Lecture 9: Time-Domain Analysis of Discrete-Time Systems

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Outline

- Discrete-Time System Equations
- E operator
- Response of Linear Discrete-Time Systems
- Useful Signal Operations
- Response of Linear Discrete-Time Systems
- Zero-Input and Zero-State Response
- Convolution Sum
- Graphical Procedure for the Convolution Sum
- Classical Solution of Linear Difference Equations

Discrete-Time System Equations

Advanced operator form:

$$y[k+n] + a_{n-1}y[k+n-1] + \cdots + a_1 y[k+1] + a_0 y[k] =$$
$$b_m f[k+m] + b_{m-1} f[k+m-1] + \cdot + b_1 f[k+1] + b_0 f[k]$$

The left-hand side of this form consists of the output at instants $k+n$, $k+n-1$, $k+n-2$, and so on. The right-hand side of the equation consists of the input at instants $k+m$, $k+m-1$, $k+m-2$, and so on.

The condition that the above system is causal is $m \le n$. For a general causal case, $m = n$, the above system can be expressed as

$$y[k+n] + a_{n-1}y[k+n-1] + \cdots + a_1 y[k+1] + a_0 y[k] =$$
$$b_n f[k+n] + b_{n-1} f[k+n-1] + \cdots + b_1 f[k+1] + b_0 f[k]$$

Discrete-Time System Equations

Delay operator form: In case $m = n$, we can replace k by $k - n$ throughout the equation. Such replacement yields the delay operator form.

$$y[k] + a_{n-1}y[k-1] + \cdots + a_1 y[k-n+1] + a_0 y[k-n] =$$
$$b_n f[k] + b_{n-1} f[k-1] + \cdots + b_1 f[k-n+1] + b_0 f[k-n]$$

Iterative Solution of Difference Equations

From the delay operator form

$$y[k] + a_{n-1}y[k-1] + \cdots + a_1 y[k-n+1] + a_0 y[k-n] =$$
$$b_n f[k] + b_{n-1} f[k-1] + \cdots + b_1 f[k-n+1] + b_0 f[k-n]$$

It can be expressed as

$$y[k] = -a_{n-1}y[k-1] - a_{n-2}y[k-2] - \cdots - a_0 y[k-n]$$
$$+ b_n f[k] + b_{n-1} f[k-1] + \cdots + b_0 f[k-n]$$

There are the past n values of the output: $y[k-1]$, $y[k-2]$,..., $y[k-n]$, the past n values of the input: $f[k-1]$, $f[k-2]$, ..., $f[k-n]$, and the present value of the input $f[k]$.

Initial Conditions and Iterative Solution of Difference Equations

If the input is causal, the $f[-1] = f[-2] = \ldots = f[-n] = 0$, and we need only n initial conditions $y[-1]$, $y[-2]$, …, $y[-n]$. This result allows us to compute iteratively or recursively the output $y[0]$, $y[1]$, $y[2]$, $y[3]$, …, and so on. For instant,

- to find $y[0]$ we set $k = 0$.
- the left-hand side is $y[0]$, and the right-hand side contains terms $y[-1], y[-2], \ldots, y[-n]$ and the input $f[0], f[-1], f[-2], \ldots, f[-n]$.
- Therefore, we must know the n initial conditions $y[-1], y[-2], \ldots, y[-n]$ to find $y[0], y[1], y[2], \ldots$ and so on.

Iterative Solution of Difference Equations

Examples

Solve iteratively

$$y[k] - 0.5y[k-1] = f[k]$$

with initial condition $y[-1] = 16$ and causal input $f[k] = k^2$.
Solution: Rewritten the equation in the delay operator form and move all past outputs to the left:

$$y[k] = 0.5y[k-1] + f[k]$$

We obtain

$$\begin{aligned}
y[0] &= 0.5y[-1] + f[0] = 0.5(16) + 0 = 8 \\
y[1] &= 0.5y[0] + f[1] = 0.5(8) + (1)^2 = 5 \\
y[2] &= 0.5y[1] + f[2] = 0.5(5) + (2)^2 = 6.5 \\
y[3] &= 0.5y[2] + f[3] = 0.5(6.5) + (3)^2 = 12.25 \\
y[4] &= 0.5y[3] + f[4] = 0.5(12.25) + (4)^2 = 22.125
\end{aligned}$$

Iterative Solution of Difference Equations

Examples cont.

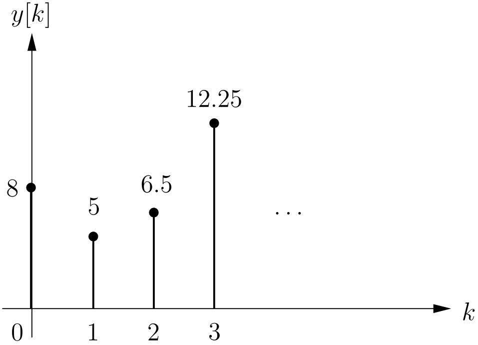


Figure: Iterative solution of a difference

Solve iteratively

$$y[k+2] - y[k+1] + 0.24y[k] = f[k+2] - 2f[k+1]$$

with initial conditions $y[-1] = 2$, $y[-2] = 1$ and a causal input $f[k] = k$.
Solution: Rewritten the equation in the delay operator form and move all past outputs to the left:

$$y[k] = y[k-1] - 0.24y[k-2] + f[k] - 2f[k-1]$$

We obtain

$$\begin{aligned}
y[0] &= y[-1] - 0.24y[-2] + f[0] - 2f[-1] = 2 - 0.24(1) + 0 - 0 = 1.76 \\
y[1] &= y[0] - 0.24y[-1] + f[1] - 2f[0] = 1.76 - 0.24(2) + 1 - 0 = 2.28 \\
y[2] &= y[1] - 0.24y[0] + f[2] - 2f[1] = 2.28 - 0.24(1.76) + 2 - 2(1) = 1.8576
\end{aligned}$$

