Milestone 2: The recombination history of the universe

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1 Problem statement and deliverables

The topic of the second milestone of the AST5220 project is the recombination history of the universe: How did the baryons go from being ionized (consisting only of free electron and protons) to being neutral (consisting of neutral hydrogen atoms)? The final goal of this part is to compute the optical depth as a function of x, the logarithm of the scale factor, a, and its derivatives, and the visibility function, g, and its derivatives (or, rather, it's scaled version, $\tilde{g} = g/\mathcal{H}$).

The deliverables are the following:

- A report defining the quantities to be computed, a short description of the algorithms used, and one plot of $\tau(x)$, $\tau'(x)$ and $\tau''(x)$ (the latter two suitably scaled to fit into the same range as τ), one plot of $\tilde{g}(x)$, $\tilde{g}'(x)$ and $\tilde{g}''(x)$ (properly scaled), and one plot of $X_e(z)$.
- A transcript of the module written for the evaluations

2 How to get started

- Start out with the source written for Milestone 1
- The new relevant module from the provided source code is called "rec_mod.f90" (short for "recombination module"), already available in your working directory.
- Read through the comments in this file. Whereever there is a comment starting with "Task", you are expected to do something :-)
- I recommend reading through Section IIB of Callin (2006). The following description follows that presentation very closely.

3 Definitions

The problem of this part of the project is to compute the optical depth, $\tau(x)$, and the so-called visibility function, $g(x)$, which is needed for integrating the Boltzmann-Einstein equations later on, and computing the CMB power spectrum. Physical descriptions and derivations will not be given here, but can be found in Callin (2006) or Dodelson. Here only the hard-core and required definitions will be presented.

First, the optical depth is defined as

$$\tau(\eta) = \int_{\eta}^{\eta_0} n_e \sigma_T a d\eta' \tag{1}$$

This quantifies the probability for scattering a photon between some previous time and today, such that if one had originally a photon beam of intensity I_0 at time η, then one would observe an intensity of $I = I_0 e^{-\tau(\eta)}$ today at η_0; the rest would have been absorbed or scattered by the medium between us.

The components involved in this expression are

- $n_e = n_e(\eta)$; the electron density (ie., the number of free electrons per cubic meter) at time η
- σ_T; the Thompson cross-section (physical constant, defined in params.f90)

- a; the scale factor.

The expression for τ may also be written on an differential form, such that

$$\tau' = -\frac{n_e \sigma_T a}{\mathcal{H}}. \tag{2}$$

This implies that we can use existing routines for solving differential equations to compute τ, if we can only somehow compute n_e at any time. And *that*'s the difficult part...

Estimating the electron density

Instead of actually computing n_e, we rather focus on the fractional electron density, $X_e \equiv n_e/n_H$, where n_H is the proton density. Here we will assume that all baryons are protons (ie., no helium or heavier elements), and therefore

$$n_H = n_b \approx \frac{\rho_b}{m_H} = \frac{\Omega_b \rho_c}{m_H a^3} \tag{3}$$

where $\rho_c \equiv \frac{3H_0^2}{8\pi G}$ is the critical density of the universe. (Note that this is the first time we actually use cosmological parameters for computing observables – from now on, we will at least be sensitive to Ω_b and H_0 :-))

Now, we have two different equations available for X_e as a function of temperature and density, namely the so-called Saha and Peebles' equations. The first one of these reads

$$\frac{X_e^2}{1-X_e} = \frac{1}{n_b}\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b}, \tag{4}$$

where T_b is the baryon temperature of the universe, and $\epsilon = 13.6eV$ is the ionization energy of hydrogen (ie., the energy a photon needs in order to rip an electron away from a proton). In principle, one would have to solve separately for both T_b and the photon temperature, T_r, but in practice it is an excellent approximation to set these equal. We can therefore assume that $T_b = T_r = T_0/a = 2.725\mathrm{K}/a$.

With the above information, Saha's equation reduces to a standard second-order equation in X_e, and can be solved directly using the normal formula, $y = (-b \pm \sqrt{b^2 - 4ac})/2a$. Note: When applying this formula, don't try to be clever, for instance by pulling out common factors. Implement it *exactly* according to this formula, and you'll be safe :-)

Saha's equation is an excellent approximation when $X_e \approx 1$. When X_e is noticeably smaller than one, better approximations are required, and one such approximation is the Peebles' equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{H}\left[\beta(T_b)(1-X_e) - n_H \alpha^{(2)}(T_b) X_e^2\right], \tag{5}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s\to1s} + \Lambda_\alpha}{\Lambda_{2s\to1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{6}$$

$$\Lambda_{2s\to1s} = 8.227\mathrm{s}^{-1} \tag{7}$$

$$\Lambda_\alpha = H\frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{8}$$

$$n_{1s} = (1-X_e)n_H \tag{9}$$

$$\beta^{(2)}(T_b) = \beta(T_b)e^{3\epsilon_0/4T_b} \tag{10}$$

$$\beta(T_b) = \alpha^{(2)}(T_b)\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b} \tag{11}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}}\frac{\alpha^2}{m_e^2}\sqrt{\frac{\epsilon_0}{T_b}}\phi_2(T_b) \tag{12}$$

$$\phi_2(T_b) = 0.448\ln(\epsilon_0/T_b). \tag{13}$$

This looks a bit scary, but it's not too bad, really. First of all, the various constants are simply physical constants describing simple atomic physics; Peebles' equation takes into account the transition rates between the ground-state

(1s) and the first excited state (2s) of the hydrogen atom. For even higher-accurate work, many more states should be included, and also other atoms, most notably the helium atom. But here we will be satisfied with the Peebles' equation. Second, the Peebles' equation is simply yet another linear first-order differential equation – which by now is something we know how to solve. The fact that the right-hand side is slightly more complicated than previous examples doesn't really make things that much harder.

