

Método clásico (1D)

$$0 \leq g(x) \leq c, \quad x \in (a, b), \quad \theta := \int_a^b g(x) dx$$

$$(X, Y) \sim \text{Unif}(a, b) \times \text{Unif}(0, c), \quad p := \mathbb{P}(Y \leq g(X))$$

$$p = \frac{1}{c(b-a)} \int_a^b g(x) dx = \frac{\theta}{c(b-a)} \implies \theta = c(b-a)p$$

$$\{(X_i, Y_i)\}_{i=1}^n \text{ i.i.d.}, \quad N := \sum_{i=1}^n \mathbf{1}\{Y_i \leq g(X_i)\} \sim \text{Bin}(n, p)$$

$$\hat{p} = \frac{N}{n}, \quad \hat{\theta} = c(b-a)\hat{p} = c(b-a)\frac{N}{n}$$

Propiedades

$$\mathbb{E}[\hat{\theta}] = \theta$$

$$\text{Var}(\hat{\theta}) = \text{Var}\left(c(b-a)\frac{N}{n}\right) = c^2(b-a)^2 \frac{p(1-p)}{n} = \frac{1}{n} \theta(c(b-a) - \theta)$$

$$\hat{\theta} \xrightarrow{\mathbb{P}} \theta \quad \text{y} \quad \frac{\hat{\theta} - \theta}{\sqrt{\text{Var}(\hat{\theta})}} \approx \mathcal{N}(0, 1)$$

Tamaño de muestra (cota tipo Chebyshev)

$$\forall \varepsilon > 0, \quad 0 < \alpha < 1: \quad n \geq \frac{c^2(b-a)^2}{4\alpha\varepsilon^2} \implies \mathbb{P}(|\hat{\theta} - \theta| < \varepsilon) \geq 1 - \alpha$$

Intervalos de confianza

(1) Forma con parámetro ε :

$$n \geq \frac{c^2(b-a)^2}{2\alpha\varepsilon^2} \implies \mathbb{P}(\hat{\theta} - \varepsilon < \theta < \hat{\theta} + \varepsilon) \geq 1 - \alpha$$

$$\varepsilon^2 = \frac{c^2(b-a)^2}{2\alpha n} \implies \mathbb{P}\left(\hat{\theta} - \frac{1}{\sqrt{\alpha}} \frac{c(b-a)}{\sqrt{2n}} < \theta < \hat{\theta} + \frac{1}{\sqrt{\alpha}} \frac{c(b-a)}{\sqrt{2n}}\right) \geq 1 - \alpha$$

(2) Aproximación normal asintótica:

$$\mathbb{P}\left(\hat{\theta} - z_{\alpha/2} \sqrt{\text{Var}(\hat{\theta})} < \theta < \hat{\theta} + z_{\alpha/2} \sqrt{\text{Var}(\hat{\theta})}\right) \approx 1 - \alpha$$

(3) Cota ampliada con $\text{Var}(\hat{\theta}) \leq \frac{c^2(b-a)^2}{2n}$:

$$\mathbb{P}\left(\hat{\theta} - z_{\alpha/2} \frac{c(b-a)}{\sqrt{2n}} < \theta < \hat{\theta} + z_{\alpha/2} \frac{c(b-a)}{\sqrt{2n}}\right) \approx 1 - \alpha$$

Caso multidimensional (2D)

$$0 \leq g(x, y) \leq c, \quad (x, y) \in (a_1, b_1) \times (a_2, b_2)$$

$$\theta := \int_{a_1}^{b_1} \int_{a_2}^{b_2} g(x, y) \, dy \, dx$$

$$(X, Y, Z) \sim \text{Unif}(a_1, b_1) \times \text{Unif}(a_2, b_2) \times \text{Unif}(0, c), \quad p = \mathbb{P}(Z \leq g(X, Y)) = \frac{\theta}{c(b_1 - a_1)(b_2 - a_2)}$$

$$N = \sum_{i=1}^n \mathbf{1}\{Z_i \leq g(X_i, Y_i)\} \sim \text{Bin}(n, p), \quad \boxed{\hat{\theta} = c(b_1 - a_1)(b_2 - a_2) \frac{N}{n}}$$

Algoritmo en símbolos

$$X_i \sim \text{Unif}(a, b), \quad Y_i \sim \text{Unif}(0, c), \quad I_i = \mathbf{1}\{Y_i \leq g(X_i)\}, \quad N = \sum_{i=1}^n I_i, \quad \hat{\theta} = c(b - a) \frac{N}{n}$$

