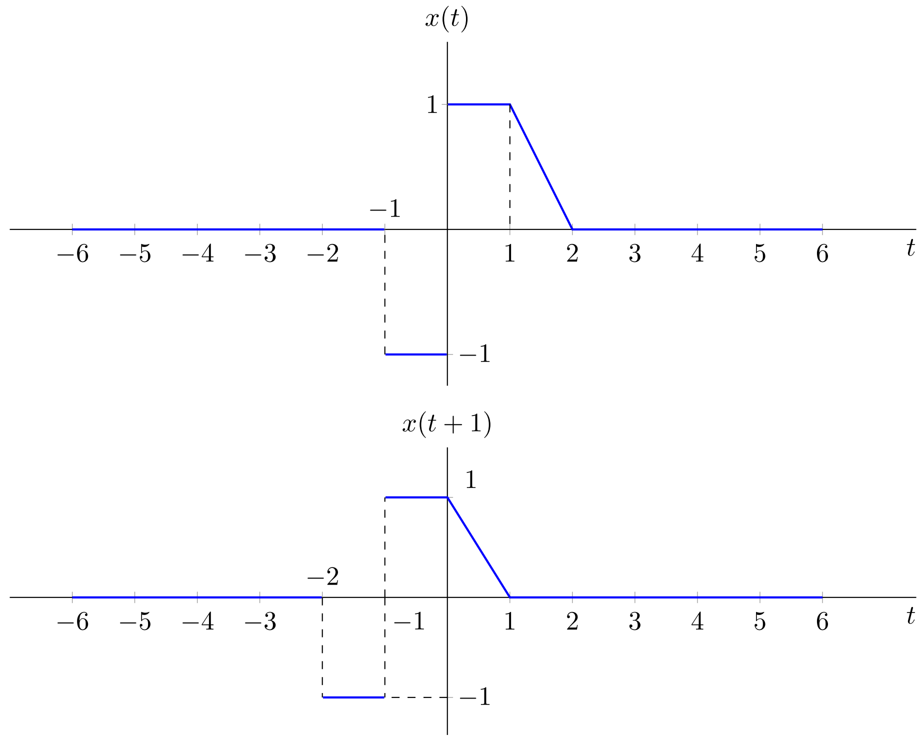
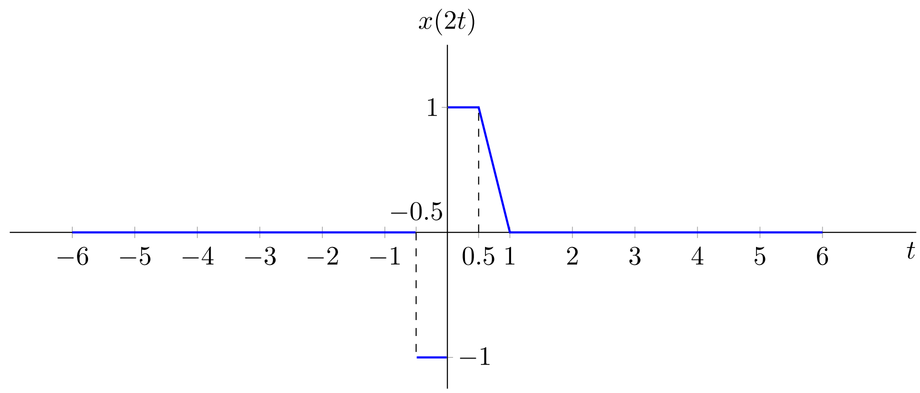
Middle East Technical University
Department of Electrical and Electronics Engineering
EE301: Signals and Systems I

Solutions for Homework 1
October 20, 2019
If you face any problem or mistake please contact Ömer Çayır, *ocayir@metu.edu.tr*, DZ-10.

