Optimización Avanzada

Exact methods

Juan Antonio Díaz García

Outline

1. Exact solution methods
   - Optimality, relaxation and bounds
   - Branch & Bound
   - Gomory's fractional cuts

Optimality and Relaxation[1]

Given an IP

$$z = \max\{c(x) : x \in X \subseteq \mathbb{Z}^n\}$$

we need to prove that a given point $x^\star$ is optimal.
We can prove it finding a lower bound $\underline{z} \leq z$ and an upper bound $\overline{z} \geq z$, such that $\underline{z} = \overline{z} = z$. In practice we need and algorithm to find a decreasing sequence

$$\overline{z}_1 > \overline{z}_2 > \ldots > \overline{z}_s \geq z$$

of upper bounds and an increasing sequence

$$\underline{z}_1 < \underline{z}_2 < \ldots < \underline{z}_t \leq z$$

of lower bounds, and stop when

$$\overline{z}_s - \underline{z}_t \leq \epsilon$$

where ϵ is some small non-negative value.

[1] Wolsey, L., Integer Programming, Wiley-Interscience Series in Discrete Mathematics ad Optimization, 1998.

Primal bounds

Every feasible solution $x^\star \in X$ provides a lower bound $\underline{z} = c(x^\star) \leq z$.

- For some IP problems, finding feasible solutions can be an easy task. for example, For the TSP, any permutation of the cities is a feasible tour.
- For other IP problems, finding feasible solutions may be very difficult.
- The most important challenge is to find good quality (optimal or near optimal) feasible solutions. Different heuristic and metaheuristic methods can be used to obtain primal bounds.

Dual bounds

Finding upper bounds for a maximization problem (or lower bounds for a minimization problem) presents a different challenge.

- Relaxation: replace a difficult max (min) IP problem by a simpler optimization problem whose optimal value is at least as large (small) as z. We have two different options to obtain a relaxed problem that satisfy the above property:
  1. Enlarge the set of feasible solutions so that one optimizes over a large set, or
  2. replace the max (min) objective function by a function that has the same or larger (smaller) value everywhere.

Definition

A problem (RP) $z^R = \max\{f(x) : x \in T \subset \mathbb{R}^n\}$ is a *relaxation* of a problem (IP) $z = \max\{c(x) : x \in X \subseteq \mathbb{R}^n\}$ if:

1. $X \subseteq T$, and
2. $f(x) \geq c(x)$ for all $x \in X$.

Dual bounds

Proposition

If RP is a relaxation of IP, $z^R \geq z$

Proof

- If $x^\star$ is an optimal solution of IP, $x^\star$ is a feasible solution for IP, then $x^\star \in X \subseteq T$. Therefore $z = c(x^\star) \leq f(x^\star)$, because $f(x) \geq c(x)$ for all $x \in X$.
- Since $x^\star \in T$, $f(x^\star)$ is a lower bound on z^R, and $z \leq f(x^\star) \leq z^R$.

Linear programming relaxations

Definition:

For the integer program $\max\{cx : x \in P \cap \mathbb{Z}^n\}$ with formulation $P = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$, the *linear programming relaxation* is the linear program $z^{LP} = \max\{cx : x \in P\}$.

Branch and bound[2]

Branch and bound is a divide and conquer strategy to solve an IP.

Consider the problem:

$$z = \max\{cx : x \in S\}$$

The strategy is to divide the problem into a series of smaller problems that are easier, solve the smaller problems, and then put the information together again to solve the original problem.

Proposition:

Let $S = S_1 \cup S_2 \cup \ldots \cup S_K$ be a decomposition of S into a smaller sets, and let $z^k = \max\{cx : x \in S_k\}$ for $k = 1, \ldots, K$. Then $z = \max_{k \in \{1,\ldots,K\}}\{z^k\}$

[2] Wolsey, L., Integer Programming, Wiley-Interscience Series in Discrete Mathematics ad Optimization, 1998.

Example

Branch & Bound using linear programming relaxations to provide bounds.

