Numerical Analysis Ex1

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I: (a)The width of the interval at the n^{th} step is $\frac{1}{2^{n-1}}$.(b)Let the midpoint be $c_n = \frac{1}{2}(b_n + a_n)$.The distance between the root r and the midpoint can be expressed as

$$|c_n - r| \leq 2^{-(n+1)}(b_0 - a_0) = 2^{-(n+1)}(2) = 2^{-n}$$

Thus, the maximum possible distance is 2^{-n}.

II:As the relative error is not greater than ϵ, we know that

$$\frac{|x_n - r|}{r} \leq \frac{|b_n - a_n|}{r} = \frac{1}{2^n r}(b_0 - a_0)$$

where r is the true solution and x_n is the approximate value.Here we take $\frac{1}{2^n r}(b_0 - a_0)$ as ϵ.Since $r \geq a_0$, we have

$$\frac{1}{2^n a_0}(b_0 - a_0) \leq \epsilon$$

$$-nlog2 - loga_0 + log(b_0 - a_0) \leq log\epsilon$$

$$n \geq \frac{log(b_0 - a_0) - log\epsilon - loga_0}{log2}$$

Proven.

III:Given that $p(x) = 4x^3 - 2x^2 + 3 = 0$ with the starting point $x_0 = -1$.We use the Newton's method stated below to get the approximate answer for 4 times.By hand calculate, $p'(x) = 12x^2 - 4x$.

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)}, n \in N$$

n \ functions	x_n	$p(x_n)$	$p'(x_n)$
$n = 0$	-1	-3	16
$n = 1$	-0.812500	-0.465820	11.171875
$n = 2$	-0.770804	-0.020136	10.212882
$n = 3$	-0.768832	-0.000040	10.168560
$n = 4$	-0.768828	$-$	$-$

IV:First let the true solution be r, and naturally we know $f(r) = 0$.As the given function f is differentiable, use Taylor's expansion at r,then we get

$$f(x) = f(r) + f'(\xi)(x - r) \ , \xi = r + \theta(x - r), 0 < \theta < 1$$

$$f(x_n)=f'(\xi)(x_n-r)\ \ ,\xi=r+\theta(x_n-r),0<\theta<1$$

Let the error of Newton's method at step n be e_n , $x_n=r+e_n$ and substitute in $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_0)}$.

$$f(x_n)=f'(\xi)(e_n)\ \ ,\xi=r+\theta(x_n-r),0<\theta<1$$

$$r+e_{n+1}=r+e_n-\frac{f'(\xi)(e_n)}{f'(x_0)}$$

$$e_{n+1}=e_n[1-\frac{f'(\xi)}{f'(x_0)}]$$

Thus,by comparing to $e_{n+1}=Ce_n^s$, $C=1-\frac{f'(\xi)}{f'(x_0)},\xi=r+\theta(x_n-r),0<\theta<1,\ \ s=1$

V:By comparing to the fixed-point iteration method, take $g(x_n)=tan^{-1}x_n$. As we know $g(x_n)$ is continuous on $(-\frac{\pi}{2},\frac{\pi}{2})$,now we prove the contraction on it. $\forall x,y\in(-\frac{\pi}{2},\frac{\pi}{2})$,we have

$$|g(x)-g(y)|=|tan^{-1}x-tan^{-1}y|=|(tan^{-1}\xi)'(x-y)|\le\lambda|x-y|$$

where λ satisfies $0<\lambda=\frac{1}{1+(\xi)^2}<1$.By Theorem 1.38, there is a unique fixed point and the iteration will converge to it.

VI:Let $x_{n+1}=g(x_n)=\frac{1}{p+x_n}$.As $p>1$, $\forall n\in N$, $0<x_n<1$, thus $g(x_n)$ is continuous. $\forall x,y$

$$|g(x)-g(y)|=|\frac{1}{p+x}-\frac{1}{p+y}|=|\frac{x-y}{(p-x)(p-y)}|\le\lambda|x-y|$$

where λ satisfies $0<\lambda=\frac{1}{(p-x)(p-y)}<1$.By Theorem 1.38, there is a unique fixed point and the iteration will converge to it. Now we start to find the value of the fixed point. As $\lim\limits_{n\to\infty}x_{n+1}=\lim\limits_{n\to\infty}x_n:=x$,

$$x=\frac{1}{p+x}$$

$$\frac{1}{x}=p+x$$

$$x^2+px-1=0$$

$$x=\frac{-p\pm\sqrt{p^2+4}}{2}$$

VII:The relative error is not an appropriate measure method.By Theorem 1.13, $c_n=\frac{1}{2^{n-1}}(b_0+a_0)$,let α be the true solution.

When $\alpha\ne0$

$$|c_n-\alpha|=|\frac{1}{2^{n-1}}(b_0+a_0)-\alpha|\le2^{-(n+1)}(b_0-a_0)$$

$$\frac{2}{2^n}(b_0+a_0)-|\alpha|\le\frac{1}{2}\cdot\frac{1}{2^n}(b_0-a_0)$$

$$|\alpha|\ge\frac{1}{2^n}(\frac{3}{2}b_0+\frac{5}{2}a_0)$$

$$n\ge\frac{log(\frac{3}{2}b_0+\frac{5}{2}a_0)-log|\alpha|}{log2}$$

When $\alpha=0$,we may consider it independently by setting a condition in the program. The condition could be

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if(r==0&&f(r)==0){
return r;
}
else
(The recurrence of bisection)
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