

Competitive Games — Cheat Sheet

Game types and representation

- **Strategic settings:** interdependence of actions and payoffs.
- **Noncooperative vs. cooperative:** individual actions vs. joint actions as shorthand for negotiated contracts.
- **Normal form:** $\Gamma = (N, \{S_i\}_{i=1}^n, \{v_i\}_{i=1}^n)$ with $v_i : S_1 \times \dots \times S_n \rightarrow \mathbb{R}$.
- **Static game, complete information, common knowledge:** simultaneous moves, known actions/outcomes/preferences.
- **Matrix form (2 players, finite):** rows S_1 , columns S_2 , entry (v_1, v_2) .

Zero-sum (matrix) games

- **Definition:** two-player zero-sum \Rightarrow a single payoff matrix $A = (a_{ij})$ for row player; column gets $-a_{ij}$.
- **Saddle point and value:** if $\max_i \min_j a_{ij} = \min_j \max_i a_{ij} = v$, then (i^*, j^*) is a pure NE and v the value.
- **Constant-sum \rightarrow zero-sum:** subtract $k/2$ from both components if $v_1 + v_2 = k$ (equilibria unchanged).

Example: Battle of the Bismarck Sea. Model (row=Kenney, col=Imamura, payoff=days of bombing for row):

	North	South
North	2	2
South	1	3

Row minima: $(2, 1) \Rightarrow \max \min = 2$ at row North. Column maxima: $(2, 3) \Rightarrow \min \max = 2$ at column North. **Saddle point** (North, North), **value** $v = 2$.

Nonzero-sum canonical games

Prisoner's Dilemma.

	C	D
C	$(-1, -1)$	$(-10, 0)$
D	$(0, -10)$	$(-9, -9)$

D strictly dominates C for both \Rightarrow unique NE (D, D) , which is not Pareto-optimal (since (C, C) Pareto-dominates).

Battle of the Sexes.

	A	B
A	$(2, 1)$	$(0, 0)$
B	$(0, 0)$	$(1, 2)$

No dominance. Best-response intersections \Rightarrow two pure NE: (A, A) and (B, B) (coordination problem).

Cournot duopoly (no costs, inverse demand $p = 1 - Q$). Players choose $q_i \geq 0$. Profits $K_i(q_1, q_2) = q_i(1 - q_1 - q_2)$ for $q_1 + q_2 < 1$, else 0. Best-responses: $q_i = \frac{1-q_i}{2}$. Cournot NE: $q_1 = q_2 = \frac{1}{3}$. Not Pareto-optimal: $(\frac{1}{4}, \frac{1}{4})$ raises both firms' profits.

Nash equilibrium (pure) and how to find it

Definition. $s^* = (s_1^*, \dots, s_n^*)$ is NE if each $s_i^* \in \arg \max_{t_i \in S_i} v_i(t_i, s_{-i}^*)$.

3-step marking for 2×2 (pure NE):

- 1) For each column, underline the largest row payoff (row best responses).
- 2) For each row, overline the largest column payoff (column best responses).
- 3) Any cell with both marks is a pure NE. If strict-dominant profile exists or is the unique IESDS survivor, it is the unique NE.

Pareto optimality

Definition. x Pareto-dominates y if $v_i(x) \geq v_i(y) \forall i$ and $>$ for at least one i . An outcome is Pareto-optimal if it is not Pareto-dominated by any other outcome. Note: do not confuse Pareto-optimal with “equal payoffs” or symmetry.

Dominance and IESDS

Strict dominance. s_i strictly dominates s'_i if $v_i(s_i, s_{-i}) > v_i(s'_i, s_{-i})$ for all $s_{-i} \in S_{-i}$. Rational players never play strictly dominated strategies.

IESDS algorithm. Repeatedly delete strictly dominated strategies for any player. The surviving strategy sets S_i^k define the reduced game at round k . If a strict dominant-strategy equilibrium exists, it uniquely survives IESDS. Profiles that survive IESDS are “iterated-elimination equilibria.”

