Lecture 3

June 23, 2025

Restricted null space property (RNS)

- Define

$$\mathbb{C}(S) := \{\Delta \in \mathbb{R}^d : \|\Delta_{S^c}\|_1 \leq \|\Delta_S\|_1\}.$$

Theorem

The following two are equivalent:

- *For any $\theta^* \in \mathbb{R}^d$ with support $\subset S$, the basis pursuit program applied to the data $(X, y = X\theta^*)$ has unique solution $\hat{\theta} = \theta^*$.*
- *The restricted null space (RNS) property holds, i.e.,*

$$\mathbb{C}(S) \cap \ker(X) = \{0\}.$$

Recall that $\ker(X) = \{\theta \in \mathbb{R}^d : X\theta = 0\}$.

Sufficient conditions for restricted nullspace

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$$\delta_{PW}(X) = \max_{i,j\in[d]} \left| \frac{\langle X_i, X_j \rangle}{n} - 1_{\{i=j\}} \right| .$$

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$$\delta_{PW}(X) = \max_{i,j\in[d]} \left| \frac{\langle X_i, X_j \rangle}{n} - 1_{\{i=j\}} \right| .$$

- If $\delta_{PW}(X)$ is small then

$$\langle X_i/\sqrt{n}, X_j/\sqrt{n} \rangle \approx \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{otherwise.} \end{cases}$$

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- **Proposition.** (Uniform) restricted nullspace holds for all S with $|S| \leq s$ if

$$\delta_{PW}(X) \;\leq\; \frac{1}{3s}.$$

- **Proof.** For a vector $\theta \in \mathbb{R}^d$ and set $S \subset \{1, \ldots, d\}$, let $\theta_S \in \mathbb{R}^d$ be given as

$$(\theta_S)_i = \begin{cases} \theta_i & \text{if } \; i \in S \\ 0 & \text{Otherwise.} \end{cases}$$

- **Proof.** For a vector $\theta \in \mathbb{R}^d$ and set $S \subset \{1, \ldots, d\}$, let $\theta_S \in \mathbb{R}^d$ be given as

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where the first inequality follows from the ineqaulity $u^T M v \leq \|M\|_\infty \|u\|_1 \|v\|_1$, and the third from the inequality $\|\theta_S\|_1 \leq \sqrt{s}\|\theta_S\|_2$.

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- Hence,

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- **Definition.** $X \in \mathbb{R}^{n \times d}$ satisfies a restricted isometry property (RIP) of order s with constant $\delta_s(X) > 0$ if

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- PW incoherence is close to RIP with $s = 2$.
- In general, for any $s \geq 2$, it holds that

$$\delta_{PW}(X) \;\le\; \delta_s(X) \;\le\; s\delta_{PW}(X).$$

RIP gives sufficient conditions:

- **Proposition (HDS Prop. 7.2).** (Uniform) restricted null space holds for all S with $|S| \leq s$ if

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- Consider a sub-Gaussian matrix X with i.i.d. entries and $\mathbb{E}(X_{ij}) = 0$ and $\mathbb{E}(X_{ij}^2) = 1$ (Exercise 7.7):
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  - Sample complexity requirement for RIP is milder.

Noisy sparse regression

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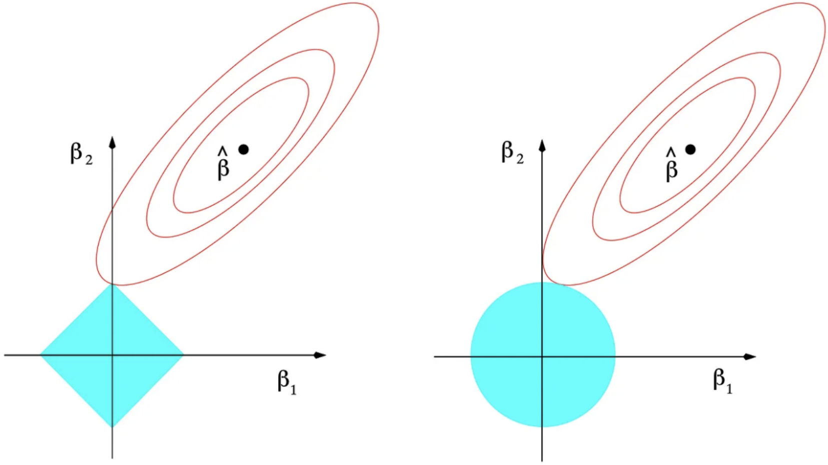
- The idea: minimizing ℓ_1 norm leads to sparse solutions.
- (1) is a convex program; global solution can be obtained efficiently.
- Other options: constrained form of lasso
$$\min_{\|\theta\|_1 \leq R} \frac{1}{2n} \|y - X\theta\|_2^2$$