E operator

In continuous-time system we used the operator D to denote the operation of differentiation. For discrete-time systems we use the operator E to denote the operation for advancing the sequence by one time unit. Thus

$$\begin{aligned} Ef[k] &= f[k+1] \\ E^2 f[k] &= f[k+2] \\ &\vdots \\ E^n f[k] &= f[k+n] \end{aligned}$$

For example

$$\begin{aligned} y[k+1] - ay[k] &= f[k+1] \\ Ey[k] - ay[k] &= Ef[k] \\ (E-a)y[k] &= Ef[k] \end{aligned}$$

E operator
cont.

For the second-order difference equation

$$y[k+2] + \frac{1}{4}y[k+1] + \frac{1}{16}y[k] = f[k+2]$$
$$\left(E^2 + \frac{1}{4}E + \frac{1}{16}\right)y[k] = E^2 f[k]$$

A general nth-order difference equation $(n = m)$can be expressed as

$$\begin{aligned}&\left(E^n + a_{n-1}E^{n-1} + \cdots + a_1E + a_0\right)y[k] = \\ &\qquad\qquad \left(b_nE^n + b_{n-1}E^{n-1} + \cdots + b_1E + b_0\right)f[k] \\ &Q[E]y[k] = P[E]f[k]\end{aligned}$$

where

$$Q[E] = E^n + a_{n-1}E^{n-1} + \cdots + a_1E + a_0$$
$$P[E] = b_nE^n + b_{n-1}E^{n-1} + \cdots + b_1E + b_0$$

Similar to the continuous-time case,

$$\boxed{\text{Total response} = \text{zero-input response} + \text{zero-state response}}$$

The zero-input response $y_0[k]$ is the solution of the system with $f[k] = 0$; that is,

$$Q[E]y_0[k] = 0$$

or

$$\left(E^n + a_{n-1}E^{n-1} + \cdots + a_1 E + a_0\right) y_0[k] = 0$$
$$y_0[k+n] + a_{n-1}y_0[k+n-1] + \cdots + a_1 y_0[k+1] + a_0 y_0[k] = 0$$

The equation states that a linear combination of $y_0[k]$ and advanced $y_0[k]$ is zero not for some values of k but for all k. Such situation is possible if and only if $y_0[k]$ and advanced $y_0[k]$ have the same form. This is true only for an exponential function γ^k. Since

$$\gamma^{k+m} = \gamma^m \gamma^k$$

Therefore, if $y_0[k] = c\gamma^k$ we have

$$\begin{aligned}
Ey_0[k] &= y_0[k+1] = c\gamma^{k+1} = c\gamma\gamma^k \\
E^2 y_0[k] &= y_0[k+2] = c\gamma^{k+2} = c\gamma^2\gamma^k \\
&\vdots \\
E^n y_0[k] &= y_0[k+n] = c\gamma^{k+n} = c\gamma^n\gamma^k
\end{aligned}$$

Substitution of these results to the system equation yields

$$c\left(\gamma^n + a_{n-1}\gamma^{n-1} + \cdots + a_1\gamma + a_0\right)\gamma^k = 0$$

For a nontrivial solution of this equation

$$\left(\gamma^n + a_{n-1}\gamma^{n-1} + \cdots + a_1\gamma + a_0\right) = 0 \text{ or } Q[\gamma] = 0$$

$Q[\gamma]$ is an nth-order polynomial and can be expressed in the factorized form (assuming all distinct roots):

$$(\gamma - \gamma_1)(\gamma - \gamma_2)\cdots(\gamma - \gamma_n) = 0$$

Clearly, γ has n solutions $\gamma_1, \gamma_2, \cdots, \gamma_n$ and, the system has n solutions $c_1\gamma_1^k, c_2\gamma_2^k, \ldots, c_n\gamma_n^k$.

The zero-input response is

$$y_0[k] = c_1\gamma_1^k + c_2\gamma_2^k + \cdots + c_n\gamma_n^k$$

where $\gamma_1, \gamma_2, \ldots, \gamma_n$ are the roots of the polynomial.

- $Q[\gamma]$ is called the **characteristic polynomial** of the system.
- $Q[\gamma] = 0$ is the **characteristic equation** of the system.
- $\gamma_1, \gamma_2, \ldots, \gamma_n$ are called **characteristic roots** or **characteristic values** (also **eignevalues**) of the system.
- The exponentials $\gamma_i^k (i = 1, 2, \ldots, n)$ are **characteristic modes** or **natural modes** of the system.

Repeated Roots:

If two or more roots are repeated, the form of the characteristic modes is modified. Similar to the continuous-time case, if a root γ repeats r times, the characteristic modes corresponding to this root are γ^k, $k\gamma^k$, $k^2\gamma^k$, ..., $k^{r-1}\gamma^k$.

