Cosmology Assignment 1

Daniel Herman: daniel.herman@astro.uio.no

September 22, 2017

Exercise 1.

(1) Show that in a universe that is undergoing Hubble expansion, that $\bar{\rho}(t) = \bar{\rho}(t = t_0)a^{-3}$, where t_0 denotes the age of the Universe today:

We use the definition $\vec{v_r} = H\vec{r_r}$, and the continuity equation $\frac{d\rho_0}{dt} + \rho_0 \nabla \cdot \vec{v_0}$.

So, $\frac{d\rho_0}{dt} + 3\rho_0\frac{\dot{a}}{a} = 0 \Rightarrow \frac{d\rho_0}{dt} = -3\rho_0\frac{\dot{a}}{a}$. Now we integrate both sides.

$$\int \frac{\dot{\rho_0}}{\rho_0} = -3\int \frac{\dot{a}}{a} \Rightarrow \ln\frac{\bar{\rho}}{\rho} = -3\ln a \Rightarrow \frac{\bar{\rho}}{\rho} = a^{-3} \Rightarrow \bar{\rho} = \rho a^{-3}$$

.

(2) Derive the perturbed Poisson's and Euler's Equations.

Poisson's Equation:

$$\nabla^2(\phi_0 + \delta\phi) = 4\pi G(\rho_0 + \delta\rho) \Rightarrow \nabla^2\phi_0 + \nabla^2\delta\phi = 4\pi G\rho_0 + 4\pi G\delta\rho$$

.

Since $\nabla^2\phi_0 = 4\pi G\rho_0$,

$$\nabla^2\delta\phi = 4\pi G\delta\rho \tag{1}$$

Euler's Equation: $\frac{dv_0}{dt} = -\frac{1}{\rho_0}\nabla p_0 - \nabla\phi_0$. We need to break apart this problem to successfully derive the solution.

We utilize the definition of the time derivative for comoving space: $\frac{d}{dt} = (\frac{\partial}{\partial t} + \vec{v}\cdot\nabla)$.

We defined the perturbed quantities as follows:

Pressure: $p = p_0 + \delta p$.

Velocity: $\vec{v} = \vec{v_0} + \delta\vec{v}$

Potential: $\nabla\phi = \nabla(\phi_0 + \delta\phi) = \nabla\phi_0 + \nabla\delta\phi$

Density: $\frac{1}{\rho} = \frac{1}{\rho_0 + \delta\rho}$

And we rewrite the unperturbed Euler's equation below:

$$\frac{d\vec{v_0}}{dt} = (\frac{\partial}{\partial t} + \vec{v_0}\cdot\nabla)\vec{v_0} = \frac{\partial\vec{v_0}}{\partial t} + (\vec{v_0}\cdot\nabla)\vec{v_0} = -\frac{1}{\rho_0}\nabla p_0 - \nabla\phi_0 \tag{2}$$

First we will derive the L.H.S.:

$$\frac{d\vec{v}}{dt} = \frac{d(\vec{v_0} + \delta\vec{v})}{dt} = \frac{\partial}{\partial t}(\vec{v_0} + \delta\vec{v}) + (\vec{v_0} + \delta\vec{v})\cdot\nabla(\vec{v_0} + \delta\vec{v})$$

$$= \frac{\partial\vec{v_0}}{\partial t} + \frac{\partial\delta\vec{v}}{\partial t} + (\vec{v_0}\cdot\nabla)\vec{v_0} + (\vec{v_0}\cdot\nabla)\delta\vec{v} + (\delta\vec{v}\cdot\nabla)\vec{v_0} + (\delta\vec{v}\cdot\nabla)\delta\vec{v}. \tag{3}$$

The $(\delta\vec{v}\cdot\nabla)\delta\vec{v}$ term is equal to zero since we are taking a linear perturbation, so equation 3 becomes:

$$\frac{\partial\vec{v_0}}{\partial t} + \frac{\partial\delta\vec{v}}{\partial t} + (\vec{v_0}\cdot\nabla)\vec{v_0} + (\vec{v_0}\cdot\nabla)\delta\vec{v} + (\delta\vec{v}\cdot\nabla)\vec{v_0} \tag{4}$$

