they are the same polynomial. Also, without uniqueness the linear system (3.1.2) would not be uniquely solvable; from results in linear algebra, this would imply the existence of data vectors y for which there is no interpolating polynomial of degree $\leq n$.

The formula

$$p_n(x) = \sum_{i=0}^{n} y_i l_i(x) \tag{3.1.6}$$

is called *Lagrange's formula* for the interpolating polynomial.

Example

$$p_1(x) = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1 = \frac{(x_1 - x) y_0 + (x - x_0) y_1}{x_1 - x_0}$$

$$p_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1$$
$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

The polynomial of degree ≤ 2 that passes through the three points $(0, 1)$, $(-1, 2)$, and $(1, 3)$ is

$$p_2(x) = \frac{(x+1)(x-1)}{(0+1)(0-1)} \cdot 1 + \frac{(x-0)(x-1)}{(-1-0)(-1-1)} \cdot 2 + \frac{(x-0)(x+1)}{(1-0)(1+1)} \cdot 3$$
$$= 1 + \frac{1}{2} x + \frac{3}{2} x^2$$

If a function $f(x)$ is given, then we can form an approximation to it using the interpolating polynomial

$$p_n(x; f) \equiv p_n(x) = \sum_{i=0}^{n} f(x_i) l_i(x) \tag{3.1.7}$$

This interpolates $f(x)$ at $x_0, \ldots, x_n$. For example, we later consider $f(x) = \log_{10} x$ with linear interpolation. The basic result used in analyzing the error of interpolation is the following theorem. As a notation, $\mathscr{H}\{a, b, c, \ldots\}$ denotes the smallest interval containing all of the real numbers $a, b, c, \ldots$.

Theorem 3.2 Let $x_0, x_1, \ldots, x_n$ be distinct real numbers, and let f be a given real valued function with $n + 1$ continuous derivatives on the interval $I_t = \mathscr{H}\{t, x_0, \ldots, x_n\}$, with t some given real number.

Then there exists $\xi \in I_t$ with

$$f(t) - \sum_{j=0}^{n} f(x_j) l_j(t) = \frac{(t - x_0) \cdots (t - x_n)}{(n+1)!} f^{(n+1)}(\xi) \quad (3.1.8)$$

Proof Note that the result is trivially true if t is any node point, since then both sides of (3.1.8) are zero. Assume t does not equal any node point. Then define

$$E(x) = f(x) - p_n(x) \qquad p_n(x) = \sum_{j=0}^{n} f(x_j) l_j(x)$$

$$G(x) = E(x) - \frac{\Psi(x)}{\Psi(t)} E(t) \qquad \text{for all} \quad x \in I_t \qquad (3.1.9)$$

with

$$\Psi(x) = (x - x_0) \cdots (x - x_n)$$

The function $G(x)$ is $n + 1$ times continuously differentiable on the interval I_t, as are $E(x)$ and $\Psi(x)$. Also,

$$G(x_i) = E(x_i) - \frac{\Psi(x_i)}{\Psi(t)} E(t) = 0 \qquad i = 0, 1, \ldots, n$$

$$G(t) = E(t) - E(t) = 0$$

Thus G has $n + 2$ distinct zeros in I_t. Using the mean value theorem, G' has $n + 1$ distinct zeros. Inductively, $G^{(j)}(x)$ has $n + 2 - j$ zeros in I_t, for $j = 0, 1, \ldots, n + 1$. Let ξ be a zero of $G^{(n+1)}(x)$,

$$G^{(n+1)}(\xi) = 0$$

Since

$$E^{(n+1)}(x) = f^{(n+1)}(x)$$

$$\Psi^{(n+1)}(x) = (n + 1)!$$

we obtain

$$G^{(n+1)}(x) = f^{(n+1)}(x) - \frac{(n+1)!}{\Psi(t)} E(t)$$

Substituting $x = \xi$ and solving for $E(t)$,

$$E(t) = \frac{\Psi(t)}{(n+1)!} \cdot f^{(n+1)}(\xi)$$

the desired result.