Derivatives

Enrique Covarrubias
ecovarrubias@actinver.com.mx
Tel. 01 (55) 1103-6600
Ext. 1061

Gustavo Muñoz
gumunoz@actinver.com.mx
Tel. 01 (55) 1103-6600
Ext. 2466

Derivatives
Binomial Trees

- **The binomial tree model is one of the most popular and simple techniques for valuing options and other derivative securities. It provides a straightforward way to model the possible paths that an asset price can take over time, based on the assumption that in each small time step the price can move either up or down by a given factor. By iterating this process across multiple periods, a tree of possible future asset prices is generated.**

- **Key Assumptions of the Binomial Model:**

1. **Discrete price movements: At each time step, the underlying asset price can move only to one of two possible values: up by a factor uu or down by a factor d. (Price follows a discrete random walk at each time step)**

2. **No arbitrage: The model assumes markets are arbitrage-free, which means a unique risk-neutral probability can be defined for pricing.**

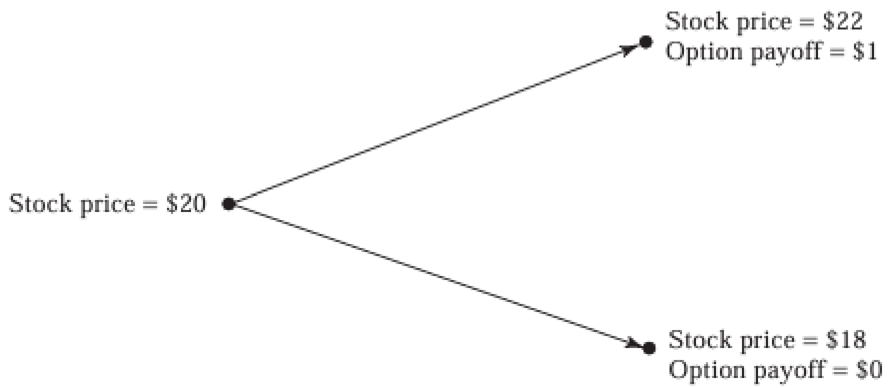
3. **Constant risk-free rate: The risk-free interest rate rr is assumed to be known and constant throughout the life of the option.**

4. **Frictionless markets: No transaction costs, no taxes, and assets are perfectly divisible.**

5. **Rational pricing: Investors are risk-neutral in valuation, using the risk-neutral measure to discount expected payoffs.**

Derivatives
One Step Binomial Tree

- **A stock price is currently $20, and it is known that at the end of 3 months it will be either $22 or $18. We are interested in valuing a European call option to buy the stock for $21 in 3 months. Assume risk free rate is 3%:**



- **We should be indifferent between earning $20 for sure and having a random amount of money with expected value of $20 and we are satisfied to earn the risk-free rate on a risky asset…**

Derivatives

Risk-Neutral Pricing

- In this RISK-NEUTRAL WORLD, we can calculate the probability of the stock going up to 22.
- Namely, the starting price of the stock must be equal to the average present value of potential payments weighted by probability…

$$S_0 = 20 = e^{-rT}\left(22p^* + 18(1 - p^*)\right) = e^{-0.03(0.25)}\left(22p^* + 18(1 - p^*)\right) \Leftrightarrow$$

$$p^* = \frac{20e^{\frac{0.03}{4}} - 18}{22 - 18} \approx 0.538$$

- Now. Let's translate this result in terms of our option price…

$S_0 = 20$

$C(K = 21, T = \frac{1}{4})$

$p^* = 0.538$

22

$\max\{22 - 21, 0\} = 1$

$1 - p^* = 0.462$

18

$\max\{18 - 21, 0\} = 0$

Derivatives
Risk-Neutral Pricing

- **So, the value of the call option is....**

$$C_0 = e^{-rT}\big((Pay\ at\ upper\ node)p^* + (Pay\ at\ lower\ node)(1-p^*)\big)$$

