# From the Chain Rule to $u$-Substitution to Change of Variables

A careful, step-by-step exposition with intuition and generalization

## 0) Core idea in one line

Differentiation: $\frac{d}{dx}F(g(x)) = F'(g(x))\,g'(x)$.
Integration in reverse: $\int F'(g(x))\,g'(x)\,dx = F(g(x)) + C$.
Rename $u = g(x)$: $\int \phi(g(x))\,g'(x)\,dx = \int \phi(u)\,du$.
With limits and bijections: push bounds via $u = g(x)$; in $\mathbb{R}^n$, use a diffeomorphism and the Jacobian determinant.

## 1) Chain rule $\Rightarrow$ reverse chain rule

Let $F$ satisfy $F' = \phi$. Then

$$\frac{d}{dx}F(g(x)) = \phi(g(x))\,g'(x) \quad\Longrightarrow\quad \int \phi(g(x))\,g'(x)\,dx = F(g(x)) + C.$$

This is the algebraic heart of $u$-substitution.

## 2) $u$-substitution (indefinite integrals)

**Definition 1** (Indefinite $u$-substitution)**.** *Choose $u = g(x)$ with $g$ differentiable. Then $du = g'(x)\,dx$, so*

$$\int \phi(g(x))\,g'(x)\,dx = \int \phi(u)\,du = F(u) + C = F(g(x)) + C.$$

**Intuition.** Relabel the inner expression $g(x)$ as $u$. The differential rescales by the chain rule: $dx = du/g'(x)$.
**Pattern recognition.** Seek $g(x)$ whose derivative $g'(x)$ multiplies the remaining factor up to a constant.

**Example 1** (Indefinite)**.** $\int \frac{5x-2}{\sqrt{7+(5x-2)^2}}\,dx$*. Set $u = 7+(5x-2)^2 \Rightarrow du = 10(5x-2)\,dx$. Then*

$$\int \frac{5x-2}{\sqrt{7+(5x-2)^2}}\,dx = \frac{1}{10}\int u^{-1/2}\,du = \frac{1}{5}\sqrt{u} + C = \frac{1}{5}\sqrt{7+(5x-2)^2} + C.$$

## 3) $u$-substitution with limits

**Theorem 1** (1D change of variable). *Let $g : [A, B] \to \mathbb{R}$ be $C^1$ and strictly monotone, and let $\phi$ be integrable. Then*

$$\int_A^B \phi(g(x))\, g'(x)\, dx = \int_{g(A)}^{g(B)} \phi(u)\, du, \quad \text{with } u = g(x).$$

**Intuition.** The map $u = g(x)$ reparametrizes the horizontal axis. The factor $g'(x)$ compensates local stretching so that area is preserved.

**Example 2** (Definite).

$$\int_0^1 \frac{5x-2}{\sqrt{7+(5x-2)^2}}\, dx = \frac{1}{10} \int_{u=7+(-2)^2}^{u=7+(3)^2} u^{-1/2}\, du = \frac{1}{5}\left(\sqrt{16}-\sqrt{11}\right) = \frac{1}{5}\,(4-\sqrt{11}).$$

Note: *transform bounds through $u = g(x)$; do not back-substitute $x$ after evaluation.*

**Remark 1** (Non-monotone $g$). *If $g$ is not monotone on $[A, B]$, split into subintervals where it is monotone and apply the theorem piecewise.*

## 4) Algebraic alignment when the match is imperfect

If the numerator is not exactly $g'(x)$, decompose it so part matches $g'(x)$ and the remainder is integrable by a known primitive.

**Example 3** (A composite definite integral with a split).

$$I = \int_0^2 \frac{6x-5}{\sqrt{10+(3x-2)^2}}\, dx.$$

*Write $6x-5 = 2(3x-2)-1$. Let $y = 3x-2$ so $dy = 3\,dx$ and bounds $y(0) = -2$, $y(2) = 4$. Then*

$$I = \frac{1}{3}\int_{-2}^{4} \frac{2y}{\sqrt{10+y^2}}\, dy - \frac{1}{3}\int_{-2}^{4} \frac{1}{\sqrt{10+y^2}}\, dy.$$

*First term: $u = 10+y^2$, $du = 2y\,dy \Rightarrow \dfrac{2}{3}\left(\sqrt{26}-\sqrt{14}\right)$.*

*Second term: $\displaystyle\int \frac{dy}{\sqrt{a^2+y^2}} = \operatorname{arsinh}(y/a)$ with $a = \sqrt{10}$:*

$$\frac{1}{3}\left[\operatorname{arsinh}\left(\frac{y}{\sqrt{10}}\right)\right]_{-2}^{4} = \frac{1}{3}\left(\operatorname{arsinh}\frac{4}{\sqrt{10}} + \operatorname{arsinh}\frac{2}{\sqrt{10}}\right).$$

