# Introduction to the Coriolis Effect

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The Coriolis Effect is foundational to the study of the ocean and atmosphere. The largest dynamical features of Earth's surface—atmospheric cells, surface currents, thermohaline circulation, etc.—are all influenced by this effect, which is characterized by the following:

1. motion at Earth's surface is *deflected* to the right in the northern hemisphere and to the left in the southern hemisphere.
2. the effect appears as acceleration which is proportional ($\propto$) and perpendicular ($\perp$) to velocity.[1]
3. the effect is greatest at the poles and 0 at the equator
4. the radius of curvature for such a deflected object is proportional to its velocity; slow motion curves more tightly than fast motion

We will start with a brief overview of the relevant mathematical constructions used to describe the Coriolis effect. This playlist is recommended for further information on the math.

## 1 Vector

A vector is a mathematical object which encodes both *direction* and *magnitude*. Magnitude may be a unitless number or a quantity (the product of a number and a set of units). Direction is usually determined relative to some position, but vectors in general do not retain this position information. We will see how this plays out later.

Here, vectors are drawn as arrows with magnitude represented by length (although the true meaning may have nothing to do with length) and tails placed at the origin. Vectors pointing toward or away from the reader are drawn as a circled dot (arrowhead) or cross (tail feathers), respectively. See Figure 1.

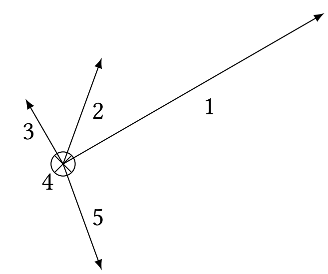


Figure 1: Geometric representation of vectors.

Some quantities are easier to conceptualize as vectors than others. For example, acceleration imposed by gravity at Earth's surface is given by the vector  with downward direction and magnitude:

$$||\,|| \approx 9.8\,\mathrm{m\,s^{-2}} \tag{1}$$

Earth's angular velocity  takes a bit more imagination to imagine as a vector. Its magnitude (angular *speed*) is the angular distance travelled per unit time, which is characteristically uniform for a rigid rotating object. We know that every point will travel once around the circle over the period $T$, so we have:

$$||\,|| = 2\pi/T \tag{2}$$

On Earth $T$ is the sidereal day 86 164 s:

$$||\Omega|| = 2\pi/86\,164\,\mathrm{s} \approx 7.29 \times 10^{-5}\,\mathrm{s^{-1}} \tag{3}$$

and its direction is parallel to Earth's axis of rotation, i.e., from Earth toward Polaris.

[1] Equivalently, as a force which is proportional and perpendicular to momentum.

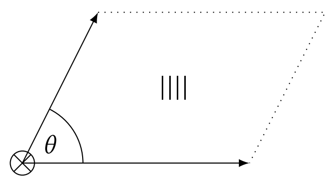


Figure 2: Geometric representation of $\times =$.

## 2 Cross Product

The cross product of two vectors $\times$ is a third vector which obeys the following rules, summarized visually in Figure 2:

1. The output magnitude $||||$ is the "area" of a parallelogram with sides & :

$$||||=||||\cdot||||\cdot\sin\theta \tag{4}$$

   The units of this "area" need not be length squared, just as the units of a vector need not be length. Notice that holding direction of and constant, scaling $||||$ or $||||$ up or down scales $||||$ by the same amount.

2. is perpendicular ($\perp$) to both and . If and are two spokes of a wheel, is the axle. Thus holding $||||$ and $||||$ constant, changing the direction of either or affects *both magnitude and direction* of . $||||$ is greatest when $\perp$; $||||=0$ when is parallel to .

