Appendix A

Theorem 1. *The DTFT of a Gaussian G_σ is approximately $\hat{G}_\sigma(\omega) \approx \sum_{k=-\tau}^{\tau} \exp(-\frac{(\omega-2\pi k)^2}{2}\sigma^2)$. For $\sigma \geq 0.4$, using $\tau = 2$ gives an ℓ^∞ error of $5.68 \cdot 10^{-9}$. Gaussians with $\sigma < 0.4$ in space can be treated as constant functions in frequency.*

Proof. The DTFT of G_σ is $\hat{G}_\sigma(\omega) = \sum_{k\in\mathbb{Z}} e^{-\frac{1}{2}(\omega-2\pi k)^2\sigma^2}$. The partial sum of the first τ positive and negative terms gives an error $\varepsilon(\sigma,\tau) = \max_{\omega\in[-\pi,\pi]} \sum_{|k|>\tau} e^{-\frac{1}{2}(\omega-2\pi k)^2\sigma^2}$. By bounding the sums $k \neq 0$ with integrals, one can obtain $\varepsilon(\sigma,\tau) \leq 2e^{-\frac{1}{2}\sigma^2\pi^2(2\tau+1)^2} + \mathrm{erfc}(\pi\sigma(2\tau+1)/\sqrt{2})/\sqrt{2\pi}\sigma$. Using $\tau = 2$ gives $\varepsilon(\sigma,2) \leq 5.68 \cdot 10^{-9}$ for $\sigma \geq 0.4$. Gaussians with $\sigma < 0.4$ can be treated as constant functions in frequency since $\hat{G}'_\sigma(\omega) \approx 0$ for $\omega \in [-\pi,\pi]$. □

Theorem 2. *The solution f to the edge-aware Tikhonov regularization problem satisfies*

$$f = \sum_{k=0}^{\infty} (\mathcal{M}_x + \mathcal{M}_y)^k [g_H] .$$

Proof. Since $(\mathcal{I} - (\mathcal{M}_x + \mathcal{M}_y))[f] = g_H$, if $\mathcal{I} - (\mathcal{M}_x + \mathcal{M}_y)$ is invertible, then

$$f = (\mathcal{I} - (\mathcal{M}_x + \mathcal{M}_y))^{-1}[g] .$$

This inverse does in fact exist since

$$\begin{aligned} \|\mathcal{M}_x + \mathcal{M}_y\|_\infty &\leq \max_{x,y}(p_F + p_B + p_U + p_D)(x,y) \\ &= \max_{x,y} \frac{H(x,y) - 1}{H(x,y)} < 1 . \end{aligned} \tag{1}$$

Then we can write the solution f as a Neumann series $f = \sum_{k=0}^{\infty} (\mathcal{M}_x + \mathcal{M}_y)^k [g_H]$. □

Theorem 3. *The filters $\mathcal{T}_x$ and $\mathcal{T}_y$ can be implemented with recursive filtering.*

Proof. Given an input $\psi(x,y)$ we want to compute a single iteration $\phi(x,y) = \mathcal{T}_x[\psi](x,y)$. This is

$$\phi = (\mathcal{M}_x + \mathcal{M}_x^2 + \mathcal{M}_x^3 + ...)[\psi] ,$$

which can be written as $\phi = (\mathcal{I} - \mathcal{M}_x)^{-1}\mathcal{M}_x[\psi]$ so that $(\mathcal{I} - \mathcal{M}_x)[\phi] = \mathcal{M}_x[\psi]$. This gives

$$\begin{aligned} \phi(x,y) &- p_B(x,y)\phi(x-1,y) \\ &- p_F(x,y)\phi(x+1,y) = \mathcal{M}_x[\psi](x,y) . \end{aligned} \tag{2}$$

We implement this filter using a normalization factor $\eta(x,y)$, and a first-order backward pass filter $\mathcal{B}_x$ followed by a first-order forward pass filter $\mathcal{F}_x$. This is $\phi = \frac{1}{\eta}\mathcal{F}_x\mathcal{B}_x[\mathcal{M}_x[\psi]]$, reversing both filters gives

$$\mathcal{B}_x^{-1}\mathcal{F}_x^{-1}[\eta \cdot \phi] = \mathcal{M}_x[\psi] . \tag{3}$$

