

# A Statistical Approach for the Fusion of Data and Finite Element Analysis in Vibroacoustics

Final Presentation

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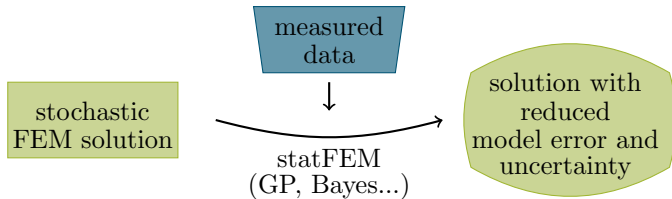
Part I

# **Objectives**

# Objectives

statFEM:

- Merge FEM and data while quantifying uncertainty: Apply statFEM to an FEM solution to reduce the model error



# Objectives

The goal of the thesis is:

- to understand statFEM
- to apply statFEM to vibroacoustics using the Helmholtz equation
- to provide a means to use sparse sensor data to improve the accuracy of an FEM solution
- to use a non-parametric approach to increase the accuracy of the solution without touching the FEM itself

## Part II

### **Prerequisites**

# FEM Solution

# Prerequisites: Data

## Statistical Model

The vector of measured data  $\mathbf{y}$  can be described as a combination of different parts:

$$\mathbf{y} = \mathbf{z} + \mathbf{e} = \rho \mathbf{P} \mathbf{u} + \mathbf{d} + \mathbf{e} \quad (1)$$

$\mathbf{z}$ : The true solution

$\mathbf{e}$ : Measurement error

$\rho \mathbf{P} \mathbf{u}$ : Scaled projected FEM solution

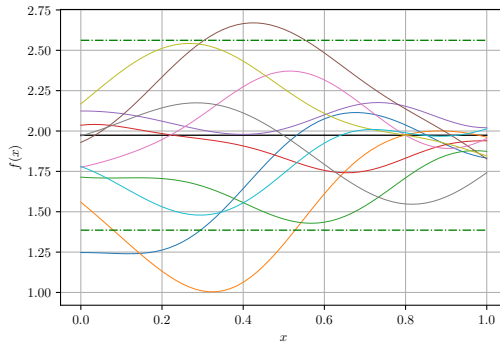
$\mathbf{d} \sim \mathcal{GP}(\bar{\mathbf{d}}, \mathbf{C}_d)$ : Model mismatch error



# Prerequisites: Gaussian Processes

## Gaussian Processes:

A GP  $\mathbf{u} \sim \mathcal{GP}(\bar{\mathbf{u}}, \mathbf{C}_u)$  is a distribution of functions. It is defined by a mean  $\bar{\mathbf{u}}$  and a covariance matrix  $\mathbf{C}_u$ . It is basically just a multivariate Gaussian distribution.



# Prerequisites: Gaussian Processes

## Squared Exponential Kernel:

A kernel is a *covariance function*. It continuously describes the covariance between different points in the calculation domain and describes the expected behaviour of the approximated function.

$$k_{SE} = \sigma \exp \left( -\frac{||x - x'||^2}{2l^2} \right) \quad (2)$$

$\sigma$ : standard deviation

$l$ : distance measure

Evaluating the kernel at a defined set of points (e.g. nodes in an FEM mesh) yields the covariance matrix.

# Prerequisites: Bayesian Inference

## Bayes' law:

Conditioning a distribution  $\mathbf{u}$  on data  $\mathbf{y}$  requires a prior for the distribution  $p(\mathbf{u})$  and a likelihood for the data  $p(\mathbf{y}|\mathbf{u})$ .

$$p(\mathbf{u}|\mathbf{y}) = \frac{p(\mathbf{y}|\mathbf{u})p(\mathbf{u})}{p(\mathbf{y})} \quad (3)$$

The prior is the stochastic FEM solution.

The likelihood can be derived from the statistical model

$(\mathbf{y} = \rho \mathbf{P} \mathbf{u} + \mathbf{d} + \mathbf{e})$  in which all components are Gaussian.

The marginal likelihood  $p(\mathbf{y})$  is the probability for the data averaged over all possible prior solutions.

# Prerequisites: Bayesian Inference

Assuming the stochastic FEM solution to be a GP, the posterior GP can be inferred using the statistical model for the data.

## Inference of the posterior GP:

The posterior solution is going to be a GP:

$$\mathbf{u}_{|y} \sim \mathcal{GP}(\bar{\mathbf{u}}_{|y}, C_{u|y}).$$

$$\bar{\mathbf{u}}_{|y} = C_{u|y} (\rho P^T (C_d + C_e)^{-1} \mathbf{y} + C_u^{-1} \bar{\mathbf{u}}) \quad (4)$$

$$C_{u|y} = (\rho^2 P^T (C_d + C_e)^{-1} P + C_u^{-1})^{-1} \quad (5)$$

Using an optimization method, the hyperparameters for  $C_d$  are determined.