$$C_0 = e^{-\frac{0.03}{4}}\big((1)p^* + (0)(1-p^*)\big)$$

$$C_0 = e^{-\frac{0.03}{4}}\big((1)0.538 + (0)0.462\big) = e^{-\frac{0.03}{4}}(1)0.538 = 0.534$$

$$\boldsymbol{C_0 = 0.534}$$

- **To give more notation…**

$$S_0 = 20,$$
$$S_u = S_0 * u = 22$$
$$S_d = S_0 * d = 18$$

Then:

$$u = 1.1, \qquad d = 0.9 \qquad \Rightarrow \qquad p^* = \frac{S_0 e^{rT} - S_d}{S_u - S_d} = \frac{S_0 e^{rT} - S_0 d}{S_0 u - S_0 d} = \frac{e^{rT} - d}{u - d}$$

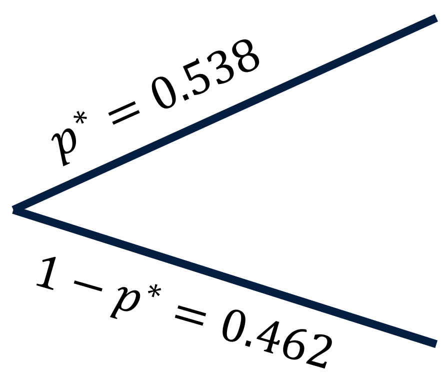
Derivatives
Risk-Neutral Pricing

Given the same exercise, calculate the Put premium (same underlying, maturity, and strike)

$$C(K,t) - P(K,t) = Se^{-\delta t} - Ke^{-rt} \Leftrightarrow$$
$$C(K,t) - Se^{-\delta t} + Ke^{-rt} = P(K,t) \Leftrightarrow$$
$$0.534 - 20 + 21e^{-\frac{0.03}{4}} = \boldsymbol{P(K,t)} = \mathbf{1.377}$$

$S_0 = 20$

$\mathrm{P}(K = 21, T = \frac{1}{4})$



22

$\max\{21\text{-}22,0\} = 0$

18

$\max\{21\text{-}18,0\} = 3$

$$\boldsymbol{P(K,t)} = \boldsymbol{e^{-\frac{0.03}{4}}} * \mathbf{0.462} * \mathbf{3} = \mathbf{1.377}$$

Notes:

- If a call option pays at both the upper and lower node, then the put option with same expiration date and strike price must be worthless.
- In this case, the value of the call option, by put-call parity is:

$$C(K,t) = Se^{-\delta t} - Ke^{-rt}$$

- Similarly, if a put option pays off at both nodes, its value is:

$$P(K,t) = Ke^{-rt} - Se^{-\delta t}$$

Derivatives
Upper and Lower Rates

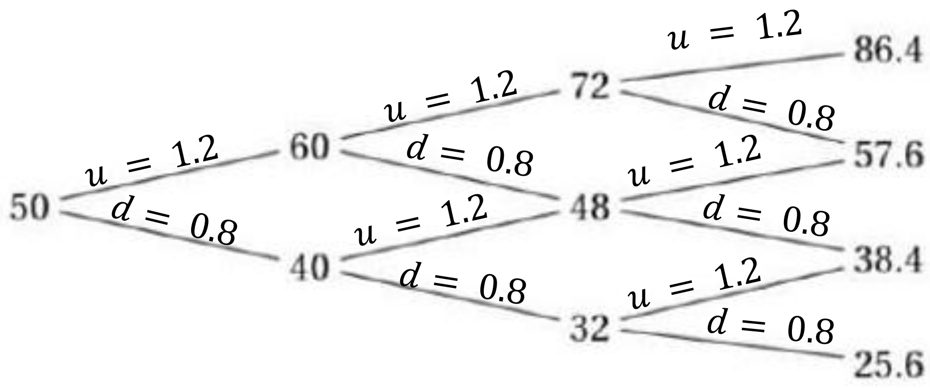
- **We now introduce some notation:**

- In the binomial tree model, u and d indicate the constants to multiply the initial value by to get the upper and lower nodes.

