Middle East Technical University
Department of Electrical and Electronics Engineering
EE301: Signals and Systems I

Solutions for Homework 3
November 24, 2019
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1 (a)

i. $x(t) = \delta(t-3) - 2\delta(t+2)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, \mathrm{d}t = \int_{-\infty}^{\infty} \left[\delta(t-3) - 2\delta(t+2)\right]\, e^{-j\omega t}\, \mathrm{d}t = e^{-j3\omega} - 2e^{j2\omega}$$

ii. $x(t) = e^{-3t}\, u(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-3t}\, u(t)\, e^{-j\omega t}\, \mathrm{d}t = \int_{0}^{\infty} e^{-(3+j\omega)t}\, \mathrm{d}t = \left(\frac{-e^{-(3+j\omega)t}}{3+j\omega}\right)\Bigg|_{t=0}^{\infty} = \frac{1}{3+j\omega}$$

iii. $x(t) = e^{-3(t-5)}\, u(t-5)$

$$e^{-3t}\, u(t) \quad \xleftrightarrow{\mathcal{F}} \quad \frac{1}{3+j\omega} \qquad \text{(Result from part-ii)}$$

$$x(t) = e^{-3(t-5)}\, u(t-5) \quad \xleftrightarrow{\mathcal{F}} \quad X(j\omega) = \frac{e^{-j5\omega}}{3+j\omega} \qquad \text{(Time shifting)}$$

iv. $x(t) = e^{-2|t|}\, \cos(t)$
Let $x_1(t) = e^{-2|t|}$. The Fourier transform $X_1(j\omega)$ of $x_1(t)$ is found as below.

$$X_1(j\omega) = \int_{-\infty}^{\infty} e^{-2|t|}\, e^{-j\omega t}\, \mathrm{d}t = \int_{-\infty}^{0} e^{2t}\, e^{-j\omega t}\, \mathrm{d}t + \int_{0}^{\infty} e^{-2t}\, e^{-j\omega t}\, \mathrm{d}t = \frac{1}{2-j\omega} + \frac{1}{2+j\omega} = \frac{4}{4+\omega^2}$$

$$\cos(t) = \frac{e^{jt} + e^{-jt}}{2} \quad \xleftrightarrow{\mathcal{F}} \quad \pi\left[\delta(\omega-1) + \delta(\omega+1)\right]$$

$$x(t) = x_1(t)\, \cos(t) \quad \xleftrightarrow{\mathcal{F}} \quad X(j\omega) = \frac{1}{2\pi}\left[X_1(j\omega) * \Big(\pi\left[\delta(\omega-1) + \delta(\omega+1)\right]\Big)\right]$$

$$X(j\omega) = \frac{1}{2}\left[X_1(j(\omega-1)) + X_1(j(\omega+1))\right] = \frac{2}{4+(\omega-1)^2} + \frac{2}{4+(\omega+1)^2}$$

(b)

i. $x(t) = \mathcal{F}^{-1}\left\{X(j\omega)\right\} = \mathcal{F}^{-1}\left\{e^{j5\omega} + e^{-j7\omega}\right\} = \delta(t+5) + \delta(t-7)$

ii. $x(t) = \mathcal{F}^{-1}\left\{X(j\omega)\right\} = \mathcal{F}^{-1}\left\{2\pi\delta(\omega-3) + 2\pi\delta(\omega+3)\right\} = e^{j3t} + e^{-j3t} = 2\cos(3t)$

iii. $x(t) = \mathcal{F}^{-1}\left\{X(j\omega)\right\} = \mathcal{F}^{-1}\left\{\cos\left(2\omega + \frac{\pi}{4}\right)\right\} = \mathcal{F}^{-1}\left\{\frac{1}{2}\left(e^{j\frac{\pi}{4}} e^{j2\omega} + e^{-j\frac{\pi}{4}} e^{-j2\omega}\right)\right\}$

$x(t) = \frac{1}{2} e^{j\frac{\pi}{4}}\delta(t+2) + \frac{1}{2} e^{-j\frac{\pi}{4}}\delta(t-2)$

2. (a) The signal $x(t)$ is neither square integrable nor absolutely integrable since it satisfies the none of conditions

$$\int_{-\infty}^{\infty} |x(t)|^2 \,\mathrm{d}t < \infty \text{ and } \int_{-\infty}^{\infty} |x(t)| \,\mathrm{d}t < \infty$$

