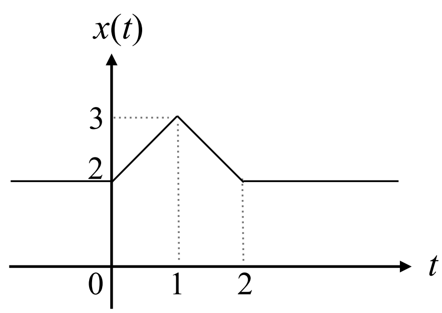
Middle East Technical University
Department of Electrical and Electronics Engineering
EE301: Signals and Systems I

Homework 3
Due: Sunday November 24, 2019

1. (a) Find the Fourier transform $X(j\omega)$ of the following signals $x(t)$:
   i. $\delta(t-3) - 2\delta(t+2)$
   ii. $e^{-3t}u(t)$
   iii. $e^{-3(t-5)}u(t-5)$ [Use (ii)].
   iv. $e^{-2|t|}\cos(t)$

   (b) Find the inverse Fourier transform $x(t)$ of the following functions $X(j\omega)$:
   i. $e^{j5\omega} + e^{-j7\omega}$
   ii. $2\pi\delta(\omega-3) + 2\pi\delta(\omega+3)$
   iii. $\cos(2\omega + \frac{\pi}{4})$

2. Consider the signal $x(t)$ plotted in the figure below.



   (a) Determine whether the signal $x(t)$ is square integrable or absolutely integrable. Does this signal satisfy the conditions that guarantee the existence and convergence of its Fourier transform $X(j\omega)$?

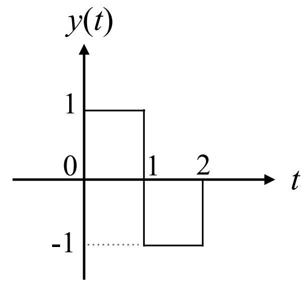
   (b) As the observations in part (a) suggest, it is not straightforward to obtain $X(j\omega)$ with the usual Fourier transform integral. Instead, let's find $X(j\omega)$ using the properties of the Fourier transform (without computing any integrals) by following the steps below:

   i. First, express $x(t)$ in the form
   $$x(t) = c + (r * r)(t)$$
   such that $c > 0$ is a constant and $r(t) > 0$ is a rectangular signal whose convolution with itself gives $x(t)$ as above.

   ii. Remember that the rectangular pulse signal
   $$p(t) = \begin{cases} 1, & \text{if } |t| < \frac{1}{2} \\ 0, & \text{otherwise} . \end{cases}$$
   has the Fourier transform $P(j\omega) = \frac{2\sin(\omega/2)}{\omega}$. Observing that $r(t)$ is a time-shifted version of $p(t)$, use this result to obtain its Fourier transform $R(j\omega)$.

   iii. Use the results in (i) and (ii) to obtain the Fourier transform $X(j\omega)$ of $x(t)$.

(c) Consider the signal $y(t)$ given below. How are the signals $x(t)$ and $y(t)$ related? Use $X(j\omega)$ to compute $Y(j\omega)$ directly (without evaluating any integrals).



3. The signal $x(t)$ has the Fourier transform $X(j\omega)$. The following are known about $x(t)$:

- $x(t)$ is real and even.
- $X(j\omega) = c$ for $\omega \in [0, 2\pi]$, where c is an unknown constant.
- $X(j\omega) = ce^{2\omega}$ for $w < -2\pi$.
- $\int_{-\infty}^{\infty} x(t)dt = 1$.

Using the given information

(a) Find the constant c (without any calculations) and plot $X(j\omega)$.

(b) Compute the signal $x(t)$.

4. The unit step function $u(t)$ is not absolutely integrable or square integrable, hence, it is not easy to compute its spectrum with the usual Fourier transform integral. In this question, we will derive the Fourier transform of $u(t)$ with the following procedure:

(a) Plot the function $x(t) = e^{-at}u(t) - e^{at}u(-t)$ where $a > 0$ is real. Observe that as $a \to 0$, $x(t)$ approaches the signum function given by

$$\mathrm{sgn}(t) = \begin{cases} -1, & \text{if } t < 0 \\ 0, & \text{if } t = 0 \\ 1, & \text{if } t > 0 \end{cases}$$

(b) Compute the Fourier transform $X(j\omega)$ of the function $x(t)$ in part (a). Evaluate the limit

$$\lim_{a \to 0} X(j\omega)$$

to show that the Fourier transform of $\mathrm{sgn}(t)$ is given by $\frac{2}{j\omega}$.

(c) Express the unit step function $u(t)$ in terms of $\mathrm{sgn}(t)$ as

$$u(t) = \alpha + \beta\,\mathrm{sgn}(t)$$

where α and β are real constants. Use this relation and the Fourier transform of $\mathrm{sgn}(t)$ to find the Fourier transform of $u(t)$.

5. The signal $x(t)$ is a periodic CT signal with the Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkt}$$

where the Fourier coefficients are given by

$$a_k = \begin{cases} \dfrac{2}{\pi}, & k = 0 \\ \\ \dfrac{1}{\pi\, k^2}, & \text{otherwise}. \end{cases}$$

(a) Find and plot the Fourier transform $X(j\omega)$ of $x(t)$.

