Overview of Newton Raphson Method

Numerical Analysis

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June 24, 2020

Background of Newton Raphson Iteration

This iterative method Newton Raphson is said to be named after Isaac Newton and Joseph Raphson. It is one of the opened Iterative method that helps in a quick way of finding the solutions of non-linear equations. This is achieved by producing successively better approximations to the actual roots of a real-valued functions $f(x) = 0$.
This iterative method is applicable when $f(x)$ and $f'(x)$ are both continuous and differentiable. The function $f(x)$ is approximated by a straight line tangential to function curve, starting with an initial guess then, the x-intercept of the tangent line gives us the next approximation.

Graphical Example

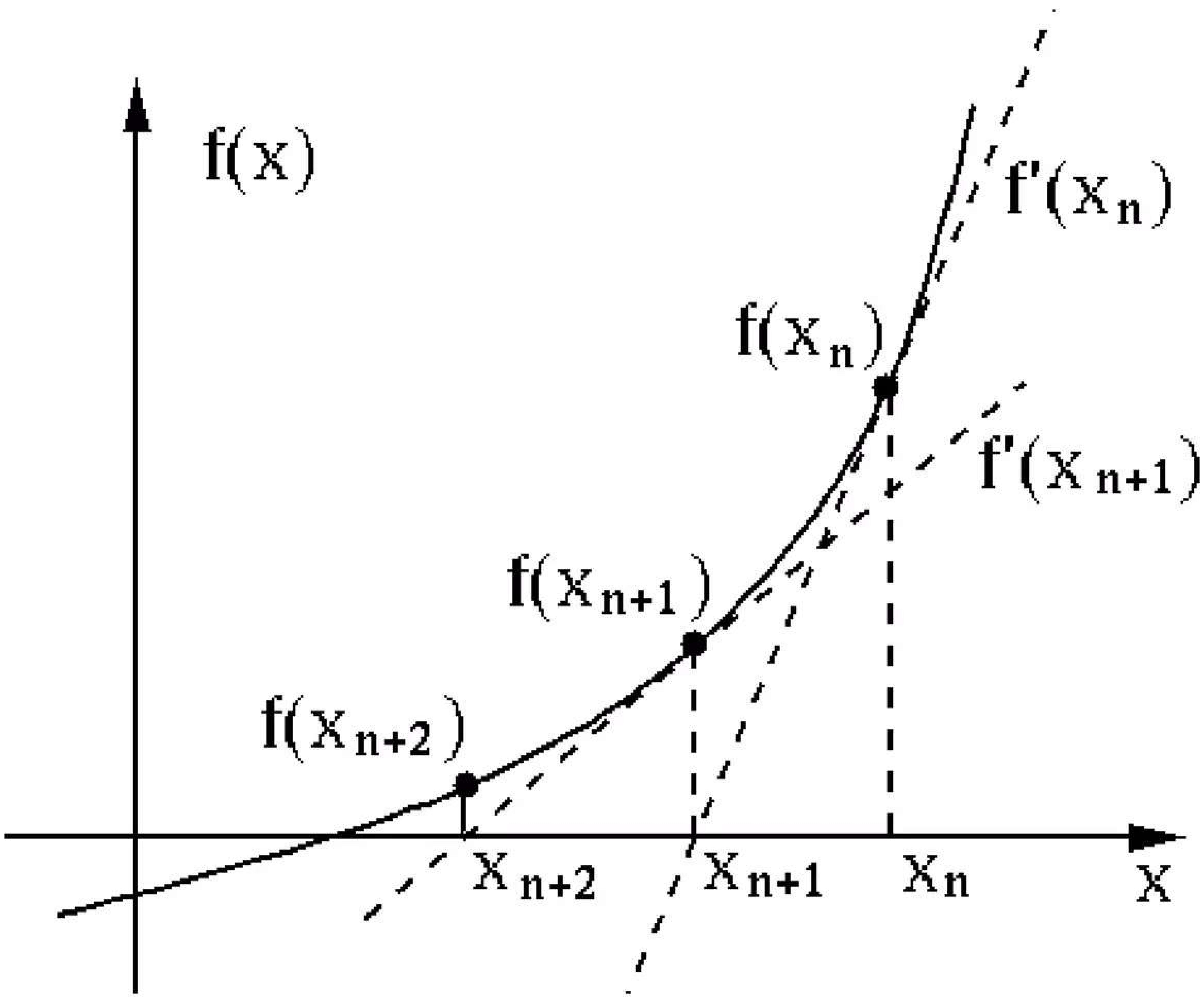

Figure: A Graphic Example of Newton Raphson's Iterations

Steps in Deriving Newton Raphson Iteration

Given $f(x) = 0$

$$tan\theta = f'(x) = \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} \tag{1}$$