If the characteristic equation of a system is

$$Q[\gamma] = (\gamma - \gamma_1)^r(\gamma - \gamma_{r+1})(\gamma - \gamma_{r+2}) \cdots (\gamma - \gamma_n)$$

the zero-input response of the system is

$$\begin{aligned} y_0[k] = {} & (c_1 + c_2 k + c_3 k^2 + \cdots + c_r k^{r-1})\gamma_1^k \\ & + c_{r+1}\gamma_{r+1}^k + c_{r+2}\gamma_{r+2}^k + \cdots + c_n\gamma_n^k \end{aligned}$$

Complex Roots:

As in the case of continuous-time systems, the complex roots of a discrete-time system must occur in pairs of conjugates so that the system equation coefficients are real. Like the case of continuous-time systems, we can eliminate dealing with complex numbers by using the real form of the solution.

- First express the complex conjugate roots γ and γ^* in polar form.

$$\gamma = |\gamma|e^{j\beta} \text{ and } \gamma^* = |\gamma|e^{-j\beta}$$

- the zero-input response is given by

$$\begin{aligned} y_0[k] &= C_1\gamma^k + C_2(\gamma^*)^k \\ &= C_1|\gamma|^k e^{j\beta k} + C_2|\gamma|^k e^{-j\beta k} \end{aligned}$$

For a real system, C_1 and C_2 must be conjugates so that $y_0[k]$ is a real function of k. Let

$$C_1 = \frac{C}{2}e^{j\theta} \text{ and } C_2 = \frac{C}{2}e^{-j\theta}$$

$$y_0[k] = \frac{C}{2}|\gamma|^k \left[e^{j(\beta k+\theta)} + e^{-j(\beta k+\theta)}\right]$$

$$= C|\gamma|^k \cos(\beta k + \theta)$$

where C and θ are arbitrary constants determined from the auxiliary conditions.

For an LTID system described by the difference equation

$$y[k+2] - 0.6y[k+1] - 0.16y[k] = 5f[k+2]$$

Find the zero-input response $y_0[k]$ of the system if the initial conditions are $y[-1] = 0$ and $y[-2] = \frac{25}{4}$.

The system equation in E operator form is

$$(E^2 - 0.6E - 0.16)y[k] = 5E^2 f[k]$$

The characteristic equation is

$$\gamma^2 - 0.6\gamma - 0.16 = (\gamma + 0.2)(\gamma + 0.8) = 0$$

The zero-input response is

$$y_0[k] = C_1(-0.2)^k + C_2(0.8)^k$$

Substitute $y_0[-1] = 0$ and $y_0[-2] = \frac{25}{4}$ we obtain

$$-5C_1 + \frac{5}{4}C_2 = 0$$
$$25C_1 + \frac{25}{16}C_2 = \frac{25}{4}$$

and $C_1 = \frac{1}{5}$ and $C_2 = \frac{4}{5}$. Therefore

$$y_0[k] = \frac{1}{5}(-0.2)^k + \frac{4}{5}(0.8)^k,\ k \geq 0.$$

A system specified by the equation

$$(E^2 + 6E + 9)y[k] = (2E^2 + 6E)f[k]$$

determine $y_0[k]$, the zero-input response, if the initial condition are $y_0[-1] = -\frac{1}{3}$ and $y_0[-2] = -\frac{2}{9}$.
The characteristic equation is

$$\gamma^2 + 6\gamma + 9 = (\gamma + 3)^2 = 0$$

and we have a repeated characteristic root at $\gamma = -3$. Hence, the zero-input response is

$$y_0[k] = (C_1 + C_2 k)(-3)^k.$$

From the initial conditions we have

$$C_1 - C_2 = 1$$
$$C_1 - 2C_2 = -2$$

and $C_1 = 4$, $C_2 = 3$. Finally, we have $y_0[k] = (4 + 3k)(-3)^k$, $k \geq 0$.

Find the zero-input response of an LTID system described by the equation

$$(E^2 - 1.56E + 0.81)y[k] = (E + 3)f[k]$$

when the initial conditions are $y_0[-1] = 2$ and $y_0[-2] = 1$.
The characteristic equation is

$$(\gamma^2 - 1.56\gamma + 0.81) = (\gamma - 0.78 - j0.45)(\gamma - 0.78 + j0.45) = 0.$$

The characteristic roots are $0.78 \pm j0.45$; that is , $0.9e^{\pm j\frac{\pi}{6}}$. Thus, $|\gamma| = 0.9$ and $\beta = \frac{\pi}{6}$, and the zero-input response is given by

$$y_0[k] = C(0.9)^k \cos(\frac{\pi}{6}k + \theta).$$

Substituting the initial conditions $y_0[-1] = 2$ and $y_0[-2] = 1$, we obtain

$$\frac{C}{0.9}\cos\left(-\frac{\pi}{6} + \theta\right) = \frac{C}{0.9}\left[\cos(-\frac{\pi}{6})\cos\theta - \sin(-\frac{\pi}{6})\sin\theta\right] = \frac{C}{0.9}\left[\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right] = 2$$

and

$$\frac{C}{(0.9)^2}\cos\left(-\frac{\pi}{3}+\theta\right)=\frac{C}{0.81}\left[\cos(-\frac{\pi}{3})\cos\theta-\sin(-\frac{\pi}{3})\sin\theta\right]=\frac{C}{0.81}\left[\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta\right]=1$$

or

$$\begin{aligned}\frac{\sqrt{3}}{1.8}C\cos\theta+\frac{1}{1.8}C\sin\theta=2\\ \frac{1}{1.62}C\cos\theta+\frac{\sqrt{3}}{1.62}C\sin\theta=1.\end{aligned}$$