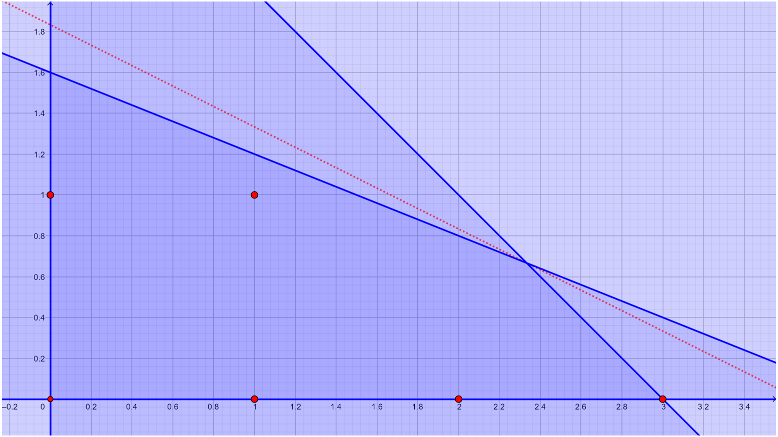
$$\begin{array}{llccccc} \max & z = & x_1 & + & 2x_2 & & \\ & & 2x_1 & + & 5x_2 & \leq & 8 \\ & & x_1 & + & x_2 & \leq & 3 \\ & & x_1 & , & x_2 & \in & \mathbb{Z}_+ \end{array}$$

Example

Relaxing integrality constraints we obtain the following linear program which provides an upper bound for the optimal solution

$$
\begin{array}{lrcrcl}
\max \quad z = & x_1 & + & 2x_2 & & \\
& 2x_1 & + & 5x_2 & \leq & 8 \\
& x_1 & + & x_2 & \leq & 3 \\
& x_1 & , & x_2 & \geq & 0
\end{array}
$$

Example



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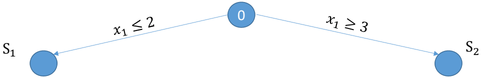
Example

Best lower bound: $\underline{z}^* = -\infty$



Optimal Solution

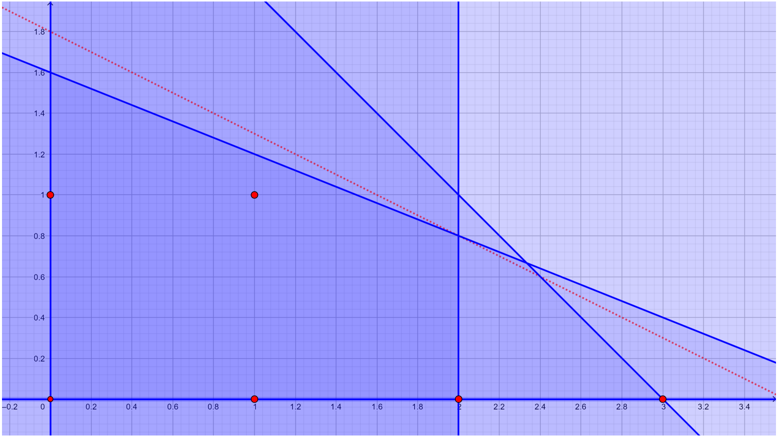
$$\begin{aligned} \overline{z} &= 3.6667 \\ x_1 &= 2.3333 \\ x_2 &= 0.6667 \end{aligned}$$

$$S = \left\{ \binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \binom{0}{1}, \binom{1}{1} \right\}$$

$$S_1 = \left\{ \binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{0}{1}, \binom{1}{1} \right\}$$

$$S_2 = \left\{ \binom{3}{0} \right\}$$

Example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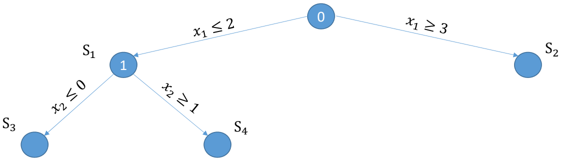
Example

Best lower bound: $\underline{z}^* = -\infty$



Optimal Solution

$$\begin{aligned} \overline{z} &= 3.6 \\ x_1 &= 2.0 \\ x_2 &= 0.8 \end{aligned}$$

$$S = \left\{\binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \binom{0}{1}, \binom{1}{1}\right\}$$

