From "Numerical Mathematics And Computing", by W. Cheney and D. Kincaid pages 130 - 131.
Note that the indexing of the ordinates starts at 1 and there are n data values.

Theorems on Interpolation Errors

It is possible to assess the errors of interpolation by means of a formula involving the nth derivative of the function being interpolated. Here is the formal statement.

THEOREM 1 ON INTERPOLATION ERRORS

If p is the polynomial of degree at most $n-1$ that interpolates f at the n distinct nodes $x_1, x_2, \ldots, x_n$ belonging to an interval $[a, b]$ and if $f^{(n)}$ is continuous, then for each x in $[a, b]$ there is a ξ in (a, b) for which

$$f(x) - p(x) = \frac{1}{n!} f^{(n)}(\xi) \prod_{i=1}^{n} (x - x_i) \tag{2}$$

Proof Observe first that (2) is obviously valid if x is one of the nodes x_i, for then both sides of the equation reduce to zero. If x is not a node, let it be fixed in the remainder of the discussion, and define

$$\begin{aligned} w(t) &= \prod_{i=1}^{n} (t - x_i) && \text{(polynomial in the variable } t\text{)} \\ c &= \frac{f(x) - p(x)}{w(x)} && \text{(constant)} \\ \phi(t) &= f(t) - p(t) - cw(t) && \text{(function in the variable } t\text{)} \end{aligned} \tag{3}$$

Observe that c is well defined because $w(x) \neq 0$ (x is not a node). Note also that ϕ takes the value 0 at the $n+1$ points $x_1, x_2, \ldots, x_n$, and x. Now invoke **Rolle's theorem**,* which states that between any two roots of ϕ there must occur a root of ϕ'. Thus ϕ' has at least n roots. By similar reasoning, ϕ'' has at least $n-1$ roots, ϕ''' has at least $n-2$ roots, and so on. Finally, it can be inferred that $\phi^{(n)}$ must have at least one root. Let ξ be a root of $\phi^{(n)}$. All the roots being counted in this argument are in (a, b). Thus

$$0 = \phi^{(n)}(\xi) = f^{(n)}(\xi) - p^{(n)}(\xi) - cw^{(n)}(\xi)$$

In this equation, $p^{(n)}(\xi) = 0$ because p is a polynomial of degree $\leq n-1$. Also, $w^{(n)}(\xi) = n!$ because $w(t) = t^n +$ (lower-order terms in t). Thus we have

$$0 = f^{(n)}(\xi) - cn! = f^{(n)}(\xi) - \frac{n!}{w(x)} [f(x) - p(x)]$$

This equation is a rearrangement of equation (2). □

*__Rolle's theorem__: Let f be a function that is continuous on $[a, b]$ and differentiable on (a,b). If $f(a) = f(b) = 0$, then $f'(c) = 0$ for some point c in (a, b).

A special case that often arises is the one in which the interpolation nodes are equally spaced. Suppose that $x_i = a + (i-1)h$ for $i = 1, 2, \ldots, n$ and that $h = (b-a)/(n-1)$. Then we can show that for any $x \in [a, b]$,

$$\prod_{i=1}^{n} |x - x_i| \leqq \frac{1}{4} h^n (n-1)! \tag{4}$$

In order to establish this inequality, fix x and select j so that $x_j \leqq x \leqq x_{j+1}$. It is an exercise in calculus (Problem 1) to show that

$$|x - x_j|\,|x - x_{j+1}| \leqq \frac{h^2}{4} \tag{5}$$

Using (5), we have

$$\prod_{i=1}^{n} |x - x_i| \leqq \frac{h^2}{4} \prod_{i=1}^{j-1} (x - x_i) \prod_{i=j+2}^{n} (x_i - x)$$

The sketch in Figure 4.3, showing a typical case, may be helpful. Since $x_j \leqq x \leqq x_{j+1}$, we have further

$$\prod_{i=1}^{n} |x - x_i| \leqq \frac{h^2}{4} \prod_{i=1}^{j-1} (x_{j+1} - x_i) \prod_{i=j+2}^{n} (x_i - x_j)$$

Now use the fact that $x_i = a + (i-1)h$. Then $x_{j+1} - x_i = (j - i + 1)h$ and $x_i - x_j = (i - j)h$. Therefore

$$\begin{aligned}\prod_{i=1}^{n} |x - x_i| &\leqq \frac{h^2}{4} h^{j-1} h^{n-(j+2)+1} \prod_{i=1}^{j-1} (j - i + 1) \prod_{i=j+2}^{n} (i - j) \\ &\leqq \frac{1}{4} h^n j!(n-j)! \leqq \frac{1}{4} h^n (n-1)!\end{aligned}$$

In the last step we use the fact that if $1 \leqq j \leqq n-1$ then $j!(n-j)! \leqq (n-1)!$. This, too, is left as an exercise (Problem 2). Hence, inequality (4) is established.

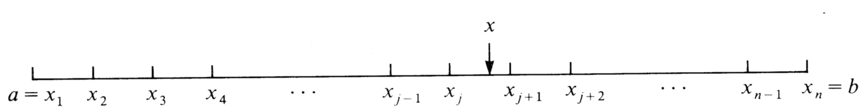

FIGURE 4.3