Now the R.H.S.:

$$-\frac{\nabla(p_0 + \delta p)}{\rho_o + \delta\rho} = -\frac{(\rho_0 - \delta\rho)\nabla(p_0 + \delta p)}{\rho_o^2 - \delta\rho^2} - \nabla\phi_0 - \nabla\delta\phi \tag{5}$$

Again, since this is a linear perturbation, $\delta\rho^2 = 0$.

$$= -\frac{\rho_0\nabla p_0 - \delta\rho\nabla p_0 - \delta\rho\nabla\delta p + \rho_0\nabla\delta p}{\rho_0^2} - \nabla\phi_0 - \nabla\delta\phi = -\frac{\nabla p_0 + \nabla\delta p}{\rho_0} - \nabla\phi_0 - \nabla\delta\phi$$

Since we exclude these terms: $(\delta\rho\nabla p_0 = \delta\rho\nabla\delta p = 0)$
The perturbed Euler equation becomes:

$$\frac{\partial\vec{v_0}}{\partial t} + \frac{\partial\delta\vec{v}}{\partial t} + (\vec{v_0}\cdot\nabla)\vec{v_0} + (\vec{v_0}\cdot\nabla)\delta\vec{v} + (\delta\vec{v}\cdot\nabla)\vec{v_0} = -\frac{\nabla p_0}{\rho_0} - \frac{\nabla\delta p}{\rho_0} - \nabla\phi_0 - \nabla\delta\phi$$
$$\Rightarrow \frac{d\vec{v_0}}{dt} + \frac{\partial\delta\vec{v}}{\partial t} + (\vec{v_0}\cdot\nabla)\delta\vec{v} + (\delta\vec{v}\cdot\nabla)\vec{v_0} = -\frac{\nabla p_0}{\rho_0} - \frac{\nabla\delta p}{\rho_0} - \nabla\phi_0 - \nabla\delta\phi$$

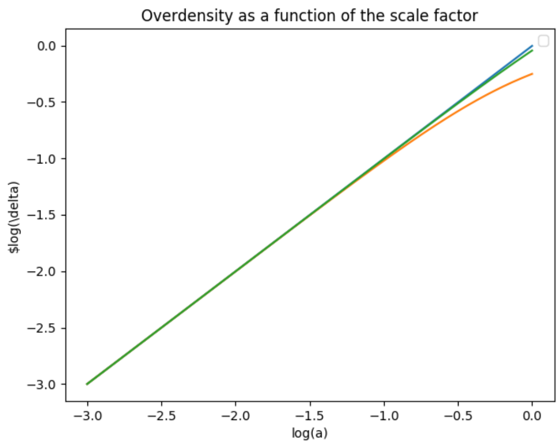
And removing the unperturbed portion (2) it simplifies to:

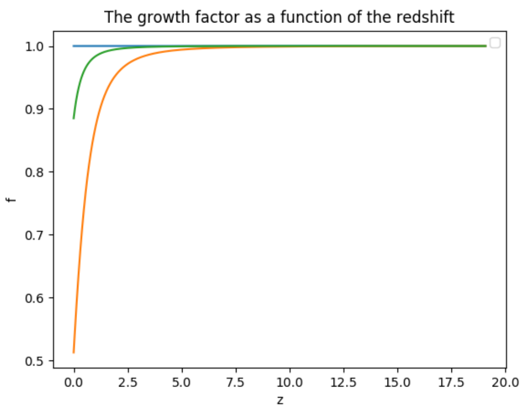
$$\frac{\partial\delta\vec{v}}{\partial t} + (\vec{v_0}\cdot\nabla)\delta\vec{v} + (\delta\vec{v}\cdot\nabla)\vec{v_0} = -\frac{\nabla\delta p}{\rho_0} - \nabla\delta\phi \tag{6}$$

Utilizing the definition of the time derivative again, $\frac{\partial\delta\vec{v}}{\partial t} + (\vec{v_0}\cdot\nabla)\delta\vec{v} = \frac{d\delta\vec{v}}{dt}$, so we finally reach our result:

$$\frac{d\delta\vec{v}}{dt} + (\delta\vec{v}\cdot\nabla)\vec{v_0} = -\frac{\nabla\delta p}{\rho_0} - \nabla\delta\phi \tag{7}$$