$$S_2 = \left\{\binom{3}{0}\right\}$$

$$S_3 = \left\{\binom{0}{0}, \binom{1}{0}, \binom{2}{0}\right\}$$

$$S_4 = \left\{\binom{0}{1}, \binom{1}{1}\right\}$$

Example

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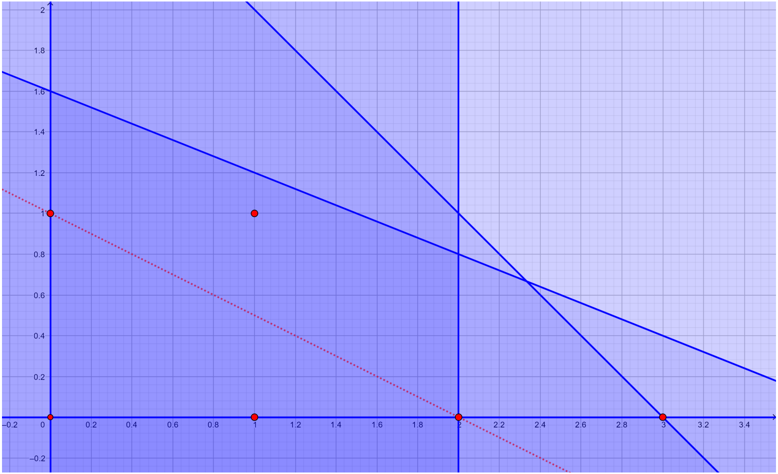
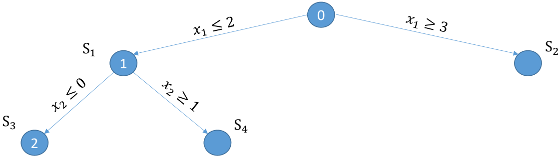


Example

Best lower bound: $\underline{z}^* = 2$



Optimality

Optimal Solution

$$\begin{aligned} \overline{z} &= 2.0 \\ x_1 &= 2.0 \\ x_2 &= 0.0 \end{aligned}$$

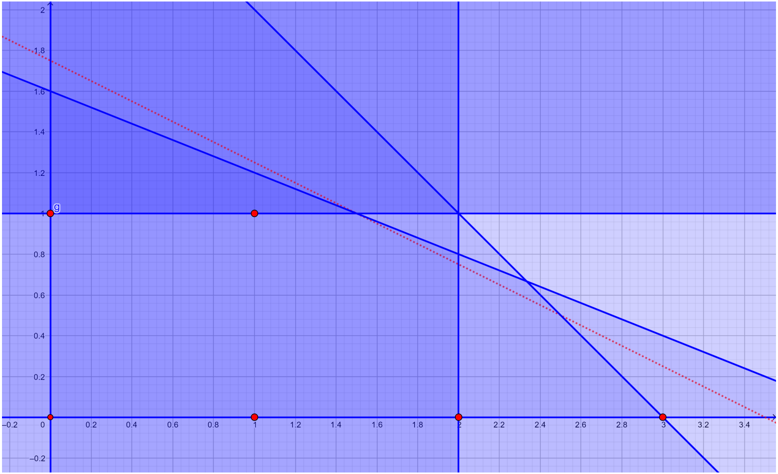
$$S = \left\{ \binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \binom{0}{1}, \binom{1}{1} \right\}$$

$$S_2 = \left\{ \binom{3}{0} \right\}$$

$$S_3 = \left\{ \binom{0}{0}, \binom{1}{0}, \binom{2}{0} \right\}$$

$$S_4 = \left\{ \binom{0}{1}, \binom{1}{1} \right\}$$

Example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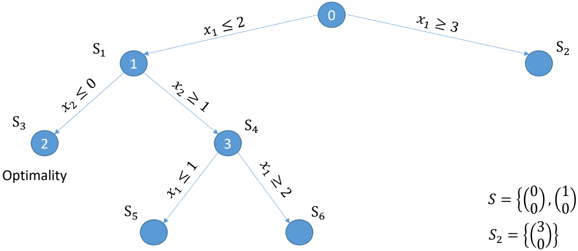
Example

Best lower bound: $\underline{z}^* = 2$



Optimal Solution

$$\begin{aligned} \overline{z} &= 3.5 \\ x_1 &= 1.5 \\ x_2 &= 1.0 \end{aligned}$$

$$S = \left\{\binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \binom{0}{1}, \binom{1}{1}\right\}$$

