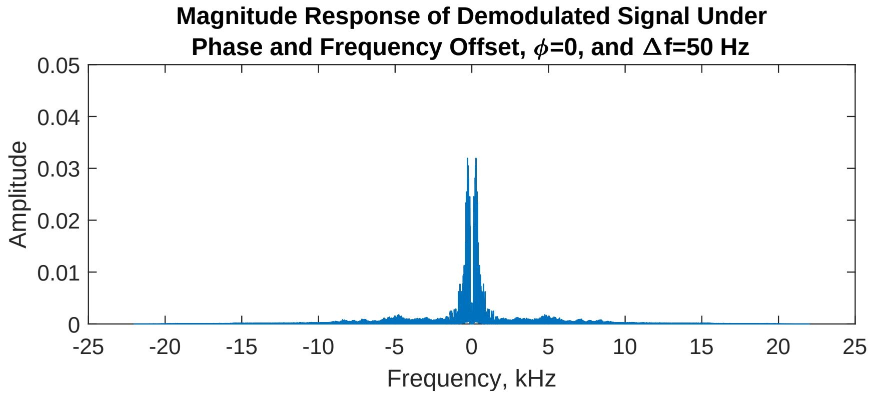
$$v_o(t) = x_c(t)\,\cos\left(2\pi \Delta f t + \phi\right)$$

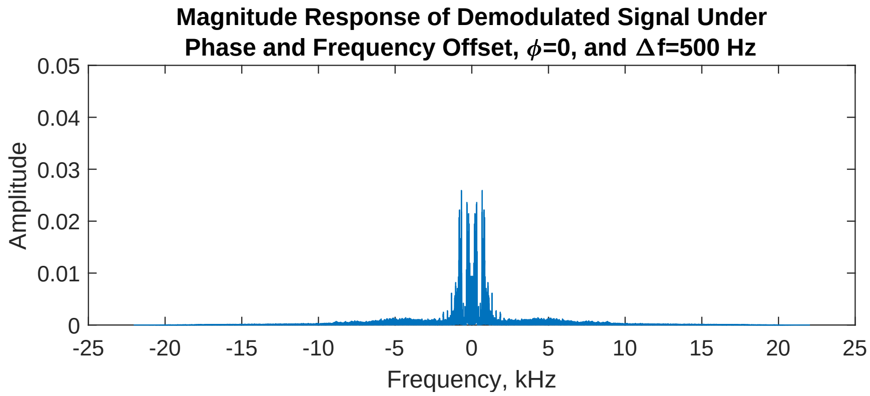
If there is no offset in phase and the frequency of the local oscillator, i.e. $\phi = 0$ and $\Delta f = 0$, the demodulator output is

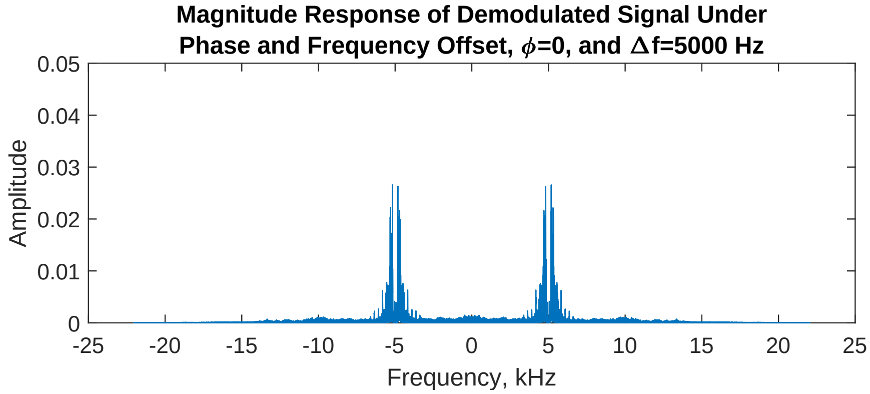
$$v_o(t) = x_c(t)\,\cos\left(0\right) = x_c(t).$$

(c) For different values of Δf given in part (a), plots of the magnitude response of the demodulated signal are given below.