From the graph, when we can consider these Cartesian coordinates: $\Big(x_{n+1}, f(x_{n+1})\Big)$ and $\Big(x_n, f(x_n)\Big)$, but $f(x_{n+1}) = 0$. Hence,

$$f'(x) = \frac{0 - f(x_n)}{x_{n+1} - x_n} \tag{2}$$

$$(x_{n+1} - x_n)f'(x) = -f(x_n) \tag{3}$$

$$(x_{n+1} - x_n) = -\frac{f(x_n)}{f'(x)} \tag{4}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_)} \quad (5)$$

Remarks

- The $x_{n+1}-$ value at which there is converges is the solution of interest.
- It takes an intial guess to start its iteration process

Second Derivation

Suppose x^* is the solution of $f(x) = 0$ and let x_n be the approximate solution of x^* s.t $|x^* - x_n| = \delta << 1$.

By the use of Taylor Series approximation, we can have:

$$0 = f(x^*) = f(x_n + \delta) = f(x_{n+1}) \tag{6}$$

$$f(x_{n+1}) = f(x_n) + \delta f'(x_n) + \frac{\delta^2}{2!} f''(x_n) + \ldots = 0 \tag{7}$$

But neglecting the powers of $\delta, n \geq 2$, we now have;

$$f(x_n) + \delta f'(x_n) \approx 0 \tag{8}$$

But $\delta = x_{n+1} - x_n$. We can write eqn(8) as:

$$f(x_n) = -(x_{n+1} - x_n) f'(x_n) \tag{9}$$

$$\frac{f(x_n)}{f'(x_n)} = x_n - x_{n+1} \tag{10}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{11}$$

Performing Newton Raphson Iteration

We start our iteration with an initial point x_0

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$\vdots$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Convergence Criteria of Newton Raphson

Using the Newton Raphson Formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$\Phi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\begin{aligned}
\Phi'(x_n) &= \frac{d}{dx}\left[x_n - \frac{f(x_n)}{f'(x_n)}\right] \\
&= 1 - \frac{d}{dx}\left[\frac{f(x_n)}{f'(x_n)}\right] \\
&= 1 - \frac{d}{dx}\left(f(x_n) \cdot f'(x_n)^{-1}\right) \\
&= 1 - \left[f'(x_n) \cdot f'(x_n)^{-1} - f(x_n) \cdot f'(x_n)^{-2} \cdot f''(x_n)\right] \\
&= 1 - \left[\frac{f'(x_n)}{f'(x_n)} - \frac{f(x_n) \cdot f''(x_n)}{f'(x_n)^2}\right]
\end{aligned}$$

Derivation continue....

$$\Phi'(x_n) = \frac{f(x_n) \cdot f''(x_n)}{f'(x_n)^2}$$

For convergences with respect to the initial guess,

$$|\Phi'(x_n)| < 1$$

We can also consider checking the convergence criterion as:

$$f(x_n) \cdot f''(x_n) < f'(x_n)^2$$

Stopping Criteria

Error Formula

$$\text{Absolute Error} = |x - x^*|$$

$$\text{Relative Error} = \frac{|x - x^*|}{|x|}$$

Where x, x^* are true-value and approximated-value respectively.

These error formula is not of direct use as the true value x is not known.