3. Of the two vectors with these properties, follows a "right-hand rule," which we will see later.

## 3 Definition

The Coriolis effect is an apparent force which influences the motion of objects relative to the spinning Earth. We now have the tools necessary to define its mathematical form, although the derivation of this form will not be covered here:

$$=-2m\times \tag{5}$$

where the Coriolis *force* depends on the mass $m$ and velocity (relative to Earth) of the moving object. Alternatively, any force expressed as a function of mass $m$ can be given as an acceleration which is independent of mass:

$$=-2\times \tag{6}$$

where is Coriolis *acceleration*. The properties of the Coriolis effect follow directly from the the cross product and vector definitions:

1. Fundamentally, is influenced by Earth's rotation and the object's velocity, but not its position.

2. The amount of acceleration is greatest when is perpendicular to the axis of rotation; it is zero when is parallel.

3. Coriolis acceleration is always perpendicular to the velocity of the moving object which means things deflect along a curved path but never speed up or slow down.

## 4 Example Calculation

Consider a penguin flying horizontally above the North Pole at speed $||||=100\,\mathrm{m\,s^{-1}}$. Notice that $\perp$ $(\theta=90°)$. How much Coriolis acceleration will it experience? From omega,mag,Ca:

$$\begin{aligned}||||&=-2\left(100\,\mathrm{m\,s^{-1}}\right)\left(7.29\times10^{-5}\,\mathrm{s^{-1}}\right)\sin 90°\\&\approx-0.0146\,\mathrm{m\,s^{-2}}\end{aligned}$$

So the exceptionally fast penguin drifts with an acceleration almost than 700 times smaller than that imposed by gravity. If our penguin has a mass of 20 kg, it feels a slight nudge of:

$$||||=20\,\mathrm{kg}\left(-0.0146\,\mathrm{m\,s^{-2}}\right)\approx-0.29\,\mathrm{N}$$

which is roughly the weight of a pencil.

## 5 Constraints

Imagine a big box on a smooth ($\mu_k=0.05$) ice rink. The vertical gravitational (and normal) force $_f = m\cdot$ is 20 times greater than horizontal friction $\mathbf{f}=\mu_k\cdot m\cdot$. So the box may slide horizontally with a gentle nudge but cannot be lifted easily. That is, vertical motion is limited more than horizontal motion. While a force may come from any direction, only the horizontal component of that force is translated into acceleration.

The ocean and atmosphere have analogous dimensional (rather than frictional) constraints. Atmospheric convection takes place primarily in the troposphere, which is ≈10 km thick. Ocean surface currents occur in the top ≈100 m; deep circulation is bounded by the abyssal plain 4 km to 6 km below the surface. By contrast, the ocean and atmosphere each span at least a majority of the Earth's $\approx 5.1 \times 10^8$ km surface area in the horizontal dimension. As we have seen, (and will soon quantify) Coriolis deflection is simply not noticeable at the scales involved in vertical motion. This means we only observe , the horizontal component of . How do we solve for ?

The relationship between the horizontal plane and varies across Earth's surface. At the poles, they are perpendicular; at the equator, parallel. This is why the observable Coriolis effect varies with latitude ($\varphi$), despite any mention of position in Cf. For our flying penguin at the North Pole ($\varphi = 90°$), horizontal motion is always perpendicular to and thus the full magnitude of the Coriolis Effect is observed. Compare this to the more complicated situation at the equator ($\varphi = 0°$).

If the penguin flies due North or South at the equator, $\|$ and thus $||||=0$. What about East or West? If the penguin continues to fly at 100 m s$^{-1}$, and $\perp$, it experiences the same as it did at the pole. But here the direction of is vertical; it is overpowered by gravity and does not cause horizontal drift. Instead, it adds or subtracts to the other acceleration vector in the vertical direction——and the penguin's *weight* changes imperceptibly. Thus we have two distinct reasons why no Coriolis effect is observed at the equator.

Any direction of intermediate between N/S and E/W results in intermediate between the two extremes given. At this location and speed, magnitude varies from 0 m s$^{-2}$ to $2.31 \times 10^{-3}$ m s$^{-2}$ but direction is always vertical, so = 0.

