Time: 1 h
Week 0

Linear Algebra Quiz

CS 188.2
September 27, 2019

The quiz is split in 2 parts: a multiple choice section, and a free-form section. For the multiple choice section, several answers might be correct.

Part 1: Multiple choice

1. Are the following set of vectors linearly independent?

$$u = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}, v = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} \; and \quad w = \begin{bmatrix} -2.5 \\ 0.5 \\ -5 \end{bmatrix} \tag{1}$$

A) No
B) Yes

Solution
A) No. It is obvious that C is a linear sum of A and B. (Note: $C = -\frac{1}{2}(A + B)$)

2. What is the dot product of vectors A and B?

$$A = \begin{bmatrix} -6 \\ 4 \\ 3 \\ 8 \end{bmatrix}, B = \begin{bmatrix} -2 \\ -1 \\ 0 \\ -1 \end{bmatrix} \tag{2}$$

A) 0
B) $\begin{bmatrix} 12 \\ -4 \\ 0 \\ -8 \end{bmatrix}$
C) 24
D) $\begin{bmatrix} -12 \\ 4 \\ 0 \\ 8 \end{bmatrix}$

Solution
A) The dot product of two vectors is a scalar and can be calculated as:
(-6)(-2) + (4)(-1)+ (3) (0) + (8)(-1) =0
Tip: If vectors are identified with row matrices, the dot product can also be written as a matrix product: $a.b = a.b^T$

3. What is the scalar projection of B into A?

$$A = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}, B = \begin{bmatrix} 3.5 \\ 4 \\ -2 \end{bmatrix} \tag{3}$$

A) 4
B) -4
C) 0.25
D) -0.25

Solution

B) The scalar projection of B onto A can be calculated as follows:
$\frac{A.B}{|A|} = \frac{(-4)(3.5)+(0)(4)+(3)(-2)}{\sqrt{16+9}} = \frac{-20}{5} = -4$

4. What is $A^T B$ given the following vectors?

$$A = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} \tag{4}$$

A) 11
B) $\begin{bmatrix} 1 & -2 \\ 4 & 4 \\ -3 & 1 \end{bmatrix}$
C) -11
D) $\begin{bmatrix} 1 & 1 \\ 4 & 4 \\ -2 & -3 \end{bmatrix}$

Solution

A) Its obvious that the transpose of A changes its shape to 1×3 and given the fact that the shape of B is 3×1, the matrix multiplication results in a scalar according to the following:

$(-2)(1) + (4)(4) + (1)(-3) = 11$

5. What is the determinant of matrix A?

$$A = \begin{bmatrix} 2 & 4 & 5 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \tag{5}$$

A) 20
B) -20
C) 0
D) -16

Solution

C) The determinant of matrix A can be calculated as follows:

$\det(A) = 2[(2)(3) - (1)(0)] - 4[(0)(3) - (1)(2)] + 5[(0)(0) - (2)(2)] = 12 + 8 - 20 = 0$

6. Given vectors a and b, and $\cos(a, b) = \pi/2$, find the projection of b along a. Here, $\cos(a, b) = \cos\theta$, where θ denotes the angle between a and b.

   A) Insufficient information
   B) 0
   C) $|a||b|$
   D) $|b|$

   Solution
   B) (Projection of b along a : $|b| \cos 90 = 0$)

7. Which of the following options give enough information to calculate the dot product of vectors **a** and **b**?

   A) $a = [2, 3, -1]$ and $b = [3, 4, 1]$
   B) $a = [2, 3, -1]$, angle between a and b given.
   C) $a = [2, 3, -1]$, projection of b onto a given.
   D) $a = [2, 3, -1]$, $b = [3, 4, 1]$, angle between a and b given.

   Solution
   A)C)D)
   A: $(2 \times 3) + (3 \times 4) + (-1 \times 1)$
   C: $|a| \times projection = |a| \times |b| \cos\theta$
   D: $|a| \times |b| \times \cos\theta$

8. Does the following matrix A have an inverse?

   $$\begin{bmatrix} 2 & -1 & 3 & 0 \\ -1 & 1 & 0 & 4 \\ -2 & 1 & 4 & 1 \\ -1 & 3 & 0 & -2 \end{bmatrix}$$

   A) Yes
   B) No
   C) Insufficient information

   Solution
   A)Yes it does since its determinant is not 0. (Review how to find Determinant of 4*4 matrix if needed)

9. What is AB given the following matrices?

   $$A = \begin{bmatrix} 1 & 5 \\ 8 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \tag{6}$$

A) $\begin{bmatrix} 1 & 10 \\ 24 & 8 \end{bmatrix}$
B) $\begin{bmatrix} 16 & 22 \\ 14 & 24 \end{bmatrix}$
C) 43
D) 78