Exercise 2. (Code included at https://github.com/hermda02/myfirstcosmology)
(1) $H^2 = \left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_\Lambda\right]$ and $\frac{\Omega_r}{a^4} \rightarrow 0$.
Let $(\Omega_m, \Omega_\Lambda) = (1.0{,}0.0)$; $\frac{\dot{a}}{a} = H_0 \cdot a^{-3/2} = H$
Let $(\Omega_m, \Omega_\Lambda) = (0.3{,}0.7)$; $\frac{\dot{a}}{a} = H_0\left[\frac{0.3}{a^3} + 0.7\right]^{1/2} = H$
Let $(\Omega_m, \Omega_\Lambda) = (0.8{,}0.2)$; $\frac{\dot{a}}{a} = H_0\left[\frac{0.8}{a^3} + 0.2\right]^{1/2} = H$





Exercise 3.
(1) Derive and plot the radiation and gas temperature as the universe expands, assuming it expands adiabatically:
Radiation Temperature (T_r)**:**

We have the equation of state for radiation pressure $p = \frac{u}{3} = \frac{\sigma T^4}{3}$ where σ is the Stephan-Boltzmann constant.
From the second law of thermodynamics we know that:

$$
\begin{aligned}
dS =& \frac{dU}{T} + p\frac{dV}{T} = 0 \\
0 =& \frac{d(uV)}{T} + \frac{u}{3}\frac{dV}{t} \\
=& du\frac{V}{T} + dV\frac{u}{T} + \frac{udV}{3T} \\
=& \frac{4u}{3T}dV + \frac{V}{T}du
\end{aligned}
$$

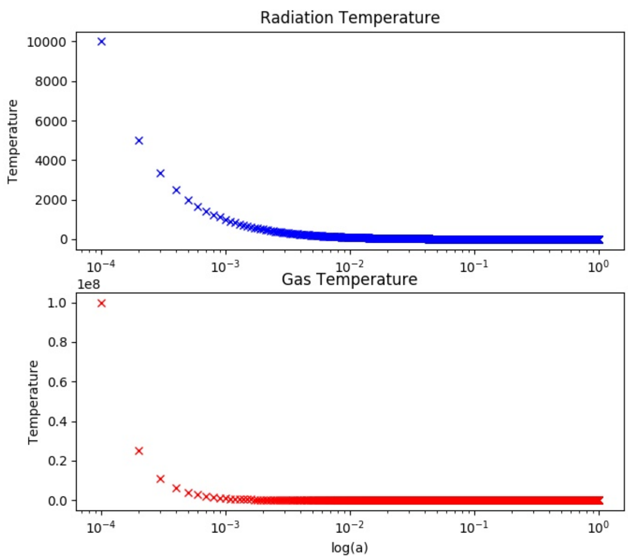
Rearranging and using $\frac{du}{dT} = 4\sigma T^3$

$$
\begin{aligned}
\frac{4u}{3T}dV =& -\frac{V}{T}\frac{du}{dT}dT \\
\frac{dV}{V} =& -\frac{4T^3\sigma \cdot 3T}{4\sigma T^4 \cdot T}dT \\
=& -\frac{3dT}{T}
\end{aligned}
$$

And using the fact that the volue of a cube of side length a is $V = a^3$, we see that $\frac{dV}{V} = -\frac{da}{a}$. Therefore $T_r \propto a^{-1} = (1+z)$.
Gas Temperature (T_m)**:**

We know for non-relativistic particles that $E = \frac{p^2}{2m}$. Utilizing the deBroglie wavelength of a particle, we can approximate the distance between particles at recombination, such that:

$$T \propto E \propto p^2 \propto \lambda_{dB}^{-2} \propto a^{-2} \propto (1+z)^2 \tag{8}$$



Exercise 4.
Assuming an Einstein deSitter Universe $[\Omega_m = 1.0, \Omega_\Lambda = 0.0]$ show that the parameterization

$$\begin{aligned} R =& A(1-\cos\theta) \\ t =& B(\theta - \sin\theta) \\ A^3 =& GMB^2 \end{aligned}$$

satisfies $\ddot{R} = -\frac{GM}{R^2}$. Let's expand the L.H.S. and R.H.S. and independently to show that they are equal.