*Thus*

$$\boxed{I = \frac{2}{3}\left(\sqrt{26}-\sqrt{14}\right) - \frac{1}{3}\left(\operatorname{arsinh}\frac{4}{\sqrt{10}} + \operatorname{arsinh}\frac{2}{\sqrt{10}}\right).}$$

## 5) Full 1D change-of-variables formula

A bijective $C^1$ map $g$ with $g'(x) \neq 0$ yields

$$\int_A^B f(x)\,dx = \int_{u=g(A)}^{u=g(B)} f\big(g^{-1}(u)\big)\,\frac{du}{g'(g^{-1}(u))}.$$

Equivalently, if $f(x) = \phi(g(x))\,g'(x)$ then

$$\int_A^B \phi(g(x))\,g'(x)\,dx = \int_{g(A)}^{g(B)} \phi(u)\,du.$$

Orientation: if $g' < 0$, the absolute value appears in the denominator form; in practice one flips the bounds.

## 6) Multivariable change of variables (Jacobian)

**Theorem 2** (Jacobian formula)**.** *Let $\Phi : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^n$ be a $C^1$ bijection with $C^1$ inverse and $\det D\Phi(\mathbf{x}) \neq 0$ on $U$. For integrable $f$,*

$$\int_{\Phi(\Omega)} f(\mathbf{u})\,d\mathbf{u} = \int_{\Omega} f\big(\Phi(\mathbf{x})\big)\,\big|\det D\Phi(\mathbf{x})\big|\,d\mathbf{x}.$$

**Intuition.** $|\det D\Phi|$ is the local $n$-D volume scale of the linearization. In 1D this reduces to $|g'(x)|$.

**Example 4** (Polar coordinates)**.** $\Phi(r, \theta) = (r\cos\theta, r\sin\theta)$.

$$D\Phi = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}, \qquad |\det D\Phi| = r.$$

*Hence*

$$\iint_D f(x, y)\,dx\,dy = \int_{\Phi^{-1}(D)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

## 7) When to use what

Use $u$-sub if you can isolate $g(x)$ with a multiple of $g'(x)$. If not, try algebraic alignment, integration by parts, completing the square with trig/hyperbolic substitutions, rational substitutions, or partial fractions.

## 8) Common errors

(i) Not transforming bounds in definite integrals. (ii) Choosing $u$ so $du$ does not appear up to a constant. (iii) Ignoring non-monotonicity. (iv) In $\mathbb{R}^n$, omitting $|\det D\Phi|$.

## 9) Worked example connecting all levels (summary of Example 3)

$$
\begin{aligned}
I &= \int_0^2 \frac{6x-5}{\sqrt{10+(3x-2)^2}}\,dx = \frac{1}{3}\int_{-2}^{4} \frac{2y}{\sqrt{10+y^2}}\,dy - \frac{1}{3}\int_{-2}^{4} \frac{1}{\sqrt{10+y^2}}\,dy \\
&= \frac{2}{3}(\sqrt{26}-\sqrt{14}) - \frac{1}{3}\left(\operatorname{arsinh}\frac{4}{\sqrt{10}} + \operatorname{arsinh}\frac{2}{\sqrt{10}}\right).
\end{aligned}
$$

Outer affine change $x \mapsto y$ is the 1D theorem; inner $y \mapsto u$ is pure reverse chain rule.

## 10) Minimal recipes

**Indefinite.** Pick $u = g(x)$ so $du$ appears, integrate in $u$, back-substitute.
**Definite.** Replace bounds by $g(A), g(B)$, evaluate in $u$, no back-substitution.
**Multivariable.** Choose $\Phi$ simplifying the domain, multiply integrand by $|\det D\Phi|$, transform the region.

## 11) Two quick generalizations

Trig and hyperbolic substitutions are structured changes of variables that linearize radicals: $x = a\tan\theta \Rightarrow 1+\tan^2\theta = \sec^2\theta$ and $x = a\sinh t \Rightarrow \sqrt{a^2+x^2} = a\cosh t$.

## 12) Targeted practice

(a) $\displaystyle\int_{-1}^{3} \frac{4x+1}{\sqrt{5+(2x+1)^2}}\,dx$ (b) $\displaystyle\int_0^{\pi/4} \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}\,d\theta$ (c) $\displaystyle\iint_{x^2+y^2\le 4} (x^2+y^2)\,dx\,dy.$

Hints: (a) $u = 5+(2x+1)^2$. (b) use $\sqrt{1+\tan^2\theta} = \sec\theta$ or $u = 1+\tan^2\theta$. (c) polar with Jacobian $r$.