Bisection (Dichotomy) Method

Numerical Analysis

Isaac Amornortey Yowetu

NIMS-Ghana

August 29, 2020

Outline

Background of Bisection Method

Introduction

- This method is also know as the Interval halving method.
- It is method based on division of halves.
- It is one of the bracketing method in finding roots on a nonlinear equations.
- It is noted to be based on Bolzano's theorem for continuous functions.

Theorem (Bolzano)

If a function $f(x)$ is continuous on an interval $[a, b]$ and $f(a) \cdot f(b) < 0$, then a value $c \in (a, b)$ exists for which $f(c) = 0$.

Graphical Example

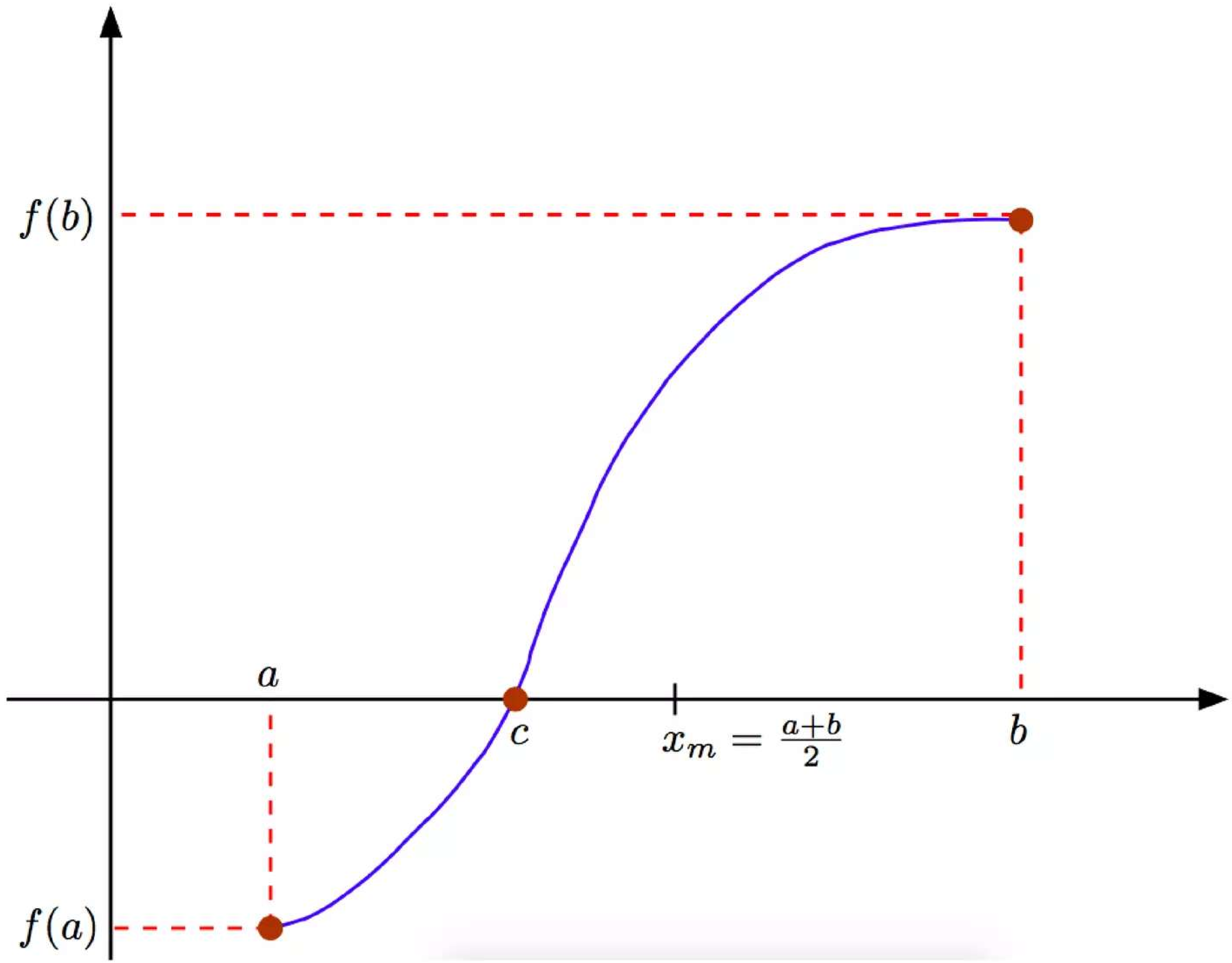

Figure: A Graphic Example of Bisection Method

Bisection Algorithm

if $f(a) \cdot f(b) < 0$:
  root exists
else:
  root doesn't exist

Iteration Processes when root exists

1. Let $c = \frac{a+b}{2}$
2. If $f(c) = 0$, stop! c is the root.
3. if $f(a) \cdot f(c) < 0$:
   set $b \leftarrow c$
4. else if:
   set $a \leftarrow c$
5. Go to the beginning and repeat till convergence.

