v.6.1.1.8

V Generative Recursion

If you follow the design recipe of the first four parts, you either turn domain knowledge into code or you exploit the structure of the data definition to organize your code. The latter functions typically decompose their arguments into their immediate structural components and then process those components. If one of these immediate components belongs to the same class of data as the input, the function is *structurally recursive*. While structurally designed functions make up the vast majority of code in the real world, some problems cannot be solved with a structural approach to design.

To solve such complicated problems, programmers use *generative recursion*, a form of recursion that is strictly more powerful than structural recursion. The study of generative recursion is as old as mathematics and is often called the study of *algorithms*. The inputs of an algorithm represent a problem. An algorithm tends to re-arrange a problem into set of several problems, solve those, and combine their solutions into one overall solution. Often some of these newly **generated** problems are the same kind of problem as the given one, in which case the algorithm can be re-used to solve them. In these cases, the algorithm is recursive but its recursion use newly generated data not immediate parts of the input data.

Some functions merely compose such functions; we group those functions also with the “structural” group.

From the very description of generative recursion, you can tell that designing a generative recursive function is more of an ad hoc activity than designing a structurally recursive function. Still, many elements of the general design recipe apply to the design of algorithms, too, and this part of the book illustrates how and how much the design recipe helps. The key to designing algorithms is the “generation” step, which often means dividing up the problem. And figuring out a novel way of dividing a problem requires insight. Sometimes very little insight is required. For example, it might just require a bit of common-sense knowledge about breaking up sequences of letters. At other times, it may rely on deep mathematical theorems about numbers. In practice, programmers design simple algorithms on their own and rely on domain specialists for their complex brethren. For either kind, programmers must thoroughly understand the underlying ideas so that they can code up algorithms and have the program communicate with future readers. The best way to get acquainted with the idea is to study a wide range of examples and to develop a sense for the kinds of generative recursions that may show up in the real world.

Greek calls it a “eureka.”

29 Non-standard Recursion

At this point you have designed numerous functions that employ structural recursion. When you design a function, you know you need to look at the data definition for its major input. If this input is described by a self-referential data definition, you end up with a function that refers to itself basically where the data definition refers to itself.

This chapter presents two sample programs that use recursion differently. They are illustrative of the problems that require some eureka, ranging from the obvious idea to the sophisticated insight.

29.1 Recursion without Structure

Imagine you have joined the DrRacket team. The team is working on a sharing service to support collaborations among programmers. Concretely, the next revision of DrRacket is going to enable ISL programmers to share the content of their DrRacket's definition area across several computers. Each time one programmer modifies the buffer, the revised DrRacket broadcasts the content of the definitions area to the instances of DrRacket that participate in the sharing session.

> **Sample Problem:** Your task is to design the function `bundle`, which prepares the content of the definitions area for broadcasting. DrRacket hands over the content as a list of 1Strings. The function's task is to bundle up sub-sequences of individual "letters" into chunks and to thus produce a list of strings—called **chunks**—of a given length, called **chunk size**.

As you can see, the problem basically spells out the signature and there is no need for any problem-specific data definition:

```
; [List-of 1String] N -> [List-of String]
; bundles sub-sequences of s into strings of length n
(define (bundle s n)
  '())
```

The purpose statement reformulates a sentence fragment from the problem statement and uses the parameters from the dummy function header.

The third step calls for function examples. Here is a list of 1Strings:

```
(list "a" "b" "c" "d" "e" "f" "g" "h")
```

If we tell `bundle` to bundle this list into pairs—that is, $n = 2$—then the following list is the expected result:

```
(list "ab" "cd" "ef" "gh")
```

Now if `n` is `3` instead, there is a left-over "letter". Since the problem statement does not tell us which of the characters is left over, we can imagine at least two valid scenarios:

- The function produces `(list "ab" "cd" "ef" "g")`, that is, it considers the last letter as the left-over letter.
- Or it produces `(list "a" "bc" "de" "fg")`, which represents a different choice, namely, packing the lead character into a string by itself.

