European Journal of Operational Research 236 (2014) 820–832

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Requiem for the Miller–Tucker–Zemlin subtour elimination constraints?

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ARTICLE INFO

Article history:
Received 25 August 2012
Accepted 23 July 2013
Available online 30 July 2013

Keywords:
Asymmetric traveling salesman problem
Vehicle routing
Miller–Tucker–Zemlin constraints
Projection
Subtour elimination constraints

ABSTRACT

The Miller–Tucker–Zemlin (MTZ) Subtour Elimination Constraints (SECs) and the improved version by Desrochers and Laporte (DL) have been and are still in regular use to model a variety of routing problems. This paper presents a systematic way of deriving inequalities that are more complicated than the MTZ and DL inequalities and that, in a certain way, "generalize" the underlying idea of the original inequalities. We present a polyhedral approach that studies and analyses the convex hull of feasible sets for small dimensions. This approach allows us to generate generalizations of the MTZ and DL inequalities, which are "good" in the sense that they define facets of these small polyhedra. It is well known that DL inequalities imply a subset of Dantzig–Fulkerson–Johnson (DFJ) SECs for two-node subsets. Through the approach presented, we describe a generalization of these inequalities which imply DFJ SECs for three-node subsets and show that generalizations for larger subsets are unlikely to exist. Our study presents a similar analysis with generalizations of MTZ inequalities and their relation with the lifted circuit inequalities for three node subsets.

1. Introduction

Given a loop-free directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ where $\mathcal{V} = \{1, \ldots, n\}$ is the set of nodes, $\mathcal{A} = \{(i,j) : i,j \in \mathcal{V}, i \neq j\}$ is the set of arcs and c_{ij} is the cost (e.g., distance) associated with each arc $(i,j) \in \mathcal{A}$, the asymmetric traveling salesman problem (ATSP) consists of finding the least cost Hamiltonian circuit, i.e., an elementary circuit traversing all nodes contained in the graph (Lawler, Lenstra, Rinnooy Kann, & Shmoys, 1985). Several known formulations for the ATSP are based on the following general scheme (see, for instance, Langevin, Soumis, & Desrosiers, 1990):

$$(ATSP) \quad \text{minimize} \quad \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij}, \tag{1.1}$$

$$\text{subject to} \quad \sum_{i\in\mathcal{V}:(i,j)\in\mathcal{A}} x_{ij} = 1 \quad \forall j \in \mathcal{V}, \tag{1.2}$$

$$\sum_{j\in\mathcal{V}:(i,j)\in\mathcal{A}} x_{ij} = 1 \quad \forall i \in \mathcal{V}, \tag{1.3}$$

$$\{(i,j) \in \mathcal{A} : x_{ij} = 1\} \text{ do not contain subtours}, \tag{1.4}$$

$$x_{ij} \in \{0,1\} \quad \forall (i,j) \in \mathcal{A}, \tag{1.5}$$

where x_{ij} is a binary variable equal to 1 if arc (i,j) is in the tour, or equal to 0 otherwise. Node 1 is the depot where the salesman begins and ends the tour. Constraints (1.2), (1.3) and (1.5) define the usual assignment relaxation for the ATSP. Constraints (1.4) prevent the formation of subtours not containing node 1 and can be written in several different forms. We will refer to these constraints as Subtour Elimination Constraints (SECs) and elaborate on such constraints shortly. Given an integer linear programming formulation $\mathcal{P}$, we shall henceforth denote by $\mathcal{P}_L$ its linear programming (LP) relaxation and by $v(\mathcal{P})$ the value of its optimal solution. The LP relaxation of any of the models discussed in this paper is defined by replacing constraints (1.5) with

$$0 \leqslant x_{ij} (\leqslant 1), \quad \forall (i,j) \in \mathcal{A}. \tag{1.6}$$

Different ways of modeling constraints (1.4) have been proposed in the literature. These can be divided into two families. The first family of *natural* formulations uses inequalities involving only the x_{ij} variables, as they contain one and only one variable corresponding to each arc included in the underlying graph. The other family uses inequalities that involve additional variables and formulations in this class are usually referred to as *extended* formulations. The information attached to these additional variables may considerably reduce the number of constraints involved in the model and quite often allow for deriving compact formulations (i.e., formulations involving a polynomial number of constraints and variables). By projecting the set of feasible solutions of the LP relaxation of an extended formulation into the subspace defined

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http://dx.doi.org/10.1016/j.ejor.2013.07.038

by the x_{ij} variables, one obtains a natural formulation which is equivalent to the former in the sense that both formulations produce the same LP bound. However, such natural formulations usually involve an exponential number of constraints. The reader is referred to Padberg and Sung (1991), Wong (1980), Langevin et al. (1990), Gouveia and Pires (1999), Öncan, Altinel, and Laporte (2009) and Roberti and Toth (2012), who describe and introduce several extended formulation together with the corresponding "equivalent" natural formulations. Some examples to these formulations will be given in the following sections.

One of the strongest and well known representations of (1.4) is due to Dantzig, Fulkerson, and Johnson (1954), who proposed the following inequalities:

$$\sum_{i\in\mathcal{S}}\sum_{j\in\mathcal{S}} x_{ij} \leqslant |\mathcal{S}|-1 \quad \forall \mathcal{S} \subseteq \{2,\ldots,n\}, \quad |\mathcal{S}| \geqslant 2. \tag{1.7}$$

These constraints are exponential in number and are known to be facet defining for the ATSP polytope under mild conditions (see Grötschel & Padberg, 1985). These inequalities imply that the number of arcs which can be packed in the clique defined by the set of nodes $\mathcal{S}$ cannot exceed $|\mathcal{S}|-1$. We will refer to constraints (1.7) as CLIQUE constraints. A weaker version of constraints (1.7) are the CIRCUIT inequalities which are given as follows:

$$\sum_{(i,j)\in\mathcal{C}} x_{ij} \leqslant |\mathcal{C}|-1 \quad \forall \mathcal{C} \in \mathcal{G}_{\mathcal{V}_1}, \tag{1.8}$$

where $\mathcal{G}_{\mathcal{V}_1}$ is the complete graph induced by the set of nodes $\mathcal{V}_1 = 2,\ldots,n$ and $\mathcal{C}$ is any set of arcs defining an elementary circuit (i.e., no nodes are repeated in the circuit) in $\mathcal{G}_{\mathcal{V}_1}$. These constraints have been described by Grötschel and Padberg (1985) and state that the number of arcs which can be packed in a given circuit $\mathcal{C}$ cannot exceed $|\mathcal{C}|-1$.

1.1. The Miller–Tucker–Zemlin and Desrochers–Laporte formulations

A well-known example of an extended formulation for the ATSP is due to Miller, Tucker, and Zemlin (1960), who proposed the following Subtour Elimination Constraints (SECs):

$$u_i - u_j + (n-1)x_{ij} \leqslant n-2 \quad \forall i \neq j = 2,\ldots,n, \tag{1.9}$$

$$u_i \in \mathbb{R} \quad \forall i = 2,\ldots,n. \tag{1.10}$$

Formulation *ATSP* where (1.4) is replaced by (1.9) and (1.10) will henceforth be referred to as *F*(*MTZ*) and constraints (1.9) by MTZ. The additional variables u_i in these constraints are used to give an ordering to all nodes excluding the depot to prevent the formation of illegal subtours. This is ensured by having $u_j \geqslant u_i + 1$ when $x_{ij} = 1$. Notice also that constraints (1.10) can be replaced by

$$0 \leqslant u_i \leqslant n-2 \quad \forall i = 2,\ldots,n, \tag{1.11}$$

without altering $\upsilon(F(MTZ)_L)$. Constraints (1.11) help to give a precise meaning to each variable u_i, i.e., with the inclusion of (1.11), the value given by u_i indicates the position of node i in the tour, or more precisely the number of intermediate nodes in the path between node 1 and node i in the optimal tour.

It is known that *F*(*MTZ*) has a weak LP relaxation (Gouveia & Pires, 1999; Langevin et al., 1990) and that the projection of the set of feasible solutions defined by $F(MTZ)_L$ into the space defined by the x variables is given by (1.2), (1.3) and (1.6) (Padberg & Sung, 1991) and the following inequalities

$$\sum_{(i,j)\in\mathcal{C}} x_{ij} \leqslant |\mathcal{C}| - \frac{|\mathcal{C}|}{n-1} \quad \forall \mathcal{C} \in \mathcal{G}_{\mathcal{V}_1}, \tag{1.12}$$

which are weaker versions of the CIRCUIT inequalities (since $|\mathcal{C}| \leqslant n-1$).

One ingenious way of improving MTZ constraints is due to Desrochers and Laporte (1991), who proposed the following lifted version of (1.9):

$$u_i - u_j + (n-1)x_{ij} + (n-3)x_{ji} \leqslant n-2 \quad \forall i \neq j = 2,\ldots,n. \tag{1.13}$$

We will further refer to inequalities (1.13) as DL. These constraints are interesting for two reasons: (i) they make the relation between the u_i and x_{ij} variables stronger since when $x_{ij} = 1$, a combined use of constraints (1.13) for pairs (i, j) and (j, i) imply $u_j = u_i + 1$, and (ii) they imply the CLIQUE constraints for sets $\mathcal{S}$ with two nodes. This is easy to see by adding two DL inequalities, one for a given pair (i, j) and the other for the reversed pair (j, i), which results in the CLIQUE constraint for the set $\{i, j\}$.

Reformulation techniques using other sets of variables and which imply MTZ and DL constraints have been proposed in Gouveia and Pires (1999, 2001), Sherali and Driscoll (2002), Marcotte, Savard, and Semet (2004), Sarin, Sherali, and Bhootra (2005), Gouveia and Pesneau (2006), Sherali, Sarin, and Tsai (2006) and Yaman (2006).

Of the one that is relevant to our discussion here is by Sherali and Driscoll (2002), who propose an integer linear programming formulation of the TSP obtained by applying a Reformulation–Linearization Technique (RLT), which we will refer to as SD. They show that formulation SD implies the Desrochers and Laporte (1991) for two-node subsets. Their formulation uses, in addition to the u_i variables, new variables y_{ij} representing the position of arc $(i,j) \in \mathcal{A}$ in the tour. Later, Godinho, Gouveia, and Pesneau (2011, chap. 7) showed that the SD formulation is the flow-based formulation by Gavish and Graves (1978) augmented with several sets of constraints. An interesting application of the RLT presented by Sherali and Driscoll (2002) is to extend the idea to three-node subsets, similar to the aims of this paper. The application results in the following MTZ-like valid inequalities for the ATSP:

$$\begin{aligned} u_k \geqslant\ & u_i + 2 - (n-1)(2 - x_{ij} - x_{jk}) + (n-2)(1-x_{ij})(1-x_{jk}) \\ & + (n-1)x_{ik} + (n-4)(x_{ki}x_{ij} + x_{jk}x_{ki} + x_{kj}x_{ji}) \\ \forall i \neq\ & j \neq k = 2,\ldots,n. \end{aligned} \tag{1.14}$$

Inequalities (1.14) are interesting in that they provide information on the relationship between the u_i and u_k variables, in a same way as (1.13), but for three nodes. In particular, the constraints imply $u_k = u_i + 2$ when $x_{ij} = x_{jk} = 1$ for $i \neq j \neq k = 2,\ldots,n$. However, as it is clear from their structure, constraints (1.14) are multilinear (quadratic).

