EE 301 Signals & Systems
Homework 2

Solutions

Problem 1:

$$a_k = \frac{1}{T_0}\int_0^{T_0} \mathrm{x}(t)\, e^{-jk\omega_0 t} dt = \frac{1}{2}\int_0^{\frac{1}{2}} \cos(\pi t)\, e^{-jk(\frac{2\pi}{2})t} dt$$

$$= \frac{1}{2}\int_0^{\frac{1}{2}} \frac{1}{2}\left(e^{j\pi t} + e^{-j\pi t}\right) e^{-jk\pi t} dt$$

$$= \frac{1}{4}\int_0^{\frac{1}{2}} e^{-j(k-1)\pi t} dt + \frac{1}{4}\int_0^{\frac{1}{2}} e^{-j(k+1)\pi t} dt$$

$$= \frac{1}{4}\left(\frac{e^{-j(k-1)\pi t}}{-j(k-1)\pi}\right)\Big|_{t=0}^{1/2} + \frac{1}{4}\left(\frac{e^{-j(k+1)\pi t}}{-j(k+1)\pi}\right)\Big|_{t=0}^{1/2}$$

$$= \frac{1}{4}\left(\frac{e^{-\frac{j(k-1)\pi}{2}} - 1}{-j(k-1)\pi} + \frac{e^{-\frac{j(k+1)\pi}{2}} - 1}{-j(k+1)\pi}\right)$$

$$= \frac{1}{4}\left(e^{-\frac{j(k-1)\pi}{4}} \frac{e^{-\frac{j(k-1)\pi}{4}} - e^{+\frac{j(k-1)\pi}{4}}}{-j(k-1)\pi} + e^{-\frac{j(k+1)\pi}{4}} \frac{e^{-\frac{j(k+1)\pi}{4}} - e^{+\frac{j(k+1)\pi}{4}}}{-j(k+1)\pi}\right)$$

$$= \frac{1}{4}\left(e^{-\frac{j(k-1)\pi}{4}} \frac{-2j\sin\left((k-1)\frac{\pi}{4}\right)}{-j(k-1)\pi} + e^{-\frac{j(k+1)\pi}{4}} \frac{-2j\sin\left((k+1)\frac{\pi}{4}\right)}{-j(k+1)\pi}\right)$$

$$= \frac{1}{4}\left(e^{-\frac{j(k-1)\pi}{4}} \frac{2\sin\left((k-1)\frac{\pi}{4}\right)}{4(k-1)\frac{\pi}{4}} + e^{-\frac{j(k+1)\pi}{4}} \frac{2\sin\left((k+1)\frac{\pi}{4}\right)}{4(k+1)\frac{\pi}{4}}\right)$$

$$= \frac{1}{4}\left(e^{-\frac{j(k-1)\pi}{4}} \frac{\mathrm{sinc}\left(\frac{(k-1)}{4}\right)}{2} + e^{-\frac{j(k+1)\pi}{4}} \frac{\mathrm{sinc}\left(\frac{(k+1)}{4}\right)}{2}\right)$$

$$a_k = \frac{1}{8} e^{-\frac{j(k-1)\pi}{4}} \mathrm{sinc}\left(\frac{(k-1)}{4}\right) + \frac{1}{8} e^{-\frac{j(k+1)\pi}{4}} \mathrm{sinc}\left(\frac{(k+1)}{4}\right)$$

Note: Consider the periodic signal $y(t)$ defined as

$$y(t) = \begin{cases} 1\ for\ 0 \le t \le \frac{1}{2} \\ 0\ for \frac{1}{2} \le t \le 2 \end{cases}$$

over a single period. Notice that $x(t) = \cos(\pi t) y(t)$. Also, it can be shown that the Fourier series coefficients of y(t) are

$$b_k = \frac{1}{4} e^{-\frac{jk\pi}{4}} sinc\left(\frac{k}{4}\right)$$

Then, recall that $\cos(\pi t) = \frac{1}{2} e^{j\pi t} + \frac{1}{2} e^{-j\pi t}$ and it can be considered as a periodic signal with fundamental frequency $\omega_0 = \pi$. Therefore, its Fourier series coefficients are

$$c_k = \begin{cases} \frac{1}{2}\ for\ k = 1\ and\ k = -1 \\ 0\ otherwise \end{cases}$$

Observe that $a_k = b_k \star c_k$, because $x(t) = \cos(\pi t) y(t)$.

