Derivatives

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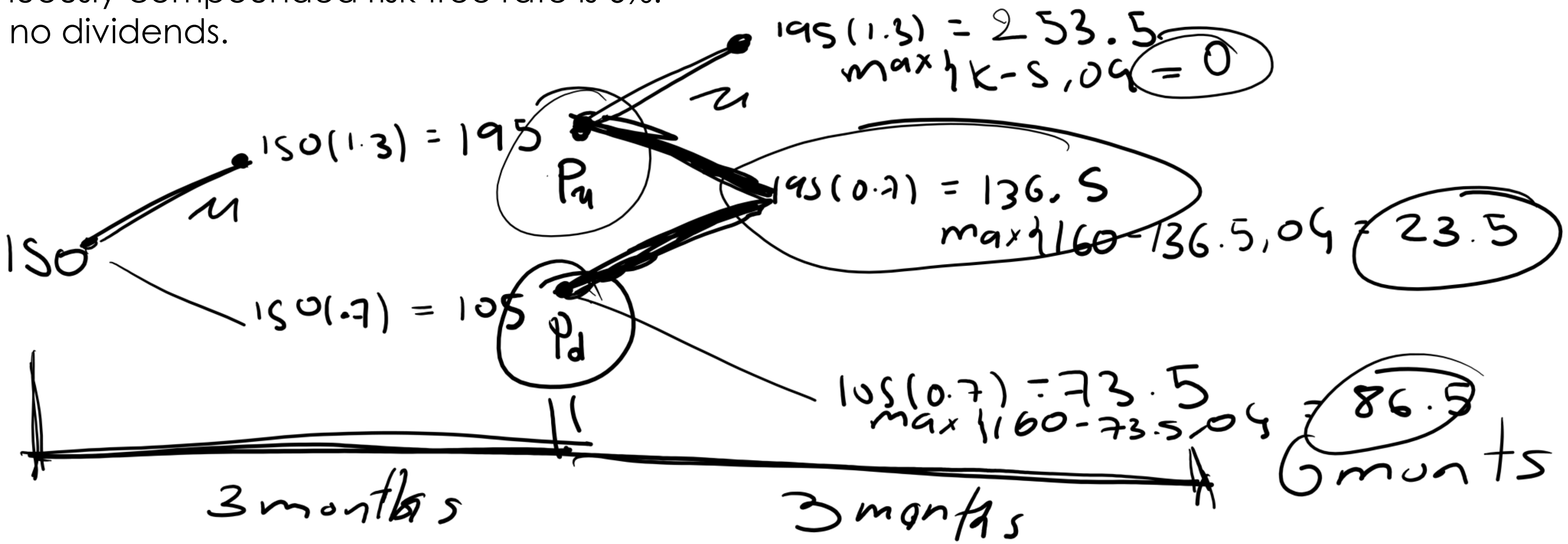
Derivatives
Binomial Trees – General

- **In multiple step cases, An option is priced by starting at the end of the tree and recursicvely moving backwards**

- **Example:** For a 6-month European put option on a stock, you are given:

(i) The stock price is 150.
(ii) The strike price is 160.
(iii) u = 1.3 and d=0.7
(iv) The continuously compounded risk-free rate is 6%.
(v) There are no dividends.



expiration T, $n = \#$ steps $\Rightarrow h = \frac{T}{n}$

- **Example (developed in class):**

In the 1-step $P_0 = e^{-rT}\{[p^* \cdot \max\{K - S_u, 0\} + (1-p^*) \cdot \max\{K - S_d, 0\}]$

$$p^* = \frac{e^{rh} - d}{u - d} = \frac{e^{0.06(\frac{1}{4})} - 0.7}{1.3 - 0.7} = 0.5251$$

$$P_u = e^{-0.06(\frac{1}{4})}\{0.5251(10) + (1-0.5251)23.5\} = 10.99195$$

$$P_d = e^{-0.06(\frac{1}{4})}\{0.5251(23.5) + (1-0.5251)86.5\} = 52.6179$$

$$P_0 = e^{-0.06(\frac{1}{4})}\{0.5251(10.99195) + (1-0.5251)52.6179\} = \underline{30.2985}$$

