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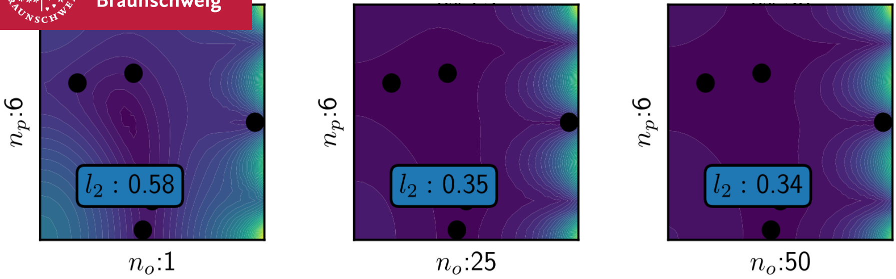

A Statistical Approach for the Fusion of Data and Finite Element Analysis in Vibroacoustics

Defense of the Master’s Thesis

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Lucas Hermann, October 28, 2021

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- statFEM
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- 2D Helmholtz example
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Part I

Objectives

Objectives

The goals of the thesis are:

- to understand and explain statFEM (statistical FEM) using a practical example
- to implement a statFEM approach for vibroacoustics using the Helmholtz equation
- to provide a means to use sparse sensor data to improve the accuracy of an FEM solution and to quantify uncertainty
- to gain insight on the limits and possibilities of statFEM in vibroacoustics

Objectives

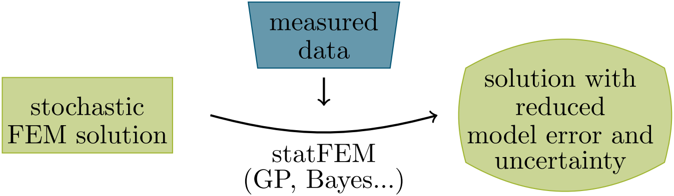
Hypotheses:

- uncertainty in parameters of the PDE will lead to uncertainty in the solution
- introducing observation data will decrease the overall variance
- the more data is introduced, the less probability mass will be on the FEM solution prior

Objectives

Motivation:

- Merge FEM and data while quantifying uncertainty: Apply statFEM to an FEM solution to reduce the model error



Part II

Prerequisites

Prerequisites: Data

Statistical Model

The vector of measured data $\boldsymbol{y}$ can be described as a combination of different parts:

$$\boldsymbol{y} = \boldsymbol{z} + \boldsymbol{e} = \rho \boldsymbol{P}\boldsymbol{u} + \boldsymbol{d} + \boldsymbol{e}$$

$\boldsymbol{z}$: The true solution
$\boldsymbol{e}$: Measurement error
$\boldsymbol{u}$: Original FEM solution
$\rho \boldsymbol{P}\boldsymbol{u}$: Scaled projected FEM solution ($\boldsymbol{P}$ is the projection matrix)
$\boldsymbol{d} \sim \mathcal{GP}(\bar{\boldsymbol{d}}, \mathbf{C}_d)$: Model mismatch

Prerequisites: Gaussian Processes

Gaussian Processes:

A GP $\boldsymbol{u} \sim \mathcal{GP}(\bar{\boldsymbol{u}}, \mathbf{C}_u)$ is a distribution of functions. It is defined by a mean $\bar{\boldsymbol{u}}$ and a covariance matrix $\mathbf{C}_u$. It is the limit of a multivariate Gaussian distribution (i.e. one with infinitely many dimensions).

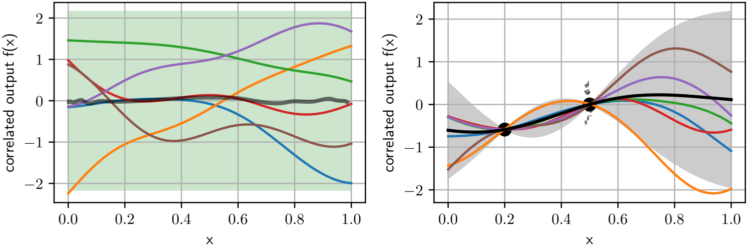


Figure 1: Conditioning a GP on data (black dots) yields a new GP with decreased variance around the sensor positions.

