From “Numerical Mathematics And Computing”, by W. Cheney and D. Kincaid pages 113-115.
Note that the indexing of the ordinates starts at 1 and there are n data values.

THEOREM ON RECURSIVE PROPERTY OF DIVIDED DIFFERENCES

The divided differences obey the formula

$$f[x_1, x_2, \ldots, x_n] = \frac{f[x_2, x_3, \ldots, x_n] - f[x_1, x_2, \ldots, x_{n-1}]}{x_n - x_1} \tag{9}$$

Proof Since $f[x_1, x_2, \ldots, x_n]$ was defined to be equal to the coefficient a_n in the Newton interpolating polynomial p_{n-1} of equation (2), we can say that $f[x_1, x_2, \ldots, x_n]$ is Likewise, $f[x_1, x_2, \ldots, x_{n-1}]$ is the coefficient of x^{n-2} in the polynomial p_{n-2} of degree $\leqq n-2$, which interpolates f at $x_1, x_2, \ldots, x_{n-1}$. The three polynomials p_{n-1}, q, and p_{n-2} are intimately related. In fact,

$$p_{n-1}(x) = q(x) + \frac{x - x_n}{x_n - x_1}[q(x) - p_{n-2}(x)] \tag{10}$$

In order to establish (10), observe that the right side is a polynomial of degree $\leqq n-1$. Evaluating it at x_1 gives $f(x_1)$:

$$\begin{aligned} q(x_1) + \frac{x_1 - x_n}{x_n - x_1}[q(x_1) - p_{n-2}(x_1)] &= q(x_1) - [q(x_1) - p_{n-2}(x_1)] \\ &= p_{n-2}(x_1) = f(x_1) \end{aligned}$$

Evaluating it at x_i $(2 \leqq i \leqq n-1)$ results in $f(x_i)$:

$$\begin{aligned} q(x_i) + \frac{x_i - x_n}{x_n - x_1}[q(x_i) - p_{n-2}(x_i)] &= f(x_i) + \frac{x_i - x_n}{x_n - x_1}[f(x_i) - f(x_i)] \\ &= f(x_i) \end{aligned}$$

Similarly, at x_n we get $f(x_n)$:

$$q(x_n) + \frac{x_n - x_n}{x_n - x_1}[q(x_n) - p_{n-2}(x_n)] = q(x_n) = f(x_n)$$

By the uniqueness of interpolating polynomials, the right side of (10) must be $p_{n-1}(x)$, and equation (10) is established.

Completing the argument to justify (9), take the coefficient of x^{n-1} on both sides of equation (10). The result is equation (9). Indeed, $f[x_2, x_3, \ldots, x_n]$ is the coefficient of x^{n-2} in q, and $f[x_1, x_2, \ldots, x_{n-1}]$ is the coefficient of x^{n-2} in p_{n-2}. □

Notice that $f[x_1, x_2, \ldots, x_k]$ is not changed if the nodes $x_1, x_2, \ldots, x_k$ are permuted; thus, for example, $f[x_1, x_2, x_3] = f[x_2, x_3, x_1]$. The reason is that $f[x_1, x_2, x_3]$ is the coefficient of x^2 in the quadratic polynomial interpolating f at x_1, x_2, x_3, whereas $f[x_2, x_3, x_1]$ is the coefficient of x^2 in the quadratic polynomial interpolating f at x_2, x_3, x_1. These two polynomials are, of course, the same. A formal statement in mathematical language is as follows:

INVARIANCE THEOREM
The divided difference $f[x_1, x_2, \ldots, x_k]$ is invariant under all permutations of the arguments $x_1, x_2, \ldots, x_k$.

Since the variables $x_1, x_2, \ldots, x_k$ and k are arbitrary, the recursive formula (9) can also be written

(11)
$$f[x_i, x_{i+1}, \ldots, x_{j-1}, x_j] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_j] - f[x_i, x_{i+1}, \ldots, x_{j-1}]}{x_j - x_i}$$

The first three divided differences are thus

$$f[x_i] = f(x_i)$$
$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$
$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

Using formula (11), it is possible to construct a divided-difference table for a function f. It is customary to arrange it as follows (here $n = 4$):

x	$f[\ ]$	$f[\ ,\ ]$	$f[\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ]$
x_1	$f[x_1]$			
		$f[x_1, x_2]$		
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		$f[x_1, x_2, x_3, x_4]$
x_3	$f[x_3]$		$f[x_2, x_3, x_4]$	
		$f[x_3, x_4]$		
x_4	$f[x_4]$			

The coefficients encircled in the top diagonal are the ones needed to form the Newton interpolating polynomial (6).