Stopping Criteria

Number of Iteration Formula

$$n \geq \frac{\log(b-a) - \log \epsilon}{\log(2)}$$

suppose that $|c_n - c| \leq \epsilon$ where:

- c_n is the approximate root
- c is the actual root
- ϵ is certain tolerance or error.

Graphical Example

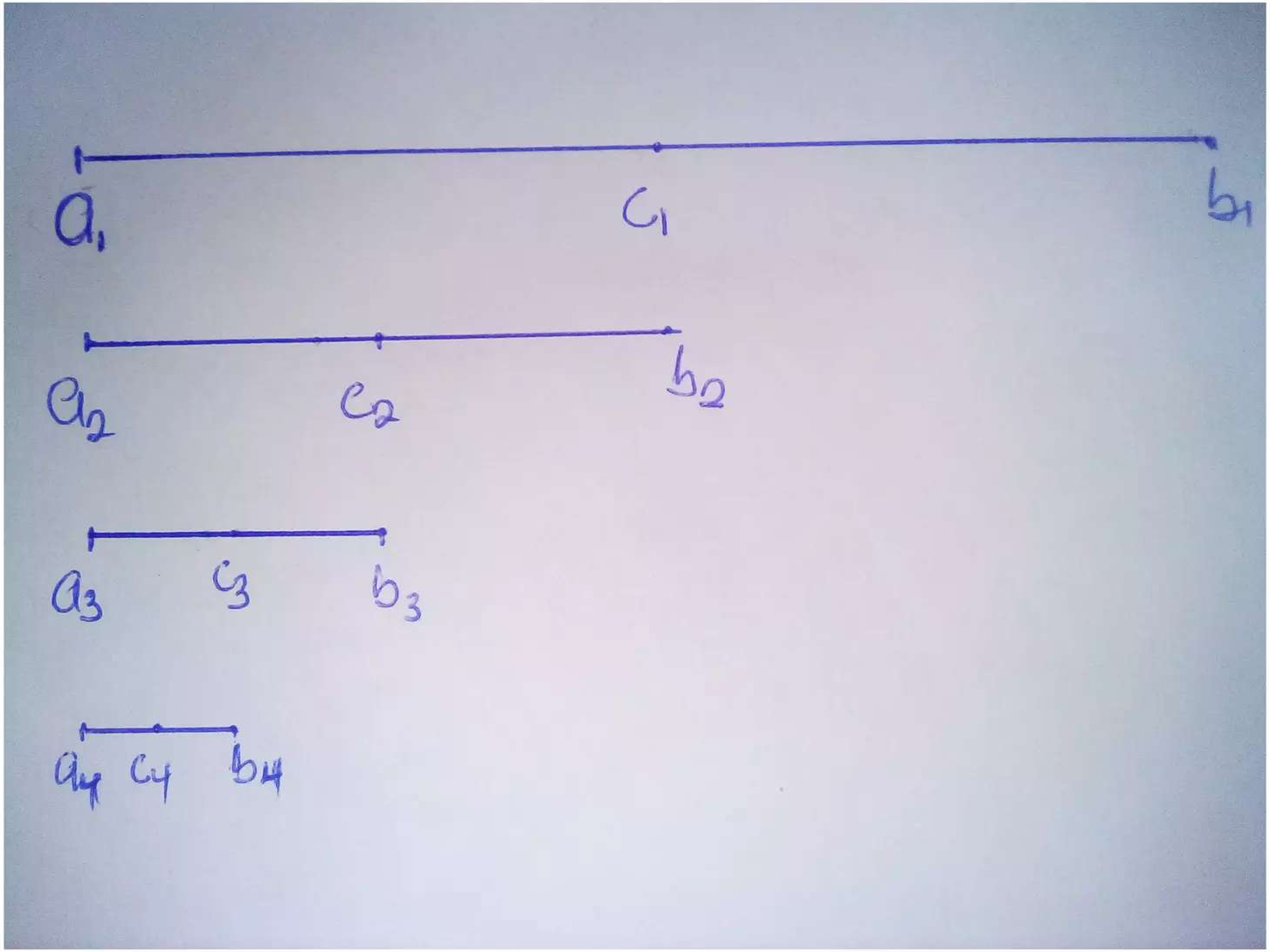

Figure: A Graphic Example of Bisection number of iteration

Derivation of the number of iteration for convergence.