Prerequisites: Gaussian Processes

A kernel is a *covariance function*. It continuously describes the covariance between different points in the calculation domain and describes the expected behaviour of the approximated function.

Matérn Kernel:

$$k_M(r) = \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{\sqrt{2\nu} r}{l} \right)^{\nu} K_{\nu} \left(\frac{\sqrt{2\nu} r}{l} \right)$$

σ: standard deviation
l: correlation length
ν: free parameter
Choosing a kernel means giving prior information on the expected shape of the resulting functions.

Prerequisites: Bayesian Inference

Bayes' law:

Conditioning a distribution $\boldsymbol{u}$ on data $\boldsymbol{y}$ requires a prior for the distribution $p(\boldsymbol{u})$ and a likelihood for the data $p(\boldsymbol{y}|\boldsymbol{u})$.

$$p(\boldsymbol{u}|\boldsymbol{y}) = \frac{p(\boldsymbol{y}|\boldsymbol{u})p(\boldsymbol{u})}{p(\boldsymbol{y})}$$

The prior is the stochastic FEM solution.
The likelihood can be derived from the statistical model $(\boldsymbol{y} = \rho\boldsymbol{P}\boldsymbol{u} + \boldsymbol{d} + \boldsymbol{e})$ in which all components are Gaussian.
The marginal likelihood $p(\boldsymbol{y})$ is the probability for the data averaged over all possible prior solutions.

Part III

The statFEM procedure

statFEM: Bayesian Inference

Assuming the stochastic FEM solution to be a GP, the posterior GP can be infered using the statistical model for the data.

Inference of the posterior GP:

The posterior solution is going to be a GP:
$\boldsymbol{u}_{|y} \sim \mathcal{GP}(\bar{\mathbf{u}}_{|y}, \mathbf{C}_{u|y})$.

$$\bar{\mathbf{u}}_{|y} = \mathbf{C}_{\boldsymbol{u}|\boldsymbol{y}} \left(\rho \boldsymbol{P}^T (\mathbf{C}_d + \mathbf{C}_e)^{-1} \boldsymbol{y} + \mathbf{C}_u^{-1} \bar{\mathbf{u}}\right)$$

$$\mathbf{C}_{\boldsymbol{u}|\boldsymbol{y}} = \left(\rho^2 \boldsymbol{P}^T (\mathbf{C}_d + \mathbf{C}_e)^{-1} \boldsymbol{P} + \mathbf{C}_u^{-1}\right)^{-1}$$

Using an optimization method, the hyperparameters for $\mathbf{C}_d$ are determined.

statFEM: Estimation of Hyperparameters

Bayes' rule for the hyperparameters:

$$p(\boldsymbol{w}|\boldsymbol{y}) = \frac{p(\boldsymbol{y}|\boldsymbol{w})p(\boldsymbol{w})}{\int p(\boldsymbol{y}|\boldsymbol{w})p(\boldsymbol{w})\,\mathrm{d}\boldsymbol{w}}$$

with $\boldsymbol{w}$ the vector of hyperparameters and $\boldsymbol{y}$ the observations vector. For an uninformative prior there holds $p(\boldsymbol{w}|\boldsymbol{y}) \propto p(\boldsymbol{y}|\boldsymbol{w})$ and it can be shown that ($\boldsymbol{K}$ is a covariance matrix, n the number of observations)

negative log likelihood:

$$-\log p(\boldsymbol{y}|\boldsymbol{w}) = \frac{n}{2}\log(2\pi) + \frac{1}{2}\log|\boldsymbol{K}| + \frac{1}{2}(\rho\boldsymbol{P}\bar{\boldsymbol{u}} - \boldsymbol{y})^T\boldsymbol{K}^{-1}(\rho\boldsymbol{P}\bar{\boldsymbol{u}} - \boldsymbol{y}) \;,$$

which is minimized (with e.g. MCMC or L-BFGS) to find the optimal set of hyperparameters.

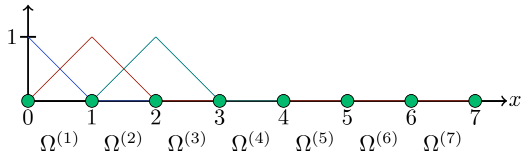
Setting: 1D Poisson

To test the methods and code, the well-known Poisson example was solved in 1D.

