COURSEWORK 2

1. By definition (from lecture notes),

$$\begin{aligned} T_k(x) &= \cos(k\cos^{-1}(x)) \\ &= cos(kx), cos(\theta) = x \end{aligned}$$

Also note that:

$$\begin{aligned} &\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \\ \implies &\begin{cases} \cos(k\theta + \theta) = \cos(k\theta)\cos(\theta) - \sin(k\theta)\sin(\theta) \\ \cos(k\theta - \theta) = \cos(k\theta)\cos(\theta) + \sin(k\theta)\sin(\theta) \end{cases} \\ \implies &\cos(k\theta + \theta) + \cos(k\theta - \theta) = 2\cos(k\theta)\cos(\theta) \\ \implies &T_{k+1}(x) + T_{k-1}(x) = 2T_k(x)x \\ \implies &T_{k+1}(x) = 2T_k(x) - T_{k-1}(x) \end{aligned}$$

We then proved in lectures:

$$T'_{k+1}(x) = 2(k+1)T_k(x) + \frac{k+1}{k-1}T'_{k-1}(x)$$

Hence, let's compute its $(p-1)^{th}$ derivatives, noting the coefficients do not depend on x (so they don't change):

$$T^{(p)}_{k+1}(x) = 2(k+1)T^{(p-1)}_k(x) + \frac{k+1}{k-1}T^{(p)}_{k-1}(x)$$

2. Let's use our above recurrence relation in order to solve the boundary value problem of the following Airy equation:

$$y''(x) - xy(x) = 0, -40 \leq x \leq 40$$

With conditions as follows:

$$\begin{aligned} y(40) &= 0 \\ y'(-40) &= 1 \end{aligned}$$

Let's use the collocation technique that we studied in lectures in order to solve this problem, we have the following estimation using Chebyshev polynomials:

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n T_n(x) \\ y'(x) &= \sum_{n=0}^{\infty} a_n T'_n(x) \\ y''(x) &= \sum_{n=0}^{\infty} a_n T''_n(x) \end{aligned}$$

We then replace ∞ by N and eliminate x-dependence by choosing a sequence of N+1 unknowns in [-1,1], here corresponding to our Gauss-Lobatto points $x_j = cos(\frac{j\pi}{N}), j = 0, ..., N$.

$$\implies \sum_{j=0}^{N}\sum_{k=0}^{N} a_k(T_k''(x_j) - x_j T_k(x_j)) = 0 \tag{1}$$

Regarding the range for which x is defined, we have seen in lectures that $-1 \leq x_j = \cos(\frac{j\pi}{N}) \leq 1$. Hence, in order to compute our boundary conditions, we need to make a substitution as follows: $-40 \leq x \leq 40 \implies -1 \leq \frac{x}{40} \leq 1$.

$$\begin{aligned} y''(x) - xy(x) = 0 & \\ &= [x_j = \frac{x}{40}] \\ &= \frac{1}{40^2}y''(x_j) - 40x_j y(x_j) \end{aligned}$$

This is because:

$$\begin{aligned} u = \frac{x}{40} &\implies \frac{du}{dx} = \frac{1}{40} \\ &\implies \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{40}\frac{dy}{du} \\ &\implies \frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(\frac{1}{40}\frac{dy}{du}) = \frac{1}{40}\frac{d^2y}{dxdu} = \frac{1}{40}\frac{d}{du}(\frac{dy}{dx}) = \frac{1}{40^2}\frac{d^2y}{du^2} \end{aligned}$$

We then have, taking into account that substitution into equation (1):

$$\begin{aligned} \sum_{j=0}^{N}\sum_{k=0}^{N} a_k(\frac{1}{40^2}T_k''(x_j) - 40x_j T_k(x_j)) = 0 & \\ & \sum_{j=0}^{N}\sum_{k=0}^{N} a_k D_{jk} = 0, \text{ with } D_{jk} = \frac{1}{40^2}T_k''(x_j) - 40x_j T_k(x_j) \end{aligned}$$

This matrix $(D_{jk})_{j,k=0,1,...,N}$ has rows corresponding to collocation points and columns corresponding to degree of polynomial.