Derivatives
Binomial Trees – General

- **For European options with no early expiration, there is a way in which we can compute the price of the option without computing the intermediate nodes.**
- **The distribution of the ending nodes is a binomial distribution with parameters n as the numbre of periods and $q = p^*$ the risk neutral probability of an up-move.**
- **Thus, we calculate the expected value of the ending nodes, using this distribution and then discount it for the full period….**

$$X \sim Bin(n, p)$$

$$f_X(x) = P[X = x] = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f_X(x) = P[X = x] = \frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x}$$

For a 6-month European put option on a non dividend paying stock:

(i) The stock's price is currently 150.
(ii) $u = 1.3$ and $d = 0.7$.
(iii) The strike price is 160.
(iv) The continuously compounded risk-free rate is 6%.
(v) The option is modeled with a 2-period binomial tree

Derivatives
Binomial Trees – General

- **Example (developed in class):** $n = 2 \quad p = 0.52519$

Case Scenario	Probability
uu	$\binom{2}{2}(0.52519)^2(1-0.52519)^0 = 0.27582$
ud	$\binom{2}{1}(0.52519)(1-0.52519) = 0.49873$
dd	$\binom{2}{0}(1-0.52519)^2 = 0.225446$

$uu \to 0$

$ud \to 23.5$

$dd \to 86.5$

- **The expected value of the ending put value is:**

$$E[P] = 0.27582(0) + 0.49873(23.5) + 0.225446(86.5) = 31.22126$$

- **So, the put premium is…**

$$P_0 = e^{-0.06\left(\frac{1}{2}\right)}(31.22126) = 30.2985$$

Derivatives
Binomial Trees – General

- It is also possible to compute Δ and B at each node. These calculations are done using the option values at the nodes. For example, at the initial node,

$$\Delta = \frac{P_u - P_d}{S_u - S_d}$$

$$\mathrm{B} = e^{rh}\frac{(uP_d - dP_u)}{u - d}$$

$$\mathrm{P}_0 = \Delta S_0 + B$$

Derivatives
Volatility

- So far, we have picked arbitrary u and d
- We can´t make u too low or d too high. The upper node must be higher than the result of a risk-free investment, and the lower node must be lower, otherwise arbitrage would be possible.

- A risk –free investment of S produces Se^{rh}, whereas the stock produces $Se^{\delta h}$, so…

$\ln(d)$
$\ln(u)$

$$d < e^{(r-\delta)h} < u$$

- The further apart u and d are, the greater the variance in the stock's price.

- Rather than considering the variance of a variable assuming values u and d, let´s consider a r.v assuming values $ln(u)$ and $ln(d)$; (i.e. the variance of the continuously compounded rate of return of the stock)

- To simplify calculation, remember that a Bernoulli r.v. with equall probability of assuming values 0 and 1 has variance 0.25.

$p = 0.5$ $\quad Var_{1,0}(Bll_i) = p(1-p)$; $p :=$ the prob. of success

- Hence, a r.v. equally likely to assume the values $ln(u)$ and $ln(d)$ has variance…

$v^2 = 0.25$ $\quad \ln(u)$, $\ln(d)$ $\Rightarrow$ $v^2 = 0.25(\ln(u) - \ln(d))^2$

Derivatives
Volatility

- **While our trees does not always have a 50/50 probability of an up or down move, we will calculate the variance as if the probability is 0.5.**

- **Then, solving for d in terms of u:**

$$v^2 = \frac{1}{4}\left(\ln(u) - \ln(d)\right)^2 \iff v = \frac{1}{2}\left(\ln(u) - \ln(d)\right)$$

$$2v = \ln\left(\frac{u}{d}\right) \iff e^{2v} = \frac{u}{d} \iff \boxed{u = de^{2v}}$$

$$\sigma = \frac{v}{\sqrt{h}} \qquad \sqrt{h} = \sqrt{\frac{t}{n}} \underset{n=1}{=} \sqrt{t} \qquad \boxed{u = de^{2\sqrt{h}\,\sigma}}$$

- v is the square root of the variance, or the **volatility** of one period, we annualize it by dividing it by $\sqrt{h}$ where h is the size of the period. Thus $\sigma = v/\sqrt{h}$ is the annual volatility.

