Milestone 3: The evolution of structures in the universe

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1 Problem statement and deliverables

The topic of the second milestone of the AST5220 project is the evolution of structure in the universe: How did small fluctuations in the baryon-photon-dark-matter fluid grow from shortly after inflation until today? The ultimate goal of this part is to construct a two-dimensional grid in time and Fourier scale, x and k, for each of the main physical quantities of interest, $\Phi(x,k)$, $\Psi(x,k)$, $\delta(x,k)$, $\delta_b(x,k)$, $v(x,k)$, $v_b(x,k)$, $\Theta_l(x,k)$, and their derivatives.

The deliverables are the following:

- A report defining the quantities to be computed, a short description of the algorithms used
- One plot for each of the physical quantities as function of x, for six different k's within the interval of interest. (For the radiation quantities, plot only for $l = 0$.) Choose values of k such that each of the three main regimes are shown; large-scales (ie., unaffected by causal physics), small-scales (ie., early oscillations with subsequent damping), and intermediate scales (ie., scales that have just undergone a few oscillations).
- A transcript of the module written for the evaluations

Note 1: At this stage, the project splits into two branches, one for Ph.D. students and one for Master students. The difference is that Ph. D. students have to consider also neutrinos and polarization, while Master students only need to take into account photons, baryons and dark matter, and only temperature fluctuations. So make sure to choose the one appropriate for you. (But of course, if you are a Master student and want to go for the more advanced problem, you are more than welcome to do so ;-))

Note 2: These notes were written *after* coding up the corresponding equations myself. They are therefore not properly "debugged", and typos may exist. Similarly, there may be missing pieces of information. So if you do find any such problems, please let me know as soon as possible :-)

2 How to get started

- Start out with the source written for Milestones 1 and 2
- Take a look at F90 template module called "evolution_mod.f90"
- Read through the comments in this file, and look for things marked by "Task"
- If you haven't done so already, I now *strongly* recommend reading through Callin (2006). This part of the project represents a significant step up in complexity from the two previous ones, and now you will actually have to understand what's going on. So start reading :-)

3 Overview

During the lectures, we have derived (or will derive) the linearized Einstein and Boltzmann equations for photons, baryons and dark matter, and their respective inflationary initial conditions. The task of the current part of the project is to solve these equations numerically. The good news is that the numerical solution of these equations follows in exactly the same path as when solving for instance the Peebles' equation or the equation for the conformal time; we use ode_solver with the Bulirsch-Stoer stepper. The bad news is that the expressions for the equations are somewhat more complicated. But if one is just a little careful about typing these in correctly, everything should work just fine :-)

But before we write down the equations, there are a few issues that should be pointed out. First, at early times the optical depth, τ, is very large. This means that electrons at a given place only observe temperature fluctions that are very nearby. This, in turn, implies that it will only see very smooth fluctuations, since the full system is in thermodynamic equilibrium, and all gradients are efficiently washed out. The only relevant quantities in this regime are therefore 1) the monopole, Θ_0, which measures the mean temperature at the position of the electron, 2) the dipole, Θ_1, which is given by the velocity of the fluid due to the Doppler effect, and 3) the quadrupole, Θ_2, which is the only relevant source of polarization signals. The regime where this is the case is called *tight coupling*.

At later times, though, the fluid becomes thinner, and the electrons start "seeing" further away, and then become sensitive to higher-ordered multipoles, Θ_l. Fortunately, because of a very nice computational trick due to Zaldarriaga and Seljak called "line of sight integration", we only need to take into account a relatively small number of these (six to be exact), and so the system of relevant equations is therefore still tractable. (Note that before 1996 or so, people actually included thousands of variables, to trace multipoles for the full range. Needless to say, this was *slow*, and other approximations were required. Thank you, Mattias and Uros! :-))

A second issue is the very large value of τ' at early times, which multiplies $(3\Theta_1 - v_b)$ in the Boltzmann equations. The latter factor is very small early on, and the product of the two is therefore numerically extremely unstable. The result is that the standard Boltzmann equation set is completely unstable if one simply implements the full expressions at early times. The solution to this problem is to use a proper approximation for $(3\Theta_1 - v_b)$ at early times, as described in detail by both Callin (2006) and Dodelson.

So, to summarize: Your problem is to solve the Einstein-Boltzmann equations from some early time (we start at $a_{\text{init}} = 10^{-8}$ here) until today. You have to do this by solving one set of equations from a_{init} until the end of tight coupling, a_{tc}, and a different set of equations from a_{tc} until today. These equation sets are provided below.

The variables you need to include in your system are $\Phi(x,k)$, $\Psi(x,k)$, $\delta(x,k)$, $\delta_b(x,k)$, $v(x,k)$, $v_b(x,k)$ and $\Theta_l(x,k)$ for $\ell = 0,\ldots,6$, and, if you are a Ph. d. student, also $\Theta_l^P(x,k)$ and $\mathcal{N}_l(x,k)$. The latter two describes polarization fluctuations in the photon fluid and neutrinos. Note that for neutrinos, you need to include slightly more multipoles than for photons, $l_{\max,\nu} = 10$. Note also that the expression for Ψ is an algebraic equation in the other variables, not a differential expression, and it should therefore not be included in the differential equation system as such. Instead it should be evaluated directly whenever needed.

