TTK4190 - Assignment 1

Group 27
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1 Problem 1 - Attitude Control of Satellite

1.1 Problem 1.1

Since the state η is a function of $\boldsymbol{\epsilon}$, it is not necessary to include the state in the analysis. From Equation (2.77) in [1], we have

$$\dot{\boldsymbol{q}} = \begin{bmatrix} \dot{\eta} \\ \dot{\boldsymbol{\epsilon}} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -\boldsymbol{\epsilon}^T \\ \eta \boldsymbol{I}_3 + \boldsymbol{S}(\boldsymbol{\epsilon}) \end{bmatrix} \boldsymbol{\omega} \tag{1}$$

Our system, given in Equation (1) in the assignment, can then be reduced to

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\boldsymbol{\epsilon}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\eta \boldsymbol{I}_3 + \boldsymbol{S}(\boldsymbol{\epsilon}))\boldsymbol{\omega} \\ \\ \boldsymbol{I}_g^{-1}\boldsymbol{S}(\boldsymbol{I}_g\boldsymbol{\omega})\boldsymbol{\omega} + \boldsymbol{I}_g^{-1}\boldsymbol{\tau} \end{bmatrix} \tag{2}$$

The equilibrium point $\boldsymbol{x}_0$ of the open-loop system is found when $\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\boldsymbol{\epsilon}} & \dot{\boldsymbol{\omega}} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}$. Applying this, and the assumptions given in the assignment, (2) yields

$$\begin{aligned} \frac{1}{2}(\eta \boldsymbol{I}_3 + \boldsymbol{S}(\boldsymbol{\epsilon}))\boldsymbol{\omega} = 0 \\ \boldsymbol{I}_g^{-1}\boldsymbol{S}(\boldsymbol{I}_g\boldsymbol{\omega})\boldsymbol{\omega} = 0 \end{aligned} \tag{3}$$

which implies $\boldsymbol{\omega} = 0$. In the assignment, it was further given that $\boldsymbol{\epsilon} = 0$. Hence, the equilibrium point is given as

$$\boldsymbol{x}_0 = \begin{bmatrix} \boldsymbol{\epsilon}_0 & \boldsymbol{\omega}_0 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}} \tag{4}$$

The $\boldsymbol{A}$ and $\boldsymbol{B}$ matrices are then given as

$$\boldsymbol{A} = \begin{bmatrix} \frac{\partial f_1}{\partial \boldsymbol{\epsilon}} & \frac{\partial f_1}{\partial \boldsymbol{\omega}} \\ \\ \frac{\partial f_2}{\partial \boldsymbol{\epsilon}} & \frac{\partial f_2}{\partial \boldsymbol{\omega}} \end{bmatrix}_{|\boldsymbol{x}=\boldsymbol{x}_0} \qquad \boldsymbol{B} = \begin{bmatrix} \frac{\partial f_1}{\partial \boldsymbol{\tau}} \\ \\ \frac{\partial f_2}{\partial \boldsymbol{\tau}} \end{bmatrix}_{|\boldsymbol{x}=\boldsymbol{x}_0} \tag{5}$$

where

$$\begin{aligned} f_1 &= \frac{1}{2}(\eta \boldsymbol{I}_3 + \boldsymbol{S}(\boldsymbol{\epsilon}))\boldsymbol{\omega} \\ f_2 &= \boldsymbol{I}_g^{-1}\boldsymbol{S}(\boldsymbol{I}_g\boldsymbol{\omega})\boldsymbol{\omega} + \boldsymbol{I}_g^{-1}\boldsymbol{\tau} \end{aligned} \tag{6}$$

Due to the inertia matrix $\boldsymbol{I}_g$ only being the identity matrix times a constant, the cross-product in f_2 will be zero. The expression can therefore be reduced to

$$f_2 = \boldsymbol{I}_g^{-1}\boldsymbol{\tau} \tag{7}$$

Then

$$\boldsymbol{A} = \begin{bmatrix} -\frac{1}{2}\boldsymbol{S}(\boldsymbol{w}) & \frac{1}{2}(\eta \boldsymbol{I}_3 + \boldsymbol{S}(\boldsymbol{\epsilon})) \\ 0 & 0 \end{bmatrix}_{|\boldsymbol{x}=\boldsymbol{x}_0} \tag{8}$$