Solution
B)Answer should be 2x2 matrix.
Top left element: $1 \times 1 + 5 \times 3 = 16$
Top right element: $1 \times 2 + 5 \times 4 = 22$
Bottom left element: $8 \times 1 + 2 \times 3 = 14$
Bottom right element: $8 \times 2 + 2 \times 4 = 24$

10. What is $\lim_{x \to 0} \frac{\sin 2x}{x}$?

A) 0
B) -2
C) 2
D) 1

Solution
C) 2. Use L'hopital's rule: $\lim_{x\to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \frac{2 \cos 2x}{1} = 2$. Alternate solution: use Taylor series of $\sin x = x - x^3/3! + x^5/5! - ...$ and substitute 2x for x in Taylor series. We see that $\lim_{x\to 0} \frac{\sin 2x}{x} = \lim_{x\to 0} \frac{2x}{x} = 2$, where we can ignore higher order terms in Taylor expansion.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Which of the following sets of conditions imply that x is a local minimum of f?

A) $f'(x) = 0$
B) $f'(x) = 0$ and there exists $\epsilon > 0$ such that $f'(y) > 0$ for all y in $(x, x + \epsilon)$ and $f'(y) < 0$ for all y in $(x - \epsilon, x)$
C) $f'(x) = 0$ and $f''(x) < 0$ (for this choice, assume f is twice differentiable at x)
D) $f'(x) = 0$ and $f''(x) > 0$ (for this choice, assume f is twice differentiable at x)

Solution
B) D)
This question might be kind of tricky (especially answer choice B). Answer choice A is not sufficient; take $f(x) = x^3$ at $x = 0$, which is a saddle point.
Answer choice B is sufficient by the first derivative test. This might be kind of tricky to recognize due to the ϵ's. Intuitively, this choice is saying that for a small region to the left of x ($x - \epsilon < y < x$), f is decreasing ($f'(y) < 0$) and for a small region to the right of x ($x < y < x + \epsilon$), f is increasing ($f'(y) > 0$). Combined with f being a critical point ($f'(x) = 0$), this is sufficient to conclude that x is a local minimum.

More formally, the mean value theorem can be used to prove that x is a local minimum of f. The mean value theorem states that if f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point c in (a, b) such that:
$f'(c) = \frac{f(b)-f(a)}{b-a}$.
As f is differentiable everywhere, it is continuous everywhere and we can use the mean value theorem. Hence, as $f'(y) > 0$ for all y in $(x, x+\epsilon)$, we have that $\frac{f(y)-f(x)}{y-x} > 0$ and $f(y) - f(x) > 0 \quad \forall y \in (x, x+\epsilon)$. Similarly, as $f'(y) < 0$ for all y in $(x-\epsilon, x)$, we have that $f(y) - f(x) > 0 \quad \forall y \in (x-\epsilon, x)$. Combining the two cases, we have that $\forall y \in (x-\epsilon, x+\epsilon)$, $f(y) \geq f(x)$, proving that x is a local minimum of f.
Answer choice C actually shows that x is a local maximum (second derivative test).
Answer choice D is sufficient by second derivative test.

12. Let A, B be 2 $n \times n$ real matrices. Does $e^{A+B} = e^A e^B$ hold?

A) Yes
B) No

Solution
B) No. This statement does not necessarily hold for two general matrices. The statement is true if A and B commute (i.e. $AB = BA$) as can be shown by Taylor expanding both sides and looking at each term of e^{A+B}.

13. What are the eigenvalues of $\begin{bmatrix} 4 & 7 & 1 \\ 0 & -3 & 8 \\ 0 & 0 & 2 \end{bmatrix}$?

A) 1, 4, 7
B) 0, 4
C) -3, 2, 4
D) -3, 0 , 1

Solution
C) The eigenvalues of a triangular matrix are on the diagonal.

14. $\begin{bmatrix} -5 \\ -5 \\ -5 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$. What is the corresponding eigenvalue?

A) 0
B) 3
C) 4
D) -5

Solution
B) The eigenvectors v of matrix A satisfy $Av = \lambda v$, where λ is the eigenvalue associated with v.

Part 2: Free-form questions

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T(e_1) = u_1$ and $T(e_2) = u_2$, where e_1 and e_2 are the standard unit vectors of $\mathbb{R}^2$ and:

$$u_1 = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}, u_2 = \begin{bmatrix} 8 \\ 2 \\ 6 \end{bmatrix} \tag{7}$$

   Find $T(\begin{bmatrix} 3 \\ -2 \end{bmatrix})$

   Solution

$$T(\begin{bmatrix} 3 \\ -2 \end{bmatrix}) = T(3e_1 - 2e_2) = 3T(e_1) - 2T(e_2) = 3 \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 8 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 15 \\ 3 \\ 6 \end{bmatrix} - \begin{bmatrix} 16 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -6 \end{bmatrix} \tag{8}$$

2. Let $S = v_1, v_2$ be the set of following vectors in $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \tag{9}$$

   Find an orthogonal basis of the subspace $Span(S)$ of $\mathbb{R}^4$

   Solution
   Gram-Schmidt: The dot product of $v1$ and $v2$ is nonzero, so $(v1, v2)$ is not an orthogonal basis of S. We define $(u1, u2)$ through G-S s.t. $(u1, u2)$ is an orthogonal basis of S.