As a matter of fact, `(list "ab" "c" "de" "fg")` is also a legitimate result according to the problem statement.

To make things simple, we pick the first choice as the desired result and say so by writing down a corresponding test:

```
(check-expect (bundle (explode "abcdefg") 3) (list "abc" "def" "g"))
```

To keep things concise, this test uses `explode`.

Examples and tests must also describe what happens at the boundary of data definitions. In this context, boundary clearly means `bundle` is given a list that is too short for the given chunk size:

```
(check-expect (bundle '("a" "b") 3) (list "ab"))
```

It also means we should consider what happens when `bundle` is given `'()`. For simplicity, we choose `'()` as the desired result:

```
(check-expect (bundle '() 3) '())
```

One natural alternative is to ask for `'("")`. Can you see others?

```
; N as the driving data definition, s considered atomic
; according to Processing Two Lists Simultaneously: Case 1
(define (bundle s n)
  (cond

    [(zero? n) (...)]
    [else (... s ... n ... (bundle s (sub1 n)))]))

; [List-of 1String] as the driving data definition, n considered atomic
; according to Processing Two Lists Simultaneously: Case 1
(define (bundle s n)
  ; s as the driving data definition
  (cond
    [(empty? s) (...)]
    [else (... s ... n ... (bundle (rest s) n))]))

; [List-of 1String] and N are on equal footing
; according to Processing Two Lists Simultaneously: Case 2
(define (bundle s n)
  ; lock step
  (cond
    [(and (empty? s) (zero? n)) (...)]
    [else (... s ... n ... (bundle (rest s) (sub1 n)))]))

; the cross-product of possibilities,
; according to Processing Two Lists Simultaneously: Case 3
(define (bundle s n)
  (cond
    [(and (empty? s) (zero? n)) (...)]
    [(and (cons? s) (zero? n)) (...)]
    [(and (empty? s) (positive? n)) (...)]
    [else (... s ... n ...
            (bundle (rest s) (sub1 n)) ...
            (bundle s (sub1 n)) ...
            (bundle (rest s) n) ...)]))
```

Figure 76: A useless template for breaking up strings into chunks

The template step reveals that a structural approach cannot work. Figure 76 shows **four** possible templates given that both arguments to are complex arguments. The first two consider of the arguments atomic, but that clearly cannot be the case because the function has to understand each argument. The third template is based on the assumption that the two arguments are processed in lock step, which is close—except that `bundle` clearly has to reset the chunk size to its original value at regular intervals. The final template says that the two arguments are independent and must be processed in this manner, meaning there are four possibilities to proceed at each stage. But this case split decouples the arguments too much because the list and the counting number must be processed together. In short, we are forced to admit that the structural templates appear to be useless for this design problem.

```
; [List-of 1String] N -> [List-of String]
; bundles sub-sequences of s into strings of length n
; idea take and drop n items at a time

(check-expect (bundle (explode "abcdefg") 3) (list "abc" "def" "g"))
(check-expect (bundle (explode "ab") 3) (list "ab"))
(check-expect (bundle '() 3) '())

(define (bundle s n)
  (cond
    [(empty? s) '()]
    [else (cons (implode (take s n)) (bundle (drop s n) n))]))

; [List-of X] N -> [List-of X]
; retrieves the first n items in l if possible or everything
(define (take l n)
  (cond
    [(zero? n) '()]
    [(empty? l) '()]
    [else (cons (first l) (take (rest l) (sub1 n)))]))

; [List-of X] N -> [List-of X]
; remove the first n items from l if possible or everything
(define (drop l n)
  (cond

    [(zero? n) l]
    [(empty? l) l]
    [else (drop (rest l) (sub1 n))]))
```

Figure 77: Generative recursion

Figure 77 shows a complete definition for `bundle`. The definition uses the `drop` and `take` functions requested in exercise 333; these functions are also available in standard libraries. For completeness, the figure comes with their definitions: `drop` eliminates up to `n` items from the front of the list, `take` returns up to that many items. Using these functions, it is quite straightforward to define `bundle`:

- if the given list is `'()`, the result is `'()` as decided upon;
- otherwise `bundle` uses `take` to grab the first `n` 1Strings from `s` and `implode`s them into a plain String;
- it then recurs with a list that is shortened by `n` items, which is accomplished with `drop`;
- finally, `cons` combines the string from 2 with the list of strings from 3 to create the result for the complete list.