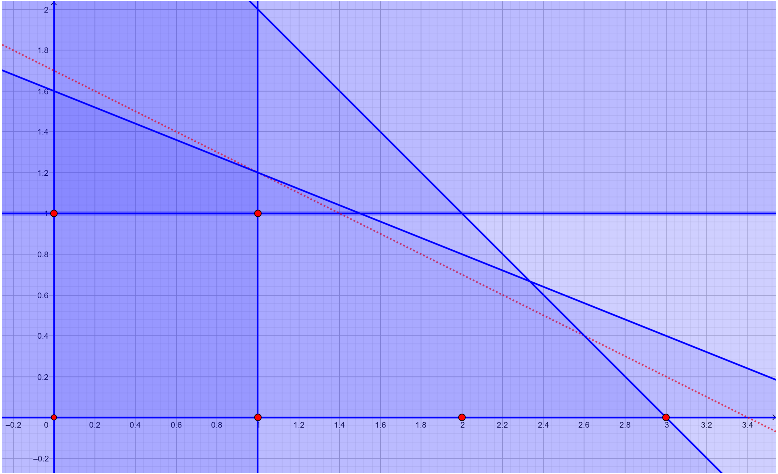
$$S_2 = \left\{\binom{3}{0}\right\}$$

$$S_3 = \left\{\binom{0}{0}, \binom{1}{0}, \binom{2}{0}\right\}$$

$$S_5 = \left\{\binom{0}{1}, \binom{1}{1}\right\}$$

$$S_6 = \emptyset$$

Example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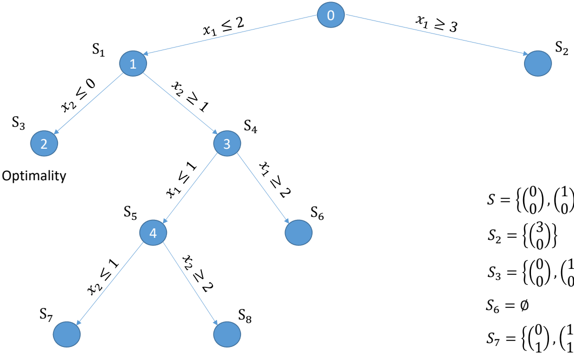
Example

Best lower bound: $\underline{z}^* = 2$



Optimal Solution

$$\begin{aligned} \overline{z} &= 3.4 \\ x_1 &= 1.0 \\ x_2 &= 1.2 \end{aligned}$$

$$S = \left\{\binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \binom{0}{1}, \binom{1}{1}\right\}$$

$$S_2 = \left\{\binom{3}{0}\right\}$$

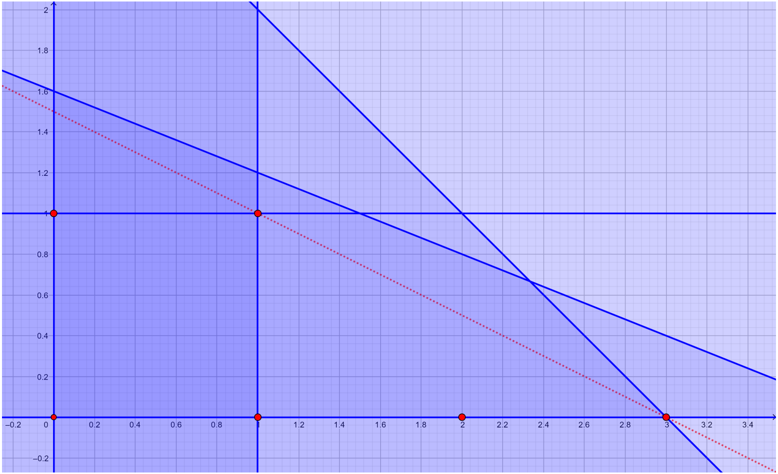
$$S_3 = \left\{\binom{0}{0}, \binom{1}{0}, \binom{2}{0}\right\}$$

$$S_6 = \emptyset$$

$$S_7 = \left\{\binom{0}{1}, \binom{1}{1}\right\}$$

$$S_8 = \emptyset$$

Example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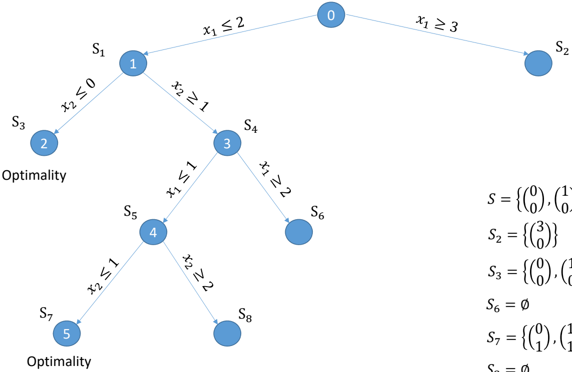
Example

Best lower bound: $\underline{z}^* = 3$



Optimal Solution

$$\begin{aligned} \overline{z} &= 3.0 \\ x_1 &= 1.0 \\ x_2 &= 1.0 \end{aligned}$$

$$S = \left\{ \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}3\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}$$

$$S_2 = \left\{ \begin{pmatrix}3\\0\end{pmatrix} \right\}$$

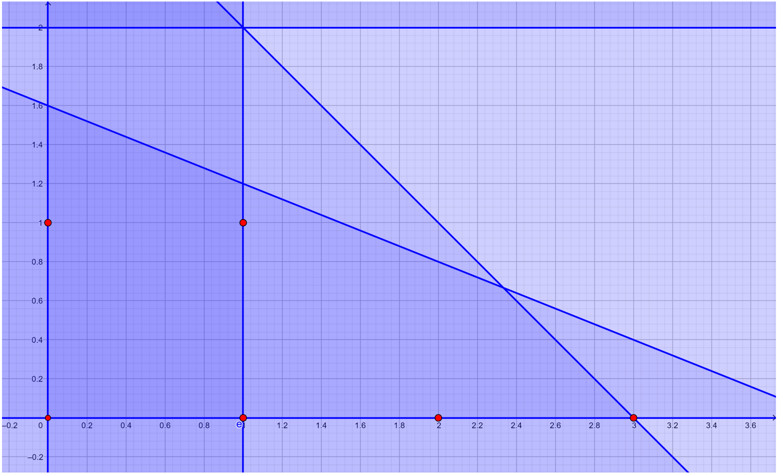
$$S_3 = \left\{ \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix} \right\}$$