Ejercicio con cómputo a mano

$$g(x) = \sin x + 1, \quad a = 0, \quad b = \pi, \quad c = 2$$

$$\theta = \int_0^\pi (\sin x + 1) \, dx = \left[-\cos x \right]_0^\pi + \left[x \right]_0^\pi = 2 + \pi$$

$$p = \frac{\theta}{c(b - a)} = \frac{\pi + 2}{2\pi}$$

Escenario muestral con $n = 1000$, $N = 818$: (aprox. $\hat{p} = 0.818$)

$$\hat{\theta} = c(b - a) \frac{N}{n} = 2\pi \cdot 0.818$$

$$\hat{\theta} \approx 2\pi \cdot 0.818 \approx 5.14 \quad \text{y} \quad \theta = \pi + 2 \approx 5.14$$

$$\text{Var}(\hat{\theta}) = \frac{1}{n} \theta (c(b - a) - \theta) = \frac{1}{1000} (\pi + 2)(2\pi - (\pi + 2))$$

$$\text{IC}_{95} \text{ (aprox. normal)}: \hat{\theta} \pm 1.96 \sqrt{\text{Var}(\hat{\theta})}$$

===== BLOQUE 2 (OCGX2) =====

General

$$\theta = \int_a^b g(x) \, dx, \quad \theta = \int_a^b \frac{g(x)}{f(x)} f(x) \, dx = \mathbb{E} \left[\frac{g(X)}{f(X)} \right].$$

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^n \frac{g(X_i)}{f(X_i)}, \quad \mathbb{E}[\hat{\theta}] = \theta.$$

Uniforme

$$X \sim \text{Unif}(a, b), \quad f(x) = \frac{1}{b - a}.$$

$$\theta = (b - a) \mathbb{E}[g(X)], \quad \hat{\theta}_u = (b - a) \frac{1}{n} \sum_{i=1}^n g(X_i).$$

$$\text{Var}(\hat{\theta}_u) = \frac{1}{n} \left[(b - a) \int_a^b g^2(x) \, dx - \theta^2 \right].$$

$$\text{Var}(\hat{\theta}_{\text{u}}) \leq \text{Var}(\hat{\theta}_{\text{cl}}).$$

Exponencial

$$X \sim \text{Exp}(\lambda), \quad f(x) = \lambda e^{-\lambda x}, \quad \lambda = 1.$$

$$\theta = \mathbb{E}[e^X g(X)], \quad \hat{\theta}_{\text{exp}} = \frac{1}{n} \sum_{i=1}^n e^{x_i} g(x_i).$$

Normal

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad X \sim \mathcal{N}(0, 1).$$

$$\theta = \sqrt{2\pi} \mathbb{E}[e^{X^2/2} g(X)], \quad \hat{\theta}_{\text{norm}} = \sqrt{2\pi} \frac{1}{n} \sum_{i=1}^n e^{x_i^2/2} g(x_i).$$

Ejercicio

$$g(x) = x, \quad a = 0, \quad b = 1, \quad x_1 = \frac{1}{4}, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{3}{4}, \quad n = 3.$$

$$\theta = \int_0^1 x \, dx = \frac{1}{2}.$$

$$\hat{\theta}_{\text{u}} = (b - a) \frac{1}{n} \sum_{i=1}^n g(x_i) = 1 \cdot \frac{1}{3} \left(\frac{1}{4} + \frac{1}{2} + \frac{3}{4} \right) = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}.$$

$$|\hat{\theta}_{\text{u}} - \theta| = 0.$$

===== BLOQUE 3: MUESTREO CONDICIONAL =====

Varianza y esperanza condicional

Definición. Sea X con $\mathbb{E}[X^2] < \infty$ y Y una v.a. cualquiera. Se define

$$\text{Var}(X \mid Y) = \mathbb{E}[(X - \mathbb{E}[X \mid Y])^2 \mid Y].$$

Propiedades.