Consider the filters defined by the recursions

$$\begin{aligned} \mathcal{B}_x[u](x,y) &= u(x,y) + B(x,y)\mathcal{B}_x[u](x+1,y) \\ \mathcal{F}_x[u](x,y) &= u(x,y) + F(x,y)\mathcal{F}_x[u](x-1,y) . \end{aligned} \tag{4}$$

We can also compute the inverse of the filters $\mathcal{B}_x$ and $\mathcal{F}_x$ by

$$\begin{aligned} \mathcal{B}_x^{-1}[u](x,y) &= u(x,y) - B(x,y)u(x+1,y) \\ \mathcal{F}_x^{-1}[u](x,y) &= u(x,y) - F(x,y)u(x-1,y) . \end{aligned} \tag{5}$$

Substituting Eq. 5 in Eq. 3 yields

$$\begin{aligned} \eta(x,y)\phi(x,y)\,(1 &+ F(x+1,y)B(x,y)) \\ &- F(x,y)\eta(x-1,y)\phi(x-1,y) \\ &- B(x,y)\eta(x+1,y)\phi(x+1,y) = \mathcal{M}_x[\psi](x,y) . \end{aligned} \tag{6}$$

Equating Eq. 6 and Eq. 2 gives us the system of equations

$$\begin{cases} \eta(x,y) + \eta(x,y)F(x+1,y)B(x,y) &= 1 \\ F(x,y)\eta(x-1,y) &= p_B(x,y) \\ B(x,y)\eta(x+1,y) &= p_F(x,y) . \end{cases} \tag{7}$$

The last two equations can be used to obtain in the first one

$$\eta(x,y) = 1 - \frac{p_B(x+1,y)p_F(x,y)}{\eta(x+1,y)} ,$$

so that η can be computed by recursion. The terms B and F can be obtained from η. Finally, using Eq. 4 we can compute $\phi = \mathcal{F}_x\mathcal{B}_x[\mathcal{M}_x[\psi]]/\eta$ by

$$\mathrm{bwd}(x,y) = \mathcal{M}_x[\psi](x,y) + B(x,y)\mathrm{bwd}(x+1,y) ,$$

$$\mathrm{fwd}(x,y) = \mathrm{bwd}(x,y) + F(x,y)\mathrm{fwd}(x-1,y) ,$$

and

$$\phi(x,y) = \mathrm{fwd}(x,y)/\eta(x,y) ,$$

where bwd and fwd are the functions $\mathrm{bwd}(x,y) = \mathcal{B}_x[\mathcal{M}_x[\psi]]$ and $\mathrm{fwd}(x,y) = \mathcal{F}_x\mathcal{B}_x[\mathcal{M}_x[\psi]]$. Therefore we can apply $\mathcal{T}_x$ in $O(N)$. Implementing $\mathcal{T}_y$ is completely analogous. □

Theorem 4. *Consider the matrices $A_{cg} = \mathcal{D}(\mathcal{I} - \mathcal{M}_x - \mathcal{M}_y)$ and $A_{rec} = \mathcal{I} - \mathcal{T}_y\mathcal{T}_x$. If the weights are $W_x = \lambda_x$ and $W_y = \lambda_y$ and the boundary conditions are circular, then the condition numbers of the matrices are given by $\kappa_{cg} = 1 + 4\lambda_x + 4\lambda_y$ and*

$$\kappa_{rec} = \max\left\{\frac{(2\lambda_x+1)(4\lambda_y+1)}{2\lambda_x + 4\lambda_y + 1}, \frac{(2\lambda_y+1)(4\lambda_x+1)}{2\lambda_y + 4\lambda_x + 1}\right\} .$$

Proof. The matrix A_{cg} is normal since it is equal to its transpose, so that $\kappa_{\mathrm{cg}} = \sigma_{\max}(A_{\mathrm{cg}})/\sigma_{\min}(A_{\mathrm{cg}})$, where $\sigma_{\max}(A_{\mathrm{cg}})$ and $\sigma_{\min}(A_{\mathrm{cg}})$ are the maximal, and minimal eigenvalues of A_{cg}. Since the weights are homogeneous, A_{cg} is a LTI filter with kernel K_d satisfying $A_{\mathrm{cg}}[\psi] = K_{\mathrm{cg}} * \psi$, with DTFT

$$\hat{K}_{\mathrm{cg}}(\omega,\xi) = 1 + 2\lambda_x(1-\cos(\omega)) + 2\lambda_y(1-\cos(\xi)) .$$

