Lecture 2

June 30, 2025

Detour: Martingales

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- One of the most general dependence structures in probability.
- Allow relaxing independence assumptions in classical limit theorems.

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- Likelhood ratio process is an example. Say the X_i are i.i.d from the pdf f_0. Then for any pdf f_1 we have that

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- **Theorem 7 (Azuma–Hoeffding).** Let $X = (X_1, \ldots, X_n)^\top$ be a random vector and let $Z = f(X)$. Consider the Doob's martingale

$$Y_i := \mathbb{E}_i(Z) := \mathbb{E}(Z|X_1, \ldots, X_i)$$

and let $\Delta_i = Y_i - Y_{i-1}$. Assume that

$$\mathbb{E}_{i-1}\left(e^{\lambda \Delta_i}\right) \leq e^{\sigma_i^2 \lambda^2/2}, \quad \forall \lambda \in \mathbb{R} \tag{1}$$

almost surely for all $i = 1, \ldots, n$. Then, $Z - \mathbb{E}(Z)$ is sub-Gaussian with parameter $\sigma = \sqrt{\sum_{i=1}^n \sigma_i^2}$.

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almost surely for all $i = 1, \ldots, n$. Then, $Z - \mathbb{E}(Z)$ is sub-Gaussian with parameter $\sigma = \sqrt{\sum_{i=1}^n \sigma_i^2}$.

- In particular, we have the tail bound

$$\mathbb{P}\left(|Z - \mathbb{E}(Z)| \geq t\right) \; \leq \; 2 \exp\left(-\frac{t^2}{2\sigma^2}\right).$$

Proof

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- By properties of conditional expectation, and assumption (1)

$$\mathbb{E}_{n-1}(e^{\lambda S_n}) \; = \; e^{\lambda S_{n-1}} \mathbb{E}_{n-1}\left(e^{\lambda \Delta_n}\right)$$

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and the claim follows.

Bounded difference inequality

- Conditional sub-G. assump. holds under bounded difference property:

$$\left| f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n) \right| \\ \leq L_i \tag{2}$$

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- **Example.** $f(X) = \sum_{i=1}^n X_i$, $X_i \in [a_i, b_i]$.

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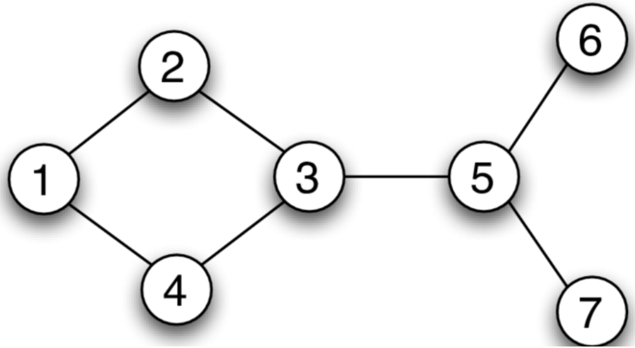

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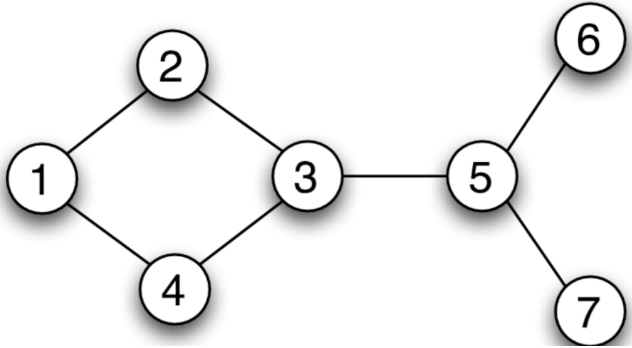

- The edge set is
  $E(G) := \{\{1,2\},\{1,4\},\{2,3\},\{3,4\},\{3,5\},\{5,6\},\{5,7\}\}$.

