

Notation and Basic Facts

EMALCA 2025: High-dimensional Statistics

Class of June 30

General Notation

1. For sequences of real numbers a_n and b_n we write $a_n = O(b_n)$ if $a_n \leq Cb_n$ for all n and for some constant $C > 0$. An equivalent notation is sometimes used and this is expressed as $a_n \lesssim b_n$.
2. The ℓ_p for $p > 0$ norm in \mathbb{R}^d is defined as $\|v\|_p = (\sum_{i=1}^d v_i^p)^{1/p}$ for $v \in \mathbb{R}^d$. When the context is clear, sometimes for the case $p = 2$ we drop the index p and simply denote the resulting norm as $\|v\|$. The ℓ_∞ norm of a vector is defined as

$$\|x\|_\infty := \max_{i=1,\dots,d} |x_i|$$

for $x \in \mathbb{R}^d$.

3. For a matrix $A \in \mathbb{R}^{n \times m}$ its Frobenius norm is defined as

$$\|A\|_F := \sqrt{\sum_{i=1}^n \sum_{j=1}^m A_{i,j}^2},$$

where $A_{i,j}$ is the entry in the i th row and j th column of A .

4. Throughout, we think of vectors as column vectors, thus $x \in \mathbb{R}^n$ means

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

and $x^\top = (x_1, \dots, x_n)$. The inner product in \mathbb{R}^n is written as

$$x^\top y = \sum_{i=1}^n x_i y_i, \quad \forall x, y \in \mathbb{R}^n$$

though sometimes we denote this as $\langle x, y \rangle$.

5. For an event \mathcal{A}_n depending on n , we say that \mathcal{A}_n holds with high probability if

$$a_n := \mathbb{P}(\mathcal{A}_n)$$

satisfies $\lim_{n \rightarrow \infty} a_n = 1$.

6. For a collection of vectors $v_1, \dots, v_n \subset \mathbb{R}^d$, the notation $\text{span}\{v_1, \dots, v_n\}$ refers to the vector subspace generated by v_1, \dots, v_n .

7. For a matrix $A \in \mathbb{R}^{m \times n}$ we define its kernel as

$$\text{Ker}(A) := \{v \in \mathbb{R}^n : Av = 0\}.$$

The column space of is defined as

$$\mathcal{C}(A) := \{w \in \mathbb{R}^m : w = Av \text{ for some } v \in \mathbb{R}^n\}.$$

8. For a function $f : \mathcal{C} \rightarrow \mathbb{R}$ with $\mathcal{C} \subset \mathbb{R}^n$, the notation

$$x_0 = \arg \min_{x \in \mathcal{C}} f(x)$$

indicates that $f(x_0) \leq f(x)$ for all $x \in \mathcal{C}$.

Random Variables

- Let X be a continuous random variable (r.v.) $-\infty < X < \infty$.
- The function $f(x)$ is the so called probability density function (pdf) if

$$\text{a) } f(x) \geq 0$$

$$\text{b) } \int_{-\infty}^{\infty} f(x)dx = 1,$$

$$\text{c) } \mathbb{P}(a < X < b) = \int_a^b f(x)dx, \mathbb{P}(X < b) = \int_{-\infty}^b f(x)dx, \mathbb{P}(a < X) = \int_a^{\infty} f(x)dx.$$

Note that for a continuous random variable

$$\mathbb{P}(X \leq a) = \mathbb{P}(X < a),$$

but this is not true for a discrete random variable.

- Cumulative distribution function (cdf):

$$F(x) = \mathbb{P}(X \leq x).$$

Also,

$$F'(x) = f(x).$$

4. Compute probabilities using cdf:

$$\mathbb{P}(a < X < b) = \mathbb{P}(X \leq b) - \mathbb{P}(X \leq a) = F(b) - F(a).$$

5. Mean of a continuous r.v.

$$\mu := E(X) = \int_{-\infty}^{\infty} xf(x)dx.$$

6. For a random variable X with pdf f and a function g , we have that

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx.$$

7. Variance of continuous r.v.

$$\text{Var}(X) := E((X - \mu)^2) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2.$$

Sometimes we denote $\sigma^2 = \text{Var}(X)$.