$$S_6 = \emptyset$$

$$S_7 = \left\{ \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}$$

$$S_8 = \emptyset$$

Example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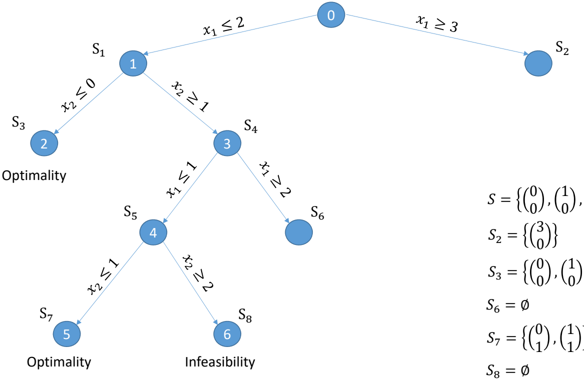
Example

Best lower bound: $\underline{z}^* = 3$



$S = \left\{ \binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \binom{0}{1}, \binom{1}{1} \right\}$

$S_2 = \left\{ \binom{3}{0} \right\}$

$S_3 = \left\{ \binom{0}{0}, \binom{1}{0}, \binom{2}{0} \right\}$

$S_6 = \emptyset$

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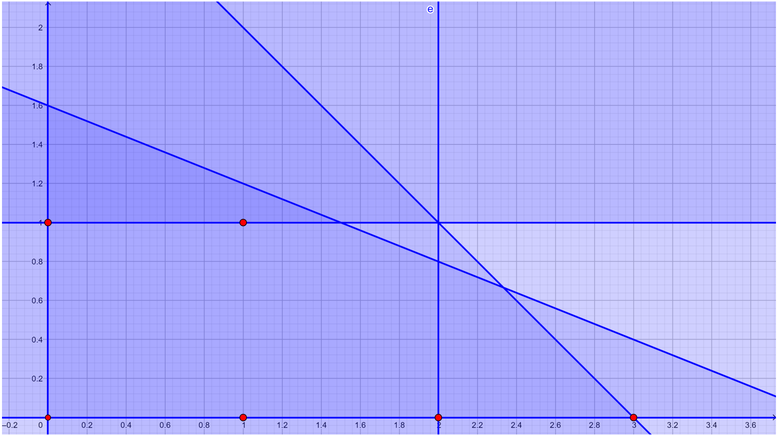
$S_8 = \emptyset$

Example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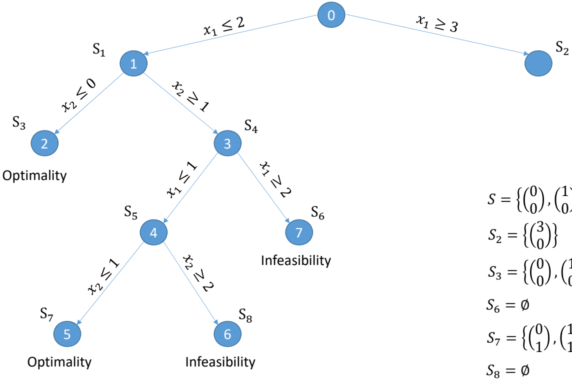
Example

Best lower bound: $\underline{z}^* = 3$



$$S = \left\{\binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \binom{0}{1}, \binom{1}{1}\right\}$$

$$S_2 = \left\{\binom{3}{0}\right\}$$

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$$S_6 = \emptyset$$

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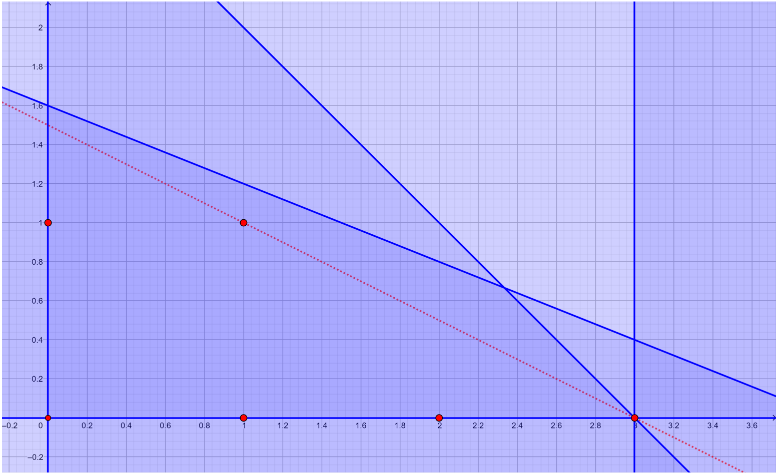
$$S_8 = \emptyset$$