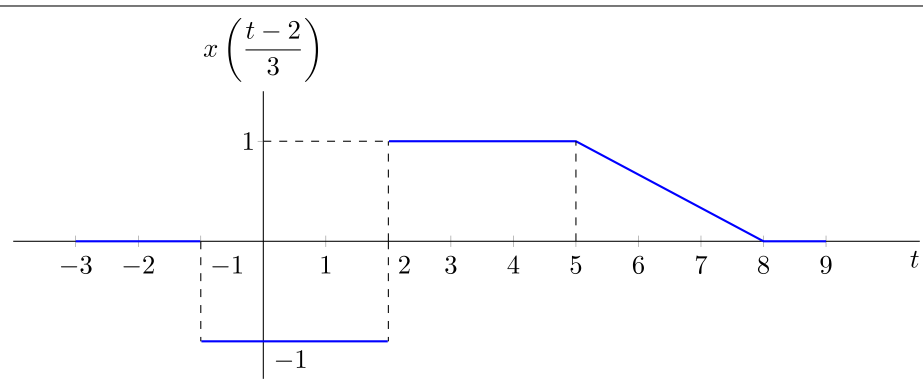
1. i.



ii.

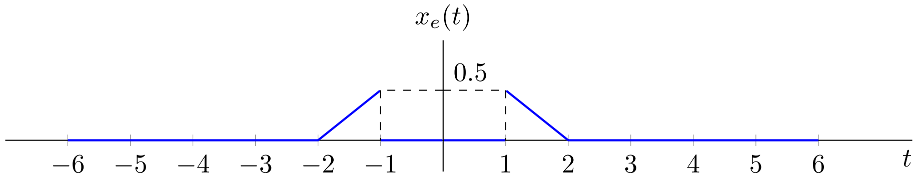


iii.

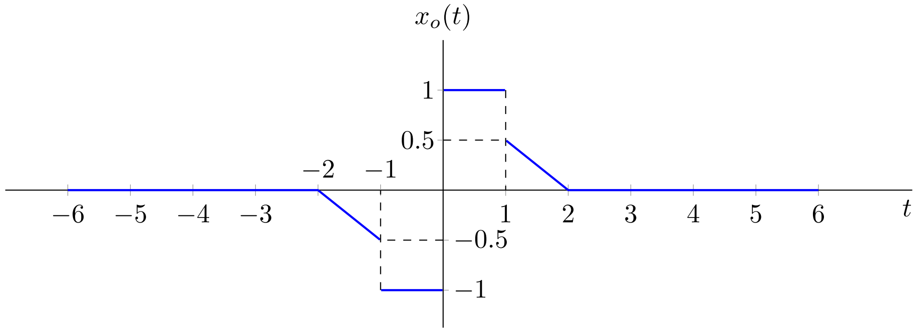


iv.

$$x_e(t) = \mathcal{E}\nu\left\{x(t)\right\} = \frac{1}{2}\left(x(t) + x(-t)\right)$$



$$x_o(t) = \mathcal{O}d\left\{x(t)\right\} = \frac{1}{2}\left(x(t) - x(-t)\right)$$



2.

- If a CT signal $x(t)$ is periodic, there exists a positive number T such that $x(t+T) = x(t)$ and its fundamental period T_0 is the smallest nonzero period T satisfying this relation.
- If a DT signal $x[n]$ is periodic, there exists a positive integer N such that $x[n+N] = x[n]$ and its fundamental period N_0 is the smallest nonzero period N satisfying this relation.

i.

$$x(t) = 3\cos\left(4t + \frac{\pi}{3}\right)$$

is **periodic** with the fundamental period $T_0 = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$.

ii.

$$x(t) = \left[\cos\left(2t - \frac{\pi}{3}\right)\right]^2 = \frac{1}{2}\left[1 + \cos\left(4t - \frac{2\pi}{3}\right)\right]$$

is **periodic** and its fundamental period is $T_0 = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$.

iii.