Weak form:

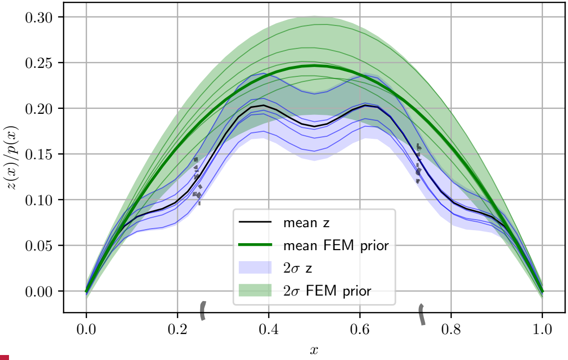
The uncertainty is in the source term: $\boldsymbol{f} \sim \mathcal{GP}(\bar{\boldsymbol{f}}, \mathbf{C}_f)$

$$-\int_\Omega \nabla p \cdot \nabla v \, \mathrm{d}x = \int_\Omega \boldsymbol{f} v \, \mathrm{d}x$$



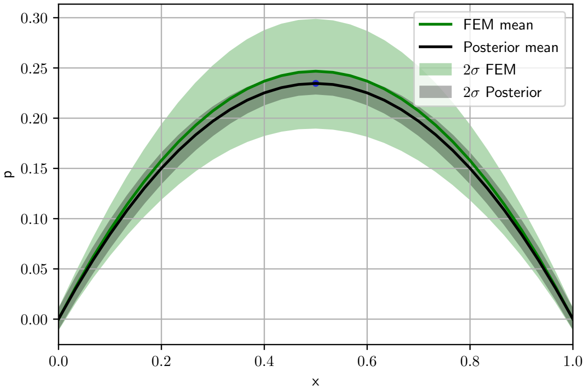
FEM solution: 1D Poisson

The FEM solution serves as a prior and is modeled as a GP with a mean and variance. Arbitrarily many samples can be drawn from it. A GP for the 'true' solution (here z) is created.



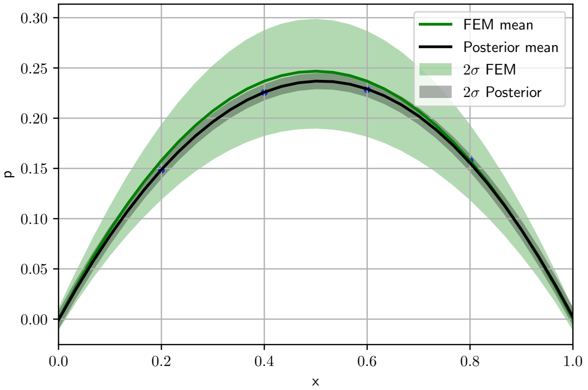
Infered solution: 1D Poisson

1 sensor, 1 observation



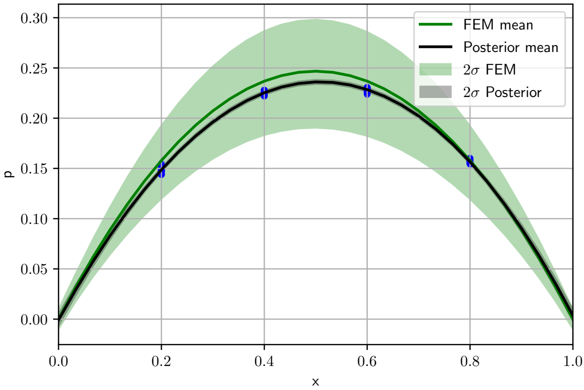
Infered solution: 1D Poisson

4 sensors, 1 observation each



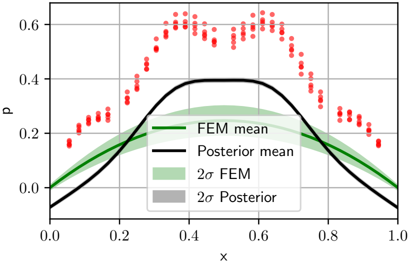
Infered solution: 1D Poisson

4 sensors, 20 observations each

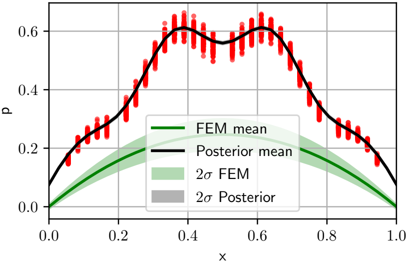


Infered solution: 1D Poisson

Prior-data conflict: The further away from the prior mean data is observed, the more observations are needed to match the mean.