Example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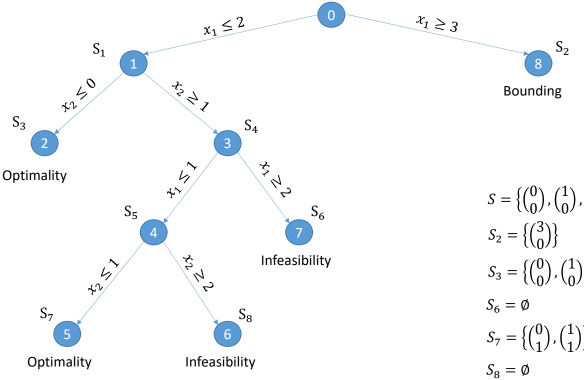
Example

Best lower bound: $\underline{z}^* = 3$



Optimal Solution

$$\begin{aligned} \overline{z} &= 3.0 \\ x_1 &= 3.0 \\ x_2 &= 0.0 \end{aligned}$$

$$S = \left\{ \binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \binom{0}{1}, \binom{1}{1} \right\}$$

$$S_2 = \left\{ \binom{3}{0} \right\}$$

$$S_3 = \left\{ \binom{0}{0}, \binom{1}{0}, \binom{2}{0} \right\}$$

$$S_6 = \emptyset$$

$$S_7 = \left\{ \binom{0}{1}, \binom{1}{1} \right\}$$

$$S_8 = \emptyset$$

Gomory's fractional cuts

Consider the following linear programming problem

$$\begin{array}{llrcl} (IP) & \max & \mathbf{cx} & & \\ & \text{s.t} & \mathbf{Ax} & = & \mathbf{b} \\ & & \mathbf{x} & \in & \mathbb{Z}_+^n \end{array}$$

where $\mathbf{A}$ is a $m \times n$ matrix with rank m, $\mathbf{c}$ is an n row vector, $\mathbf{x}$ an n column vector and $\mathbf{b}$ an m row vector. Let the following simplex tableau be the tableau associated with an optimal basic solution of the linear programming relaxation of (IP).

	z	x_1	x_2	$\ldots$	x_{B_i}	$\ldots$	x_n	RHS
z	1	$z_1 - c_1$	$z_2 - c_2$	$\ldots$	$z_{B_i} - c_{B_i}$	$\ldots$	$z_n - c_n$	z_0
x_{B_1}	0	y_{11}	y_{12}	$\ldots$	0	$\ldots$	y_{1n}	$\bar{b}_1$
x_{B_2}	0	y_{21}	y_{22}	$\ldots$	0	$\ldots$	y_{2n}	$\bar{b}_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{B_i}	0	y_{i1}	y_{i2}	$\ldots$	1	$\ldots$	y_{in}	$\bar{b}_i$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{B_m}	0	y_{m1}	y_{m2}	$\ldots$	0	$\ldots$	y_{mn}	$\bar{b}_m$

Gomory's fractional cuts

Suppose that for row i, $\bar{b}_i \notin \mathbb{Z}$. Choosing row i we have that

$$x_{B_i} + \sum_{j \in R} y_{ij} x_j = \bar{b}_i \tag{1}$$

we can rewrite (1) in the following way

$$x_{B_i} + \sum_{j \in R} \left(\lfloor y_{ij} \rfloor + f_{ij} \right) x_j = \lfloor \bar{b}_i \rfloor + F_i \tag{2}$$

where $f_{ij} = y_{ij} - \lfloor y_{ij} \rfloor$, $F_i = \bar{b}_i - \lfloor \bar{b}_i \rfloor$, R is the index set of non-basic variables, and $\lfloor \alpha \rfloor$ is the greatest integer that is less than or equal to α. Gomory's fractional cut associated with row i is

$$F_i - \sum_{j \in R} f_{ij} x_j \leq 0 \quad or \quad \sum_{j \in R} f_{ij} x_j \geq F_i$$

Gomory's fractional cuts

Proposition

The Gomory's fractional cut, $\sum_{j\in R} f_{ij}x_j \geq F_i$, cuts off the fractional solution associated with the optimal feasible solution of the linear programming relaxation of (IP). That is, the current optimal solution to the linear programming relaxation of (IP) will not satisfy the cut.

