Numerical Analysis Ex2

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I:(a)Given that $f(x)=\frac{1}{x}$ and its second derivative is $f''(x)=\frac{2}{x^3}$.Use Lagrange Formula , we can get

$$\begin{aligned}p_1(f;x)&=1\cdot\frac{x-2}{1-2}+\frac{1}{2}\cdot\frac{x-1}{2-1}\\&=-\frac{1}{2}x+\frac{3}{2}\end{aligned}$$

then substitute into the linear interpolation, we can get

$$\begin{aligned}L.H.S&=\frac{1}{x}+\frac{1}{2}x-\frac{3}{2}\\&=\frac{(x-1)(x-2)}{2x}\\R.H.S&=f''(\xi(x))\frac{(x-1)(x-2)}{2}\\&=\frac{2}{(\xi(x))^3}\frac{(x-1)(x-2)}{2}\end{aligned}$$

By comparing, we can determine $\xi(x)$ explicitly as $\xi(x)=\sqrt[3]{2x}$

(b)For $x\in[1,2]$,as $\xi(x)$ is a monotonically increasing function, $max\xi(x)=\sqrt[3]{4}$,$min\xi(x)=\sqrt[3]{2}$ and $minf''(\xi(x))=1$.

II:Let $h(x)\in P_m^+$,it's particularly in the form of $\sqrt{f(x)}$,as we want to make sure $h(x)\geq 0$.Given that $p(x)\in P_{2n}^+$ such that $p(x_i)=f_i$,thus we can directly let $p(x_i)=q^2(x_i)=f_i$.By Lagrange Formula, we can get

$$\begin{aligned}&q(x_i)=\sum_{i=0}^{n}\sqrt{f_i}\prod_{j=0,j\neq i}^{n}\frac{x-x_j}{x_i-x_j}\\&p(x_i)=q^2(x_i)\\&p(x_i)=(\sum_{i=0}^{n}\sqrt{f_i}\prod_{j=0,j\neq i}^{n}\frac{x-x_j}{x_i-x_j})^2\end{aligned}$$

III:(a)For the first term,

$$f[t,t+1]=\frac{f(t+1)-f(t)}{t+1-t}=e^{t+1}-e^t=\frac{(e-1)^1}{1!}e^t$$

hence ,it is true for the case of first term.Now assume that it is true for the $(n-1)$th term.

$$f[t,t+1,t+2,...,t+n-1]=\frac{(e-1)^{n-1}}{(n-1)!}e^t$$

For the case of nth term ,

$$f[t,t+1,t+2,...,t+n] = \frac{f[t+1,t+2,...,t+n]-f[t,t+1,t+2,...,t+n-1]}{t+n-t}$$
$$= \frac{\frac{(e-1)^n}{(n-1)!}e^{t+1} - \frac{(e-1)^{n-1}}{(n-1)!}e^t}{n}$$
$$= \frac{(e-1)^{n-1}e^t(e-1)}{n!} = \frac{(e-1)^n}{n!}e^t$$

it is true for the case of nth term. In conclusion, it is universally true for $n \in N$.

(b) $\exists \xi \in (0,n), f[0,1,2,...,n] = \frac{1}{n!}f^{(n)}(\xi)$,and after calculate we have $f^{(n)}(x) = e^x$.By the induction in (a),take $t = 0$,

$$\frac{(e-1)^n}{n!} = \frac{e^\xi}{n!}$$
$$(e-1)^n = e^\xi$$
$$\xi = nln(e-1) > \frac{n}{2}$$

Hence, ξ is located to the right of the midpoint.

IV:(a)Use the given results $f(0) = 5, f(1) = 3, f(3) = 5, f(4) = 12$ to construct a table From the

$x_0 = 0$	5	-	-	-
$x_1 = 1$	3	-2	-	-
$x_2 = 3$	5	1	1	-
$x_3 = 4$	12	7	2	$\frac{1}{4}$

table above, we know that

$$p_3(f;x) = 5 - 2x + x(x-1) + \frac{1}{4}x(x-1)(x-3)$$
$$= \frac{1}{4}x^3 - \frac{9}{4}x + 5$$