$$
\begin{align}
|c - c_1| &\leq \frac{b_1 - a_1}{2} = \frac{1}{2}(b_1 - a_1) \tag{1}\\
|c - c_2| &\leq \frac{b_2 - a_2}{2} = \frac{1}{2}(b_2 - a_2) \tag{2}\\
&= \frac{1}{2}\left(\frac{b_1 + a_1}{2} - a_1\right) \tag{3}\\
&= \frac{1}{2^2}(b_1 - a_1) \tag{4}\\
|c - c_3| &\leq \frac{b_3 - a_3}{2} = \frac{1}{2}(b_3 - a_3) \tag{5}\\
&= \frac{1}{2}\left(\frac{b_2 + a_2}{2} - a_2\right) \tag{6}
\end{align}
$$

Derivation Continues...

$$
\begin{aligned}
&= \frac{1}{2^2}(b_2 - a_2) && (7) \\
&= \frac{1}{2^2}(\frac{b_1 + a_1}{2} - a_1) && (8) \\
&= \frac{1}{2^3}(b_1 - a_1) && (9) \\
&\vdots && (10) \\
|c - c_n| &\le \frac{1}{2^n}(b_1 - a_1) && (11)
\end{aligned}
$$

Finding n the number of iteration

$$
\begin{aligned}
\epsilon \geq \frac{b_1 - a_1}{2^n} &\geq |c_n - c| \\
2^n \cdot \epsilon &\geq (b_1 - a_1) \\
\log(2^n \cdot \epsilon) &\geq \log(b_1 - a_1) \\
\log(2^n) + \log(\epsilon) &\geq \log(b_1 - a_1) \\
n \cdot \log(2) &\geq \log(b_1 - a_1) - \log(\epsilon) \\
n &\geq \frac{\log(b_1 - a_1) - \log(\epsilon)}{\log(2)}
\end{aligned}
$$

Application of Bisection Method

Example 1

Find a root of $xe^x = 1$ on $I = [0, 1]$ using Bisection method.

Considering our $f(x) = xe^x - 1 = 0$

Solution

$f(0) = 0 \cdot e^0 - 1 = -1$
$f(1) = 1 \cdot e^1 - 1 = 1.7183$
$f(0) \cdot f(1) < 0$, hence $c \in [0, 1]$.

$$c_1 = \frac{1+0}{2} = 0.5 \tag{12}$$

$$f(c_1) = -0.1756 \tag{13}$$

choose $[a_2, b_2] = [0.5, 1]$

Solution Continue...

$$c_2 = \frac{0.5 + 1}{2} = 0.75 \tag{14}$$

$$f(c_2) = 0.5878 \tag{15}$$

choose $[a_3, b_3] = [0.5, 0.75]$

$$c_3 = \frac{0.5 + 0.75}{2} = 0.625 \tag{16}$$

$$f(c_3) = 0.1677 \tag{17}$$

choose $[a_4, b_4] = [0.5, 0.625]$

Solution summary

iteration	a	b	c	f(c)
1	0	1	0.5	-0.1756
2	0.5	1	0.75	0.5878
3	0.5	0.75	0.625	0.1677
4	0.5	0.625	0.5625	-0.0128
5	0.5625	0.625	0.5938	0.0751
6	0.5625	0.5938	0.5781	0.0306
7	0.5625	0.5781	0.5703	0.0088
8	0.5625	0.5703	0.5664	-0.0020
9	0.5664	0.5703	0.5684	0.0034
10	0.5664	0.5684	0.5684	0.0007

Conclusion: The approximate solution $c_n = 0.56714$

Question: Determining the number of iterations

Problem

One root of the equation $e^x - 3x^2 = 0$ lies in the interval (3,4). Find the least number of iterations of the bisection method so that $|\epsilon| < 10^{-3}$.