By choosing row i associated with a non-integer basic variable and since $0 \leq f_{ij} < 1$, $0 < F_i < 1$, and $x_j = 0, \forall j \in R$, these inequality cuts off the optimal point of the linear programming relaxation.

Gomory's fractional cuts

Proposition

All feasible points of (IP) satisfy te The Gomory's fractional cut $\sum_{j\in R} f_{ij}x_j \geq F_i$.

We can rewrite equation (2) in the following way

$$x_{B_i} + \sum_{j\in R} \lfloor y_{ij} \rfloor x_j - \lfloor \bar{b}_i \rfloor = F_i - \sum_{j\in R} f_{ij}x_j, \qquad (3)$$

and we can observe that for any feasible integer point for (IP), the left hand side of (3) is an integer value that is less than 1. Therefore the cut $\sum_{j\in R} f_{ij}x_j \geq F_i$ will not cut any feasible point for (IP).

Gomory's fractional cuts

Consider the following integer programming problem and the simplex tableau of its linear programming relaxation:

$$\begin{array}{lrcrcl} \max \; z = & 4x_1 & - & x_2 & & \\ & 7x_1 & - & 2x_2 & \leq & 14 \\ & & & x_2 & \leq & 3 \\ & 2x_1 & - & 2x_2 & \leq & 3 \\ & x_1 & , & x_2 & \in & \mathbb{Z}^n_+ \end{array}$$

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	0	$\frac{4}{7}$	$\frac{1}{7}$	0	$\frac{59}{7}$
x_2	0	0	1	0	1	0	3
x_5	0	0	0	$-\frac{2}{7}$	$\frac{10}{7}$	1	$\frac{23}{7}$
x_1	0	1	0	$\frac{1}{7}$	$\frac{2}{7}$	0	$\frac{20}{7}$

Gomory's fractional cuts

We can use the rows associated with fractional basic variables x_1 and x_5,

$$x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 = \frac{20}{7} \tag{4}$$

$$x_5 - \frac{2}{7}x_3 + \frac{10}{7}x_4 = \frac{23}{7} \tag{5}$$

to generate Gomory's fractional cuts.
The Gomory's fractional cut associated with (4) is

$$\frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{6}{7},$$

and the Gomory's fractional cut associated with (5) is

$$\frac{5}{7}x_3 + \frac{3}{7}x_4 \geq \frac{2}{7}$$

Gomory's fractional cuts

Definition[a]

[a]Taha, H. Investigación de Operaciones, 5a. Edición, Alfaomega, 1998.

We say that a cut

$$\sum_{j\in R} f_{ij}x_j \geq F_i$$

is stronger thant the cut

$$\sum_{j\in R} f_{kj}x_j \geq F_k$$

si $F_i \geq F_k$ and $f_{ij} \leq f_{kj}$ for all $j \in R$ and at least one of these inequalities is strictly satisfied.

Gomory's fractional cuts

From the above definition, we observe that

$$\frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{6}{7}, \tag{6}$$

is stronger than

$$\frac{5}{7}x_3 + \frac{3}{7}x_4 \geq \frac{2}{7} \tag{7}$$

Substituting for $x_3 = 14 - 7x_1 + 2x_2$ and $x_4 = 3 - x_2$ in (6) and (7) we obtain