- To determine u and d we first specify the volatility of the stock, then, given a d and set for u or voce-versa

- A specific way to determine, by an arbitrage argument, we can show that the price at the upper node must be at least as high as the forward Price and the lower node must be no higher.

- To guarantee that u and d satisfy this property, we set:

$$u = \frac{F_{0,h}(S)}{S_0} e^{\sigma\sqrt{h}}, \qquad d = \frac{F_{0,h}(S)}{S_0} e^{-\sigma\sqrt{h}}$$

$F_{0,h}(S) = S_0 e^{(r-\delta)h}$

- So…

$u = \frac{S_0 e^{(r-\delta)h}}{S_0} \cdot e^{\sigma\sqrt{h}} = \boxed{u = e^{(r-\delta)h + \sigma\sqrt{h}}}$

$\boxed{d = e^{(r-\delta)h - \sigma\sqrt{h}}}$

$(u = d e^{2\sigma\sqrt{h}}$

Derivatives
Binomial Trees with forward prices

- Example (developed in class):
- 3 month put option on a stock
- Stock Price is 75
- Strike is 80
- r = 0.08
- δ=0.02
- $\sigma = 0.3$

$$75 \begin{cases} \xrightarrow{u} 88.42 & \max\{0, 80 - 88.42\} = 0 \\ \xrightarrow{d} 65.5275 & \max\{0, 80 - 65.52\} = 14.47 \end{cases}$$

- Premium of the put = 7.62

$$u = e^{(r-\delta)h + \sigma\sqrt{h}} = e^{(0.08-0.02)(\frac{1}{4}) + 0.3\sqrt{\frac{1}{4}}} = 1.1793$$

$$d = 0.8737$$

$$C = 3.83$$

$$C - P = S e^{-\delta h} - K e^{-rh}$$

1.01511

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.06)(\frac{1}{4})} - d}{u - d} = \frac{0.14141}{0.30567}$$

$$\boxed{p^* = 0.479}$$

$$p^* = 0.4626$$

$$p^* = 0.4627$$

$$[p^*(0) + (1-p^*)14.47]\, e^{-0.08(\frac{1}{4})}$$

- Shortcuts…

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}$$

$$\left(\begin{aligned} u &= e^{(r-\delta)h + \sigma\sqrt{h}} = F \cdot w \\ d &= e^{(r-\delta)h - \sigma\sqrt{h}} = e^{(r-\delta)h} \cdot e^{-\sigma\sqrt{h}} = \frac{e^{(r-\delta)h}}{e^{\sigma\sqrt{h}}} = \frac{F}{w} \end{aligned} \right)$$

let's say:

$$F = e^{(r-\delta)h} \qquad w = e^{\sigma\sqrt{h}}$$

$$p^* = \frac{F - \frac{F}{w}}{Fw - \frac{F}{w}} = \frac{1 - 1/w}{w - 1/w} = \frac{\frac{w-1}{w}}{\frac{w^2-1}{w}} = \frac{w-1}{w^2-1} = \frac{\cancel{(w-1)}}{\cancel{(w-1)}(w+1)}$$

$$p^* = \frac{1}{1+w} = \frac{1}{1+e^{\sigma\sqrt{h}}}$$

Derivatives
American Options

- **American options are easily accommodated. We assume that exercise is only possible at the end of a period of the tree. The only difference between a tree for an American and a European option is that when pulling back for an American option, the value of the option is the higher of the pulled back value and the exercise value. If the exercise value is higher, then it is optimal to exercise…**

