

Financial Engineering & Risk Management

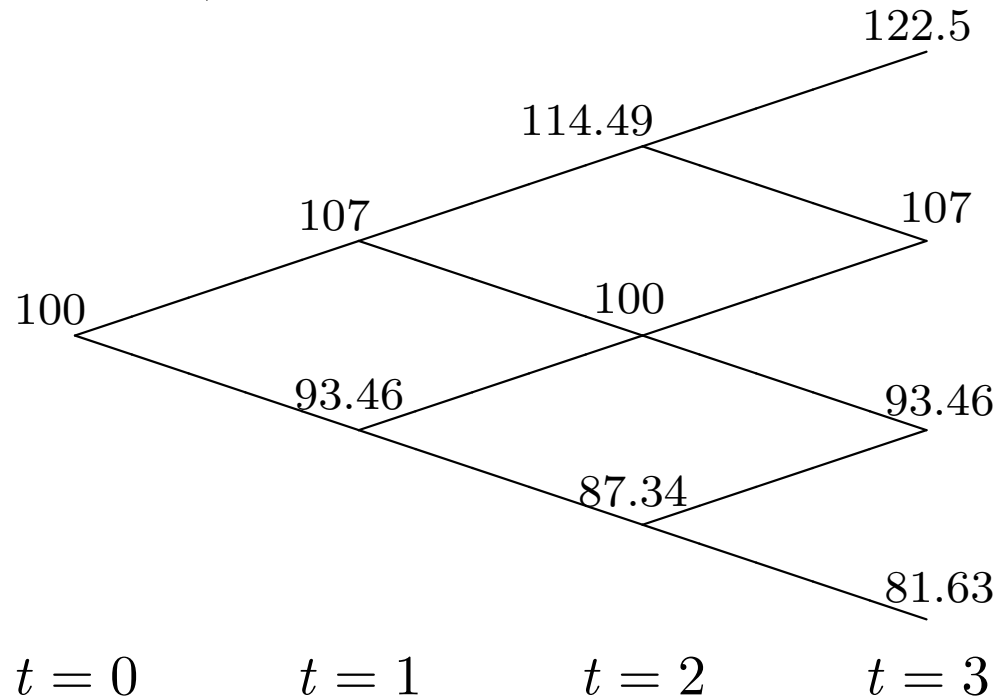
The Multi-Period Binomial Model

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A 3-period Binomial Model

Recall $R = 1.01$ and $u = 1/d = 1.07$.