$$x(t) = 2\cos\left(10t + 1\right) - \sin\left(4t - 1\right)$$

$\cos\left(10t+1\right)$ is periodic with period $\dfrac{2\pi}{10} = \dfrac{\pi}{5}$ and $\sin\left(4t-1\right)$ is periodic with period $\dfrac{2\pi}{4} = \dfrac{\pi}{2}$.
Hence, $x(t)$ is **periodic** and its fundamental period is $T_0 = \mathrm{LCM}\left(\dfrac{\pi}{5}, \dfrac{\pi}{2}\right) = \pi$.

iv.

$$x(t) = \mathcal{E}\nu\left\{\cos\left(4\pi\, t\right)\, u(t)\right\} = \frac{1}{2}\left[\cos\left(4\pi\, t\right)\, u(t) + \cos\left(-4\pi\, t\right)\, u(-t)\right] = \frac{1}{2}\cos\left(4\pi\, t\right)$$

(Remember that $\cos\left(-\theta\right) = \cos\left(\theta\right)$.)

The signal $x(t)$ is **periodic** and its fundamental period is $T_0 = \dfrac{2\pi}{4\pi} = \dfrac{1}{2}$.

v.

$$x[n] = j^n = \exp\left(j\frac{\pi}{2}n\right)$$

$$x[n+N] = x[n] \implies \exp\left(j\frac{\pi}{2}N\right) = 1 = \exp(j2\pi k) \text{ for } k \in \mathbb{Z}$$

Then, we obtain $N = 4k$. Hence, $x[n]$ is **periodic** and its fundamental period is $N_0 = 4$ (for $k = 1$).

vi.

$$x[n] = (1+j)^n = \left(\sqrt{2}\exp\left(j\frac{\pi}{4}\right)\right)^n = \left(\sqrt{2}\right)^n \exp\left(j\frac{\pi}{4}n\right)$$

Owing to $\left(\sqrt{2}\right)^n$, we cannot find a nonzero N satisfying $x[n+N] = x[n]$, and the signal $x[n]$ is **not periodic**.

vii.

$$x[n] = \exp\left(j\frac{25}{4}\pi n\right)$$

$$x[n+N] = x[n] \implies \exp\left(j\frac{25}{4}\pi N\right) = 1 = \exp(j2\pi k) \text{ for } k \in \mathbb{Z}$$

Then, we obtain $N = \frac{8}{25}k$. Hence, $x[n]$ is **periodic** and its fundamental period is $N_0 = 8$ (for $k = 25$).

3.

(1) A system is *memoryless* (instantaneous) if the output at any time instant depends only on the value of the input at that particular instant.

(2) A system is *causal* if its output at any time depends only on the values of the input at present time and in the past.

(3) A system is *stable* if bounded inputs lead to bounded outputs (BIBO stable).

(4) A system is *time-invariant* if a time shift in the input signal causes same amount of time shift in the output signal.

(5) A system is *linear* if it possesses the superposition property:

$$\text{CT:} \quad a\,x_1(t) + b\,x_2(t) \to a\,y_1(t) + b\,y_2(t) \quad \Longleftarrow \quad x_1(t) \to y_1(t) \text{ and } x_2(t) \to y_2(t)$$

$$\text{DT:} \quad a\,x_1[n] + b\,x_2[n] \to a\,y_1[n] + b\,y_2[n] \quad \Longleftarrow \quad x_1[n] \to y_1[n] \text{ and } x_2[n] \to y_2[n]$$

System	(1) Memoryless	(2) Causal	(3) Stable	(4) Time-Invariant	(5) Linear
i. $y[n] = x[n-2] + x[2-n]$	✗	✗	✓	✗	✓
ii. $y(t) = \cos(3t)\ x(t)$	✓	✓	✓	✗	✓
iii. $y[n] = \sum_{k=-\infty}^{2n} x[k]$	✗	✗	✗	✗	✓
iv. $y(t) = \frac{dx(t)}{dt}$	✗	✓	✗	✓	✓

i.

$$y[n] = x[n-2] + x[2-n]$$

The system is **not causal**. For instance, when $n = 0$, we see that $y[0] = x[-2] + x[2]$, and the current value of the output depends on the future value of the input.