$$\text{Var}(X \mid Y) \geq 0 \quad \text{c.s.}$$

$$\text{Var}(X \mid Y) = \mathbb{E}[X^2 \mid Y] - \left(\mathbb{E}[X \mid Y] \right)^2.$$

$$\boxed{\text{Var}(X) = \mathbb{E}[\text{Var}(X \mid Y)] + \text{Var}(\mathbb{E}[X \mid Y])} \quad (\text{ley de la varianza total}).$$

$$\text{Var}(\mathbb{E}[X \mid Y]) \leq \text{Var}(X).$$

===== MUESTREO CONDICIONAL: FORMULAS ===== Marco básico

$$\theta = \int_a^b g(x) \, dx = \mathbb{E} \left[\frac{g(X)}{f(X)} \right], \quad X \sim f$$

$$\hat{\theta}_1 = \frac{1}{n} \sum_{i=1}^n \frac{g(X_i)}{f(X_i)}, \quad \hat{\theta}_2 = \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[\frac{g(X)}{f(X)} \mid Y_i \right]$$

$$\mathbb{E}[\hat{\theta}_1] = \mathbb{E}[\hat{\theta}_2] = \theta$$

$$\text{Var}(\hat{\theta}_1) = \frac{1}{n} \text{Var} \left(\frac{g(X)}{f(X)} \right), \quad \text{Var}(\hat{\theta}_2) = \frac{1}{n} \text{Var} \left(\mathbb{E} \left[\frac{g(X)}{f(X)} \mid Y \right] \right)$$

$$\text{Var}(X) = \mathbb{E}[\text{Var}(X \mid Y)] + \text{Var}(\mathbb{E}[X \mid Y])$$

$$\text{Var}(\hat{\theta}_2) \leq \text{Var}(\hat{\theta}_1)$$

$$\text{===== EJEMPLO 1 =====} \quad \theta = \iint (x + y)$$

$$\theta = \int_0^1 \int_0^1 (x + y) \, dx \, dy = 1$$

$$\hat{\theta}_1 = \frac{1}{n} \sum_{i=1}^n (X_i + Y_i), \quad (X_i, Y_i) \stackrel{\text{i.i.d.}}{\sim} U([0, 1]^2)$$

$$\mathbb{E}[X \mid Y = y] = \frac{1}{2} \Rightarrow \hat{\theta}_2 = \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{2} + Y_i \right), \quad Y_i \stackrel{\text{i.i.d.}}{\sim} U(0, 1)$$

$$\text{===== EJEMPLO 2 =====} \quad \theta = P(X + Y \leq u)$$

$$\theta = \mathbb{P}(X + Y \leq u) = \mathbb{E}[\mathbf{1}_{\{X+Y \leq u\}}]$$

$$\hat{\theta}_1 = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{X_i + Y_i \leq u\}}, \quad (X_i, Y_i) \stackrel{\text{i.i.d.}}{\sim} F$$

$$\theta = \mathbb{E}[\mathbb{E}(\mathbf{1}_{\{X \leq u - Y\}} \mid Y)] = \mathbb{E}[F_X(u - Y)]$$

$$\hat{\theta}_2 = \frac{1}{n} \sum_{i=1}^n F_X(u - Y_i), \quad Y_i \stackrel{\text{i.i.d.}}{\sim} F_Y$$

$$\text{===== EJEMPLO 3 =====} \quad \text{Suma compuesta SN}$$

$$S_N = \sum_{i=1}^N X_i, \quad N \perp\!\!\!\perp \{X_i\}, \quad X_i \stackrel{\text{i.i.d.}}{\sim} F$$

$$\theta = \mathbb{P}(S_N \leq x) = \mathbb{E}[\mathbf{1}_{\{S_N \leq x\}}]$$

$$\hat{\theta}_1 = \frac{1}{n} \sum_{k=1}^n \mathbf{1}_{\{S_{N_k}^{(k)} \leq x\}}, \quad \text{Var}(\hat{\theta}_1) = \frac{1}{n} \theta (1 - \theta)$$

$$F^{*n}(x) = \mathbb{P} \left(\sum_{i=1}^n X_i \leq x \right) = \mathbb{E} \left[F \left(x - \sum_{i=2}^n X_i \right) \right]$$

$$F^{*N}(x) = \mathbb{E} \left[F \left(x - \sum_{i=2}^N X_i \right) \right]$$

$$\hat{\theta}_2 = \frac{1}{n} \sum_{k=1}^n F \left(x - \sum_{i=2}^{N_k} X_i^{(k)} \right)$$