Problem 2:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_0^T \mathrm{x}(t)\, e^{-jk\omega_0 t} dt$$

$$\omega_0 = \frac{2\pi}{T}$$

a)

$$x(t - t_0) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (t - t_0)} = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$$x(t - t_0) \overset{FS}{\leftrightarrow} b_k = e^{-jk\omega_0 t_0} a_k$$

$$x(t - t_0) + x(t + t_0) \overset{FS}{\leftrightarrow} c_k$$

$$c_k = e^{-jk\omega_0 t_0} a_k + e^{+jk\omega_0 t_0} a_k$$

$$c_k = 2\cos(k\omega_0 t_0) a_k$$

b)

$$\frac{x(t) - x\left(t - \frac{T}{2}\right)}{2} \overset{FS}{\leftrightarrow} b_k$$

$$b_k = \frac{a_k}{2} - \frac{e^{-\frac{jk\omega_0 T}{2}} a_k}{2}$$

$$b_k = \frac{a_k}{2} - \frac{e^{-jk\pi} a_k}{2} = \frac{a_k}{2} - \frac{(-1)^k a_k}{2} = \frac{(1 - (-1)^k)}{2} a_k$$

$$b_k = \begin{cases} a_k & k \text{ is odd} \\ 0 & k \text{ is even} \end{cases}$$

c)

$$Ev\{x(t)\} = \frac{x(t) + x(-t)}{2}$$

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(-t)} = \sum_{k=-\infty}^{\infty} a_k e^{j(-k)\omega_0 t}$$

$$x(-t) \overset{FS}{\leftrightarrow} b_k = a_{-k}$$

$$Ev\{x(t)\} \overset{FS}{\leftrightarrow} c_k = \frac{a_k + a_{-k}}{2}$$

d)

$$Re\{x(t)\} = \frac{x(t) + x^*(t)}{2}$$

$$x^*(t) = \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right)^* = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$

$$x^*(t) \overset{FS}{\leftrightarrow} b_k = a_{-k}^*$$

$$Re\{x(t)\} \overset{FS}{\leftrightarrow} c_k = \frac{a_k + a_{-k}^*}{2}$$

e)

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right) = \sum_{k=-\infty}^{\infty} a_k \frac{d}{dt}\left(e^{jk\omega_0 t}\right) = \sum_{k=-\infty}^{\infty} jk\omega_0 a_k e^{jk\omega_0 t}$$

$$\frac{dx(t)}{dt} \overset{FS}{\leftrightarrow} b_k = jk\omega_0 a_k$$

$$\frac{d^2x(t)}{dt^2} = \frac{d}{dt}\left(\frac{dx(t)}{dt}\right) \overset{FS}{\leftrightarrow} c_k = jk\omega_0 b_k = -k^2\omega_0^2 a_k$$

f)

$$b_k = \frac{1}{T/3}\int_0^{T/3} \mathrm{x}(3t-1)\, e^{-jk3\omega_0 t} dt$$

Let $t' = 3t - 1$.

$$b_k = \frac{1}{T/3}\int_{-1}^{T-1} \mathrm{x}(t')\, e^{-jk3\omega_0\left(\frac{t'+1}{3}\right)} \left(\frac{1}{3}dt'\right)$$

$$b_k = e^{-jk\omega_0}\frac{1}{T}\int_{-1}^{T-1} \mathrm{x}(t')\, e^{-jk\omega_0 t'} dt'$$

$$b_k = e^{-jk\omega_0} a_k$$

Note that the fundamental frequency of the resultant signal is $3\omega_0$.

g)

$$x^2(t) = x(t)x(t) = \left(\sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}\right)\left(\sum_{m=-\infty}^{\infty} a_m e^{jm\omega_0 t}\right)$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_n e^{jn\omega_0 t} a_m e^{jm\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_n a_m e^{j(n+m)\omega_0 t}$$

$$k = n + m, \qquad m \overset{\Delta}{\rightarrow} k - n$$

$$= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_n a_{k-n} e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} a_n a_{k-n}\right) e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$$b_k = \left(\sum_{n=-\infty}^{\infty} a_n a_{k-n}\right)$$

$$b_k = a_k \star a_k$$

Problem 3:

