Homework One

EMALCA 2025: High-dimensional Statistics

Due in class on July 1 (Submit in groups of 2 or 3)

1. Consider $X \sim \text{Exp}(1)$, i.e., an exponential random variable with rate 1.

    a) (**2.5/3 points**) Verify that X is not sub-Gaussian by showing that

    $$\frac{\|X\|_p}{\sqrt{p}} \to \infty, \text{ as } p \to \infty.$$

    Here, $\|X\|_p = (\mathbb{E}(|X|^p))^{1/p}$.

    b) (**2.5/3 points**) Show that X is sub-Exponential.

    c) (**2.5/3 points**) Show that $\sqrt{X}$ is sub-Gaussian.

2. (**2.5 points**) Show that exponential, Pareto and Cauchy distributions are not sub-gaussian.

3. (**2.5 points**) Show that $\|\cdot\|_{\psi_2}$ satisfies the triangle inequality. Hint: Use the convexity of the function $f(x) = e^{x^2}$.

4. (**2.5 points**) Let $X_1, \ldots, X_n$ be mean zero, independent sub-Exponential random variables with parameters $\nu, \alpha > 0$. For any vector $a = (a_1, \ldots, a_n)^\top \in \mathbb{R}^n$ define the weighted sum

    $$S(a) = \sum_{i=1}^{n} a_i X_i.$$

    Show that for any $t > 0$, we have that

    $$\mathbb{P}(|S(a)| > t) \le 2\exp\left(-C\min\left\{\frac{t^2}{\nu^2\|a\|_2^2}, \frac{t}{\nu\|a\|_\infty}\right\}\right)$$

    for some positive constant C.