- In the example above, $u = 1.1$ and $d = 0.9$

- In multi period trees, u and d will not vary

- **Example ($u = 1.2$ and $d = 0.8$)** (This is a 3-Step/Period Binomial Tree):



Derivatives
Replicating Portfolio

- **Originally, the binomial model was priced by creating a replicating portfolio for the option, one that has the same outcomes in all scenarios as the option. As we shall see, no assumption about probability is needed to construct this portfolio.**

- **The law of one price says that if two portfolios lead to the same outcomes in all scenarios, they must have the same price.**

- **In a universe with no arbitrage, the law of one price must hold; otherwise one could buy the cheaper portfolio and sell the more expensive portfolio.**

- **It turns out that using risk-neutral pricing is equivalent to pricing with replicating portfolio; this is how the validity of risk-neutral pricing was proved.**

$$\underbrace{C(K,t) - P(K,t) = Se^{-\delta t} - Ke^{-rt}}$$

- We replicate the option with a portfolio of a number of shares, which we call Δ, of the stock, and an investment in the amount of B in a risk-free bond.(B stands for bond, not for borrowing—in fact, it is the amount we lend, not borrow!)

For a 1-year European call option on a non dividend paying stock:

(i) The stock's price is currently 50.
(ii) The stock's price will be either 60 or 40 at the end of the year.
(iii) The strike price is 55.
(iv) The continuously compounded risk-free rate is 5%.

Derivatives
Replicating Portfolio

- **Solution:**

The option pays 5 if the stock hoses down in price. We shall create a portfolio of stocks and bonds that also pays 5 if the stock goes up in price and 0 if it goes down

$$Increase: \Delta * 60 + e^{0.05}B = 5$$

$$Decrease: \Delta * 40 + e^{0.05}B = 0$$

$$\begin{cases} \Delta * 60 + e^{0.05}B = 5 \\ \Delta * 40 + e^{0.05}B = 0 \end{cases} \Rightarrow \begin{cases} B = \dfrac{5 - \Delta 60}{e^{0.05}} \\ \Delta * 40 + e^{0.05}\dfrac{5 - \Delta 60}{e^{0.05}} = 0 \end{cases} \Rightarrow \begin{cases} - \\ \Delta * 40 + 5 - \Delta 60 = 0 \end{cases} \Rightarrow \begin{cases} B = -9.512 \\ \Delta = \dfrac{5}{20} \end{cases}$$

$$C_0 = \Delta S_0 + \mathrm{B} = \left(\frac{1}{4}\right) 50 \; - 9.512 = = 2.988$$

- For a general formula, if S is the initial stock price C_u and C_d the value of the call (or put) at the upper and lower nodes respectively, t is the number of year for expiration…

$$Increase: \Delta * S_u + e^{rt}B = C_u$$

$$Decrease: \Delta * S_d + e^{rt}B = C_d$$

$$\begin{cases} \Delta S_u + e^{rt}B = C_u \\ \Delta S_d + e^{rt}B = C_d \end{cases} \Rightarrow \begin{cases} B = e^{-rt}(C_u - \Delta S_u) \\ \Delta S_d + e^{rt}e^{-rt}(C_u - \Delta S_u) = C_d \end{cases} \Rightarrow \begin{cases} B = e^{-rt}(C_u - \Delta S_u) \\ \Delta S_d + C_u - \Delta S_u = C_d \end{cases} \Rightarrow$$

$$\begin{cases} B = e^{-rt}\left(C_u - (\frac{C_u - C_d}{S_u - S_d})S_u\right) \\ \Delta = \frac{C_u - C_d}{S_u - S_d} \end{cases} \Rightarrow \begin{cases} B = e^{-rt}\left(C_u - \left(\frac{C_u - C_d}{S_0(u-d)}\right)S_0 u\right) \\ \Delta = \frac{C_u - C_d}{S_0(u-d)} \end{cases} \Rightarrow \begin{cases} B = e^{-rt}\left(\frac{uC_d - dC_u}{u-d}\right) \\ \Delta = \frac{C_u - C_d}{S_0(u-d)} \end{cases}$$