Inserting $\eta = 1$ and $\boldsymbol{x} = \boldsymbol{x}_0$ yields

$$\boldsymbol{A} = \begin{bmatrix} 0 & \frac{1}{2}\boldsymbol{I}_3 \\ 0 & 0 \end{bmatrix} \tag{9}$$

Lastly

$$\boldsymbol{B} = \begin{bmatrix} 0 \\ \boldsymbol{I}_g^{-1} \end{bmatrix} \tag{10}$$

1.2 Problem 1.2

The closed-loop system is

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\boldsymbol{\epsilon}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\eta \boldsymbol{I}_3 + \boldsymbol{S}(\boldsymbol{\epsilon}))\boldsymbol{\omega} \\ \boldsymbol{I}_g^{-1}\boldsymbol{S}(\boldsymbol{I}_g\boldsymbol{\omega})\boldsymbol{\omega} - \boldsymbol{I}_g^{-1}\boldsymbol{K}_d\boldsymbol{\omega} - \boldsymbol{I}_g^{-1}k_p\boldsymbol{\epsilon} \end{bmatrix} \tag{11}$$

The linearized system matrix around the equilibrium is then

$$\boldsymbol{A} = \begin{bmatrix} 0 & \frac{1}{2}\boldsymbol{I}_3 \\ -\boldsymbol{I}_g^{-1}k_p & -\boldsymbol{I}_g^{-1}\boldsymbol{K}_d \end{bmatrix} \tag{12}$$

The eigenvalues of the system are found from the roots of the characteristic polynomial

$$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \lambda^2 + \boldsymbol{I}_g^{-1}\boldsymbol{K}_d\lambda + \frac{1}{2}k_p\boldsymbol{I}_3\boldsymbol{I}_g^{-1} = 0 \tag{13}$$

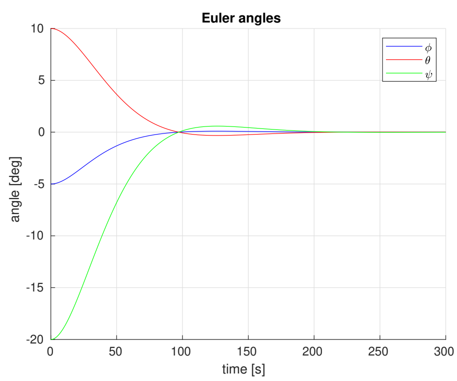
The roots can then be found using the quadratic formula

$$\lambda_{1,2} = \frac{-\boldsymbol{I}_g^{-1}\boldsymbol{K}_d \pm ...}{2} \tag{14}$$

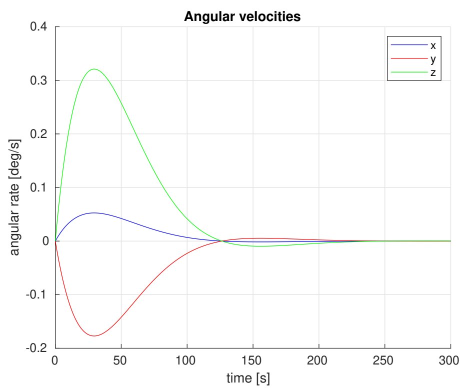
Since both the inertia matrix $\boldsymbol{I}_g$ and the gain matrix $\boldsymbol{K}_d$ are positive definite, the real part of the eigenvalues are negative. It can therefore be concluded that the linearized closed-loop system is stable.

Since this is a physical system, real poles would be preferred. Complex poles would give the system an oscillatory behaviour, which could damage the physical components of the system.

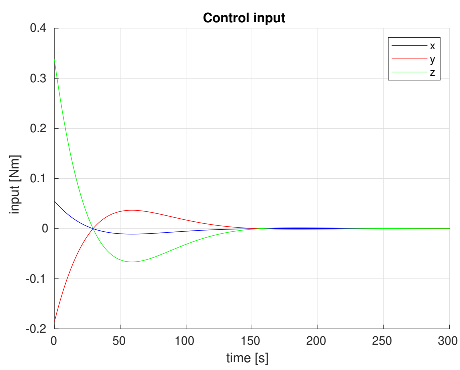
1.3 Problem 1.3



(a) Euler angles



(b) Angular velocities



(c) Control inputs

Figure 1: Relevant plots for Problem 1.3

We see that with the controller, all Euler angles (and therefore also the angular velocities) converges at the stable equilibrium point $\begin{bmatrix}\phi & \theta & \psi\end{bmatrix}^T_{|eq} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^T$. Because this is a stable point we will also have zero control input when the stable point is reached. However we see that the system is relatively slow for its size (mass $180kg$). We would perhaps prefer a faster system (reaches the equilibrium point at $t \approx 150s$. If we look closer at the system's eigenvalues defined in eq. (14) we see that the eigenvalues are small (close to zero), suggesting a slow response of the controller.

