Summary of equations for CMB power spectrum calculations

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1 Definitions and background cosmology

- Four "time" variables: t = physical time, $\eta = \int_0^t ca^{-1}(t)dt$ = conformal time, a = scale factor, $x = \ln a$
- Metric: $ds^2 = -c^2dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$
- Friedmann's equations:

$$H \equiv \frac{1}{a}\frac{da}{dt} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda} \tag{1}$$

$$\mathcal{H} \equiv \frac{c}{a}\frac{da}{d\eta} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2} \tag{2}$$

- Conformal time as a function of scale factor (to be computed numerically),

$$\eta(a) = \int_0^a \frac{c}{a'\mathcal{H}(a')}da', \qquad \frac{d\eta}{da} = \frac{c}{a\mathcal{H}} \tag{3}$$

2 The perturbation equations

2.1 Overview of involved quantities

All quantities are functions of wavenumber k and time x.

- δ – dark matter density perturbation
- δ_b – baryonic matter density perturbation
- v – dark matter velocity
- v_b – baryonic matter velocity
- Θ_l – l'th photon moment
- Θ_l^P – l'th polarized photon moment
- $\mathcal{N}_l$ – l'th neutrino moment
- Φ – gravitational curvature potential
- Ψ – Newtonian gravitational potential

2.2 Boltzmann-Einstein equations without neutrinos or polarization

All dynamic quantities are functions of x and k. Derivatives are with respect to x:

$$\Theta_0' = -\frac{ck}{\mathcal{H}}\Theta_1 - \Phi', \tag{4}$$

$$\Theta_1' = \frac{ck}{3\mathcal{H}}\Theta_0 - \frac{2ck}{3\mathcal{H}}\Theta_2 + \frac{ck}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{5}$$

$$\Theta_l' = \frac{lck}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)ck}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau'\left[\Theta_l - \frac{1}{10}\Theta_l\delta_{l,2}\right], \qquad 2 \le l < l_{\max} \tag{6}$$

$$\Theta_l' = \frac{ck}{\mathcal{H}}\Theta_{l-1} - \frac{(l+1)c}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \qquad l = l_{\max} \tag{7}$$

$$\delta' = \frac{ck}{\mathcal{H}}v - 3\Phi' \tag{8}$$

$$v' = -v - \frac{ck}{\mathcal{H}}\Psi \tag{9}$$

$$\delta_b' = \frac{ck}{\mathcal{H}}v_b - 3\Phi' \tag{10}$$

$$v_b' = -v_b - \frac{ck}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{11}$$

$$\Phi' = \Psi - \frac{c^2k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{12}$$

$$\Psi = -\Phi - \frac{12H_0^2}{c^2k^2a^2}\Omega_r\Theta_2 \tag{13}$$

$$R = \frac{4\Omega_r}{3\Omega_b a} \tag{14}$$

In the tight coupling regime, the equation for v_b' must be approximated, and reads

$$q = \frac{-[(1-2R)\tau' + (1+R)\tau''](3\Theta_1 + v_b) - \frac{ck}{\mathcal{H}}\Psi + (1 - \frac{\mathcal{H}'}{\mathcal{H}})\frac{ck}{\mathcal{H}}(-\Theta_0 + 2\Theta_2) - \frac{ck}{\mathcal{H}}\Theta_0'}{(1+R)\tau' + \frac{\mathcal{H}'}{\mathcal{H}} - 1} \tag{15}$$

$$v_b' = \frac{1}{1+R}\left[-v_b - \frac{ck}{\mathcal{H}}\Psi + R(q + \frac{ck}{\mathcal{H}}(-\Theta_0 + 2\Theta_2) - \frac{ck}{\mathcal{H}}\Psi)\right] \tag{16}$$

The photon moment equations in the same regime read

$$\Theta_1' = \frac{1}{3}(q - v_b') \tag{17}$$

$$\Theta_2 = -\frac{4ck}{9\mathcal{H}\tau'}\Theta_1 \tag{18}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{ck}{\mathcal{H}\tau'}\Theta_{l-1} \tag{19}$$

2.3 Boltzmann-Einstein equations with neutrinos and polarization

$$\Theta_0' = -\frac{ck}{\mathcal{H}}\Theta_1 - \Phi', \tag{20}$$

$$\Theta_1' = \frac{ck}{3\mathcal{H}}\Theta_0 - \frac{2ck}{3\mathcal{H}}\Theta_2 + \frac{ck}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{21}$$

$$\Theta_l' = \frac{lck}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)ck}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau'\left[\Theta_l - \frac{1}{10}\Pi\delta_{l,2}\right], \qquad 2 \le l < l_{\max} \tag{22}$$

$$\Theta_l' = \frac{ck}{\mathcal{H}}\Theta_{l-1} - \frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \qquad l = l_{\max} \tag{23}$$

$$\Theta_{P0}' = -\frac{ck}{\mathcal{H}}\Theta_{P1} + \tau'\left[\Theta_{P0} - \frac{1}{2}\Pi\right] \tag{24}$$

