Introduction

Since 2003, the Robert Gillespie Academic Skills Centre has offered Peer Facilitated Study Groups, in collaboration with faculty and students, with the purpose of enhancing the student learning experience at UTM. The first of their kind at the University of Toronto, the Peer Facilitated Study Groups are based on the Supplemental Instruction model developed by the University of Missouri at Kansas City. The study groups give students the opportunity to study discipline-related material in an organized, regularly-scheduled format. The study groups practice skills such as comparing and reviewing study notes; discussing and clarifying concepts with peers; developing and evaluating study strategies; preparing for tests and exams. (From `http://goo.gl/x5muc1`)

Example: Proofs Bingo!

Prove that for two real numbers x and y: $$\lvert x-y\rvert \geq \lvert x\rvert - \lvert y\rvert$$	Free Space :)	Consider the sets: $$\begin{cases} A = (0,3) \cap \mathbb{Z} \\ B = [0,3] - (0,3) \end{cases}$$ Find $A \times B$.	Write this set as a union of intervals: $$A = \{x \in \mathbb{R} \mid x^2 - 3x + 2 < 0\}$$
Find a function whose domain is the set of all even integers and whose image is the set of all odd integers.	Let C and D be sets in the domain of g. Prove: $$g(C \cap D) \subseteq g(C) \cap g(D)$$	Prove the following for three real numbers x, y, and z: $$\lvert x+y+z\rvert \leq \lvert x\rvert + \lvert y\rvert + \lvert z\rvert$$	Free Space :)
Free Space :)	Prove the image of the function f is the set (0, 1], for: $$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{x^2+1}$$	Show that for any two sets A and B: $$(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$$	Let C and D be subsets of the domain of $f : A \to B$. If $f(A) = f(B)$, is it necessarily true $A = B$? Provide a counterexample!
When does equality hold for the triangle inequality? In other words, for what values of x and y does $$\lvert x+y\rvert = \lvert x\rvert + \lvert y\rvert$$	Define the following sets: $$\begin{cases} C = [1,2] \times [2,3] \\ D = \mathbb{Z} \times \mathbb{Z} \end{cases}$$ Find $C \cap D$	Free Space :)	Find the image of the function below: $$h : \mathbb{N} \to \mathbb{Z}, \quad h(n) = \lvert n-3\rvert$$

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Notes and Rough Work

All About Fields

In a field F, with operations $+$ and $\cdot$ and distinguished identity elements 0 and 1, the following properties hold for all $x, y, z \in F$:

A0 $x + y \in F$	**M0** $x \cdot y \in F$	*Closure*
A1 $(x + y) + z = x + (y + z)$	**M1** $(x \cdot y) \cdot z = x \cdot (y \cdot z)$	*Associativity*
A2 $x + y = y + x$	**M2** $x \cdot y = y \cdot x$	*Commutativity*
A3 $x + 0 = x$	**M3** $x \cdot 0 = x$	*Identity*
A4 For any x, there exists a $w \in F$ such that $x + w = 0$	**M4** For any $x \neq 0$, there exists a $w \in F$ such that $x \cdot w = 1$	*Inverse*
	DL $x \cdot (y + z) = x \cdot y + x \cdot z$	*Distributivity*

Some Questions

Field Construction

Consider the three element field $F = \{0, 1, \omega\}$ with operations $+$ and $\cdot$. Fill out the addition and multiplication tables below:

$+$	0	1	ω
0			
1			
ω			

$\cdot$	0	1	ω
0			
1			
ω			

Field Conjectures

Perplexed Pat thinks he has all of this figured out. For each of his statements below, figure out whether or not Pat is correct, and either provide a proof or counterexample. He says that for a field $(S, +, \cdot, 0, 1)$...

1. For any $x \in S$, we have $x \cdot 0 = 0$
2. $y \cdot 1^{-1} = y^{-1}$ for any $y \neq 0$ in S

Field Logic

Let $(F, +, \cdot, 0, 1)$ be a field. Figure out what each of the following statements and whether or not they are true. A proof is unnecessary, but you need to be able to justify your answer.

1. $(\exists v \in F)(\forall w \in F - \{0\})(v \cdot w = 1)$
2. $(\forall v \in F - \{0\})(\exists w \in F)(v \cdot w = 1)$

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1 Induction in a Nutshell

In order to prove some statement $P(n)$ for $n \in \mathbb{N}$, either weak induction or strong induction can be used.

Weak Induction

1. *Base Case:* The statement $P(1)$ is true.
2. *Induction Step:* If the statement $P(k)$ is true for any $k \in \mathbb{N}$, then $P(k+1)$ must be true.

Strong Induction

1. *Base Case:* The statement $P(1)$ is true.
2. *Induction Step:* If the statements $P(1), P(2), \ldots, P(k)$ are true, then $P(k+1)$ must be true.

