

# Financial Engineering and Risk Management

Review of matrices

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# Matrices

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- Matrices are rectangular arrays of real numbers

- Examples:

- $\mathbf{A} = \begin{bmatrix} 2 & 3 & 7 \\ 1 & 6 & 5 \end{bmatrix} : 2 \times 3 \text{ matrix}$

- $\mathbf{B} = \begin{bmatrix} 2 & 3 & 7 \end{bmatrix} : 1 \times 3 \text{ matrix} \equiv \text{row vector}$

- $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} : \mathbf{m} \times \mathbf{n} \text{ matrix} \dots \mathbb{R}^{\mathbf{m} \times \mathbf{n}}$

- $\mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \dots n \times n \text{ Identity matrix}$

- Vectors are clearly also matrices

# Matrix Operations: Transpose

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- Transpose:  $\mathbf{A} \in \mathbb{R}^{m \times d}$

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1d} \\ a_{21} & a_{22} & \dots & a_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{md} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{d2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1d} & a_{2d} & \dots & a_{md} \end{bmatrix} \in \mathbb{R}^{d \times m}$$

- Transpose of a row vector is a column vector
- **Example:**
  - $\mathbf{A} = \begin{bmatrix} 2 & 3 & 7 \\ 1 & 6 & 5 \end{bmatrix}$ :  $2 \times 3$  matrix ...  $\mathbf{A}^T = \begin{bmatrix} 2 & 1 \\ 3 & 6 \\ 7 & 5 \end{bmatrix}$ :  $3 \times 2$  matrix
  - $\mathbf{v} = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$ : column vector ...  $\mathbf{v}^T = [2 \quad 6 \quad 4]$ : row vector

# Matrix Operations: Multiplication

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- Multiplication:  $\mathbf{A} \in \mathbb{R}^{m \times d}$ ,  $\mathbf{B} \in \mathbb{R}^{d \times p}$  then  $\mathbf{C} = \mathbf{AB} \in \mathbb{R}^{m \times p}$

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{id} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{dj} \end{bmatrix}$$

- row vector  $\mathbf{v} \in \mathbb{R}^{1 \times d}$  times column vector  $\mathbf{w} \in \mathbb{R}^{d \times 1}$  is a scalar.
- Identity times any matrix  $\mathbf{A}\mathbf{I}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$

- **Examples:**

- $\begin{bmatrix} 2 & 3 & 7 \\ 1 & 6 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 2(2) + 3(6) + 7(4) \\ 1(2) + 6(6) + 5(4) \end{bmatrix} = \begin{bmatrix} 50 \\ 58 \end{bmatrix}$

- $\ell_2$  norm:  $\left\| \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\|_2 = \sqrt{1^2 + (-2)^2} = \sqrt{\begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}} = \sqrt{\begin{bmatrix} 1 \\ -2 \end{bmatrix}^\top \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$

- inner product:  $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^\top \mathbf{w}$

# Linear functions

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- A function  $f : \mathbb{R}^d \mapsto \mathbb{R}^m$  is **linear** if

$$f(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \alpha, \beta \in \mathbb{R}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^d$$

- A function  $f$  is linear if and only if  $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$  for matrix  $\mathbf{A} \in \mathbb{R}^{m \times d}$

- **Examples**

- $f(\mathbf{x}) : \mathbb{R}^3 \mapsto \mathbb{R}: f(\mathbf{x}) = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2x_1 + 3x_2 + 4x_3$

- $f(\mathbf{x}) : \mathbb{R}^3 \mapsto \mathbb{R}^2: f(\mathbf{x}) = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ x_1 + 2x_3 \end{bmatrix}$

- Linear **constraints** define sets of vectors that satisfy linear relationships
  - Linear equality:  $\{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\}$  ... line, plane, etc.
  - Linear inequality:  $\{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$  ... half-space

# Rank of a matrix

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- **column** rank of  $\mathbf{A} \in \mathbb{R}^{m \times d}$  = number of linearly independent **columns**
  - $\text{range}(\mathbf{A}) = \{\mathbf{y} : \mathbf{y} = \mathbf{A}\mathbf{x} \text{ for some } \mathbf{x}\}$
  - **column** rank of  $\mathbf{A}$  = size of basis for  $\text{range}(\mathbf{A})$
  - **column** rank of  $\mathbf{A} = m \Rightarrow \text{range}(\mathbf{A}) = \mathbb{R}^m$
- **row** rank of  $\mathbf{A}$  = number of linearly independent **rows**
- **Fact**: row rank = column rank  $\leq \min\{m, d\}$
- **Example**:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}, \quad \text{rank} = 1, \quad \text{range}(\mathbf{A}) = \left\{ \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix} : \lambda \in \mathbb{R} \right\}$$

- $\mathbf{A} \in \mathbb{R}^{n \times n}$  and  $\text{rank}(\mathbf{A}) = n \Rightarrow \mathbf{A}$  invertible, i.e.  $\mathbf{A}^{-1} \in \mathbb{R}^{n \times n}$

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$