Math 151a week 1 discussion notes

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Logistics

I am your TA! My name is Jean-Michel Maldague. Welcome to numerical analysis.
OH: Tuesdays 11-12, Thursdays 12-1, and by appointment, all in MS 2963.
email: jmmaldague@gmail.com ← email me if you have questions about anything

Plan for today:
1. Introductions
2. Scraps from the calculus boneyard
3. MATLAB demo

Introductions

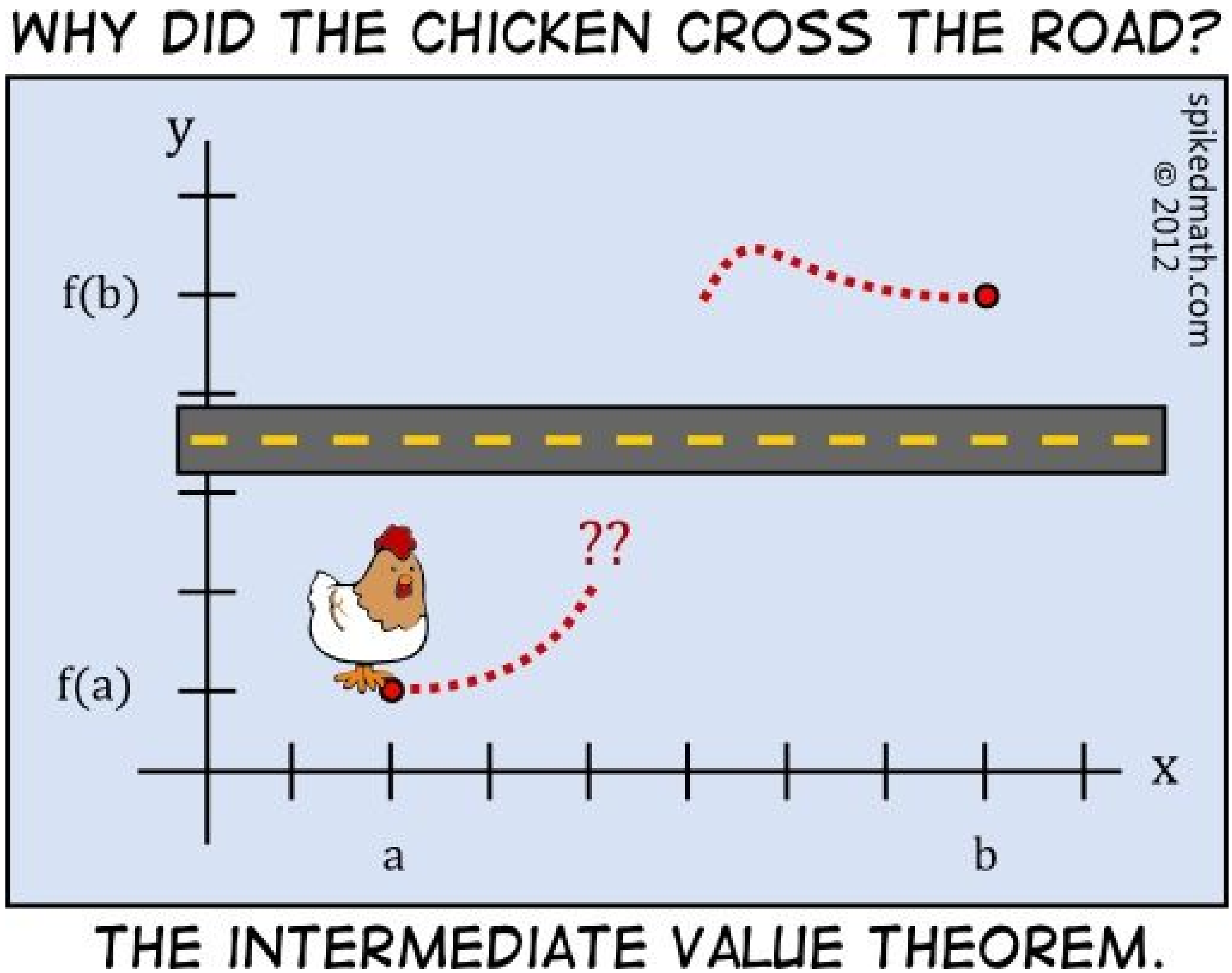
Get into pairs and interview your partner. Ask them their name, year, a fun fact about them, and what the first thing they'll do after graduating is. Then introduce your partner to the class, and they'll introduce you.