For images with bounded support $[0,M]\times[0,M]$, we can use the functions $e^{i\omega_j x + i\xi_j y}$ with $j = 0,...,M-1$ as a basis by Discrete Fourier Transform inversion, with $\omega_j = 2\pi j/M$ and $\xi_j = 2\pi j/M$. Further, $A_{\mathrm{cg}}[e^{i\omega_j x + i\xi_j y}] = \hat{K}_{\mathrm{cg}}(\omega_j,\xi_j)e^{i\omega x + i\xi y}$, so that $\hat{K}_{\mathrm{cg}}$ gives all the possible eigenvalues of A_{cg}. It is then easy to see that $\kappa_{\mathrm{cg}} \leq 1 + 4\lambda_x + 4\lambda_y$. Equality holds for even-sized images and is approximate for odd-sized images due to aliasing.

We proceed similarly to compute κ_{rec}. Using that $\mathcal{T}_x[\psi] = \phi$ satisfies

$$\phi(x,y) - p_B(x,y)\phi(x-1,y) - p_F(x,y)\phi(x+1,y) = \mathcal{M}_x[\psi](x,y)\,.$$

(and the analogous equation for $\mathcal{T}_y$), we can apply the DTFT to obtain $A_{\text{rec}}[\psi] = K_{\text{rec}} * \psi$ where

$$\hat{K}_{\text{rec}}(\omega,\xi) = \left(1 - \frac{4\lambda_x\lambda_y\cos(\omega)\cos(\xi)}{(1+2\lambda_x(1-\cos(\omega))+2\lambda_y)(1+2\lambda_x+2\lambda_y(1-\cos(\xi)))}\right)^{-1}\,.$$

It is clear that $\hat{K}_{\text{rec}}(\omega,\xi) \le \hat{K}_{\text{rec}}(0,0)$. For the lower bound, we can manually compute the derivatives of $\hat{K}_{\text{rec}}$ and see that the minima can only occur in $(0,\pi)$ or $(\pi,0)$ so that $\hat{K}_{\text{rec}}(\omega,\xi) \ge \max\{\hat{K}_{\text{rec}}(0,\pi), \hat{K}_{\text{rec}}(\pi,0)\}$. Using that $\hat{K}_{\text{rec}}$ gives the eigenvalues of A_{rec} we finish the proof. □

Theorem 5. *Let X^i, for $i = 0, 1, ...$ be a random walk Markov chain process of a particle in the 2D discrete plane $\mathbb{Z}\times\mathbb{Z}$ with probability $p_B(x,y)$ of moving backward, $p_F(x,y)$ of moving forward, $p_D(x,y)$ of moving downward, $p_U(x,y)$ of moving upwards, and $\frac{1}{H(x,y)}$ of terminating the random walk at each iteration. Let the impulse $\delta(x-u,y-v)$ be the input to the non-homogeneous problem. Then, the impulse response is given by the probability of the random walk terminating at (u,v) given that it started at (x,y)*

$$K(x,y,u,v) = Prob\{X^{final} = (u,v)|X^0 = (x,y)\}\,.$$

This means that the filtered image is given by $f(x,y) = \sum_{u,v\in\mathbb{Z}} K(x,y,u,v)g(u,v)$. Further, K is symmetrical, so that

$$K(x,y,u,v) = Prob\{X^{final} = (x,y)|X^0 = (u,v)\}\,.$$

Proof. By Theorem 2 we have

$$K(x,y,u,v) = \sum_{j=0}^{\infty}(\mathcal{M}_x+\mathcal{M}_y)^j\mathcal{D}^{-1}\delta(x-u,y-v)\,. \tag{8}$$

Let $K_n(x,y,u,v) = (\mathcal{M}_x+\mathcal{M}_y)^n\ \mathcal{D}^{-1}\delta(x-u,y-v)$. Denote X^{n*} the random variable that gives the last position where the particle was in before it disappeared. We will show that

$$K_n(x,y,u,v) = \text{Prob}\{X^{n^*} = (u,v)|X^0 = (x,y)\}\,. \tag{9}$$

It is easy to see that the statement is true for $K_0(x,y,u,v) = 1/H(x,y)$. Suppose the statement is true for K_{n-1} then

$$\begin{aligned} K_n(x,y,u,v) = \\ &p_B(x,y)K_{n-1}(x-1,y,u,v) \\ &+ p_F(x,y)K_{n-1}(x+1,y,u,v) \\ &+ p_U(x,y)K_{n-1}(x,y+1,u,v) \\ &+ p_D(x,y)K_{n-1}(x,y-1,u,v)\,. \end{aligned} \tag{10}$$