For a 6-month European put option on a non dividend paying stock: $\delta = 0$

(i) The stock's price is currently 150. $S_0 = 150$

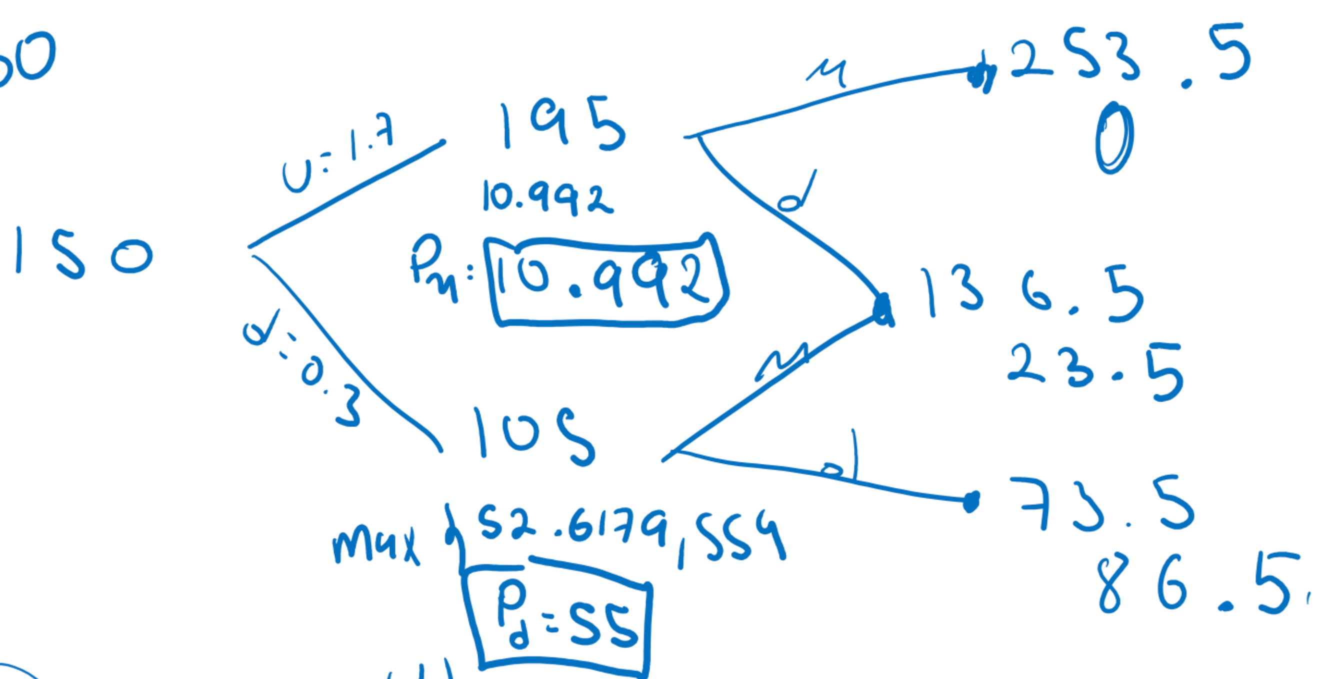
(ii) $u = 1.3$ and $d = 0.7$.

(iii) The strike price is 160. $K = 160$

(iv) The continuously compounded risk-free rate is 6%. $r = 0.06$

(v) The option is modeled with a 2-period binomial tree $n = 2$ $h = \frac{\frac{1}{2}}{2} = h = \frac{1}{4}\ (0.25)$

$K = 160$



$$p^* = \frac{e^{rh} - d}{u - d} = 0.52519 \approx 0.5252$$

$$P_u = e^{-0.06\left(\frac{1}{4}\right)}\{p^*(0) + (1-p^*)23.5\} = 10.9919$$

$$\boxed{P_d = 52.6179}$$

$$\max\{0, 160 - 105\} = 55$$

max{Payoff at t, expected price}

$$\max\{55, 52.6179\} = 55$$

- Example (developed in class):

$$P_0^{amer} = e^{-0.06\left(\frac{1}{4}\right)} \{ p^*(10.992) + (1-p^*)55 \}$$

$$= 31.413$$

$$\boxed{\begin{aligned} P_0^{eur} &= e^{-0.06\left(\frac{1}{4}\right)} \{ p^*(10.992) + (1-p^*)52.6179 \} \\ &= 30.299 \end{aligned}}$$