Scraps from the calculus boneyard

Intermediate value theorem (IVT), Mean value theorem (MVT), and Taylor's error formula. We'll add more to this list as the course goes on. You won't need to memorize the proofs of these theorems, but you will need to know how/when to use them.

Intermediate value theorem (IVT): if $f : [a, b] \to \mathbb{R}$ is continuous, then its image is an interval.

Intuitively, this means that if two numbers are in f's image, then every number between them also lies in f's image. By possibly shrinking the domain interval, the theorem (as stated) implies that:
If $c_1 < c_2$ and $f(c_i) = d_i$ for $i = 1, 2,$ then for any $y \in (d_1, d_2) \cup (d_2, d_1)$, there exists $x \in (c_1, c_2)$ such that $f(x) = y$.



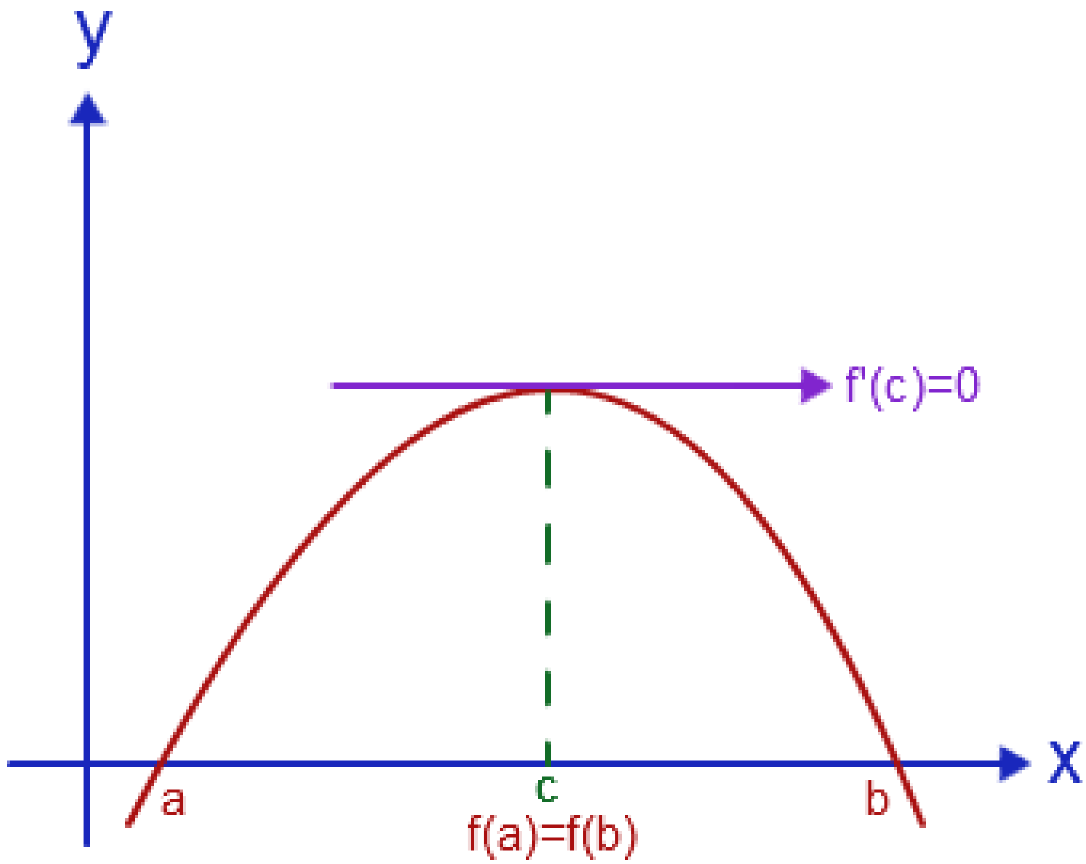
Proof.
(1) The continuous image of a connected set is connected, and
(2) the connected sets of $\mathbb{R}$ are precisely the intervals. $\square$