The initial conditions

All relevant initial conditions for our system may be derived from inflationary theory, and they turn out to all boil down to a choice of the gravitational curvature potential, $\Phi(k, a=0)$. This will in general be a function of k. In particular, inflation predicts that the power spectrum of Φ, which is denoted P_k, should look like $P_k = A\,k^{n_s}$, where A is an overall amplitude and n_s is called the spectral index of scalar perturbations. However, because our equations are linear, we don't have to think about this when actually solving the equations: We can instead solve the equations setting $\Phi(k,0) = 1$, and then simply multiply the solution at the end with whatever P_k we happen to be interested in. In other words, we can solve the system of all choices of P_k in a single shot. Nice!

So, with this convention the full set of initial conditions read

$$\Phi = 1 \tag{1}$$

$$\delta = \delta_b = \frac{3}{2}\Phi \tag{2}$$

$$v = v_b = \frac{ck}{2\mathcal{H}}\Phi \tag{3}$$

$$\Theta_0 = \frac{1}{2}\Phi \tag{4}$$

$$\Theta_1 = -\frac{ck}{6\mathcal{H}}\Phi \tag{5}$$

$$\Theta_2 = \begin{cases} -\frac{8ck}{15\mathcal{H}\tau'}\Theta_1, & \text{(with polarization)} \\ -\frac{20ck}{45\mathcal{H}\tau'}\Theta_1, & \text{(without polarization)} \end{cases} \tag{6}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{ck}{\mathcal{H}\tau'}\Theta_{l-1} \tag{7}$$

If you additionally need polarization and neutrino initial conditions, these are

$$\Theta_0^P = \frac{5}{4}\Theta_2 \tag{8}$$

$$\Theta_1^P = -\frac{ck}{4\mathcal{H}\tau'}\Theta_2 \tag{9}$$

$$\Theta_2^P = \frac{1}{4}\Theta_2 \tag{10}$$

$$\Theta_l^P = -\frac{l}{2l+1}\frac{ck}{\mathcal{H}\tau'}\Theta_{l-1}^P \tag{11}$$

$$\mathcal{N}_0 = \frac{1}{2}\Phi \tag{12}$$

$$\mathcal{N}_1 = -\frac{ck}{6\mathcal{H}}\Phi \tag{13}$$

$$\mathcal{N}_2 = -\frac{c^2k^2a^2\Phi}{12H_0^2\Omega_\nu}\frac{1}{\frac{5}{2f_\nu}+1} \tag{14}$$

$$\mathcal{N}_l = \frac{ck}{(2l+1)\mathcal{H}}\mathcal{N}_{l-1}, \qquad l \geq 3 \tag{15}$$

where $f_\nu = \frac{\rho_\nu}{\rho_r+\rho_\nu} \approx 0.405$ for $N = 3$ neutrino species.

The Einstein-Boltzmann equations without polarization or neutrinos

The full equation set

Without more ado:

$$\Theta_0' = -\frac{ck}{\mathcal{H}}\Theta_1 - \Phi', \tag{16}$$

$$\Theta_1' = \frac{ck}{3\mathcal{H}}\Theta_0 - \frac{2ck}{3\mathcal{H}}\Theta_2 + \frac{ck}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{17}$$

$$\Theta_l' = \frac{lck}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)ck}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau'\left[\Theta_l - \frac{1}{10}\Theta_l\delta_{l,2}\right], \qquad 2 \leq l < l_{\max} \tag{18}$$

$$\Theta_l' = \frac{ck}{\mathcal{H}}\Theta_{l-1} - c\frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \qquad l = l_{\max} \tag{19}$$

$$\delta' = \frac{ck}{\mathcal{H}}v - 3\Phi' \tag{20}$$

$$v' = -v - \frac{ck}{\mathcal{H}}\Psi \tag{21}$$

$$\delta_b' = \frac{ck}{\mathcal{H}}v_b - 3\Phi' \tag{22}$$

$$v_b' = -v_b - \frac{ck}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{23}$$

$$\Phi' = \Psi - \frac{c^2k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{24}$$

$$\Psi = -\Phi - \frac{12H_0^2}{c^2k^2a^2}\Omega_r\Theta_2 \tag{25}$$

$$R = \frac{4\Omega_r}{3\Omega_b a} \tag{26}$$

Easy!