We have $C\cos\theta=2.308$ and $C\sin\theta=-0.397$. Then

$$\theta=\tan^{-1}\frac{-0.397}{2.308}=-0.17 \text{ rad}$$

Substituting $\theta=-0.17$ radian in $C\cos\theta=2.308$ yields $C=2.34$ and

$$y_0[k]=2.34(0.9)^k\cos\left(\frac{\pi}{6}k-0.17\right),\ k\geq 0$$

The Unit Impulse Response $h[k]$

Consider an nth-order system specified by the equation

$$\left(E^n + a_{n-1}E^{n-1} + \cdots + a_1E + a_0\right) y[k] = \\ \left(b_nE^n + b_{n-1}E^{n-1} + \cdots + b_1E + b_0\right) f[k]$$

or

$$Q[E]y[k] = P[E]f[k]$$

The unit impulse response $h[k]$ is the solution of this equation for the input $\delta[k]$ with all the initial conditions zero; that is

$$Q[E]h[k] = P[E]\delta[k]$$

subject to initial conditions

$$h[-1] = h[-2] = \cdots = h[-n] = 0$$

The Unit Impulse Response $h[k]$

$h[k]$ is the system response to the input $\delta[k]$, which is zero for $k > 0$. We know that when the input is zero, only the characteristic modes can be sustained by the system. Therefore, $h[k]$ must be made up of characteristic modes for $k > 0$. At $k = 0$, it may have some nonzero value, and $h[k]$ can be expressed as

$$h[k] = \frac{b_0}{a_0}\delta[k] + y_n[k]u[k].$$

The n unknown coefficients in $y_n[k]$ can be determined from a knowledge of n values of $h[k]$. It is a straightforward task to determine values of $h[k]$ iteratively.

For a discrete-time system specified above, we have

$$h[k] = A_0\delta[k] + y_n[k]u[k]$$

Then

$$Q[E](A_0\delta[k] + y_n[k]u[k]) = P[E]\delta[k]$$

because $y_n[k]u[k]$ is a sum of characteristic modes

$$Q[E](y_n[k]u[k]) = 0,\ k \geq 0$$

The above equation reduces to

$$A_0 Q[E]\delta[k] = P[E]\delta[k],\ k \geq 0$$

or

$$
\begin{aligned}
&A_0\left(E^n + a_{n-1}E^{n-1} + \cdots + a_1E + a_0\right)\delta[k] \\
&\qquad\qquad = \left(b_nE^n + b_{n-1}E^{n-1} + \cdots + b_1E + b_0\right)\delta[k] \\
&A_0(\delta[k+n] + a_{n-1}\delta[k+n-1] + \cdots + a_1\delta[k+1] + a_0\delta[k]) \\
&\qquad\qquad = b_n\delta[k+n] + b_{n-1}\delta[k+n-1] + \cdots + b_1\delta[k+1] + b_0\delta[k]
\end{aligned}
$$

If we set $k=0$ in the equation and recognize that $\delta[0]=1$ and $\delta[m]=0$ when $m \neq 0$, all but the last terms vanish on both sides, yielding

$$A_0a_0 = b_0 \quad \text{and} \quad A_0 = \frac{b_0}{a_0}$$

Note: for the special case $a_0 = 0$ see the reference.

The Unit Impulse Response $h[k]$

Example

Determine the unit impulse response $h[k]$ for a system specified by the equation

$$y[k] - 0.6y[k-1] - 0.16y[k-2] = 5f[k]$$

This equation can be expressed in the advance operator form as

$$y[k+2] - 0.6y[k+1] - 0.16y[k] = 5f[k+2]$$

or

$$\left(E^2 - 0.6E - 0.16\right) y[k] = 5E^2 f[k]$$

The characteristic equation is

$$\gamma^2 - 0.6\gamma - 0.16 = (\gamma + 0.2)(\gamma - 0.8) = 0.$$

Therefore

$$y_n[k] = C_1(-0.2)^k + C_2(0.8)^k$$

The Unit Impulse Response $h[k]$

Example cont.

From the system we have $a_0 = -0.16$ and $b_0 = 0$. Therefore

$$h[k] = \frac{0}{-0.16}\delta[k] + \left[C_1(-0.2)^k + C_2(0.8)^k\right]u[k] = \left[C_1(-0.2)^k + C_2(0.8)^k\right]u[k]$$

To determine C_1 and C_2, we need to find two values of $h[k]$ iteratively. To do this, we must let the input $f[k] = \delta[k]$ and the output $y[k] = h[k]$ in the system equation. The resulting equation is

$$h[k] - 0.6h[k-1] - 0.16h[k-2] = 5\delta[k]$$

subject to zero initial state; that is , $h[-1] = h[-2] = 0$.
Setting $k = 0$ in this equation yields

$$h[0] - 0.6(0) - 0.16(0) = 5(1) \Longrightarrow h[0] = 5$$

Next, setting $k = 1$ and using $h[0] = 5$, we obtain

$$h[1] - 0.6(5) - 0.16(0) = 5(0) \Longrightarrow h[1] = 3$$

The Unit Impulse Response $h[k]$

Example cont.