Interestingly enough, the fact that the connected sets of $\mathbb{R}$ are the intervals is often proven using an argument that looks like the bisection algorithm, though in this course we will use the IVT to show that the bisection algorithm works.

To prove the MVT, let's have a lemma first.

Rolle's theorem: Suppose $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = 0 = f(b)$, then $\exists\ c \in (a,b)$ with $f'(c) = 0$.

Intuitively, if the function hits the x-axis twice, then its graph has a horizontal tangent somewhere between the roots.



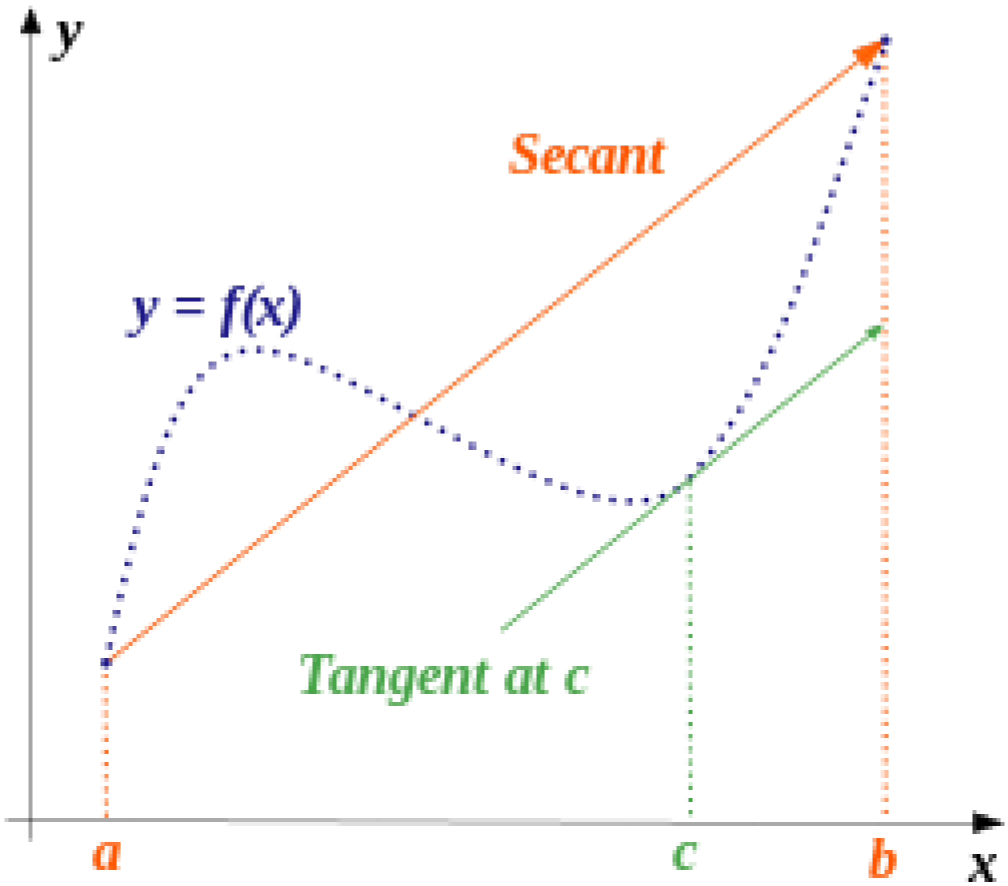
Proof. The compact sets of $\mathbb{R}$ are precisely those which are closed and bounded. Thus $[a,b]$ is compact. The continuous image of a compact set is compact, so $f([a,b])$ is compact in $\mathbb{R}$. Then $f([a,b])$ is closed and bounded, so it contains its (finite) supremum and infimum. This means f achieves its min and max over $[a,b]$. So $\exists\ c,d \in [a,b]$ s.t. $\forall\ x \in [a,b], f(c) \leq f(x) \leq f(d)$.