The system is **not memoryless** since it is not causal.

The system is **not time-invariant** owing to $x[2-n]$. Let $x_2[n] = x_1[n-n_0]$ for $x_1[n] \to y_1[n]$ and $x_2[n] \to y_2[n]$, then check whether $y_2[n] = y_1[n-n_0]$ or not.

$$\left.\begin{aligned} y_2[n] = x_2[n-2] + x_2[2-n] = x_1[n-2-n_0] + x_1[2-n-n_0] \\ y_1[n-n_0] = x_1[n-n_0-2] + x_2[2-n+n_0] \end{aligned}\right\} \implies y_2[n] \neq y_1[n-n_0]$$

ii.

$$y(t) = \cos(3t)\; x(t)$$

The system is **not time-invariant** owing to $\cos(3t)$. Let $x_2(t) = x_1(t-t_0)$ for $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$, then check whether $y_2(t) = y_1(t-t_0)$ or not.

$$\left.\begin{aligned} y_2(t) = \cos(3t)\; x_2(t) = \cos(3t)\; x_1(t-t_0) \\ y_1(t-t_0) = \cos(3t-3t_0)\; x_1(t-t_0) \end{aligned}\right\} \implies y_2(t) \neq y_1(t-t_0)$$

iii.

$$y[n] = \sum_{k=-\infty}^{2n} x[k]$$

The system is **not causal**. For instance, when $n = 1$, we see that $y[1] = x[2]+x[1]+x[0]+\ldots$, and the current value of the output depends on the future value of the input.

The system is **not memoryless** since it is not causal.

The system is **not time-invariant** owing to the upper limit of the summation $2n$. Assuming $x_2[n] = x_1[n-n_0]$ for $x_1[n] \to y_1[n]$ and $x_2[n] \to y_2[n]$, we obtain $y_2[n] \neq y_1[n-n_0]$.

$$\left.\begin{aligned} y_2[n] = \sum_{k=-\infty}^{2n} x_2[k] = \sum_{k=-\infty}^{2n} x_1[k-n_0] \overset{\ell \triangleq k-n_0}{=} \sum_{\ell=-\infty}^{2n-n_0} x_2[\ell] \\ y_1[n-n_0] = \sum_{k=-\infty}^{2n-2n_0} x_1[k] \end{aligned}\right\} \implies y_2[n] \neq y_1[n-n_0]$$

The system is **not stable**. Let $x[n] = u[n]$. Then, we have

$$y[n] = \sum_{k=-\infty}^{2n} u[k] = \sum_{k=0}^{2n} u[k] = (2n+1)\, u[n],$$

and $y[n] \to \infty$ as $n \to \infty$.

iv.

$$y(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$$

The derivative of $x(t)$ can be expressed as

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lim_{\Delta \to 0} \frac{x(t) - x(t-\Delta)}{\Delta},$$

see *Section 2.5.3 from Oppenheim*. Thus, the system is **causal**, but it is **not memoryless** since the current value of the output depends on the past value of the input.

The system is **not stable**. Let $x(t) = \sqrt{4-t^2}$ if $|t| < 2$, and $x(t) = 0$ otherwise. Then, we have

$$y(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{t}{\sqrt{4-t^2}}$$

if $|t| < 2$, and it is zero otherwise. As $t \to 2$, $y(t) \to \infty$, while $x(t) \leqslant 2$ for all t.