I. $x(t) = a_{-1}e^{-j\omega_0 t} + a_1 e^{j\omega_0 t}$

II. $a_{-1} = a_1^*, \quad x(t) = a_1^* e^{-j\omega_0 t} + a_1 e^{j\omega_0 t}$

III. $a_{-1} = a_1, \quad x(t) = a_1 e^{-j\omega_0 t} + a_1 e^{j\omega_0 t} = a_1\left(e^{-j\omega_0 t} + e^{j\omega_0 t}\right) = 2a_1\cos(\omega_0 t)$

IV. $\omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}, \quad x(t) = 2a_1\cos\left(\frac{\pi}{3}t\right)$

V. $-x(t-3) = -2a_1\cos\left(\frac{\pi}{3}(t-3)\right) = -2a_1\cos\left(\frac{\pi}{3}t - \pi\right) = 2a_1\cos\left(\frac{\pi}{3}t\right) = x(t)$

VI. $\frac{1}{6}\int_{-3}^{3}|x(t)|^2 dt = \sum_{k=-\infty}^{\infty}|a_k|^2 = |a_{-1}|^2 + |a_1|^2 = |a_1|^2 + |a_1|^2 = 2|a_1|^2 = \frac{1}{2}$

$$|a_1| = \frac{1}{2}$$

Since it is given that a_1 is a positive and real number,

$$a_1 = \frac{1}{2}$$

$$x(t) = 2a_1\cos\left(\frac{\pi}{3}t\right) = \cos\left(\frac{\pi}{3}t\right)$$

$$A = 1, \qquad B = \frac{\pi}{3}, \qquad C = 0$$

Problem 4:

a)

$$z(t) = x(t)y(t) = \left(\sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}\right)\left(\sum_{m=-\infty}^{\infty} b_m e^{jm\omega_0 t}\right)$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_n e^{jn\omega_0 t} b_m e^{jm\omega_0 t} = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_n b_m e^{j(n+m)\omega_0 t}$$

$$k = n + m, \qquad m \overset{\Delta}{\rightarrow} k - n$$

$$z(t) = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_n b_{k-n} e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} a_n b_{k-n}\right) e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}$$

$$c_k = a_k \star b_k$$

b)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$y(t) = x^*(t) = \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right)^* = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$

$$b_k = a_{-k}^*$$

$$c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n} = \sum_{n=-\infty}^{\infty} a_n a_{n-k}^*$$

$$\frac{1}{T_0} \int_0^{T_0} |\mathrm{x}(t)|^2 dt = \frac{1}{T_0} \int_0^{T_0} x(t) x^*(t) dt = \frac{1}{T_0} \int_0^{T_0} x(t) y(t) dt = \frac{1}{T_0} \int_0^{T_0} \left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right) dt$$

$$\frac{1}{T_0} \int_0^{T_0} |\mathrm{x}(t)|^2 dt = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \left(c_k \int_0^{T_0} e^{jk\omega_0 t} dt \right)$$

Calculate the integral $\int_0^{T_0} e^{jk\omega_0 t} dt$;

$$\int_0^{T_0} e^{jk\omega_0 t} dt = \frac{e^{jk\omega_0 T_0} - 1}{jk\omega_0} = \frac{e^{jk2\pi} - 1}{jk\omega_0} = \frac{1-1}{jk\omega_0} = 0, \qquad for\ k \neq 0$$

If $k = 0$;

$$\int_0^{T_0} e^{jk\omega_0 t} dt = \int_0^{T_0} (1) dt = T_0, \qquad for\ k = 0$$

Then;

$$\frac{1}{T_0} \int_0^{T_0} |\mathrm{x}(t)|^2 dt = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \left(c_k \int_0^{T_0} e^{jk\omega_0 t} dt \right) = \frac{1}{T_0} \mathrm{c}_0 T_0 = c_0 = \sum_{n=-\infty}^{\infty} a_n a_{n-0}^*$$

$$\frac{1}{T_0} \int_0^{T_0} |\mathrm{x}(t)|^2 dt = \sum_{n=-\infty}^{\infty} |a_n|^2$$