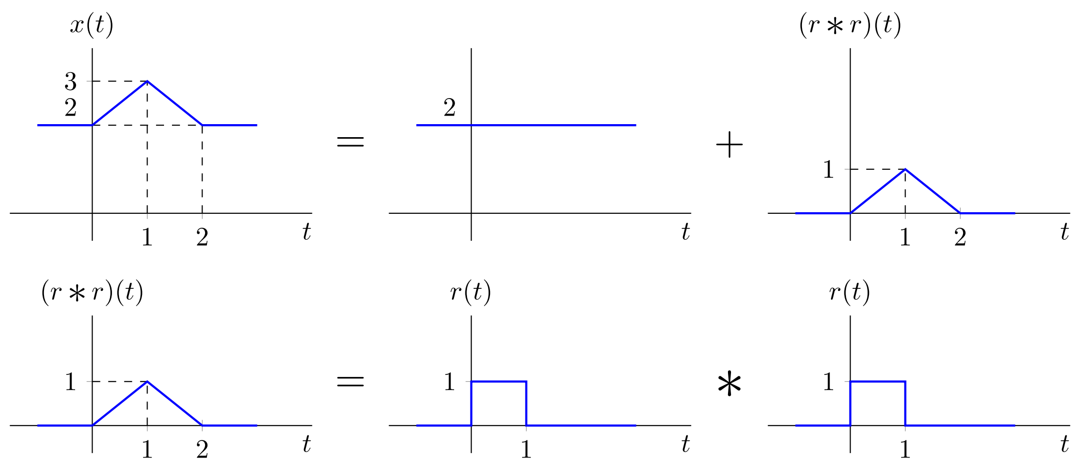
necessary for being square integrable and absolutely integrable, respectively.

The signal does not satisfy the conditions that guarantee the existence and convergence of its Fourier transform $X(j\omega)$.

(b) i. We can express $x(t)$ in the form $x(t) = c + (r * r)(t)$, where $c = 2$ is a constant and $r(t)$ is a rectangular signal given by

$$r(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

as shown below.



ii.

$$p(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases} \quad \xleftrightarrow{\mathcal{F}} \quad P(j\omega) = \frac{2\sin(\omega/2)}{\omega}$$

$$r(t) = p\left(t - \tfrac{1}{2}\right) \quad \xleftrightarrow{\mathcal{F}} \quad R(j\omega) = e^{-j\omega/2} P(j\omega) = \frac{2e^{-j\omega/2}\sin(\omega/2)}{\omega} \quad \text{(Time shifting)}$$

iii.

$$2 \quad \xleftrightarrow{\mathcal{F}} \quad 4\pi\,\delta(\omega)$$

$$(r*r)(t) \quad \xleftrightarrow{\mathcal{F}} \quad R(j\omega)R(j\omega) = \frac{4e^{-j\omega}(\sin(\omega/2))^2}{\omega^2}$$

$$x(t) = 2 + (r*r)(t) \quad \xleftrightarrow{\mathcal{F}} \quad X(j\omega) = 4\pi\,\delta(\omega) + \frac{4e^{-j\omega}(\sin(\omega/2))^2}{\omega^2} \quad \text{(Linearity)}$$

(c)

$$y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t) \quad \xleftrightarrow{\mathcal{F}} \quad Y(j\omega) = j\omega X(j\omega) \quad \text{(Differentiation in time)}$$

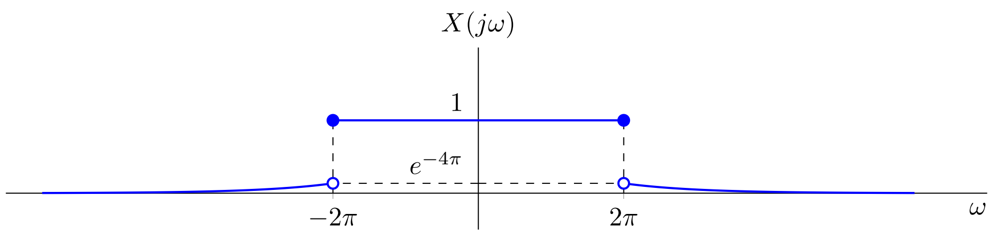
$$Y(j\omega) = \underbrace{j4\pi\omega\,\delta(\omega)}_{j4\pi\,0\,\delta(\omega)=0} + \frac{j4e^{-j\omega}(\sin(\omega/2))^2}{\omega} = \frac{j4e^{-j\omega}(\sin(\omega/2))^2}{\omega}$$

3. (a)

$$X(j0) = c \text{ and } X(j0) = \int_{-\infty}^{\infty} x(t)\,\mathrm{d}t = 1 \implies c = 1$$