(a) With model error. $n_p = 30$, $n_o = 5$



(b) With model error. $n_p = 30$, $n_o = 100$

Figure 2: Posterior of the 1D example for observations far outside the FEM prior variance band

Setting: 2D Helmholtz

Weak form:

The uncertainty is in the boundary term.

$$-\int_\Omega \nabla p \cdot \nabla v \, \mathrm{d}\Omega + \oint_\Gamma v \nabla p \, \mathrm{d}\Gamma + \int_\Omega k^2 p v \, \mathrm{d}\Omega = 0$$

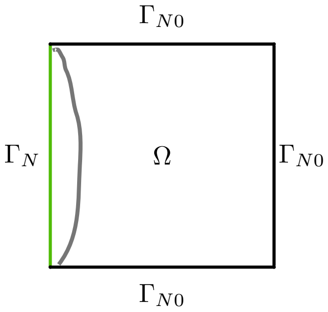
Neumann boundary:

$\oint_{\Gamma_N} v(\rho\omega^2 \vec{D}\vec{n}) \, \mathrm{d}\Gamma$

$\vec{D}$: Piston displacement

1D boundary function $D(y)$:

$D \sim \mathcal{GP}(\bar{\mathbf{g}}, \mathbf{C}_g)$



FEM solution: 2D Helmholtz

The FEM solution serves as a prior and is modeled as a GP with a mean and variance. Arbitrarily many samples can be drawn from it.

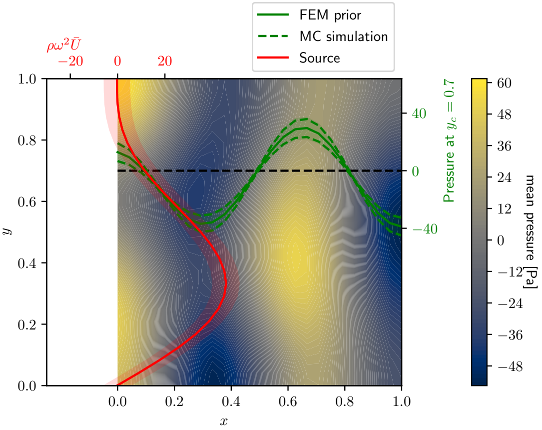


Figure 3: Overview of the source and prior GP

Inferred solution: 2D Helmholtz, Variance fields

Variance drops with increasing number of sensors and/or number of observations.

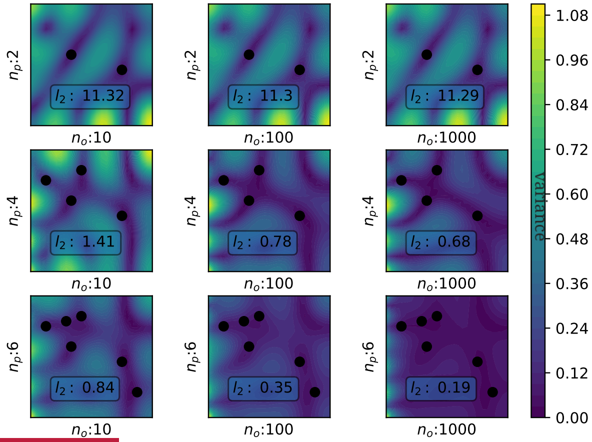


Figure 4: Posterior of the 2D Helmholtz equation example for different numbers of sensors and observations

Inferred solution: observations with error

When observing on a GP which is not a realization of the prior GP, statFEM can create a new GP which fits the data. The model error is greater than zero.

