# Financial Engineering & Risk Management

## Introduction to Brownian Motion

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# Brownian Motion

**Definition.** We say that a random process, $\{X_t : t \geq 0\}$, is a Brownian motion with parameters $(\mu, \sigma)$ if

1. For $0 < t_1 < t_2 < \ldots < t_{n-1} < t_n$

$$(X_{t_2} - X_{t_1}),\ (X_{t_3} - X_{t_2}),\ \ldots,\ \ (X_{t_n} - X_{t_{n-1}})$$

   are mutually independent.

2. For $s > 0$, $X_{t+s} - X_t \ \sim\ \mathsf{N}(\mu s, \sigma^2 s)$ and
3. $X_t$ is a continuous function of $t$.

We say that $X_t$ is a $B(\mu, \sigma)$ Brownian motion with drift $\mu$ and volatility $\sigma$

Property $\#1$ is often called the independent increments property.

**Remark.** Bachelier (1900) and Einstein (1905) were the first to explore Brownian motion from a mathematical viewpoint whereas Wiener (1920's) was the first to show that it actually exists as a well-defined mathematical entity.

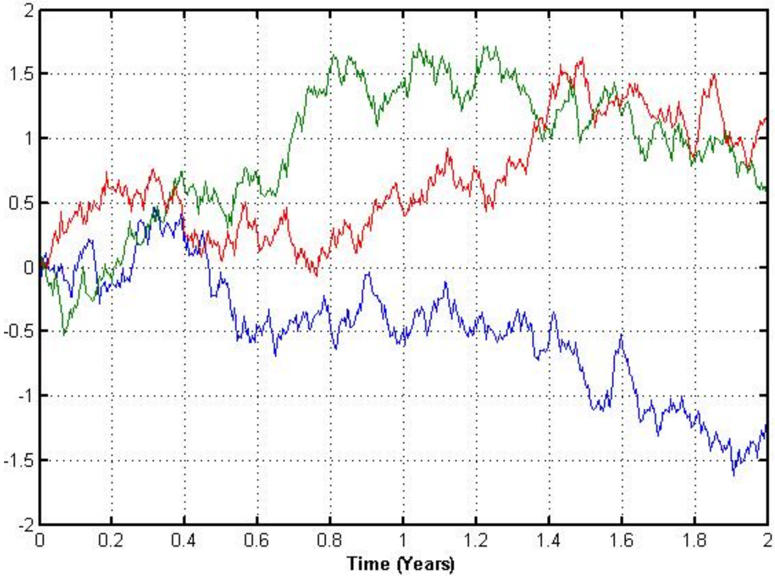
# Standard Brownian Motion

- When $\mu = 0$ and $\sigma = 1$ we have a standard Brownian motion (SBM).
- We will use $W_t$ to denote a SBM and we always assume that $W_0 = 0$.
- Note that if $X_t \sim B(\mu, \sigma)$ and $X_0 = x$ then we can write

$$X_t = x + \mu t + \sigma W_t \tag{8}$$

where $W_t$ is an SBM. Therefore see that $X_t \sim \mathsf{N}\left(x + \mu t,\, \sigma^2 t\right)$.

# Sample Paths of Brownian Motion



# Information Filtrations

- For any random process we will use $\mathcal{F}_t$ to denote the information available at time $t$
  - the set $\{\mathcal{F}_t\}_{t\geq 0}$ is then the information filtration
  - so $\mathsf{E}[\cdot \mid \mathcal{F}_t]$ denotes an expectation conditional on time $t$ information available.
- Will usually write $\mathsf{E}[\cdot \mid \mathcal{F}_t]$ as $\mathsf{E}_t[\,\cdot\,]$.

**Important Fact:** The independent increments property of Brownian motion implies that any function of $W_{t+s} - W_t$ is independent of $\mathcal{F}_t$ and that

$$(W_{t+s} - W_t) \sim \mathsf{N}(0,\ s).$$

# A Brownian Motion Calculation

Question: What is $\mathsf{E}_0[W_{t+s} W_s]$?

