Appendix A

Theorem 1. *The DTFT of a Gaussian G_σ is approximately $\hat{G}_\sigma(\omega) \approx \sum_{k=-\tau}^{\tau} \exp(-\frac{(\omega-2\pi k)^2}{2}\sigma^2)$. For $\sigma \geq 0.4$, using $\tau = 2$ gives an ℓ^∞ error of $5.68{\cdot}10^{-9}$. Gaussians with $\sigma < 0.4$ in space can be treated as constant functions in frequency.*

Proof. The DTFT of G_σ is $\hat{G}_\sigma(\omega) = \sum_{k\in\mathbb{Z}} e^{-\frac{1}{2}(\omega-2\pi k)^2\sigma^2}$. The partial sum of the first τ positive and negative terms gives an error $\varepsilon(\sigma,\tau) = \max_{\omega\in[-\pi,\pi]} \sum_{|k|>\tau} e^{-\frac{1}{2}(\omega-2\pi k)^2\sigma^2}$. By bounding the sums $k \neq 0$ with integrals, one can obtain $\varepsilon(\sigma,\tau) \leq 2e^{-\frac{1}{2}\sigma^2\pi^2(2\tau+1)^2} + \operatorname{erfc}(\pi\sigma(2\tau+1)/\sqrt{2})/\sqrt{2\pi}\sigma$. Using $\tau = 2$ gives $\varepsilon(\sigma, 2) \leq 5.68{\cdot}10^{-9}$ for $\sigma \geq 0.4$. Gaussians with $\sigma < 0.4$ can be treated as constant functions in frequency since $\hat{G}'_\sigma(\omega) \approx 0$ for $\omega \in [-\pi,\pi]$. $\square$

Appendix B

$$\begin{array}{llll} a_0 = 1.6800 & a_1 = 3.7350 & w_0 = 0.6318 & b_0 = 1.7830 \\ c_0 = -0.6803 & c_1 = -0.2598 & w_1 = 1.9970 & b_1 = 1.7230 \end{array}$$

$$\begin{aligned}
n_0 &= a_0 + c_0 \\
n_1 &= e^{-\frac{b_1}{\sigma}}\left(c_1 \sin(\tfrac{w_1}{\sigma}) - (c_0 + 2a_0)\cos\left(\tfrac{w_1}{\sigma}\right)\right) + \\
&\quad e^{-\frac{b_0}{\sigma}}\left(a_1 \sin\left(\tfrac{w_0}{\sigma}\right) - (2c_0 + a_0)\cos\left(\tfrac{w_0}{\sigma}\right)\right) \\
n_2 &= 2e^{-\frac{b_0}{\sigma}-\frac{b_1}{\sigma}}\Big((a_0 + c_0)\cos\left(\tfrac{w_1}{\sigma}\right)\cos\left(\tfrac{w_0}{\sigma}\right) - \\
&\quad \cos\left(\tfrac{w_1}{\sigma}\right) a_1 \sin\left(\tfrac{w_0}{\sigma}\right) - \cos\left(\tfrac{w_0}{\sigma}\right) c_1 \sin\left(\tfrac{w_1}{\sigma}\right)\Big) \\
&\quad + c_0 e^{-2\frac{b_0}{\sigma}} + a_0 e^{-2\frac{b_1}{\sigma}} \\
n_3 &= e^{-\frac{b_1}{\sigma}-2\frac{b_0}{\sigma}}\left(c_1 \sin\left(\tfrac{w_1}{\sigma}\right) - \cos\left(\tfrac{w_1}{\sigma}\right) c_0\right) + \\
&\quad e^{-\frac{b_0}{\sigma}-2\frac{b_1}{\sigma}}\left(a_1 \sin\left(\tfrac{w_0}{\sigma}\right) - \cos\left(\tfrac{w_0}{\sigma}\right) a_0\right) \\
d_1 &= -2e^{-\frac{b_1}{\sigma}}\cos\left(\tfrac{w_1}{\sigma}\right) - 2e^{-\frac{b_0}{\sigma}}\cos\left(\tfrac{w_0}{\sigma}\right) \\
d_2 &= 4\cos\left(\tfrac{w_1}{\sigma}\right)\cos\left(\tfrac{w_0}{\sigma}\right) e^{-\frac{b_0}{\sigma}-\frac{b_1}{\sigma}} + e^{-2\frac{b_1}{\sigma}} + e^{-2\frac{b_0}{\sigma}} \\
d_3 &= -2\cos\left(\tfrac{w_0}{\sigma}\right) e^{-\frac{b_0}{\sigma}-2\frac{b_1}{\sigma}} - 2\cos\left(\tfrac{w_1}{\sigma}\right) e^{-\frac{b_1}{\sigma}-2\frac{b_0}{\sigma}} \\
d_4 &= e^{-2\frac{b_0}{\sigma}-2\frac{b_1}{\sigma}} \\
\eta_0 &= n_0 - d_1^2 n_0 - d_2^2 n_0 - d_3^2 n_0 \\
&\quad - d_4^2 n_0 + 2d_1 n_1 + 2d_2 n_2 + 2d_3 n_3 \\
\eta_1 &= -d_1 d_2 n_0 - d_2 d_3 n_0 - d_3 d_4 n_0 + d_2 n_1 + d_1 n_2 \\
&\quad + d_3 n_2 + d_2 n_3 + d_4 n_3 + n_1 \\
\eta_2 &= -d_1 d_3 n_0 - d_2 d_4 n_0 + d_3 n_1 + d_4 n_2 + d_1 n_3 + n_2 \\
\eta_3 &= -d_1 d_4 n_0 + d_4 n_1 + n_3
\end{aligned}$$

