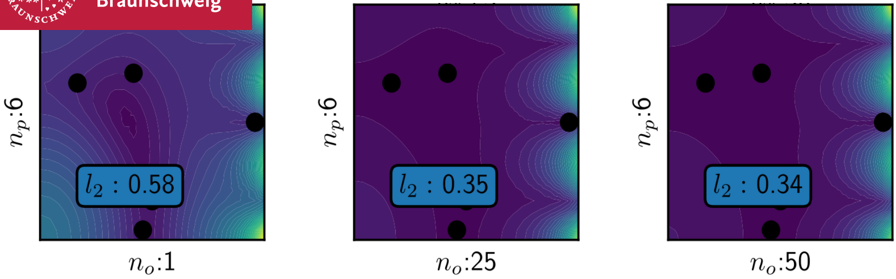


A Statistical Approach for the Fusion of Data and Finite Element Analysis in Vibroacoustics

Half-time report

Lucas Hermann, July 27, 2021

Content

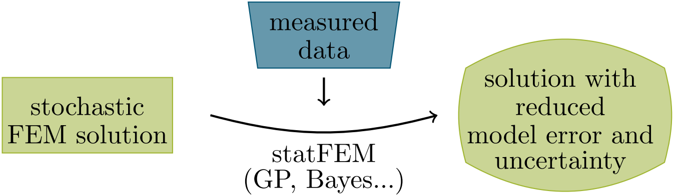
Part I

Objectives

Objectives

statFEM:

- Merge FEM and data while quantifying uncertainty: Apply statFEM to an FEM solution to reduce the model error



Objectives

The goal of the thesis is:

- to understand statFEM
- to apply statFEM to vibroacoustics using the Helmholtz equation
- to provide a means to use sparse sensor data to improve the accuracy of an FEM solution
- to use a non-parametric approach to increase the accuracy of the solution without touching the FEM itself

Part II

Prerequisites

Prerequisites: Data

Statistical Model

The vector of measured data $\boldsymbol{y}$ can be described as a combination of different parts:

$$\boldsymbol{y} = \boldsymbol{z} + \boldsymbol{e} = \rho \boldsymbol{P}\boldsymbol{u} + \boldsymbol{d} + \boldsymbol{e} \tag{1}$$

$\boldsymbol{z}$: The true solution

$\boldsymbol{e}$: Measurement error

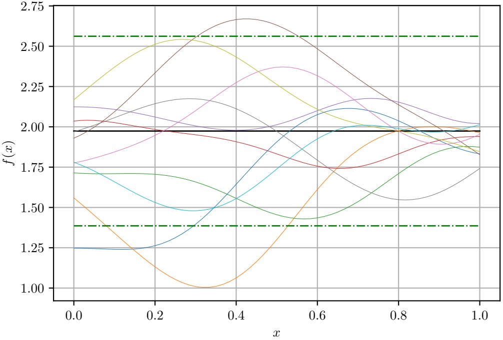
$\rho \boldsymbol{P}\boldsymbol{u}$: Scaled projected FEM solution

$\boldsymbol{d} \sim \mathcal{GP}(\bar{\boldsymbol{d}}, C_d)$: Model mismatch error

Prerequisites: Gaussian Processes

Gaussian Processes:

A GP $\boldsymbol{u} \sim \mathcal{GP}(\bar{\boldsymbol{u}}, C_u)$ is a distribution of functions. It is defined by a mean $\bar{\boldsymbol{u}}$ and a covariance matrix C_u. It is basically just a multivariate Gaussian distribution.



Prerequisites: Gaussian Processes

Squared Exponential Kernel:

A kernel is a *covariance function*. It continuously describes the covariance between different points in the calculation domain and describes the expected behaviour of the approximated function.

$$k_{SE} = \sigma \exp\left(-\frac{||x - x'||^2}{2l^2}\right) \tag{2}$$

σ: standard deviation
l: distance measure
Evaluating the kernel at a defined set of points (e.g. nodes in an FEM mesh) yields the covariance matrix.