Extensive form (sequential) pointer

Sequential Battle of the Sexes: first-mover observed by second; analyze via the game tree (backward reasoning) to select credible outcomes.

Normal-Form Games — Cheat Sheet

Core object

Normal-form game:

$$\Gamma = (N, \{S_i\}_{i=1}^n, \{v_i\}_{i=1}^n), \quad v_i : S_1 \times \cdots \times S_n \rightarrow \mathbb{R}.$$

Components:

- **Players:** $N = \{1, 2, \dots, n\}$.
- **Pure-strategy sets:** $\{S_1, \dots, S_n\}$.
- **Payoff functions:** $\{v_1, \dots, v_n\}$ mapping each joint choice to a real payoff.

Simultaneity and outcomes

- **Simultaneous choice:** each player i selects $s_i \in S_i$ without knowing others' selections.
- **Strategy profile:** $s = (s_1, \dots, s_n) \in S_1 \times \cdots \times S_n$.
- **Realized payoffs:** after choices, player i receives $v_i(s_1, \dots, s_n)$.

Strategies

- **Strategy (informal):** plan of action aimed at a goal.
- **Pure strategy (formal):** deterministic plan; the set for player i is S_i .
- **Profile of pure strategies:** $s = (s_1, \dots, s_n)$ with $s_i \in S_i$ for all i .

Canonical example: Prisoner's Dilemma (normal form)

Players $N = \{1, 2\}$; strategies $S_i = \{C, D\}$. Representative payoffs:

$$v_1(C, C) = v_2(C, C) = -1, \quad v_1(D, D) = v_2(D, D) = -9,$$

$$v_1(D, C) = v_2(C, D) = 0, \quad v_1(C, D) = v_2(D, C) = -10.$$

(Displayed as a 2×2 matrix with entries (v_1, v_2) .)

Finite games

A game is **finite** if the player set is finite and each S_i is finite.

Matrix representation (two-player finite)

- **Rows:** strategies of player 1; k strategies $\Rightarrow k$ rows.
- **Columns:** strategies of player 2; m strategies $\Rightarrow m$ columns.
- **Cells:** ordered pair (v_1, v_2) at each (row, column).

Quick workflow (encode any normal-form problem)

- 1) Identify N (decision-makers).
- 2) Enumerate S_i (feasible pure actions).
- 3) Specify $v_i(\cdot)$ on $S_1 \times \cdots \times S_n$.
- 4) If $n = 2$ and finite, build the payoff matrix with cells (v_1, v_2) .

Dominance & IESDS — Cheat Sheet

Notation

Profile $s = (s_1, \dots, s_n)$ with payoff $v_i(s) = v_i(s_i, s_{-i})$. Opponents' strategy set $S_{-i} = S_1 \times \dots \times S_{i-1} \times S_{i+1} \times \dots \times S_n$ and opponents' profile $s_{-i} \in S_{-i}$.

Pareto baseline

x Pareto-dominates y if $v_i(x) \geq v_i(y)$ for all i and $>$ for some i . Pareto-optimal \iff not dominated.

Strict dominance

Definition. $s'_i \in S_i$ is strictly dominated by $s_i \in S_i$ if

$$v_i(s_i, s_{-i}) > v_i(s'_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}.$$

Implication. A rational player never plays a strictly dominated strategy.

Strictly dominant strategy and equilibrium

Strictly dominant strategy. s_i is strictly dominant for i if it strictly dominates every other $s'_i \neq s_i$.

Strict dominant-strategy equilibrium (DS). $s^D = (s_1^D, \dots, s_n^D)$ where each s_i^D is strictly dominant.

Uniqueness. If a game has a strict DS equilibrium s^D , then s^D is the unique DS equilibrium.

Weak dominance

s'_i is weakly dominated by s_i if $v_i(s_i, s_{-i}) \geq v_i(s'_i, s_{-i})$ for all s_{-i} with strict inequality for some s_{-i} . A weakly dominant equilibrium need not be unique.

