Milestone 4: The CMB power spectrum

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1 Problem statement and deliverables

Finally! After spending a lot of time on understanding the evolution of the background properties of the universe, its ionization history, and the growth of structure in the early universe, we are now ready to pull it all together, and compute the CMB power spectrum! Cool! :-)

The deliverables for this project are the following:

- A report defining the quantities to be computed, and a short description of the algorithms used
- One plot of the transfer function $\Theta_l(k)$ for six (significantly) different values of k, one plot of the spectrum integrand, $\Theta_l(k)^2/k$, for six values of k, and finally, the actual CMB power spectrum.
- Data file with $\{l, C_l\}$. We will spend (at least) one lecture simply on comparing these from all of you, and talk about what these actually mean physically.
- A transcript of the module written for the evaluations
- For AST9420 students: An additional plot of C_ℓ^{EE} and C_ℓ^{TE}, overplotted on Planck measurements.

2 How to get started

- Start out with the source written for Milestones 1, 2 and 3
- Look at the new F90 modules called "cl_mod.f90", "sphbess_mod.f90" and "spline_2D_mod.f90" from HKE's home directory:
  ~hke/AST5220/v10/src/cmbspec_mk4/. Copy over the WMAP spectrum file as well, which may be used for comparison reasons.
- Read through the comments in the new "evolution_mod.f90" routine and the "cl_mod.f90" file, and look for things marked by "Task"
- If you haven't done so already, I now *strongly* recommend reading through Callin (2006). This part of the project represents a significant step up in complexity from the two previous ones, and now you will actually have to understand what's going on. So start reading :-)

3 Overview

Finally, we now are in the position to actually compute the CMB power spectrum: We know how the ionization history of the universe, and we know how structure have grown from the very earliest epochs. Now we just have to pull it all together, in order to derive our preferred observable, namely the CMB power spectrum.

In order to understand how to get to the CMB power spectrum from our computed quantities, let us first recall the definition of the spherical harmonics transform of the CMB temperature field,

$$T(\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n}), \tag{1}$$

where $\hat{n}$ is the direction on the sky, a_{lm} are the spherical harmonics coefficients, and Y_{lm} are the spherical harmonics themselves. The CMB power spectrum, on the other hand, is simply defined as the expectation value of the square of the spherical harmonics coefficients,

$$C_l \equiv \left\langle |a_{lm}|^2 \right\rangle = \langle a_{lm} a_{lm}^* \rangle \,. \tag{2}$$

Note that, in principle, this function should have two subscripts, C_{lm}, but because we assume that the universe is isotropic, it must have the same power spectrum towards both the x, y and z directions, and this implies full rotational invariance. As a result, there is no m dependence in the power spectrum, and we simply average over m, and only call the spectrum C_l.

So, in order to get to the power spectrum, we need to know the temperature field we observe around us today, $T(\hat{n}, x=0)$. But fortunately, we already have this information (more or less), as we have already computed the evolution of the temperature field in the form of $\Theta_l(k,x)$. So all we have to do, is to read off the values of these functions at $x=0$, Fourier transform the resulting coefficients (to get $T(\vec{n})$ instead of $T(\vec{k})$), and read off the correct values.

"But", you object, "we only computed these functions up to $l=6$, and we are surely interested in smaller scales than that!". And you are completely right: We want to know the power spectrum to at least $l=1200$. So what you have to then, is to rerun the code, but this time with $lmax = 1200$ instead of 6, and proceed. The only problem with that is that it will take a *very* long time to complete.

This is where the brilliance of Zaldarriaga and Seljak comes to our rescue, through what they call the "line-of-sight integration approach". What we really need to know, is $\Theta(k,\mu,x=0)$. But instead of first expanding the full temperature field in multipoles and then solve the coupled equations, one can start by *formally* integrating the original equation for $\dot{\Theta}$, and then expand in multipoles at the end. For the details of this process, read Section IVA in Callin (2005), or Chapter 8 in Dodelson. However, the final expression is simply

$$\Theta_l(k, x=0) = \int_{-\infty}^{0} \tilde{S}(k,x) j_l[k(\eta_0-\eta)] dx, \tag{3}$$

where the *source function* is defined as

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Pi\right] + e^{-\tau}\left[\Psi' - \Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Pi)\right]. \tag{4}$$

where you have computed all quantities in earlier milestones, and $\Pi = \Theta_2 + \Theta_0^P + \Theta_2^P$. (If you skipped polarization in the previous case, just set the two latter terms to zero.) For full expressions for the last second derivative, see section IVA in Callin (2005).