As $x(t)$ is real and even, $X(j\omega)$ is real and even. Using the relation $X(j\omega) = X(-j\omega)$ and the given information, $X(j\omega) = c = 1$ for $\omega \in [0, 2\pi]$ and $X(j\omega) = ce^{2\omega} = e^{2\omega}$ for $\omega < -2\pi$, we find $X(j\omega)$ as below.

$$X(j\omega) = \begin{cases} e^{2\omega}, & \omega < -2\pi \\ 1, & |\omega| \leqslant 2\pi \\ e^{-2\omega}, & \omega > 2\pi \end{cases}$$



(b) We can express $X(j\omega)$ found in part (a) as below.

$$X(j\omega) = \underbrace{e^{-2\omega}\, u(\omega - 2\pi)}_{X_1(j\omega)} + \underbrace{u(\omega + 2\pi) - u(\omega - 2\pi)}_{X_2(j\omega)} + \underbrace{e^{2\omega}\, u(-\omega - 2\pi)}_{X_3(j\omega)}$$

$$x_1(t) = \mathcal{F}^{-1}\{X_1(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-2\omega}\, u(\omega - 2\pi)\, e^{j\omega t}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{2\pi}^{\infty} e^{-\omega(2-jt)}\,\mathrm{d}\omega = \frac{e^{-2\pi(2-jt)}}{2\pi(2-jt)}$$

$$x_2(t) = \mathcal{F}^{-1}\{X_2(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} [u(\omega + 2\pi) - u(\omega - 2\pi)]\, e^{j\omega t}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} e^{j\omega t}\,\mathrm{d}\omega = \frac{\sin(2\pi t)}{\pi t}$$

$$x_3(t) = \mathcal{F}^{-1}\{X_3(j\omega)\} = \mathcal{F}^{-1}\{X_1(-j\omega)\} = x_1(-t) = \frac{e^{-2\pi(2+jt)}}{2\pi(2+jt)} = x_1^*(t)$$

$$x(t) = \mathcal{F}^{-1}\{X(j\omega)\} = \mathcal{F}^{-1}\{X_1(j\omega) + X_2(j\omega) + X_3(j\omega)\} = x_1(t) + x_2(t) + x_1^*(t)$$

$$x_1(t) + x_1^*(t) = 2\,\mathrm{Re}\{x_1(t)\} = 2\,\mathrm{Re}\left\{\frac{e^{-2\pi(2-jt)}}{2\pi(2-jt)}\right\} = \frac{e^{-4\pi}}{\pi}\,\mathrm{Re}\left\{\frac{e^{j2\pi t}}{2-jt}\right\} = \frac{e^{-4\pi}}{\pi}\,\mathrm{Re}\left\{\frac{e^{j2\pi t}(2+jt)}{4+t^2}\right\}$$

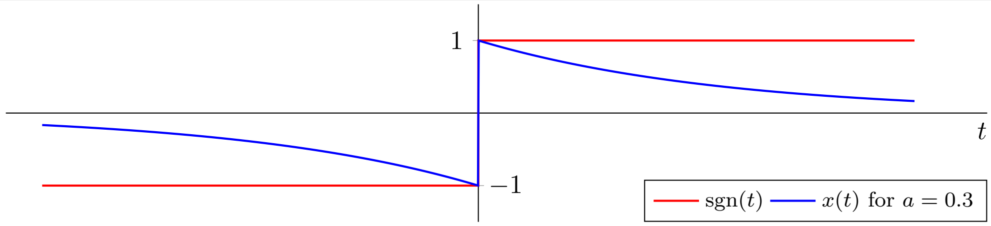
The denominator of the term in $\mathrm{Re}\{\cdot\}$ is real, and we need only to find the real part of the numerator.

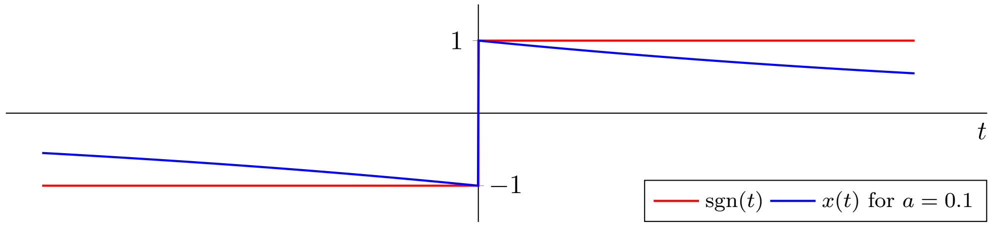
$$\mathrm{Re}\left\{e^{j2\pi t}(2+jt)\right\} = \mathrm{Re}\left\{e^{j2\pi t}\right\}\mathrm{Re}\{2+jt\} - \mathrm{Im}\left\{e^{j2\pi t}\right\}\mathrm{Im}\{2+jt\} = 2\cos(2\pi t) - t\sin(2\pi t)$$