MATLAB Assignment

Solutions

Part 1:

ii)

$$a_k = \frac{1}{T}\int_0^T x_p(t)\, e^{-jk\frac{2\pi}{T}t} dt$$

$$a_k = \frac{1}{T}\int_0^2 e^{-jk\frac{2\pi}{5}t} dt = \frac{1}{5}\frac{e^{-jk\frac{2\pi}{5}t}}{-jk\,\frac{2\pi}{5}}\Big|_{t=0}^{2} = \frac{1}{5}\frac{e^{-jk\frac{2\pi}{5}2}-1}{-jk\,\frac{2\pi}{5}}$$

$$= \frac{e^{-jk\frac{2\pi}{5}}\left(e^{-jk\frac{2\pi}{5}} - e^{+jk\frac{2\pi}{5}}\right)}{-jk2\pi} = \frac{e^{-jk\frac{2\pi}{5}}\left(-2jsin\left(k\,\frac{2\pi}{5}\right)\right)}{-jk2\pi} = 2e^{-jk\frac{2\pi}{5}}\frac{sin\left(k\,\frac{2\pi}{5}\right)}{k2\pi}$$

$$a_k = \frac{2}{5}e^{-jk\frac{2\pi}{5}}sinc(k\,\frac{2}{5})$$

$$a_0 = 0.4, \qquad a_1 = 0.0935 - \,j0.2879, \qquad a_2 = -0.0757\, + \,j0.0550$$

```
a(1:3)

ans =

   0.4000 + 0.0000i   0.0937 - 0.2879i  -0.0756 - 0.0551i
```

They are almost the same.

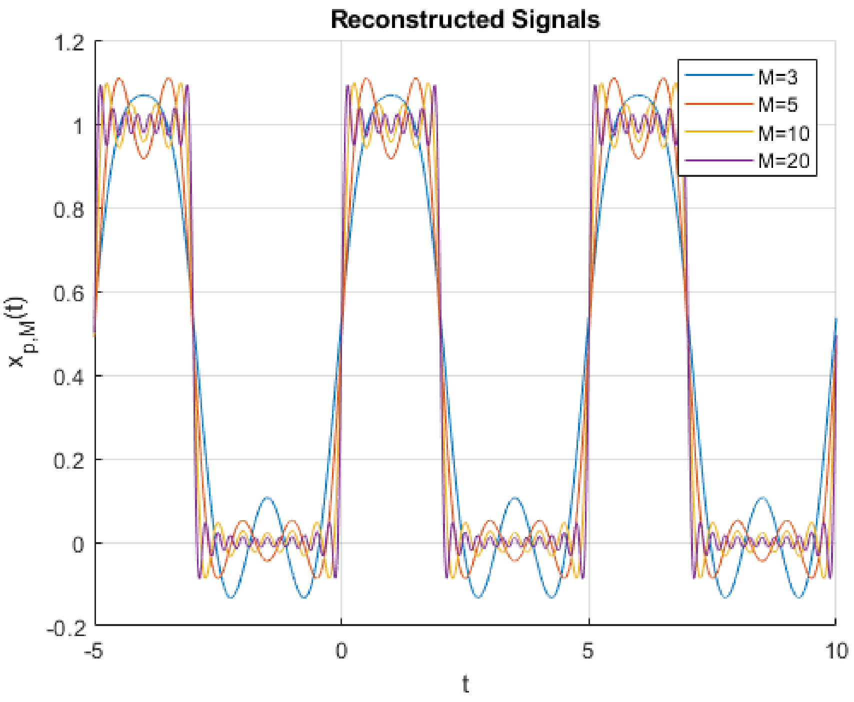
iii)

$$x_p(t) = \sum_{k=-\infty}^{\infty} a_k\, \mathrm{e}^{\mathrm{jk}\frac{2\pi}{T}t}$$