Prerequisites: Bayesian Inference

Bayes' law:

Conditioning a distribution $\boldsymbol{u}$ on data $\boldsymbol{y}$ requires a prior for the distribution $p(\boldsymbol{u})$ and a likelihood for the data $p(\boldsymbol{y}|\boldsymbol{u})$.

$$p(\boldsymbol{u}|\boldsymbol{y}) = \frac{p(\boldsymbol{y}|\boldsymbol{u})p(\boldsymbol{u})}{p(\boldsymbol{y})} \tag{3}$$

The prior is the stochastic FEM solution.
The likelihood can be derived from the statistical model ($\boldsymbol{y} = \rho \boldsymbol{P}\boldsymbol{u} + \boldsymbol{d} + \boldsymbol{e}$) in which all components are Gaussian.
The marginal likelihood $p(\boldsymbol{y})$ is the probability for the data averaged over all possible prior solutions.

Prerequisites: Bayesian Inference

Assuming the stochastic FEM solution to be a GP, the posterior GP can be infered using the statistical model for the data.

Inference of the posterior GP:

The posterior solution is going to be a GP:
$\boldsymbol{u}_{|y} \sim \mathcal{GP}(\bar{\boldsymbol{u}}_{|y}, C_{u|y})$.

$$\bar{\boldsymbol{u}}_{|y} = C_{\boldsymbol{u}|\boldsymbol{y}} \left(\rho P^T (C_d + C_e)^{-1} y + C_u^{-1} \bar{u}\right) \tag{4}$$

$$C_{\boldsymbol{u}|\boldsymbol{y}} = \left(\rho^2 P^T (C_d + C_e)^{-1} P + C_u^{-1}\right)^{-1} \tag{5}$$

Using an optimization method, the hyperparameters for C_d are determined.

Part III

The statFEM procedure

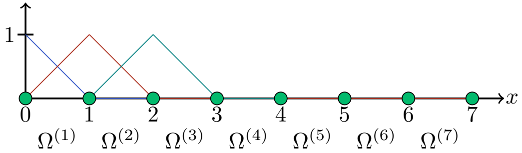
Setting: 1D Poisson

To test the methods and code, the well-known Poisson example was solved in 1D.

Weak form:

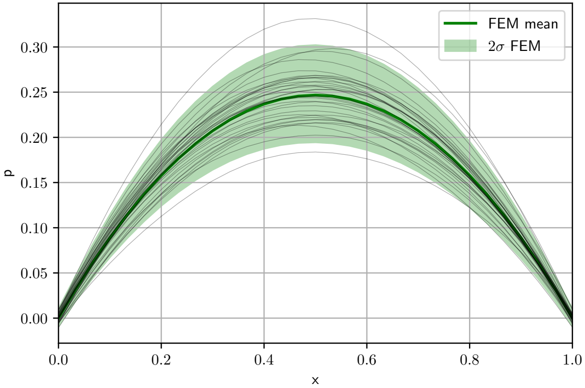
The uncertainty is in the source term: $f \sim \mathcal{GP}(\bar{f}, C_f)$

$$-\int_\Omega \nabla p \cdot \nabla v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x \tag{6}$$



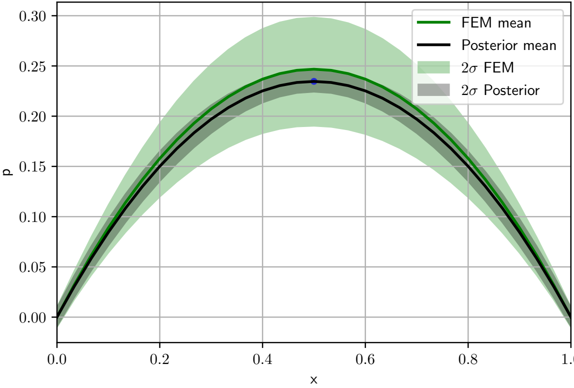
FEM solution: 1D Poisson

The FEM solution serves as a prior and is modeled as a GP with a mean and variance. Arbitrarily many samples can be drawn from it.