Answer: We can use a version of the conditional expectation identity to obtain

$$\begin{aligned} \mathsf{E}_0[W_{t+s} W_s] &= \mathsf{E}_0[(W_{t+s} - W_s + W_s) W_s] \\ &= \mathsf{E}_0[(W_{t+s} - W_s) W_s] + \mathsf{E}_0\left[W_s^2\right]. \end{aligned} \tag{9}$$

Now we know (why?) $\mathsf{E}_0[W_s^2] = s$.

To calculate first term on r.h.s. of (9) a version of the conditional expectation identity implies

$$\begin{aligned} \mathsf{E}_0[(W_{t+s} - W_s) W_s] &= \mathsf{E}_0[\mathsf{E}_s[(W_{t+s} - W_s) W_s]] \\ &= \mathsf{E}_0[W_s \mathsf{E}_s[(W_{t+s} - W_s)]] \\ &= \mathsf{E}_0[W_s 0] \\ &= 0. \end{aligned}$$

Therefore obtain $\mathsf{E}_0[W_{t+s} W_s] = s$.

# Financial Engineering & Risk Management

## Geometric Brownian Motion

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# Geometric Brownian Motion

**Definition.** We say that a random process, $X_t$, is a geometric Brownian motion (GBM) if for all $t \geq 0$

$$X_t = e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

where $W_t$ is a standard Brownian motion.

We call $\mu$ the drift, $\sigma$ the volatility and write $X_t \sim \text{GBM}(\mu, \sigma)$.

Note that

$$\begin{aligned} X_{t+s} &= X_0 \, e^{\left(\mu - \frac{\sigma^2}{2}\right)(t+s) + \sigma W_{t+s}} \\ &= X_0 \, e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)s + \sigma(W_{t+s} - W_t)} \\ &= X_t \, e^{\left(\mu - \frac{\sigma^2}{2}\right)s + \sigma(W_{t+s} - W_t)} \end{aligned} \tag{10}$$

– a representation that is very useful for simulating security prices.

# Geometric Brownian Motion

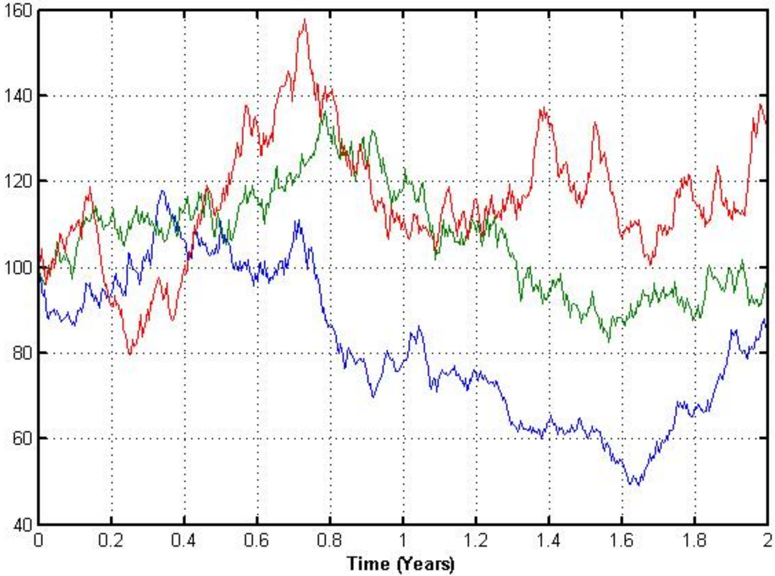
Question: Suppose $X_t \sim \mathsf{GBM}(\mu, \sigma)$. What is $\mathsf{E}_t[X_{t+s}]$?

Answer: From (10) we have

$$
\begin{aligned}
\mathsf{E}_t[X_{t+s}] &= \mathsf{E}_t\left[X_t\, e^{\left(\mu - \frac{\sigma^2}{2}\right)s \;+\; \sigma(\,W_{t+s} - W_t)}\right] \\
&= X_t\, e^{\left(\mu - \frac{\sigma^2}{2}\right)s}\; \mathsf{E}_t\left[e^{\sigma(\,W_{t+s} - W_t)}\right] \\
&= X_t\, e^{\left(\mu - \frac{\sigma^2}{2}\right)s}\;\; e^{\frac{\sigma^2}{2}s} \\
&= e^{\mu s} X_t
\end{aligned}
$$

– so the expected growth rate of $X_t$ is $\mu$.

