Cosmology Assignment 2

Daniel Herman: daniel.herman@astro.uio.no

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(Code included for exercises 1 and 2 at https://github.com/hermda02/Cosmology-Assignment-2)

Exercise 1. Compute and plot the Fourier conjugate of the top-hat smoothing function where $W(x) = 1$ when $x \leq |R|$ and $W(x) = 0$ elsewhere.

The computed Fourier transform is plotted in figure 1. The number of harmonics and FWHM of the transform are shown on the plots.

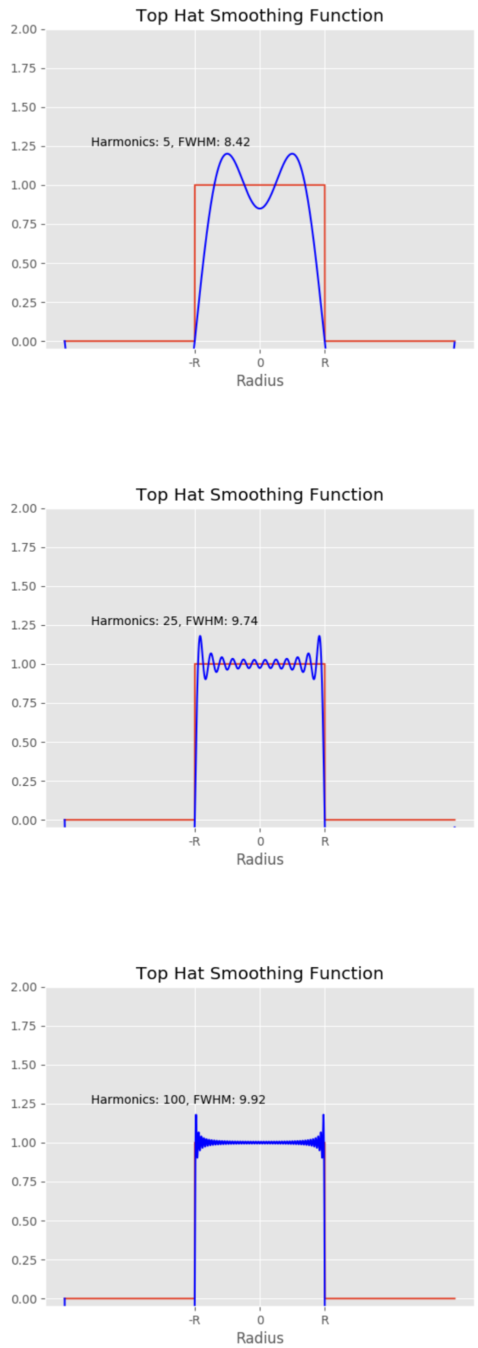

Figure 1: The Fourier transform of the top-hat for 5, 25, and 100 harmonics

Exercise 2.

To numerically solve the overdensity random walk, we utilize a recursive algorith where the probability distribution we draw from changes with the smoothing factor S_c, which we decrease by ϵ at the end of each step. We define $S_c = 2\pi/k$ and $\sigma^2(S_c) = \pi/S_c^4$ where $\sigma^2(S_c)$ is the variance of the function $P(k) = k$. We start with a radius S_c such that $\sigma^2(S_c) < 10^{-4}$. So, we start with $k = 0.47$, giving us an inital $S_c = 13.368$. We set $\epsilon = 0.03$. We stop taking random steps when $S_c = 1.0$, or after 4000 steps. We see the densities in figure. The probability distribution as given in equation 1 is over-plotted for comparison.

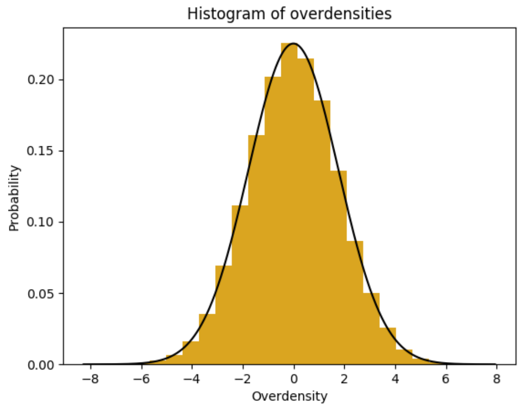

Figure 2: The over/under-densities for 10^5 random walks with $k = 0.47$ and $\epsilon = 0.03$.

$$P(\delta|M) = \frac{1}{\sqrt{2\pi}\sigma(M)} \exp\Big[-\frac{\delta^2}{2\sigma^2(M)}\Big] \tag{1}$$

Removing all random walks where δ becomes greater than δ_{crit} at any point along the walk, we see the following histogram of final δ values. The analytical expression for the distribution shown for δ provided that δ was never larger than δ_{crit} is given by equation 2 and is over-plotted for comparison.