## Part III

# **The statFEM procedure**

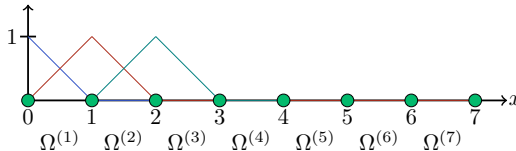
# Setting: 1D Poisson

To test the methods and code, the well-known Poisson example was solved in 1D.

## Weak form:

The uncertainty is in the source term:  $f \sim \mathcal{GP}(\bar{f}, C_f)$

$$-\int_{\Omega} \nabla p \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \quad (6)$$

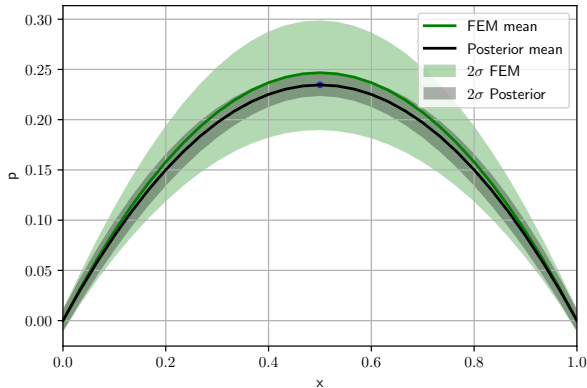


# FEM solution: 1D Poisson

The FEM solution serves as a prior and is modeled as a GP with a mean and variance. Arbitrarily many samples can be drawn from it.

# Inferred solution: 1D Poisson

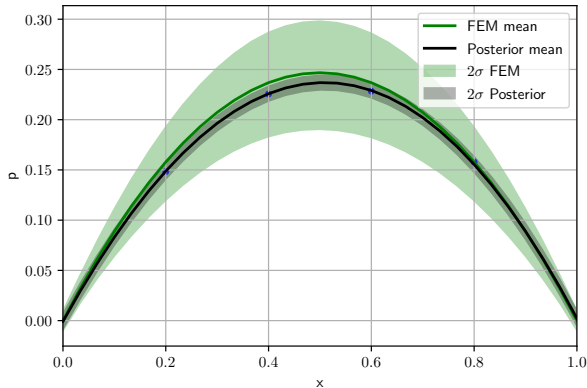
1 sensor, 1 observation





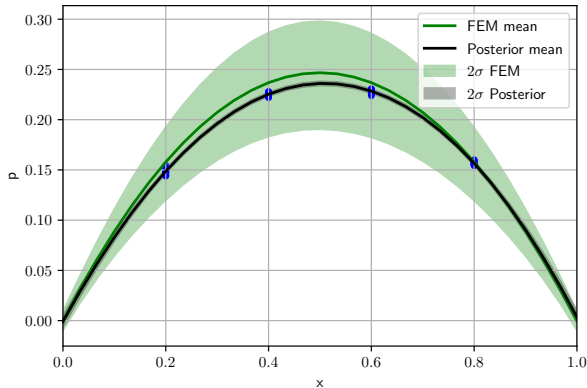
# Inferred solution: 1D Poisson

4 sensors, 1 observation each



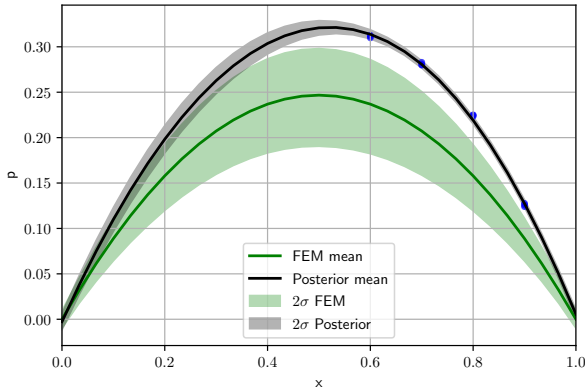
# Inferred solution: 1D Poisson

4 sensors, 20 observations each



# Inferred solution: 1D Poisson

Only partial data available: FEM prior still determines overall shape



# Setting: 2D Helmholtz

## Weak form:

The **uncertainty** is in the boundary term.

$$-\int_{\Omega} \nabla p \cdot \nabla v \, d\Omega + \oint_{\Gamma} v \nabla p \, d\Gamma + \int_{\Omega} k^2 p v \, d\Omega = 0 \quad (7)$$