IESDS: Iterated Elimination of Strictly Dominated Strategies

Rationality premises. (i) Rational players never play strictly dominated strategies. (ii) Common knowledge of rationality lets players iteratively delete dominated strategies and analyze the reduced game.

Algorithm. Let $S_i^0 = S_i$. For $k = 0, 1, 2, \dots$:

- 1) If some $s_i \in S_i^k$ is strictly dominated, remove all such strategies for all players and set S_i^{k+1} to the survivors.
- 2) Stop when no further strictly dominated strategies remain. The product $S^K = \prod_i S_i^K$ is the IESDS survivor set.

Concept. Any profile s^{ES} that survives IESDS is an *iterated-elimination equilibrium*.

Link to DS. If s^* is a strict DS equilibrium, then s^* uniquely survives IESDS.

Worked elimination pattern (from slides)

- Step 1: In a 3×3 example, column C is strictly dominated by R for player 2 since $(2 > 1, 6 > 4, 8 > 6)$ rowwise. Delete C .
- Step 2: In the reduced 3×2 game, rows M and D are strictly dominated by U for player 1. Delete M, D .
- Step 3: The residual 1×2 game leaves column L strictly dominating R for player 2. Unique survivor profile (U, L) with payoffs $(4, 3)$.

Workflow on any matrix

- 1) Scan each player's strategies for strict dominance; delete.
- 2) Repeat on the reduced game until no deletions remain (IESDS).
- 3) If a single profile survives, that is the prediction; if it is also strict DS, it will be the unique Nash equilibrium.
- 4) Check Pareto efficiency separately; do not confuse *strategies* (solutions) with *payoffs*.

Nash Equilibrium — Cheat Sheet

Core definitions

Belief-based: A Nash equilibrium is a system of beliefs and an action profile where each player plays a best response to their beliefs and beliefs are correct.

Strategy-based (pure): A profile $s^* = (s_1^*, \dots, s_n^*) \in S_1 \times \dots \times S_n$ is a Nash equilibrium iff each s_i^* is a best response to s_{-i}^* :

$$v_i(s_i^*, s_{-i}^*) \geq v_i(s'_i, s_{-i}^*) \quad \forall s'_i \in S_i, \forall i.$$

Best responses

$$BR_i(s_{-i}) \in \arg \max_{t_i \in S_i} v_i(t_i, s_{-i}).$$

In continuous-action games, compute first-order conditions (FOC) of v_i w.r.t t_i to obtain BR_i , then solve the n equations $\{t_i = BR_i(t_{-i})\}_{i=1}^n$.

Matrix recipe for pure NE (2 players)

- 1) For each *column* (player 2 fixed), underline the largest payoff of player 1 (row best responses).
- 2) For each *row* (player 1 fixed), overline the largest payoff of player 2 (column best responses).
- 3) Any cell with both an under- and an overline is a pure-strategy NE.

Links to other solution concepts

Uniqueness via dominance/IESDS: If a profile is a strict dominant-strategy equilibrium or the unique survivor of IESDS, then it is the unique NE.

NE vs. Pareto: NE need not be Pareto-optimal; individual best replies can block social efficiency.

Canonical illustrations (from slides)

Stag Hunt (Two Kinds of Societies). Two pure NE: (S, S) and (H, H) ; (S, S) Pareto-dominates (H, H) (coordination and trust).

Tragedy of the Commons (FOC & BR method). Common resource K . Each player i chooses $k_i \geq 0$.

$$v_i(k_i, k_{-i}) = \ln k_i + \ln \left(K - \sum_{j=1}^n k_j \right).$$

FOC:

$$\frac{\partial v_i}{\partial k_i} = \frac{1}{k_i} - \frac{1}{K - \sum_{j=1}^n k_j} = 0 \quad \Rightarrow \quad k_i = \frac{K - \sum_{j \neq i} k_j}{2}.$$

For $n = 2$:

$$k_1 = \frac{K - k_2}{2}, \quad k_2 = \frac{K - k_1}{2} \quad \Rightarrow \quad k_1^* = k_2^* = \frac{K}{3} \quad (\text{unique NE}).$$

Pareto benchmark (planner): maximize $w(k_1, k_2) = \sum_{i=1}^2 v_i$. FOC yield $k_1^\dagger = k_2^\dagger = \frac{K}{4}$. Conclusion: the NE overuses the commons relative to Pareto optimum.