Bullet 3 highlights the key difference between `bundle` and any function in the first four parts of this book. Because the definition of List-of `cons`es an item onto a list to create another one, all functions in the first four parts use `first` and `rest` to deconstruct a non-empty list. In contrast, `bundle` uses `drop`, which removes not just one but `n` items at once.

While the definition of `bundle` is unusual, the underlying ideas are intuitive and not too different from the functions seen so far. Indeed, if the chunk size `n` is `1`, `bundle` specializes to a structurally recursive definition. Also, `drop` is guaranteed to produce an integral part of the given list, not some arbitrarily rearranged version. And this idea is

precisely what the next section presents.

Exercise 341. Is `(bundle "abc" 0)` a proper use of the `bundle` function? What does it produce? Why? ∎

Exercise 342. Define the function `list->chunks`. It consumes a list `l` of arbitrary data and a natural number `n`. The function's result is a list of list chunks of size `n`. Each chunk represents a sub-sequence of items in `l`.

Use `list->chunks` to define `bundle` via function composition. ∎

Exercise 343. Define the function `partition`. It consumes a `String` `s` and a natural number `n`. The function produces a list of string chunks of size `n`.

For non-empty strings `s` and positive natural numbers `n`,

```
(equal? (partition s n) (bundle (explode s) n))
```

is `#true`. But don't use this equality as the definition for `partition`; use `substring` instead.

Hint Have `partition` produce its "natural" result for the empty string. For the case where `n` is `0`, see exercise 341.

Note The `partition` function is somewhat closer to what a cooperative DrRacket environment would need than `bundle`. ∎

29.2 Recursion that Ignores Structure

Recall that the `sort>` function from section Design By Composition consumes a list of, say, numbers and re-arranges the same list of numbers in some order, typically ascending or descending. It proceeds by inserting the first number into the appropriate position of the sorted rest of the list. Put differently, it is a structurally recursive function that re-processes the result of the natural recursions.

Hoare's quick-sort algorithm goes about sorting lists in a radically different manner and is the classic example of generative recursion. The underlying generative step uses the time-honored strategy of divide-and-conquer. That is, it divides the non-trivial instances of the problem into two smaller, related problems, solves those smaller problems, and combines their solutions into a solution for the original problem. In the case of the quick-sort algorithm, the intermediate goal is to divide the list of numbers into two lists:

- one that contains all the numbers that are strictly smaller than the first item
- and another one with all those items that are strictly larger than the first.

Then the two smaller lists are sorted via the quick-sort algorithm. Once the two lists are sorted, it composes the two pieces with the first item in the middle. Owing to its special role, the first item on the list is called *pivot item*.

