Graphs

Basic notation and definitions

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Contents

Cardinality of a set[1]

Definition

Given a set $V = \{1, \ldots, n\}$, we denote by $|V| = n$, the *cardinality (number of elements)* of V (i.e., n is the number of elements of set V).

Definition

Given a set $V = \{1, \ldots, n\}$, we define the *power set* of V, denoted as 2^V, as the set of all subsets of V, including the empty set $\emptyset$ and the set V itself.

[1] Resende, M.G.C. and Ribeiro, C.C., *Optimization by GRASP: Greedy Randomized Adaptive Search Procedures*, Springer, 2013, Chapter 1

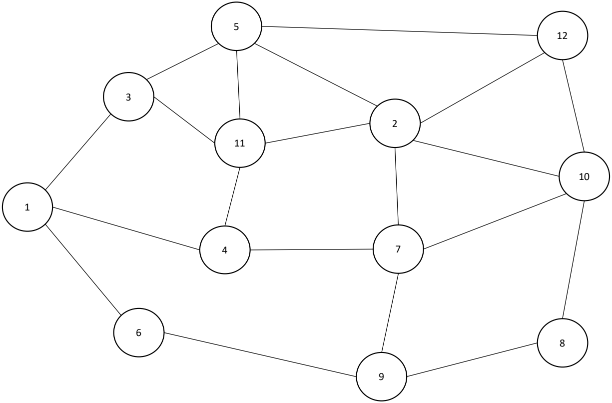
Graph[2]

Definition

A *graph* $G = (V, U)$ is defined by a set $V = \{1, \ldots, n\}$ of *vertices* or *nodes* and a set $U \subseteq V \times V$ of unordered pairs $i, j \in V$ called *edges*. Therefore, both pairs $\{i, j\}$ or $\{j, i\}$ can be used to represent the same edge between $i, j \in V$ in U. A *graph* can also be referred to as an *undirected graph*.

[2] Resende, M.G.C. and Ribeiro, C.C., *Optimization by GRASP: Greedy Randomized Adaptive Search Procedures*, Springer, 2013, Chapter 1

Representación gráfica de un grafo



$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

$$\begin{aligned} U \quad = \quad & \{\{1,3\},\{1,4\},\{1,6\},\{2,5\},\{2,7\},\{2,10\},\{2,11\}, \\ & \{2,12\},\{3,5\},\{3,11\},\{4,7\},\{4,11\},\{5,11\},\{5,12\}, \\ & \{6,9\},\{7,9\},\{7,10\},\{8,9\},\{8,10\},\{10,12\}\} \end{aligned}$$

Complete graph[3]

Definition

A graph $G = (V, U)$ is said to be a *complete graph* or *complete undirected graph* if there is and edge between any two distinct nodes in V, that is, $\{i, j\} \in U$ for each pair $i, j \in V, i \neq j$.

[3] Resende, M.G.C. and Ribeiro, C.C., *Optimization by GRASP: Greedy Randomized Adaptive Search Procedures*, Springer, 2013, Chapter 1

Path[4]

Definition

Given a graph $G = (V, U)$ a *path* $P_{st}(G)$ in an undirected graph G, from a node $s \in V$ to a node $t \in V$, is defined as a sequence of nodes $i_1, i_2, \ldots i_{q-1}, i_q \in V$ where $i_1 = s$, $i_q = t$, and each $\{i_k, i_{k+1}\} \in U$, for $k = 1, \ldots, q-1$. The number of edges in a path is given by $q - 1$

[4] Resende, M.G.C. and Ribeiro, C.C., *Optimization by GRASP: Greedy Randomized Adaptive Search Procedures*, Springer, 2013, Chapter 1

Connected[5]

Definition

Given a graph $G = (V, U)$ we say that the graph is *connected* if there is at least one path $P_{st}(G)$ for every pair of distinct nodes $s, t \in V$.

[5] Resende, M.G.C. and Ribeiro, C.C., *Optimization by GRASP: Greedy Randomized Adaptive Search Procedures*, Springer, 2013, Chapter 1

Subgraph[6]

Definition

Given a graph $G = (V, U)$, a *subgraph* $G' = (V', U')$ is such that for any pair of nodes $i, j \in V'$, edge $\{i, j\} \in U'$ if and only if $\{i, j\} \in U$, and therefore, $V' \subseteq V$ and $U' \subseteq U$.

[6] Resende, M.G.C. and Ribeiro, C.C., *Optimization by GRASP: Greedy Randomized Adaptive Search Procedures*, Springer, 2013, Chapter 1

Spanning tree[7]

Definition

A *spanning tree* of a graph $G = (V, U)$ is a connected subgraph of G with the same node set V and whose edge set $U' \subseteq U$ has exactly $n - 1$ edges, where $n = |V|$.