1.2. Motivation

MTZ-like constraints, such as (1.9) and (1.13), although originally proposed for the ATSP, have also extensions to a variety of Vehicle Routing Problems. A brief overview of the recent literature indicates that, even though these constraints produce weak relaxations, they are still popular and are being used to model various types of conditions for similar problems. One reason for this situation is the flexibility of these constraints, whereas another reason can be stated as their capability in solving small to medium sized problems to optimality via general easily available linear integer programming codes.

However, as mentioned before we should not ignore that these constraints produce weak relaxations. Also, since the work by Desrochers and Laporte (1991) no significant improvement has been done with respect to MTZ-like formulations in the space of the (u, x) variables. This motivates the main research question that we try to answer here:

Can we find generalizations of the DL inequalities in the (x, u) space that would imply all CLIQUE inequalities?

We emphasize that an affirmative answer to question would lead to a model with a strong LP relaxation, at least as strong as CLIQUE. Furthermore, we think that such a model would be attractive to the past users of MTZ like formulations even though this hypothetical model may turn out to be exponential in size and may need routines for constraint separation to be used in practice. On the other hand we should note that this question is not straightforward to address. In particular, it is not easy to find a systematic method that shows how to generalize the MTZ and DL inequalities or, alternatively, show that it cannot be done. Furthermore, generalizing DL inequalities "by hand" in such a way that by projection we would obtain CLIQUE inequalities for sets of cardinality as small as three nodes already appears to be a nontrivial task.

In this paper, we provide and illustrate a "polyhedral approach" that corresponds to finding all (or at least, interesting) facet defining inequalities of several polyhedra that are obtained by considering the integer solutions for a sequence of problems defined in increasingly larger subsets of nodes and arcs involving the two sets of variables. We first show that this approach, when used for two node sets, allows us to reinvent the MTZ and DL inequalities. This positive outcome suggests that extending the approach for larger sets might reveal what we are looking for.

As we shall show in our investigations, we will be able to answer affirmatively to the question in the case of three nodes by presenting *linear* inequalities. We will also give some valid arguments to conclude that such inequalities do not exist for bigger sets showing that the hypothetical good model is unlikely to exist. Our conclusion is that although the MTZ and DL like inequalities definitely have a most deserved Oscar for the most compact and easier to handle category in the TSP extended formulations awards, it is not clear whether much more can be done with the sets of variables u_i and x_{ij} in terms of improving the model further.

The rest of the paper is structured as follows. In Section 2, we describe the polyhedral approach to "reinvent" the DL inequalities. In Section 3 we use the same approach to discover more complicated inequalities that in a certain sense generalize the DL inequalities. Section 4 presents projection results with the new inequalities. Computational results are presented in Section 5. Conclusions are stated in Section 6.

2. A polyhedral derivation of the MTZ and the DL inequalities

One way of viewing and motivating the MTZ constraints is as follows. It is well known that inequalities (1.9) guarantee that, for two arcs (i, j) and (j, i), $i, j = 2, \ldots, n$; $i \neq j$ and for two variables $0 \leqslant u_i, u_j \leqslant n-2$, the following conditions hold:

- (A1) if $x_{ij} = 1$ and $x_{ji} = 0$ then $u_j \geqslant u_i + 1$,
- (A2) if $x_{ij} = 0$ and $x_{ji} = 1$ then $u_i \geqslant u_j + 1$,
- (A3) if $x_{ij} = 0$ and $x_{ji} = 0$ then $|u_i - u_j| \leqslant n-2$,
- (A4) $0 \leqslant u_i \leqslant n-2$, $0 \leqslant u_j \leqslant n-2$, $x_{ij}, x_{ji} \in \{0,1\}$.

In this section, we will examine the reversed argument. Suppose we consider a polyhedron $\mathcal{P}^2_{MTZ}$ on the four variables u_i, u_j, x_{ij} and x_{ji} and defined by the convex hull of the set of points satisfying conditions (A1)–(A4). Several questions arise at this point: Which set of inequalities describe this polyhedron? Does this set include the previous known MTZ constraints? Or can we do better?

In fact, for small sets of nodes ($n = 4, \ldots, 7$) we can easily give a complete description of $\mathcal{P}^2_{MTZ}$ using PORTA (Christof & Löbel, 2004):

$$x_{ij}, x_{ji} \geqslant 0, \tag{2.1}$$
$$u_i \geqslant x_{ji}, \tag{2.2}$$
$$u_j \geqslant x_{ij}, \tag{2.3}$$
$$x_{ij} + x_{ji} \leqslant 1, \tag{2.4}$$
$$u_i + x_{ij} \leqslant n-2, \tag{2.5}$$
$$u_j + x_{ji} \leqslant n-2, \tag{2.6}$$
$$u_i - u_j + (n-1)x_{ij} \leqslant n-2, \tag{2.7}$$
$$u_j - u_i + (n-1)x_{ji} \leqslant n-2. \tag{2.8}$$

We note that this description was obtained for small values of n and we have no proof of completeness for any value of n. However, the listed inequalities are valid for any value of n. We denote by $\mathcal{F}^2_{MTZ}$ the model (2.1)–(2.8). In $\mathcal{F}^2_{MTZ}$, besides rediscovering the MTZ constraints (2.7) and (2.8), constraints (2.2) and (2.3) give lower bounds on the values of the two node variables u_i and u_j, depending on the inclusion, or not, of arcs (j, i) and (i, j), respectively. Similarly, constraints (2.5) and (2.6) provide upper bounds on the same variables. We note that similar constraints have already been proposed in the literature (Desrochers & Laporte, 1991; Gouveia, 1996). Constraints (2.1) are the boundary conditions on variables x_{ij} and x_{ji}. One interesting aspect of the polyhedron $\mathcal{P}^2_{MTZ}$ is constraints (2.4), which are two-node versions of CLIQUE constraints (1.7). In fact, as the following proposition shows, these inequalities are facet defining for $\mathcal{P}^2_{MTZ}$:

Proposition 1. *Let $F_{2CLIQUE}$ denote a face of $\mathcal{P}^2_{MTZ}$ shown as $\{(u_i, u_j, x_{ij}, x_{ji}) \in \mathcal{P}^2_{MTZ} : x_{ij} + x_{ji} = 1\}$. Then, $F_{2CLIQUE}$ is a facet of $\mathcal{P}^2_{MTZ}$ for $n \geqslant 4$.*

Proof. It is easy to see that $F_{2CLIQUE}$ is a valid inequality for and a proper face of $\mathcal{P}^2_{MTZ}$. We now observe that $\mathcal{P}^2_{MTZ}$ is full-dimensional as $(n-2-2\epsilon, n-2-2\epsilon, \epsilon, \epsilon)$ is a set of interior points for a sufficiently small constant $\epsilon > 0$. Therefore, $\dim(\mathcal{P}^2_{MTZ}) = 4$ and $\dim(F_{2CLIQUE}) \leqslant 3$. To prove that $F_{2CLIQUE}$ defines a facet of the underlying polyhedron, it suffices to show the existence of four affinely independent points satisfying this constraint as an equality. Consider the set of points $(\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}) = \{(u_i, u_i + 1, 1, 0), (u_i, u_i + 2, 1, 0), (u_i + 1, u_i, 0, 1), (u_i + 2, u_i, 0, 1)\}$ for $0 \leqslant u_i \leqslant n-2$. For scalars λ_i, $i = 1, \ldots, 4$, $\lambda_1\mathbf{u_1} + \lambda_2\mathbf{u_2} + \lambda_3\mathbf{u_3} + \lambda_4\mathbf{u_4} = 0$ and $\sum_{i=1}^{4}\lambda_i = 0$ result in the unique solution $\lambda_i = 0$ for $i = 1, \ldots, 4$, hence proving that the four points are affinely independent. Then, $\dim(F_{2CLIQUE}) \geqslant 3$ which implies $\dim(F_{2CLIQUE}) = 3$ and the proof follows. □

Consider, now, the case where we complicate the original polyhedron by considering the following conditions, resulting in polyhedron $\mathcal{P}^2_{DL}$ on two nodes i and j, as the convex hull of the points satisfying the conditions below:

- (A5) if $x_{ij} = 1$ and $x_{ji} = 0$ then $u_j = u_i + 1$,
- (A6) if $x_{ij} = 0$ and $x_{ji} = 1$ then $u_i = u_j + 1$,
- (A7) if $x_{ij} = 0$ and $x_{ji} = 0$ then $|u_i - u_j| \leqslant n-2$,
- (A8) $0 \leqslant u_i \leqslant n-2$, $0 \leqslant u_j \leqslant n-2$, $x_{ij}, x_{ji} \in \{0,1\}$.

The four conditions (A5)–(A8) are very similar to (A1)–(A4) apart from the relationship between variables u_i and u_j being stated as an *equality*, rather than an *inequality*. Let $\mathcal{F}^2_{DL}$ denote the following model:

$$x_{ij}, x_{ji} \geqslant 0, \tag{2.9}$$
$$u_i \geqslant x_{ji}, \tag{2.10}$$
$$u_j \geqslant x_{ij}, \tag{2.11}$$
$$u_i + x_{ij} \leqslant n-2, \tag{2.12}$$
$$u_j + x_{ji} \leqslant n-2, \tag{2.13}$$
$$u_i - u_j + (n-1)x_{ij} + (n-3)x_{ji} \leqslant n-2, \tag{2.14}$$
$$u_j - u_i + (n-1)x_{ji} + (n-3)x_{ij} \leqslant n-2. \tag{2.15}$$

Based on an analysis with PORTA, $\mathcal{F}^2_{DL}$ gives a complete description of $\mathcal{P}^2_{DL}$ for $n = 4, \ldots, 7$. $\mathcal{F}^2_{DL}$ includes similar inequalities to $\mathcal{F}^2_{MTZ}$

as well as the "reinvented" DL inequalities (1.13) listed as (2.14) and (2.15). Note that this description does not include the CLIQUE constraints (1.7) for $S = \{i, j\}$, which, although valid for $\mathcal{P}^2_{DL}$, are implied by the DL inequalities (2.14) and (2.15). Thus, we conclude that specifying $u_j = u_i + 1$, instead of the weaker version $u_j \geqslant u_i + 1$, when $x_{ij} = 1$ has a strong effect on the description of the corresponding polyhedron.

We now show that inequalities (2.14) define facets of $\mathcal{P}^2_{DL}$.