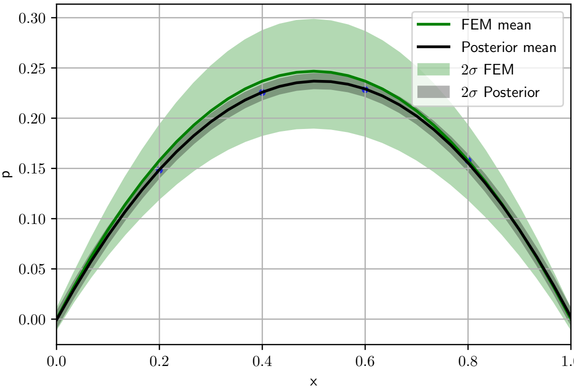
Infered solution: 1D Poisson

1 sensor, 1 observation



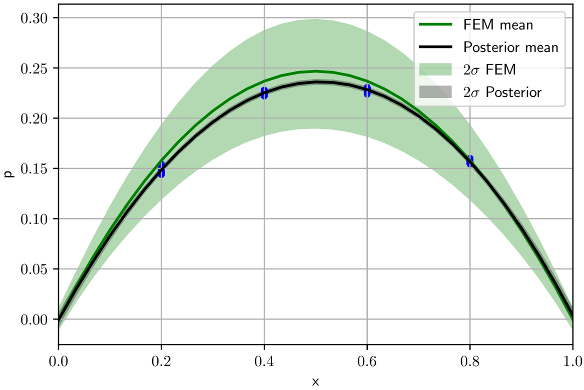
Infered solution: 1D Poisson

4 sensors, 1 observation each



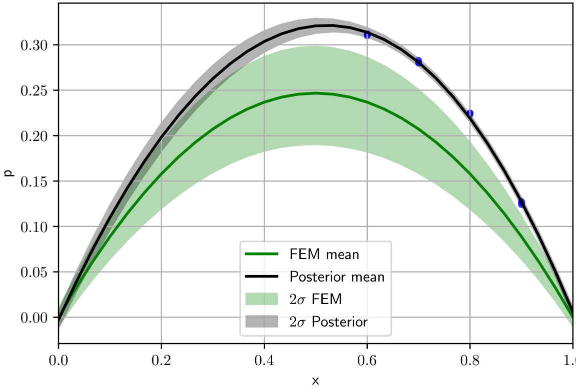
Infered solution: 1D Poisson

4 sensors, 20 observations each



Infered solution: 1D Poisson

Only partial data available: FEM prior still determines overall shape



Setting: 2D Helmholtz

Weak form:

The uncertainty is in the boundary term.

$$-\int_{\Omega} \nabla p \cdot \nabla v \, \mathrm{d}\Omega + \oint_{\Gamma} v \nabla p \, \mathrm{d}\Gamma + \int_{\Omega} k^2 p v \, \mathrm{d}\Omega = 0 \tag{7}$$

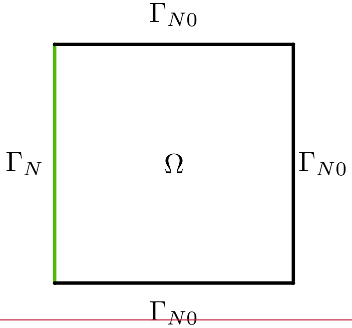
Neumann boundary:

$\oint_{\Gamma_N} v(\rho\omega^2 \vec{D}\vec{n}) \, \mathrm{d}\Gamma$

$\vec{D}$: Piston displacement

1D boundary function $D(y)$:

$D \sim \mathcal{GP}(\bar{g}, C_g)$



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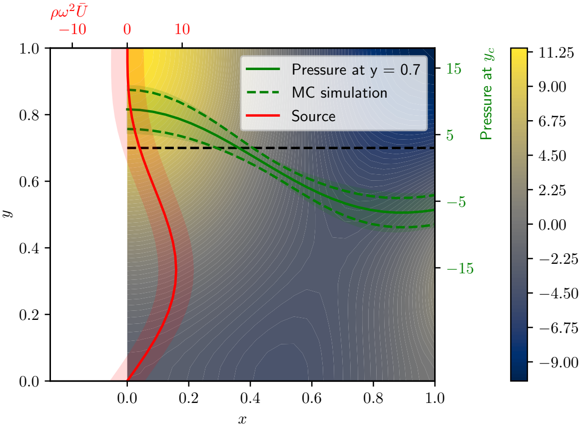
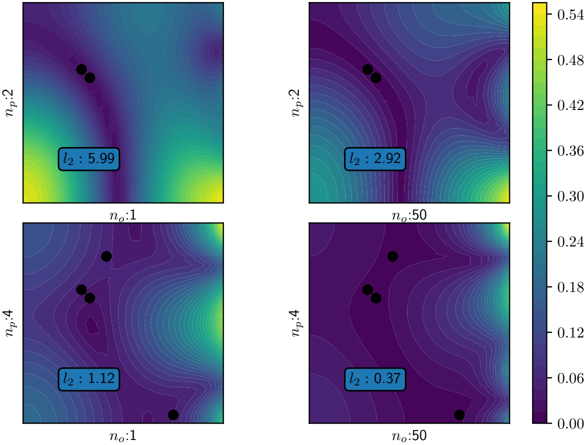