(b) The periodic signal $x(t)$ is provided as input to an LTI system with the following impulse response $h(t)$:

$$h(t) = \frac{\sin(t)\cos(\frac{5}{2}t)}{\pi t}$$

Find the Fourier transform $H(j\omega)$ of $h(t)$. (Hint: Express $h(t)$ as the product of a sinc signal and a cosine signal, then use the multiplication property of the Fourier transform.)

(c) Let $y(t)$ be the output of the LTI system when $x(t)$ is provided as input. Use the relation $Y(j\omega) = X(j\omega)H(j\omega)$ to find the spectrum $Y(j\omega)$ of the output.

(d) Use your result in (c) to find the time-domain representation $y(t) = (x * h)(t)$ of the output signal.

6. In this question, we will work on a noisy speech signal and enhance its quality by removing the noise with a low-pass filter.

(a) The signal $x(t)$ is a noisy speech signal that is severely contaminated with additive high-frequency noise. The signal $x(t)$ is available in ODTUClass under the file name `x.mat`. Load the signal in MATLAB and obtain its length using the following commands:

```
load x.mat
T=length(x);
```

Plot the noisy signal $x(t)$. Listen to the signal with the following command:

```
sound(x,44100);
```

(The number 44100 we specify here is a parameter related to the recording settings of the signal). Comment on what you hear. Is it possible to recognize the sentence in the signal?

(b) We will now compute the Fourier transform $X(j\omega)$ of the signal $x(t)$. Unfortunately, it is not possible to exactly represent and process continuous-time signals in MATLAB. However, for the moment, we can think of the `fft` function as a tool that takes the Fourier transform of signals. (We will learn its real meaning later in the course when we study Chapter 5). Compute the Fourier transform $X(j\omega)$ of the signal $x(t)$ with the following command:

```
X_jw=fftshift(fft(x));
```

Then generate the frequency variable ω with the following commands:

```
w=linspace(-pi,pi,T+1);
w=w(2:end);
```

i. Plot the magnitude $|X(j\omega)|$ of the Fourier transform of $x(t)$ as a function of ω by typing

```
figure; plot(w,abs(X_jw));
xlabel('w');
ylabel('|X(jw)|');
```

Here, it is important to remember that we should use the `abs` function to take the magnitude of the Fourier transform. The signal $x(t)$ is known to contain a low-frequency component corresponding to the original clean speech signal, whose magnitude decays as the frequency increases. On the other hand, the noise component of $x(t)$ has a strong magnitude and a flat spectrum that covers high-frequencies. By examining the spectrum $|X(j\omega)|$ of the noisy signal, can you guess within which frequency range $w \in [-B, B]$ the original speech signal lies?

ii. Plot also the real part $Re\{X(j\omega)\}$ and the imaginary part $Im\{X(j\omega)\}$ of the spectrum of the noisy signal $x(t)$. Noticing that $x(t)$ is a real signal, what kind of symmetry do we expect $Re\{X(j\omega)\}$ and $Im\{X(j\omega)\}$ to have? Verify this by examining your plots and comment.

(c) Our purpose now is to design a low-pass filter that will remove the high-frequency noise component of $x(t)$, so that we obtain back the clean speech signal. The low-pass filtering is done by passing the signal $x(t)$ through an LTI system with impulse response $h(t)$, so that we obtain the clean signal $y(t)$ as

$$y(t) = (x * h)(t).$$

As we have learned in class, the input signal $x(t)$ and the output signal $y(t)$ are related in the frequency domain as

$$Y(j\omega) = X(j\omega)H(j\omega)$$

where $H(j\omega)$ and $Y(j\omega)$ are the Fourier transforms of $h(t)$ and $y(t)$.
We will use an ideal low-pass filter $H(j\omega)$ given by

$$H(j\omega) = \begin{cases} 1, & \text{if } |\omega| < B \\ 0, & \text{otherwise .} \end{cases}$$

Choose the cut-off frequency B of your low-pass filter based on the guess you made in part (a). You can generate your ideal low-pass filter in the frequency domain using the commands:

```
H_jw=(abs(w)<B);
H_jw=double(H_jw);
```

Plot the spectrum $H(j\omega)$ of the low-pass filter.

(d) We will now remove the noise in the noisy signal $x(t)$ by passing it through the ideal low-pass filter we generated in part (c). Perform the filtering operation $Y(j\omega) = X(j\omega)H(j\omega)$ in the frequency domain as

```
Y_jw=H_jw.*X_jw;
```

Plot the magnitude $|Y(j\omega)|$ of the spectrum of the output signal $y(t)$ against the frequency variable ω. Try different values for the cut-off frequency B and observe its effect on the spectrum of the output signal.

(e) We can finally obtain the time-domain representation of the output signal $y(t)$ by taking the inverse Fourier transform of $Y(j\omega)$. The following commands give $y(t)$ by taking the inverse Fourier transform:

```
y=ifft(ifftshift(Y_jw));
```

Plot the signal $y(t)$. (Note that you might observe a small imaginary part due to numerical errors, so make sure to plot only the real part of the signal.) Compare the output signal $y(t)$ to the noisy input signal $x(t)$ and comment on the results.

(f) Now it is time to listen to our output signal $y(t)$. Type the following command:

```
sound(real(y),44100);
```

If you have chosen B properly, you should now be able to recognize the sentence in the speech signal. What does it say?