Then we have

$$h[0] = C_1(-0.2)^0 + C_2(0.8)^0 = C_1 + C_2 = 5$$
$$h[1] = C_1(-0.2)^1 + C_2(0.8)^1 = -0.2C_1 + 0.8C_2 = 3$$

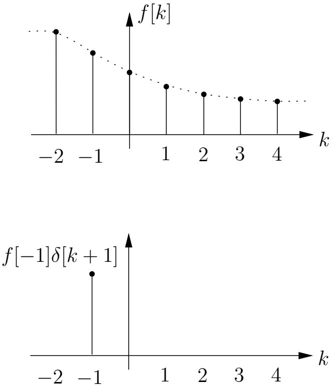
and $C_1 = 1$, $C_2 = 4$. Therefore

$$h[k] = \left[(-0.2)^k + 4(0.8)^k\right] u[k]$$

System Response to External Input

The Zero-State Response

The zero-state response $y[k]$ is the system response to an input $f[k]$ when the system is in zero state. Like in the continuous-time case an arbitrary input $f[k]$ can be expressed as a sum of impulse components.



The previous page shows how a signal $f[k]$ can be expressed as a sum of impulse components. The component of $f[k]$ at $k = m$ is $f[m]\delta[k-m]$, and $f[k]$ is the sum of all these components summed from $m = -\infty$ to ∞. Therefore

$$
\begin{aligned}
f[k] &= f[0]\delta[k] + f[1]\delta[k-1] + f[2]\delta[k-2] + \cdots \\
&\qquad + f[-1]\delta[k+1] + f[-2]\delta[k+2] + \cdots \\
&= \sum_{m=-\infty}^{\infty} f[m]\delta[k-m]
\end{aligned}
$$

If we knew the system response to impulse $\delta[k]$, the system response to any arbitrary input could be obtained by summing the system response to various impulse components.

If

$$\delta[k] \implies h[k]$$

then

$$\begin{aligned} \delta[k-m] &\implies h[k-m] \\ f[m]\delta[k-m] &\implies f[m]h[k-m] \\ \underbrace{\sum_{m=-\infty}^{\infty} f[m]\delta[k-m]}_{f[k]} &\implies \underbrace{\sum_{m=-\infty}^{\infty} f[m]h[k-m]}_{y[k]} \end{aligned}$$

System Response to External Input

The Zero-State Response cont.

We have the response $y[k]$ to input $f[k]$ as

$$y[k] = \sum_{m=-\infty}^{\infty} f[m]h[k-m].$$

This summation on the right-had side is known as the **convolution sum** of $f[k]$ and $h[k]$, and is represented symbolically by $f[k] * h[k]$

$$f[k] * h[k] = \sum_{m=-\infty}^{\infty} f[m]h[k-m]$$

The Commutative Property

$$f_1[k] * f_2[k] = f_2[k] * f_1[k]$$

This can be proved as follow:

$$\begin{aligned} f_1[k] * f_2[k] &= \sum_{m=-\infty}^{\infty} f_1[m] f_2[k-m] \\ &= -\sum_{w=\infty}^{-\infty} f_1[w-k] f_2[w], \qquad w = k - m \\ &= \sum_{w=-\infty}^{\infty} f_2[w] f_1[w-k] \\ &= f_2[k] * f_1[k] \end{aligned}$$

The Distributive Property

$$f_1[k] * (f_2[k] + f_3[k]) = f_1[k] * f_2[k] + f_1[k] * f_3[k]$$

The proof is as follow:

$$\begin{aligned} f_1[k] * (f_2[k] + f_3[k]) &= \sum_{m=-\infty}^{\infty} f_1[m] \left(f_2[k-m] + f_3[k-m]\right) \\ &= \sum_{m=-\infty}^{\infty} f_1[m] f_2[k-m] + \sum_{m=-\infty}^{\infty} f_1[m] f_3[k-m] \\ &= f_1[k] * f_2[k] + f_1[k] * f_3[k] \end{aligned}$$

The Associative Property

$$f_1[k] * (f_2[k] * f_3[k]) = (f_1[k] * f_2[k]) * f_3[k]$$

The proof is as follow:

$$\begin{aligned} f_1[k] * (f_2[k] * f_3[k]) &= \sum_{m_1=-\infty}^{\infty} f_1[m_1] \left(f_2[k-m_1] * f_3[k-m_1] \right) \\ &= \sum_{m_1=-\infty}^{\infty} f_1[m_1] \sum_{m_2=-\infty}^{\infty} f_2[m_2] f_3[k-m_1-m_2] \\ &= \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} f_1[\lambda-m_2] f_2[m_2] f_3[k-\lambda] \end{aligned}$$

,where $\lambda = m_1 + m_2$.

Then we have

$$f_1[k] * (f_2[k] * f_3[k]) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} f_1[\lambda - m_2] f_2[m_2] f_3[k - \lambda]$$
$$= (f_1[k] * f_2[k]) * f_3[k]$$

The Convolution with an Impulse

$$f[k] * \delta[k] = \sum_{m=-\infty}^{\infty} f[m]\delta[k-m]$$

Since $\delta[k-m] = 1$, if $k - m = 0$ or $m = k$, then

$$f[k] * \delta[k] = f[k].$$

The shifting Property

If

$$f_1[k] * f_2[k] = c[k]$$

then

$$\begin{aligned}
f_1[k] * f_2[k-n] &= f_1[k] * f_2[k] * \delta[k-n] \\
&= c[k] * \delta[k-n] = c[k-n] \\
f_1[k-n] * f_2[k] &= f_1[k] * \delta[k-n] * f_2[k] \\
&= f_1[k] * f_2[k] * \delta[k-n] \\
&= c[k] * \delta[k-n] = c[k-n] \\
f_1[k-n] * f_2[k-l] &= f_1[k] * \delta[k-n] * f_2[k] * \delta[k-l] \\
&= c[k] * \delta[k-n] * \delta[k-l] = c[k-n-l]
\end{aligned}$$

The Width Property

If $f_1[k]$ and $f_2[k]$ have lengths of m and n elements respectively, then the length of $c[k]$ is $m + n - 1$ elements.