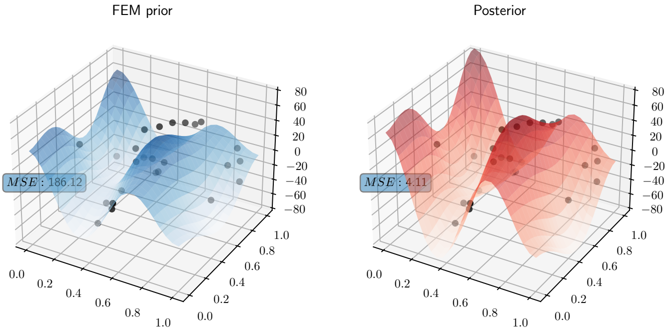


Figure 5: Comparison of prior and posterior in a 3D view for the observations with model error. Mean Squared Error (MSE) for comparison.

Inferred solution: prior-data conflict

If data is observed far outside the confidence intervals of the prior, the procedure is not able to match the posterior perfectly to the data. More sensor locations and observations are necessary.

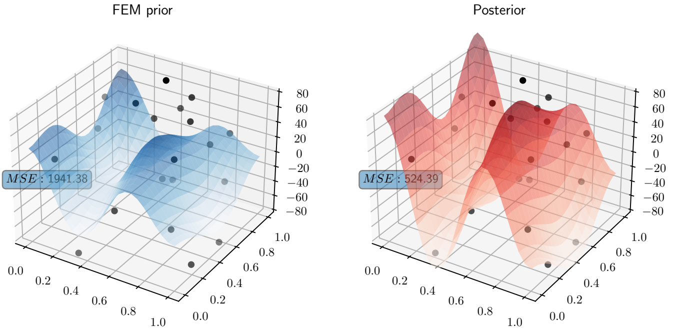


Figure 6: Comparison of prior and posterior for the observations with a prior-data conflict

Inferred solution: constrained measurement domain

For observations only in a part of the domain, the variance becomes lower in the whole domain but especially in regions where sensors are placed.

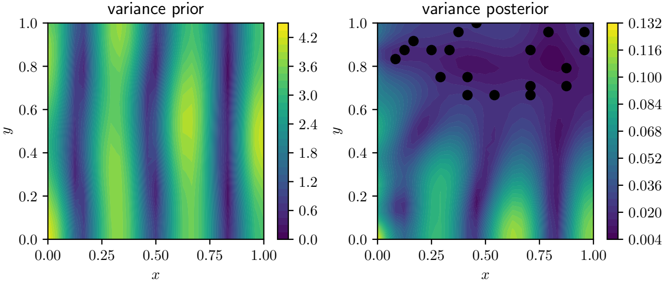


Figure 7: Prior and posterior variance compared for 500Hz and sensor locations only in the upper half of the domain.

Inferred solution: constrained measurement domain

The variance can still be lowered throughout the domain by increasing the number of sensors and observations.

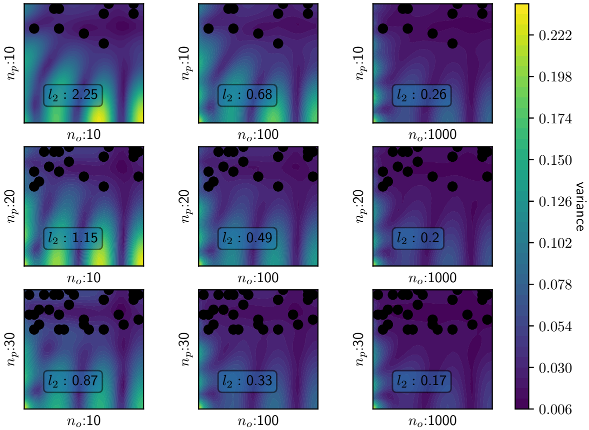


Figure 8: Posterior variance for 500 Hz *and sensor locations only in the upper half of the domain.*

Part IV

Outlook

Conclusion

- a working statFEM approach for vibroacoustics was developed
- small variances in the Neumann BC lead to strong variances in the solution
- by introducing observations, the variance can be lowered drastically
- the FEM prior is properly scaled to given data
- limits of the method lie in poorly scaled FEM priors and sparse available data

Outlook

Only very few possibilities have been show here. It is important to find out more about:

- uncertainties in different classes of boundaries such as impedance boundary conditions and in material parameters
- different choices of kernels for the boundary function
- kernels with frequency dependence
- convergence behavior of the method
- sensor placement algorithm

Thank you for your attention and feel free to ask questions!

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