(For AST9420 students: The corresponding expression for the polarization transfer function reads

$$\Theta_l^E(k,\eta_0) = \sqrt{\frac{(l+2)!}{(l-2)!}} \int_0^{\eta_0} \tilde{S}_E(k,\eta) j_l[k(\eta_0-\eta)] d\eta, \tag{5}$$

where the *polarization source function* now is defined as

$$\tilde{S}(k,\eta) = \frac{3g\Pi}{4k^2(\eta_0-\eta)^2}. \tag{6}$$

Note also that the integral is defined in terms of conformal time rather than x.)

So, the intuition behind this approach is that the CMB radiation we observe in a given direction on the sky, is basically the integral of the local CMB monopole (weighted by the visibility function) along the line of sight from us to infinity. That's the first Θ_0 term in the source function. However, there are a number of corrections to this effect. First, the Ψ term encodes the fact that the photons have to climb out of a graviational potential, and therefore lose energy on its way to us. Π is a small quadrupolar (+ polarization) correction to the original monopole contribution.

The next main term is essentially the so-called Integrated Sachs-Wolfe contribution, which describes the fact that graviational potentials actually change *while* the photons are moving. The third term is a Doppler term (I think), while I'm not sure what the last term is right now: A task for you is to figure out what that one means :-)

So, this is a much more beautiful approach: Instead of evaluating what every single photon moment is at our position today, we can compute the monopole at all positions and times, and then do the integral through space. Much faster, and also quite intuitive.

By the way, the $j_l(x)$'s in the above expression are the so-called spherical Bessel functions, and take into account the projection of the 3D field (characterized by k) onto a 2D sphere (characterized by l).

But we're not quite done yet, even if we now know how to compute $\Theta_l(k)$ – we still need to go to actual C_l's. This corresponds to 1) take the square of $\Theta_l(k)$ (since C_l is the square of the "Fourier" coefficients), 2) multiply with the primordial power spectrum $P(k)$ coming from inflation (recall that we originally set $\Phi = 1$ for all modes; this is now corrected by rescaling everything by $P(k)$ instead, which is perfectly valid, since all our equations are linear.), and 3) integrate over all three spatial directions, instead of just the z direction (but since we assume isotropy, we can use the same derived functions for all three directions). The CMB power spectrum then reads

$$C_l = \int \frac{d^3k}{(2\pi)^3} P(k)\Theta_l^2(k). \tag{7}$$

However, this can be massaged a bit further, by noting that most inflation models predict a so-called Harrison-Zel'dovich spectrum, for which

$$\frac{k^3}{2\pi^2} P(k) = \left(\frac{ck}{H_0}\right)^{n-1}, \tag{8}$$

where n is the spectral index of scalar perturbations, and expected to be close to unity. The final expression for the spectrum is therefore

$$C_l = \int_0^\infty \left(\frac{ck}{H_0}\right)^{n-1} \Theta_l^2(k) \frac{dk}{k}. \tag{9}$$

For AST9420 students: The corresponding EE and TE power spectra are given by

$$C_l^{EE} = \int_0^\infty \left(\frac{ck}{H_0}\right)^{n-1} (\Theta_l^E)^2(k) \frac{dk}{k}. \tag{10}$$

and

$$C_l^{TE} = \int_0^\infty \left(\frac{ck}{H_0}\right)^{n-1} \Theta_l(k)\Theta_l^E(k) \frac{dk}{k}. \tag{11}$$

Two final comments: First, the power spectrum is most often plotted in units of $l(l+1)/2\pi$ in $\mu\mathrm{K}^2$, because it's overall trend is to drop as l^{-2}. It is therefore easier to see features when plotted in these units. Second, in this course we don't care about the normalization of the spectrum, except that we want it to match the observed spectrum. For simplicity, we therefore say that the maximum of the power spectrum (which should happen around $l = 220$) is $5775\mu\mathrm{K}^2$, when measured in the above units.