We can then group the terms, for example, notice that our Markov chain definition gives $p_B(x,y) = \text{Prob}\{X^{\text{next}} = (x-1,y)|X^{\text{prev}} = (x,y)\}$. Therefore

$$\begin{aligned} p_B(x,y)K_{n-1}(x-1,y,u,v) = \\ &\text{Prob}\{X^{\text{next}} = (x-1,y)|X^{\text{prev}} = (x,y)\} \\ &\text{Prob}\{X^{(n-1)^*} = (u,v)|X^0 = (x-1,y)\}\,. \end{aligned} \tag{11}$$

By shifting the indexes, this is the same as

$$\begin{aligned} p_B(x,y)K_{n-1}(x-1,y,u,v) = \\ &\text{Prob}\{X^1 = (x-1,y)|X^0 = (x,y)\} \\ &\text{Prob}\{X^{n^*} = (u,v)|X^1 = (x-1,y)\}\,, \end{aligned} \tag{12}$$

which is $\text{Prob}\{X^{n^*} = (u,v), X^1 = (x-1,y)|X^0 = (x,y)\}$. This is the probability of, given that the random walk started in (x,y), passing through $(x-1,y)$ on the first iteration, and then passing through (u,v) on the n-th iteration, and finally terminating the random walk. If we include the other terms from Eq. 10, we are actually summing over all possible directions that the first iteration could have gone through, so that their sum is simply

$$K_n(x,y,u,v) = \text{Prob}\{X^{n^*} = (u,v)|X^0 = (x,y)\}\,.$$

Eq. 8 gives $K(x,y,u,v) = \sum_{n=0}^{\infty} K_n(x,y,u,v)$ so that we are summing over all the paths that terminate in (x,y) independently of the number of steps taken. This gives

$$K(x,y,u,v) = \text{Prob}\{X^{n^*} = (u,v), \text{for some } n|X^0 = (x,y)\}\,.$$

Which can be written as $K(x,y,u,v) = \text{Prob}\{X^{\text{final}} = (u,v)|X^0 = (x,y)\}$. Finally, to see that the kernel is symmetrical, note that it is the inverse of a symmetrical matrix so it must also be symmetrical. This concludes the proof □

Theorem 6. *The condition number κ_{cg} satisfies*

$$\kappa_{cg} \le \max_{x,y}\left|H(x,y)(1+p_B(x,y)+p_F(x,y)+p_U(x,y)+p_D(x,y))\right|,$$

which does not depend on the dimensions of A_{cg}. Therefore, the convergence rate of the CG method with A_{cg} has a bound which does not depend on the size of the input images. Similarly, κ_{rec} can be bounded independently of the matrix dimensions.

Proof. By definition, $\kappa_{\text{cg}} = \|A_{\text{cg}}\|_2\|A_{\text{cg}}^{-1}\|_2$. We observe that

- For any matrix A, $\|A\|_2 \le \sqrt{\|A\|_\infty\|A\|_1}$;
- A_{cg} and A_{cg}^{-1} are symmetric;
- $\|A\|_\infty$ is the maximum absolute column sum, and $\|A\|_1$ is the maximum absolute row sum ;
- Denoting the matrix row i corresponding to (x,y) and the column j corresponding to (u,v),

$$A_{\text{cg}\,i,j}^{-1} = \text{Prob}\{X^{\text{final}} = (u,v)|X^0 = (x,y)\}\,.$$

Since both $A_{\rm cg}$ and $A_{\rm cg}^{-1}$ are symmetrical, their respective ℓ_1 and ℓ_∞ norms are equal. We get $\kappa_{\rm cg} \le \|A_{\rm cg}\|_\infty \|A_{\rm cg}^{-1}\|_\infty$

$$\begin{aligned}\kappa_{\rm cg} \le \max_{x,y} \Big\{ |H(x,y)(1 + p_B(x,y) + p_F(x,y) \\ + p_U(x,y) + p_D(x,y))| \Big\} \\ \max_{x,y} \sum_{u,v} \text{Prob}\{X^{\rm final} = (u,v) | X^0 = (x,y)\}\,.\end{aligned}$$