Commonly Use Stopping Criteria

- $|x_{n+1} - x_n| < \varepsilon$
- $\frac{|x_{n+1} - x_n|}{|x_{n+1}|} < \varepsilon$ or $\frac{|x_n - x_{n+1}|}{|x_n|} < \varepsilon$

Application of Newton Raphson Iteration

Example 1

Find a root of $xe^x = 1$ using Newton Raphson iteration. Take $x_0 = 0.5$

Considering our $f(x) = xe^x - 1 = 0$

Solution

Checking the convergence at $x_0 = 0.5$

$$f'(x) = e^x + xe^x = e^x(1 + x) \tag{12}$$

$$f''(x) = e^x + e^x + xe^x = e^x(2 + x) \tag{13}$$

$$|\Phi'(0.5)| = \left| \frac{(0.5 \cdot e^{0.5} - 1) \cdot e^{0.5}(2+0.5)}{\left(e^{0.5}(1+0.5)\right)^2} \right| = 0.1184 < 1.$$

Hence, there will be convergence

Solution Continue...

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{0.5e^{0.5} - 1}{e^{0.5}(1 + 0.5)} = 0.5710$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.5672$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.5671$$

$$\vdots$$

$$x_{100} = x_{99} - \frac{f(x_{99})}{f'(x_{99})} = 0.5671 - \frac{0.5671e^{0.5671} - 1}{e^{0.5671}(1 + 0.5671)} = 0.5671$$

After successive iterations, the approximated value converges $x^* = 0.5671$

Example 2

Consider the nonlinear equation $x^3 = 2x + 1$ with a solution with the interval $I = [1.5, 2.0]$ using Newton Raphson iteration with initial guess $x_0 = 1.5$, find the approximated root.

Considering our $f(x) = x^3 - 2x - 1 = 0$

Solution

Checking the convergence at $x_0 = 1.5$

$$f'(x) = 3x^2 - 2 \tag{14}$$

$$f''(x) = 6x \tag{15}$$

$$|\Phi'(1.5)| = \left| \frac{\left((1.5)^3 - 2(1.5) - 1\right) \cdot \left(6(1.5)\right)}{\left(3(1.5)^2 - 2\right)^2} \right| = 0.2493 < 1.$$

Hence, there will be convergence

Solution Continue...

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.5 - \frac{(1.5)^3 - 2(1.5) - 1}{3(1.5)^2 - 2} = 1.6316$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.6182$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.6180$$

$$\vdots$$

$$x_{100} = x_{99} - \frac{f(x_{99})}{f'(x_{99})} = 1.6180 - \frac{(1.6180)^3 - 2(1.6180) - 1}{3(1.6180)^2 - 2} = 1.6180$$

After successive iterations, the approximated value converges $x^* = 1.6180$

End

THANK YOU

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Overview of Secant Method

Numerical Analysis

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June 27, 2020

Background of Secant Iteration

- Secant is one of the opened-iterative method for finding root of a given function $f(x) = 0$.
- This method assumes a function should be approximately linear to the area under consideration.
- It requires two initial approximations x_0 and x_1 to start its iterations.
- It retains only the most recent approximations in its iterative process.