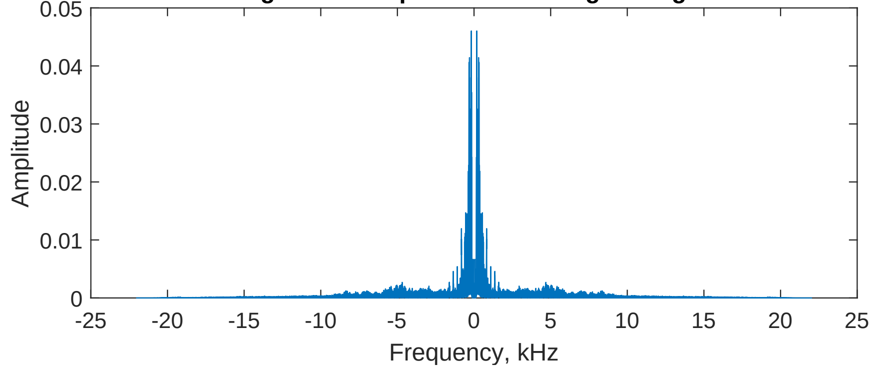
To obtain these plots, the MATLAB code is given below.

```
Fs = 441e3;
fir_order = 128; % The low-pass filter for demodulation will be
                 % an FIR filter of order 128
downsmp_rate = 10;
Fs = Fs / downsmp_rate;

fDELTA = [0,50,500,5000];
phi = 0;
for i=1:numel(fDELTA)
    fdelta = fDELTA(i);
        v = 2*s.*cos(2*pi*(fc+fdelta)*t'+phi);
        demodulated_signal = decimate(v, downsmp_rate, ...
            fir_order, 'fir');
        w_axis = (-Fs/2:Fs/(voice_len-1):Fs/2)/1e3;

        % Observe magnitude response of the demodulated signal
        % under phase and frequency offset
        figure, plot(w_axis, ...
            fftshift(abs(fft(demodulated_signal)))/Fs);

        title(['Magnitude Response of Demodulated Signal ' ...
            'Under' newline 'Phase and Frequency Offset, ' ...
            '\phi=' num2str(phi) ', and \Deltaf=' ...
            num2str(fdelta) ' Hz']);
        xlabel('Frequency, kHz'), ylabel('Amplitude');
        set(gca,'ylim',[0, 0.05]);
end
```

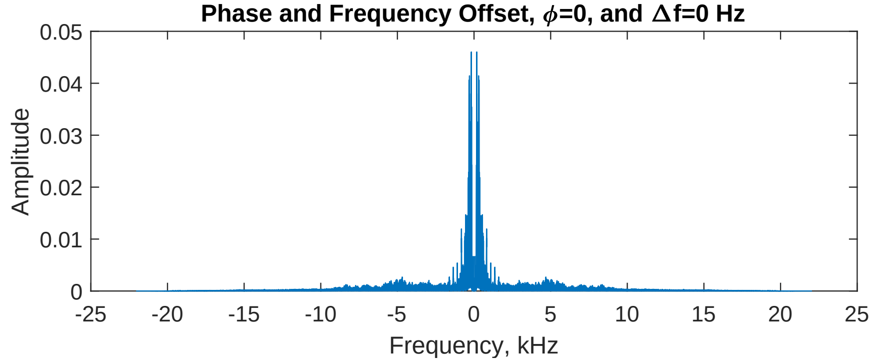
When $\phi \neq 0$ and $\Delta f \neq 0$, the demodulated signal has perceptional distortion.

As verified in part (b), the demodulated signal has not any perceptional distortion, if there is no offset in phase and the frequency of the local oscillator, i.e., $\phi = 0$ and $\Delta f = 0$. The magnitude response of the original signal is given below with the magnitude response of the demodulated signal for $\phi = 0$ and $\Delta f = 0$.

Magnitude Response of the Original Signal

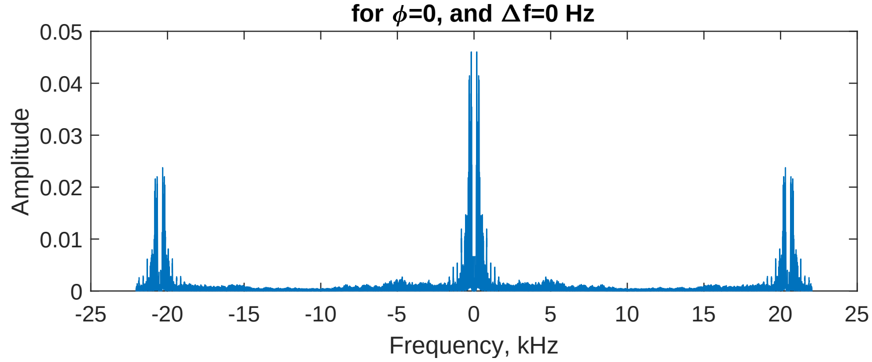


Magnitude Response of Demodulated Signal Under Phase and Frequency Offset, ϕ=0, and Δf=0 Hz

