$$I = \int_0^2 \frac{6x-5}{\sqrt{10+(3x-2)^2}}\,dx$$

Change-of-variable theorem (definite integrals). If $g : [A,B] \to \mathbb{R}$ is C^1 and strictly monotone and ϕ is integrable, then

$$\int_A^B \phi\big(g(x)\big)\, g'(x)\,dx = \int_{g(A)}^{g(B)} \phi(t)\,dt, \qquad t = g(x),\ dt = g'(x)\,dx.$$

This is the rigorous form of u-substitution for definite integrals.

Solution linking u-sub and change of variable. Apply the affine substitution

$$y = g(x) = 3x-2, \qquad g'(x) = 3, \qquad dx = \frac{dy}{3}, \qquad g(0) = -2,\ g(2) = 4.$$

Rewrite the numerator to expose y:

$$6x-5 = 2(3x-2) - 1 = 2y - 1.$$

Thus

$$\begin{aligned} I &= \frac{1}{3}\int_{-2}^{4} \frac{2y-1}{\sqrt{10+y^2}}\,dy \\ &= \frac{1}{3}\left(\int_{-2}^{4} \frac{2y}{\sqrt{10+y^2}}\,dy - \int_{-2}^{4} \frac{1}{\sqrt{10+y^2}}\,dy\right). \end{aligned}$$

First piece (u-sub via the theorem): with $t = 10+y^2$, $dt = 2y\,dy$,

$$\int \frac{2y}{\sqrt{10+y^2}}\,dy = \int t^{-1/2}\,dt = 2\sqrt{t} = 2\sqrt{10+y^2}.$$

Evaluating on $[-2,4]$ and including the prefactor $1/3$ gives

$$\frac{1}{3}\cdot 2\big(\sqrt{26}-\sqrt{14}\big) = \frac{2}{3}\big(\sqrt{26}-\sqrt{14}\big).$$

Second piece (standard primitive):

$$\int \frac{dy}{\sqrt{a^2+y^2}} = \operatorname{arsinh}\Big(\frac{y}{a}\Big) = \ln\Big(y + \sqrt{y^2+a^2}\Big) \quad (a>0).$$

With $a = \sqrt{10}$,

$$\frac{1}{3}\int_{-2}^{4} \frac{dy}{\sqrt{10+y^2}} = \frac{1}{3}\left[\operatorname{arsinh}\Big(\frac{y}{\sqrt{10}}\Big)\right]_{-2}^{4} = \frac{1}{3}\left(\operatorname{arsinh}\frac{4}{\sqrt{10}} + \operatorname{arsinh}\frac{2}{\sqrt{10}}\right).$$

Combine:

$$\boxed{I = \frac{2}{3}\big(\sqrt{26}-\sqrt{14}\big) - \frac{1}{3}\left[\operatorname{arsinh}\frac{4}{\sqrt{10}} + \operatorname{arsinh}\frac{2}{\sqrt{10}}\right]}.$$

Equivalent logarithmic form. Using $\operatorname{arsinh}(x) = \ln\bigl(x + \sqrt{1+x^2}\bigr)$,

$$\operatorname{arsinh}\frac{4}{\sqrt{10}} + \operatorname{arsinh}\frac{2}{\sqrt{10}} = \ln\left(\frac{4+\sqrt{26}}{\sqrt{10}}\right) + \ln\left(\frac{2+\sqrt{14}}{\sqrt{10}}\right) = \ln\left(\frac{(4+\sqrt{26})(2+\sqrt{14})}{10}\right).$$

Hence

$$I = \frac{2}{3}\bigl(\sqrt{26} - \sqrt{14}\bigr) - \frac{1}{3}\ln\left(\frac{(4+\sqrt{26})(2+\sqrt{14})}{10}\right).$$

Derivative check. Define

$$F(x) = \frac{2}{3}\sqrt{10 + (3x-2)^2} - \frac{1}{3}\operatorname{arsinh}\frac{3x-2}{\sqrt{10}}.$$

Then

$$F'(x) = \frac{2}{3}\cdot\frac{(3x-2)\cdot 3}{\sqrt{10+(3x-2)^2}} - \frac{1}{3}\cdot\frac{3}{\sqrt{10}}\cdot\frac{1}{\sqrt{1+\frac{(3x-2)^2}{10}}} = \frac{6x-5}{\sqrt{10+(3x-2)^2}}.$$