Proposition 2. *Let F_{DL} denote a face of $\mathcal{P}^2_{DL}$ shown as $\{(u_i, u_j, x_{ij}, x_{ji}) \in \mathcal{P}^2_{DL} : u_i - u_j + (n-1)x_{ij} + (n-3)x_{ji} = n-2\}$. Then, F_{DL} is a facet of $\mathcal{P}^2_{DL}$ for $n \geqslant 4$.*

Proof. It is easy to see that F_{DL} is a (proper) face of $\mathcal{P}^2_{DL}$ by checking for all points satisfying conditions (A5)–(A8) and that F_{DL} holds with equality for at least one point in $\mathcal{P}^2_{DL}$, e.g., $(u_i, u_j, x_{ij}, x_{ji}) = (n-2, n-1, 1, 0)$. Similar to Proposition 1, it is easy to see that $\mathcal{P}^2_{DL}$ is full-dimensional so $\dim(\mathcal{P}^2_{DL}) = 4$ and $\dim(F_{DL}) \leqslant 3$. To prove that (2.14) define facets of the underlying polyhedron, notice that $(u_i, u_j, x_{ij}, x_{ji}) = \{(0, 1, 1, 0), (n-3, n-2, 1, 0), (n-2, n-3, 0, 1), (n-2, 0, 0, 0)\}$ are four affinely independent points satisfying F_{DL} at equality, hence $\dim(F_{DL}) \geqslant 3$. This implies $\dim(F_{DL}) = 3$. □

Before concluding the section we review and compare, from a theoretical point of view, the two polyhedra $\mathcal{P}^2_{MTZ}$ and $\mathcal{P}^2_{DL}$. First, we observe that allowing $u_j = u_i + 1$ leads to the strong DL constraints (2.14) and (2.15) which imply CLIQUE constraints (1.7) for $\mathcal{S} = \{i, j\}$. The DL inequalities are not valid for $\mathcal{P}^2_{MTZ}$ and therefore CLIQUE constraints for two nodes need to be included in its description.

The analysis of the polyhedra $\mathcal{P}^2_{MTZ}$ and $\mathcal{P}^2_{DL}$ presented in this section for a subset of two nodes i and j produce inequalities which are already known, namely the MTZ and the DL inequalities, for a subset of two nodes i and j. In order to find generalizations of these inequalities, we complicate the previous polyhedra by "enlarging" the set of nodes and arcs in our initial polyhedron. This is presented in the next section.

3. Polyhedral investigation for larger sets

This section extends the analysis in the previous section to three and four node sets.

3.1. Polyhedra on three nodes

In the previous section, we have distinguished between the "$\leqslant$" polyhedron (where the relationship between the u variables is an inequality) named $\mathcal{P}^2_{MTZ}$ and the "=" polyhedron named $\mathcal{P}^2_{DL}$ (where the relationship between the u variables is an equality), and we have shown that stronger inequalities can be obtained by using the latter. It is for this reason that the analysis presented in the following will only consider the "=" polyhedra, although for comparison purposes we will comment on the "$\leqslant$" at the end of this section.

We now extend the previous analysis to a set with three nodes $S = \{i, j, k\}$. Let $x(S : S')\sum_{i \in S, j \in S'} x_{ij}$. For notational convenience, we will write $x(S)$ instead of $x(S: S)$. Consider now the polyhedron $\mathcal{P}^3_{DL}$ defined by the convex hull of the set of points satisfying the conditions below, with $x_{ab} \in \{0,1\}$ for all distinct a, $b \in S$ and $0 \leqslant u_a \leqslant n-2$ for all $a \in S$:

- $x(\{t\}: S \backslash \{t\}) \leqslant 1$, for $t \in S$,
- $x(S \backslash \{t\}:\{t\}) \leqslant 1$, for $t \in S$,
- for all $a \neq b \neq c \in \{i, j, k\}$, if $x_{ab} + x_{bc} = 2$ then $u_c = u_a + 2$ and $u_b = u_a + 1$, otherwise $|u_c - u_a| \leqslant n-2$.
- for all $a \neq b \in \{i, j, k\}$, if $x_{ab} = 1$ then $u_b = u_a + 1$, otherwise $|u_b - u_a| \leqslant n-2$.

We started our analysis by using PORTA (Christof & Löbel, 2004) for $4 \leqslant n \leqslant 7$ to list a complete description of the convex hull of solutions satisfying the conditions above. Unfortunately, even for these small values of n the list turned out to be too long. However, a "search" of the list has provided some interesting inequalities, presented below.

3.1.1. CLIQUE inequalities for three nodes

The first set of constraints (3.1) are similar in structure to DL inequalities in that they include two variables u_i and u_k along with a set of variables associated with every arc in the three-node set:

$$u_i - u_k + (n-1)(x_{ij} + x_{jk}) + (n-3)(x_{kj} + x_{ji}) + nx_{ik} + (n-4)x_{ki} \leqslant 2n-4. \tag{3.1}$$

These constraints are clearly valid for $4 \leqslant n \leqslant 7$ (as we obtained them by using PORTA) and here we provide a proof of their validity for any value of n:

Proposition 3. *For every node triple $i \neq j \neq k \neq 1$, constraints (3.1) are valid for the polyhedron $\mathcal{P}^3_{DL}$.*

Proof. To see that constraints (3.1) are valid we consider the six maximal mutually exclusive cases:

1. $x_{ij} = x_{jk} = 1$ and $x_{ab} = 0$ for the four remaining arcs (a, b) where a, $b \in S$ implies $u_i - u_k + 2(n-1) \leqslant 2n-4$ leading to $u_i - u_k \leqslant -2$ which is valid.
2. $x_{jk} = x_{ki} = 1$ and $x_{ab} = 0$ for the four remaining arcs (a, b) where a, $b \in S$ implies $u_i - u_k \leqslant 1$ which is valid.
3. $x_{ki} = x_{ij} = 1$ and $x_{ab} = 0$ for the four remaining arcs (a, b) where a, $b \in S$ is similar to Case 2 above implying $u_i - u_k \leqslant 1$.
4. $x_{ik} = x_{kj} = 1$ and $x_{ab} = 0$ for the four remaining arcs (a, b) where a, $b \in S$ implies $u_i - u_k \leqslant -1$.
5. $x_{ji} = x_{ik} = 1$ and $x_{ab} = 0$ for the four remaining arcs (a, b) where a, $b \in S$ implies $u_i - u_k \leqslant -1$.
6. $x_{kj} = x_{ji} = 1$ and $x_{ab} = 0$ for the four remaining arcs (a, b) where a, $b \in S$ implies $u_i - u_k \leqslant -2$. □

A few interesting remarks on these inequalities are given next:

- We first observe that these inequalities are also valid for the ATSP polyhedron. This follows from the result of Proposition 3.
- Constraints (3.1) imply CLIQUE constraints for three node sets. Such a constraint for the set $\{i, j, k\}$ results from adding up two inequalities (3.1) for the triples (i, j, k) and (k, j, i). In fact, the same inequality can also be obtained by adding three of such constraints, namely for the triples (i, j, k), (k, i, j) and (j, k, i). Thus, the new constraints (3.1) can be considered as a generalization of DL inequalities for three nodes. Fig. 1 shows the support graph of constraints (3.1).
- Combining inequality (3.1) for the triples $\{i, j, k\}$ and $\{k, j, i\}$ when $x_{ij} = 1$ and $x_{jk} = 1$ (or $x_{kj} = 1$ and $x_{ji} = 1$) we obtain the intended interpretation $u_i - u_k = 2$. However, in contrast to the DL inequalities, constraints (3.1) alone in place of (1.4) do not provide a valid model for the ATSP since they only guarantee $u_i - u_k \leqslant n-4$ when $x_{ik} = 1$. Thus, they provide the correct information on the u variables for paths of length two, but not for a single arc. Therefore, these constraints must only be considered as valid inequalities to an already valid formulation, such as DL. In fact, the DL inequalities also appear in the description obtained by PORTA which guarantee the correct information on the u_i variables for single arcs.

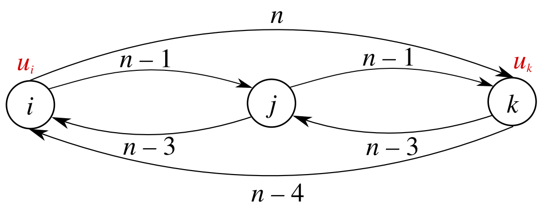

Fig. 1. The support graph of constraints (3.1).

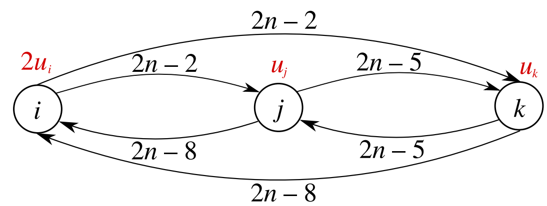

Fig. 2. The support graph of constraints (3.3).

We can also give some intuition on how to obtain inequality (3.1) using the ones obtained from the two node analysis presented in the previous section. We start by adding two DL constraints for the pairs (i, j) and (j, k) leading to,

$$u_i - u_k + (n-1)(x_{ij} + x_{jk}) + (n-3)(x_{kj} + x_{ji}) \leqslant 2n - 4. \tag{3.2}$$

Now, consider the lifting of inequality (3.2) by inclusion of the term $\alpha_{ik}x_{ik}$, as a result of which we obtain $\alpha_{ik} = n$ as a valid lifting coefficient for x_{ik}. Next, we consider the inclusion of the term $\alpha_{ki}x_{ki}$ to the lifted version of inequality (3.2). Again, one can easily compute $\alpha_{ki} = n - 4$ as a valid lifting coefficient for x_{ik}. Note that this procedure gives an alternate proof of the validity of (3.2). The proof that the lifting coefficient are correct is omitted, but a sketch follows setting $x_{ki} = 1$ and calculating the maximum value for α_{ik} for all possible cases. We note, however, that when producing these constraints by the described procedure, the lifting sequence is not unique and that different sequences would result in different constraints. The reason why we mention this alternative derivation here is that it might suggest how to derive "similar" inequalities for larger subsets.

We now show that (3.1) are facets of $\mathcal{P}^3_{DL}$.