The tight coupling equations

In the tight coupling regime, the only differences are 1) that one should only include $\ell = 0$ and 1 for Θ_l – all higher moments are given by those, and 2) that the expressions for Θ_1' and v_b' are quite a bit more involved:

$$q = \frac{-[(1-2R)\tau' + (1+R)\tau''](3\Theta_1 + v_b) - \frac{ck}{\mathcal{H}}\Psi + (1 - \frac{\mathcal{H}'}{\mathcal{H}})\frac{ck}{\mathcal{H}}(-\Theta_0 + 2\Theta_2) - \frac{ck}{\mathcal{H}}\Theta_0'}{(1+R)\tau' + \frac{\mathcal{H}'}{\mathcal{H}} - 1} \tag{27}$$

$$v_b' = \frac{1}{1+R}\left[-v_b - \frac{ck}{\mathcal{H}}\Psi + R(q + \frac{ck}{\mathcal{H}}(-\Theta_0 + 2\Theta_2) - \frac{ck}{\mathcal{H}}\Psi)\right] \tag{28}$$

$$\Theta_1' = \frac{1}{3}(q - v_b') \tag{29}$$

The expressions for the higher-ordered photon moments during tight coupling are simply the same as the initial conditions,

$$\Theta_2 = -\frac{20ck}{45\mathcal{H}\tau'}\Theta_1 \tag{30}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{ck}{\mathcal{H}\tau'}\Theta_{l-1} \tag{31}$$

The Einstein-Boltzmann equations with polarization and neutrinos

The full equation set

Including polarization and neutrinos, the full equation set becomes slightly longer,

$$\Theta_0' = -\frac{ck}{\mathcal{H}}\Theta_1 - \Phi', \tag{32}$$

$$\Theta_1' = \frac{ck}{3\mathcal{H}}\Theta_0 - \frac{2ck}{3\mathcal{H}}\Theta_2 + \frac{ck}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{33}$$

$$\Theta_l' = \frac{lck}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)ck}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau'\left[\Theta_l - \frac{1}{10}\Pi\delta_{l,2}\right], \qquad 2 \leq l < l_{\max} \tag{34}$$

$$\Theta_l' = \frac{ck}{\mathcal{H}}\Theta_{l-1} - c\frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \qquad l = l_{\max} \tag{35}$$

$$\Theta_{P0}' = -\frac{ck}{\mathcal{H}}\Theta_{P1} + \tau'\left[\Theta_{P0} - \frac{1}{2}\Pi\right] \tag{36}$$

$$\Theta_{Pl}' = \frac{lck}{(2l+1)\mathcal{H}}\Theta_{l-1}^P - \frac{(l+1)ck}{(2l+1)\mathcal{H}}\Theta_{l+1}^P + \tau'\left[\Theta_l^P - \frac{1}{10}\Pi\delta_{l,2}\right], \qquad 1 \leq l < l_{\max} \tag{37}$$

$$\Theta_{P,l}' = \frac{ck}{\mathcal{H}}\Theta_{l-1}^P - c\frac{l+1}{\mathcal{H}\eta(x)}\Theta_l^P + \tau'\Theta_l^P, \qquad l = l_{\max} \tag{38}$$

$$\mathcal{N}_0' = -\frac{ck}{\mathcal{H}}\mathcal{N}_1 - \Phi', \tag{39}$$

$$\mathcal{N}_1' = \frac{ck}{3\mathcal{H}}\mathcal{N}_0 - \frac{2ck}{3\mathcal{H}}\mathcal{N}_2 + \frac{ck}{3\mathcal{H}}\Psi \tag{40}$$

$$\mathcal{N}_l' = \frac{lck}{(2l+1)\mathcal{H}}\mathcal{N}_{l-1} - \frac{(l+1)ck}{(2l+1)\mathcal{H}}\mathcal{N}_{l+1}, \qquad 2 \leq l < l_{\max,\nu} \tag{41}$$

$$\mathcal{N}_l' = \frac{ck}{\mathcal{H}}\mathcal{N}_{l-1} - c\frac{l+1}{\mathcal{H}\eta(x)}\mathcal{N}_l, \qquad l = l_{\max,\nu} \tag{42}$$

$$\delta' = \frac{ck}{\mathcal{H}}v - 3\Phi' \tag{43}$$

$$v' = -v - \frac{ck}{\mathcal{H}}\Psi \tag{44}$$

$$\delta_b' = \frac{ck}{\mathcal{H}}v_b - 3\Phi' \tag{45}$$

$$v_b' = -v_b - \frac{ck}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{46}$$

$$\Phi' = \Psi - \frac{c^2k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0 + 4\Omega_\nu a^{-2}\mathcal{N}_0\right] \tag{47}$$

$$\Psi = -\Phi - \frac{12H_0^2}{c^2k^2a^2}\left[\Omega_r\Theta_2 + \Omega_\nu\mathcal{N}_2\right] \tag{48}$$

$$R = \frac{4\Omega_r}{3\Omega_b a} \tag{49}$$

$$\Pi = \Theta_2 + \Theta_0^P + \Theta_2^P \tag{50}$$

Still easy!

In the tight coupling regime, we again get the same expressions for the higher-ordered photon moments as given by the initial conditions,

$$\Theta_2 = -\frac{8ck}{15\mathcal{H}\tau'}\Theta_1 \tag{51}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{ck}{\mathcal{H}\tau'}\Theta_{l-1} \tag{52}$$

$$\Theta_2^P = \frac{1}{4}\Theta_2 \tag{53}$$

$$\Theta_l^P = -\frac{l}{2l+1}\frac{ck}{\mathcal{H}\tau'}\Theta_{l-1}^P \tag{54}$$