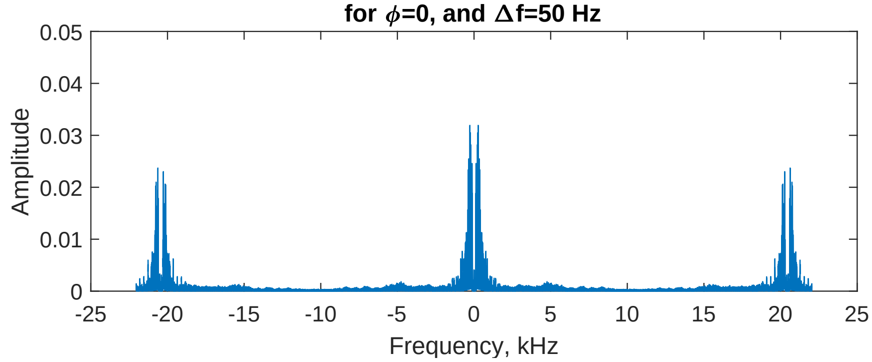


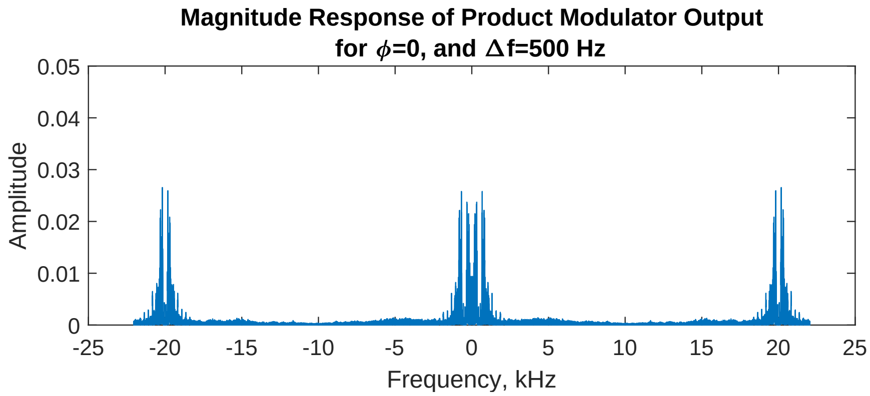
(d) For $F_s = 44.1$ kHz, $\phi = 0$ and $\Delta f = 0, 50, 500, 5000$ Hz, plots of the magnitude response of $v[n] = v(n/F_s)$ are given below.

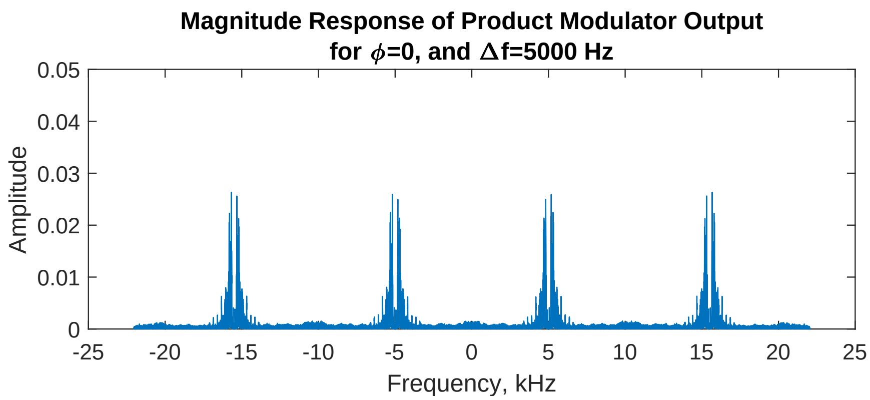
Magnitude Response of Product Modulator Output for ϕ=0, and Δf=0 Hz



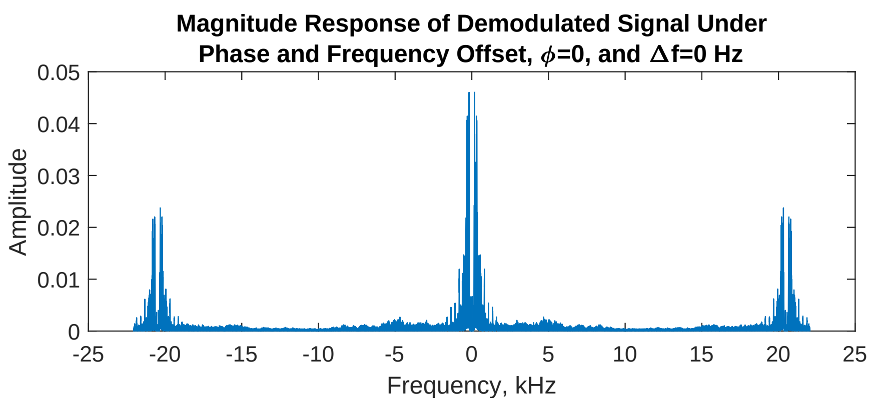
Magnitude Response of Product Modulator Output for ϕ=0, and Δf=50 Hz