Proposition 4. *Let F_{3DL} denote a face of $\mathcal{P}^3_{DL}$ shown as $\{(u_i, u_j, u_k, x_{ij}, x_{jk}, x_{kj}, x_{ji}, x_{ik}, x_{ki}) \in \mathcal{P}^3_{DL} : u_i - u_k + (n-1)(x_{ij} + x_{jk}) + (n-3)(x_{kj} + x_{ji}) + nx_{ik} + (n-4)x_{ki} = 2n - 4\}$. Then, F_{3DL} is a facet of $\mathcal{P}^3_{DL}$ for $n \geqslant 6$.*

Proof. We first observe that F_{3DL} is a face of $\mathcal{P}^3_{DL}$ using the result of Proposition 3, which also provides three points satisfying F_{3DL} with equality. It is easy to see that $\mathcal{P}^3_{DL}$ is full-dimensional by considering any point with $x_{ab} = x_{bc} = x_{ca} = x_{ba} = x_{cb} = x_{ac} = \epsilon$ for small enough $\epsilon > 0$ and with $u_a, u_b, u_c > 0$ satisfying $|u_b - u_a| < n - 2$ for all $a, b, c \in \{i, j, k\}$. Therefore, $\dim(\mathcal{P}^3_{DL}) = 8$ and $\dim(F_{3DL}) \leqslant 7$. Consider, now the following set of points induced by all possible two-cycles $\mathcal{C}_2$ on $\mathcal{S} = \{i, j, k\}$, i.e., $(u_a, u_b, x_{ij}, x_{jk}, x_{kj}, x_{ji}, x_{ik}, x_{ki}) \in \mathcal{P}^3_{DL}$ such that, if $x_{ab} = 1$ for $a, b, c \in \mathcal{S}$ then $u_b = u_a + 1$ and if $x_{ab} + x_{bc} = 2$ then $u_c = u_a + 2$. It is easy to see that there are six of these points corresponding to the six two-cycles on S. Consider, now, the two points induced by $x_{ij} = 1$ or $x_{jk} = 0$ with all other $x_{ab} = 0$ for $a, b \in S$. The former implies $u_i \leqslant n - 3$ as node i cannot be the last node in a route and induces the point $(n - 3, 0, 1, 0, 0, 0, 0, 0)$. The latter implies $u_k \geqslant 1$ as node k cannot be the first node on a route and induces the point $(n - 2, 1, 0, 1, 0, 0, 0, 0)$. It is easy to prove that the eight points just defined are affinely independent (see Lemma A.1 for a full description of these points and its proof in Appendix A). Hence $\dim(F_{3DL}) \geqslant 7$, which implies $\dim(F_{3DL}) = 7$, and the proof follows. □

The constraints we will now present, also taken from the list generated by PORTA, are somewhat more radical in that they include all the three variables u_i, u_j, and u_k, which is not usual and is in contrast to most MTZ-based constraints which only include two.

Proposition 5. *The following set of constraints are valid constraints for the polyhedron $\mathcal{P}^3_{DL}$:*

$$2u_i - u_j - u_k + (2n-2)(x_{ij} + x_{ik}) + (2n-8)(x_{ji} + x_{ki}) + (2n-5)(x_{jk} + x_{kj}) \leqslant 4n - 10, \tag{3.3}$$

$$-2u_i + u_j + u_k + (2n-8)(x_{ij} + x_{ik}) + (2n-2)(x_{ji} + x_{ki}) + (2n-5)(x_{jk} + x_{kj}) \leqslant 4n - 10. \tag{3.4}$$

Proof. To prove the validity of constraint (3.3) we consider the three maximal cases stated below associated with all the possible traversal of 3-node paths:

1. $x_{ij} = x_{jk} = 1$ or $x_{ik} = x_{kj} = 1$: In this case, the right hand side of constraint (3.3) becomes -3. Since $u_j + u_k = 2u_i + 3$, $2u_i - u_j - u_k = -3 \leqslant -3$ which is valid.
2. $x_{ji} = x_{ik} = 1$ or $x_{ki} = x_{ij} = 1$: In this case, the right hand side of constraint (3.3) becomes 0. Since $u_j + u_k = 2u_i$, $2u_i - u_j - u_k = 0 \leqslant 0$ which is valid.
3. $x_{jk} = x_{ki} = 1$ or $x_{kj} = x_{ji} = 1$: In this case, the right hand side of constraint (3.3) becomes 3. Since $u_j + u_k = 2u_i - 3$, $2u_i - u_j - u_k = 3 \leqslant 3$ which is valid.

Note that two node paths with only one variable equal to 1 are trivially satisfied by constraint (3.3). This proves the validity of constraint (3.3). The validity of constraint (3.4) can be proved in a similar manner. □

The support graph of these constraints is slightly different as compared to the previous one and is as shown in Fig. 2.

Again, we observe that these inequalities are also valid for the ATSP polyhedron. This follows from the previously stated result. One can easily see that by adding up (3.3) for the triple (i, j, k) to (3.4) for the same triple gives the CLIQUE constraints for the set $|S| = 3$. Thus, these constraints may also be considered as a generalization of the DL inequalities, although in a more radical way. However, unlike constraints (3.1), a lifting procedure that will result in constraints (3.3) is far from obvious.

Similar to what happens with constraints (3.1), the new constraints (3.3) and (3.4) alone in place of (1.4) do not provide a valid model for the ATSP. The set of constraints presented in the next subsection, however, do.

3.1.2. 2PATH inequalities and an alternative formulation for the ATSP

A different class of inequalities obtained from the listing provided by PORTA is presented below.

Proposition 6. *The following set of constraints, named 2PATH constraints, are valid constraints for the polyhedron $\mathcal{P}^3_{DL}$:*

$$u_i - u_k + (2n-3)x_{ik} + (n-4)x_{ki} + (n-1)(x_{ij} + x_{jk}) \leqslant 2n - 4, \tag{3.5}$$

$$u_k - u_i + (2n-7)x_{ik} + (n-1)x_{ki} + (n-4)(x_{ij} + x_{jk}) \leqslant 2n - 6. \tag{3.6}$$

Proof. We only show the validity of (3.5). The proof for (3.6) is similar. For constraints (3.5) we consider the three maximal mutually exclusive cases:

1. $x_{ik} = 1$ and $x_{ki} = x_{ij} = x_{jk} = 0$ implies $u_i - u_k \leqslant -1$ which is valid.
2. $x_{ij} = x_{jk} = 1$ and $x_{ik} = x_{ki} = 0$ implies $u_i - u_k \leqslant -2$ which is valid.
3. $x_{ki} = x_{ij} = 1$, $x_{jk} = x_{ik} = 0$ or $x_{ki} = x_{jk} = 1$, $x_{jk} = x_{ik} = 0$ both imply $u_i - u_k \leqslant 1$ which is valid. □

A few interesting remarks on these inequalities are given next:

- As before, these inequalities are also valid for the ATSP polyhedron.
- Constraints (3.5) and (3.6) imply a lifted version of the circuit inequalities (1.8) that are not implied by the CLIQUE inequalities (1.7). As mentioned before, the CLIQUE inequalities (1.7) can be seen as lifted versions of (1.8). However, as pointed out by Grötschel and Padberg (1985) and Fischetti (1991), sequential lifting can be used to obtain "asymmetric" lifted versions of the CIRCUIT inequalities (1.8). For sets of three nodes, there is only one such inequality given by $2x_{ik} + x_{ij} + x_{jk} + x_{ki} \leqslant 2$ which results from lifting the circuit inequality $x_{ij} + x_{jk} + x_{ki} \leqslant 2$. In fact by adding (3.5) and (3.6) for $\{i, j, k\}$, we obtain the lifted CIRCUIT inequalities (1.8) as described above for these three nodes. Support graphs of lifted CIRCUIT and the 2PATH inequalities for three nodes are given in Figs. 3 and 4, respectively.
- Constraints (3.5) can be obtained by adding two MTZ constraints of (2.7) for an arc pair (i, j) and (j, k) and lifting with x_{ik} and x_{ki} in the given order. A similar observation holds for the second set of constraints. We can view them as generalizations of the DL inequalities in the sense they provide the right information on the u variables not only for two consecutive arcs, as with constraints (3.1), but, at the same time, also on a single arc, which constraints (3.1) alone do not guarantee. In other words, for a solution with $x_{ij} = x_{jk} = 1$ $(x_{ik} = x_{ki} = 0)$, the same constraints imply $u_k = u_i + 2$. Similarly, for a solution with $x_{ik} = 1$ $(x_{ki} = x_{ij} = x_{jk} = 0)$, constraints (3.5) and (3.6) imply $u_k = u_i + 1$. Furthermore, considering only the x_{ik} term on the left-hand side of (3.5) one obtains a MTZ like constraint (1.9) albeit with weaker coefficients $2n - 3$ and $2n - 4$ as opposed to $n - 1$ and $n - 2$, but which still guarantee $u_k \geqslant u_i + 1$ when arc (i, k) appears in the solution. Thus, subtours are prevented and we may conclude that these constraints, when used in place of (1.4) in the generic model *ATSP*, result in a valid model for the ATSP.
- For a subset of three nodes $\{i, j, k\}$, constraints (3.5) provide the same information as constraints (1.14) proposed by Sherali and Driscoll (2002). This is easy to see by checking the three maximal mutually exclusive cases shown in the proof of Proposition 6 in constraints (1.14). However, constraints (3.5) represent this information as a *linear* set of inequalities whereas constraints (1.14) are *quadratic*.

We conclude this section by referring to a similar analysis of a polyhedron on three nodes with a "$\leqslant$" relationship between the u variables. This analysis, together with the provided analysis for the "=" polyhedron, yields the following observations: (i) the new inequalities described in this section are not valid for the "$\leqslant$" polyhedron, which is easy to see since these inequalities are derived from the assumptions that equalities between the u_i variables hold and (ii) the "$\leqslant$" polyhedron has, in place of the new inequalities, CLIQUE constraints and lifted CIRCUIT inequalities for three nodes (which are only defined in the x_{ij} subspace). Note that this phenomenon is exactly the same as the one observed in the two node set polyhedral analysis presented in Section 2. However, this does not extend itself in an analysis for four nodes, as discussed in the following section.

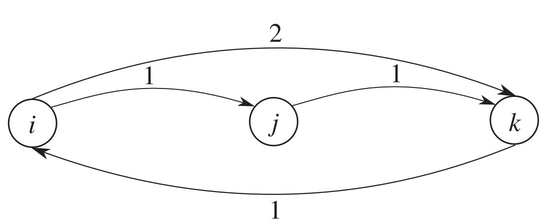

Fig. 3. A lifted 3-cycle inequality.

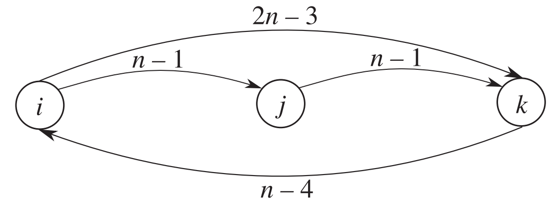

Fig. 4. 2PATH Inequalities.