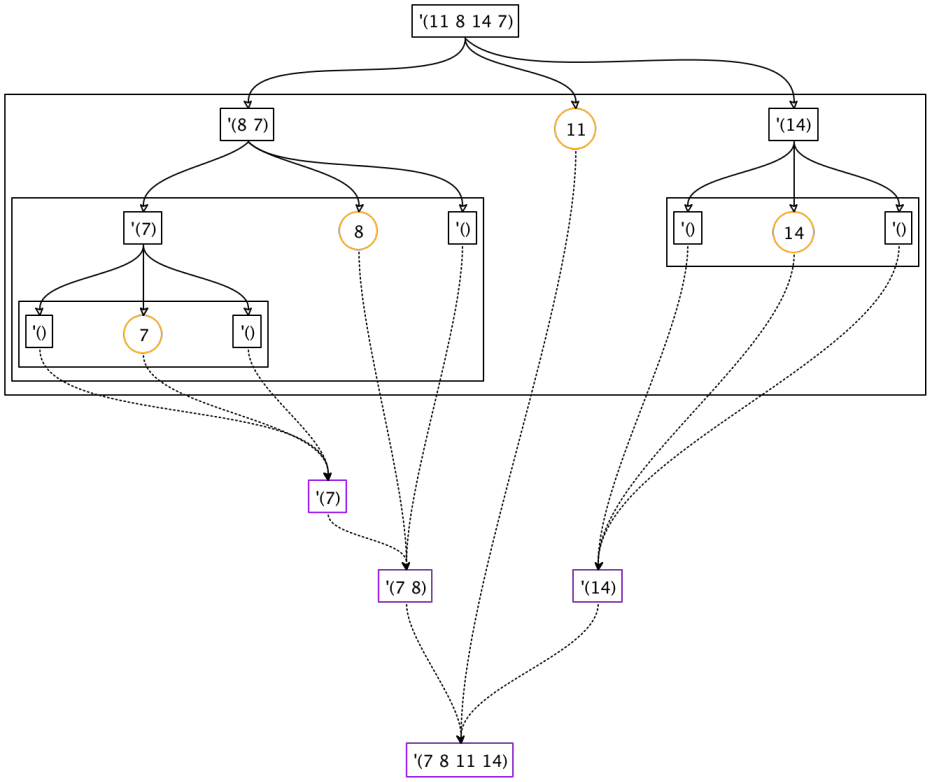


Figure 78: A graphical illustration of the quick-sort algorithm

To develop an understanding of how the quick-sort algorithm works, let's walk through an example, quick-sorting `(list 11 8 14 7)`. Figure 78 illustrates the process in a graphical way. The figure consists of a top half, the divide phase, and the bottom half, the conquer phase.

The partition phase is represented with boxes and solid arrows. Three arrows emerge from each boxed list and go to a box with three pieces: the circled pivot element in the middle, to its left the boxed list of numbers smaller than the pivot, and to its right the boxed list of those numbers that are larger than the pivot. Each of these steps isolates at least one number as the pivot, meaning the two neighboring lists are shorter than the given list. Consequently, the overall process terminates too.

Consider the first step where the input is `(list 11 8 14 7)`. The pivot item is `11`. Partitioning the list into items larger and smaller than `11` produces `(list 8 7)` and `(list 14)`. The remaining steps of the partitioning phase work in an analogous way. Partitioning ends when all numbers have been isolated as pivot elements. At this point, you can already read off the final result by reading the pivots from left to right.

The conquering phase is represented with dashed arrows and boxed lists. Three arrows enter each result box: the middle one from a pivot, the left one from the boxed result of sorting the smaller numbers, and the right one from the boxed result of sorting the larger ones. Each step adds at least one number to the result list, the pivot, meaning the lists grow toward the bottom of the diagram. The box at the bottom is a sorted variant of the given list at the top.

Take a look at the left-most, upper-most conquer step. It combines the pivot `7` with two empty lists, resulting in `'(7)`. The next one down corresponds to the partitioning step

that isolated `8` and thus yields `'(7 8)`. Each level in the conquering phase mirrors a corresponding level from the partitioning phase. After all, the overall process is recursive.

Exercise 344. Draw a quick-sort diagram for `(list 11 9 2 18 12 14 4 1)`.

Now that we have a good understanding of the quick-sort idea, we can translate it into ISL+. Clearly, `quick-sort` distinguishes two cases. If the input is `'()`, it produces `'()` because this list is sorted already; otherwise, it performs a generative recursion. This case split suggests the following `cond` expression:

```
; [List-of Number] -> [List-of Number]
; creates a list of numbers with the same numbers as
; alon, sorted in ascending order
(define (quick-sort alon)
  (cond
    [(empty? alon) '()]
    [else ...]))
```

The answer for the first case is given. For the second case, when `quick-sort`'s input is a non-empty list, the algorithm uses the first item to partition the rest of the list into two sublists: a list with all items smaller than the pivot item and another one with those larger than the pivot item.