Causality and Zero-State Response

- We assumed the system to be linear and time-invariant.
- In practice, almost all of the input signals are causal, and a majority of the system are also causal.
- If the input $f[k]$ is causal, then $f[m] = 0$ for $m < 0$.
- Similarly, if the system is causal, then $h[x] = 0$ for negative x, so that $h[k-m] = 0$ when $m > k$.
- Therefore, if $f[k]$ and $h[k]$ are both causal, the product $f[m]h[k-m] = 0$ for $m < 0$ and for $m > k$, and it is nonzero only for the range $0 \leq m \leq k$. Therefore, the convolution sum is reduced to

$$y[k] = \sum_{m=0}^{k} f[k]h[k-m]$$

Convolution Sum

Analytical Method Example

Determine $c[k] = f[k] * g[k]$ for

$$f[k] = (0.8)^k u[k] \text{ and } g[k] = (0.3)^k u[k]$$

we have

$$c[k] = \sum_{m=0}^{k} f[m]g[k-m]$$

since both signals are causal.

$$c[k] = \begin{cases} \displaystyle\sum_{m=0}^{k} (0.8)^m (0.3)^{k-m} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

$$c[k] = (0.3)^k \sum_{m=0}^{k} \left(\frac{0.8}{0.3}\right)^m u[k] = (0.3)^k \frac{(0.8)^{k+1} - (0.3)^{k+1}}{(0.3)^k(0.8-0.3)} u[k]$$

$$= 2\left[(0.8)^{k+1} - (0.3)^{k+1}\right] u[k]$$

Zero-State Response

Analytical Method Example

Find the zero-state response $y[k]$ of an LTID system described by the equation

$$y[k+2] - 0.6y[k+1] - 0.16y[k] = 5f[k+2]$$

if the input $f[k] = 4^{-k}u[k]$ and $h[k] = \left[(-0.2)^k + 4(0.8)^k\right]u[k]$.
We have

$$\begin{aligned} y_i[k] &= f[k] * h[k] \\ &= (4)^{-k}u[k] * \left[(-0.2)^k u[k] + 4(0.8)^k u[k]\right] \\ &= (4)^{-k}u[k] * (-0.2)^k u[k] + (4)^{-k}u[k] * 4(0.8)^k u[k] \\ &= (0.25)^k u[k] * (-0.2)^k u[k] + 4(0.25)^k u[k] * (0.8)^k u[k] \end{aligned}$$

Using Pair 4 from the convolution sum table:

$$y[k] = \left[\frac{(0.25)^{k+1} - (-0.2)^{k+1}}{0.25 - (-0.2)} + 4\frac{(0.25)^{k+1} - (0.8)^{k+1}}{0.25 - 0.8}\right]u[k]$$

Zero-State Response

Analytical Method Example cont.

$$
\begin{aligned}
y[k] &= \left(2.22\left[(0.25)^{k+1} - (-0.2)^{k+1}\right] - 7.27\left[(0.25)^{k+1} - (0.8)^{k+1}\right]\right) u[k] \\
&= \left[-5.05(0.25)^{k+1} - 2.22(-0.2)^{k+1} + 7.27(0.8)^{k+1}\right] u[k]
\end{aligned}
$$

Recognizing that

$$\gamma^{k+1} = \gamma(\gamma)^k$$

We can express $y[k]$ as

$$
\begin{aligned}
y[k] &= \left[-1.26(0.25)^k + 0.444(-0.2)^k + 5.81(0.8)^k\right] u[k] \\
&= \left[-1.26(4)^{-k} + 0.444(-0.2)^k + 5.81(0.8)^k\right] u[k]
\end{aligned}
$$

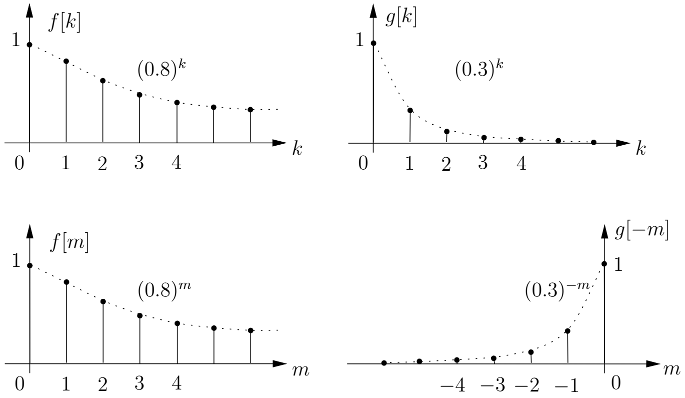
Graphical Procedure for the Convolution Sum