$$\lambda_{1,2} = \frac{-\boldsymbol{I}_g^{-1}\boldsymbol{K}_d \pm \sqrt{(-\boldsymbol{I}_g^{-1}\boldsymbol{K}_d)^2 - 4\frac{1}{2}\boldsymbol{I_3}\boldsymbol{I_g^{-1}}k_p}}{2} \tag{15}$$

With $\boldsymbol{I_g} = mR_{33}\boldsymbol{I_3}$, $m = 180$kg, $R_{33} = 2.0$m, $\boldsymbol{K_d} = 40\boldsymbol{I_3}$ and $k_p = 2$ we get that the eigenvalues are:

$$\begin{aligned}
\lambda_1 &= \begin{bmatrix} -0.0278 + 0.0248i & 0 & 0 \\ 0 & -0.0278 + 0.0248i & 0 \\ 0 & 0 & -0.0278 + 0.0248i \end{bmatrix} \\
\lambda_2 &= \begin{bmatrix} -0.0278 - 0.0248i & 0 & 0 \\ 0 & -0.0278 - 0.0248i & 0 \\ 0 & 0 & -0.0278 - 0.0248i \end{bmatrix}
\end{aligned} \tag{16}$$

So we can confirm that even though the poles are on the left hand plane (i.e. we have stability) they will have a slow response due to being close to zero. We can change this by tuning K_d and k_p. We can also verify the small oscillating behavior of the complex conjugates ($\pm 0.0248i$).

To follow a nonzero constant reference signal we could introduce another nonzero component in the control law equation, i.e. $\boldsymbol{\Theta_{ref}}$. In other words, we implement error coordinates. The equilibrium point is given at zero, however we can shift this point to a nonzero constant reference.

1.4 Problem 1.4

The quaternion error $\tilde{\boldsymbol{q}}$ is defined as

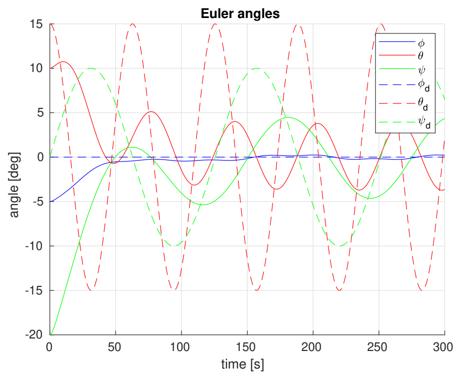
$$\begin{aligned}
\tilde{\boldsymbol{q}} := \begin{bmatrix}\tilde{\eta} \\ \tilde{\boldsymbol{\epsilon}}\end{bmatrix} &= \bar{\boldsymbol{q}}_d \otimes \boldsymbol{q} \\
&= \begin{bmatrix} \eta_d\eta + \boldsymbol{\epsilon}_d^T\boldsymbol{\epsilon} \\ \eta_d\boldsymbol{\epsilon} - \eta\boldsymbol{\epsilon}_d - \boldsymbol{S}(\boldsymbol{\epsilon_d})\boldsymbol{\epsilon} \end{bmatrix} \\
&= \begin{bmatrix} \eta_d\eta + \epsilon_{d1}\epsilon_1 + \epsilon_{d2}\epsilon_2 + \epsilon_{d3}\epsilon_3 \\ \eta_d\epsilon_1 - \eta\epsilon_{d1} + \epsilon_{d3}\epsilon_2 - \epsilon_{d2}\epsilon_3 \\ \eta_d\epsilon_2 - \eta\epsilon_{d2} - \epsilon_{d3}\epsilon_1 + \epsilon_{d1}\epsilon_3 \\ \eta_d\epsilon_3 - \eta\epsilon_{d3} + \epsilon_{d2}\epsilon_1 - \epsilon_{d1}\epsilon_2 \end{bmatrix}
\end{aligned} \tag{17}$$