## 6 Latitude ($\varphi$)

What about at latitudes between the poles and equator? Consider a diatom floating in the surface ocean at $\varphi = 30°$. If is due North, the angle $\theta$ between and is 30°, that is, $\varphi = \theta$ (see Figure 3).

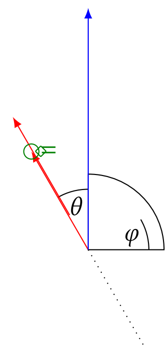


Figure 3: $\varphi = \theta$ for due-North motion

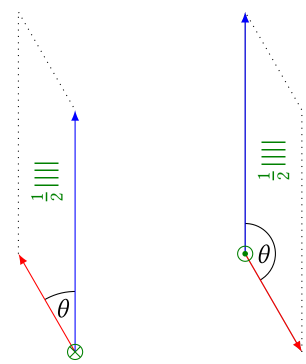


Figure 4: Equivalence of $||||$ for due-North ($\varphi = \theta$) and due-South ($\varphi = 180° - \theta$) motion.

If is due South, $\theta$ is the supplement of $\varphi$ ($180° - \varphi = \theta$). Note that $||||$ ($\propto$ area) is identical in both cases (see Figure 4).

Thus we can substitute $\sin\varphi$ for $\sin\theta$ as we calculate $||||$ for North/South motion:

$$||||=2\cdot||||\cdot||||\cdot\sin\theta$$
$$||||=||||=2\cdot||||\cdot||||\cdot\sin\varphi \tag{7}$$

What about East-West motion? Like the situation at the poles and equator, $\perp$, so $||||$ is mathematically identical to what it would be at the poles. The difference here is that the direction of now includes a vertical component which is not observed. Unlike at the equator, now

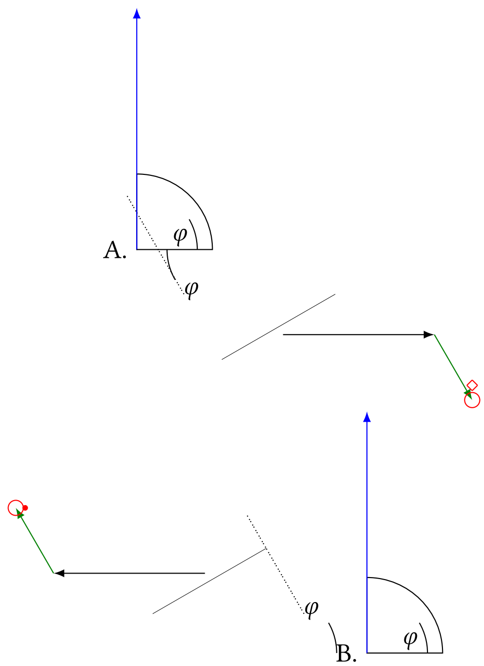


Figure 5: $|||| = |||| \cdot \sin\varphi$ for due-East (A.) and due-West (B.) motion.

also includes a horizontal component, which is observed. Figure 5 shows that we *must* include a factor of $\sin\varphi$ in our definition of  for East/West motion. Since $\perp$, we have $\sin\theta = 1$, thus:

$$\begin{aligned} |||| &= 2 \cdot |||| \cdot |||| \cdot 1 \\ |||| &= 2 \cdot |||| \cdot |||| \cdot \sin\varphi \end{aligned} \tag{7}$$

So we find that despite North/South and East/West motion influencing  very differently, the expression for $||||$ is concisely expressed for all horizontal directions a function of $\varphi$, not $\theta$. For completeness, we can give an expression for the vector  as a product of $||||$ and a unit vector  with a subscript indicating the vector to which  is perpendicular ($\perp$ in both cases):

$$= 2 \cdot |||| \cdot |||| \cdot \sin\theta \cdot_{\Omega} \tag{8}$$

$$= 2 \cdot |||| \cdot |||| \cdot \sin\varphi \cdot \tag{9}$$

# 7 Curvature ($\kappa$)

We want to know how velocity and acceleration interact to shape the path travelled by a Coriolis-affected object. The relevant parameter is curvature, defined by $\kappa = 1/R$, where $R$ is the radius of curvature. In general:

$$\kappa = \frac{|| \times ||}{||||^3} \tag{10}$$

where  is the acceleration vector.[2] For a Coriolis-affected object, the  of interest is , and since $\perp$, we can define Coriolis curvature:

$$\kappa_C = \frac{|||| \cdot |||| \cdot \sin 90^\circ}{||||^3} = \frac{||||}{||||^2} \tag{11}$$