(b)Simply denote $p_3(f;x) := p_3(x)$. $x \in (1,3)$

$$p_3'(x) = \frac{3}{4}x^2 - \frac{9}{4} = 0$$
$$x = \pm\sqrt{3}$$
$$p_3''(x) = \frac{3}{2}x$$
$$p_3''(\sqrt{3}) = \frac{3\sqrt{3}}{2} > 0(min)$$

Take $x = \sqrt{3}$.The approximate value of the minimum is $p(\sqrt{3}) = \frac{1}{4}\cdot 3^{\frac{3}{2}} - \frac{9}{4}\cdot 3^{\frac{1}{2}} + 5 \simeq 2.4019$

V:(a) Given that $f(x) = x^7$, to compute $f[0,1,1,1,2,2]$, first we should know $f[x_0,x_0,x_0,...,x_0] = \frac{1}{n!}f^{(n)}(x_0)$,where x_0 is repeated $n+1$ times ,then we may construct a table. For your reference:$f'(x) = 7x^6, f''(x) = 42x^5$

From the table above,we know that

$$p(x) = x + 6x(x-1) + 15x(x-1)^2 + 42x(x-1)^3 + 30x(x-1)^3(x-2)$$

$x_0 = 0$	0	-	-	-	-	-
$x_1 = 1$	1	1	-	-	-	-
$x_2 = 1$	1	7	6	-	-	-
$x_3 = 1$	1	7	21	15	-	-
$x_4 = 2$	128	127	120	99	42	-
$x_5 = 2$	128	448	321	201	102	30

ANS:$f[0,1,1,1,2,2] = 30$

(b)The fifth derivative of f is $f^{(5)} = 2520x^2$.We can solve ξ by

$$f^{(5)}(\xi) = 2520\xi^2 = 30$$

$$\xi^2 = \frac{1}{84}$$

$$\xi = \pm\frac{\sqrt{21}}{42}$$

As $\xi \in (0,2)$,hence the true value of $\xi = \frac{\sqrt{21}}{42}$

VI:Use the given data $f(0) = 1, f(1) = 2, f'(1) = -1, f(3) = f'(3) = 0$ to construct a table.

$x_0 = 0$	1	-	-	-	-
$x_1 = 1$	2	1	-	-	-
$x_2 = 1$	2	-1	-2	-	-
$x_3 = 3$	0	-1	0	$\frac{2}{3}$	-
$x_4 = 3$	0	0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{5}{36}$

From the table we know that

$$p(x) = 1 + x - 2x(x-1) + \frac{2}{3}x(x-1)^2 - \frac{5}{36}x(x-1)^2(x-3)$$

ANS:$f(2) \simeq p(2) = \frac{11}{18}$

(b)Maximum possible error is the expression of $f(x) - p(x) = \frac{f^{(5)}(\xi)}{5!}x(x-1)^2(x-3)^2$.Now we estimate the upper bound.

$$\begin{aligned}|f(2) - p(2)| &= |\frac{f^{(5)(\xi)}}{5!}(2)(2-1)^2(2-3)^2| \\ &= |\frac{f^{(5)}(\xi)}{60}| \\ &\leq \frac{M}{60}\end{aligned}$$

VII:Use Lagrange Formula,we can get

$$\begin{aligned} f[x_0,x_1,...,x_n] &= \sum_{i=0}^{n} \frac{f(x_i)}{\prod_{j=1,j\neq i}^{n}(x_i-x_j)} \\ &= \sum_{i=0}^{n} \frac{(-1)^{n-i}f(x+ih)}{h^n i!(n-i)!} \\ &= \sum_{i=0}^{n}(-1)^{n-i}C_n^i f(x+ih) \\ &= \frac{\Delta^n f(x)}{h^n n!} \\ h^n n! f[x_0,x_1,...,x_n] &= \Delta^n f(x) \end{aligned}$$

Now we prove $\bigtriangledown^k f(x) = k!h^k f[x_0,x_{-1},...,x_{-k}]$ by mathematical induction, the other one is similarly could be proven.For the first case,

$$\begin{aligned} \bigtriangledown f(x) &= f(x) - f(x-h) \\ &= f(x_0) - f(x_{-1}) \\ &= (x_0 - x_{-1})f([x_0,x_{-1}]) \\ &= 1!h^1 f[x_0,x_{-1}] \end{aligned}$$

it is true for the case of $n=1$.Now assume that it is true for the $(n-1)$th term.