After convergence, when $\boldsymbol{q} = \boldsymbol{q}_d$, we get

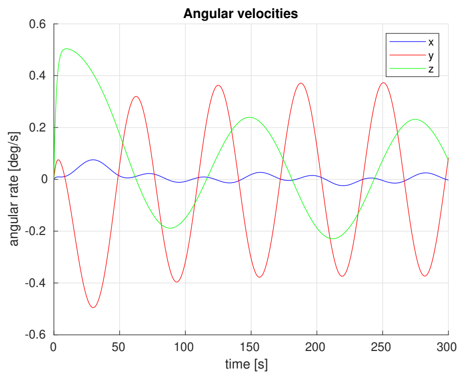
$$\begin{aligned}
\tilde{\boldsymbol{q}} &= \begin{bmatrix} \eta_d\eta_d + \epsilon_{d1}\epsilon_{d1} + \epsilon_{d2}\epsilon_{d2} + \epsilon_{d3}\epsilon_{d3} \\ \eta_d\epsilon_{d1} - \eta_d\epsilon_{d1} + \epsilon_{d3}\epsilon_{d2} - \epsilon_{d2}\epsilon_{d3} \\ \eta_d\epsilon_{d2} - \eta_d\epsilon_{d2} - \epsilon_{d3}\epsilon_{d1} + \epsilon_{d1}\epsilon_{d3} \\ \eta_d\epsilon_{d3} - \eta_d\epsilon_{d3} + \epsilon_{d2}\epsilon_{d1} - \epsilon_{d1}\epsilon_{d2} \end{bmatrix} \\
&= \begin{bmatrix} \eta_d^2 + \epsilon_{d1}^2 + \epsilon_{d2}^2 + \epsilon_{d3}^2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned} \tag{18}$$

where we in the last step used the fact that $|\boldsymbol{q}_d| = 1$. Hence, we have reached the equilibrium $\boldsymbol{q}_0 = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T$ after convergence. We can also reach equilibrium when $\boldsymbol{q} = -\boldsymbol{q}_d$ where we also have a point of attraction. This point is not desirable due to it being unstable.

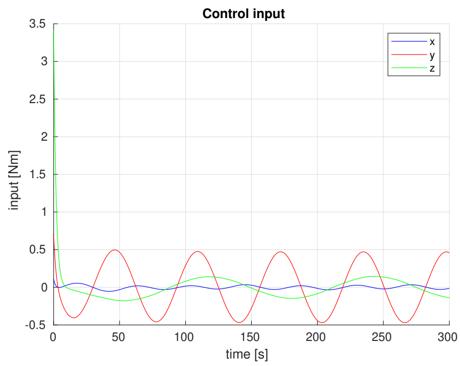
1.5 Problem 1.5



(a) Euler angles and reference signals



(b) Angular velocities and reference signals



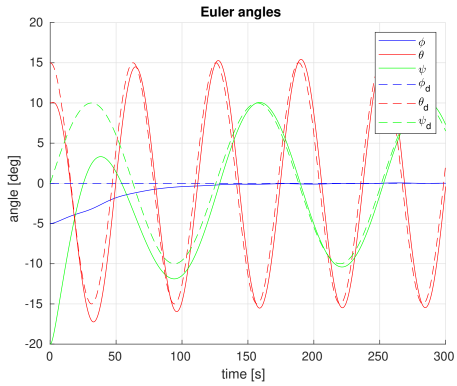
(c) Control inputs

Figure 2: Relevant plots for Problem 1.5

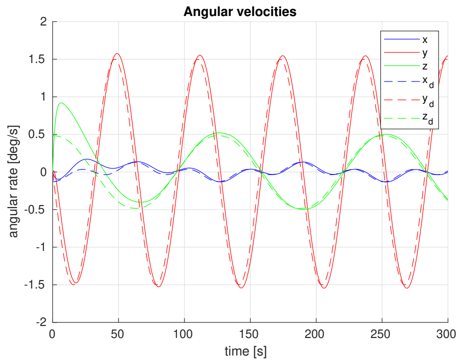
We can again see that the system is too slow to react and that the difference between K_d and k_p is too large. If we inspect further we see that we no longer have complex eigenvalues with these values for K_d and k_p. This is good in the sense that we will no longer have oscillating behavior to the desired values, but we see that we still don't get the response that we want. The reason we don't get the desired response is that we have oscillating desired signal and the controller is constructed to want zero angular velocity. We have updated the desired signal, but not the controller. Because of this we get a controller that is something "in-between", meaning that it struggles to get to zero velocity while the reference still is oscillating. To change this we have to change the controller to want oscillating angular velocity.