3.2. Polyhedra on four nodes

The polyhedral approach for three node sets is worth using but is already complicated since it needs an exhaustive and a tedious analysis of a long list of inequalities even for small values of n. A similar polyhedral approach for four node sets proved to be rather impractical due to the number of points needed to be generated in order to be able to use PORTA to generate the full description of the corresponding polytope. Despite this difficulty, we were able to produce a complete list of inequalities with four node sets and "=" conditions for $n = 4$. Such a list contains just over 25,500 inequalities and an exhaustive search of the inequalities in it would prove to be quite complicated in order to find generalizations of the inequalities (3.1), (3.5) and (3.6). Unfortunately or fortunately (depending on the point of view), and remarkably at the same time, the list contained the easier to find CLIQUE constraints and the whole set of lifted CIRCUIT inequalities for four nodes in the x_{ij} subspace. This is remarkable because when the analysis is extended from two or three node sets for bigger sets, there is not a similar behavior to what was observed in the previous section. CLIQUE constraints for two or three nodes sets were obtained in the "$\leqslant$" polyhedra. The corresponding analysis for the "=" polyhedra provides more information and produces the MTZ-based inequalities described previously in this paper that imply these small CLIQUE constraints. For four node subsets, the "=" analysis for $n = 4$ already includes the corresponding CLIQUE inequalities and, as we shall show next, CLIQUE constraints for four nodes turn out to be facets of the corresponding equality polyhedron for any value of n. To see this, consider the following polyhedron $\mathcal{P}^4_{DL}$ on the node set $S = \{i, j, k, l\}$, defined by the convex hull of the set of points satisfying the conditions below:

- $x(\{t\}: S\backslash\{t\}) \leqslant 1$, for $t \in S$,
- $x(S\backslash\{t\}:\{t\}) \leqslant 1$, for $t \in S$,
- for all $a \neq b \neq c \neq d \in \{i, j, k\}$, if $x_{ab} + x_{bc} + x_{cd} = 3$ then $u_d = u_a + 3$, $u_c = u_a + 2$ and $u_b = u_a + 1$, otherwise $|u_d - u_a| \leqslant n - 2$,
- for all $a \neq b \neq c \in \{i, j, k\}$, if $x_{ab} + x_{bc} = 2$ then $u_c = u_a + 2$ and $u_b = u_a + 1$, otherwise $|u_c - u_a| \leqslant n - 2$,
- for all $a \neq b \in \{i, j, k\}$, if $x_{ab} = 1$ then $u_b = u_a + 1$, otherwise $|u_b - u_a| \leqslant n - 2$.

We now show how CLIQUE inequalities (1.7) written for four nodes are related to $\mathcal{P}^4_{DL}$.

Proposition 7. *Let (u, x) denote a point in $\mathcal{P}^4_{DL}$ and $F_{4CLIQUE}$ be a face of $\mathcal{P}^4_{DL}$ shown as $\left\{(u,x) \in \mathcal{P}^4_{DL} : \sum_{i\in\mathcal{S}}\sum_{j\in\mathcal{S}} x_{ij} = |\mathcal{S}| - 1\right\}$ for $S = \{i, j, k, l\}$. Then, $F_{4CLIQUE}$ is a facet of $\mathcal{P}^4_{DL}$ for $n \geqslant 5$.*

Proof. We first note that $\mathcal{P}^4_{DL}$ is full-dimensional therefore $\dim(\mathcal{P}^4_{DL}) = 16$, hence and dim $(F_{4CLIQUE}) \leqslant 15$. Consider the fully connected graph K_4 on S and let $\mathcal{P} = \{P_1, P_2, \ldots, P_{24}\}$ bet the set of all directed simple paths of length three on K_4, where each path is characterized by a sequence of four nodes $\{a, b, c, d\}$. We will denote the generalized incidence vector $(\chi^{P_4}, \upsilon^{P_4})$ induced by path P_i, $i = 1,\ldots,24$, where $\chi^{P_i}(\{a,b\}) = 1$ for any edge $\{a, b\} \in P_i$ and $\upsilon^{P_i} = (u_a, u_b, u_c, u_d) = (u, u+1, u+2, u+3)$ for $P_i \in \mathcal{P}$ and $u \leqslant n-5$. A full characterization of the matrix A of the set of these 24 points is given in Appendix B. It can be verified through any numerical package (e.g., Matlab) that rank $(A) = 15$, which shows the existence of 15 linearly independent points satisfying $F_{4CLIQUE}$ at equality, hence dim $(F_{4CLIQUE}) \geqslant 15$ and the proof follows. □

The result of Proposition 4 does not mean that several generalized DL-like inequalities cannot be obtained for larger sets of four nodes or more. In fact, the reason why we have provided ways of obtaining constraints (3.1) and (3.5) based on adding smaller inequalities and then use lifting is so that similar ideas can be applied to bigger sets. What Proposition 4 does imply, however, is that one will not be able to obtain inequalities for larger sets that would, at the same time, imply inequalities (1.7) for $|S| = 4$. At this time we conjecture that the same applies for larger sets and in fact, our new inequalities (3.1), (3.5) and (3.6) and the corresponding projection result could be considered as an exception and not the rule.

4. Projection results

In this section we give a framework for obtaining all the interesting projected inequalities in the space of the x_{ij} variables from the linear programming feasible set of many of the MTZ-like models described before. In order for our framework to be valid, all the MTZ inequalities in these models must be of the form $u_i + Exp(X) \leqslant u_j$ where $Exp(X)$ is an expression involving some x_{ij} variables and possibly a constant. Note that for a given pair (i, j) there may be more than one such inequality, e.g., the model with 2PATH inequalities includes several such constraints.

For the purpose of describing the framework we also define a multigraph $MG = (\mathcal{V}, A)$ where $\mathcal{V}$ corresponds to the node set of the original graph and A is built as follows: Any inequality of the form $u_i + Exp(X) \leqslant u_j$ gives rise to an arc (i, j). This is the reason that such a construction may lead to a multigraph.

Now we claim that for a model under these conditions, all interesting projected inequalities are obtained by adding the inequalities associated to the arcs defining an elementary circuit in MG.

We give some examples starting from the simplest model:

- Consider the MTZ inequalities (1.9). In this case the projected inequalities are of the following form,

$$\sum_{(i,j)\in\mathcal{C}} x_{ij} \leqslant |\mathcal{C}| - \frac{|\mathcal{C}|}{n-1} \qquad \forall \mathcal{C} \in \mathcal{G}_{\mathcal{V}_1}. \tag{4.1}$$

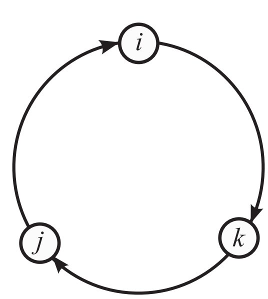

Fig. 5. An elementary circuit on the multigraph MG.

- Consider the DL inequalities (1.13). We obtain the 2-CLIQUE constraints $x_{ij} + x_{ji} \leqslant 1$ when we consider 2-circuits in the corresponding multigraph MG and the following more general set of inequalities,

$$\sum_{(i,j)\in\mathcal{C}} x_{ij} + \sum_{(i,j)\in\mathcal{C}^R} \left(\frac{n-3}{n-1}\right) x_{ij} \leqslant |\mathcal{C}| - \frac{|\mathcal{C}|}{n-1} \quad \forall \mathcal{C} \in \mathcal{G}_{\mathcal{V}_1}, \tag{4.2}$$

obtained by considering k-circuits in MG with $k > 2$, where $\mathcal{C}^R$ denotes the reversed circuit in which arc $(i,j) \in \mathcal{C}^R$ if and only if $(i,j) \in \mathcal{C}$.

In the previous two cases, the associated MG is a simple graph since we only have a single inequality for each pair (i, j). Next, we consider more complicated cases.

- Consider one of the 2PATH inequalities, namely (3.5). For this case, we have a multigraph since for each pair (i, k) there are $n-3$ different inequalities obtained by choosing a different middle node from $j \in \mathcal{V} \setminus \{i,k\}$. Thus two different k-circuits in the multigraph MG for the same value of k lead to different inequalities. Consider the case $k = 2$; here it is important to distinguish the case where the two inequalities generated from a given pair (i, k) and (k, i) have the same middle node, leading to the following inequalities for triples $\{i, j, k\}$,

$$(3n-7)(x_{ik} + x_{ki}) + (n-1)(x_{ij} + x_{jk} + x_{kj} + x_{ji}) \leqslant 4n-8, \tag{4.3}$$

or having a different middle node, leading to the following different inequalities for 4-tuples $\{i, j, l, k\}$

$$(3n-7)(x_{ik} + x_{ki}) + (n-1)(x_{ij} + x_{jk} + x_{kl} + x_{li}) \leqslant 4n-8. \tag{4.4}$$

The case with $k = 3$ leads to several inequalities. The following example in Fig. 5 shows an elementary circuit defined on three nodes $\{i, j, k\}$ on a multigraph MG. Fig. 6 shows all possible "mappings" of this elementary circuit on the original graph $\mathcal{G}$ where the middle nodes of inequalities (3.5) are chosen from the set $\{a, b\}$. The general expression for the projected inequalities in the 2PATH model is given by

$$\sum_{(i,k)\in\mathcal{C}} x_{ik} + \left(\frac{n-4}{2n-3}\right) \sum_{(i,k)\in\mathcal{C}^R} x_{ik} + \left(\frac{n-1}{2n-3}\right) \sum_{(i,k)\in\mathcal{C}} (x_{iv_{ik}} + x_{v_{ik}k}) \leqslant |\mathcal{C}| - \frac{|\mathcal{C}|}{2n-3}, \tag{4.5}$$

where $v_{ik} \in \mathcal{V} \setminus \{i,k\}$ for each $(i,k) \in \mathcal{C}$.
One particular case of the inequalities (4.5) is

$$\sum_{(i,k)\in\mathcal{C}} x_{ik} + \left(\frac{3n-6}{2n-3}\right) \sum_{(i,k)\in\mathcal{C}^R} x_{ik} \leqslant |\mathcal{C}| - \frac{|\mathcal{C}|}{2n-3}. \tag{4.6}$$

Inequalities (4.6) can be obtained from the previous set (4.5) by considering a circuit $\mathcal{C}$ where for each $(i,j) \in \mathcal{C}$, the middle node $v_{ij} \in \mathcal{V} \setminus \{i,k\}$ is identical, e.g., a unique node j.

$$\sum_{(i,k)\in\mathcal{C}} x_{ik} + \left(\frac{n-4}{2n-3}\right) \sum_{(i,k)\in\mathcal{C}^R} x_{ik} + 2\left(\frac{n-1}{2n-3}\right) \sum_{(i,k)\in\mathcal{C}} (x_{iv_{ik}} + x_{v_{ik}k}) \leqslant |\mathcal{C}| - \frac{|\mathcal{C}|}{2n-3}, \tag{4.7}$$

which, after rearranging the terms, yields (4.6). Note that if we remove the second summation on the lefthand side of (4.6), we obtain a cycle breaking constraint similar to the cycle constraints (4.1) projected from the MTZ inequalities but with a weaker right-hand side. This gives an alternative proof that one of the 2PATH inequalities (3.5) in the place of (1.4) also yield a valid model for the ATSP (recall that an alternative and similar argument using the 2PATH inequalities directly was given in Section 3.1).