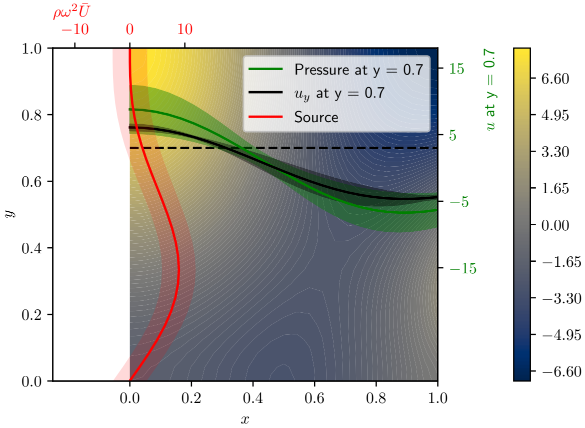
Inferred solution: 2D Helmholtz, Variance fields

Variance drops with increasing number of sensors and/or number of observations



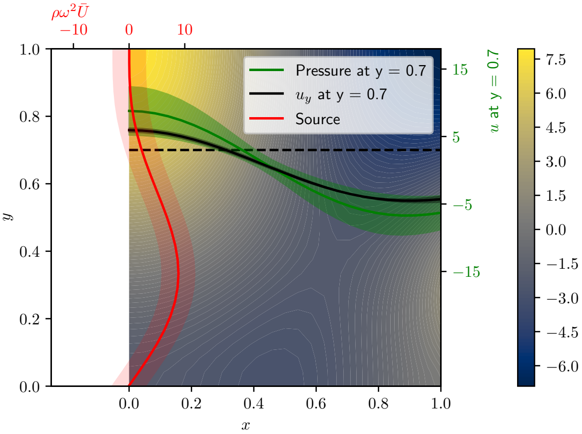
Inferred solution: 2D Helmholtz

1 sensor, 1 observation. Data "observed" on a 75% scaling of the FEM prior.



Inferred solution: 2D Helmholtz

4 sensors, 50 observations. Data "observed" on a 75% scaling of the FEM prior.



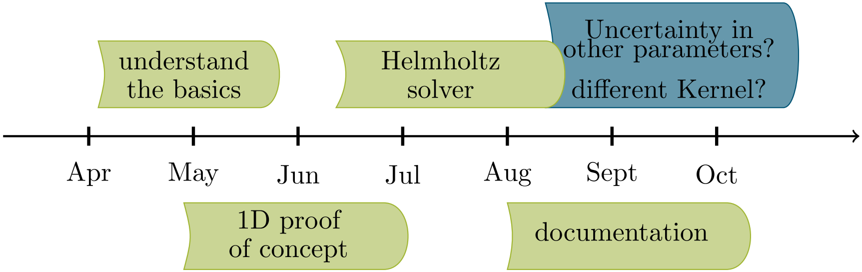
Part IV

Outlook

Status

- I understood the basics and began documenting
- A 1D proof of concept works well
- The code for the Helmholtz solver works but needs some iterations:
  - plausibility checks
  - numerical experiments
  - explore and find limits of the method
- documentation is at $\sim$ 50%
- further work could be done, if there is time

Timeline


understand the basics
Helmholtz solver
Uncertainty in other parameters?
different Kernel?
Apr
May
Jun
Jul
Aug
Sept
Oct
1D proof of concept
documentation