Appendix C

Here we show in detail how to derive the elliptical symmetry for the kernel K in the spatial domain. We have

$$\hat{K}(\omega,\xi) = \frac{1}{1 + 2\lambda_x(1-\cos(\omega)) + 2\lambda_y(1-\cos(\xi))}\,.$$

Consider the change of variables

$$\begin{cases} l(\omega,\xi) = \lambda_x\cos(\omega) + \lambda_y\cos(\xi), \\ \theta(\omega,\xi) = \tan^{-1}\left(\xi\,/\,\omega\right). \end{cases} \tag{1}$$

Using the inverse function theorem we have

$$\begin{bmatrix} \frac{\partial\theta}{\partial\omega} & \frac{\partial\theta}{\partial\xi} \\ \frac{\partial l}{\partial\omega} & \frac{\partial l}{\partial\xi} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial\omega}{\partial\theta} & \frac{\partial\omega}{\partial l} \\ \frac{\partial\xi}{\partial\theta} & \frac{\partial\xi}{\partial l} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2}$$

then we can obtain the derivatives of ω and ξ with respect to l and θ by reversing the first matrix in the LHS of Eq. 2

$$\begin{bmatrix} \frac{\partial\omega}{\partial\theta} & \frac{\partial\omega}{\partial l} \\ \frac{\partial\xi}{\partial\theta} & \frac{\partial\xi}{\partial l} \end{bmatrix} = \frac{\omega\lambda_x\sin(\omega) + \xi\lambda_y\sin(\xi)}{\omega^2+\xi^2} \cdot \begin{bmatrix} -\lambda_y\sin(\xi) & -\frac{\omega}{\omega^2+\xi^2} \\ \lambda_x\sin(\omega) & -\frac{\xi}{\omega^2+\xi^2} \end{bmatrix}. \tag{3}$$

The function $\hat{K}$ can be written as a function of l and θ as

$$\hat{K}(\omega,\xi) = \frac{1}{1 + 2(\lambda_x + \lambda_y + l(\omega,\xi))}$$

To see that $\frac{d\hat{K}}{d\theta} = 0$, notice that $\frac{dl}{d\theta} = 0$ because

$$\frac{dl}{d\theta} = \frac{\partial l}{\partial\omega}\frac{\partial\omega}{\partial\theta} + \frac{\partial l}{\partial\xi}\frac{\partial\xi}{\partial\theta}$$

and this is 0 according to Eq. 2 (see the second row and first column of the RHS). Therefore

$$\frac{d\hat{K}}{d\theta} = \frac{\partial\hat{K}}{\partial\omega}\frac{\partial\omega}{\partial\theta} + \frac{\partial\hat{K}}{\partial\xi}\frac{\partial\xi}{\partial\theta} = 0. \tag{4}$$

Using Eq. 3, Eq. 4 can be written as

$$-\lambda_y\frac{\partial\hat{K}}{\partial\omega}\sin(\xi) + \lambda_x\frac{\partial\hat{K}}{\partial\xi}\lambda_x\sin(\omega) = 0\,. \tag{5}$$

Using the identities

$$\text{DTFT}\left[\delta(x+1) - \delta(x+1)\right](\omega) = 2\sin(\omega)\,,$$
$$\text{DTFT}\left[xf(x)\right](\omega) = \sqrt{-1}\hat{f}(\omega)\,,$$

and the convolution property

$$\text{DTFT}\left[f * g\right](\omega) = \hat{f}(\omega)\hat{g}(\omega)\,,$$

we apply the inverse DTFT in Eq. 5 to obtain

$$\begin{aligned} &\lambda_x(yK(x,y)) * (\delta(x+1) - \delta(x-1)) \\ &\quad - \lambda_y(xK(x,y)) * (\delta(y+1) - \delta(y-1)) = 0 \end{aligned} \tag{6}$$

finally, expanding the convolution in Eq. 6 gives

$$\begin{aligned} &\lambda_x\, y\left(K(x+1,y) - K(x-1,y)\right) \\ &-\lambda_y\, x\left(K(x,y+1) - K(x,y-1)\right) = 0. \end{aligned}$$