The convolution sum of causal signals $f[k]$ and $g[k]$ is given by

$$c[k] = \sum_{m=0}^{k} f[k]g[k-m]$$

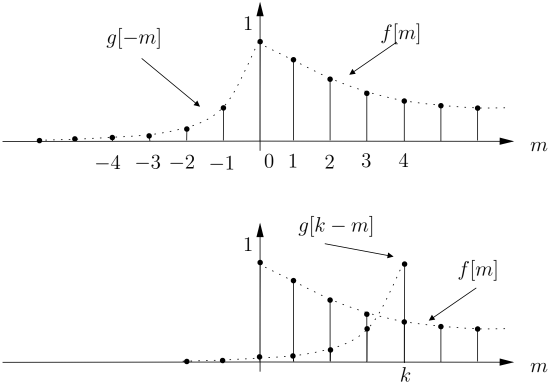
- Invert $g[m]$ about the vertical axis $(m = 0)$ to obtain $g[-m]$.
- Time shift $g[-m]$ by k units to obtain $g[k-m]$. For $k > 0$, the shift is to the right (delay); for $k < 0$, the shift is to the left (advance).
- Next we multiply $f[m]$ and $g[k-m]$ and add all the products to obtain $c[k]$. The procedure is repeated to each value of k over the range $-\infty$ to ∞.

Find $c[k] = f[k] * g[k]$, where $f[k]$ and $g[k]$ are depicted in the Figures.



Graphical Procedure for the Convolution Sum

Example



The two functions $f[m]$ and $g[k-m]$ overlap over the interval $0 \le m \le k$.

Therefore

$$
\begin{aligned}
c[k] &= \sum_{m=0}^{k} f[m]g[k-m] \\
&= \sum_{m=0}^{k} (0.8)^m (0.3)^{k-m} \\
&= (0.3)^k \sum_{m=0}^{k} \left(\frac{0.8}{0.3}\right)^m \\
&= 2\left[(0.8)^{k+1} - (0.3)^{k+1}\right], \qquad k \geq 0
\end{aligned}
$$

For $k < 0$, there is no overlap between $f[m]$ and $g[k-m]$, so that $c[k] = 0 \qquad k < 0$ and

$$
c[k] = 2\left[(0.8)^{k+1} - (0.3)^{k+1}\right] u[k].
$$

Graphical Procedure for the Convolution Sum

Sliding Tape Method

Using the sliding tape method, convolve the two sequences $f[k]$ and $g[k]$.

- write the sequences $f[k]$ and $g[k]$ in the slots of two tapes
- leave the f tape stationary (to correspond to $f[m]$). The $g[-m]$ tape is obtained by time inverting the $g[m]$
- shift the inverted tape by k slots, multiply values on two tapes in adjacent slots, and add all the products to find $c[k]$.

Graphical Procedure for the Convolution Sum

Sliding Tape Method cont.

For the case of $k = 0$,

$$c[0] = 0 \times 1 = 0$$

For $k = 1$

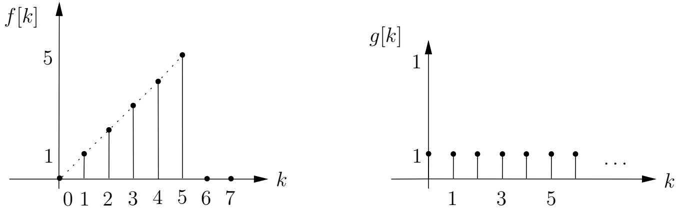
$$c[1] = (0 \times 1) + (1 \times 1) = 1$$

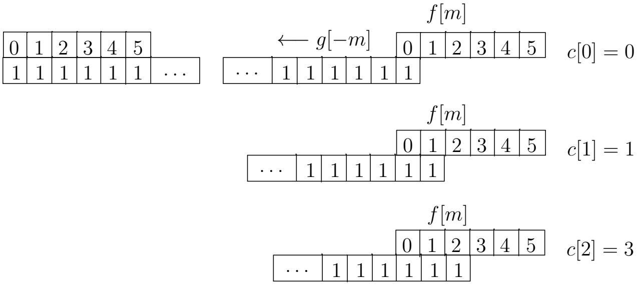
Similarly,

$$\begin{aligned}
c[2] &= (0 \times 1) + (1 \times 1) + (2 \times 1) = 3 \\
c[3] &= (0 \times 1) + (1 \times 1) + (2 \times 1) + (3 \times 1) = 6 \\
c[4] &= (0 \times 1) + (1 \times 1) + (2 \times 1) + (3 \times 1) + (4 \times 1) = 10 \\
c[5] &= (0 \times 1) + (1 \times 1) + (2 \times 1) + (3 \times 1) + (4 \times 1) + (5 \times 1) = 15 \\
c[6] &= (0 \times 1) + (1 \times 1) + (2 \times 1) + (3 \times 1) + (4 \times 1) + (5 \times 1) = 15 \\
&\vdots
\end{aligned}$$

Graphical Procedure for the Convolution Sum

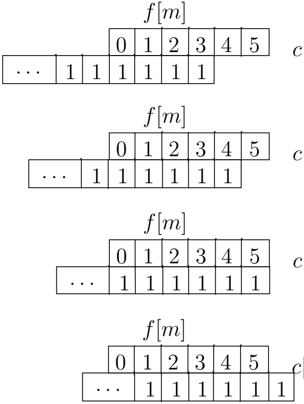
Sliding Tape Method cont.





Graphical Procedure for the Convolution Sum

Sliding Tape Method cont.