Noisy sparse regression



```r
library(glmnet)

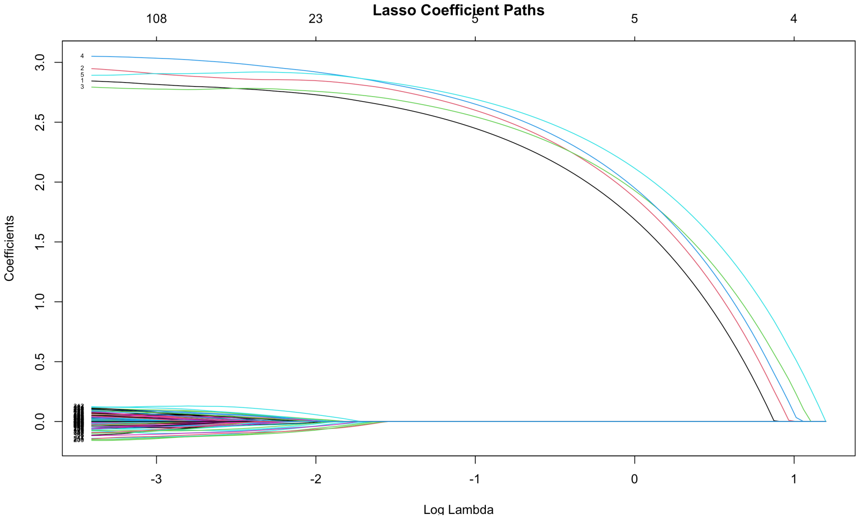
# Simulate data
set.seed(42)
n <- 200  # number of observations
p <- 500   # number of predictors

X <- matrix(rnorm(n * p), nrow = n, ncol = p)
beta <- c(rep(3, 5), rep(0, p - 5))  # sparse true coefficients
y <- X %*% beta + rnorm(n)

# Fit Lasso regression (alpha=1 for Lasso)
lasso_fit <- glmnet(X, y, alpha = 1)

# Plot solution paths
plot(lasso_fit, xvar = "lambda", label = TRUE)
title("Lasso Coefficient Paths")

# Cross-validation to select best lambda
cv_fit <- cv.glmnet(X, y, alpha = 1)
plot(cv_fit)
title("Cross-Validation Error")
```



Lasso Coefficient Paths
108
23
5
5
4
Coefficients
Log Lambda
-3
-2
-1
0
1
0.0
0.5
1.0
1.5
2.0
2.5
3.0

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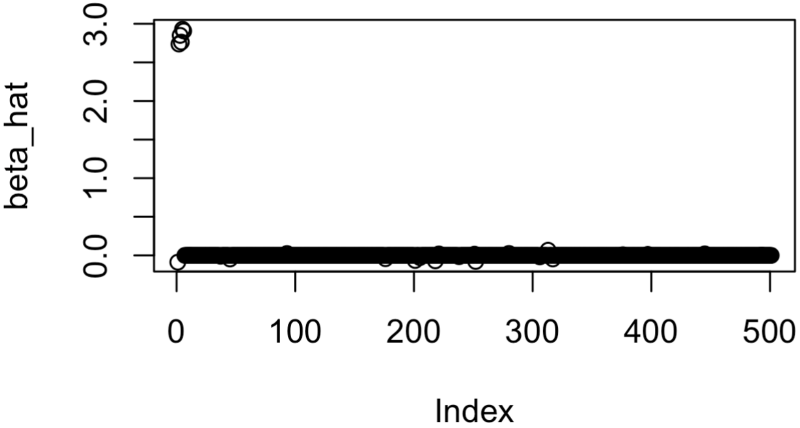
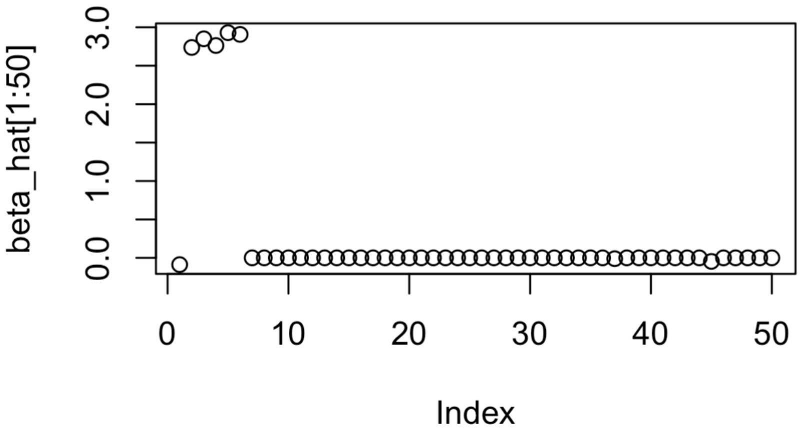
# Cross-validation to select best lambda
cv_fit <- cv.glmnet(X, y, alpha = 1)
plot(cv_fit)
title("Cross-Validation Error")