Case 1: Both c and d are endpoints. Then $0 = f(c) \leq f(x) \leq f(d) = 0$ for all $x \in [a,b]$, so $f \equiv 0$ on $[a,b]$, and thus $f' \equiv 0$ on (a,b). Then $f'\left(\frac{a+b}{2}\right) = 0$, so $c = \frac{a+b}{2}$ satisfies the conclusion of Rolle's theorem.

Case 2: At least one of c, d is not an endpoint; say WLOG c isn't (the case that d isn't is similar). Further, suppose $f'(c) > 0$. Then since $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c)$, using the definition of the limit ensures $\exists\, \delta > 0$ with $(c-\delta, c+\delta) \subset (a,b)$ s.t. $\forall\ x \in (c-\delta, c+\delta)$, we have $\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < \frac{|f'(c)|}{2}$. This last inequality implies $\frac{f(x)-f(c)}{x-c} - f'(c) > -\frac{f'(c)}{2}$, and if we choose $x = c - \delta/2$, we get $f(c-\delta/2) < f(c) - \frac{\delta f'(c)}{4} < f(c)$, violating that $f(c)$ was smallest. We have arrived at a contradiction. A similar argument shows that $f'(c) < 0$ is also impossible. So c works in the conclusion of Rolle's theorem. $\square$

Mean value theorem (MVT): Suppose $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then $\exists\ c \in (a,b)$ with $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Intuitively, the graph of f has a tangent line whose slope is the same as that of the secant line over the whole interval.



Proof. Define $g : [a,b] \to \mathbb{R},\ g(x) = f(x) - f(a) - \dfrac{f(b)-f(a)}{b-a} \cdot (x-a)$. Then g satisfies the hypotheses of Rolle's theorem, so $\exists\ c \in (a,b)$ with $0 = g'(c) = f'(c) - \dfrac{f(b)-f(a)}{b-a}$. This implies the conclusion of the MVT. $\square$

To prove Taylor's error formula let's first have a lemma.

Cauchy's MVT: Suppose $F, G : [a,b] \to \mathbb{R}$, both continuous on $[a,b]$, both differentiable on (a,b), and G' never equal to 0 on (a,b). Then $\exists\ \xi \in (a,b)$ s.t. $\dfrac{F'(\xi)}{G'(\xi)} = \dfrac{F(b)-F(a)}{G(b)-G(a)}$.

Note that $G(b) - G(a) \neq 0$, or the MVT would violate the assumption on G'.

Proof. Put $h : [a,b] \to \mathbb{R},\ h(x) = F(x) - \dfrac{F(b)-F(a)}{G(b)-G(a)} \cdot G(x)$. Then h is continuous on $[a,b]$ and differentiable on (a,b), so the MVT gives us that

$$0 = \frac{h(b)-h(a)}{b-a} = h'(\xi) = F'(\xi) - \frac{F(b)-F(a)}{G(b)-G(a)} \cdot G'(\xi)$$

for some $\xi \in (a,b)$, and then some algebra gives us the result. □

Taylor's error formula (rough): Suppose $f : (a,b) \to \mathbb{R}$ has $(k+1)$ derivatives (we do not assume the $(k+1)^{th}$ derivative is continuous). Fix x and x_0 in (a,b). Then $\exists\ \xi \in (x,x_0) \cup (x_0,x)$ s.t.

$$\begin{aligned} f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f'''(x_0)}{6}(x-x_0)^3 + \cdots \\ + \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(k+1)}(\xi)}{(k+1)!}(x-x_0)^{k+1}. \end{aligned}$$

Mnemonically, we can remember that the last term takes the form of the first skipped term but with the derivative evaluated at an arbitrary point in between x_0 and x. Since the theorem doesn't tell us what that point is, we often have to maximize that derivative over some interval, inducing some error. That is why this is the "rough" version of the theorem.