4. i.

$$y_i(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(t-\tau)\, x_2(\tau)\, \mathrm{d}\tau = \int_0^{\infty} \underbrace{[u(t-\tau-3) - u(t-\tau-5)]}_{\text{nonzero only for } t-5<\tau<t-3} e^{-3\tau}\, \mathrm{d}\tau$$

For $t \leqslant 3$, we have $y_i(t) = 0$.
For $3 < t \leqslant 5$, we have

$$y_i(t) = \int_0^{t-3} e^{-3\tau}\, \mathrm{d}\tau = \left(\frac{-1}{3}\, e^{-3\tau}\right)\Bigg|_{\tau=0}^{t-3} = \frac{1-e^{-3(t-3)}}{3}.$$

For $t > 5$, we have

$$y_i(t) = \int_{t-5}^{t-3} e^{-3\tau}\, \mathrm{d}\tau = \left(\frac{-1}{3}\, e^{-3\tau}\right)\Bigg|_{\tau=t-5}^{t-3} = \frac{e^{-3(t-5)} - e^{-3(t-3)}}{3} = \frac{\left(1-e^{-6}\right)\, e^{-3(t-5)}}{3}.$$

$$y_i(t) = \begin{cases} 0, & t \leqslant 3 \\ \dfrac{1-e^{-3(t-3)}}{3}, & 3 < t \leqslant 5 \\ \dfrac{\left(1-e^{-6}\right)\, e^{-3(t-5)}}{3}, & t > 5 \end{cases}$$

ii.

$$y_{ii}(t) = \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} * x_2(t) = [\delta(t-3) - \delta(t-5)] * x_2(t) = x_2(t-3) - x_2(t-5)$$

$$y_{ii}(t) = e^{-3(t-3)}\, u(t-3) - e^{-3(t-5)}\, u(t-5)$$

iii.

$$y_{ii}(t) = \frac{\mathrm{d}y_i(t)}{\mathrm{d}t} = \begin{cases} 0, & t \leqslant 3 \\ e^{-3(t-3)}, & 3 < t \leqslant 5 \\ \left(e^{-6}-1\right)\, e^{-3(t-5)}, & t > 5 \end{cases}$$

5. We find the output signals for the given input signals and $h[n] = \alpha^n\, u[n]$, where $\alpha \in \mathbb{C}$, as follows:

$$y_1[n] = h[n] * x_1[n] = \sum_{k=-\infty}^{\infty} x_1[n-k]\, h[k] = \sum_{k=-\infty}^{\infty} \alpha^k\, u[k] = \sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}, \text{ if } |\alpha| < 1$$

$$y_2[n] = \sum_{k=-\infty}^{\infty} x_2[n-k]\, h[k] = \sum_{k=-\infty}^{\infty} u[n-k]\, \alpha^k\, u[k] = u[n] \sum_{k=0}^{n} \alpha^k = \begin{cases} (n+1)\, u[n], & \alpha = 1 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha}\, u[n], & \alpha \neq 1 \end{cases}$$

$$y_3[n] = \sum_{k=-\infty}^{\infty} x_3[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} u[-k-1]\, \alpha^{n-k}\, u[n-k] = \sum_{k=-\infty}^{-1} \alpha^{n-k}\, u[n-k]$$

This summation is bounded only for $|\alpha| < 1$, and it can be computed separately for $n < -1$ and $n \geqslant -1$, since $u[n-k]$ is nonzero only when $k \leqslant n$.

$$y_3[n] = \sum_{k=-\infty}^{-1} \alpha^{n-k}\, u[n-k] = \begin{cases} \displaystyle\sum_{k=-\infty}^{n} \alpha^{n-k} \overset{\ell \triangleq n-k}{=} \sum_{\ell=0}^{\infty} \alpha^{\ell} = \frac{1}{1-\alpha}, & n < -1 \text{ and } |\alpha| < 1 \\ \displaystyle\sum_{k=-\infty}^{-1} \alpha^{n-k} \overset{\ell \triangleq -k-1}{=} \sum_{\ell=0}^{\infty} \alpha^{n+\ell+1} = \frac{\alpha^{n+1}}{1-\alpha}, & n \geqslant -1 \text{ and } |\alpha| < 1 \end{cases}$$

As $x_3[n] = x_1[n] - x_2[n]$, we have $y_3[n] = y_1[n] - y_2[n]$ owing to the distributive property of LTI systems shown below.