===== EJERCICIO (CÁLCULO A MANO) ===== Uniformes y u=1

$$X, Y \stackrel{\text{i.i.d.}}{\sim} U(0, 1), \quad u = 1$$

$$\theta = \mathbb{P}(X + Y \leq 1) = \int_0^1 \int_0^1 \mathbf{1}_{\{x+y \leq 1\}} dx dy = \int_0^1 \left(\int_0^{1-y} dx \right) dy = \int_0^1 (1-y) dy = \frac{1}{2}$$

$$F_X(1-Y) = (1-Y) \mathbf{1}_{\{0 \leq Y \leq 1\}}$$

$$\theta = \mathbb{E}[F_X(1-Y)] = \mathbb{E}[1-Y] = 1 - \mathbb{E}[Y] = 1 - \frac{1}{2} = \frac{1}{2}$$

===== BLOQUE 4: MUESTREO POR IMPORTANCIA =====

=== Muestreo por Importancia (resumen solo fórmulas) ===

Objetivo

$$\theta = \int_a^b g(x) dx$$

$$\theta = \int_a^b \frac{g(x)}{f(x)} f(x) dx = \mathbb{E} \left[\frac{g(X)}{f(X)} \right]$$

Estimador Monte Carlo

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^n \frac{g(X_i)}{f(X_i)}$$

$$\mathbb{E}[\hat{\theta}] = \theta$$

$$\text{Var}(\hat{\theta}) = \frac{1}{n} \text{Var} \left(\frac{g(X)}{f(X)} \right)$$

Cota por Cauchy-Schwarz y f óptima

$$\text{Var}(\hat{\theta}) \geq \frac{1}{n} \left[\left(\int_a^b |g(x)| dx \right)^2 - \theta^2 \right]$$

$$f^*(x) = \frac{|g(x)|}{\int_a^b |g(t)| dt}$$

$$\text{Var}(\hat{\theta}; f^*) = \frac{1}{n} \left[\left(\int_a^b |g(x)| dx \right)^2 - \theta^2 \right]$$

Requisito de integrabilidad

$$\int_a^b |g(x)| dx < \infty$$

Aproximación por partición (discreta por tramos) Partición: $a = u_0 < u_1 < \dots < u_m = b$, $\Delta_i = u_i - u_{i-1}$, $x_i \in (u_{i-1}, u_i)$ $\bar{g}(x) = \sum_{i=1}^m g_i \mathbf{1}_{(u_{i-1}, u_i)}(x)$, $g_i := g(x_i)$

$$p_i = \mathbb{P}(X = x_i) = \frac{|g_i| \Delta_i}{\sum_{j=1}^m |g_j| \Delta_j}$$

$$\theta = \int_a^b g(x) dx \approx \mathbb{E} \left[\frac{g(X) \Delta(X)}{p(X)} \right]$$

$$\hat{\theta}_\Delta = \frac{1}{n} \sum_{k=1}^n \frac{g(X_k) \Delta(X_k)}{p(X_k)}$$

Caso multidimensional

$$\theta = \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} g(\mathbf{x}) d\mathbf{x} = \mathbb{E} \left[\frac{g(\mathbf{X})}{f(\mathbf{X})} \right]$$

$$f^*(\mathbf{x}) = \frac{|g(\mathbf{x})|}{\int |g(\mathbf{t})| d\mathbf{t}}$$

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^n \frac{g(\mathbf{X}_i)}{f(\mathbf{X}_i)}$$

===== Ejercicio simple con cálculos a mano ===== Integral objetivo y partición m=2 Ejercicio: $\theta = \int_0^1 e^{-x} dx$, $m = 2$, $u_0 = 0$, $u_1 = \frac{1}{2}$, $u_2 = 1$, $\Delta_1 = \Delta_2 = \frac{1}{2}$ $x_1 = \frac{1}{4}$, $x_2 = \frac{3}{4}$, $g_1 = e^{-1/4}$, $g_2 = e^{-3/4}$

$$p_1 = \frac{|g_1| \Delta_1}{|g_1| \Delta_1 + |g_2| \Delta_2} = \frac{e^{-1/4} \cdot \frac{1}{2}}{e^{-1/4} \cdot \frac{1}{2} + e^{-3/4} \cdot \frac{1}{2}} = \frac{e^{-1/4}}{e^{-1/4} + e^{-3/4}}, \quad p_2 = \frac{e^{-3/4}}{e^{-1/4} + e^{-3/4}}$$