Proof. (Keeping $x \,\&\, x_0 \in (a,b)$ fixed). Define $F, G : (a,b) \to \mathbb{R}$ by

$$\begin{aligned} &F(t) = -f(x) + f(t) + f'(t)(x-t) + \frac{f''(t)}{2}(x-t)^2 + \cdots + \frac{f^{(k)}(t)}{k!}(x-t)^k, \\ &G(t) = (x-t)^{k+1}. \end{aligned}$$

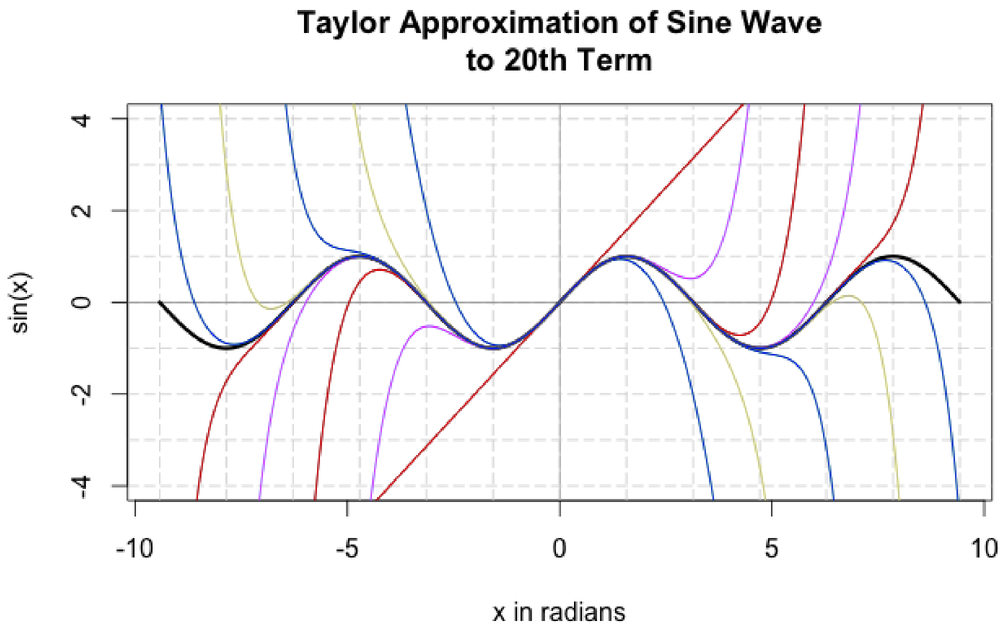
Then both F and G are continuous on $[x,x_0] \cup [x_0,x]$ and differentiable on $(x,x_0) \cup (x_0,x)$. Also, $G'(t) = -(k+1)(x-t)^k$ is never 0 on $(x,x_0) \cup (x_0,x)$, so Cauchy's MVT applies. Note that

$$\begin{aligned} F'(t) = f'(t) - f'(t) + f''^{(t)}(x-t) - f''(t)(x-t) + \ \cdots\ + \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} \\ - \ \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} + \frac{f^{(k+1)}(t)}{k!}(x-t)^k \\ = \frac{f^{(k+1)}(t)}{k!}(x-t)^k, \end{aligned}$$