Quick checklist

- Define N, S_i, v_i ; write BR_i .
- Discrete case: use 3-step underline/overline test for pure NE.
- Continuous case: FOC $\Rightarrow BR_i$ equations \Rightarrow solve system.
- If strict dominance exists for all players, the dominant profile is the unique NE.
- Do not equate NE with efficiency; check Pareto separately.

Game Theory Cheat Sheet

Normal-form game: $\Gamma = (N, \{S_i\}_{i \in N}, \{v_i\}_{i \in N})$, with $v_i : S_1 \times \dots \times S_n \rightarrow \mathbb{R}$.

Pure strategies $s = (s_1, \dots, s_n)$, $s_i \in S_i, \forall i$

opponents' set $S_{-i} = \prod_{j \neq i} S_j$

profile $s_{-i} \in S_{-i}$.

Payoff functions $v_i(s) = v_i(s_i, s_{-i})$.

Matrix entries: Each entry in this matrix is an ordered pair $(v_1(s_1 = \{S_i\}, s_2), v_2(s))$, where v_1 is the payoff to the row player and v_2 is the payoff to the column player.

Static, complete-information, common knowledge: choices simultaneous; actions, outcomes, and preferences known.

Zero-sum vs. nonzero-sum

Zero-sum (2p): one matrix $A = (a_{ij})$ for row; column payoff $-a_{ij}$. *Saddle point/value:* if $\max_i \min_j a_{ij} = \min_j \max_i a_{ij} = v$, the corresponding cell is a pure NE with game value v . Constant-sum reduces to zero-sum by affine shift. In a zero-sum game, the row player maximizes his minimal payoff $\min_j a_{ij}$ (maximin), while the column player minimizes the maximal drawdown $\max_i a_{ij}$ (minimax). If $\max_i \min_j a_{ij} < \min_j \max_i a_{ij}$, no pure NE exists. E. g., Battle of the Bismarck Sea, for the row player (Kenney) vs. column (Imamura): For Kenney the solution is $\max(\min(2, 2), \min(1, 3)) = 2$ at North; for Imamura it is $\min(\max(2, 1), \max(2, 3)) = 2$ at North. Thus (North, North) is a saddle point with value 2.

Dominance and IESDS (Iterated Elimination of Strictly Dominated Strategies)

s'_i is strictly dominated by s_i if $v_i(s'_i, s_{-i}) < v_i(s_i, s_{-i})$ for all s_{-i} . Rational players never use strictly dominated strategies. We say that $s_i \succ s'_i$.

Strictly dominant strategy: $s_i \in S_i$ is a strictly dominant strategy for player i if every other strategy is strictly dominated by it $s'_i \neq s_i$. $v_i(s_i, s_{-i}) > v_i(s'_i, s_{-i})$ for all $s'_i \neq s_i$, all s_{-i} , all i . A profile $s^D = (s_1^D, \dots, s_n^D)$ where each s_i^D is strictly dominant is a strict DS equilibrium. If a game has a strict DS equilibrium, the profile is unique if it exists.

Weak dominance: s_i weakly dominates s'_i if $v_i(s_i, s_{-i}) \geq v_i(s'_i, s_{-i})$ for all s_{-i} .

IESDS: iteratively delete strictly dominated strategies in the reduced game until none remain; the survivors S_i^K define the IESDS survivor set. Any profile s^{ES} that survives IESDS is an iterated-elimination equilibrium. If a strict DS equilibrium s^* exists, it uniquely survives IESDS.

Nash equilibrium

Definition (pure): s^* is a NE iff $v_i(s_i^*, s_{-i}^*) \geq v_i(t_i, s_{-i}^*)$ for all $t_i \in S_i$, all i . Equivalently $s_i^* \in \text{BR}_i(s_{-i}^*)$, where $\text{BR}_i(s_{-i}) = \arg \max_{t_i \in S_i} v_i(t_i, s_{-i})$.