Graphical Example

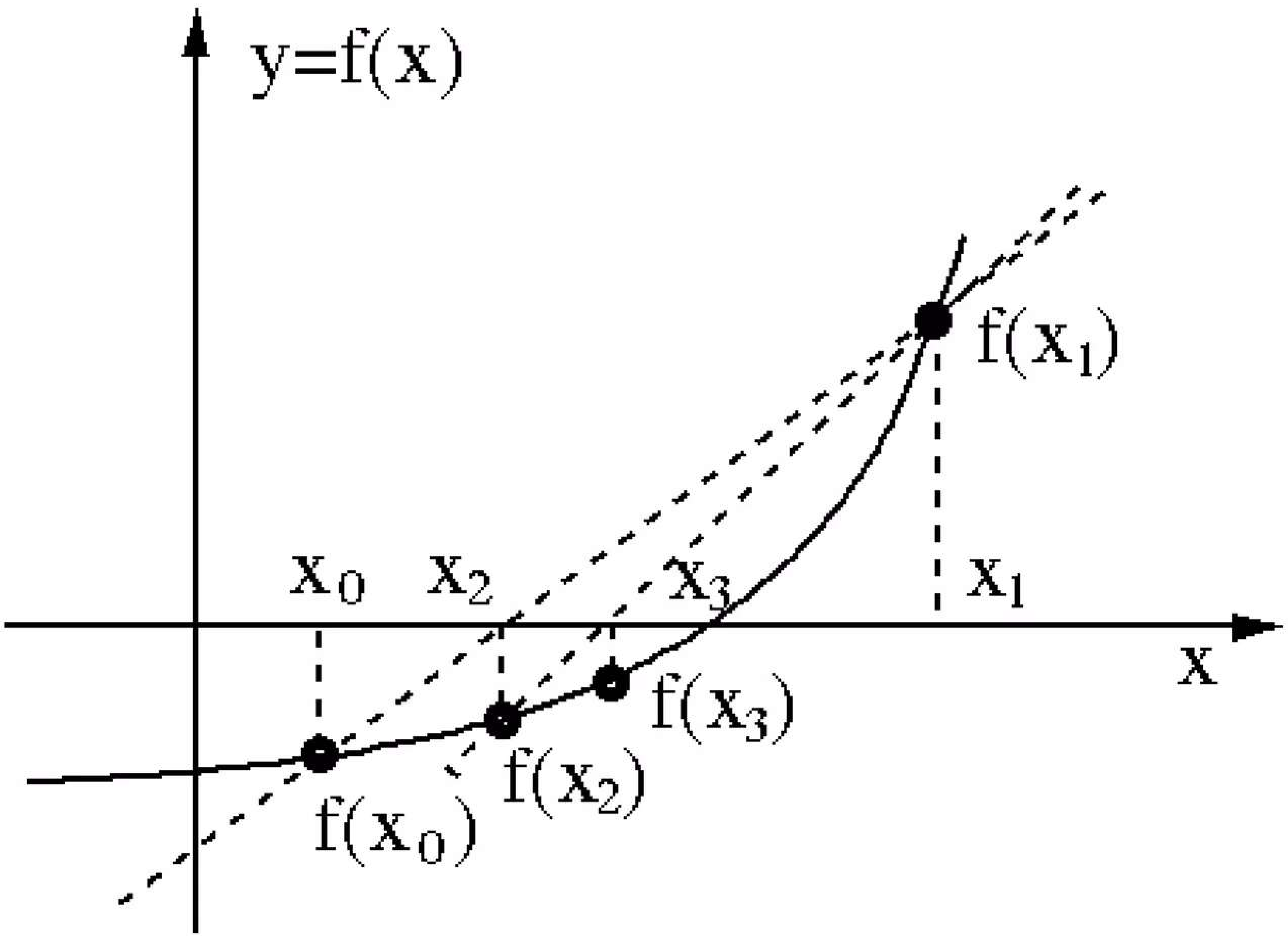

Figure: A Graphical Example of Secant Iteration

Steps in Deriving Secant Method

Given $f(x) = 0$ and Newton Raphson Method as this:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

But from the eqn(1), when we can consider expressing $f'(x)$ as:

$$f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \tag{2}$$

Then,

$$x_{n+1} = x_n - \frac{f(x_n) \cdot [x_n - x_{n-1}]}{f(x_n) - f(x_{n-1})} \tag{3}$$

$$x_{n+1} = \frac{x_n[f(x_n) - f(x_{n-1})] - x_n \cdot f(x_n) + x_{n-1} \cdot f(x_n)}{f(x_n) - f(x_{n-1})} \tag{4}$$

$$x_{n+1} = \frac{x_n \cdot f(x_n) - x_n \cdot f(x_{n-1}) - x_n \cdot f(x_n) + x_{n-1} \cdot f(x_n)}{f(x_n) - f(x_{n-1})} \tag{5}$$

$$x_{n+1} = \frac{x_{n-1} \cdot f(x_n) - x_n \cdot f(x_{n-1})}{f(x_n) - f(x_{n-1})} \tag{6}$$

Remarks

- Secant Method ignores finding derivatives of functions by falling on Backward Divide Difference.
- The $x_{n+1} -$ value at which there is converges is the solution of interest.
- It takes 2 approximate values to start its iteration process.
- Secant method uses the 2 most current approximations in its iterative process.

