Math 151A
Numerical Integration
Supplementary material from "An Introduction to Numerical Analysis" by Atkinson.

Background facts

Theorem 1.3 (Integral Mean Value) Let $w(x)$ be nonnegative and integrable on $[a, b]$, and let $f(x)$ be continuous on $[a, b]$. Then

$$\int_a^b w(x)f(x)\,dx = f(\xi)\int_a^b w(x)\,dx$$

for some $\xi \in [a, b]$.

An interpolation error formula using divided differences Let t be a real number, distinct from the node points $x_0, x_1, \ldots, x_n$. Construct the polynomial interpolating to $f(x)$ at $x_0, \ldots, x_n$, and t:

$$\begin{aligned} p_{n+1}(x) &= f(x_0) + (x - x_0)f[x_0, x_1] + \ldots + (x - x_0)\ldots(x - x_{n-1})f[x_0, \ldots, x_n] \\ &\quad + (x - x_0)\ldots(x - x_n)f[x_0, x_1, \ldots, x_n, t] \\ &= p_n(x) + (x - x_0)\ldots(x - x_n)f[x_0, \ldots, x_n, t] \end{aligned}$$

Since $p_{n+1}(t) = f(t)$, we have

$$f(t) = p_n(t) + (t - x_0)\ldots(t - x_n)f[x_0, \ldots, x_n, t] \tag{3.22}$$

and this gives us another formula for the error $f(t) - p_n(t)$. Comparing this with the earlier error formula (3.8), and canceling the multiplying polynomial $\Psi_n(t)$, we have

$$f[x_0, x_1, \ldots, x_n, t] = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

for some $\xi \in \mathcal{H}\{x_0, x_1, \ldots, x_n, t\}$. To make this result symmetric in the arguments,

Error relations for the Trapezoidal method and the Composite Trapezoidal method

The trapezoidal rule The simple trapezoidal rule is based on approximating $f(x)$ by the straight line joining $(a,f(a))$ and $(b,f(b))$. By integrating the formula for this straight line, we obtain the approximation

$$I_1(f)=\left(\frac{b-a}{2}\right)\left[f(a)+f(b)\right] \tag{5.10}$$

This is of course the area of the trapezoid shown in Figure 5.1. To obtain an error formula, we use the interpolation error formula (3.22),

$$f(x)-\frac{(b-x)f(a)+(x-a)f(b)}{b-a}=(x-a)(x-b)f[a,b,x]$$

We also assume for all work with the error for the trapezoidal rule in this section that $f(x)$ is twice continuously differentiable on $[a,b]$. Then

$$\begin{aligned} E_1(f)&=\int_a^b f(x)dx-\frac{(b-a)}{2}\left[f(a)+f(b)\right] \\ &=\int_a^b (x-a)(x-b)f[a,b,x]\,dx \end{aligned} \tag{5.11}$$

Using the integral mean-value theorem (Theorem 1.3 of Chapter 1),

$$\begin{aligned} E_1(f)&=f[a,b,\xi]\int_a^b (x-a)(x-b)\,dx \qquad \text{some } a\leqq\xi\leqq b \\ &=\left[\tfrac{1}{2}f''(\eta)\right]\left[-\tfrac{1}{6}(b-a)^3\right] \qquad \text{some } \eta\in[a,b] \end{aligned}$$

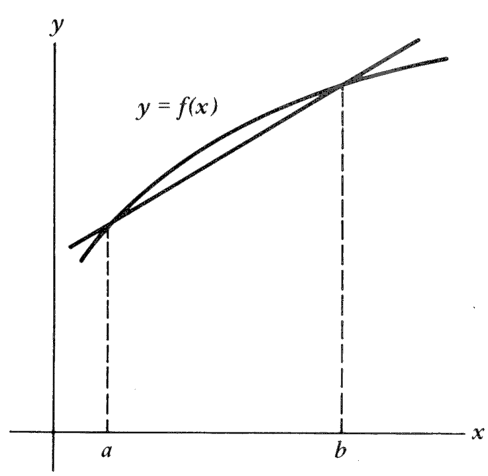

Figure 5.1 Illustration of trapezoidal rule (5.10).