If s^* is either s^D or the unique s^{ES} survivor, then s^* is the unique NE. If multiple s^{ES} survive, they are all NE, but there may be other NE as well.

If no s^{ES} survives, NE may or may not exist, If a NE exists, it may or may not be Pareto-optimal. If NE exists, a strict DS equilibrium need not exist.

If a strict DS equilibrium exists, it is the unique NE. If a NE exists, the IESDS survivor set may be empty, a singleton, or contain multiple profiles.

Matrix recipe (2p): underline row best replies column-by-column; overline column best replies row-by-row; double-marked cells are pure NE. A strict DS profile or a unique IESDS survivor is the unique NE.

Continuous actions: compute first-order conditions in t_i to get BR_i , then solve the system $t_i = \text{BR}_i(t_{-i})$.

Efficiency

s **Pareto-dominates** s' if $v_i(s) \geq v_i(s') \forall i$ and $>$ for some i , in which case s' is **Pareto-dominated by** s . An outcome is **Pareto-optimal** if it is not Pareto-dominated by any other outcome.

Canonical micro-examples (minimal forms)

Bismarck Sea (zero-sum).

	North	South
North	2	2
South	1	3

Row max min = 2 at North; column min max = 2 at North \Rightarrow saddle point (North, North), value 2.

Prisoner's Dilemma.

	C	D
C	$(-1, -1)$	$(-10, 0)$
D	$(0, -10)$	$(-9, -9)$

D strictly dominates C for both. Unique NE (D, D) , not Pareto-optimal.

Battle of the Sexes.

	A	B
A	$(2, 1)$	$(0, 0)$
B	$(0, 0)$	$(1, 2)$

Two pure NE (A, A) and (B, B) (coordination).

Cournot (costless, $p = 1 - Q$). $K_i(q) = q_i(1 - \sum_j q_j)$ for $\sum q_j < 1$. $BR_i(q_{-i}) = \frac{1-q_{-i}}{2}$. For $n = 2$: $q_1^* = q_2^* = \frac{1}{3}$ (Cournot NE), not Pareto-optimal.

Tragedy of the Commons (2p). $v_i(k) = \ln k_i + \ln(K - \sum_j k_j)$. FOC $\Rightarrow BR_i(k_{-i}) = \frac{K - \sum_{j \neq i} k_j}{2}$. Solve $\Rightarrow k_1^* = k_2^* = \frac{K}{3}$ (NE).
Planner max $\sum_i v_i$ yields $k_1^\dagger = k_2^\dagger = \frac{K}{4}$ (overuse at NE).

Workflow (one page)

1. Specify $(N, \{S_i\}, \{v_i\})$; build matrix if $n = 2$ and finite.
2. Delete strictly dominated strategies; iterate (IESDS).
3. Find pure NE by best-reply markings; for continuous actions solve $\{t_i = BR_i(t_{-i})\}$.
4. Compare NE to Pareto set.

Common Knowledge and Complete Information

Definition (Common Knowledge): An event E is *common knowledge* if:

- 1) Everyone knows E ,
- 2) Everyone knows that everyone knows E ,
- 3) Everyone knows that everyone knows that everyone knows E ,
- 4) and so on ad infinitum.

Definition (Complete Information): A game has *complete information* if the following four components are common knowledge among all players:

- All possible actions of all players,
- All possible outcomes,
- How each combination of actions affects which outcome will materialize,
- The preferences of each and every player over outcomes.

Players are rational: A rational player is one who chooses his action, $s_i \square S_i$, to maximize his payoff consistent with his beliefs about what is going on in the game. Players are intelligent: An intelligent player knows everything about the game: the actions, the outcomes, and the preferences of all the players. Common knowledge: The fact that players are rational and intelligent is common knowledge among the players of the game. Self-enforcement: It is the core of noncooperative game theory. Each player is in control own action, and he will stick to an action only if he finds it to be in his best interest