We then compute the matrices corresponding to the degree of differentiation of our Chebyshev:

$$\begin{aligned} T^0(x_j) &= \begin{pmatrix} T_0(x_j) & T_1(x_j) & \dots & T_N(x_j) \end{pmatrix} \\ T^1(x_j) &= \begin{pmatrix} T_0'(x_j) & T_1'(x_j) & \dots & T_N'(x_j) \end{pmatrix} \\ T^2(x_j) &= \begin{pmatrix} T_0''(x_j) & T_1''(x_j) & \dots & T_N''(x_j) \end{pmatrix} \end{aligned}$$

Now, let's compute the first entries of these matrices by hand because our recurrence relation can only start when k=2.

$$\begin{aligned} T_k(x_j) &= \cos(k\cos^{-1}(x_j)) \\ &= \cos(\frac{kj\pi}{N}) \end{aligned}$$

$$
\begin{aligned}
T_k'(x_j) &= \sin(k\cos^{-1}(x_j))\frac{k}{\sqrt{1-x_j^2}} \\
\implies T_0'(x_j) &= 0 \\
\implies T_1'(x_j) &= \frac{\sin(\frac{j\pi}{N})}{\sin(\frac{j\pi}{N})} = 1 \\
\implies T_2'(x_j) &= 2\frac{\sin(\frac{2j\pi}{N})}{\sin(\frac{j\pi}{N})} = 4\cos(\frac{j\pi}{N})
\end{aligned}
$$

$$
\begin{aligned}
T_k''(x_j) &= \sin(k\cos^{-1}(x_j))\frac{kx_j}{(1-x_j^2)^{\frac{3}{2}}} - \cos(k\cos^{-1}(x_j))\frac{k^2}{1-x_j^2} \\
&= \sin(\frac{kj\pi}{N})\frac{kx_j}{\sin^3(\frac{j\pi}{N})} - \cos(\frac{kj\pi}{N})\frac{k^2}{\sin^2(j\pi/N)} \\
\implies T_0''(x_j) &= 0 \\
\implies T_1''(x_j) &= \frac{\sin(\frac{j\pi}{N})}{\sin(\frac{j\pi}{N})} = 1 \\
\implies T_2''(x_j) &= 4
\end{aligned}
$$

The rest can be just calculated using the recurrence relation we computed in part 1; adapted as follows:

$$
\begin{aligned}
T_{k+1}'(x_j) &= 2(k+1)T_k(x_j) + \frac{k+1}{k-1}T_{k-1}'(x_j) \\
T_{k+1}''(x_j) &= 2(k+1)T_k'(x_j) + \frac{k+1}{k-1}T_{k-1}''(x_j)
\end{aligned}
$$

Here's the whole code on Matlab with comments:

```
% STEP 1: Compute values of collocation points x_j=[x_0, x_1, ..., x_n]=cos(j*pi/N)
    % STEP 1.1: Initialize N and vector of zeros for x to fill
N = 200;
x = zeros(1, N+1);
    % STEP 1.2: Loop to update each entry of x_j
for j = 1:N+1
    x(j) = cos((j-1)*pi/N);
end

% STEP 2: Create matrices for the degrees of which our Chebyshev polynomial
% is evaluated
    % STEP 2.1:  T_0 is the 0th derivative of T_k(x_j)
T_0 = zeros(N+1,N+1);
for k = 1:N+1
    for j = 1:N+1
        T_0(j,k) = cos((k-1)*(j-1)*pi/N);
    end
end
```