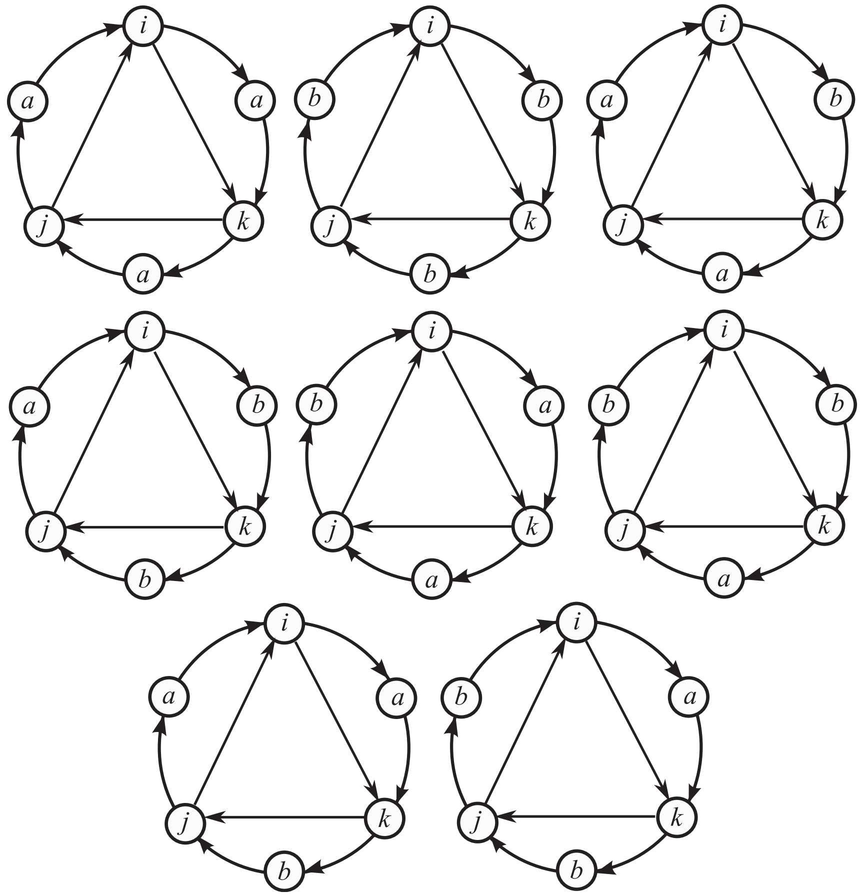

Fig. 6. All decompositions of the elementary circuit shown in Fig. 5 where the middle nodes are chosen from the set $\{a, b\}$.

This process of obtaining projected inequalities can become more complicated, but would still be within the same framework, if we consider the second family of 2PATH inequalities (3.5) together with the inequalities (3.6) (in which case the multigraph *MG* has more parallel arcs) or if we consider other sets of unrelated inequalities such as the DL inequalities. For instance, in the corresponding multigraph MG, one inequality corresponding to a 2-circuit may be obtained by adding the DL inequality for (i, k) with a 2PATH inequality for (k, i) with node j in the middle. Another inequality corresponding to a 3-circuit can be obtained by adding the DL inequalities for the pairs (i, j) and (j, k) and a 2PATH inequality for the pair (k, i). Here depending on the middle node of the 2PATH inequality, a different inequality is obtained.

- Other similar examples can be obtained with the CLIQUE inequalities for $|S| = 3$ alone or combined with the DL or the 2PATH inequalities. Adding to the models under study, inequalities defined in the x_{ij} space do not alter the framework. For instance, if we include the assignment constraints in the original model, then these inequalities will also appear in the projected model (recall the projection characterization of MTZ given in Section 1).

Inequalities obtained in this circuit fashion are clearly implied by the given model and that non-elementary cycles lead to implied inequalities. The hard part is to prove "sufficiency".

We will provide now a proof for this general case. Let us assume that the generic model $\mathcal{G}$ contains Q_{ij} inequalities for each ordered pair i, j, as follows:

$$u_i + \sum_{(k,l)\in\mathcal{A}} \alpha_{kl}^{q,ij} x_{kl} \leqslant u_j + \beta^{q,ij} \qquad \forall i,j,q = 1,\ldots,Q_{ij}, \tag{4.8}$$

where $\alpha_{kl}^{q,ij}$ is the coefficient of variable x_{ij} in the qth constraint associated to the pair i, j and $\beta^{q,ij}$ is the corresponding right-hand side. We assume that the constraints are ordered for each pair i, j. As noted for many of the cases described before, for a given index (q, ij), many of the coefficients are equal to zero.

Each of the projected inequalities is obtained by adding several of the previous inequalities in circuit fashion as explained above. Thus, a generic projected inequality is as follows:

$$\sum_{(i,j)\in\mathcal{C}} \sum_{(k,l)\in\mathcal{A}} \alpha_{kl}^{q,ij} x_{kl} \leqslant \sum_{(i,j)\in\mathcal{C}} \beta^{q,ij}, \tag{4.9}$$

where C is a cycle in the multigraph *MG*.

Proposition 8. *All the inequalities of the form (4.9) are sufficient to describe the projection of (4.8) into the space of the x_{ij} variables.*

Proof. Consider any assignment of values A_{ij} to the variables x_{ij} satisfying all the inequalities following the generic form (4.9) as well as (1.6). One has to show that there exist variables u_i, $i = 2,\ldots,n$ such that:

$$u_i - u_j \leqslant \beta^{q,ij} - \sum_{(k,l)\in\mathcal{A}} \alpha_{kl}^{q,ij} A_{kl} \quad \forall i,j,q = 1,\ldots,Q_{ij}. \tag{4.10}$$

By assigning nonnegative dual variables v_{ij}^q to each constraint (4.10) Farkas lemma shows that the system (4.10) has a solution if and only if

$$\sum_{i,j\in\mathcal{V}} \sum_{q=1,\ldots,Q_{ij}} \left(\beta^{q,ij} - \sum_{(k,l)\in\mathcal{A}} \alpha_{kl}^{q,ij} A_{kl} \right) \geqslant 0 \tag{4.11}$$

for

Table 1
Comparison of LP relaxations of enhanced MTZ and DL formulations on TSP instances.

Instance	n	MTZ + 2CLQ	T(MTZ + 2CLQ)	DL	T(DL)
ftv33.tsp	34	2.32	0.01	2.48	0
ftv35.tsp	36	2.07	0.01	2.07	0
ftv38.tsp	39	2.49	0.01	2.53	0
p43.tsp	43	22.90	0.01	22.90	0
ftv44.tsp	45	3.04	0.01	3.04	0.01
ftv47.tsp	48	4.21	0.01	4.22	0.01
ry48p.tsp	48	9.90	0.01	9.90	0.01
ft53.tsp	53	1.27	0.01	1.28	0.01
ftv55.tsp	56	5.04	0.01	5.04	0.01
ftv64.tsp	65	2.21	0.02	2.21	0.01
ftv70.tsp	71	5.04	0.02	5.04	0.01
ft70.tsp	70	0.32	0.03	0.32	0.03
kro124p.tsp	100	2.84	0.07	2.85	0.05
ftv170.tsp	171	2.55	0.19	2.55	0.15
Average		4.73	0.03	4.74	0.02

$$\sum_{i\in\mathcal{V}}\sum_{q=1,\ldots,Q_{ij}} \upsilon_{ij}^{q} - \sum_{i\in\mathcal{V}}\sum_{q=1,\ldots,Q_{ij}} \upsilon_{ji}^{q} = 0 \quad \forall j \in \mathcal{V} \tag{4.12}$$

$$\upsilon_{ij}^{q} \geqslant 0 \quad \forall i,j,q = 1,\ldots,Q_{ij}. \tag{4.13}$$

We observe that constraints (4.12) and (4.13) are those associated to a flow problem on the corresponding multigraph MG. Any feasible solution in this multigraph can be decomposed into elementary circuits in such a way that each arc of each circuit has the same value of flow V. Consider such a circuit $\mathcal{C}$ with cost (value) given by

$$\sum_{i,j\in\mathcal{C}}\left(\beta^{q,ij} - \sum_{(k,l)\in\mathcal{A}} \alpha_{kl}^{q,ij} A_{kl}\right) V. \tag{4.14}$$

Since (4.13) implies that V is nonnegative and the inequality (4.9) associated to the same circuit implies that $\sum_{i,j\in\mathcal{C}}\left(\beta^{q,ij} - \sum_{(k,l)\in\mathcal{A}}\alpha_{kl}^{q,ij}A_{kl}\right)$ is nonnegative, we con conclude that (4.14) is nonnegative for the same circuit. Thus the contribution of (4.14) to the left-hand side of (4.11) is nonnegative. Since this holds for any circuit obtained after decomposing any feasible solution of (4.12) and (4.13) we conclude that (4.11) always holds.

Now, since (4.11) holds for any solution satisfying (4.12) and (4.13) iff (4.9) holds for all cycles $\mathcal{C}$ and by Farkas lemma, (4.11) holds for any solution satisfying (4.11) and (4.11) iff the values A_{ij} satisfy the inequalities (4.10) we obtain the desired result. $\square$

5. Computational results

Although the main emphasis of this paper is a theoretical treatment of the MTZ constraints and to present ways in which they can (or cannot) be extended and generalized, we provide some computational results in this section to assess the strength of the LP relaxation of the new constraints and to compare their use with the use of corresponding projected inequalities. Besides the TSP, we also provide a set of similar experiments with the precedence-constrained TSP (PCTSP) (see, e.g., Sherali & Driscoll, 2002 for a description). The reason for this choice is that the PCTSP is easily modeled with the new set of variables, and permits us to evaluate the behaviour of the new inequalities in a different setting.

We emphasize that our aim behind the computational experiments is not to show the superiority of the new constraints or to use them in solving the TSP or the PCTSP to optimality (for which there exists a wide range of specialized methods), but rather to support and numerically confirm the theoretical findings. The

Table 2
Deviation of LP relaxation values from the best known or optimal solution values on TSP instances.

Instance	n	DL + 3CLQ	DL + NR	DL + L3	DL + 2P	DL + R	R + 2P	NR + 2P	NR + R + 2P	DL + NR + R + 2P
ftv33.tsp	34	1.19	1.19	1.22	1.20	1.19	1.19	1.19	1.19	1.19
ftv35.tsp	36	1.83	1.83	1.83	1.76	1.83	1.77	1.77	1.77	1.75
ftv38.tsp	39	1.76	1.76	1.76	1.69	1.76	1.71	1.71	1.71	1.69
p43.tsp	43	96.26	96.26	96.44	96.43	96.26	96.28	96.28	96.28	96.26
ftv44.tsp	45	1.85	1.85	1.85	1.85	1.85	1.85	1.85	1.85	1.85
ftv47.tsp	48	2.13	2.13	2.13	2.13	2.13	2.46	2.46	2.46	2.13
ry48p.tsp	48	4.23	4.22	4.19	4.16	4.22	4.16	4.16	4.16	4.15
ft53.tsp	53	12.94	12.84	12.94	12.71	12.85	12.67	12.67	12.67	12.67
ftv55.tsp	56	4.63	4.49	4.62	4.49	4.52	4.61	4.60	4.60	4.47
ftv64.tsp	65	3.86	3.86	3.86	3.85	3.86	3.85	3.85	3.85	3.85
ft70.tsp	70	1.46	1.45	1.46	1.45	1.45	1.45	1.45	1.45	1.45
ftv70.tsp	71	4.33	4.33	4.33	4.31	4.33	4.31	4.31	4.31	4.31
Average		11.37	11.35	11.39	11.34	11.36	11.36	11.36	11.36	11.31

Table 3
Computational times (in seconds) to solve the LP relaxations on TSP instances.