Since the rest of the list is of unknown size, we leave the task of partitioning the list to two auxiliary functions: `smaller-items` and `larger-items`. They process the list and filter out those items that are smaller and larger, respectively, than the pivot. Hence each auxiliary function accepts two arguments, namely, a list of numbers and a number. Designing these two functions is an exercise in structural recursion. Try on your own or read the definitions shown in figure 79.

```
; [List-of Number] -> [List-of Number]
; creates a list of numbers with the same numbers as
; alon, sorted in ascending order
; assume the numbers are all distinct
(define (quick-sort alon)
  (cond
    [(empty? alon) '()]
    [else (local ((define pivot (first alon)))
            (append (quick-sort (smaller-items alon pivot))
                    (list pivot)
                    (quick-sort (larger-items alon pivot)))))]))

; [List-of Number] Number -> [List-of Number]
; creates a list with all those numbers on alon
; that are larger than n
(define (larger-items alon n)
  (cond
    [(empty? alon) '()]
    [else (if (> (first alon) n)
              (cons (first alon) (larger-items (rest alon) n))
              (larger-items (rest alon) n))]))

; [List-of Number] Number -> [List-of Number]
; creates a list with all those numbers on alon
; that are smaller than n
```

```
(define (smaller-items alon n)
  (cond
    [(empty? alon) '()]
    [else (if (< (first alon) n)
              (cons (first alon) (smaller-items (rest alon) n))
              (smaller-items (rest alon) n))]))
```

Figure 79: The quick-sort algorithm

Each of these list is sorted separately, using `quick-sort`, which implies the use of recursion, specifically the following two expressions:

1. `(quick-sort (smaller-items alon pivot))`, which sorts the list of items smaller than the pivot; and
2. `(quick-sort (larger-items alon pivot))`, which sorts the list of items larger than the pivot.

Once `quick-sort` has the sorted versions of the two lists, it must combine the two lists and the pivot in the proper order: first all those items smaller than `pivot`, then `pivot`, and finally all those that are larger. Since the first and last list are already sorted, `quick-sort` can simply use `append`:

```
(append (quick-sort (smaller-items alon pivot))
        (list (first alon))
        (quick-sort (larger-items alon pivot)))
```

Figure 79 contains the full program; read it before proceeding.

Now that we have an actual function definition, we can evaluate the example from above by hand:

```
(quick-sort (list 11 8 14 7))
=
(append (quick-sort (list 8 7))
        (list 11)
        (quick-sort (list 14)))
=
(append (append (quick-sort (list 7))
                (list 8)
                (quick-sort '()))
        (list 11)
        (quick-sort (list 14)))
=
(append (append (append (quick-sort '())
                        (list 7)
                        (quick-sort '()))
                (list 8)
                (quick-sort '()))
        (list 11)
        (quick-sort (list 14)))
=
(append (append (append '()
                        (list 7)
                        '())
                (list 8)
                '())
        (list 11)
```

```
        (quick-sort (list 14)))
=
(append (append (list 7)
                (list 8)
                '())
        (list 11)
        (quick-sort (list 14)))
...
```

The calculation shows the essential steps of the sorting process, that is, the partitioning steps, the recursive sorting steps, and the concatenation of the three parts. From this calculation, it is easy to see how `quick-sort` implements the process illustrated in figure 78.

Both figure 78 and the calculation also show how `quick-sort` completely ignores the structure of the given list. The first recursion works on two distant numbers from the originally given list and the second one on the list's third item. These recursions aren't random but they are certainly not relying on the structure of the data definition.

Contrast `quick-sort`'s organization with that of the `sort>` function from Design By Composition. The design of the latter follows the structural design recipe, yielding a program that processes a list item by item. By splitting the list, `quick-sort` can speed up the process of sorting the list, though at the cost of not using plain `first` and `rest`.

Exercise 345. Complete the hand evaluation.

The hand evaluation of `(quick-sort (list 11 8 14 7))` suggests an additional trivial case for `quick-sort`. Every time `quick-sort` consumes a list of one item, it produces the very same list. After all, the sorted version of a list of one item is the list itself.