$$y_3[n] = h[n] * x_3[n] = h[n] * (x_1[n] - x_2[n]) = h[n] * x_1[n] - h[n] * x_2[n] = y_1[n] - y_2[n]$$

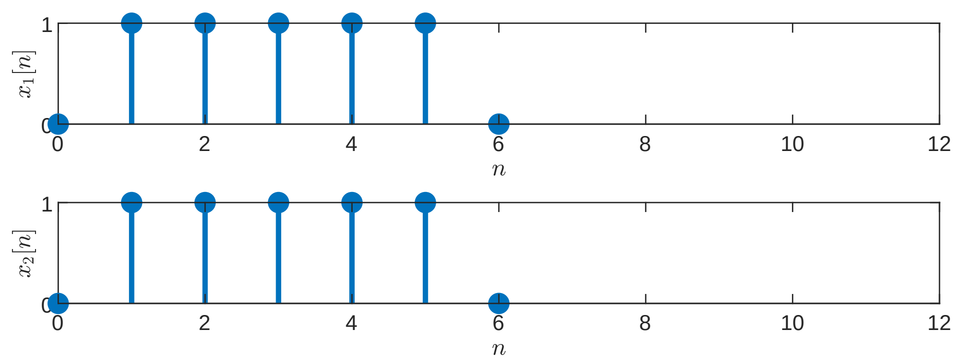
$$y_3[n] = y_1[n] - y_2[n] = \begin{cases} \dfrac{1}{1-\alpha}, & n < 0 \text{ and } |\alpha| < 1 \\ \dfrac{\alpha^{n+1}}{1-\alpha}, & n \geqslant 0 \text{ and } |\alpha| < 1 \end{cases}$$

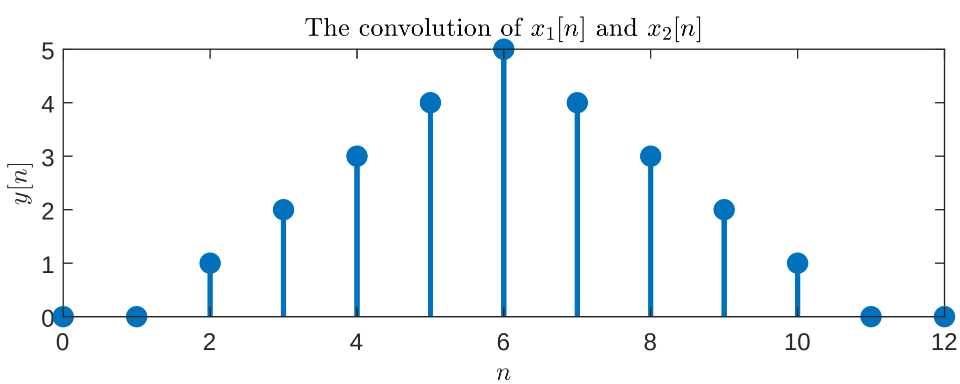
$$\left.\frac{\alpha^{n+1}}{1-\alpha}\right|_{n=-1} = \frac{1}{1-\alpha} \implies y_3[n] = y_1[n] - y_2[n] = \begin{cases} \dfrac{1}{1-\alpha}, & n < -1 \text{ and } |\alpha| < 1 \\ \dfrac{\alpha^{n+1}}{1-\alpha}, & n \geqslant -1 \text{ and } |\alpha| < 1 \end{cases}$$

6. i.

$$x_1[n] = x_2[n] = \begin{cases} 1, & 1 \leqslant n \leqslant 5 \\ 0, & \text{otherwise} \end{cases}$$

$$y[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k] = \sum_{k=1}^{5} x_2[n-k] = \begin{cases} n-1, & 1 < n \leqslant 6 \\ 11-n, & 6 < n \leqslant 10 \\ 0, & \text{otherwise} \end{cases}$$





To verify our result $x_1[n] * x_2[n]$ in MATLAB, the sample code is given below.