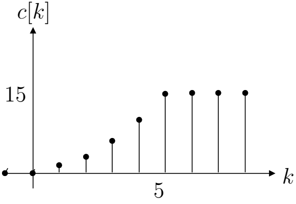


$c[3] = 6$

$c[4] = 10$

$c[5] = 15$

$c[6] = 15$



Total Response

The total response of an LTID system can be expressed as a sum of the zero-input and zero-state components:

$$\boxed{\text{Total response } y[k] = \underbrace{\sum_{j=1}^{n} c_j \gamma_j^k}_{\text{Zero-input component}} + \underbrace{f[k] * h[k]}_{\text{Zero-state component}}}$$

From the previous example, the system described by the equation

$$y[k+2] - 0.6y[k+1] - 0.16y[k] = 5f[k+2]$$

with initial conditions $y[-1] = 0, y[-2] = \frac{25}{4}$ and input $f[k] = (4)^{-k}u[k]$. We have

$$y[k] = \underbrace{0.2(-0.2)^k + 0.8(0.8)^k}_{\text{Zero-input component}} \underbrace{-1.26(4)^{-k} + 0.444(-0.2)^k + 5.81(0.8)^k}_{\text{Zero-state component}}$$

Classical solution of Linear Difference Equations

If $y_n[k]$ and $y_\phi[k]$ denote the natural and the forced response respectively, the the total response is given by

$$\boxed{\text{Total response } y[k] = \underbrace{y_n[k]}_{\text{modes}} + \underbrace{y_\phi[k]}_{\text{nonmodes}}}$$

Because $y_n[k] + y_\phi[k]$ is a solution of the system, we have

$$Q[E](y_n[k] + y_\phi[k]) = P[E]f[k]$$

$y_n[k]$ is made up of characteristic modes,

$$Q[E]y_n[k] = 0$$

Substitution of this equation yields

$$Q[E]y_\phi[k] = P[E]f[k]$$

By definition, the forced response contains only nonmode terms and and the list of the inputs and the corresponding forms of the forced function is show below:

Input $f[k]$	Forced Response $y_\phi[k]$
1. r^k $r \neq \gamma_i$ $(i = 1, 2, \cdots, n)$	cr^k
2. r^k $r = \gamma_i$	ckr^k
3. $\cos(\beta k + \theta)$	$c\cos(\beta k + \phi)$
4. $\left(\sum_{i=0}^{m} \alpha_i k^i\right) r^k$	$\left(\sum_{i=0}^{m} c_i k^i\right) r^k$

Note: By definition $y_\phi[k]$ cannot have any characteristic mode terms.

Determine the total response $y[k]$ of a system

$$(E^2 - 5E + 6)y[k] = (E - 5)f[k]$$

if the input $f[k] = (3k + 5)u[k]$ and the auxiliary conditions are $y[0] = 4, y[1] = 13$.
The characteristic equation is

$$\gamma^2 - 5\gamma + 6 = (\gamma - 2)(\gamma - 3) = 0$$

Therefore, the natural response is

$$y_n[k] = B_1(2)^k + B_2(3)^k$$

To find the form of forced response $y_\phi[k]$, we use above Table, Pair 4 with $r = 1, m = 1$.
This yields

$$y_\phi[k] = c_1 k + c_0$$

Classical solution of Linear Difference Equations

Forced Response example cont.

Therefore

$$
\begin{aligned}
y_\phi[k+1] &= c_1(k+1) + c_0 = c_1 k + c_1 + c_0 \\
y_\phi[k+2] &= c_1(k+2) + c_0 = c_1 k + 2c_1 + c_0
\end{aligned}
$$

Also

$$f[k] = 3k + 5$$

and

$$f[k+1] = 3(k+1) + 5 = 3k + 8$$

Substitution of the above results yiels

$$
\begin{aligned}
c_1 k + 2c_1 + c_0 - 5(c_1 k + c_1 + c_0) + 6(c_1 k + c_0) &= 3k + 8 - 5(3k+5) \\
2c_1 k - 3c_1 + 2c_0 &= -12k - 17
\end{aligned}
$$

Classical solution of Linear Difference Equations

Forced Response example cont.

Comparison of similar terms on the two sides yields

$$\begin{aligned} 2c_1 &= -12 \\ -3c_1 + 2c_0 &= -17 \end{aligned}$$

and $c_1 = -6,\ c_2 = -\frac{35}{2}$. Therefore

$$y_\phi[k] = -6k - \frac{35}{2}$$

The total response is

$$\begin{aligned} y[k] &= y_n[k] + y_\phi[k] \\ &= B_1(2)^k + B_2(3)^k - 6k - \frac{35}{2},\ k \geq 0 \end{aligned}$$

Classical solution of Linear Difference Equations

Forced Response example cont.

To determine arbitrary constants B_1 and B_2 we set $k = 0$ and 1 and substitute the initial conditions $y[0] = 4$, $y[1] = 13$ to obtain

$$B_1 + B_2 - \frac{35}{2} = 4$$
$$2B_1 + 3B_2 - \frac{47}{2} = 13$$

and $B_1 = 28,\ B_2 = -\frac{13}{2}$.
Therefore

$$y_n[k] = 28(2)^k - \frac{13}{2}(3)^k$$

and

$$y[k] = \underbrace{28(2)^k - \frac{13}{2}(3)^k}_{y_n[k]} \underbrace{-6k - \frac{35}{2}}_{y_\phi[k]}$$

Reference

1. Lathi, B. P., *Signal Processing & Linear Systems*, Berkeley-Cambridge Press, 1998.