Performing Secant Iteration

We start our iteration with initial points x_0 and x_1

$$x_2 = \frac{x_0 \cdot f(x_1) - x_1 \cdot f(x_0)}{f(x_1) - f(x_0)} \tag{7}$$

$$x_3 = \frac{x_1 \cdot f(x_2) - x_2 \cdot f(x_1)}{f(x_2) - f(x_1)} \tag{8}$$

$$x_4 = \frac{x_2 \cdot f(x_3) - x_3 \cdot f(x_2)}{f(x_3) - f(x_2)} \tag{9}$$

$$\vdots \tag{10}$$

$$x_{n+1} = \frac{x_{n-1} \cdot f(x_n) - x_n \cdot f(x_{n-1})}{f(x_n) - f(x_{n-1})} \tag{11}$$

Stopping Criteria

Error Formula

$$\text{Absolute Error} = |x - x_{n+1}|$$

$$\text{Relative Error} = \frac{|x - x_{n+1}|}{|x|}$$

Where x, x_{n+1} are true-value and approximated-value respectively.

These error formula is not of direct use as the true value x is not known.

Commonly Use Stopping Criteria

- $|x_{n+1} - x_n| < \varepsilon$
- $\frac{|x_{n+1} - x_n|}{|x_{n+1}|} < \varepsilon$ or $\frac{|x_n - x_{n+1}|}{|x_n|} < \varepsilon$

Application of Secant Iteration

Example 1

Find a root of $xe^x = 1$ using Secant iteration. Take $x_0 = 0.45$ and $x_1 = 0.5$

Considering our $f(x) = xe^x - 1 = 0$

Solution

Checking the convergence at $x_0 = 0.45$ and $x_1 = 0.5$

$$f(0.45) = 0.45(e^{0.45}) - 1 = -0.2943 \tag{12}$$

$$f(0.50) = 0.50(e^{0.50}) - 1 = -0.1756 \tag{13}$$

$$x_2 = \frac{0.45(-0.1756) - 0.50(-0.2943)}{(-0.1756) - (-0.2943)} = 0.5740 \tag{14}$$

Solution Continue...

We ignore $x_0 = 0.45$

$$f(0.50) = 0.50(e^{0.50}) - 1 = -0.1756 \tag{15}$$

$$f(0.5740) = 0.5740(e^{0.5740}) - 1 = 0.0191 \tag{16}$$

$$x_3 = \frac{0.50(0.0191) - 0.5740(-0.1756)}{(0.0191) - (-0.1756)} = 0.5668 \tag{17}$$

$$f(0.5740) = 0.5740(e^{0.5740}) - 1 = 0.0191 \tag{18}$$

$$f(0.5668) = 0.5668(e^{0.5668}) - 1 = -0.0011 \tag{19}$$

$$x_4 = \frac{0.5740(-0.0011) - 0.5668(0.0191)}{(-0.0011) - (0.0191)} = 0.5671 \tag{20}$$

After successive iterations, the approximated value converges $x^* = 0.5671$

Example 2

Consider the nonlinear equation $x^3 = 2x + 1$ with a solution with the interval $I = [1.5, 2.0]$ using Newton Raphson iteration with initial approximations $x_0 = 1.5$ and $x_1 = 1.6$, find the approximated root.

Considering our $f(x) = x^3 - 2x - 1 = 0$

Solution

Checking the convergence at $x_0 = 1.5$ and $x_1 = 1.6$

$$f(1.5) = (1.5)^3 - 2(1.5) - 1 = -0.625 \tag{21}$$

$$f(1.6) = (1.6)^3 - 2(1.6) - 1 = -0.104 \tag{22}$$

$$x_2 = \frac{1.5(-0.104) - 1.6(-0.625)}{(-0.104) - (-0.625)} = 1.6200 \tag{23}$$

Solution Continue...

We ignore $x_0 = 1.5$

$$f(1.6) = (1.6)^3 - 2(1.6) - 1 = -0.104 \tag{24}$$

$$f(1.62) = (1.62)^3 - 2(1.62) - 1 = 0.0113 \tag{25}$$

$$x_3 = \frac{1.6(0.0113) - 1.62(-0.104)}{(0.0113) - (-0.104)} = 1.6180 \tag{26}$$

$$f(1.62) = (1.62)^3 - 2(1.62) - 1 = 0.0113 \tag{27}$$

$$f(1.618) = (1.618)^3 - 2(1.618) - 1 = -0.0002 \tag{28}$$

$$x_4 = \frac{1.62(-0.0002) - 1.618(0.0113)}{(-0.0002) - (0.0113)} = 1.6180 \tag{29}$$

After successive iterations, the approximated value converges $x^* = 1.6180$

End

THANK YOU