# Sample Paths of Geometric Brownian Motion



# Geometric Brownian Motion

The following properties of GBM follow immediately from the definition of BM:

1. Fix $t_1, t_2, \ldots, t_n$. Then $\frac{X_{t_2}}{X_{t_1}},\ \frac{X_{t_3}}{X_{t_2}},\ \ldots, \frac{X_{t_n}}{X_{t_{n-1}}}$ are mutually independent.

   (For a period of time t, consider 0<t1 <t2 <t3 <t4 …..tn < t)

2. Paths of $X_t$ are continuous as a function of $t$, i.e., they do not jump.

3. For $s > 0$, $\log\left(\frac{X_{t+s}}{X_t}\right) \sim \mathsf{N}\left((\mu - \frac{\sigma^2}{2})s,\ \sigma^2 s\right)$.

# Modeling Stock Prices as GBM

Suppose $X_t \sim \text{GBM}(\mu, \sigma)$. Then clear that:

1. If $X_t > 0$, then $X_{t+s}$ is always positive for any $s > 0$.
   - so limited liability of stock price is not violated.
2. The distribution of $X_{t+s}/X_t$ only depends on $s$ and not on $X_t$

These properties suggest that GBM might be a reasonable model for stock prices.

Indeed it is the underlying model for the famous Black-Scholes option formula.

# Financial Engineering and Risk Management

## Review of vectors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# Reals numbers and vectors

- We will denote the set of real numbers by $\mathbb{R}$
- Vectors are finite collections of real numbers
- Vectors come in two varieties
  - Row vectors: $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$
  - Column vectors $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$
  - By default, vectors are column vectors
- The set of all vectors with **n** components is denoted by $\mathbb{R}^{\mathbf{n}}$

# Linear independence

- A vector $\mathbf{w}$ is linearly dependent on $\mathbf{v}_1, \mathbf{v}_2$ if

$$\mathbf{w} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 \text{ for some } \alpha_1, \alpha_2 \in \mathbb{R}$$

  Example:

$$\begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

- Other names: linear combination, linear span

- A set $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ are linearly independent if **no** $\mathbf{v}_i$ is linearly dependent on the others, $\{\mathbf{v}_j : j \neq i\}$

# Basis

- Every $\mathbf{w} \in \mathbb{R}^n$ is a linear combination of the linearly independent set

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\} \qquad \mathbf{w} = w_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\mathbf{e}_1} + w_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}}_{\mathbf{e}_2} + \ldots + w_n \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{\mathbf{e}_n}$$

- Basis $\equiv$ any linearly independent set that spans the entire space
- Any basis for $\mathbb{R}^n$ has exactly $n$ elements

# Norms

- A function $\rho(\mathbf{v})$ of a vector $\mathbf{v}$ is called a norm if
  - $\rho(\mathbf{v}) \geq 0$ and $\rho(\mathbf{v}) = 0$ implies $\mathbf{v} = \mathbf{0}$
  - $\rho(\alpha\mathbf{v}) = |\alpha|\, \rho(\mathbf{v})$ for all $\alpha \in \mathbb{R}$
  - $\rho(\mathbf{v}_1 + \mathbf{v}_2) \leq \rho(\mathbf{v}_1) + \rho(\mathbf{v}_2)$ (triangle inequality)

  $\rho$ generalizes the notion of "length"

- **Examples**:
  - $\ell_2$ norm: $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$ ... usual length
  - $\ell_1$ norm: $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$
  - $\ell_\infty$ norm: $\|\mathbf{x}\|_\infty = \max_{1\leq i\leq n} |x|_i$
  - $\ell_p$ norm, $1 \leq p < \infty$: $\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x|_i^p\right)^{\frac{1}{p}}$

# Inner product

- The inner-product or dot-product of two vector $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ is defined as

$$\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} v_i w_i$$

- The $\ell_2$ norm $\|\mathbf{v}\|_2 = \sqrt{\mathbf{v} \cdot \mathbf{v}}$
- The angle $\theta$ between two vectors $\mathbf{v}$ and $\mathbf{w}$ is given by

$$\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|_2 \, \|\mathbf{w}\|_2}$$

- Will show later: $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^\top \mathbf{w} =$ product of $\mathbf{v}$ transpose and $\mathbf{w}$