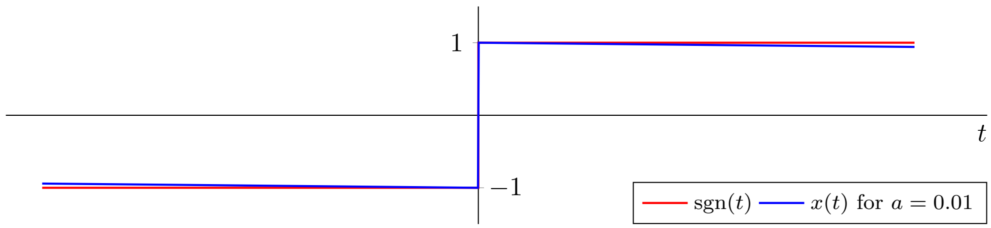
After adding the signals found above, we obtain the signal $x(t)$.

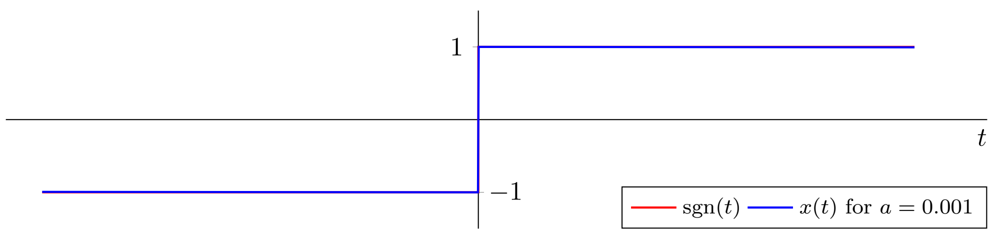
$$x(t) = 2\,\mathrm{Re}\{x_1(t)\} + x_2(t) = \frac{e^{-4\pi}\,(2\cos(2\pi t) - t\sin(2\pi t))}{\pi(4+t^2)} + \frac{\sin(2\pi t)}{\pi t}$$

4. (a)









From these plots, we observe that $x(t) = e^{-at}\,u(t) - e^{at}\,u(-t)$, where $a > 0$ is real, approaches the signum function given by

$$\mathrm{sgn}(t) = \begin{cases} -1, & t < 0 \\ 0, & t = 0 \\ 1, & t > 0 \end{cases}$$

as $a \to 0$.

(b)

$$e^{-at}\,u(t), \quad a > 0 \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{1}{a + j\omega}, \quad a > 0$$

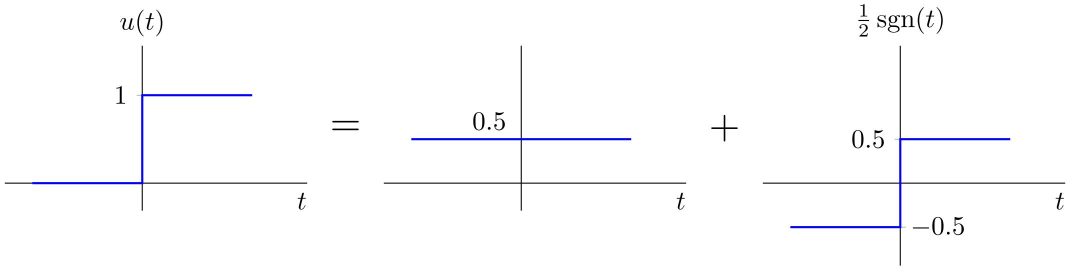
$$e^{at}\,u(-t), \quad a > 0 \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{1}{a - j\omega}, \quad a > 0$$

$$x(t) = e^{-at}\,u(t) - e^{at}\,u(-t), \quad a > 0 \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X(j\omega) = \frac{1}{a + j\omega} - \frac{1}{a - j\omega} = \frac{-j2\omega}{a^2 + \omega^2}, \quad a > 0$$

$$\lim_{a \to 0} X(j\omega) = \lim_{a \to 0} \frac{-j2\omega}{a^2 + \omega^2} = \frac{-j2\omega}{\omega^2} = \frac{2}{j\omega} = \mathcal{F}\left\{\mathrm{sgn}(t)\right\}$$

(c)

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



$$u(t) = \alpha + \beta\,\mathrm{sgn}(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t) \implies \alpha = \frac{1}{2} \text{ and } \beta = \frac{1}{2}$$

$$\frac{1}{2} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \pi\,\delta(\omega)$$