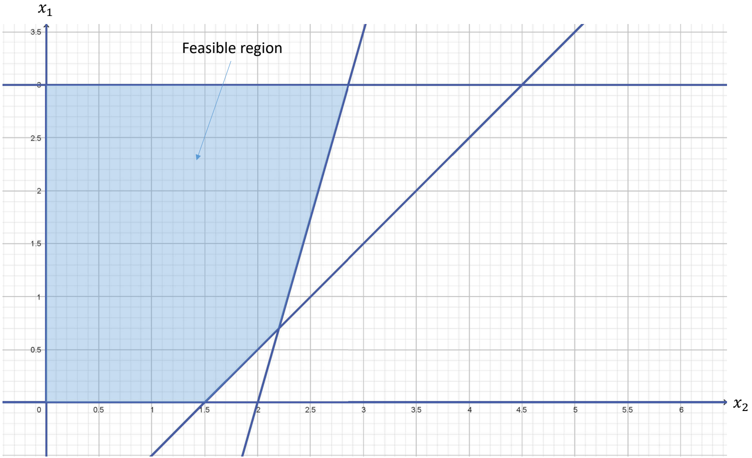
$$x_1 \leq 2 \tag{8}$$

and

$$5x_1 - x_2 \leq 11 \tag{9}$$

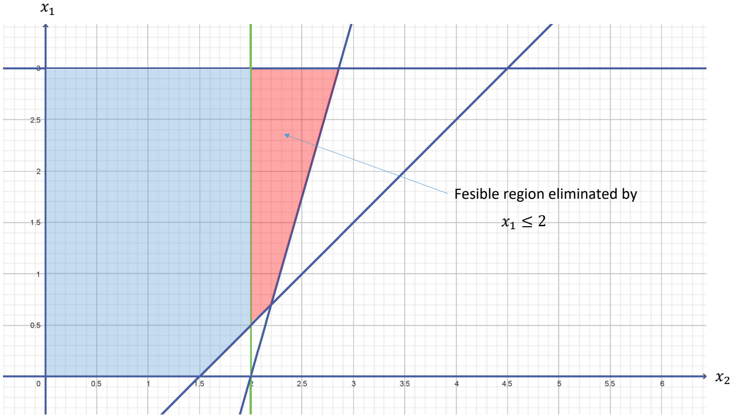
Gomory's fractional cut

Initial feasible region:



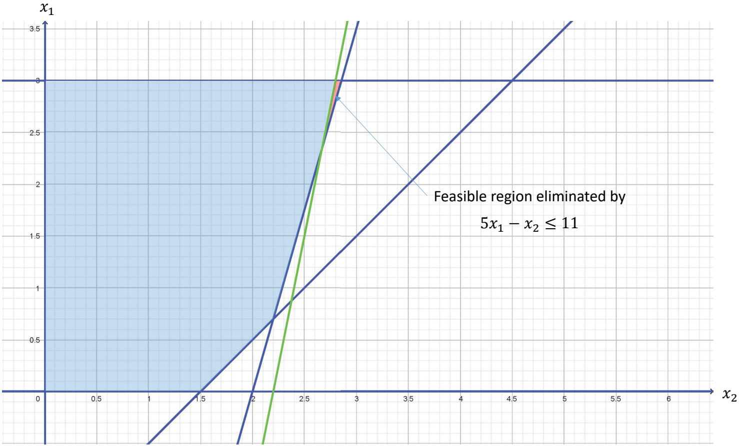
Gomory's fractional cut

After adding Gomory's fractional cut $x_1 \leq 2$:



Gomory's fractional cut

Initial feasible region:



Gomory's fractional cuts

Empirical rules when to select a Gomory's fractional cut when none of the cuts is stronger than the others[3]:

Rule 1:

Select the row $i^\star$ such that

$$i^\star \in \arg\max_{i \in 1,\ldots,m: \bar{b}_i \notin \mathbb{Z}} \{F_i\}$$

Rule 2:

Select the row $i^\star$ such that

$$i^\star \in \arg\max_{i \in 1,\ldots,m: \bar{b}_i \notin \mathbb{Z}} \left\{ \frac{F_i}{\sum_{j \in R} f_{ij}} \right\}$$

[3] Taha, H. Investigación de Operaciones, 5a Edición, Alfaomega, 1998.

Gomory's fractional cuts

Since in our example $\frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{6}{7}$ is stronger than $\frac{5}{7}x_3 + \frac{3}{7}x_4 \geq \frac{2}{7}$ we add the Gomory's fractional cut to the optimal tableau of the linear programming relaxation. We first write the cut in standard form by including the slack variable x_6

$$\frac{1}{7}x_3 + \frac{2}{7}x_4 - x_6 = \frac{6}{7}$$

and multiply in both sides by -1

$$-\frac{1}{7}x_3 - \frac{2}{7}x_4 + x_6 = -\frac{6}{7}$$

After adding the above inequality to the optimal simplex tableau of the linear programming relaxation, the new optimal solution is obtained using the dual simplex method.

Gomory's fractional cuts

Gomory's fractional cutting plane algorithm

$Stop \leftarrow$ **false**
repeat
  Solve the linear programming relaxation of (IP).
  if $(x \notin \mathbb{Z}^n_+)$ **then**
    Generate Gomory's fractional cuts associated with non-integer basic variables.
    Select one of the Gomory's fractional cuts an add it to (IP) and reoptimize using dual simplex method.
  else
    $Stop \leftarrow$ **true**
  end if
until $(Stop)$

Activities

1. Continue solving the sample problem until you get the optimal solution to the problem. In each iteration:
   - Provide all Gomory's fractional cuts associated with non-integer basic variables in the optimal solution of the linear programming relaxation of (IP)
   - Select one of the Gomory's fractional cut.
   - Show the feasible region of the linear programming relaxation of the current (IP) eliminated by the cut.