$$x_{p,M}(t) = \sum_{k=-M}^{M} a_k\, \mathrm{e}^{\mathrm{jk}\frac{2\pi}{T}t}$$

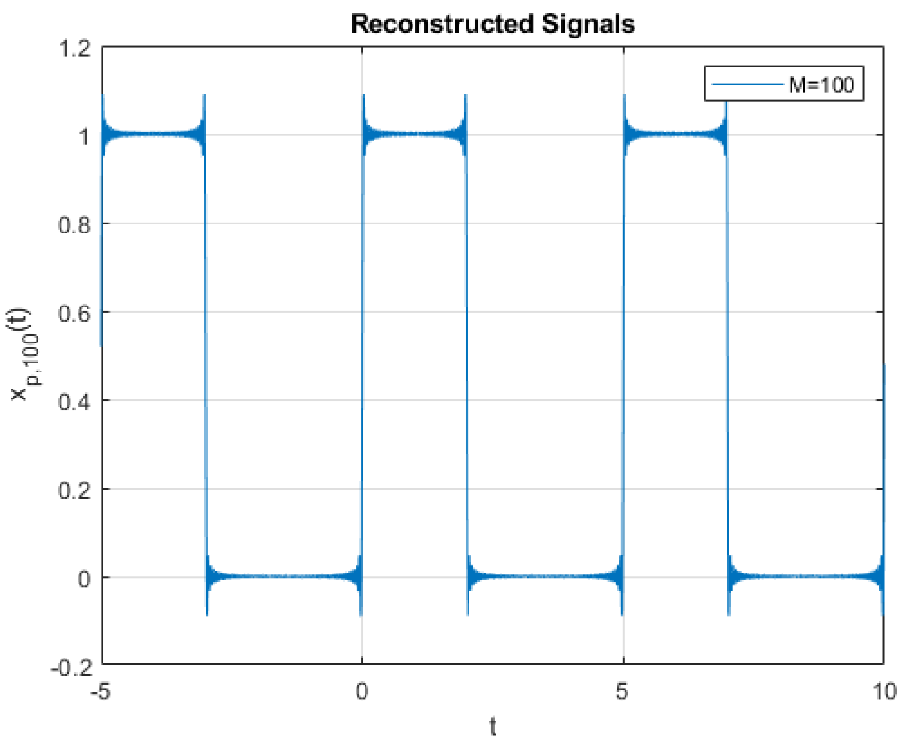
```
period=5;
Length=2;
signal_type=1;
figureonoff=1;
[a_k , x_p]=signal_generator(signal_type,period,Length,figureonoff);
dt=0.001;
n=-period:dt:(2*period-dt);
M=3;
x_pM = a_k(1)*ones(1,length(n));
for k=1:M
x_pM = x_pM + a_k(k+1)*exp(1i*k*(2*pi/period)*n) ...
   ... + conj(a_k(k+1))*exp(-1i*k*(2*pi/period)*n);
end
```

iv)

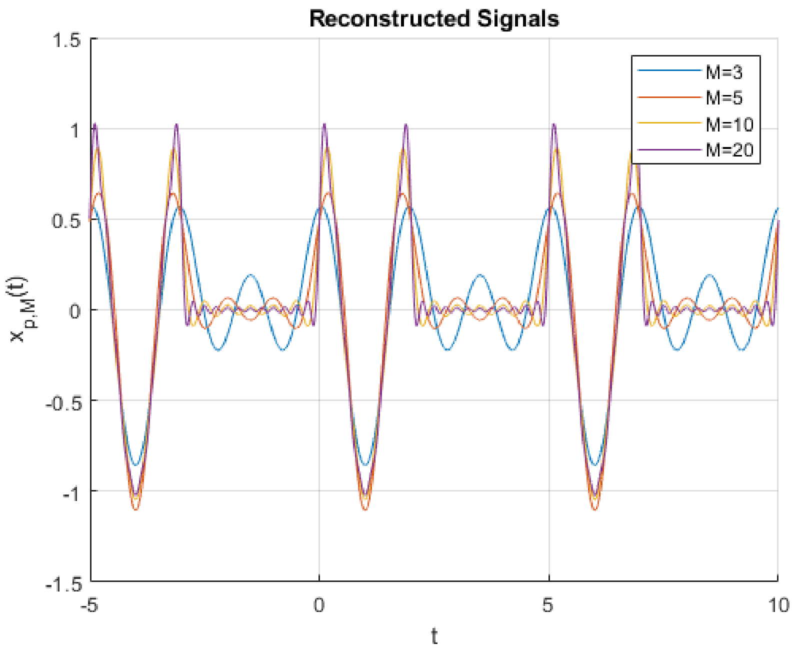


$x_{p,M}(t)$ converges to $x(t)$ as M increases.