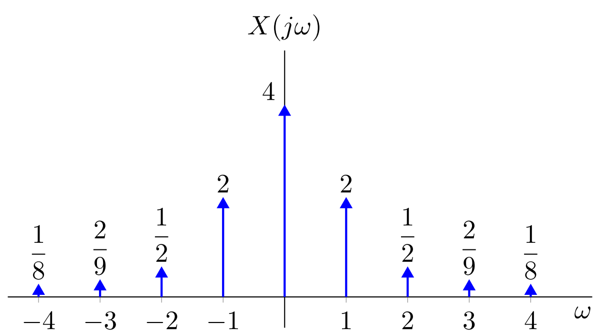
$$\frac{1}{2}\,\mathrm{sgn}(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{1}{j\omega}$$

$$u(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad U(j\omega) = \pi\,\delta(\omega) + \frac{1}{j\omega} \quad \text{(Linearity)}$$

5. (a)

$$\sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad 2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta\left(\omega - k\omega_0\right)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jkt} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta\left(\omega - k\right) = 4\delta(\omega) + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{2}{k^2}\, \delta\left(\omega - k\right)$$



(b)

$$\cos\left(\frac{5}{2}t\right) = \frac{e^{j\frac{5}{2}t} + e^{-j\frac{5}{2}t}}{2} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \pi\left[\delta\left(\omega - \frac{5}{2}\right) + \delta\left(\omega + \frac{5}{2}\right)\right]$$

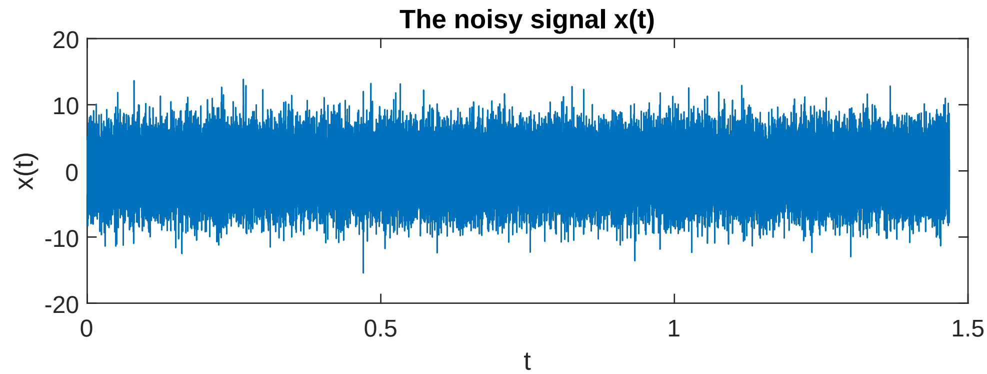
$$g(t) = \frac{\sin(t)}{\pi t} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad G(j\omega) = \begin{cases} 1, & |\omega| < 1 \\ 0, & |\omega| > 1 \end{cases}$$

$$h(t) = g(t)\,\cos\left(\frac{5}{2}t\right) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad H(j\omega) = \frac{1}{2\pi}\left[G(j\omega) * \left\{\pi\left[\delta\left(\omega - \frac{5}{2}\right) + \delta\left(\omega + \frac{5}{2}\right)\right]\right\}\right]$$

$$H(j\omega) = \frac{1}{2}\left[G\left(j\left(\omega - \frac{5}{2}\right)\right) + G\left(j\left(\omega + \frac{5}{2}\right)\right)\right] = \begin{cases} \frac{1}{2}, & \frac{3}{2} < |\omega| < \frac{7}{2} \\ 0, & |\omega| < \frac{3}{2} \text{ and } |\omega| > \frac{7}{2} \end{cases}$$

	(c)	$$Y(j\omega) = X(j\omega)\,H(j\omega) = \frac{1}{4}\left[\delta(\omega-2)+\delta(\omega+2)\right] + \frac{1}{9}\left[\delta(\omega-3)+\delta(\omega+3)\right]$$
	(d)	$$y(t) = \mathcal{F}^{-1}\left\{Y(j\omega)\right\} = \frac{1}{4\pi}\cos(2t) + \frac{1}{9\pi}\cos(3t)$$

6. (a)