Solution

$$n \geq \frac{\log(4-3) - \log(10^{-3})}{\log(2)} = 9.9658$$

$\therefore n \approxeq 10$

Question: Determining the number of iterations

Problem

One root of the equation $xe^x - 1 = 0$ lies in the interval (0,1). Find the least number of iterations of the bisection method so that $|\epsilon| < 10^{-5}$.

Solution

$$n \geq \frac{\log(1-0) - \log(10^{-5})}{\log(2)} = 16.6096$$

$\therefore n \approxeq 17$

End

THANK YOU

Regula Falsi (False Position) Method

Numerical Analysis

Isaac Amornortey Yowetu

NIMS-Ghana

September 12, 2020

Application of Regula Falsi Method

Question

Find a root of $xe^x = 1$ on $I = [0, 1]$ using Regula Falsi method.

Outline

Background of Regula Falsi Method

Introduction

- It is one of the bracketing iterative methods in finding roots of a nonlinear equations.
- The approximated root is found by the use of straight lines or slopes.
- It is also noted to be based on Bolzano's theorem for continuous functions.

Theorem (Bolzano)

If a function $f(x)$ is continuous on an interval $[a, b]$ and $f(a) \cdot f(b) < 0$, then a value $c \in (a, b)$ exists for which $f(c) = 0$.