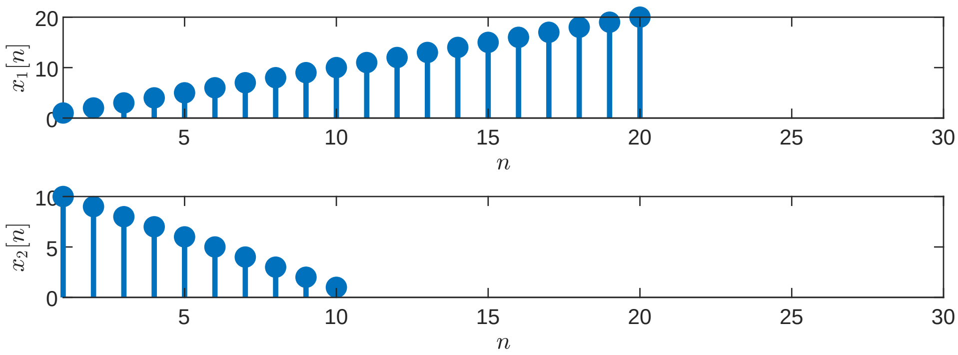
```
x = @(n) (n>=1 & n<=5)*1; % x[n]=1 if 1<=n<=5, x[n]=0 otherwise
nx = 0:6; % sample index for x1 and x2
ny = 2*min(nx):2*max(nx); % sample index for the convolution

% obtain x1 and x2
xOne = x(nx);
xTwo = x(nx);
```

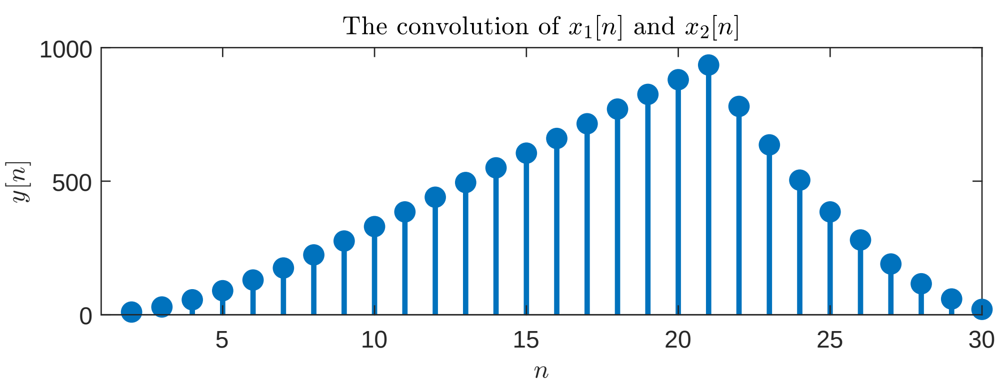
```
% compute the convolution of x1 and x2 by using 'conv' function
y = conv(xOne,xTwo);

% visualization of the DT signals
figure(1);subplot(211); % x1
stem(nx,xOne,'filled','LineWidth',2);
set(gca,'XLim',[min(ny) max(ny)],'YTick',[0 1])
xlabel('$$n$$','Interpreter','latex');
ylabel('$$x_1[n]$$','Interpreter','latex')

subplot(212); % x2
stem(nx,xTwo,'filled','LineWidth',2);
set(gca,'XLim',[min(ny) max(ny)],'YTick',[0 1])
xlabel('$$n$$','Interpreter','latex');
ylabel('$$x_2[n]$$','Interpreter','latex')

figure(2); % y
stem(ny,y,'filled','LineWidth',2);
title('The convolution of $$x_1[n]$$ and $$x_2[n]$$', ...
    'Interpreter','latex');
xlabel('$$n$$','Interpreter','latex');
ylabel('$$y[n]$$','Interpreter','latex')
```