Since the sum is over a probability function, it is less than or equal to 1. So that

$$\begin{aligned}\kappa_{\rm cg} \le \max_{x,y} \Big(|H(x,y)(1 + p_B(x,y) + p_F(x,y) \\ + p_U(x,y) + p_D(x,y))| \Big)\,.\end{aligned}$$

Therefore, as long as the weights are bounded, the condition number does depend on the size of the matrices. We conclude the proof by observing that the convergence rate of the CG method only depends on $\kappa_{\rm cg}$. The proof that $\kappa_{\rm rec}$ does not depend on the size is similar but with more specific details. □

Appendix B

$$\begin{array}{llll} a_0 = 1.6800 & a_1 = 3.7350 & w_0 = 0.6318 & b_0 = 1.7830 \\ c_0 = -0.6803 & c_1 = -0.2598 & w_1 = 1.9970 & b_1 = 1.7230 \end{array}$$

$$\begin{aligned}
n_0 &= a_0 + c_0 \\
n_1 &= e^{-\frac{b_1}{\sigma}} \left(c_1 \sin(\tfrac{w_1}{\sigma}) - (c_0 + 2a_0)\cos\left(\tfrac{w_1}{\sigma}\right)\right) + \\
&\quad e^{-\frac{b_0}{\sigma}} \left(a_1 \sin\left(\tfrac{w_0}{\sigma}\right) - (2c_0 + a_0)\cos\left(\tfrac{w_0}{\sigma}\right)\right) \\
n_2 &= 2e^{-\frac{b_0}{\sigma}-\frac{b_1}{\sigma}} \left((a_0 + c_0)\cos\left(\tfrac{w_1}{\sigma}\right)\cos\left(\tfrac{w_0}{\sigma}\right) - \right. \\
&\quad \left. \cos\left(\tfrac{w_1}{\sigma}\right) a_1 \sin\left(\tfrac{w_0}{\sigma}\right) - \cos\left(\tfrac{w_0}{\sigma}\right) c_1 \sin\left(\tfrac{w_1}{\sigma}\right)\right) \\
&\quad + c_0 e^{-2\frac{b_0}{\sigma}} + a_0 e^{-2\frac{b_1}{\sigma}} \\
n_3 &= e^{-\frac{b_1}{\sigma}-2\frac{b_0}{\sigma}} \left(c_1 \sin\left(\tfrac{w_1}{\sigma}\right) - \cos\left(\tfrac{w_1}{\sigma}\right) c_0\right) + \\
&\quad e^{-\frac{b_0}{\sigma}-2\frac{b_1}{\sigma}} \left(a_1 \sin\left(\tfrac{w_0}{\sigma}\right) - \cos\left(\tfrac{w_0}{\sigma}\right) a_0\right) \\
d_1 &= -2e^{-\frac{b_1}{\sigma}} \cos\left(\tfrac{w_1}{\sigma}\right) - 2e^{-\frac{b_0}{\sigma}} \cos\left(\tfrac{w_0}{\sigma}\right) \\
d_2 &= 4\cos\left(\tfrac{w_1}{\sigma}\right)\cos\left(\tfrac{w_0}{\sigma}\right) e^{-\frac{b_0}{\sigma}-\frac{b_1}{\sigma}} + e^{-2\frac{b_1}{\sigma}} + e^{-2\frac{b_0}{\sigma}} \\
d_3 &= -2\cos\left(\tfrac{w_0}{\sigma}\right) e^{-\frac{b_0}{\sigma}-2\frac{b_1}{\sigma}} - 2\cos\left(\tfrac{w_1}{\sigma}\right) e^{-\frac{b_1}{\sigma}-2\frac{b_0}{\sigma}} \\
d_4 &= e^{-2\frac{b_0}{\sigma}-2\frac{b_1}{\sigma}} \\
\eta_0 &= n_0 - d_1^2 n_0 - d_2^2 n_0 - d_3^2 n_0 \\
&\quad - d_4^2 n_0 + 2d_1 n_1 + 2d_2 n_2 + 2d_3 n_3 \\
\eta_1 &= -d_1 d_2 n_0 - d_2 d_3 n_0 - d_3 d_4 n_0 + d_2 n_1 + d_1 n_2 \\
&\quad + d_3 n_2 + d_2 n_3 + d_4 n_3 + n_1 \\
\eta_2 &= -d_1 d_3 n_0 - d_2 d_4 n_0 + d_3 n_1 + d_4 n_2 + d_1 n_3 + n_2 \\
\eta_3 &= -d_1 d_4 n_0 + d_4 n_1 + n_3
\end{aligned}$$