8. Some properties. Let a and b be constants and X, Y r.v. When the quantities exist, we have that:

a) $E(X + a) = a + E(X)$.

b) $E(X + Y) = E(X) + E(Y)$.

c) $\text{var}(X + a) = \text{var}(X)$.

d) $\text{var}(aX) = a^2 \text{var}(X)$.

e) If X and Y are independent then

$$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y).$$

9. If X is a continuous non-negative random variable, then

$$\mathbb{E}(X) = \int_0^{\infty} \mathbb{P}(X \geq u)du.$$

10. For a collection of events A_1, \dots, A_n the union bound inequality states that:

$$\mathbb{P}(A_1 \cup \dots \cup A_n) \leq \sum_{i=1}^n \mathbb{P}(A_i).$$

11. The moment generating function of random variable X is defined as

$$M_X(t) = \mathbb{E}(\exp(tX)).$$

Informal: Therefore, If X is discrete

$$M_X(t) = \sum_x e^{tx} P(x).$$

If X is continuous

$$M_X(t) = \int_x e^{tx} f(x) dx.$$

Aside

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Similarly,

$$e^{tx} = 1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots$$

Let X be a discrete random variable. Then

$$\mathbb{E}(e^{tX}) = \sum_x e^{tx} P(x) = \sum_x \left[1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots \right] P(x).$$

Or

$$M_X(t) = \sum_x P(x) + \frac{t}{1!} \sum_x xP(x) + \frac{t^2}{2!} \sum_x x^2 P(x) + \dots$$

To find the k th moment simply evaluate the k th derivative of the $M_X(t)$ at $t = 0$.

Thus

$$\mathbb{E}(X^k) = M_X^{(k)}(0).$$

For example: since

$$M_X(t) = \sum_x P(x) + \frac{t}{1!} \sum_x xP(x) + \frac{t^2}{2!} \sum_x x^2 P(x) + \dots,$$

first moment:

$$\begin{aligned} M_X'(t) &= \frac{d}{dt} \left(\sum_x P(x) \right) + \frac{d}{dt} \left(\frac{t}{1!} \sum_x xP(x) \right) + \frac{d}{dt} \left(\frac{t^2}{2!} \sum_x x^2 P(x) \right) + \dots \\ &= \sum_x xP(x) + \frac{2t}{2!} \sum_x x^2 P(x) + \dots \end{aligned}$$

This implies

$$M_X'(0) = \mathbb{E}(X).$$

Similarly, for the second moment

$$M_X''(t) = \sum_x x^2 P(x) + \frac{6t}{3!} \sum_x x^3 P(x) + \dots$$

Therefore,

$$M_X''(0) = \mathbb{E}(X^2).$$

Example: Poisson. Suppose that $X \sim \text{Poisson}(\lambda)$. Then

$$\begin{aligned}
 M_X(t) &= \sum_{x=0}^{\infty} e^{tx} \mathbb{P}(X = x) \\
 &= \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!} \\
 &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t \lambda)^x}{x!} \\
 &= \exp(-\lambda) \exp(e^t \lambda) \\
 &= \exp(-\lambda + e^t \lambda).
 \end{aligned}$$

12. Central Limit Theorem:

Lindeberg–Lévy CLT—Suppose $X_1, X_2, X_3 \dots$ is a sequence of i.i.d. random variables with $\mathbb{E}[X_i] = \mu$ and $\text{Var}[X_i] = \sigma^2 < \infty$. Then, as n approaches infinity, the random variables $\sqrt{n}(\bar{X}_n - \mu)$ converge in distribution to a normal $\mathcal{N}(0, \sigma^2)$.^[4]

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2).$$