Modify the definition of `quick-sort` to take advantage of this observation.

Hand evaluate the example again. How many steps does the extended algorithm save? ∎

Exercise 346. While `quick-sort` quickly reduces the size of the problem in many cases, it is inappropriately slow for small problems. Hence people use `quick-sort` to reduce the size of the problem and switch to a different sort function when the list is small enough.

Develop a version of `quick-sort` that uses `sort>` from Recursive Auxiliary Functions if the length of the input is below some threshold. ∎

Exercise 347. If the input to `quick-sort` contains the same number several times, the algorithm returns a list that is strictly shorter than the input. Why? Fix the problem so that the output is as long as the input. ∎

Exercise 348. Use `filter` to define `smaller-items` and `larger-items` as one-liners. ∎

Exercise 349. Develop a variant of `quick-sort` that uses only one comparison function, say, `<`. Its partitioning step divides the given list `alon` into a list that contains the items of `alon` smaller than `(first alon)` and another one with those that are not smaller.

Use `local` to package up the program as a single function: Abstract this function so that it consumes a list and a comparison function. ∎

The overview for this part already explains that the design of generative recursion functions is more ad hoc than structural design. As the first chapter shows, two generative recursions can radically differ in how they process functions. Both `bundle` and `quick-sort` process lists, but while the former at least respects the sequencing in the given list, the latter re-arranges its given list at will. The question is whether a single design recipe can help with the creation of such widely differing functions.

The first section of this chapter shows how to adapt the process dimension of the design recipe to generative recursion. The second section hones in on another new phenomenon: an algorithm may fail to produce an answer for some of its inputs. Programmers must therefore analyze their programs and supplement the design information with a comment on termination. The remaining sections in this chapter compare and contrast structural and generative recursion.

30.1 Adapting the Design Recipe

Let's examine the six general steps of our structural design recipe in light of the examples in the preceding chapter:

- As before, we must represent the problem information as data in our chosen programming language. The choice of a **data representation** for a problem affects our thinking about the computational process, so some planning ahead is necessary. Alternatively, be prepared to backtrack and to explore different data representations. Regardless, we must analyze the problem information and define data collections.

- We also need a signature, a function header, and a purpose statement. Since the generative step has no connection to the structure of the data definition, the purpose statement must go beyond **what** the function is to compute and also explain **how** the function computes its result.

- It is useful to explain the "how" with function examples, the way we explained `bundle` and `quick-sort` in the previous chapter. That is, while function examples in the structural world merely specify which output the function is to produce for which input, the purpose of examples in the world of generative recursion is to explain the underlying idea behind the computational process.

  For `bundle`, the examples specify how the function acts in general and in certain boundary cases. For `quick-sort`, the example in figure 78 illustrates how the function partitions the given list with respect to the pivot item. By adding such worked examples to the purpose statement, we—the designers—gain an improved understanding of the desired process, and we communicate this understanding to future readers of this code.

- Our discussion suggests a general template for algorithms. Roughly speaking, the design of an algorithm distinguishes two kinds of problems: those that are *trivially solvable* and those that are not. If a given problem is trivially solvable, an algorithm produces the matching solution. For example, the problems of sorting an empty list or a one-item list are trivially solvable. A list with many items is a non-trivial problem. For these non-trivial problems, algorithms commonly generate new problems of the same kind as the given one, solve those recursively, and combine the solutions into an overall solution.

  Based on this rough sketch, all algorithms have roughly this organization:

For this part of the book,

```
(define (generative-recursive-fun problem)
  (cond
    [(trivially-solvable? problem)
     (determine-solution problem)]
    [else
     (combine-solutions
       ... problem ...
       (generative-recursive-fun (generate-problem-1 problem))
       ...
       (generative-recursive-fun (generate-problem-n problem)))]))
```

"trivial" is a technical term.

The original problem is occasionally needed to combine the solutions for the newly generated problems, which is why it is handed over to `combine-solutions`.