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    % STEP 2.2: T_1 is the 1st derivative of T_k(x_j)
T_1 = zeros(N+1,N+1);
for j = 1:N+1
    T_1(j,1) = 0;
    T_1(j,2) = 1;
    T_1(j,3) = 4*x(j);
end
for k = 3:N
    for j = 1:N+1
        T_1(j,k+1) = 2*k*T_0(j,k) + (k/(k-2)) * T_1(j,k-1);
    end
end
    % STEP 2.3: T_2 is the 2nd derivative of T_k(x_j)
T_2 = zeros(N+1,N+1);
for j = 1:N+1
    T_2(j,1) = 0;
    T_2(j,2) = 0;
    T_2(j,3) = 4;
end
for k = 3:N
    for j = 1:N+1
        T_2(j,k+1) = 2*k*T_1(j,k) + (k/(k-2)) * T_2(j,k-1);
    end
end

% STEP 3: Compute our collocation matrix D_jn (j rows: collocation
    points, k columns: degree)
    % STEP 3.1: Initialize matrix to fill
D = zeros(N+1,N+1);
    % STEP 3.2: Loop
for k = 1 : N+1
    for j = 1 : N+1
        D(j,k) = (1/(40^2))*T_2(j,k) - 40*x(j).*T_0(j,k);
    end
end

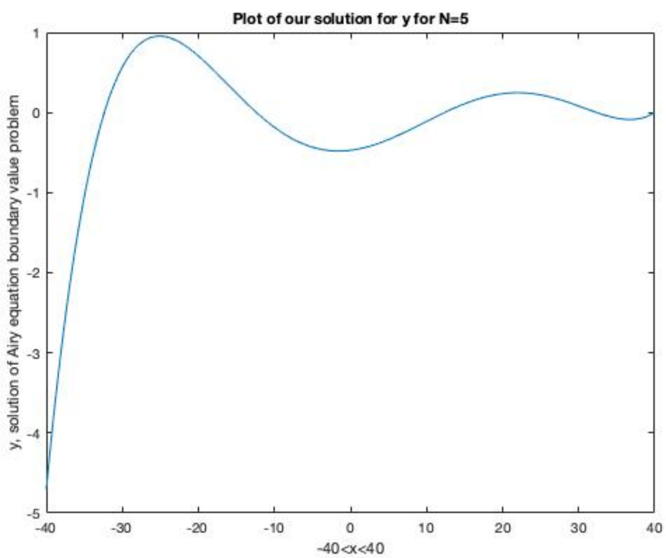
% STEP 4: Update values of D_jn with boundary conditions
    % STEP 4.1: Change 1st row
D(1,:) = 1 ; % as acos(1)=0 =>cos(acos(1))=1=T_n(1)
    % STEP 4.2: Update last row
D(N+1,:) = T_1(N+1,:) ; % as T'_n(-1)=x_N

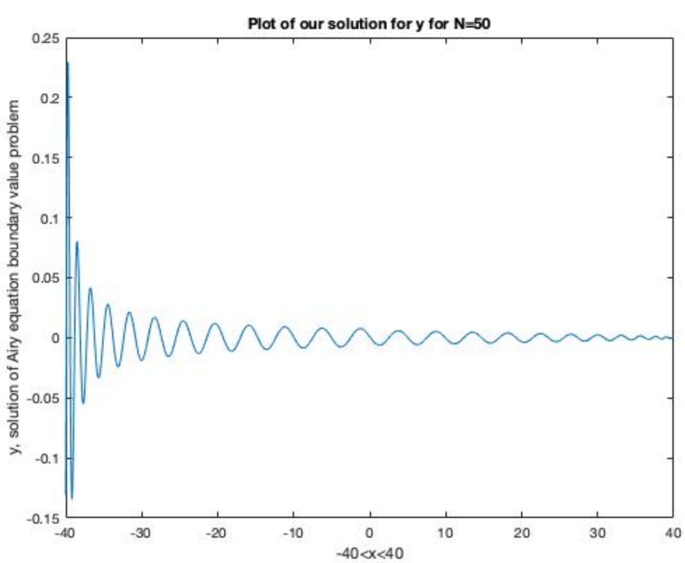
% STEP 5: Compute our coefficients a_n st sum(a_n*D_jn)=0
    % STEP 5.1: Create right hand side matrix for equation D_jn*a_n=b
b = zeros(N+1, 1);
    % STEP 5.2: Update value in rhs matrix
b(N+1)=40;
    % STEP 5.3: Solve equation
a = linsolve(D,b);

