Solution (d). By the definition of p_{n+1} as the zero of the tangent line approximation at p_n, the fact $f(p_*) = 0$, and the Lagrange form of the Taylor remainder for the tangent line approximation of f at p_n, we have

$$\begin{gathered} 0 = f(p_n) + f'(p_n)(p_{n+1} - p_n)\,, \\ 0 = f(p_*) = f(p_n) + f'(p_n)(p_* - p_n) + \tfrac{1}{2} f''(q_n)(p_* - p_n)^2\,, \\ \text{for some } q_n \text{ between } p_n \text{ and } p_*\,. \end{gathered}$$

Upon subtracting the second equation from the first we see that

$$0 = f'(p_n)(p_{n+1} - p_*) - \tfrac{1}{2} f''(q_n)(p_* - p_n)^2\,.$$

Upon solving for $p_{n+1} - p_*$ we obtain the general relation

$$p_{n+1} - p_* = \frac{f''(q_n)}{2f'(p_n)}(p_n - p_*)^2\,.$$

For $f(x) = x^3 - 6$ we have $f'(x) = 3x^2$ and $f''(x) = 6x$, which are both increasing functions over $[6^{\frac{1}{3}}, 2]$. We thereby have the bounds

$$3 \cdot 6^{\frac{2}{3}} \le f'(x)\,, \qquad 0 < f''(x) \le 6 \cdot 2 \qquad \text{for every } x \in [6^{\frac{1}{3}}, 2]\,.$$

$3 \cdot (6^{1/3})^2$

Therefore because p_n and q_n lie in $[6^{\frac{1}{3}}, 2]$ while $p_* = 6^{\frac{1}{3}}$, we obtain the bound

$3 \cdot 6^{2/3}$

$$\left|p_{n+1} - 6^{\frac{1}{3}}\right| = \frac{f''(q_n)}{2f'(p_n)}\left|p_n - 6^{\frac{1}{3}}\right|^2 \le \frac{12}{2 \cdot 3 \cdot 6^{\frac{2}{3}}}\left|p_n - 6^{\frac{1}{3}}\right|^2,$$

which reduces to the desired bound. $\square$