EE 301 Signals and Systems I

Homework 0

(due Oct. 4, 2019)

This homework contains some useful background on complex calculus to be exercised upon.

Question 1

Calculate the following complex numbers in Cartesian or polar form

a. $3\exp\left(j\dfrac{\pi}{2}\right)+3\exp\left(-j\dfrac{\pi}{2}\right)+\exp\left(j\pi\right)-4\exp\left(-j\dfrac{\pi}{2}\right)+4j+2\exp\left(j\dfrac{\pi}{6}\right)$

b. $\dfrac{(1-j)(5-5j)\left(\sqrt{3}-j\right)}{10j\left(5-j5\sqrt{3}\right)}$

c. $\left(\dfrac{1}{\sqrt{2}}-\dfrac{1}{\sqrt{2}}j\right)^5$

d. $\left(\cos\dfrac{\pi}{6}-j\sin\dfrac{\pi}{6}\right)^6$.

Hint: Let $z_1 = x_1 + jy_1 = r_1\exp\left(j\theta_1\right)$ and $z_2 = x_2 + jy_2 = r_2\exp\left(j\theta_2\right)$ be two complex numbers. It is clear that addition (or, subtraction) is simpler with Cartesian representations:

$$z_1 \pm z_2 = (x_1 \pm x_2) + j(y_1 \pm y_2)$$

However, in multiplication or division polar representations are easier to handle:

$$z_1 z_2 = r_1 r_2 \exp\left(j\left(\theta_1+\theta_2\right)\right)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\exp\left(j\left(\theta_1-\theta_2\right)\right) \text{ for } z_2 \neq 0$$

Question 2

Prove Euler's formula, i.e. $\exp\left(j\theta\right)=\cos\theta + j\sin\theta$. Hint: Remember that for the real variable x, $\exp(x)=\sum_{n=0}^{\infty}\dfrac{x^n}{n!}$. Assume (without proof) that this series expansion is valid for any complex variable z, and finally let $z = j\theta$.

Question 3

Show that for the complex variable $z = x + jy$, $\exp(z) = \exp(x)\left[\cos(y) + j\sin(y)\right]$.

Question 4

All values of the complex variable $z = x + jy$ can be represented as points on the *complex plane* using the x-axis as the real axis and y-axis as the imaginary axis. This plane is also called the *Argand plane* or *Gauss plane*. Plot the following subsets of the complex plane:

a. $S_1 = \left\{ z = x + jy \middle| x > 0 \right\}$, equivalently $S_1 = \left\{ z \middle| \mathrm{Re}(z) > 0 \right\}$.

b. $S_2 = \left\{ z = x + jy \middle| x \geq 0 \right\}$, equivalently $S_2 = \left\{ z \middle| \mathrm{Re}(z) \geq 0 \right\}$.

   i. Compare the sets S_1 and S_2. Which one includes the imaginary axis?

   ii. Express the set of complex numbers on the imaginary axis by the set notation. How can you relate this set and S_1 to S_2.

c. $S_3 = \left\{ z \middle| -1 < \mathrm{Re}(z) < 1 \right\}$.

d. $S_4 = \left\{ z \middle| -1 < \mathrm{Im}(z) < 1 \right\}$.

e. $S_5 = \left\{ z \middle| \, |z| < 1 \right\}$, where $|z|$ is called the "modulus" of z, and defined as $|z| = \sqrt{x^2 + y^2}$

   That is, if we use the polar representation $z = r\exp(j\theta)$, then $r = |z|$.

f. $S_6 = \left\{ z \middle| \, 1 < |z| < 2 \right\}$.

g. $S_7 = \left\{ z \middle| \, |z| = 1 \right\}$. S_7 is called the *unit circle*. Why?

   i. Express the unit disk (set of all complex numbers z with modulus less than or equal to 1) in set notation. Note that the unit circle is the *boundary* of the unit disk.

   ii. The set $S_8 = \left\{ z \middle| \, 0 < |z| \leq 1 \right\}$ is called the *punctured unit disk*. Explain why.

Question 5

a. Show that $\left(r\exp\left(j\theta\right)\right)^n = r^n \exp\left(jn\theta\right)$ for any integer n. Remark: The Cartesian form of this expression, $\left(r\cos\theta + jr\sin\theta\right)^n = r^n\left(\cos n\theta + j\sin n\theta\right)$, is known as *de Moivre*'s formula.

b. By using the expression given above, verify the following trigonometric identities:
   $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$
   $\sin 3\theta = -\sin^3\theta + 3\cos^2\theta\sin\theta$.

c. Suppose that we want to solve the equation $z^n = 1$, where n is a positive integer. This equation can be written as $r^n \exp\left(jn\theta\right) = \exp\left(j2\pi k\right)$ for $k = 0, \pm 1, \pm 2, \ldots$

i. Explain why this equation yields $r=1$ and $\theta=\dfrac{2\pi k}{n}$ for $k=0,1,\dots,n-1$.

ii. The complex numbers $\cos\left(\dfrac{2\pi k}{n}\right)+j\sin\left(\dfrac{2\pi k}{n}\right)$ are called the n-th roots of unity for $k=0,1,\dots,n-1$. Plot the roots of unity for $n=2,3,4,5,6$ on the complex plane. Note that they are located on the *unit circle.*

iii. How can you generalize this derivation for finding the n-th roots of an arbitrary complex number $w=\rho\exp(j\phi)$? Evaluate the cube roots of $27j$.

Question 6

a. For any complex number z, establish the identity $1+z+z^2+\cdots+z^n=\dfrac{1-z^{n+1}}{1-z}$. Hint: Let $S=1+z+z^2+\cdots+z^n$ and consider $S-zS$.

b. Show that $\lim_{n\to\infty} z^{n+1}=0$, provided that $|z|<1$, and $\lim_{n\to\infty} S=\frac{1}{1-z}$.

c. Derive Lagrange's trigonometric identity

$$1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta=\frac{1}{2}+\frac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right]}{2\sin\left(\frac{1}{2}\theta\right)}.$$

Hint: Use de Moivre's formula.