Instance	n	DL + 3CLQ	DL + NR	DL + L3	DL + 2P	DL + R	R + 2P	NR + 2P	NR + R + 2P	DL + NR + R + 2P
ftv33.tsp	34	0.37	0.49	0.38	8.48	0.48	9.08	8.88	8.48	8.84
ftv35.tsp	36	0.45	0.57	0.51	12.01	0.57	10.62	11.19	9.91	12.27
ftv38.tsp	39	0.58	0.79	0.59	15.42	0.75	22.26	16.19	16.87	19.97
p43.tsp	43	0.87	1.51	0.94	34.49	1.35	35.69	32.89	38.68	30.64
ftv44.tsp	45	1.03	1.32	1.08	28.04	1.23	30.84	27.83	27.66	27.97
ftv47.tsp	48	1.46	1.71	1.32	35.26	1.91	55.63	37.6	40.32	38.69
ry48p.tsp	48	1.29	1.67	1.51	38.15	1.82	48.13	39.17	38.71	35.13
ft53.tsp	53	1.9	3.1	2.09	81.91	3.32	75.75	74.45	72.42	84.34
ftv55.tsp	56	2.25	3.93	2.63	71.53	3.39	69.03	69.78	83.25	74.39
ftv64.tsp	65	3.43	4.4	3.83	154.76	4.64	178.22	122.91	176.62	172.28
ft70.tsp	70	5.37	9.8	6.23	232.16	9.11	256.3	253.29	238.15	273.45
ftv70.tsp	71	4.47	6.71	4.72	229.13	6.9	266.39	268.25	315.73	234.68
Average		1.96	3.00	2.15	78.45	2.96	88.16	80.20	88.90	84.39

Table 4
Deviation of LP relaxation values from the best known or optimal solution values on TSP instances.

Instance	SD	SD + 3CLQ	SD + NR	SD + R	SD + 2P	SD + L3
ftv33.tsp	4.78	1.06	1.06	1.06	1.58	1.18
ftv35.tsp	3.90	1.65	1.64	1.64	1.58	1.64
ftv38.tsp	3.27	1.59	1.58	1.58	1.53	1.58
p43.tsp	84.62	84.16	84.16	84.16	84.39	84.39
ftv44.tsp	2.43	1.76	1.73	1.73	1.75	1.76
ft53.tsp	11.39	11.39	11.39	11.39	11.36	11.39
ftv55.tsp	5.89	4.48	4.47	4.47	4.41	4.45
ft70.tsp	0.80	0.80	0.80	0.80	0.80	0.80
ftv70.tsp	4.64	4.16	4.15	4.15	4.14	4.16
Average	13.52	12.34	12.33	12.33	12.39	12.37

Table 5
Computational times (in seconds) to solve the LP relaxations on TSP instances.

Instance	SD	SD + 3CLQ	SD + NR	SD + R	SD + 2P	SD + L3
ftv33.tsp	0.15	0.31	0.73	1.05	0.93	0.58
ftv35.tsp	0.09	0.48	0.97	1.08	0.88	0.54
ftv38.tsp	0.79	0.45	1.38	1.47	1.46	0.92
p43.tsp	0.9	2.07	5.15	9.41	4.7	3.48
ftv44.tsp	0.05	0.67	2.27	2.76	2.08	1.13
ft53.tsp	2.42	4.46	14.03	32.32	15.22	8.67
ftv55.tsp	0.35	1.63	4.85	4.65	3.17	1.7
ft70.tsp	80.32	282.49	1283.86	1561.14	738.61	332.01
ftv70.tsp	0.41	1.95	5.53	8.89	6.78	3.61
Average	9.50	32.72	146.53	180.31	85.98	39.18

following sections present the results obtained on the TSP and the PCTSP, respectively.

5.1. Experiments with the traveling salesman problem

The first set of experiments aims to computationally test the difference between $\mathcal{P}^2_{MTZ}$ and $\mathcal{P}^2_{DL}$. The tests are conducted on a number of asymmetric TSPLIB (1997) instances with the number of nodes ranging from 34 to 171. The LP relaxations are solved using CPLEX 12 using a computing cluster with 2.4 gigahertz and 64 gigabytes RAM (single core). The first set of results are presented in Table 1 in which two formulations are compared: (i) *ATSP* using constraints (1.9) coupled with two node CLIQUE constraints in place of constraints (1.4), denoted by $F(\text{MTZ} + 2\text{CLQ})$ and (ii) formulation *ATSP* using constraints (1.13) instead of (1.4), denoted $F(\text{DL})$. The first two columns of this table show the instances tested and the number n of nodes contained in each. The remaining columns show the % improvement in LP relaxation each constraint brings over formulation *ATSP* using MTZ constraints alone. In particular, the improvement is calculated as $100\frac{v(F(\text{MTZ}+2\text{CLQ})_L)-v(F(\text{MTZ})_L)}{v(F(\text{MTZ})_L)}$ for formulation $F(\text{MTZ} + 2\text{CLQ})$ and similarly for $F(\text{DL})$. For simplicity, the columns are titled with the names of the constraints rather than those of the full formulations. Computational times (in seconds) required to solve the corresponding LP relaxation to optimality is denoted $\text{T}(\cdot)$.

As seen in Table 1, the average improvement of the use of CLIQUE constraints for $|\mathcal{S}| = 2$ over the classical MTZ formulation is 4.73%, whereas the same value for the DL constraints is 4.74%. No major difference is seen with respect to computational times required by both formulations despite the fact that $F(\text{DL})$ uses one set of constraints (1.13) instead of two as in the model $F(\text{MTZ} + 2\text{CLQ})$.

The second set of experiments aims to measure the strength of the LP relaxations with the new constraints as compared to the value of the optimal solution for the instances at hand. The results of these experiments are presented in Tables 2 and 3. The first two columns indicate, as before, the instance and the corresponding number of nodes. The following columns indicate the various constraints that formulation *ATSP* is supplemented with. Apart from the ones already mentioned in previous sections, 3CLQ denotes constraints (1.8) written for three nodes. NR (from non-radical) and R (from radical) denote the new sets of constraints (3.1) and (3.3), respectively. L3 denotes the lifted CIRCUIT inequalities writ-

Table 6
Deviation of LP relaxation values from the best known or optimal solution values on PCTSP instances.

Instance	n	DL + 3CLQ	DL + NR	DL + L3	DL + 2P	DL + R	DL + R + 2P	NR + 2P	NR + R + 2P	DL + NR + R + 2P
ESC07.sop	9	31.43	31.27	31.35	31.26	31.22	31.11	31.16	31.11	31.11
ESC12.sop	14	11.55	11.45	11.55	11.48	11.47	11.46	11.44	11.44	11.44
ESC25.sop	27	19.40	19.35	19.40	19.23	19.35	19.23	19.23	19.23	19.23
ESC47.sop	49	27.14	27.14	27.14	27.14	27.14	27.14	27.14	27.14	27.14
ry48p.1.sop	49	12.61	12.60	12.57	12.54	12.60	12.54	12.54	12.54	12.54
ry48p.2.sop	49	17.10	17.09	17.07	17.05	17.09	17.04	17.04	17.04	17.04
ry48p.3.sop	49	28.95	28.89	28.90	28.87	28.90	28.84	28.84	28.84	28.84
ry48p.4.sop	49	44.12	44.10	44.10	44.07	44.11	44.07	44.07	44.07	44.07
rbg048a.sop	50	6.84	6.84	6.84	6.84	6.84	6.84	6.84	6.84	6.84
rbg050c.sop	52	6.30	6.29	6.29	6.29	6.29	6.29	6.29	6.29	6.29
ft53.1.sop	54	20.16	20.03	20.16	19.94	20.05	19.87	19.87	19.87	19.87
ft53.2.sop	54	27.82	27.67	27.82	27.61	27.69	27.53	27.53	27.53	27.53
ft53.3.sop	54	43.68	43.59	43.67	43.50	43.60	43.47	43.47	43.47	43.47
ft53.4.sop	54	44.36	44.25	44.33	44.29	44.30	44.23	44.20	44.20	44.20
ESC63.sop	65	11.29	11.29	11.29	11.29	11.29	11.29	11.29	11.29	11.29
ft70.1.sop	71	3.06	3.06	3.06	3.06	3.06	3.06	3.06	3.06	3.06
ft70.2.sop	71	5.54	5.54	5.54	5.54	5.54	5.54	5.54	5.54	5.54
ft70.3.sop	71	8.63	8.63	8.62	8.60	8.63	8.60	8.60	8.60	8.60
ft70.4.sop	71	19.92	19.91	19.91	19.91	19.91	19.91	19.91	19.91	19.91
ESC78.sop	80	48.46	48.46	48.46	48.46	48.46	48.46	48.46	48.46	48.46
kro124p.1.sop	101	11.98	11.95	11.95	11.93	11.95	11.92	11.92	11.92	11.92
kro124p.2.sop	101	14.73	14.70	14.60	14.58	14.70	14.57	14.56	14.56	14.56
kro124p.3.sop	101	29.68	29.63	29.54	29.50	29.63	29.47	29.47	29.47	29.47
kro124p.4.sop	101	44.96	44.90	44.96	44.90	44.91	44.88	44.87	44.87	44.87
rbg109a.sop	111	8.02	7.98	8.02	8.00	7.99	7.98	7.98	7.98	7.98
rbg150a.sop	152	5.85	5.83	5.85	5.84	5.83	5.83	5.83	5.83	5.83
Average		21.29	21.25	21.27	21.22	21.25	21.20	21.20	21.20	21.20

Table 7
Computational times (in seconds) to solve the LP relaxations on PCTSP instances.