using (3.23). Thus

$$E_1(f) = -\frac{(b-a)^3}{12} f''(\eta) \qquad \eta \in [a,b] \tag{5.12}$$

If $b-a$ is not sufficiently small, the trapezoidal rule (5.10) is not of much use. For such an integral, we break it into a sum of integrals over small subintervals, and then we apply (5.10) to each of these smaller integrals. Let $n \geqq 1$, $h=(b-a)/n$, and $x_j = a+jh$ for $j=0,1,\ldots,n$. Then

$$\begin{aligned} I(f) &= \int_a^b f(x)dx = \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(x)dx \\ &= \sum_{j=1}^{n} \left\{ \left(\frac{h}{2}\right)[f(x_{j-1}) + f(x_j)] - \frac{h^3}{12} f''(\eta_j) \right\} \end{aligned}$$

with $x_{j-1} \leqq \eta_j \leqq x_j$. There is no reason why the subintervals $[x_{j-1}, x_j]$ must all have equal length, but it is customary to first introduce the general principles involved in this way. Although this is also the customary way in which the method is applied, there are situations in which it is desirable to vary the spacing of the nodes.

The first terms in the sum can be combined to give the composite trapezoidal rule,

$$I_n(f) = h\left\{ \tfrac{1}{2} f_0 + f_1 + f_2 + \ldots + f_{n-1} + \tfrac{1}{2} f_n \right\} \qquad n \geqq 1 \tag{5.13}$$

with $f(x_j) \equiv f_j$. For the error in $I_n(f)$,

$$\begin{aligned} E_n(f) = I(f) - I_n(f) &= \sum_{j=1}^{n} -\frac{h^3}{12} f''(\eta_j) \\ &= -\frac{h^3 n}{12} \left[\frac{1}{n} \sum_{j=1}^{n} f''(\eta_j) \right] \end{aligned} \tag{5.14}$$

For the term in brackets,

$$\operatorname*{Min}_{a \leqq x \leqq b} f''(x) \leqq M \equiv \frac{1}{n} \sum_{j=1}^{n} f''(\eta_j) \leqq \operatorname*{Max}_{a \leqq x \leqq b} f''(x)$$

Since $f''(x)$ is continuous for $a \leqq x \leqq b$, it must attain all values between its minimum and maximum at some point of $[a,b]$; and thus $f''(\eta) = M$ for some $\eta \in [a,b]$. Thus we can write

$$E_n(f) = -\frac{(b-a)h^2}{12} f''(\eta) \qquad \text{some } \eta \in [a,b] \tag{5.15}$$

Euler-Maclaurin Formula

The Euler-MacLaurin formula The following theorem gives a very detailed asymptotic error formula for the trapezoidal rule; and it is at the heart of much of the asymptotic error analysis of this section. The connection with some other integration formulas appears later in the section.

Theorem 5.3 (Euler-MacLaurin formula) Let $m \geqq 0$, $n \geqq 1$, and define $h=(b-a)/n$, $x_j = a + jh$ for $j = 0, 1, \ldots, n$. Further assume that $f(x)$ is $2m+2$ times continuously differentiable on $[a,b]$. Then for the error in the trapezoidal rule,

$$E_n(f) = \int_a^b f(x)\,dx - h\sum_{j=0}^{n}{}'' f(x_j)$$

$$= -\sum_{i=1}^{m} \frac{B_{2i}}{(2i)!} h^{2i} \left[f^{(2i-1)}(b) - f^{(2i-1)}(a) \right]$$

$$+ \frac{h^{2m+2}}{(2m+2)!} \int_a^b \bar{B}_{2m+2}\left(\frac{x-a}{h}\right) f^{(2m+2)}(x)\,dx \qquad (5.90)$$

Proof A complete proof is given in Ralston [R1, pp. 131–133]; and a more general development is given in Steffensen [S1, Chapter 14]. The proof in [R1] is short and correct, making full use of the special properties of the Bernoulli polynomials. Here we give a simpler, but less general, version of that proof, showing it to be based on integration by parts with a bit of clever algebraic manipulation.