# Best lambda value
best_lambda <- cv_fit$lambda.min
cat("Best lambda:", best_lambda, "\n")

# Coefficients at best lambda
beta_hat = coef(cv_fit, s = "lambda.min")
plot(beta_hat)
plot(beta_hat[1:50])
```


beta_hat
0.0
1.0
2.0
3.0
0
100
200
300
400
500
Index



beta_hat[1:50]
0.0
1.0
2.0
3.0
0
10
20
30
40
50
Index

Restricted eigenvalue condition

- For a constant $\alpha \geq 1$, we define

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- **Definition.** A matrix X satisfies the restricted eigenvalue (RE) condition over S with parameters (κ, α) if

$$\frac{1}{n}\|X\Delta\|_2^2 \geq \kappa \|\Delta\|_2^2 \;\; \forall\, \Delta \in \mathbb{C}_\alpha(S).$$

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Recall that RNS corresponds to $\mathbb{C}_1(S) \cap \ker(X) = \{0\}$. Thus,

$$\frac{1}{n}\|X\Delta\|_2^2 > 0$$

for all $\Delta \in \mathbb{C}_1(S) \backslash \{0\}$.

Deviation bounds under RE

Theorem

Assume that $y = X\theta^* + w$*, where* $X \in \mathbb{R}^{n\times d}$*,* $\theta^* \in \mathbb{R}^d$ *and*

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Let us define $z = X^\top w/n$. Then we have the following:

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$$\|\hat{\theta} - \theta^*\|_2 \le \frac{3}{\kappa}\sqrt{s}\lambda.$$

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- *Any solution of constrained Lasso with* $R = \|\theta^*\|_1$ *satisfies*

$$\|\hat{\theta} - \theta^*\|_2 \leq \frac{4}{\kappa}\sqrt{s}\|z\|_\infty.$$

Example (fixed design regression)

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$$\max_{j=1,\dots,d} \frac{\|X_j\|}{\sqrt{n}} \leq C$$

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- Recall $z = X^\top w/n$.
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- Thus, setting $\lambda = 2C\sigma\left(\sqrt{\frac{2\log d}{n}} + \delta\right)$, Lasso solution satisfies

$$\|\hat{\theta} - \theta^*\|_2 \le \frac{6C\sigma}{\kappa}\sqrt{s}\left(\frac{2\log d}{n} + \delta\right)$$

  w.p at least $1 - 2\exp(-n\delta^2/2)$.

- Taking $\delta = \sqrt{2 \log d / n}$, we have

$$\|\hat{\theta} - \theta^*\|_2 \lesssim \sigma \sqrt{\frac{s \log d}{n}}$$

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- The log d factor is the price for not knowing the support.

Proof of Theorem

- Let us simplify the loss $L(\theta) = \frac{1}{2n}\|y - X\theta\|_2^2$.

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where $z = X^\top w/n$. Hence,

$$
L(\theta) - L(\theta^*) := \frac{1}{2n}\|X\Delta\|^2 - \langle \Delta, z\rangle.
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Proof (constrained version)

- By optimality of $\hat{\theta}$ and feasibility of θ^*,

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- Since $\hat{\Delta} = \hat{\theta} - \theta^* \in \mathbb{C}_1(S)$,