Next, we must decide when to switch from Saha's equation to Peebles' equation. For simplicity, we simply say that when $X_e > 0.99$, we use Saha; when $X_e < 0.99$, we use Peebles.

Hint: To check that your results are OK, you should compare your derived X_e to Figure 1 of Callin (2006).

When you are happy with your n_e function, the next step is to spline it, such that you can evaluate it at arbitrary values of x. One thing: Because n_e varies over many, many orders of magnitude, it is useful to spline $\log n_e$ rather than n_e itself; this function varies much more slowly with x, and is therefore easier to interpolate. But do remember to exponentiate the resulting splined function value, after interpolating to the desired x.

3.1 Computing τ and $\tilde{g}$ and their derivatives

You are now ready to compute the optical depth, by solving Equation 2. For initial conditions, remember that the optical depth at our place in the universe today is precisely zero.

Once you have done this, you need spline this function as well, as done for n_e. However, later on you will also need τ' and τ''. The former of these is easy, once you already have the regular splined τ: In spline_1D_mod.f90, there is a routine called "splint_deriv", which works precisely as "splint", but gives you the first-order derivative instead. For the second-order derivatives, you need to do something slightly different: This time you have to spline the second-order derivatives that are provided to you from "spline" itself. (Holding the second-derivatives of these second-derivatives is what the arrays ending with "22" are intended for.)

Next, you should complete the small routines called something like "get_tau(x)" to return the respective value, given the precomputed splines.

The final item to compute is the so-called visibility function,

$$g(\eta) = -\dot{\tau}e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \tag{14}$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \tag{15}$$

$$\int_0^{\eta_0} g(\eta)d\eta = \int_{-\infty}^{0} \tilde{g}(x)dx = 1. \tag{16}$$

This function is actually a true probability distribution, and describes the probability density for a given photon to scatter at time x. As you will see, this function is sharply peaked around $x = -7$, corresponding to redshifts of $z \sim 1100$. The fact that this is so sharply peaked is the reason why we often refer to the recombination period as the "surface of last scattering": This process happened during a very thin shell around us, at a redshift of $z \sim 1100$.

What will be used later in the project is not really g, but rather $\tilde{g}$. This is therefore also what you should compute for this milestone. However, the names in the template code only refers to "g" and not "g_tilde"; this is just a short-hand, and it is understood that "g" everywhere really means "g_tilde". Hope that won't be too confusing..

Finally, first- and second-order derivatives of $\tilde{g}$ will be also required, and so these must also be properly splined, just as τ was.

4 Final comment: Units. Hrmpf..

One final note: I wasn't completely honest when describing the Saha and Peebles equations above. They are not quite complete as they are written here. Specifically, they have been written with funny "theoretical physicist units", where they set $c = k_B = \hbar = 1$. And that makes life a pain for observational physicists, like us.

So one of your tasks will be to re-insert the missing constants whereever necessary. To do so, remember the following points:

- Since c, k_B and $\hbar$ are all equal to 1, you are free to multiply or divide with these whereever you want; they are all just one in the above equations. Except that they are not really, because they correspond to different physical units. In particular,
  - $[c] = \mathrm{m/s}$
  - $[k_b] = \mathrm{m^2kg/s^2/K} = \mathrm{J/K}$
  - $[\hbar] = \mathrm{m^2kg/s} = \mathrm{Js}$
- Most equations are given in terms of dimensionless quantities, such as $e^{-\epsilon_0/T_b}$. To figure out whether some combination of c, k_b and $\hbar$ is missing from such an expression, look at the units of whatever is actually present in the expression, and multiply or divide by some combination of the three magic numbers to *make* the full expression dimensionless. (However, in some cases one may find equations that do have dimensions; the only really important thing is that both sides of a given equation have the *same* dimensions.)
- Example: Since the exponential only works on real numbers without units, we see that something is missing from the exponential in Saha's equation, $e^{-\epsilon_0/T_b}$: First, ϵ_0 is an energy, and is therefore measured in Joules (J). Second, T_b is a temperature, and is therefore measured in Kelvin (K). So the current dimension of the exponent is therefore J/K. In this case, the missing piece is quite simple to figure out – just divide by k_b, since that also has units J/K. The true expression, before theorists messed it up, should therefore look like $e^{-\epsilon_0/(k_bT_b)}$.

You will have to work through each of the expressions above in a similar way, and see whether any constants are missing, using similar logic. A good thing, though, is that for the next two milestones, I'll do this for you, so this is the only time you actually need to do this for the current project. But it's useful to have done this at least once, so one gets the hang of it :-)

Final note: If you get thoroughly confused, don't forget that I am considered a legal aid. So come and ask me, if you can't seem to make sense of this!