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Discussion 1B

Homework 8

1. Hall's theorem

Hall's theorem stats that in a collection of sets $S_1, S_2, \ldots S_k$, the collection has a system of distinct representatives if and only if $|S_{i1} \cup S_{i2} \cup \ldots \cup S_{im}| \geq$ m, for every subset of $(i_1, i_2, \ldots, i_m)$ of {1, 2, 3, ... , k}. In other words, every subcollection of m sets must contain at least m distinct elements for every 1<=m<=k.

In the context of this problem, we consider a bipartite graph such that the (2n-1) days match to (2n) teams. There is a match from a day to a team if that team wins on the day. A system of distinct representatives is the distinct winning team in each day. In order to show we can successfully find the system of distinct representatives, we have to show for every subcollection of the (2n-1) days, the number of winning teams is larger or equal to the cardinality of the set.

We will prove by contradiction. Take any set of days W, and let's say |W| = m. Assume not all teams won in some day in W. There is at least one team who is lost each day. Suppose we can't find a system of distinct winners: there exists a situation where the number of winning teams in a set of m days W is less than m. Let us denote t_w as the only team which did not win in any day in W. Therefore, t_w has lost to m teams in m days. However, it also means that there are m teams who has won t_w in m days. Since every team plays every other team exactly once the m winner must all be distinct.

We have shown that m distinct winners exist in a set of m days. However, we have assumed the number of winning teams is less than m. Contradiction.

2. Independent set reduced from Clique

First, we need to show that finding an independent set of at least size k' is a NP problem. The solution to the independent-set problem contains k' points. For each pair of points in these k' points, we need to show there is no edge existed between them in G. Since there are k'^2 pairs of vertices in the solution, then it is solvable in polynomial time $O(n^2)$, where n is the number of vertices in G. The independent-set problem belongs to NP.

Then, we show the problem is NP-complete by reduction from Clique problem which is known as NP-complete.

If the graph G has a clique of size k, it means that for every pair of vertices in the clique, there is an edge existed between them. On the other hand, an independent set requires there is at most one vertex which is indent to an edge. Therefore, if we take $\bar{G}$, the compliment of G, the clique in G will become an independent set in $\bar{G}$.

We can show the independent set is reducible from clique problem if it is solvable in polynomial time. Finding the compliment of G can be solved by examining the vertex matrix of G. If there is an edge between u and v, matrix[u][v]=1. Taking the compliment requires transforming every matrix entry to its compliment. Therefore, the time complexity is $O(n^2)$. We have shown the independent set is reducible from the clique in polynomial time.

Finally, since the independent set problem is NP and reducible from the Clique problem in polynomial time, the independent set problem is NP complete.

3. NP completeness reduction

Reduction from set cover to hitting set

First, we need to show the hitting set problem is in NP. Suppose we have a solution to the hitting set of size k for a collection of n sets B. To check the solution $a_1, a_2, \ldots, a_i, \ldots, a_k$, we examine each of the set $B_1, B_2, \ldots, B_n$ to see if it is hit by at least one a_i. It is easily seen that this can be done in a polynomial time O(nk), assumed checking hit can be done in O(1).

Then, we need to reduce the set cover problem to hitting set, assuming the set cover problem in NP complete. Suppose we have a collection of n sets B. A set cover of size k implies a union of k sets S_1 union S_2 union ... union S_k from B can hit every element in the union of all sets $B_1 \cup B_2 \cup \ldots \cup B_n$. If S_i contains an element in B union, then that element is a part of the hitting set. The set cover must contain all elements in the hitting set in order to cover all sets. On the other hand, the hitting set does not necessarily contain every element in the set cover to hit every set. In order to find the hitting set from the set covering, we can traverse through the set covering and mark the elements that successfully hit one or more sets. If a set is hit, we can remove the set from the collection. After all sets are removed, we have successfully found the hitting set from the set cover. We can easily see how this can be done in polynomial time.