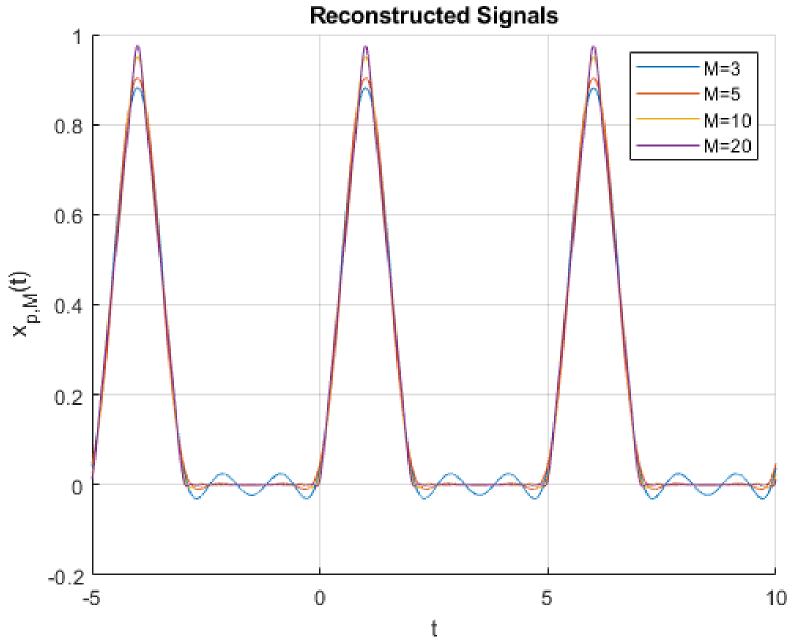
v)



Part 2:



Part 3:



Average power of the periodic signal is 0.1333. Due to Parseval's relation,

$$\frac{1}{T}\int_0^T \left|x_p(t)\right|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 = 0.1333$$

However,

$$\sum_{k=-M}^{M} |a_k|^2 \,/ \sum_{k=-\infty}^{\infty} |a_k|^2$$

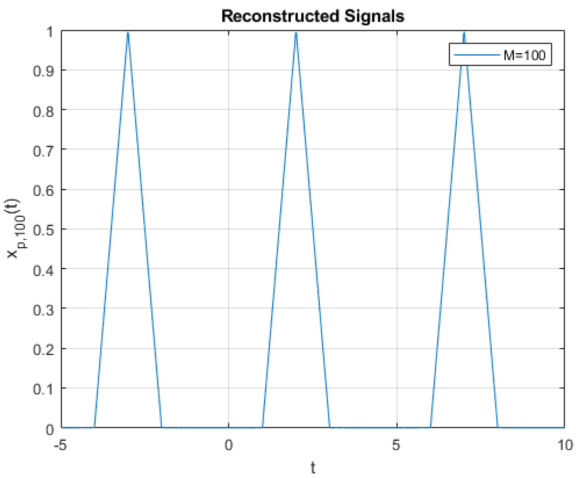
is 0.9953, 0.9971, 0.9996, 0.9999 for M=3,5,10 and 20, respectively. Therefore, further increase in the convergence is not expected.

Part 4:

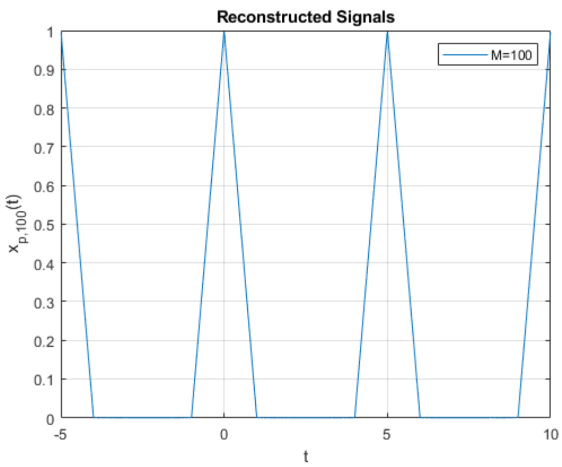
$$b_k = a_k \mathrm{e}^{-\mathrm{j}k\frac{2\pi}{T}\Delta t}$$

```
b_k =a_k.*exp(-1i*(0:100)*(2*pi/period)*delay);
```