$$w_1 = \frac{g_1 \Delta_1}{p_1} = \frac{e^{-1/4} \cdot \frac{1}{2}}{e^{-1/4} / (e^{-1/4} + e^{-3/4})} = \frac{1}{2} (e^{-1/4} + e^{-3/4}), \quad w_2 = \frac{g_2 \Delta_2}{p_2} = \frac{e^{-3/4} \cdot \frac{1}{2}}{e^{-3/4} / (e^{-1/4} + e^{-3/4})} = \frac{1}{2} (e^{-1/4} + e^{-3/4})$$

$$\hat{\theta}_{\Delta} = \frac{1}{n} \sum_{k=1}^n w_{I_k} = \frac{1}{2}(e^{-1/4} + e^{-3/4})$$

$$e^{-1/4} \approx 0.77880078, \quad e^{-3/4} \approx 0.47236655 \quad \hat{\theta}_{\Delta} \approx \frac{1}{2}(0.77880078 + 0.47236655) \approx 0.625583665$$

$$\theta_{\text{exacta}} = 1 - e^{-1} \approx 1 - 0.36787944 \approx 0.63212056$$

$$|\hat{\theta}_{\Delta} - \theta_{\text{exacta}}| \approx 0.00653690$$

===== Recap simbólico =====

$$\text{IS: } \hat{\theta} = \frac{1}{n} \sum_{i=1}^n \frac{g(X_i)}{f(X_i)}, \quad f^{\star} \propto |g|, \quad \text{Var}(\hat{\theta}) \geq \frac{1}{n} \left[\left(\int |g| \right)^2 - \theta^2 \right]$$

===== BLOQUE 5: VARIABLES COMUNES, ANTITÉTICAS Y POISSON =====

Resumen en fórmulas: Variables Comunes

Variables Comunes

$$\theta = \mathbb{E}(X - Y), \quad \hat{\theta} = \frac{1}{n} \sum_{i=1}^n (X_i - Y_i)$$

$$\text{Var}(\hat{\theta}) = \frac{1}{n} (\text{Var}(X) + \text{Var}(Y))$$

$$U \sim \text{Unif}(0, 1), \quad X = F^{-1}(U), \quad Y = G^{-1}(U)$$

$$\text{Var}(\hat{\theta}) = \frac{1}{n} [\text{Var}(X) + \text{Var}(Y) - 2 \text{Cov}(F^{-1}(U), G^{-1}(U))]$$

$$\text{Cov}(F^{-1}(U), G^{-1}(U)) \geq 0$$

Variables Antitéticas

Variables Antitéticas

$$\theta = \mathbb{E}(X + Y), \quad \hat{\theta} = \frac{1}{n} \sum_{i=1}^n (X_i + Y_i)$$

$$\text{Var}(\hat{\theta}) = \frac{1}{n} (\text{Var}(X) + \text{Var}(Y))$$

$$U \sim \text{Unif}(0, 1), \quad X = F^{-1}(U), \quad Y = G^{-1}(1 - U)$$

$$\text{Var}(\hat{\theta}) = \frac{1}{n} [\text{Var}(X) + \text{Var}(Y) + 2 \text{Cov}(F^{-1}(U), G^{-1}(1 - U))]$$

$$\text{Cov}(F^{-1}(U), G^{-1}(1 - U)) \leq 0$$

Procesos de Poisson

Procesos de Poisson

$$T \sim \text{Exp}(\lambda), \quad F_T^{-1}(U) = -\frac{1}{\lambda} \ln(1 - U) \quad (= -\frac{1}{\lambda} \ln U)$$

$$\tau_0 = 0, \quad \tau_n = \sum_{k=1}^n T_k, \quad N(s) = \max\{n : \tau_n \leq s\}$$

Ejercicio (cálculo a mano: comunes)

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$$\mu = 0.05, \quad U = 0.8, \quad Z = 0.5$$

$$T = -\frac{1}{\mu} \ln U = -20 \ln(0.8) = 4.4629$$

$$\begin{aligned}
R_A &= 0.03 + 0.01Z = 0.035, & R_B &= 0.05 + 0.01Z = 0.055 \\
V_A &= \frac{1 - e^{-R_A T}}{R_A} = \frac{1 - e^{-0.035 \cdot 4.4629}}{0.035} = \frac{1 - e^{-0.1562}}{0.035} = 4.1318 \\
V_B &= \frac{1 - e^{-R_B T}}{R_B} = \frac{1 - e^{-0.055 \cdot 4.4629}}{0.055} = \frac{1 - e^{-0.2455}}{0.055} = 3.9573 \\
\hat{\delta} &= V_B - V_A = 3.9573 - 4.1318 = -0.1744 \\
&(\text{mismo } Z \text{ en } R_A, R_B \Rightarrow \text{Cov}(V_B, V_A) > 0 \Rightarrow \text{Var}(\hat{\delta}) \text{ reducida})
\end{aligned}$$