In other words, if there is a set cover of size k, there must be a hitting set of size <= k. This is polynomial reduction.

We have shown that the hitting set is NP complete by the reduction from set cover.

Reduction from hitting set to dominating set

If a subset of vertices is a dominating set, it means that every vertex in G is either in S or has an edge connecting a vertex in S. First, we need to show the dominating set problem is in NP. To check a proposed solution S, we can examine every vertex which is not in S to see whether it has a neighbor in the solution. If all of them have a neighbor in the solution, we have verified it. We can see this is easily done in polynomial time.

Then, we need to reduce the dominating set problem from the hitting set problem. The dominating set is a set of vertices in G of size <= k such that every vertex in the graph is either in the subset or has a neighbor that is in the subset. The hitting set is a set of elements in B of size <= k such that every element hits a set in B. We can consider the graph as the collection of sets B. A vertex and its neighbors form a set within the collection. If a set of vertices has "hit" every set within the collection, this hitting set is the dominating set. If we successfully transform the graph into a collection of sets, finding a dominating set is the same as finding the hitting set. Transforming the graph into a collection of sets can be done in polynomial time (e.g. examining the adjacency matrix in O(n^2) time).

Reduction from dominating set to set cover

First, we need to show set covering problem belongs to NP. Suppose we have a collection of n sets B. If we have a solution S which contains <=k sets in which the elements can cover every element in the union of all sets in B. We can check the solution in polynomial time (e.g. O(nk), assuming we can check the existence of an element in a set B_i in O(1)).

Then we need to show the dominating set problem can be reduced to set covering. As stated in the previous proof, the dominating set is a set of vertices in G of size <= k such that every vertex in the graph is either in the subset or has a neighbor that is in the subset. We can consider the graph as the collection of sets B in which a vertex and its neighbors form a set within the collection. If we have found the dominating set in this collection, each vertex in this dominating set is a neighbor to one or more vertices in G, assuming a vertex is also a neighbor to itself. In other words, each element in the dominating set belongs to one or more set in B. A subset of B is a set cover if the elements in the subset include all elements in the dominating set. The size of the set cover is at most the cardinality of the dominating set. For each vertex in the dominating set, look for its presence in each set in G. If it is present, add the set to the set cover. We can find a set cover in this way in polynomial time.

4. Hamiltonian path/cycle

Prove NP

First, we show that finding Hamiltonian path belongs to NP. Suppose we have a solution P that goes from s to t. We can check whether the solution starts from s, visits each vertex in G exactly once, and ends at t. This can be done in polynomial time O(n). Similarly, we can show finding Hamiltonian cycle is in NP. Suppose we have a solution C. Choose an arbitrary node v in C. Checking whether the solution starts at v, visits every other vertex in G exactly once, and comes back to v in polynomial time.

Reduction from Hamiltonian cycle to Hamiltonian path

We need to show Hamiltonian path problem is reducible from Hamiltonian cycle problem. Given a Hamiltonian cycle C in graph G, choose an arbitrary node v in C and split it into two nodes v' and v'' to form G'. Now, there is a Hamiltonian path from v' and v'' in G', in which every other node in G' is visited exactly once according the definition of a Hamiltonian cycle. We have successfully reduced the Hamiltonian cycle problem to Hamiltonian path.

Reduction from Hamiltonian path to Hamiltonian cycle

Similarly, we can show Hamiltonian cycle is reducible from Hamiltonian path. Suppose there exists a Hamiltonian path from s to t in G'. Let's construct G'', such that s and t are 'glued' together to r. The Hamiltonian path from s to t starts at s, visits every node in G' exactly once, and ends at t. Now in G'', s and t become the same point r; the Hamiltonian path starts at r, visits every node in G'' exactly once, and ends at r. We can easily see how this forms a Hamiltonian cycle.

Thus, we have successfully shown Hamiltonian cycle are Hamiltonian path are reducible to each other.