ii. Assuming $x_1[n] = 0$ for $n \leqslant 0$ and $n > N_1$, and $x_2[n] = 0$ for $n \leqslant 0$ and $n > N_2$, we can use the MATLAB code given below to compute the convolution of $x_1[n]$ and $x_2[n]$.

```
function [y, ny] = ee301hw1q6partii(xOne,xTwo)
% get the length of each input signal
NOne = length(xOne);
NTwo = length(xTwo);

% memory allocation for the convolution of the input signals
y = zeros(1,NOne+NTwo-1);

% select the loop with less repetitions to run faster
if NOne<NTwo
    % use x1[1]*x2[n-1]+x1[2]*x2[n-2]+...+x1[N1]*x2[n-N1]
    for k=1:NOne
        ind = k+(0:NTwo-1);
        y(ind) = y(ind)+xOne(k)*xTwo;
    end
else
    % use x2[1]*x1[n-1]+x2[2]*x1[n-2]+...+x2[N2]*x1[n-N2]
    for k=1:NTwo
        ind = k+(0:NOne-1);
        y(ind) = y(ind)+xTwo(k)*xOne;
    end
end

% get the sample index for the convolution of the input signals
ny = 2:(NOne+NTwo);
```

iii. $$x_1[n] = \begin{cases} n, & 1 \leqslant n \leqslant 20 \\ 0, & \text{otherwise} \end{cases} \qquad x_2[n] = \begin{cases} 11-n, & 1 \leqslant n \leqslant 10 \\ 0, & \text{otherwise} \end{cases}$$





To compute $x_1[n] * x_2[n]$ in MATLAB, the sample code is given below.

```
NOne = 20;
nxOne = 1:NOne; % sample index for x1
NTwo = 10;
nxTwo = 1:NTwo; % sample index for x2

% obtain x1 and x2
% x1[n]=n if 1<=n<=20, x1[n]=0 otherwise
xOne = nxOne;
% x2[n]=11-n if 1<=n<=10, x2[n]=0 otherwise
xTwo = 11-nxTwo;

% compute the convolution of x1 and x2 by using 'conv' function
[y, ny] = ee301hw1q6partii(xOne,xTwo);

% get lower and upper limits of time axis for alignment
nMin = min(1,min(ny));
nMax = max(max(NOne,NTwo),max(ny));

% visualization of the DT signals
figure(1);subplot(211); % x1
stem(nxOne,xOne,'filled','LineWidth',2);
set(gca,'XLim',[nMin nMax])
xlabel('$$n$$','Interpreter','latex');
ylabel('$$x_1[n]$$','Interpreter','latex')

subplot(212); % x2
```

```
stem(nxTwo,xTwo,'filled','LineWidth',2);
set(gca,'XLim',[nMin nMax])
xlabel('$$n$$','Interpreter','latex');
ylabel('$$x_2[n]$$','Interpreter','latex')

figure(2); % y
stem(ny,y,'filled','LineWidth',2);
set(gca,'XLim',[nMin nMax])
title('The convolution of $$x_1[n]$$ and $$x_2[n]$$', ...
    'Interpreter','latex');
xlabel('$$n$$','Interpreter','latex');
ylabel('$$y[n]$$','Interpreter','latex')
```

iv.

$$\begin{aligned} x_1[n-4] * x_2[n+5] &= (x_1[n] * \delta[n-4]) * (x_2[n] * \delta[n+5]) \\ &= \underbrace{(x_1[n] * x_2[n])}_{y[n] \text{ from part (iii)}} * \underbrace{(\delta[n-4] * \delta[n+5])}_{\delta[n+1]} \\ &= y[n+1] \end{aligned}$$

Using commutative After running the code given in part (iii), we use the following lines in MATLAB to obtain the desired result.

```
figure(2); % y
stem(ny-1,y,'filled','LineWidth',2);
set(gca,'XLim',[nMin nMax])
title('The convolution of $$x_1[n-4]$$ and $$x_2[n+5]$$', ...
    'Interpreter','latex');
xlabel('$$n$$','Interpreter','latex');
ylabel('$$y[n]$$','Interpreter','latex')
```