Graphical Example

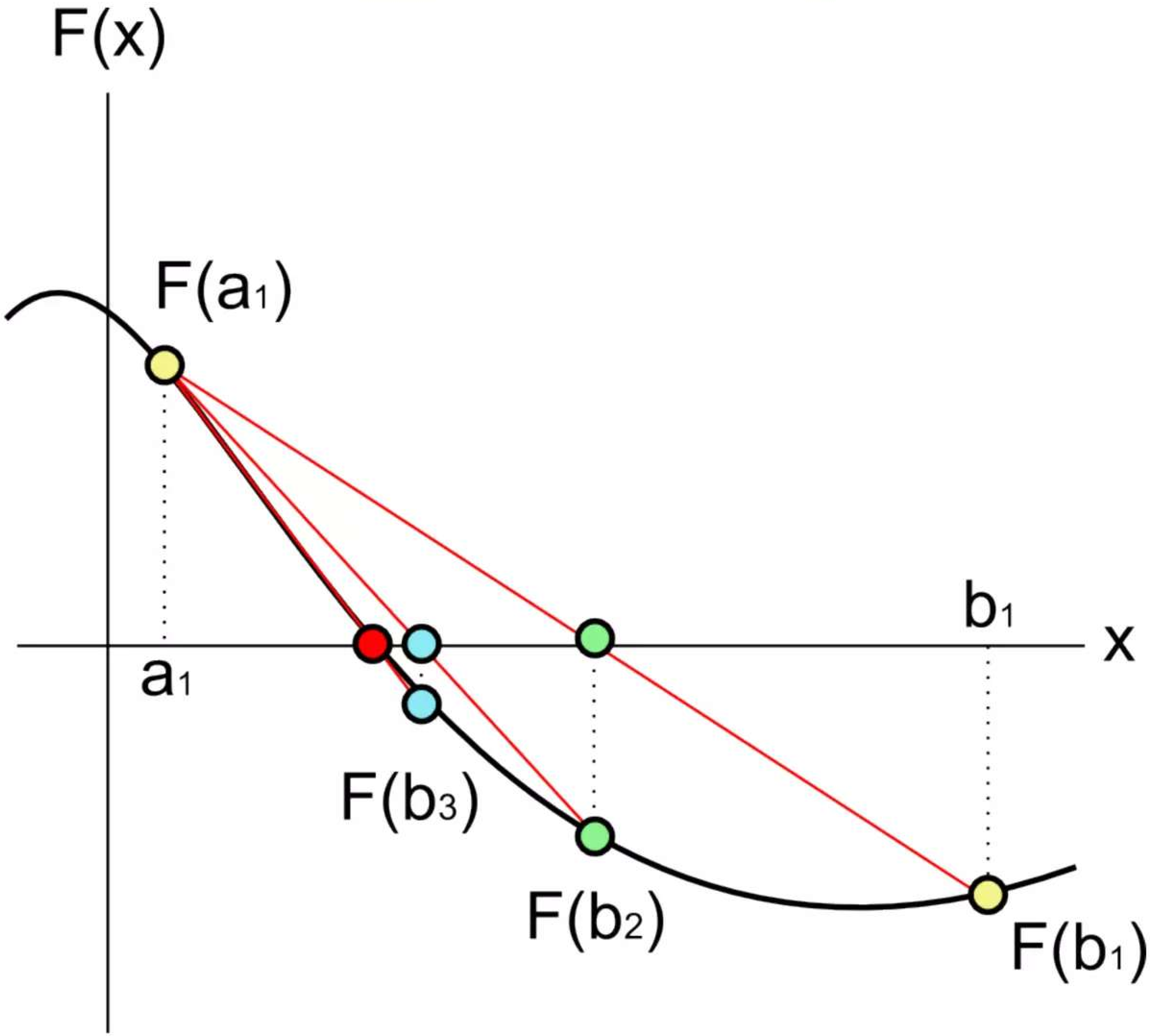

Figure: A Graphic Example of Regula Falsi Method

Derivation of the Regula Falsi Method

$$\frac{\Delta y}{\Delta x} = \frac{f(a) - 0}{a - c} = \frac{f(b) - 0}{b - c} \tag{1}$$

OR

$$\frac{\Delta y}{\Delta x} = \frac{f(a) - f(b)}{a - b} = \frac{f(a) - 0}{a - c} \tag{2}$$

OR

$$\frac{\Delta y}{\Delta x} = \frac{f(a) - f(b)}{a - b} = \frac{f(b) - 0}{b - c} \tag{3}$$

Derivation Continues...

Using any of the 3 approaches, can help us derived Regula Falsi Method. Using eqn(1):

$$\frac{f(a)-0}{a-c} = \frac{f(b)-0}{b-c} \tag{4}$$

$$\frac{f(a)}{a-c} = \frac{f(b)}{b-c} \tag{5}$$

$$f(a)(b-c) = f(b)(a-c) \tag{6}$$

$$b \cdot f(a) - c \cdot f(a) = a \cdot f(b) - c \cdot f(b) \tag{7}$$

$$c \cdot f(b) - c \cdot f(a) = a \cdot f(b) - b \cdot f(a) \tag{8}$$

$$c = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)} \tag{9}$$

Regula Falsi Algorithm

if $f(a) \cdot f(b) < 0$:

  root exists

else:

  root doesn't exist

Iteration Processes when root exists

1. Let $c = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)}$
2. If $f(c) = 0$, stop! c is the root.
3. if $f(a) \cdot f(c) < 0$:

   set $b \leftarrow c$
4. else if:

   set $a \leftarrow c$
5. Go to the beginning and repeat till convergence.

Application of Regula Falsi Method

Example 1

Find a root of $xe^x = 1$ on $I = [0, 1]$ using Regula Falsi method.

Solution

Considering our $f(x) = xe^x - 1 = 0$ and $a_1 = 0, b_1 = 1$
$f(0) = 0 \cdot e^0 - 1 = -1$
$f(1) = 1 \cdot e^1 - 1 = 1.7183$
$f(0) \cdot f(1) < 0$, hence $c \in [0, 1]$.

$$c_1 = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)} = 0.3679 \tag{10}$$

$$f(c_1) = -0.4685 \tag{11}$$

choose $[a_2, b_2] = [0.3679, 1]$

Solution Continue...

$$c_2 = \frac{a_2 \cdot f(b_2) - b_2 \cdot f(a_2)}{f(b_2) - f(a_2)} = 0.5033 \tag{12}$$

$$f(c_2) = -0.1674 \tag{13}$$

choose $[a_3, b_3] = [0.5033, 1]$

$$c_3 = \frac{a_3 \cdot f(b_3) - b_3 \cdot f(a_3)}{f(b_3) - f(a_3)} = 0.5474 \tag{14}$$

$$f(c_3) = -0.0536 \tag{15}$$

choose $[a_4, b_4] = [0.5474, 1]$

Solution summary

iteration	a	b	c	f(c)
1	0	1	0.3679	-0.4685
2	0.3679	1	0.5033	-0.1674
3	0.5033	1	0.5474	-0.0536
4	0.5474	1	0.5611	-0.0166
5	0.5611	1	0.5666	-0.0051
6	0.5666	1	0.5670	-0.0015
7	0.5670	1	0.5671	-0.0005
8	0.5671	1	0.5671	-0.0001
9	0.5671	1	0.5671	-0.00004
10	0.5671	1	0.5671	-0.00001

Conclusion: The approximate solution $c_n = 0.5671$

End

THANK YOU