2 Strategy

1. Base Case
   - Explicitly state what you are checking.
   - If your base case does not work out, you messed something up!
2. Induction Step
   - Clearly write down your assumption. "Assume for any $k \in \mathbb{N}$..."
   - It is helpful to write out what you want to show (i.e. $P(k+1)$)
   - Make sure you use the induction hypothesis: if you did not need to, you probably did it wrong.

3 Some Questions

Question 1

Find the flaw with the following proof by induction:

1. Base Case: The number 0 is even.
2. Induction Step: Assume $0, 1, \ldots k$ are even. Now consider that any $(k+1)$ can be written as a sum of two smaller integers, l and m. By the induction hypothesis, both of these are even. This means $k+1 = l+m$ is even as well.

Question 2

Prove that $6|(7^n - 1)$ for all positive integers n.

Question 3

Show that for all positive integers n:

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

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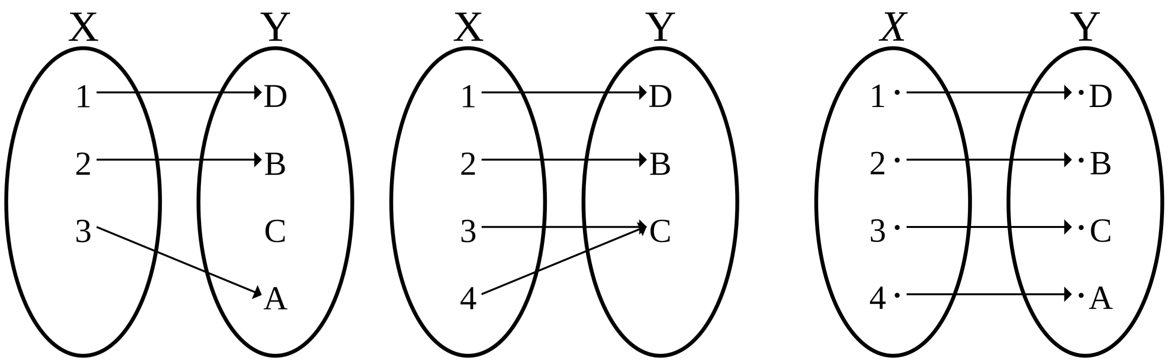

From left to right: An injective, surjective and bijection function $X \to Y$.

Related Questions

Function Composition

Assume $f : B \to C$, $g : A \to B$, and $(f \circ g) : A \to C$ are functions. Suppose we know $f \circ g$ whether injective, surjective or bijective: what can we conclude about f or g? Complete the table below:

$f \circ g$	f	g
Injective		
Surjective		
Bijective		

Induction and Injections

Assume the function $h : S \to S$ has the property that no two different elements of S have the same output. Prove that for any natural number $n \geq 2$:

$$\underbrace{(f \circ f \circ \cdots \circ f)}_{n \text{ times}} \text{ is injective}$$

In other words, prove f composed with itself n times is injective.

Strong Induction

Suppose the Canadian government considers only accepting 4 cent and 5 cent stamps to make the postage. A proponent of the legislation is quoted as saying:

> This change will not make a substantive difference for hardworking Canadians. Ultimately, any amount in postage above ___________ cents can be paid in full with only these two stamps. The existence of other unnecessary stamps is the epitome of wasteful government ineptitude.

Fill in the blank, and prove that the statement is true.

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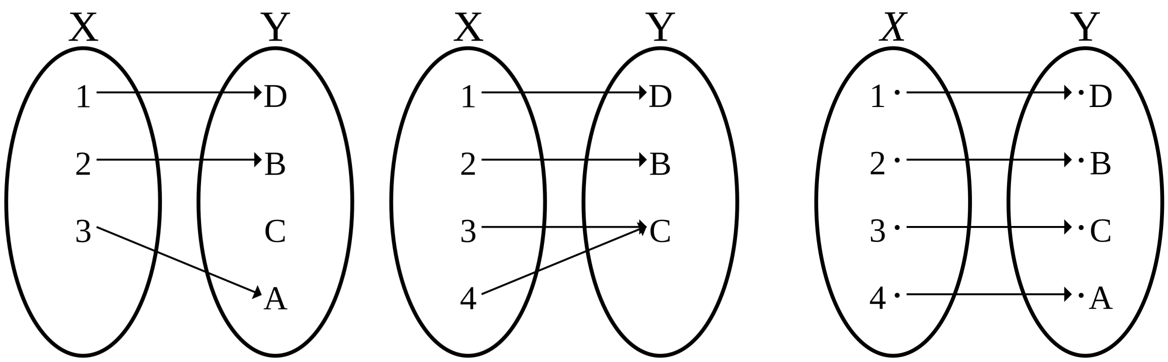