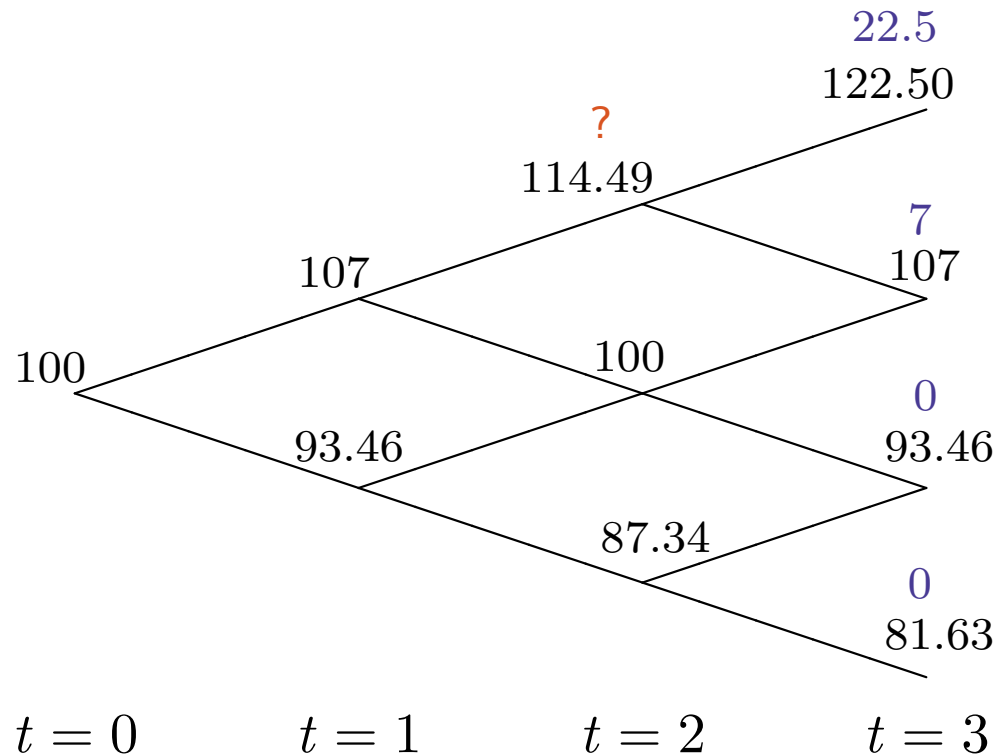


Just a series of **1-period models spliced together!**

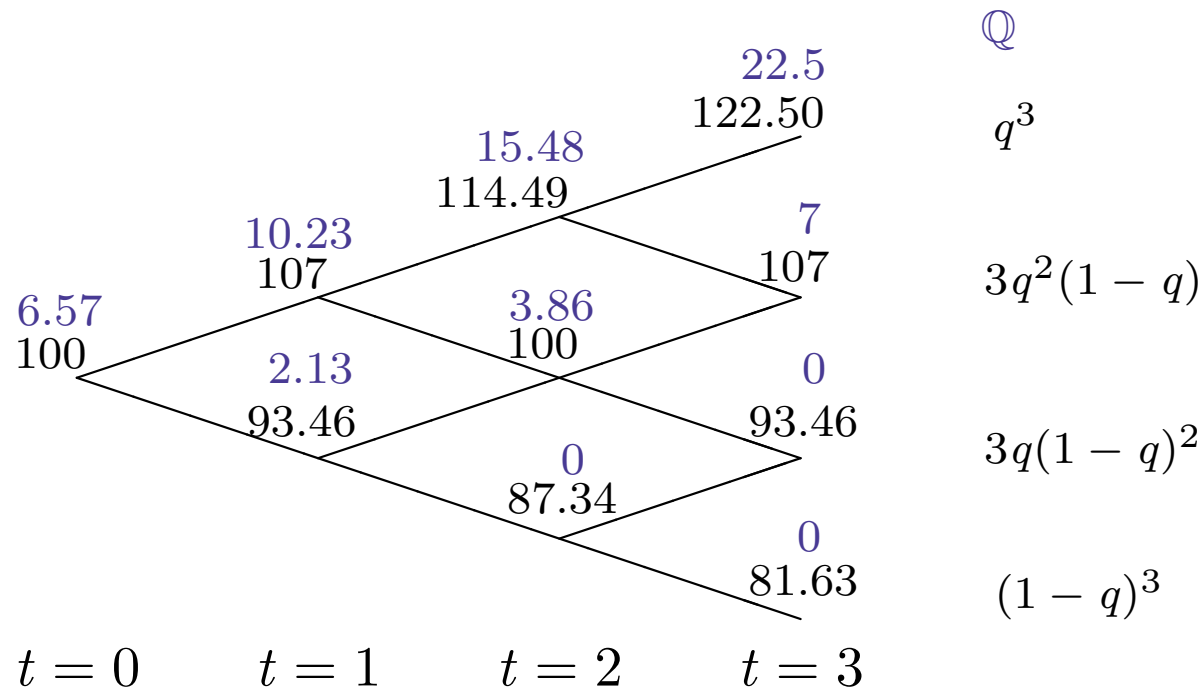
- all the results from the 1-period model apply
- just need to **multiply 1-period probabilities** along branches to get probabilities in multi-period model.

Pricing a European Call Option

Assumptions: expiration at $t = 3$, strike = \$100 and $R = 1.01$.



Pricing a European Call Option



- We can also calculate the price as

$$C_0 = \frac{1}{R^3} E_0^{\mathbb{Q}} [\max(S_T - 100, 0)] \quad (1)$$

- this is **risk-neutral pricing** in the binomial model
- avoids having to calculate the price at every node.
- How would you find a **replicating strategy**?
 - to be defined and discussed in another module.