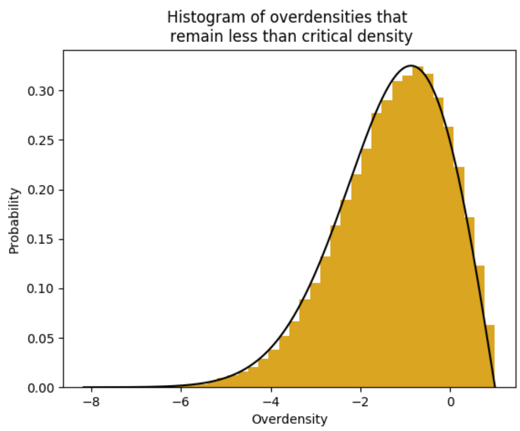

Figure 3: All random walks in which δ is less than δ_{crit} for every step.

$$P_{nc}(\delta|M) = P(\delta|M) - P([2\delta_{crit} - \delta]|M) = \frac{1}{\sqrt{2\pi}\sigma(M)}\Big(\exp\Big[-\frac{\delta^2}{2\sigma^2(M)}\Big] - \exp\Big[-\frac{[2\delta_{crit} - \delta]^2}{2\sigma^2(M)}\Big]\Big) \tag{2}$$

Exercise 3.

(1) We see in Exercise 2 that the probability that any perturbation never becomes greater than δ_{crit} is given by the difference between the two probability distribution functions (PDFs). For a mass at point $\mathbf{x}$ to be embedded within a collapsed object of mass $> M$, the mass at $\mathbf{x}$ must be in an overdense region, where $\delta > \delta_{crit}$. Therefore, the probability $P(\delta > \delta_{crit}|M) = P(> M)$ is equal to all the probability for all possible values of δ (1) subtracted by the total probability that δ is never greater than δ_{crit} (given by the integral over δ of equation 2). So we see:

$$P(> M) = 1 - \int_{-\infty}^{\delta_{crit}} d\delta P_{nc}(\delta|M) \tag{3}$$

The probability that a mass at $\mathbf{x}$ is within a collapsed object of mass M is the total probability of all δ's given mass M (i.e. 1) minus the probability that all δ's given a mass M never reach a value greater than δ_{crit}. If a δ becomes greater than δ_{crit}, then that δ will collapse, or be a part of a collapsed object. As an expression, this becomes

$$P(> M) = 1 - \int_{-\infty}^{\delta_{crit}} d\delta P_{nc}(\delta|M) \tag{4}$$

(2) Show that: $P(> M) = 2P(\delta > \delta_{crit}|M) = 1 - \mathrm{erf}(\nu/\sqrt{2})$ where $\nu = \delta/\sigma(M)$.

$$\begin{aligned}
P(> M) =& 1 - \int_{-\infty}^{\delta_{crit}} d\delta P_{nc}(\delta|M) = P(\delta > \delta_{crit}|M) \\
=& 1 - P_{nc}(\delta < \delta_{crit}|M) \\
=& 1 - P(\delta < \delta_{crit}|M) - P([2\delta_{crit} - \delta] < \delta_{crit}|M) \\
=& 1 - (P(\delta < \delta_{crit}|M) - P([\delta_{crit} - \delta] < 0|M)) \\
=& 1 - (P(\delta < \delta_{crit}|M) - P(\delta > \delta_{crit}|M)) \\
=& 1 - (1 - P(\delta > \delta_{crit}|M)) + P(\delta > \delta_{crit}|M) \\
=& 2P(\delta > \delta_{crit}|M) \\
=& 2(\frac{1}{\sqrt{2\pi}\sigma(M)})\Big(\int_0^\infty d\delta e^{-\frac{\delta^2}{2\sigma^2(M)}} - \int_{\delta_{crit}}^\infty d\delta e^{-\frac{\delta^2}{2\sigma^2(M)}}\Big)
\end{aligned}$$

The last line of the above expression is brings us to

$$1 - erf(\nu/\sqrt{2}) \tag{5}$$