4 Parameter estimation

We now have a tool that allows us to compute the angular CMB power spectrum given a set of cosmological parameters. Therefore, we obviously want to use this tool to actually constain those cosmological parameters. Normally this is done by some Markov Chain Monte Carlo method, but you will instead do this by hand, changing one parameter at a time, in order to build up intuition about how the various parameters affect the spectrum. Accuracy is not important – all we want is to produce something that looks semi-reasonable.

You will do this the following way:

- Download the latest Planck power spectrum from the Planck Legacy Archive (google it). There should be a file called something like "COM_PowerSpect_CMB_TT_*_full_R2.01.txt, and possibly one for low and one for high multipoles. If so, download both.
- Plot the spectrum including error bars; overplot your own theoretical spectrum derived using the default cosmological parameters.
- Vary each of the parameters by hand until you get a reasonable fit. Do it for as long as you feel like – you won't lose points by a poor fit. The goal here is not really to make the two match, but rather understand the effect of each parameter on the final spectrum.
- Record your final best-fit parameters in a table.

5 Specification of tasks

5.1 Changes to evolution_mod.f90

Your first task is to introduce a new routine in evolution_mod.f90 that outputs the full source function. This is done in three steps:

- Compute the 2D source function over the existing k- and x-grids
- Compute a 2D spline through this function, using splin2_full_precomp from spline_2D_mod.f90
- Resample the source function onto a uniform 5000×5000 grid in k and x using the spline, and return this

5.2 Changes to cl_mod.f90

Once you have the source function, the only remaining task is to integrate this (weighted by the Bessel functions and the power spectrum), to obtain the power spectrum. This is done for just a subset of all l's, listed in the source code, but when these calculations are done, you should compute a 1D spline through the resulting values, and plot the function for all l's.

Specifically, you should for each listed l do the following:

- Precompute the spherical Bessel functions (use sphbess for this) for each l between 0 and 5400. Compute a spline for each function.
- Compute the transfer function, $\Theta_l(k)$ using equation 3, by integrating over x
- Compute the unnormalized power spectrum by integrating $P(k)(\Theta_l^2(k)/k)$ over k, with $n = 0.96$ (corresponding roughly to the best-fit WMAP value).
- Store output value on the form $C_l l(l+1)/2\pi$ in an array

Once you've completed this, generate a spline through your sampled values, and resample onto a new l-grid with unit stepsize. Finally, normalize the spectrum to have a maximum value of $5775\mu\mathrm{K}^2$.

Bingo! You now have your very own CMB power spectrum, computed from basic principle. A big congratulations!! :-)

6 Optional: Parameter estimation by Markov Chain Monte Carlo

If you still have any energy, and want to look like a real CMB cosmologist, you can round it all off by writing your very own Markov Chain Monte Carlo sampler. It really is very simple, and the algorithm goes as follows:

- Propose a stochastic update to any one of the cosmological parameters you want to fit, e.g., $\theta_i \leftarrow \theta_i + \delta_i$, where θ_i is the parameter in question and δ_i is typically a Gaussian distributed variable with zero mean and some standard deviation. You should tune this standard deviation such that you get an accept rate between 0.3 and 0.7.
- Define a (very approximate) likelihood on the form

$$\mathcal{L}(C_\ell(\theta)) = \sum_{\ell=2}^{\ell_{\max}} \left(\frac{C_\ell - \hat{C}_\ell}{\sigma_\ell}\right)^2, \tag{12}$$

  where $\hat{C}_\ell$ is the observed Planck power spectrum, and σ_ℓ is the corresponding uncertainty.
- Apply the Metropolis accept rule: Draw a uniform random variable between 0 and 1, η, and accept the proposed change to θ_i if

$$p = \frac{\mathcal{L}(C_\ell^{\mathrm{new}})}{\mathcal{L}(C_\ell^{\mathrm{old}})} > \eta \tag{13}$$

- Print out current parameter combination to disk
- Loop over parameters, and repeat until the chain becomes stationary – and then continue for some more
- Plot one histogram for each parameter after removing burn-in, and compute the corresponding mean and standard deviation