$$\ddot{R} = \frac{d}{d\theta}(\frac{dR}{dt})(\frac{d\theta}{dt}) \tag{9}$$

Solving the parts of the R.H.S. (12) independently:

$$\begin{aligned} \frac{dR}{dt} =& \frac{dR}{d\theta} \cdot \frac{d\theta}{dt} \\ \frac{dt}{d\theta} =& \frac{d}{d\theta}(B(\theta - \sin\theta)) = B(1-\cos\theta), \frac{d\theta}{dt} = \frac{1}{\frac{dt}{d\theta}} = \frac{1}{B(1-\cos\theta)} \\ \frac{dR}{d\theta} =& \frac{d}{d\theta}(A(1-\cos\theta)) = A\sin\theta \\ \frac{dR}{dt} =& \frac{A\sin\theta}{B(1-\cos\theta)} \end{aligned}$$

Combining the terms, we see $\ddot{R}$ is equal to:

$$\begin{aligned} \ddot{R} =& \frac{A}{B}\Big[\frac{\cos\theta(1-\cos\theta) - \sin^2\theta}{(1-\cos\theta)^2}\Big] \cdot \frac{1}{B(1-\cos\theta)} && (10) \\ =& \frac{A}{B^2}\Big[\frac{\cos - \cos^2\theta - \sin^2\theta}{(1-\cos\theta)^3}\Big] && (11) \\ =& \frac{A}{B^2}\Big[\frac{\cos\theta - 1}{(1-\cos\theta)^3}\Big] && (12) \\ =& -\frac{A}{B^2(1-\cos\theta)^2} && (13) \end{aligned}$$

Now to handle the L.H.S.:

$$\begin{aligned} GM =& \frac{A^3}{B^2} \\ \frac{GM}{R^2} =& \Big(\frac{A^3}{B^2}\Big)\Big(\frac{1}{A(1-\cos\theta)}\Big)^2 \\ =& \frac{A}{B^2(1-\cos\theta)^2} \end{aligned}$$

So the L.H.S. is defined by:

$$-\frac{GM}{R^2} = -\frac{A}{B^2(1-\cos\theta)^2} \tag{14}$$

And we see that $\ddot{R} = -\frac{A}{B^2(1-\cos\theta)^2} = -\frac{GM}{R^2}$

Exercise 5.
Derive the infall velocity v_{infall} as the gas virializes. The velocity is equal to the first time derivative of the radius, so:

$$\dot{R} = \frac{dR}{dt} = \frac{dR}{d\theta} \cdot \frac{d\theta}{dt}$$

We can use values derived in exercise 4 to show that:

$$\dot{R} = \frac{A}{B} \frac{\sin\theta}{(1 - \cos\theta)} \tag{15}$$

Virialization occurs at $\theta = \frac{3\pi}{2}$, so $\dot{R}_{vir} = -\frac{A}{B}$ which shows that the velocity is directed towards the center of the sphere. And $R_{vir} = A(1 - 0) = A$.
The infall velocity (towards the center) is:

$$v_{infall} = \frac{A}{B} = \sqrt{\frac{A^2}{B^2}} = \sqrt{\frac{A^3}{B^2 \cdot A}} = \sqrt{\frac{GM}{R_{vir}}} \tag{16}$$

Exercise 6.
Show that the gravitational binding energy of a uniform sphere of radius R is $U = -\frac{3GM^2}{5R}$.
Assuming uniform density ρ, we see $m_{shell} = 4\pi r^2 \rho dr$ and the mass within the shell $m_{int} = \frac{4}{3}\pi r^3 \rho$.
Potential:

$$\begin{aligned} dU &= -\frac{G m_{int} m_{shell}}{r} = -\frac{G(4\pi r^2 \rho dr)(\frac{4}{3}\pi r^3 \rho)}{r} \\ &= -G\frac{16}{3}\pi^2 r^4 \rho^2 dr \end{aligned}$$

Integrating over the whole sphere, radially:

$$U = -G\int_0^R \frac{16}{3}\pi^2 r^4 \rho^2 dr = -G\frac{16}{3}\pi^2\rho^2 \int_0^R r^4 dr = -G\frac{16}{15}\pi^2\rho^2 R^5 \tag{17}$$

Reintroducing $\rho = \frac{M}{\frac{4}{3}\pi R^3}$, equation (17) becomes:

$$U = -G\frac{16}{15}\pi^2 \frac{M^2}{\frac{16}{9}\pi^2 R^6} R^5 = -\frac{9}{15}\frac{GM^2}{R} = -\frac{3}{5}\frac{GM^2}{R} \tag{18}$$