$$\|\hat{\Delta}\|_1 \;=\; \|\hat{\Delta}_{S^c}\|_1 + \|\hat{\Delta}_S\|_1 \;\leq\; 2\|\hat{\Delta}_S\|_1 \;\leq\; 2\sqrt{s}\|\hat{\Delta}\|_2.$$

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$$\|\hat{\Delta}\|_1 = \|\hat{\Delta}_{S^c}\|_1 + \|\hat{\Delta}_S\|_1 \leq 2\|\hat{\Delta}_S\|_1 \leq 2\sqrt{s}\|\hat{\Delta}\|_2.$$

- Combined with RE condition ($\hat{\Delta} \in \mathbb{C}_1(S) \subset \mathbb{C}_3(S)$ as well)

$$\frac{1}{2}\kappa\|\hat{\Delta}\|_2^2 \leq \frac{1}{2n}\|X\hat{\Delta}\|^2 \leq \|z\|_\infty\|\hat{\Delta}\|_1 \leq 2\sqrt{s}\|z\|_\infty\|\hat{\Delta}\|_2$$

which gives the desired result.

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- And so

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\end{aligned}$$

- We obtain that $\hat{\Delta} \in \mathbb{C}_3(S)$ ($\|\hat{\Delta}_{S^c}\|_1 \leq 3\|\hat{\Delta}_S\|_1$) and the rest of the proof follows.

Key condition: Restricted eigenvalue condition

- For a constant $\alpha \geq 1$, we define

$$\mathbb{C}_\alpha(S) := \left\{ \Delta \in \mathbb{R}^d : \|\Delta_{S^c}\|_1 \leq \alpha \|\Delta_S\|_1 \right\}.$$

- **Definition.** A matrix X satisfies the restricted eigenvalue (RE) condition over S with parameters (κ, α) if

$$\frac{1}{n}\|X\Delta\|_2^2 \geq \kappa \|\Delta\|_2^2 \quad \forall\, \Delta \in \mathbb{C}_\alpha(S).$$

  Recall that RNS corresponds to $\mathbb{C}_1(S) \cap \ker(X) = \{0\}$. Thus,

$$\frac{1}{n}\|X\Delta\|_2^2 > 0$$

  for all $\Delta \in \mathbb{C}_1(S) \backslash \{0\}$.

Deviation bounds under RE

Theorem

Assume that $y = X\theta^* + w$, where $X \in \mathbb{R}^{n\times d}$, $\theta^* \in \mathbb{R}^d$ and

- *θ^* is supported on $S \subset [d]$ with $|S| \le s$.*
- *X satisfies RE$(\kappa, 3)$ over S.*

Let us define $z = X^\top w/n$. Then we have the following:

- *Any solution of Lasso (1) with $\lambda \ge 2\|z\|_\infty$ satisfies*

$$\|\hat{\theta} - \theta^*\|_2 \le \frac{3}{\kappa}\sqrt{s}\lambda.$$

- *Any solution of constrained Lasso with $R = \|\theta^*\|_1$ satisfies*

$$\|\hat{\theta} - \theta^*\|_2 \le \frac{4}{\kappa}\sqrt{s}\|z\|_\infty.$$

RE condition for anisotropic design

- For a PSD matrix Σ, let $\rho^2(\Sigma) = \max_{i,i} \Sigma_{ii}$.

Theorem

Let $X \in \mathbb{R}^{n \times d}$ withrows i.i.d. from $N(0, \Sigma)$. Then, there exist universal constants $0 < c_1 < 1 < c_2$ such that

$$\frac{1}{n}\|X\theta\|_2^2 \geq c_1\|\sqrt{\Sigma}\theta\|_2^2 - c_2\rho^2(\Sigma)\frac{\log d}{n}\|\theta\|_1^2, \quad \forall \theta \in \mathbb{R}^d \tag{2}$$

with probability at least $1 - e^{-n/32}/(1 - e^{-n/32})$.

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- (2) implies RE condition over $\mathbb{C}_3(S)$ uniformly over all subsets of cardinality

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- In other words, $n \gtrsim s\log d \implies$ RE over $\mathbb{C}_3(S)$ for all $|S| \leq s$.