Instance	n	DL + 3CLQ	DL + NR	DL + L3	DL + 2P	DL + R	DL + R + 2P	NR + 2P	NR + R + 2P	DL + NR + R + 2P
ESC07.sop	9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.01
ESC12.sop	14	0.01	0.01	0.01	0.01	0.02	0.04	0.02	0.05	0.05
ESC25.sop	27	0.03	0.06	0.02	0.05	0.13	0.31	0.11	0.34	0.34
ESC47.sop	49	0.26	0.86	0.20	1.26	1.89	3.72	3.16	6.85	6.74
ry48p.1.sop	49	0.32	1.33	0.28	1.73	5.61	9.89	5.19	13.74	13.67
ry48p.2.sop	49	0.33	1.75	0.39	2.31	5.45	11.05	5.35	14.77	14.62
ry48p.3.sop	49	0.35	1.78	0.34	2.56	5.17	9.81	7.83	14.89	14.91
ry48p.4.sop	49	0.33	1.47	0.30	2.04	3.58	7.68	4.20	10.36	10.34
rbg048a.sop	50	0.35	0.61	0.27	0.65	1.13	2.08	1.61	2.72	2.72
rbg050c.sop	52	0.51	1.37	0.43	2.19	2.20	4.24	3.92	8.17	8.11
ft53.1.sop	54	0.34	1.03	0.28	1.66	3.07	5.86	4.61	11.90	11.95
ft53.2.sop	54	0.36	2.36	0.30	2.66	3.28	6.77	7.20	18.53	18.44
ft53.3.sop	54	0.41	2.51	0.34	2.92	2.64	5.98	7.66	14.91	15.10
ft53.4.sop	54	0.42	1.72	0.30	2.74	4.44	8.88	5.21	18.02	17.89
ESC63.sop	65	0.96	1.15	0.75	0.96	2.39	5.09	3.72	7.09	6.94
ft70.1.sop	71	1.08	3.72	0.74	7.43	11.56	24.60	13.06	30.17	22.85
ft70.2.sop	71	1.15	4.26	0.91	5.17	11.64	57.28	14.98	32.57	32.28
ft70.3.sop	71	1.18	6.00	0.95	11.72	39.71	29.78	20.23	43.46	42.11
ft70.4.sop	71	1.18	4.20	0.97	7.86	11.77	25.40	14.88	35.74	35.14
ESC78.sop	80	2.36	4.66	1.80	7.62	9.69	29.39	20.56	40.82	40.72
kro124p.1.sop	101	3.94	17.08	3.40	26.47	35.98	550.90	59.80	120.45	119.20
kro124p.2.sop	101	4.60	30.67	4.05	39.72	188.68	402.62	129.91	151.46	208.82
kro124p.3.sop	101	4.92	64.64	6.97	66.17	199.92	322.94	160.57	324.72	459.01
kro124p.4.sop	101	5.11	43.34	3.67	55.45	87.04	197.30	176.24	413.20	292.30
rbg109a.sop	111	7.41	38.93	6.27	73.11	79.57	269.05	196.90	334.72	338.78
rbg150a.sop	152	21.63	626.93	29.00	359.79	1381.85	1795.04	1420.48	2685.83	2646.15
Average		2.29	33.17	2.42	26.32	80.71	145.60	87.98	167.52	168.43

ten for three nodes and 2P is the short-hand notation for 2PATH inequalities (3.5) and (3.6). The figures reported in Table 2 shows the deviations of the LP relaxations from the optimal solution value. Table 3 shows the computational time (in seconds) required to solve the LP relaxations to optimality.

The results presented in Tables 2 and 3 have a number of interesting implications. First, DL inequalities supplemented with CLIQUE for three nodes is only slightly worse than DL supplemented with either (or both) of the new inequalities NR and R. From a theoretical perspective the bound cannot be better. Similarly, DL + 2P brings some improvements over DL + L3, albeit with a slight increase in the computational time. In summary, however, the results indicate that only marginal improvements are obtained by using the MTZ-like constraints, when compared with corresponding projected constraints in the x_{ij} space. These results conveniently fit with the theoretical findings.

We have also conducted experiments with the SD formulation and to see the amount by which the new constraints are able to improve this formulation. In particular, the SD formulation is supplemented with 3CLQ, NR, R, 2P and L3. The resulting gaps from best known or optimal solutions are presented in Table 4, and the associated computational time required to solve the LP relaxations to optimality are shown in Table 5.

The results presented in Tables 4 and 5 can be interpreted from two different perspectives: (i) they show that by using the polyhedral approach discussed in the previous sections it is easy to derive MTZ-based inequalities that significantly improve the LP bounds of formulation SD or (ii) unfortunately they yield similar conclusions as with the previous set of experiments in that the improvements obtained by new MTZ-like inequalities can also be obtained using their projected versions. It is worth noting that there are cases where NR and R perform slightly better than 3CLQ, and 2P slightly better than L3, although this comes at the expense of more computational time.

Our theoretical findings explain the reasons behind case (ii). In particular, our efforts focused on studying the polyhedra associated with the MTZ *as well as* in the x_{ij} space, showed how MTZ inequalities imply the corresponding inequalities in the x space.

5.2. Experiments with the precedence-constrained traveling salesman problem

The PCTSP is an extension of the TSP in that, for certain pairs of nodes $i, j \in \mathcal{V}$, there is an additional restriction that node i should precede node j. Using MTZ-based constraints, this requirement can easily be incorporated into formulation *ATSP* by the constraint $u_j \geqslant u_i + 1$ for every $i \in \mathcal{V}$ that should precede $j \in \mathcal{V} \setminus \{i\}$. Tables 6 and 7 show the results associated with the LP relaxations, in the same way as Tables 2 and 3. Not all the optimal solutions are available for the PCTSP, and for those instances we have compared the lower bounds with the best known upper bound that are available in the TSPLIB (1997).

The results presented for the PCTSP show that similar conclusions apply as in the previous section. In particular, the results confirm that improvements obtained by the generalized MTZ or DL constraints are marginally better than the improvements obtained by supplementing the two-node DL constraints with two or three node CLIQUE inequalities in the space of the x-variables only.

6. Conclusions

In this paper we have presented a systematic way of deriving inequalities that are more complicated than the MTZ and DL inequalities and that in a certain way "generalize" the underlying idea of the original inequalities. We described a polyhedral approach that studies and analyzes the convex hull of feasible sets for small dimensions and that allowed us to generate generalizations of the MTZ and DL inequalities, and which are "good" in the sense that they define facets of these small polyhedra. Our approach produced a generalization of these inequalities which imply CLIQUE constraints for three-node subsets. We also claim that generalizations of these inequalities implying CLIQUE inequalities for larger subsets are unlikely to exist. This claim follows the assertion that, had there been generalizations of these inequalities implying CLIQUE inequalities for four-nodes, then the CLIQUE inequalities themselves would not have been facet-defining for the corresponding polyhedron. This paper provides theoretical proofs that CLIQUE inequalities in fact define facets for the polyhedron defined on four nodes, which contradicts this assertion.

Some computational results indicate that only marginal improvements are obtained when using the new the MTZ-like constraints when compared with corresponding projected constraints in the x_{ij} space. These results could be regarded as a note of “warning” in the sense that any improvements on the MTZ constraints could also be obtained by using similar constraints in the x_{ij} space alone (if one knows which ones they are).

Acknowledgements

Thanks are due to three anonymous reviewers for their useful comments on the initial version of the paper. The work of the second author was supported by project PEst-OE/MAT/UI0152.

Appendix A. Lemma to Proposition 4

Lemma A.1. *The following eight points* $\mathbf{u}_i$, $i = 1, \ldots, 8$,

	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$	$\mathbf{u}_4$	$\mathbf{u}_5$	$\mathbf{u}_6$	$\mathbf{u}_7$	$\mathbf{u}_8$
u_i	u	u	u	u	u	u	$n-3$	$n-2$
u_k	$u+2$	$u+1$	$u-1$	$u+1$	$u-1$	$u-2$	0	1
x_{ij}	1	0	0	0	1	0	1	0
x_{jk}	1	0	1	0	0	0	0	0
x_{kj}	0	1	0	0	0	1	0	0
x_{ji}	0	0	0	1	0	1	0	0
x_{ik}	0	1	0	1	0	0	0	0
x_{ki}	0	0	1	0	1	0	0	0

are affinely independent for a given $u \in \mathbb{Z}_+$ *such that* $2 \leqslant u \leqslant n-4$.

Proof. Consider the following system of equations defined by the scalars λ_i and vectors $\mathbf{u}_i$, $i = 1, \ldots, 8$:

$$\sum_{i=1}^{8} \lambda_i = 0, \tag{A.1}$$

$$(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6)u + \lambda_7(n-3) + \lambda_8(n-2) = 0, \tag{A.2}$$

$$\begin{aligned}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6)u + 2\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 - \lambda_5 \\ - 2\lambda_6 + \lambda_8 \\ = 0,\end{aligned} \tag{A.3}$$

$$\lambda_1 + \lambda_5 + \lambda_7 = 0, \tag{A.4}$$

$$\lambda_1 + \lambda_3 + \lambda_8 = 0, \tag{A.5}$$

$$\lambda_2 + \lambda_6 = 0, \tag{A.6}$$

$$\lambda_4 + \lambda_6 = 0, \tag{A.7}$$

$$\lambda_2 + \lambda_4 = 0, \tag{A.8}$$

$$\lambda_3 + \lambda_5 = 0. \tag{A.9}$$

Eqs. (A.4) and (A.5) imply $\lambda_5 + \lambda_7 = -\lambda_1$ and $\lambda_3 + \lambda_8 = -\lambda_1$, together with Eq. (A.1) yield $\lambda_1 = \lambda_8$. Using Eqs. (A.6), (A.7) and (A.8), one calculates $\lambda_1 = \lambda_2 = \lambda_4 = \lambda_6 = 0$. These values inserted into Eq. (A.3) yields $(\lambda_3 + \lambda_5)u - \lambda_5(n-3) - \lambda_3(n-2)$, which, together with (A.9) imply $\lambda_3 = \lambda_5 = 0$. Finally, it is easy to calculate $\lambda_7 = \lambda_8 = 0$ using (A.4) and (A.5). □

Appendix B. Matrix *A* for Proposition 7

Matrix *A* mentioned in Proposition 7 is shown below, where each column corresponds to the column vector

$(x_{ij}, x_{ji}, x_{ik}, x_{ki}, x_{jk}, x_{kj}, x_{il}, x_{li}, x_{jl}, x_{lj}, x_{kl}, x_{lk}, u_i, u_j, u_k, u_l)^{\mathrm{T}}$,

$$A = \left(\begin{matrix}
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
u & u+3 & u+2 & u+1 & u & u+3 & u+2 & u+1 & u & u+3 & u+2 & u+1 \\
u+1 & u & u+3 & u+2 & u+3 & u+2 & u+1 & u & u+3 & u+2 & u+1 & u \\
u+2 & u+1 & u & u+3 & u+2 & u+1 & u & u+3 & u+1 & u & u+3 & u+2 \\
u+3 & u+2 & u+1 & u & u+1 & u & u+3 & u+2 & u+2 & u+1 & u & u+3
\end{matrix}\right.$$

$$\left.\begin{matrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
u+3 & u & u+1 & u+2 & u & u+3 & u+2 & u+1 & u+3 & u & u+1 & u+2 \\
u & u+1 & u+2 & u+3 & u+2 & u+1 & u & u+3 & u+1 & u+2 & u+3 & u \\
u+2 & u+3 & u & u+1 & u+1 & u & u+3 & u+2 & u+2 & u+3 & u & u+1 \\
u+1 & u+2 & u+3 & u3 & u+3 & u+2 & u+1 & u & u & u+1 & u+2 & u+3
\end{matrix}\right).$$

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