We could alternatively increase k_p to get a faster response that would be able to match the desired signal, however this may not be feasible for this physical system. This would essentially mean that we are not prioritizing the angular velocity as much as we did before.

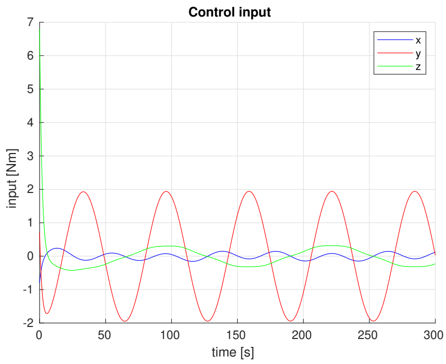
1.6 Problem 1.6



(a) Euler angles and reference signals



(b) Angular velocities and reference signals



(c) Control inputs

Figure 3: Relevant plots for Problem 1.6

We notice that the controller performs better over time. $\tilde{\omega}$ gets closer to zero as the difference between desired and current angular velocity gets smaller as time increases. We are now taking into account that the desired signal is dynamic and have changed the angular velocity accordingly. We achieve this without changing k_p and K_d, meaning there is no complex eigenvalues.

The control law can be further improved by adding a desired angular acceleration to the control law, similar to how it is done with the angular velocity.

1.7 Problem 1.7

We are given the stability function

$$V = \frac{1}{2}\tilde{\boldsymbol{\omega}}^T \boldsymbol{I}_g \tilde{\boldsymbol{\omega}} + 2k_p(1-\tilde{\eta}) \tag{19}$$

The first term of the stability function is positive due to the inertia matrix being positive and the angular velocities being squared. Equation (2.64) in [1] defines η as

$$\eta := \cos(\frac{\beta}{2}) \tag{20}$$

Hence, $\tilde{\eta}$ only has values in the interval $[-1, 1]$ making also the second term of the stability function positive. Since both terms of the stability function is positive, the stability function itself is positive.

The stability function is defined for all $\tilde{\boldsymbol{\omega}}$. Further,

$$||\tilde{\boldsymbol{\omega}}|| \to \infty \Rightarrow V(\tilde{\boldsymbol{\omega}}) \to \infty \tag{21}$$

The function is therefore radially unbounded.

Using the product rule and that $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega} - \boldsymbol{\omega}_d = \boldsymbol{\omega}$, we achieve

$$\begin{aligned}\dot{V} &= \frac{1}{2}\dot{\boldsymbol{\omega}}^T \boldsymbol{I}_g \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{I}_g \dot{\boldsymbol{\omega}} - 2k_p \dot{\eta} \\ &= \boldsymbol{\omega}^T \boldsymbol{I}_g \dot{\boldsymbol{\omega}} - 2k_p \dot{\eta}\end{aligned} \tag{22}$$

Using (2) and Equation (7) in the assignment to insert for $\dot{\boldsymbol{\omega}}$ and $\dot{\eta}$ yields

$$\begin{aligned}\dot{V} &= \boldsymbol{\omega}^T \boldsymbol{I}_g \boldsymbol{I}_g^{-1}(\tau + \boldsymbol{S}(\boldsymbol{I}_g \boldsymbol{\omega})\boldsymbol{\omega}) + 2k_p \tilde{\boldsymbol{\epsilon}}^T \boldsymbol{\omega} \\ &= \boldsymbol{\omega}^T \tau + 2k_p \tilde{\boldsymbol{\epsilon}}^T \boldsymbol{\omega}\end{aligned} \tag{23}$$

Inserting the control law given in Equation (6) in the assignment results in

$$\begin{aligned}\dot{V} &= \boldsymbol{\omega}^T(-\boldsymbol{K}_d \boldsymbol{\omega} - k_p \tilde{\boldsymbol{\epsilon}}) + k_p \tilde{\boldsymbol{\epsilon}}^T \boldsymbol{\omega} \\ &= -\boldsymbol{\omega}^T \boldsymbol{K}_d \boldsymbol{\omega}\end{aligned} \tag{24}$$

Since $\boldsymbol{K}_d \succ 0$ and that the angular velocities are squared (therefore positive), we have that $\dot{V} < 0$. Hence, the equilibrium of the closed loop system is asymptotically stable. The system is only locally asymptotically stable due to the two points of attraction in problem 1.4.

References

[1] T.I. Fossen. *Handbook of Marine Craft Hydrodynamics and Motion Control*. John Wiley & Sons, 2011.