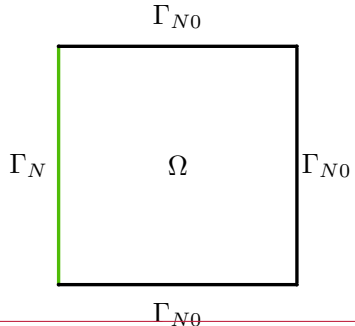
## Neumann boundary:

$$\oint_{\Gamma_N} v (\rho \omega^2 \vec{D} \vec{n}) \, d\Gamma$$

$\vec{D}$ : Piston displacement

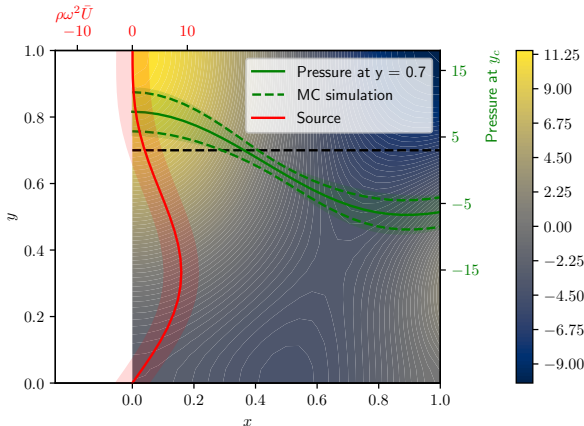
1D boundary function  $D(y)$ :

$$D \sim \mathcal{GP}(\bar{g}, C_g)$$



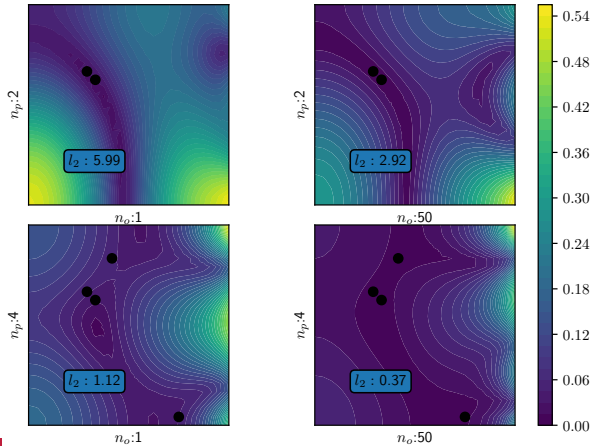
# FEM solution: 2D Helmholtz

The FEM solution serves as a prior and is modeled as a GP with a mean and variance.



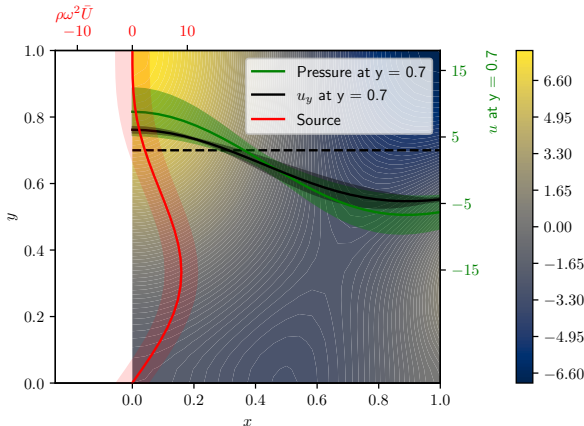
# Inferred solution: 2D Helmholtz, Variance fields

Variance drops with increasing number of sensors and/or number of observations



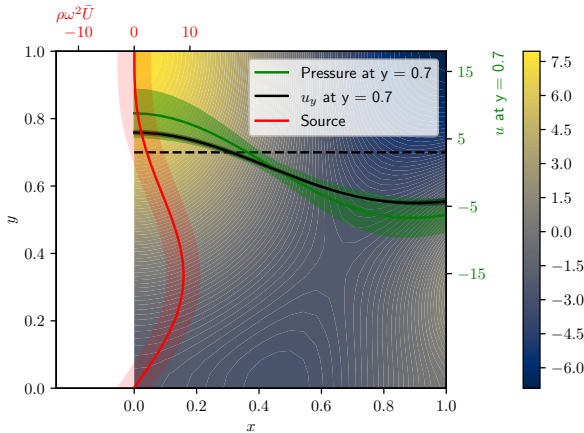
# Inferred solution: 2D Helmholtz

1 sensor, 1 observation. Data "observed" on a 75% scaling of the FEM prior.



# Inferred solution: 2D Helmholtz

4 sensors, 50 observations. Data "observed" on a 75% scaling of the FEM prior.





Part IV

## **Outlook**