Appendix C

Here we show in detail how to derive the elliptical symmetry for the kernel K in the spatial domain. We have

$$\hat{K}(\omega,\xi) = \frac{1}{1 + 2\lambda_x(1-\cos(\omega)) + 2\lambda_y(1-\cos(\xi))}\,.$$

Consider the change of variables

$$\begin{cases} l(\omega,\xi) = \lambda_x \cos(\omega) + \lambda_y \cos(\xi), \\ \theta(\omega,\xi) = \tan^{-1}(\xi \,/\, \omega). \end{cases} \tag{13}$$

Using the inverse function theorem we have

$$\begin{bmatrix} \frac{\partial\theta}{\partial\omega} & \frac{\partial\theta}{\partial\xi} \\ \frac{\partial l}{\partial\omega} & \frac{\partial l}{\partial\xi} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial\omega}{\partial\theta} & \frac{\partial\omega}{\partial l} \\ \frac{\partial\xi}{\partial\theta} & \frac{\partial\xi}{\partial l} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{14}$$

then we can obtain the derivatives of ω and ξ with respect to l and θ by reversing the first matrix in the LHS of Eq. 14

$$\begin{aligned}\begin{bmatrix} \frac{\partial\omega}{\partial\theta} & \frac{\partial\omega}{\partial l} \\ \frac{\partial\xi}{\partial\theta} & \frac{\partial\xi}{\partial l} \end{bmatrix} = \frac{\omega\lambda_x \sin(\omega) + \xi\lambda_y \sin(\xi)}{\omega^2 + \xi^2} \\ \cdot \begin{bmatrix} -\lambda_y \sin(\xi) & -\frac{\omega}{\omega^2+\xi^2} \\ \lambda_x \sin(\omega) & -\frac{\xi}{\omega^2+\xi^2} \end{bmatrix}.\end{aligned} \tag{15}$$

The function $\hat{K}$ can be written as a function of l and θ as

$$\hat{K}(\omega,\xi) = \frac{1}{1 + 2(\lambda_x + \lambda_y + l(\omega,\xi))}$$

To see that $\frac{d\hat{K}}{d\theta} = 0$, notice that $\frac{dl}{d\theta} = 0$ because

$$\frac{dl}{d\theta} = \frac{\partial l}{\partial\omega}\frac{\partial\omega}{\partial\theta} + \frac{\partial l}{\partial\xi}\frac{\partial\xi}{\partial\theta}$$

and this is 0 according to Eq. 14 (see the second row and first column of the RHS). Therefore

$$\frac{d\hat{K}}{d\theta} = \frac{\partial\hat{K}}{\partial\omega}\frac{\partial\omega}{\partial\theta} + \frac{\partial\hat{K}}{\partial\xi}\frac{\partial\xi}{\partial\theta} = 0. \tag{16}$$

Using Eq. 15, Eq. 16 can be written as

$$-\lambda_y \frac{\partial\hat{K}}{\partial\omega}\sin(\xi) + \lambda_x \frac{\partial\hat{K}}{\partial\xi}\lambda_x \sin(\omega) = 0\,. \tag{17}$$

Using the identities

$$\text{DTFT}\left[\delta(x+1) - \delta(x+1)\right](\omega) = 2\sin(\omega)\,,$$

$$\text{DTFT}\left[x f(x)\right](\omega) = \sqrt{-1}\hat{f}(\omega)\,,$$

and the convolution property

$$\text{DTFT}\left[f * g\right](\omega) = \hat{f}(\omega)\hat{g}(\omega)\,,$$

we apply the inverse DTFT in Eq. 17 to obtain

$$\begin{aligned}\lambda_x(yK(x,y)) * (\delta(x+1) - \delta(x-1)) \\ - \lambda_y(xK(x,y)) * (\delta(y+1) - \delta(y-1)) = 0\end{aligned} \tag{18}$$

finally, expanding the convolution in Eq. 18 gives

$$\begin{aligned}\lambda_x\, y\, (K(x+1,y) - K(x-1,y)) \\ -\lambda_y\, x\, (K(x,y+1) - K(x,y-1)) = 0.\end{aligned}$$