So Cauchy's MVT tells us there is a ξ between x_0 and x with

$$\frac{\dfrac{f^{(k+1)}(\xi)}{k!}(x-\xi)^k}{-(k+1)(x-\xi)^k} = \frac{F'(\xi)}{G'(\xi)} = \frac{F(x)-F(x_0)}{G(x)-G(x_0)} = \frac{f(x) - f(x_0) - f'(x_0)(x-x_0) - \cdots - \dfrac{f^{(k)}(x_0)}{k!}(x-x_0)^k}{-(x-x_0)^{k+1}},$$

and then some algebra gives us the result. □



Taylor's error formula (exact): Suppose $f : (a, b) \to \mathbb{R}$ has $(k+1)$ continuous derivatives. Fix x_0 and x in (a, b). Then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f'''(x_0)}{6}(x - x_0)^3 + \cdots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \int_{x_0}^{x} \frac{f^{(k+1)}(t)}{k!}(x - t)^k \, dt.$$

Proof. Let's repeatedly integrate by parts the "wrong" way.

$$\begin{aligned}
f(x) - f(x_0) &= \int_{x_0}^{x} f'(t)\, dt \\
&= [-f'(t)(x-t)]_{x_0}^{x} + \int_{x_0}^{x} f''(t)(x-t)\, dt \\
&= [-f'(t)(x-t)]_{x_0}^{x} + \left[\frac{-f''(t)}{2}(x-t)^2\right]_{x_0}^{x} + \int_{x_0}^{x} \frac{f'''(t)}{2}(x-t)^2\, dt \\
&= \cdots = [-f'(t)(x-t)]_{x_0}^{x} + \left[\frac{-f''(t)}{2}(x-t)^2\right]_{x_0}^{x} + \cdots \\
&\qquad + \left[\frac{-f^{(k)}(t)}{k!}(x-t)^k\right]_{x_0}^{x} + \int_{x_0}^{x} \frac{f^{(k+1)}(t)}{k!}(x-t)^k\, dt \\
&= f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f'''(x_0)}{6}(x-x_0)^3 + \cdots \\
&\qquad + \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \int_{x_0}^{x} \frac{f^{(k+1)}(t)}{k!}(x-t)^k\, dt.
\end{aligned}$$

□

Note that this technique relies on $f^{(k+1)}$ being continuous (else the final integral might not exist. Yes, there are some pathological functions that are differentiable but whose derivatives aren't Riemann integrable. See $f(x) = \begin{cases} 0 & x = 0 \\ x^2 \sin(1/x^2) & x \neq 0 \end{cases}$ if you're feeling curious. See Volterra's function if you are feeling ~~masochistic~~ adventurous).

We will use the IVT to justify why the bisection method works, and Taylor's error formula to justify the convergence rate of Newton's method. Stay tuned!

MATLAB demo

Here I will just jot down the concepts & procedures I will illustrate in class. What I skip we'll cover another time.

- installing MATLAB – pay $40 and follow the instructions on the official website, or use the computers in the PIC lab on the second floor of MS
- installing octave – on Ubuntu, it's as easy as typing `sudo apt-get install octave` into a terminal; on other distros, should be just as easy. For Mac and Windows, look for stackexchange recommendations for installation help
- running MATLAB – on linux: `./installation/path/to/matlab/bin/matlab` in terminal. On Windows and Mac: double-click the matlab executable in the file browser. The executable should be in matlab/bin
- running octave – on linux: `octave` in terminal. Will be similar for Windows and Mac
- MATLAB & octave are basically the same!!! They have very similar commands (I believe octave allows slightly more flexibility in the syntax) and run the same file types (.m), and both have all the functionality you'll need for this class. The GUI for MATLAB is slightly better but octave is free, so it's up to you.
- window panes: file browser, workspace, command history, command window, & script window/editor
- the three lines of code in Assignment1.m
- variables declared when used
- range method for initializing values – first:increment:last
- dot notation: entry-wise operations on matrices/vectors
- semicolons suppress output
- help command
- basics of the plot function

- how to run scripts
- %
- %%
- vectors
- transpose operation '
- matrices
- linspace
- accessing elements of vectors/matrices (e.g. v(4), A(3,5), v(3:end), A(4,:))
- ones, zeros, rand, NaN, inf, sum(vector)