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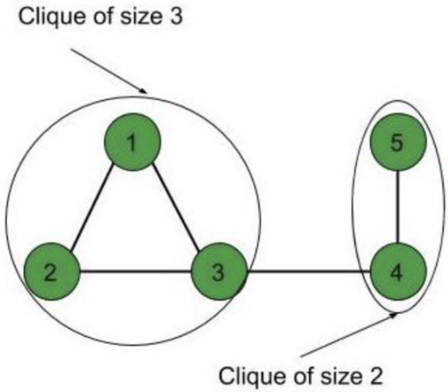

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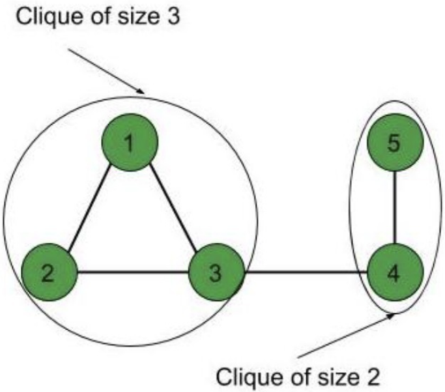

- Clique number of G—denoted as $\omega(G)$—is the size of the largest clique(s).

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- **Theorem 9 (Gaussian concentration).** Let $X \sim N(0, I_n)$ be a standard Gaussian vector and assume that $f : \mathbb{R}^n \to \mathbb{R}$ is L-Lipschitz w.r.t. the Euclidean norm. Then,

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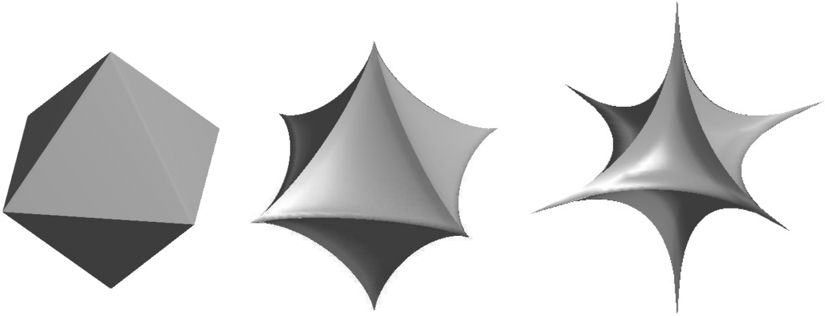
- $q = 1$ gives the ℓ_1 ball.
- $q = 0$ gives the ℓ_0 ball, same as hard sparsity:

$$\|\theta^*\|_0 := |S(\theta^*)|.$$



(a) (b) (c)

Figure 7.1 Illustrations of the ℓ_q-"balls" for different choices of the parameter $q \in (0, 1]$. (a) For $q = 1$, the set $\mathbb{B}_1(R_q)$ corresponds to the usual ℓ_1-ball shown here. (b) For $q = 0.75$, the ball is a non-convex set obtained by collapsing the faces of the ℓ_1-ball towards the origin. (c) For $q = 0.5$, the set becomes more "spiky", and it collapses into the hard sparsity constraint as $q \to 0^+$. As shown in Exercise 7.2(a), for all $q \in (0, 1]$, the set $\mathbb{B}_q(1)$ is star-shaped around the origin.

Compressed sensing or the equation $y = X\theta^*$

- When can we solve the equation $y = X\theta^*$?

The classical answer

The classical theory of linear algebra, which we learn as undergraduates, is as follows:

- If there are at least as many equations as unknowns ($n \geq d$), and X has full rank, then the problem is determined or overdetermined, and one can easily solve $y = X\theta$ uniquely (e.g. by gaussian elimination).

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- If there are at least as many equations as unknowns ($n \geq d$), and X has full rank, then the problem is determined or overdetermined, and one can easily solve $y = X\theta$ uniquely (e.g. by gaussian elimination).
- If there are fewer equations than unknowns ($n < d$), then the problem is underdetermined even when X has full rank. Knowledge of $y = X\theta$ restricts θ to an (affine) subspace of $\mathbb{R}^d$, but does not determine θ completely.

Sparse recovery

- It is thus of interest to obtain a good estimator for underdetermined problems such as $X\theta^* = y$ in the case in which θ^* is expected to be “spiky” - that is, concentrated in only a few of its coordinates.

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- It is thus of interest to obtain a good estimator for underdetermined problems such as $X\theta^* = y$ in the case in which θ^* is expected to be “spiky” - that is, concentrated in only a few of its coordinates.
- A model case occurs when θ^* is known to be s-sparse for some $1 \leq s \leq d$, which means that at most s of the coefficients of θ^* can be non-zero.

Sparse recovery

- Sparsity is a simple but effective model for many real-life signals. For instance, an image may be many megapixels in size, but when viewed in the right basis (e.g. a wavelet basis), many of the coefficients may be negligible, and so the image may be compressible into a file of much smaller size without seriously affecting the image quality. In other words, many images are effectively sparse in the wavelet basis.