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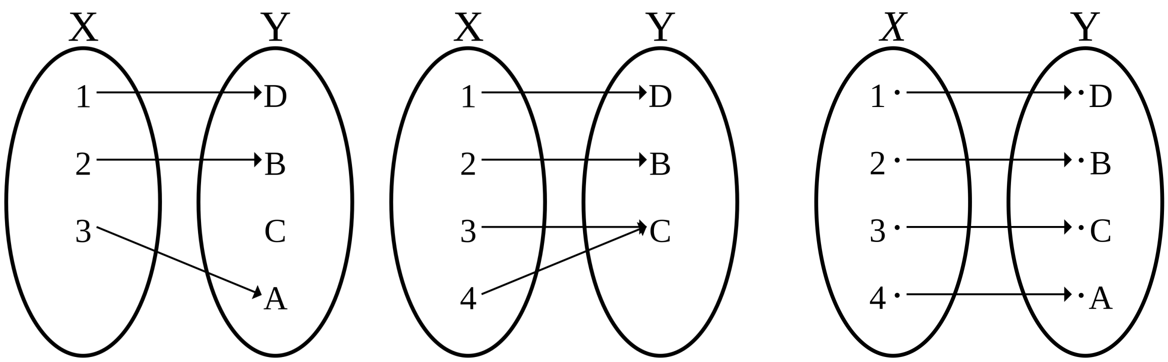

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A Proof

Statement

Prove that a function, f, mapping a set to its power set, can be injective but not surjective.

Proof

Define $f : X \to P(X)$ to be such a mapping from any set X, to its power set, $P(X)$.

1. Prove that $f : X \to P(X)$ can be an injective function.
   Strategy: Construct such a function!

2. Prove that $f : X \to P(X)$ cannot be a surjective function.
   Strategy: Use proof by contradiction.

Assume: $f : X \to P(X)$ is a surjective function. Therefore for all $Y \subseteq X$, there exists some $x \in X$ such that $f(x) = Y$. Now consider the set:

$$\{x : x \notin f(x)\}$$

More Functions

Prove that $f : X \to P(X)$ cannot be a surjective function.

Hint: Consider $f : X \to \mathcal{P}(X)$ is a surjective function. Therefore for all $Y \subseteq \mathcal{P}(X)$, there exists some $x \in X$ such that $f(x) = Y$. Now consider the set $\{x : x \notin f(x)\}$...

Fill in the Blanks

Consider some set A and its power set $\mathcal{P}(A)$. Fill in the blanks with the options provided on the next line.

1. A is a ___________ of $\mathcal{P}(A)$.
   `subset; element`

2. If A is countably infinite, then $\mathcal{P}(A)$ is ___________
   `finite; countably infinite; uncountable`

3. There ___________ bijection from $A \to \mathcal{P}(A)$.
   `cannot be; might exist some`

4. If A is an uncountable set, then there ___________ a bijective function $f : \mathbb{R} \to A$.
   `must exist; there possibly exists`

Countable Sets

Assume S is a countably infinite set. Prove the set $R = S \times \{0, 1\}$ is also a countably infinite set. To illustrate what S and P can be:

$$\text{If } S \text{ was the set } \{1, 2, 3, \dots\}, \text{ then } R \text{ would be the set } \{(1, 0), (1, 1), (2, 0), (2, 1), (3, 0), \dots\}$$

Hint: How would you construct a bijection between R and the natural numbers?

Uncoutable Sets

An *infinite integer sequence* is an infinite ordered list of integers. Some examples of infinite integer sequences are:

$$\begin{array}{rrrrrr} 0, & -3, & 6, & -9, & 12, & \dots \\ 1, & 2, & 4, & 8, & 16, & \dots \\ -1, & 1, & -1, & 1, & -1, & \dots \\ 2, & 2, & 2, & 2, & 2, & \dots \end{array}$$

Prove that the set of all possible infinite integer sequences is uncountable.
Hint: Use the diagonalization argument!

Quiz Preparation

Fill in the Blanks

Consider some set A and its power set $\mathcal{P}(A)$. Fill in the blanks with the options provided on the next line.

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   `finite; countably infinite; uncountable`

3. There ___________ bijection from $A \to \mathcal{P}(A)$.
   `cannot be; might exist some`

4. $\gcd(68, 102) =$ ___________
   `1, 17, 34, 68`

5. If A is an uncountable set, then there ___________ a bijective function $f : \mathbb{R} \to A$.
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Prove that the set of all possible infinite integer sequences is uncountable.
Hint: Use the diagonalization argument!

Divisibility

Given two positive integers a and b, prove or provide a counterexample to the claim

$$\gcd(2a, b) = 2\gcd(a, b)$$

Hint: Plug in a couple a values for a and b.

Problems

Some Questions

Find a integer solutions to the equation:

$$6x - 15y = 369369369 \tag{1}$$

Assume $\gcd(a, b) = 2$, find the following (if possible):

1. $\gcd(a^2, b^2)$
2. $\gcd(\frac{a}{2}, \frac{b}{2})$

More Questions

Which of the following relations, R, are equivalence relations:

1. On natural numbers, $(x, y) \in R$ if and only if $\gcd(x, y) = 1$
2. On all UTM students $(x, y) \in R$ if and only if x and y have the same GPA.

Prove there are no positive natural number solutions to the following equations:

$$ab + a + b = 100, \text{ for } a, b \in \mathbb{N} \tag{2}$$