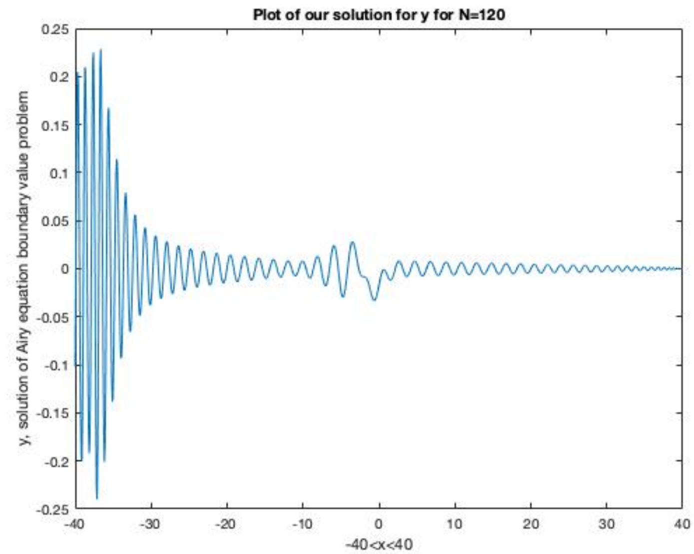
% STEP 6: Plot our solution
    % STEP 6.1: Initialize our space of existence of the function
t = linspace(-40,40,1000);
y = zeros(1,1000);
```

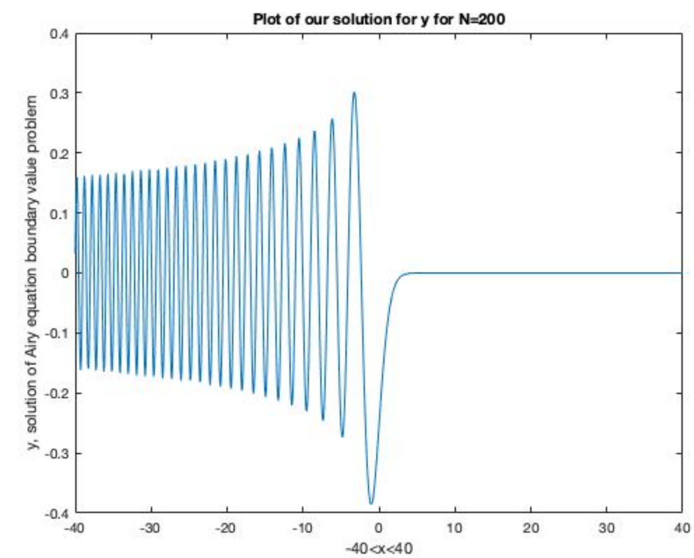
```
    % STEP 6.2: Loop to compute at each point the polynomial
for k = 1:1000
    sum = 0;
    for j = 1:N+1
        sum = sum + a(j)*cos((j-1)*acos(t/40));
    end
    y = sum;
end
    % STEP 6.3: Give title and plot our final solution
plot(t,y)
title('Plot of our solution for y for N=200')
xlabel('-40<x<40')
ylabel('y, solution of Airy equation boundary value problem')
```

Here's our solution for different values of N:






Plot of our solution for y for N=120
y, solution of Airy equation boundary value problem
-40<x<40
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
-40
-30
-20
-10
0
10
20
30
40


Plot of our solution for y for N=200
y, solution of Airy equation boundary value problem
-40<x<40
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-40
-30
-20
-10
0
10
20
30
40