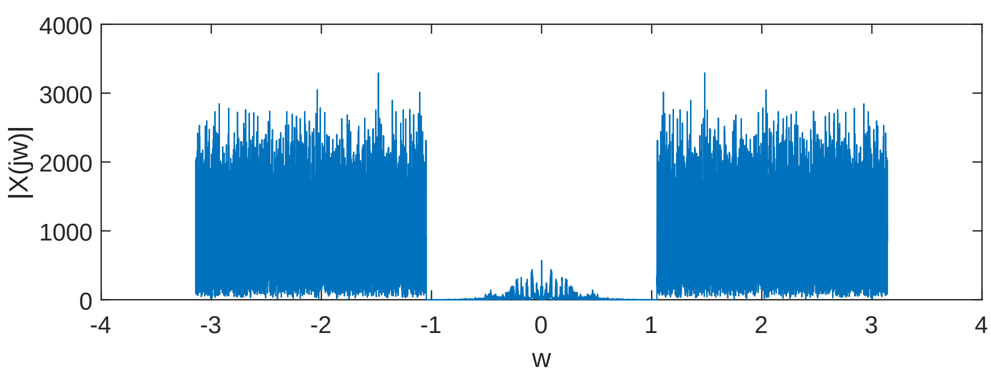
We hear only noise. No, it is not possible to recognize the sentence in the signal.

To plot the noisy signal in MATLAB, the sample code is given below.

```
load x.mat
T = length(x);
t = (1:T)/44100; % convert sample index to CT
plot(t,x);
title('The noisy signal x(t)');
xlabel('t');
ylabel('x(t)');
```

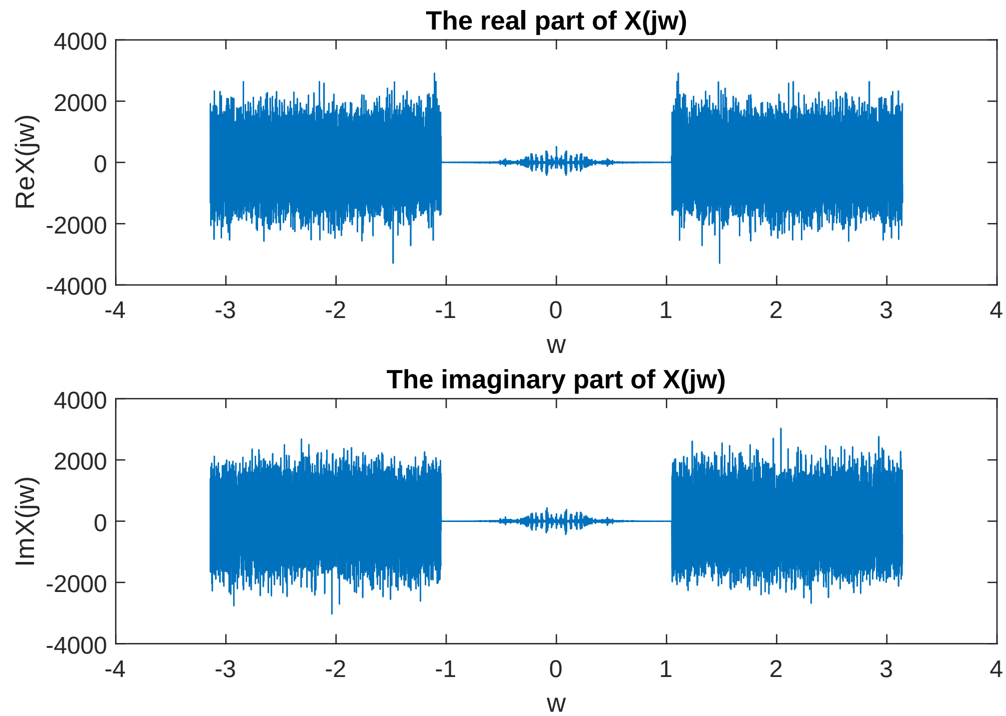
(b)

i.



The original speech signal lies within the frequency range $\omega \in [-1, 1]$, i.e., $B = 1$.

ii.