Or, recalling from Cah that $||||$ is a factor of $||||$, we can skip the  calculation altogether:

$$\begin{aligned} \kappa_C = \frac{2 \cdot |||| \cdot |||| \cdot \sin\varphi}{||||^2} \\ = \frac{2 \cdot |||| \cdot \sin\varphi}{||||} \end{aligned} \tag{12}$$

For a calculation with more practical meaning, we take the reciprocal to find the radius of curvature:

$$R = \frac{||||}{2 \cdot |||| \cdot \sin\varphi} \tag{13}$$

For one practical example, assume the Gulf Stream current has a maximum surface speed $2.5\,\mathrm{m\,s^{-1}}$. At $\varphi = 30^\circ$:

$$\begin{aligned} R = \frac{2.5\,\mathrm{m\,s^{-1}}}{2\,(7.29 \times 10^{-5}\,\mathrm{s^{-1}}) \cdot \sin 30^\circ} \\ \approx 34\,\mathrm{km} \end{aligned} \tag{14}$$

Since the actual gulf stream is part of the North Atlantic subtropical gyre (which is much larger than 34 km in radius) we can see that the Coriolis force is not the only force at play in shaping the complex dynamics of ocean circulation. For instance, what causes the current to begin flowing in the first place? Nonetheless, it's important to develop some intuition for magnitude of the effects involved in the Coriolis effect in this simplified example.

[2] Derivation

## 8 Which Way?

Another angle from which to approach the Coriolis effect gives an intuitive answer to the last big question we have put off to this point: which of the two directions is correct? We know $\perp$ and $\perp$, so from the perspective of the moving object, it must be either left or right.

Consider a "stationary" ($= 0$) iceberg at the equator. Of course, it isn't really stationary because it is moving along with the ground beneath as the Earth rotates (and orbits, etc. but we consider here the reference point of a *non-rotating* object orbiting the sun alongside Earth). The Earth's circumference is roughly 40 000 km, and it rotates once in a day (that's the period $T$ we saw in the definition of angular velocity 3), so the iceberg is traveling at a blistering $450\,\mathrm{m\,s^{-1}}$, faster than the speed of sound, relative to the nearby observer. If the iceberg has mass (it does) then it carries a non-zero momentum[3] $p = mv$ which it tends to maintain (Newton's first law of motion).

Imagine the iceberg begins to drift toward the North Pole, carrying its equator-derived momentum as it goes. As it drifts, it finds the Earth appears to move slower than it once did, approaching a point at the pole where there is no motion at all, only rotation fixed in place. Still holding on to all its momentum, the iceberg drifts in the direction of the Earth's rotation—to the East (from the iceberg's North-bound perspective, to the right). If the iceberg were to drift South again from the pole, it would experience the opposite effect. Now the Earth is moving faster relative to its original stationary momentum, so it lags behind the rotation. Now, it drifts west, which is again to the right. By symmetry, deflection is to the left in the southern hemisphere.

East/West motion is a bit trickier. If the iceberg drifts eastward, it begins to outpace the Earth's rotation. As it speeds up, it is flung outward—like an elastic cord stretching out as it is whipped in a faster circle. But since this drift isn't strong enough to overcome gravity, it simply causes a slow drift toward the equator, the latitude with the greatest distance from the axis of rotation. Likewise, westward drifting causes a slower speed and a drift toward the poles, the closest spot to the rotation axis.

[3] This velocity $v$ is *not* the same as , becuase  is relative to the Earth's spinning reference frame.