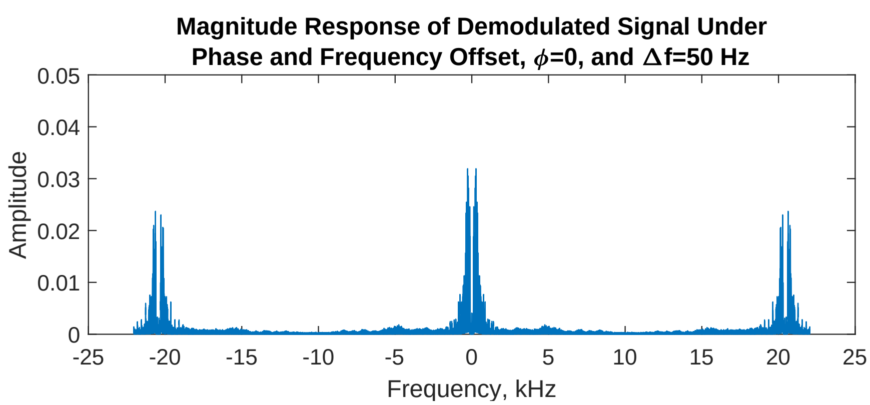


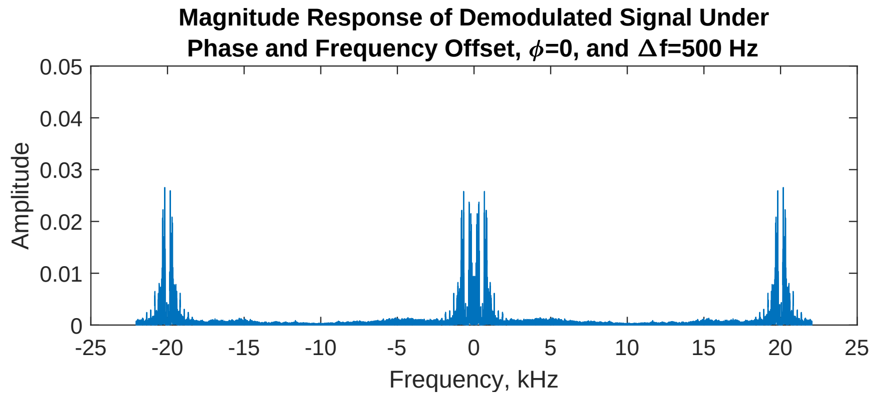


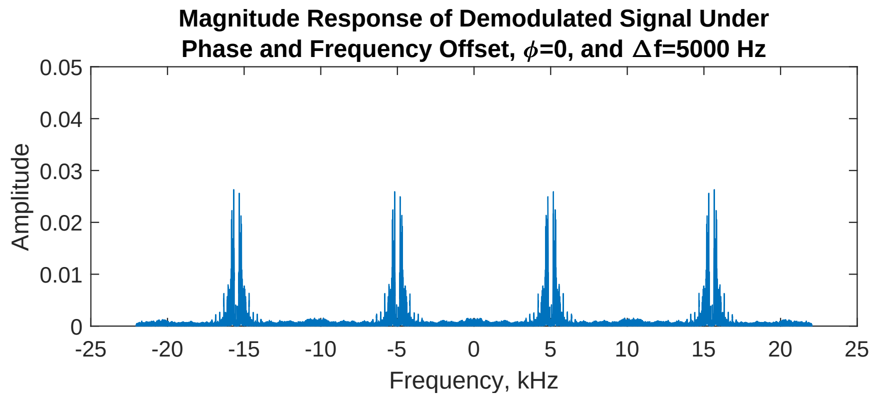


For $F_s = 44.1$ kHz and different values of Δf given in part (a), plots of the magnitude response of the demodulated signal are given below.









Aliasing occurs for all cases. When $\phi \neq 0$ and $\Delta f \neq 0$, the demodulated signal has perceptional distortion. When there is no offset in phase and the frequency of the local oscillator, i.e., $\phi = 0$ and $\Delta f = 0$, we could not observe any perceptional distortion for the demodulated signal, despite the aliasing. The original signal lies within the frequency range $f \in [-10, 10]$ kHz, while the aliasing components are dominant for $|f| > 10$ kHz, as shown below. Hence, the effect of aliasing could not be observable perceptually for this case.