Sparsity helps!

- Intuitively, if a signal $\theta^* \in \mathbb{R}^d$ is s-sparse, then it should only have s degrees of freedom rather than d. In principle, one should now only need s measurements or so to reconstruct θ^*, rather than d.

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- signals are sparse in a known basis;

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Such situations can arise in:

- Imaging.
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- MRI.
- Astronomy.

Basis pursuit

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- The resulting problem is convex.

Basis pursuit

- In fact, can be written as a linear program. Notice that the problem is equivalent to

$$\min_{s} \sum_{j=1}^{d} s_j \quad \text{subject to} \quad y = X\theta, \quad |\theta_j| \leq s_j.$$

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$$\min_{s} \sum_{j=1}^{d} s_j \quad \text{subject to} \quad y = X\theta, \quad |\theta_j| \leq s_j.$$

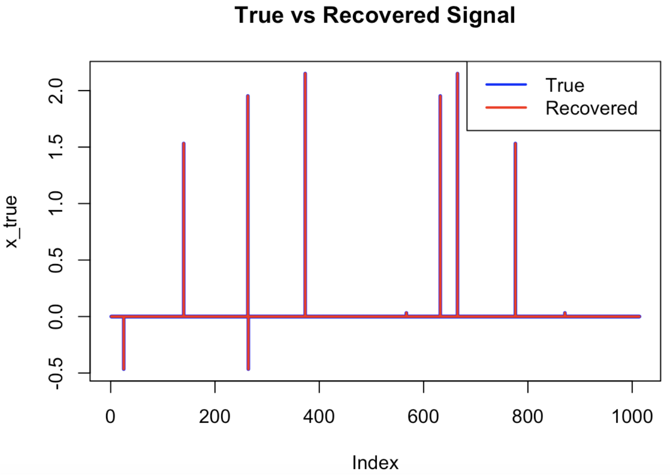
- Global solutions can be obtained very efficiently.

Basis pursuit example

```r
library(CVXR)


set.seed(123)
n <- 1014         # Signal dimension
m <- 64         # Number of measurements
A <- matrix(rnorm(m * n), m, n)  # Measurement matrix



x_true <- rep(0, n)
x_true[sample(1:n, 10)] <- rnorm(5)  # Sparse true signal
plot(x_true)
b <- A %*% x_true


x <- Variable(n)
objective <- Minimize(norm1(x))
constraints <- list(A %*% x == b)
problem <- Problem(objective, constraints)
```

```r
result <- solve(problem)
x_est <- result$getValue(x)


# Plot results
plot(x_true, type = "h", lwd = 3, col = "blue", ylim = range(c(x_true, x_est)),
     main = "True vs Recovered Signal")
lines(x_est, type = "h", col = "red", lwd = 2)
legend("topright", legend = c("True", "Recovered"), col = c("blue", "red"), lwd = 2)

mean((x_est - x_true)^2)
###########################################
```

Basis pursuit example



Terence Tao and Emmanuel Candes have been celebrated for understanding of compressed sensing

Restricted null space property (RNS)

- Define

$$\mathbb{C}(S) := \{\Delta \in \mathbb{R}^d : \|\Delta_{S^c}\|_1 \le \|\Delta_S\|_1\}.$$

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- *For any $\theta^* \in \mathbb{R}^d$ with support $\subset S$, the basis pursuit program applied to the data $(X, y = X\theta^*)$ has unique solution $\hat{\theta} = \theta^*$.*
- *The restricted null space (RNS) property holds, i.e.,*

$$\mathbb{C}(S) \cap \ker(X) = \{0\}.$$

Recall that $\ker(X) = \{\theta \in \mathbb{R}^d : X\theta = 0\}$.

Proof

- Consider the tangent cone to the ℓ_1 ball (of radius $\|\theta^*\|_1$) at θ^*:

$$\mathbb{T}(\theta^*) = \{\Delta \in \mathbb{R}^d \,:\, \|\theta^* + t\Delta\|_1 \leq \|\theta^*\|_1 \text{ for some } t > 0\}.$$

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$$\begin{aligned}
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&= \|\Delta_S\|_1
\end{aligned}$$

by the triangle inequality and setting $\theta^*_S = -\Delta_S$.