$$\Theta_{Pl}' = \frac{lck}{(2l+1)\mathcal{H}}\Theta_{l-1}^P - \frac{(l+1)ck}{(2l+1)\mathcal{H}}\Theta_{l+1}^P + \tau'\left[\Theta_l^P - \frac{1}{10}\Pi\delta_{l,2}\right], \qquad 1 \le l < l_{\max} \tag{25}$$

$$\Theta_{P,l}' = \frac{ck}{\mathcal{H}}\Theta_{l-1}^P - \frac{l+1}{\mathcal{H}\eta(x)}\Theta_l^P + \tau'\Theta_l^P, \qquad l = l_{\max} \tag{26}$$

$$\mathcal{N}_0' = -\frac{ck}{\mathcal{H}}\mathcal{N}_1 - \Phi', \tag{27}$$

$$\mathcal{N}_1' = \frac{ck}{3\mathcal{H}}\mathcal{N}_0 - \frac{2ck}{3\mathcal{H}}\mathcal{N}_2 + \frac{ck}{3\mathcal{H}}\Psi \tag{28}$$

$$\mathcal{N}_l' = \frac{lck}{(2l+1)\mathcal{H}}\mathcal{N}_{l-1} - \frac{(l+1)ck}{(2l+1)\mathcal{H}}\mathcal{N}_{l+1}, \qquad 2 \le l < l_{\max,\nu} \tag{29}$$

$$\mathcal{N}_l' = \frac{ck}{\mathcal{H}}\mathcal{N}_{l-1} - \frac{l+1}{\mathcal{H}\eta(x)}\mathcal{N}_l, \qquad l = l_{\max,\nu} \tag{30}$$

$$\delta' = \frac{ck}{\mathcal{H}}v - 3\Phi' \tag{31}$$

$$v' = -v - \frac{ck}{\mathcal{H}}\Psi \tag{32}$$

$$\delta_b' = \frac{ck}{\mathcal{H}}v_b - 3\Phi' \tag{33}$$

$$v_b' = -v_b - \frac{ck}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{34}$$

$$\Phi' = \Psi - \frac{c^2k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0 + 4\Omega_\nu a^{-2}\mathcal{N}_0\right] \tag{35}$$

$$\Psi = -\Phi - \frac{12H_0^2}{c^2k^2a^2}\left[\Omega_r\Theta_2 + \Omega_\nu\mathcal{N}_2\right] \tag{36}$$

$$R = \frac{4\Omega_r}{3\Omega_b a} \tag{37}$$

$$\Pi = \Theta_2 + \Theta_0^P + \Theta_2^P \tag{38}$$

The tight coupling equations read

$$q = \frac{-[(1-2R)\tau' + (1+R)\tau''](3\Theta_1 + v_b) - \frac{ck}{\mathcal{H}}\Psi + (1-\frac{\mathcal{H}'}{\mathcal{H}})\frac{ck}{\mathcal{H}}(-\Theta_0 + 2\Theta_2) - \frac{ck}{\mathcal{H}}\Theta_0'}{(1+R)\tau' + \frac{\mathcal{H}'}{\mathcal{H}} - 1} \tag{39}$$

$$v_b' = \frac{1}{1+R}\left[-v_b - \frac{ck}{\mathcal{H}}\Psi + R(q + \frac{ck}{\mathcal{H}}(-\Theta_0 + 2\Theta_2) - \frac{ck}{\mathcal{H}}\Psi)\right] \tag{40}$$

$$\Theta_2 = -\frac{8ck}{15\mathcal{H}\tau'}\Theta_1 \tag{41}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{ck}{\mathcal{H}\tau'}\Theta_{l-1} \tag{42}$$

$$\Theta_2^P = \frac{1}{4}\Theta_2 \tag{43}$$

$$\Theta_l^P = -\frac{l}{2l+1}\frac{ck}{\mathcal{H}\tau'}\Theta_{l-1}^P \tag{44}$$

2.4 Initial conditions

The integration of the above equations starts at $x_i = \ln a_i$, with $a_i = 10^{-8}$. Initial conditions are the following,

$$\Phi = 1 \tag{45}$$

$$\delta = \delta_b = \frac{3}{2}\Phi \tag{46}$$

$$v = v_b = \frac{ck}{2\mathcal{H}}\Phi \tag{47}$$

$$\Theta_0 = \frac{1}{2}\Phi \tag{48}$$

$$\Theta_1 = -\frac{ck}{6\mathcal{H}}\Phi \tag{49}$$

$$\Theta_2 = \begin{cases} -\frac{8ck}{15\mathcal{H}\tau'}\Theta_1, & \text{(with polarization)} \\ -\frac{4ck}{9\mathcal{H}\tau'}\Theta_1, & \text{(without polarization)} \end{cases} \tag{50}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{ck}{\mathcal{H}\tau'}\Theta_{l-1} \tag{51}$$