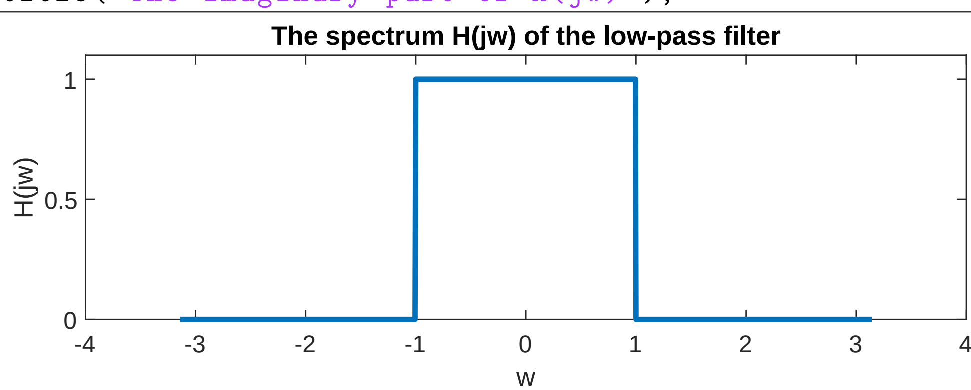


As $x(t)$ is a real signal, we expect $\text{Re}\{X(j\omega)\}$ and $\text{Im}\{X(j\omega)\}$ to have even and odd symmetry, respectively. If we compare $\text{Re}\{X(j\omega)\}$, especially for ω close to 1 and -1, we observe that $\text{Re}\{X(j\omega)\} = \text{Re}\{X(-j\omega)\}$ and $\text{Re}\{X(j\omega)\}$ has even symmetry. If we compare $\text{Im}\{X(j\omega)\}$, especially for ω close to 2 and -2, we observe that $\text{Im}\{X(j\omega)\} = -\text{Im}\{X(-j\omega)\}$ and $\text{Im}\{X(j\omega)\}$ has odd symmetry.

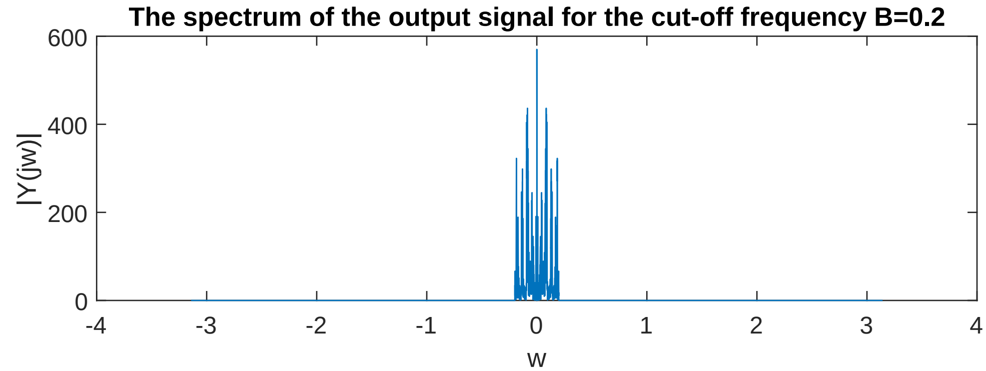
To plot the real and imaginary parts of $X(j\omega)$ in MATLAB, the sample code is given below.

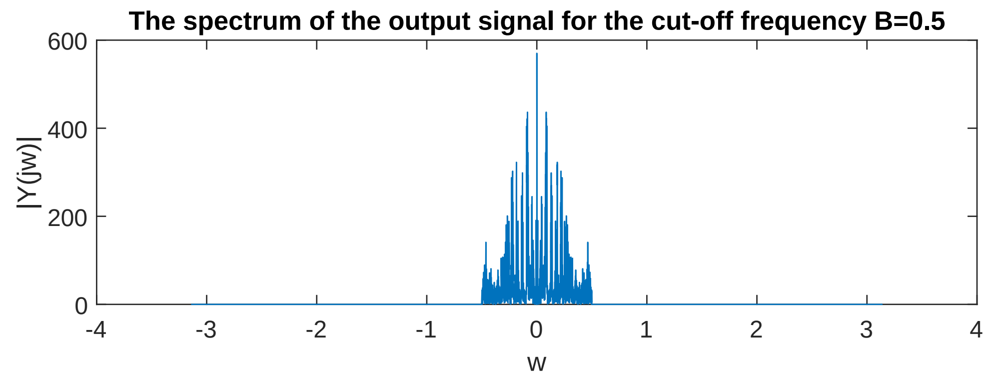
```
figure;
subplot(211);plot(w,real(X_jw));
xlabel('w');
ylabel('Re{X(jw)}');
title('The real part of X(jw)');
subplot(212);plot(w,imag(X_jw));
xlabel('w');
ylabel('Im{X(jw)}');
title('The imaginary part of X(jw)');
```