— Chuleta de distribuciones (f, F, media, var, MGF) —

$$\text{Def.: } \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-u^2/2} du, \quad B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}, \quad I_x(\alpha, \beta) = \frac{1}{B(\alpha, \beta)} \int_0^x u^{\alpha-1}(1-u)^{\beta-1} du.$$

Normal

Normal $X \sim \mathcal{N}(\mu, \sigma^2)$, $\sigma > 0$, $x \in \mathbb{R}$:

$$\begin{aligned}
f(x) &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), & F(x) &= \Phi\left(\frac{x-\mu}{\sigma}\right), \\
\mathbb{E}[X] &= \mu, & \text{Var}(X) &= \sigma^2, & M_X(t) &= \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right), \quad t \in \mathbb{R}.
\end{aligned}$$

Exponencial

Exponencial $X \sim \text{Exp}(\lambda)$, $\lambda > 0$, $x \geq 0$:

$$\begin{aligned}
f(x) &= \lambda e^{-\lambda x} \mathbf{1}_{[0, \infty)}(x), & F(x) &= \left(1 - e^{-\lambda x}\right) \mathbf{1}_{[0, \infty)}(x), \\
\mathbb{E}[X] &= \frac{1}{\lambda}, & \text{Var}(X) &= \frac{1}{\lambda^2}, & M_X(t) &= \frac{\lambda}{\lambda - t} \text{ para } t < \lambda.
\end{aligned}$$

Uniforme

Uniforme $X \sim \text{Unif}(a, b)$, $a < b$, $x \in [a, b]$:

$$\begin{aligned}
f(x) &= \frac{1}{b-a} \mathbf{1}_{[a, b]}(x), & F(x) &= \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \leq x \leq b, \\ 1, & x > b, \end{cases} \\
\mathbb{E}[X] &= \frac{a+b}{2}, & \text{Var}(X) &= \frac{(b-a)^2}{12}, & M_X(t) &= \begin{cases} \frac{e^{tb} - e^{ta}}{t(b-a)}, & t \neq 0, \\ 1, & t = 0. \end{cases}
\end{aligned}$$

Bernoulli

Bernoulli $X \sim \text{Bern}(p)$, $p \in [0, 1]$, $x \in \{0, 1\}$:

$$\begin{aligned}
\Pr(X = x) &= p^x(1-p)^{1-x}, & F(x) &= \begin{cases} 0, & x < 0, \\ 1-p, & 0 \leq x < 1, \\ 1, & x \geq 1, \end{cases} \\
\mathbb{E}[X] &= p, & \text{Var}(X) &= p(1-p), & M_X(t) &= 1 - p + pe^t, \quad t \in \mathbb{R}.
\end{aligned}$$

Binomial

Binomial $X \sim \text{Bin}(n, p)$, $n \in \mathbb{N}$, $p \in [0, 1]$, $x \in \{0, \dots, n\}$:

$$\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad F(k) = \sum_{j=0}^k \binom{n}{j} p^j (1-p)^{n-j},$$

$$\mathbb{E}[X] = np, \quad \text{Var}(X) = np(1-p), \quad M_X(t) = (1-p+pe^t)^n, \quad t \in \mathbb{R}.$$

Poisson

$$\mathbf{Poisson} \quad X \sim \text{Pois}(\lambda), \quad \lambda > 0, \quad x \in \{0, 1, 2, \dots\} :$$

$$\Pr(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad F(k) = e^{-\lambda} \sum_{j=0}^k \frac{\lambda^j}{j!},$$

$$\mathbb{E}[X] = \lambda, \quad \text{Var}(X) = \lambda, \quad M_X(t) = \exp(\lambda(e^t - 1)), \quad t \in \mathbb{R}.$$

Beta

$$\mathbf{Beta} \quad X \sim \text{Beta}(\alpha, \beta), \quad \alpha, \beta > 0, \quad x \in (0, 1) :$$

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, \quad F(x) = I_x(\alpha, \beta),$$

$$\mathbb{E}[X] = \frac{\alpha}{\alpha + \beta}, \quad \text{Var}(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)},$$

$$M_X(t) = {}_1F_1(\alpha; \alpha + \beta; t), \quad t \in \mathbb{R}.$$