$\Delta t = 1$



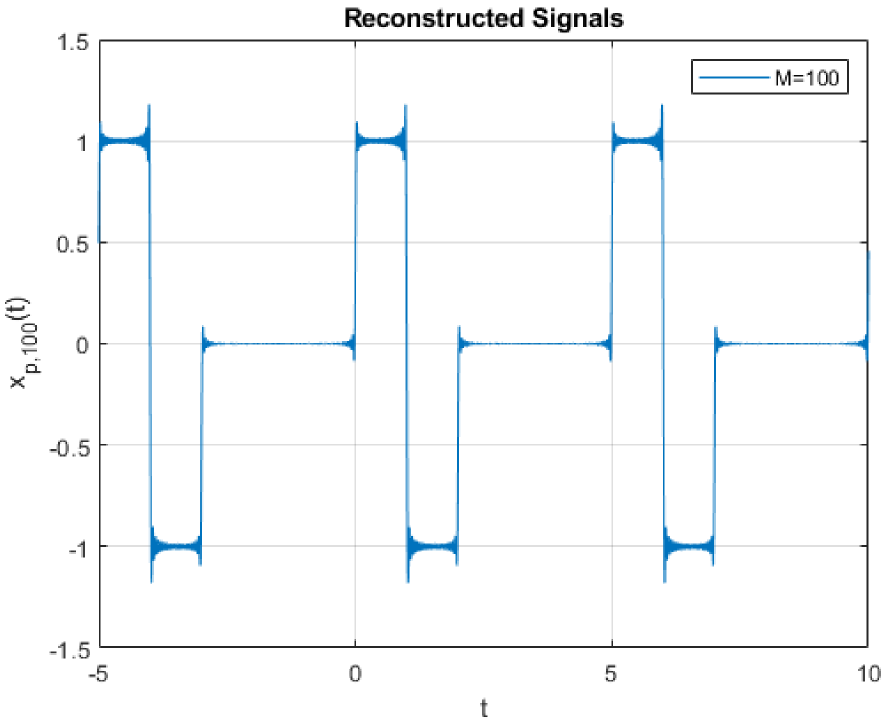
Δt=4



Part 5:

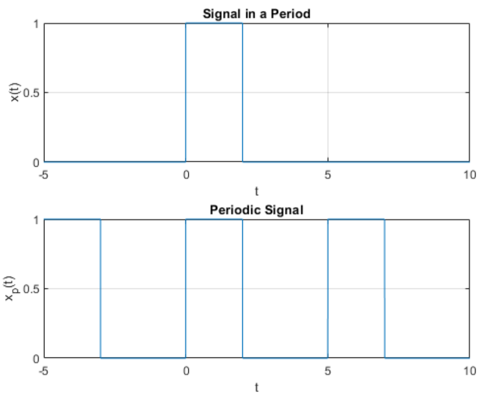
$$c_k = jk\frac{2\pi}{T}a_k$$

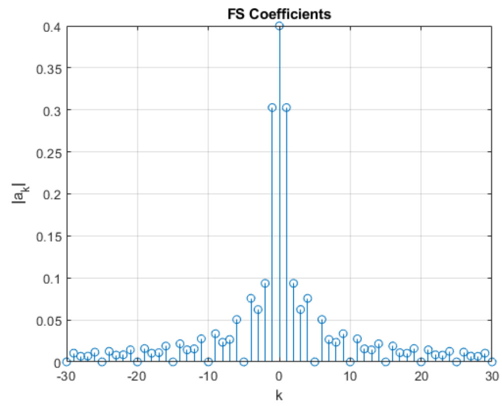
```
c_k =a_k.*(1i*(0:100)*(2*pi/period));
```



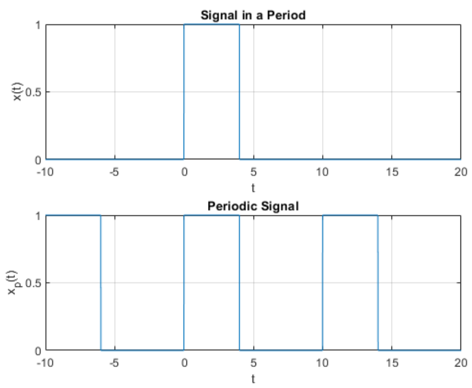
Part 6:

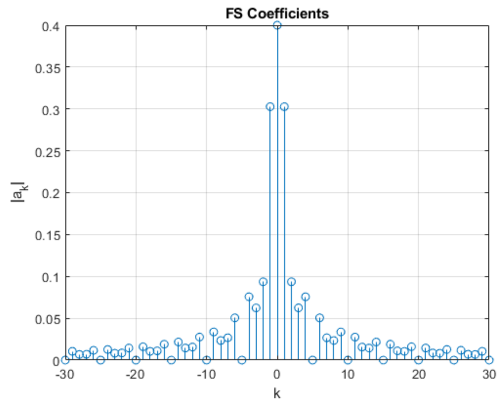
$L = 2$ and $T = 5$.





$L = 4$ and $T = 10$.





Signals are different but the FS coefficients are the same. It is the time scaling property of CTFS. However, note that the fundamental period is different. Therefore, the synthesis equation changes.