[7] Resende, M.G.C. and Ribeiro, C.C., *Optimization by GRASP: Greedy Randomized Adaptive Search Procedures*, Springer, 2013, Chapter 1

Induced graph[8]

Definition

Given a graph $G = (V, U)$ and a subset V' of its node set V, the graph $G(V') = (V', U')$ *induced* in G by V' has $U' = \{\{i, j\} \in U : \{i, j\} \in V'\}$ as its edge set.

[8] Resende, M.G.C. and Ribeiro, C.C., *Optimization by GRASP: Greedy Randomized Adaptive Search Procedures*, Springer, 2013, Chapter 1

Clique[9]

Definition

A *clique* of graph $G = (V, U)$ is a subset of nodes $C \subseteq V$ such that $\{i, j\} \in U$ for every nodes $i, j \in C$, with $i \neq j$. Alternatively, we can say that C is a *clique* if the graph $G(C)$ induced in G by C is a complete graph. The *size* of a *clique* is defined by $|C|$.

[9] Resende, M.G.C. and Ribeiro, C.C., *Optimization by GRASP: Greedy Randomized Adaptive Search Procedures*, Springer, 2013, Chapter 1

Independent set[10]

Definition

Given a graph $G = (V, U)$, a subset $I \subseteq V$ of the nodes in G is said to be an *independent set* or a *stable set* if every two nodes in I are not directly connected by an edge, i.e., if $\{i, j\} \notin U$ for all $i, j \in I$ such that $i \neq j$.

[10] Resende, M.G.C. and Ribeiro, C.C., *Optimization by GRASP: Greedy Randomized Adaptive Search Procedures*, Springer, 2013, Chapter 1

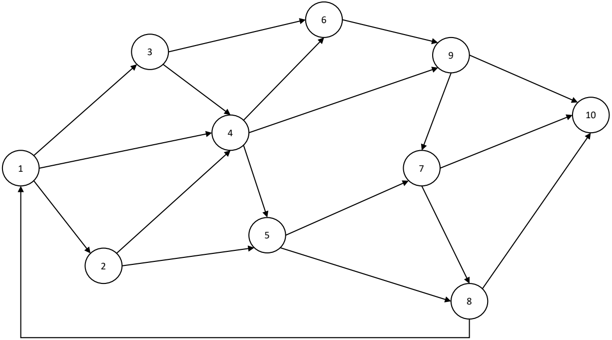
Directed graph or digraph[11]

Definition

A *directed graph* or *digraph* $G = (V, A)$ is defined by a set $V = \{1, \ldots, n\}$ of nodes and a set $A \subseteq V \times V$ of ordered pairs (i, j) of nodes $i, j \in V$ called arcs.

[11] Resende, M.G.C. and Ribeiro, C.C., *Optimization by GRASP: Greedy Randomized Adaptive Search Procedures*, Springer, 2013, Chapter 1

Representación gráfica de un grafo



$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$\begin{aligned} A \;=\; & \{(1,2),(1,3),(1,4),(2,4),(2,5),(3,4),(3,6),\\ & (4,5),(4,6),(4,9),(5,7),(5,8),(6,9),(7,8),\\ & (7,10),(8,1),(8,10),(9,10)\} \end{aligned}$$

Directed path[12]

Definition

Given a digraph $G = (V, A)$, a *directed path* $P_{st}(G)$ in a directed graph G from $s \in V$ to $t \in V$ is defined as a sequence of nodes $i_1, i_2, \ldots, i_{q-1}, i_q \in V$, where $i_1 = s, i_q = t$, and each arc $(i_k, i_{k+1}) \in A$, for any $k = 1, \ldots, q-1$.

[12] Resende, M.G.C. and Ribeiro, C.C., *Optimization by GRASP: Greedy Randomized Adaptive Search Procedures*, Springer, 2013, Chapter 1

Strongly connected graph[13]

Definition

A directed graph $G = (V, A)$ is said to be a *strongly connected graph* if there is at least one path $P_{st}(G)$ connecting node s to node t and another path $P_{ts}(G)$ connecting node s to node t, for every pair of nodes $s, t \in V$.

[13] Resende, M.G.C. and Ribeiro, C.C., *Optimization by GRASP: Greedy Randomized Adaptive Search Procedures*, Springer, 2013, Chapter 1

Hamiltonian paths and cycles[14]

Definition

A *Hamiltonian path* in a directed or undirected graph is a path between two nodes that visits each node of the graph exactly once. A *Hamiltonian cycle* in a directed or undirected graph is a *Hamiltonian path* that is also a cycle, i.e., its extremities coincide. Every *Hamiltonian cycle* corresponds to a circular permutation of the nodes of the graph. A *Hamiltonian cycle* is also known as a *Hamiltonian tour* or, simply, as a tour.

[14] Resende, M.G.C. and Ribeiro, C.C., *Optimization by GRASP: Greedy Randomized Adaptive Search Procedures*, Springer, 2013, Chapter 1