   $u1 = v1$
   $u2 = v2 - (\frac{u1.v2}{u1.u1})u1$
   $u1.u1 = v1.v1 = 2$
   $u1.v2 = v1.v2 = 1$

$$u2 = v2 - \frac{1}{2}u1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} \tag{10}$$

3. Show that the product of 2 orthogonal matrices is orthogonal

   Solution
   If A and B are orthogonal, then $A^{-1} = A^T$ and $B^{-1} = B^T$ hence:
   $(AB)^T = B^T A^T = B^{-1}A^{-1} = (AB)^{-1}$

4. Is there an orthogonal transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

$$T(\begin{bmatrix}1\\3\end{bmatrix}) = \begin{bmatrix}-3\\1\end{bmatrix} \; and \quad T(\begin{bmatrix}1\\2\end{bmatrix}) = \begin{bmatrix}-2\\-1\end{bmatrix} \tag{11}$$

   Solution
   Orthogonal transformations preserve dot products. $Tx \cdot Ty = x \cdot y \quad \forall x, y \in \mathbb{R}^{2\times 2}$
   $T\begin{bmatrix}1\\3\end{bmatrix} \cdot T\begin{bmatrix}1\\2\end{bmatrix} = 6 - 1 = 5$
   $\begin{bmatrix}1\\3\end{bmatrix} \cdot \begin{bmatrix}1\\2\end{bmatrix} = 1 + 6 = 7$
   There is no such T.

5. Let $f(x) = x^2 \sin 2x$. Compute $f'(x)$.

   Solution
   First, apply product rule for derivatives:
   $f'(x) = 2x \sin 2x + x^2(\sin 2x)'$
   Apply chain rule to compute derivative of $\sin 2x$. $f'(x) = 2x \sin 2x + 2x^2 \cos 2x$

6. Compute

$$\int_0^{\pi/2} \sin^2 x \cos x dx$$

   .

   Solution
   Use u-substitution: $u = \sin x$, $du = \cos x dx$.

$$\int_0^{\pi/2} \sin^2 x \cos x dx = \int_0^1 u^2 du = 1^3/3 - 0^3/3 = 1/3$$

   .

7. Let $A = \begin{bmatrix}1 & 2 & 1\\3 & 6 & 4\end{bmatrix}$ and let

   $a = \begin{bmatrix}-3\\1\\1\end{bmatrix}, b = \begin{bmatrix}-2\\1\\0\end{bmatrix}, c = \begin{bmatrix}1\\1\end{bmatrix}$

For each vector (a, b, c), determine whether the vector is in the null space of A. Do the same for the range of A.

Solution

The nullspace is a subset of $\mathbb{R}^3$, and the range a subset of $\mathbb{R}^2$. a and b are not in the range, and c is not in the nullspace.

$Aa = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so $a \notin \operatorname{null} A$

$Ab = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $b \in \operatorname{null} A$

To determine if c is in the range, we have to check that $Ax = c$ is consistent. We write the system matrix:

$[A|c] = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 6 & 4 & 1 \end{bmatrix}$

We apply the simple transformations $R_2 - 3R_1$ and $R_1 - R_2$ which leads to $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

The solution is therefore:

$x_1 = -2x_2 + 3$

$x_3 = -2$. c is in the range of A.

8. Consider the overdetermined system $Ax = b$ where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$. Derive the normal equation for this linear system - that is, find $x_{closest}$ which minimizes the sum-of-squares-error E :

$$E(x) = \sum_i [(Ax)_i - b_i]^2 \tag{12}$$

Solution

$$E(x) = (Ax - b)^T (Ax - b) = x^T A^T Ax - x^T A^T b - b^T Ax + b^T b \tag{13}$$

and since $x^T A^T b = b^T Ax$ (a scalar):

$$E(x) = x^T A^T Ax - 2x^T A^T b + b^T b \tag{14}$$

we want to minimize $E(x)$, which is a convex function of x. We therefore want:

$$\frac{dE}{dx} = 2x^T A^T A - 2b^T A = 0 \tag{15}$$

which leads to the normal equation:

$$A^T A x_{closest} = A^T b \tag{16}$$