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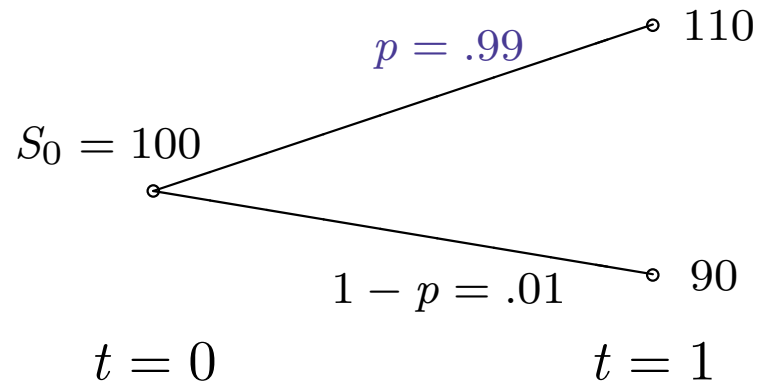
What's Going On?

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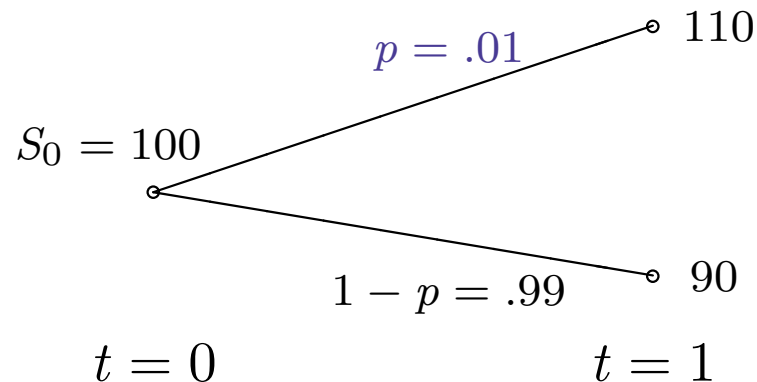
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What's Going On?

- Stock ABC



- Stock XYZ



Question: What is the price of a call option on ABC with strike $K = \$100$?

Question: What is the price of a call option on XYZ with strike $K = \$100$?

What's Going On?

- Saw earlier

$$\begin{aligned}C_0 &= \frac{1}{R} \left[\frac{R-d}{u-d} C_u + \frac{u-R}{u-d} C_d \right] \\&= \frac{1}{R} [q C_u + (1-q) C_d] \\&= \frac{1}{R} \mathbb{E}_0^{\mathbb{Q}}[C_1]\end{aligned}$$

- So it appears that p doesn't matter!
- This is true ...
- ... but it only appears surprising because we are asking the **wrong question!**

Another Surprising Result?

$$R = 1.02$$

Stock Price				European Option Price: K = 95			
			119.10				24.10
		112.36	106.00			19.22	11.00
	106.00	100.00	94.34		14.76	7.08	0.00
100.00	94.34	89.00	83.96	11.04	4.56	0.00	0.00
t=0	t=1	t=2	t=3	t=0	t=1	t=2	t=3

$$R = 1.04$$

Stock Price				European Option Price: K = 95			
			119.10				24.10
		112.36	106.00			21.01	11.00
	106.00	100.00	94.34		18.19	8.76	0.00
100.00	94.34	89.00	83.96	15.64	6.98	0.00	0.00
t=0	t=1	t=2	t=3	t=0	t=1	t=2	t=3

Question: So the option price increases when we increase R . Is this surprising?

(See “*Investment Science*” (OUP) by D. G. Luenberger for additional examples on the binomial model.)

Existence of Risk-Neutral Probabilities \Leftrightarrow No-Arbitrage

Recall our analysis of the binomial model:

- no arbitrage $\Leftrightarrow d < R < u$
- any derivative security with time T payoff, C_T , can be priced using

$$C_0 = \frac{1}{R^n} \mathbb{E}_0^{\mathbb{Q}}[C_T] \quad (2)$$

where $q > 0$, $1 - q > 0$ and $n = \#$ of periods.

(If Δt is the length of a period, then $T = n \times \Delta t$.)

In fact for any model if there exists a risk-neutral distribution, \mathbb{Q} , such that (2) holds, then arbitrage cannot exist. Why?

Reverse is also true: if there is no arbitrage then a risk-neutral distribution exists.

Together, these last two statements are often called the **first fundamental theorem of asset pricing**.