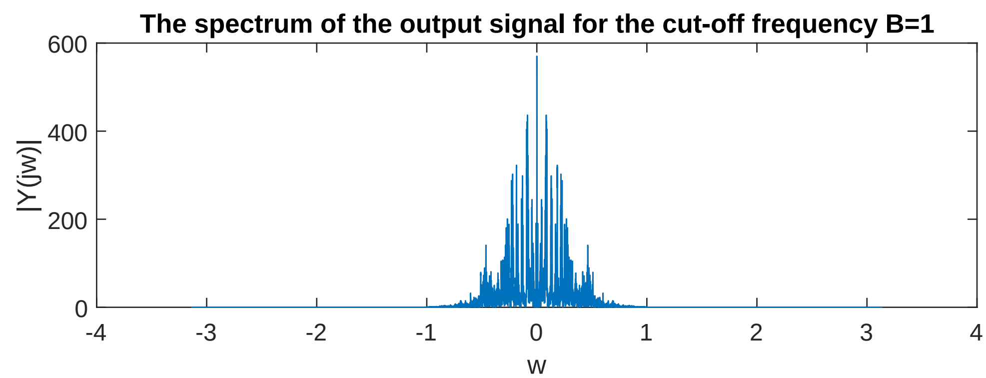
(c)

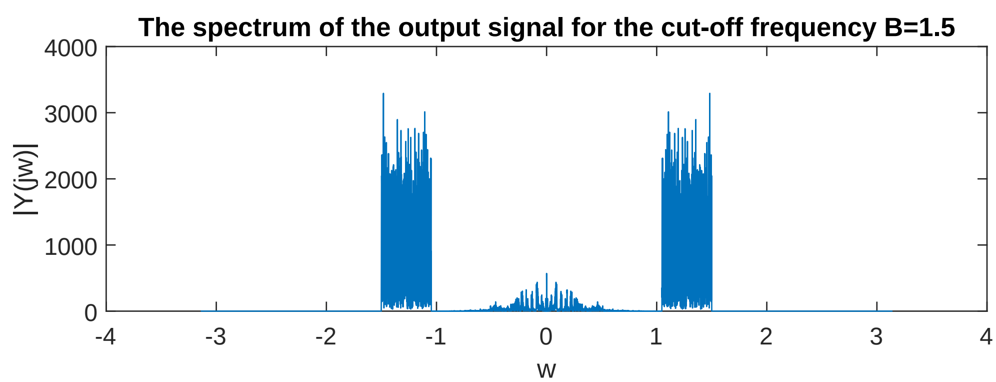


(d)







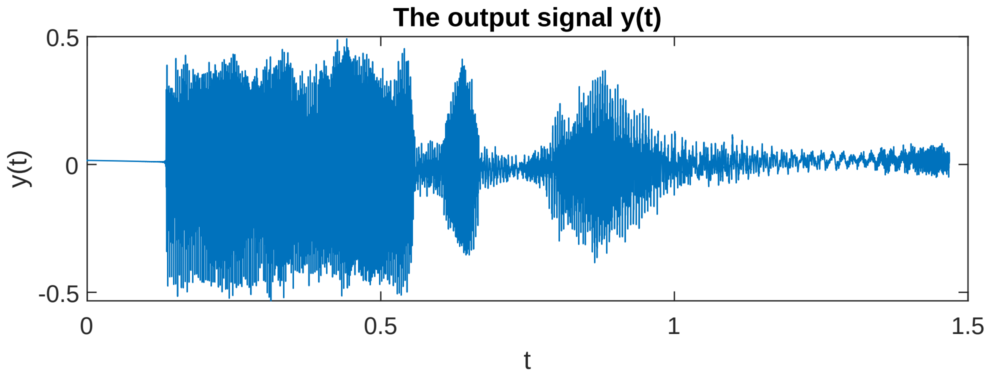


To plot the spectrum of the output signal in MATLAB, the sample code is given below.

```
B = [0.2 0.5 1 1.5];
for b=1:length(B)
    H_jw=(abs(w)<B(b));
    H_jw=double(H_jw);

    Y_jw=H_jw.*X_jw;
    figure; plot(w,abs(Y_jw));
```

```
    xlabel('w');
    ylabel('|Y(jw)|');
    title (['The spectrum of the output signal for the' ...
       ' cut-off frequency B=' num2str(B(b))]);
end
```

(e)	

From the output signal $y(t)$, we can observe the silence period before speech and the time intervals corresponding to the softer or louder speech. On the other hand, we cannot observe such information from the noisy input signal $x(t)$.

To plot the output signal in MATLAB, the sample code is given below.

```
B = 1;
H_jw=(abs(w)<B);
H_jw=double(H_jw);
Y_jw=H_jw.*X_jw;
y = ifft(ifftshift(Y_jw));
plot(t,real(y));
title('The output signal y(t)');
xlabel('t');
ylabel('y(t)');
```

(f) The sentence in the speech signal is “I have a dream”.