$$\bigtriangledown^{n-1} f(x) = (n-1)!h^{n-1}f[x_0,x_{-1},...,x_{-(n-1)}]$$

For the case of nth term

$$\begin{aligned} \bigtriangledown^n f(x) &= \bigtriangledown^{n-1} f(x) - \bigtriangledown^{n-1} f(x-h) \\ &= (n-1)!h^{n-1}f[x_0,x_{-1},...,x_{-(n-1)}] - (n-1)!h^{n-1}f[x_{-1},...,x_{-n}] \\ &= (n-1)!h^{n-1}(x_0 - x_{-n})f[x_0,x_{-1},...,f_{-n}] \\ &= (n-1)!h^{n-1}(x-(x-nh))f[x_0,x_{-1},...,f_{-n}] \\ &= n!h^n f[x_0,x_{-1},...,x_{-n}] \end{aligned}$$

it is true of the $n$term.In conclusion,it is universally true for $k=n\in N$.

VIII:For your reference:$f[x_0] = f(x_0)$, $\frac{\partial}{\partial x_0}f[x_0] = f'(x_0) = f[x_0,x_0]$.Prove by induction: For the first term:

$$\begin{aligned} \frac{\partial}{\partial x_0}f[x_0,x_1] &= \frac{\partial}{\partial x_0}(\frac{f[x_1]-f[x_0]}{x_1-x_0}) \\ &= \frac{(x_1-x_0)(\frac{\partial}{\partial x_0}(-f[x_0])) - (f[x_1]-f[x_0])(-1)}{(x_1-x_0)^2} \\ &= -\frac{(x_1-x_0)f[x_0,x_0]}{(x_1-x_0)^2} + \frac{f[x_0,x_1]}{x_1-x_0} \\ &= \frac{f[x_0,x_1]-f[x_0,x_0]}{x_1-x_0} \\ &= f[x_0,x_0,x_1] \end{aligned}$$

it is true for the case of $n=1$.Now assume that the $(n-1)$th term is true.

$$\frac{\partial}{\partial x_0}f[x_0,x_1,...,x_{n-1}]=f[x_0,x_0,x_1,...,x_{n-1}]$$

For the case of nth term:

$$\begin{aligned}
\frac{\partial}{\partial x_0}f[x_0,x_1,...,x_n] &= \frac{\partial}{\partial x_0}(\frac{f[x_1,...,x_n]-f[x_0,x_1,...,x_{n-1}]}{x_n-x_0}) \\
&= \frac{(x_n-x_0)\frac{\partial}{\partial x_0}(-f[x_0,x_1,...,x_{n-1}])-(f[x_1,...,x_n]-f[x_0,x_1,...,x_{n-1}])(-1)}{(x_n-x_0)^2} \\
&= -\frac{(x_n-x_0)f[x_0,x_0,x_1,...,x_{n-1}]}{(x_n-x_0)^2}+\frac{f[x_0,x_1,...,x_n]}{x_n-x_0} \\
&= \frac{f[x_0,x_1,...,x_n]-f[x_0,x_0,x_1,...,x_{n-1}]}{x_n-x_0} \\
&= f[x_0,x_0,x_1,...,x_n]
\end{aligned}$$

it is true for the nth term.In conclusion, it is universally true for $n\in N$.

IX:Let

$$\begin{aligned}
p(x) &= a_0x^n+a_1x^{n-1}+...+a_n \\
\frac{1}{a_0}p(x) &= x^n+\frac{a_1}{a_0}x^{n-1}+...+\frac{a_n}{a_0}\quad (a_0\neq 0) \\
\frac{1}{a_0}p(x) &:= x^n+b_1x^{n-1}+...+b_n
\end{aligned}$$

then we simply denote $q(x)=\frac{1}{a_0}p(x)$.For your reference, $T_n(x)=cos(narccosx)$,By Chebyshev, we can get

$$\max_{x\in[-1,1]}|\frac{T_n(x)}{2^{n-1}}|\leq\max_{x\in[-1,1]}|q(x)|<\frac{1}{2^{n-1}}$$

hence,by Corollary 2.45. we have

$$\begin{aligned}
\max_{x\in[-1,1]}|q(x)| &\geq \frac{1}{2^{n-1}} \\
\frac{1}{a_0}\max_{x\in[-1,1]}|p(x)| &\geq \frac{1}{2^{n-1}} \\
\max_{x\in[-1,1]}|p(x)| &\geq \frac{a_0}{2^{n-1}} \\
\min\max_{x\in[-1,1]}|p(x)| &= \frac{a_0}{2^{n-1}}
\end{aligned}$$