For polarization and neutrinos, the corresponding initial conditions are

$$\Theta_0^P = \frac{5}{4}\Theta_2 \tag{52}$$

$$\Theta_1^P = -\frac{ck}{4\mathcal{H}\tau'}\Theta_2 \tag{53}$$

$$\Theta_2^P = \frac{1}{4}\Theta_2 \tag{54}$$

$$\Theta_l^P = -\frac{l}{2l+1}\frac{ck}{\mathcal{H}\tau'}\Theta_{l-1}^P \tag{55}$$

$$\mathcal{N}_0 = \frac{1}{2}\Phi \tag{56}$$

$$\mathcal{N}_1 = -\frac{ck}{6\mathcal{H}}\Phi \tag{57}$$

$$\mathcal{N}_2 = -\frac{c^2k^2a^2\Phi}{12H_0^2\Omega_\nu}\frac{1}{\frac{5}{2f_\nu} + 1} \tag{58}$$

$$\mathcal{N}_l = \frac{ck}{(2l+1)\mathcal{H}}\mathcal{N}_{l-1}, \qquad l \geq 3 \tag{59}$$

3 Recombination and the visibility function

(Note: Units of c, k_B and $\hbar$ are missing in this section – it is your job to get these right!)

- Define the optical depth,

$$\tau(\eta) = \int_{\eta}^{\eta_0} n_e \sigma_T a d\eta' \tag{60}$$

$$\tau' = -\frac{n_e \sigma_T a}{\mathcal{H}} \tag{61}$$

- The visibility function is

$$g(\eta) = -\dot{\tau} e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \tag{62}$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \tag{63}$$

$$\int_0^{\eta_0} g(\eta) d\eta = \int_{-\infty}^{0} \tilde{g}(x) dx = 1. \tag{64}$$

(Need the free electron fraction, $X_e \equiv \frac{n_e}{n_H} = \frac{n_e}{n_b}$, as a function of time.)

- At early times, when $X_e > 0.99$, use the Saha equation,

$$\frac{X_e^2}{1 - X_e} = \frac{1}{H} \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0 / T_b}, \tag{65}$$

where $n_b = \frac{\Omega_b \rho_c}{m_h a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\mathrm{K}/a$, and $\epsilon_0 = 13.605698\mathrm{eV}$.

- During and after recombination, use the Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{H} \left[\beta(T_b)(1 - X_e) - n_H \alpha^{(2)}(T_b) X_e^2 \right], \tag{66}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s\to 1s} + \Lambda_\alpha}{\Lambda_{2s\to 1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{67}$$

$$\Lambda_{2s\to 1s} = 8.227 \mathrm{s}^{-1} \tag{68}$$

$$\Lambda_\alpha = H \frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{69}$$

$$n_{1s} = (1 - X_e) n_H \tag{70}$$

$$\beta^{(2)}(T_b) = \beta(T_b) e^{3\epsilon_0 / 4T_b} \tag{71}$$

$$\beta(T_b) = \alpha^{(2)}(T_b) \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0 / T_b} \tag{72}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}} \frac{\alpha^2}{m_e^2} \sqrt{\frac{\epsilon_0}{T_b}} \phi_2(T_b) \tag{73}$$

$$\phi_2(T_b) = 0.448 \ln(\epsilon_0 / T_b) \tag{74}$$

4 The CMB power spectrum

1. Define
$$\Pi = \Theta_2 + \Theta_2^P + \Theta_0^P \tag{75}$$
and set $\Theta_l^P = 0$ if you are not interested in polarization.

2. Compute the *source function*,
$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Pi\right] + e^{-\tau}\left[\Psi' + \Phi'\right] - \frac{1}{ck}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4c^2k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Pi)\right] \tag{76}$$
$$\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Pi)\right] = \frac{d(\mathcal{H}\mathcal{H}')}{dx}\tilde{g}\Pi + 3\mathcal{H}\mathcal{H}'(\tilde{g}\Pi + \tilde{g}\Pi') + \mathcal{H}^2(\tilde{g}''\Pi + 2\tilde{g}'\Pi' + \tilde{g}\Pi''), \tag{77}$$
$$\Pi'' = \frac{2ck}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_1 + \Theta_1'\right] + \frac{3}{10}\left[\tau''\Pi + \tau'\Pi'\right] - \tag{78}$$
$$\frac{3ck}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}(\Theta_3 + \Theta_1^p + \Theta_3^P) + (\Theta_3' + \Theta_{P1}' + \Theta_{P3}')\right] \tag{79}$$

3. Compute the multipole expansion of the transfer function observed today
$$\Theta_l(k, x=0) = \int_{-\infty}^{0} \tilde{S}(k,x) j_l[k(\eta_0 - \eta(x))]dx \tag{80}$$

4. Compute the CMB power spectrum (for a power law inflationary spectrum with spectral index n),
$$C_l = \int_0^\infty \left(\frac{k}{H_0}\right)^{n-1} \Theta_l^2(k)\frac{dk}{k} \tag{81}$$

5. Normalize spectrum such that maximum value (in units of $l(l+1)/2\pi$) equals the WMAP best-fit spectrum of $5570\mu\mathrm{K}^2$.