The statement above has assumed that $[a,b]=[-1,1]$.If there is an interval which is not general as $[-1,1]$, we may use a contraction mapping to standardize it.As well as the mapping is a bijection,we know that isomorphism takes the same answer.

X:By the definition of $||f||_\infty=\max_{x\in[-1,1]}|f(x)|$,and $T_n(x)=cos(narccosx)$,we can observe that $||\hat{p}_n||_\infty=\frac{1}{|T_n(a)|}$.The equal sign holds when $x=a$.Now prove $||\hat{p}_n(x)||_\infty<||p(x)||_\infty$ when $x\neq a$.
When $x=cos\frac{k\pi}{n},k\in Z$,we have $\hat{p}_n(cos\frac{k\pi}{n})=\frac{(-1)^k}{T_n(a)}$.Let $h(x_k)=p(x_k)-p_n(x_k)$.If $||p||_\infty<\frac{1}{|T_n(a)|}$,then $h(x_0)h(x_1)<0$, $h(x_1)h(x_2)<0,...,h(x_{n-1})h(x_n)<0$,thus there exist n of roots.But $x=a$ is a root too,it contradicts with the definition of P_n^a.Thus, the statement is true.
(Discussed with Ngoo Ling Hui 3200300299)

XI:First proof:$\forall k=0,1,...,n, \forall t \in (0,1), b_{n,k} > 0$
From the definition of $b_{n,k}(t) = C_n^k t^k (1-t)^{n-k}$

$$\begin{aligned} C_n^k = \frac{n!}{(n-k)!k!} &> 0 \\ t^k &> 0 \\ (1-t) &> 0 \\ (1-t)^{n-k} &> 0 \\ b_{n,k}(t) &> 0 \end{aligned}$$

Second proof:$\sum_{k=0}^{n} b_{n,k}(t) = 1$

$$\begin{aligned} 1 &= t+1-t \\ &= [(t+1)-t]^n \\ &= \sum_{k=0}^{n} C_n^k t^k (1-t)^k \\ &= \sum_{k=0}^{n} b_{n,k}(t) \end{aligned}$$

Third proof:$\sum_{k=0}^{n} k b_{n,k}(t) = nt$

$$(x+y)^n = \sum_{k=0}^{n} C_n^k x^k y^{n-k}$$

Differentiate with respect to x, and multiply through both sides with x,we can get

$$nx(x+y)^{n-1} = \sum_{k=0}^{n} C_n^k k x^k y^{n-k}$$

Let $x=t, y=1-t$,then we proved.

$$nt = \sum_{k=0}^{n} C_n^k k t^k (1-t)^{n-k} = \sum_{k=0}^{n} k b_{n,k}(t)$$

Fourth proof:$\sum_{k=0}^{n} (k-nt)^2 b_{n,k}(t) = nt(1-t)$
From the third proof,

$$nx(x+y)^{n-1} = \sum_{k=0}^{n} C_n^k k x^k y^{n-k}$$

Differentiate with respect to x and multiply through both sides with x,we can get

$$nx(x+y)^{n-1} + n(n-1)x^2(x+y)^{n-2} = \sum_{k=0}^{n} C_n^k k^2 x^k y^{n-k}$$

Let $x=t, y=1-t$,then we get

$$nt + n(n-1)t^2 = \sum_{k=0}^{n} k^2 b_{n,k}(t)$$

$$\begin{aligned} \sum_{k=0}^{n} (k-nt)^2 b_{n,k}(t) &= \sum_{k=0}^{n} k^2 b_{n,k}(t) - 2nt \sum_{k=0}^{n} k b_{n,k}(t) + \sum_{k=0}^{n} (nt)^2 b_{n,k}(t) \\ &= nt + n(n-1)t^2 - 2(nt)^2 + (nt)^2 \\ &= nt(1-t) \end{aligned}$$