- This template is only a suggestive blueprint, not a definitive shape. Each piece of the template is to remind us to think about the following four questions:
  - What is a trivially solvable problem?
  - How are trivial solutions solved?
  - How does the algorithm generate new problems that are more easily solvable than the original one? Is there one new problem that we generate or are there several?
  - Is the solution of the given problem the same as the solution of (one of) the new problems? Or, do we need to combine the solutions to create a solution for the original problem? And, if so, do we need anything from the original problem data?

  To define the algorithm as a function, we must express the answers to these four questions as functions and expressions in terms of the chosen data representation.
- Once the function is complete, it is time to test it. As before, the goal of testing is to discover and eliminate bugs. Remember that testing cannot validate that the function works correctly for all possible inputs.

Exercise 350. Formulate informal answers to the four key questions for `bundle`. ∎

Exercise 351. Formulate informal answers to the four key questions for the `quick-sort` problem. How many instances of `generate-problem` are there? ∎

30.2 Termination

Generative recursion adds an entirely new aspect to computations: non-termination. A function such as `bundle` may never produce a value nor signal an error for certain inputs. Exercise 341 asks what the result of `(bundle '("a" "b" "c") 0)` is, and here is an explanation of why it does not have a result:

```
(bundle '("a" "b" "c") 0)
= (cons (implode (take  '("a" "b" "c") 0)) (bundle (drop  '("a" "b" "c") 0)))
= (cons (implode '()) (bundle (drop  '("a" "b" "c") 0)))
= (cons "" (bundle (drop  '("a" "b" "c") 0)))
= (cons "" (bundle  '("a" "b" "c") 0))
```

The calculation shows that determining the result of `(bundle '("a" "b" "c") 0)` requires having a result for the very same expression. In the context of ISL+ this means the evaluation does not stop. Computer scientists say that `bundle` does not *terminate*

when the second argument is `0`; they also say that the function *loops* or that the computation is stuck in an *infinite loop*.

Contrast this insight with the designs presented in the first four parts. Every function designed according to the recipe either produces an answer or raises an error signal for every input. After all, the recipe dictates that each natural recursion consumes an immediate piece of the input, not the input itself. Because data is constructed in a hierarchical manner, input shrinks at every stage. Eventually the function is applied to an atomic piece of data, and the recursion stops.

This reminder also explains why generative recursive functions may diverge. According to the design recipe for the latter, an algorithm may generate new problems without any limitations. If the design recipe required the designer to guarantee that the new problem were “smaller” than the given one, it would terminate. But, imposing such a restriction would needlessly complicate the design of functions such as `bundle`.

The theory of computation actually shows that we must lift these restrictions eventually.

In this book, we therefore keep the first six steps of the design recipe intact and supplement them with a seventh step: the *termination argument*. Figure 80 summarize this decision in a table. It shows the new design recipe in the conventional tabular form. The unmodified steps come with – in the **activity** column. Others come with comments on how the design recipe for generative recursion differs from the one for structural recursion. The last row is completely new.

steps	**outcome**	**activity**
problem analysis	data representation and definition	---
header	a purpose statement concerning the ``how'' of the function	supplement the explanation of **what** the function computes with a one-liner on **how** it computes the result
examples	examples and tests	explain the ``how'' with several examples
template	fixed template	---
definition	full-fledged definition	formulate conditions for trivially solvable problems; formulate answers for these trivial cases; determine how to generate new problems for non-trivial problems, possibly using auxiliary functions; determine how to combine the solutions of the generate problems into a solution for the given problem
tests	discover mistakes	---
termination	(1) a size argument for each recursive call or (2) examples of exceptions to termination	investigate whether the problem data for each recursive data is smaller than the given data; find

examples that cause the function to loop

Figure 80: Designing algorithms

A termination argument comes in one of two forms. The first one argues why each recursive call works on a problem that is smaller than the given one. Often this argument is straightforward; on rare occasions, you will need to work with a mathematician to prove a new theorem for such arguments. The second kind of termination argument illustrates with an example that the function may not terminate. Ideally it should also describe the class of data for which the function may loop. In some extremely rare cases, you may not be able to make either argument because computer science does not know enough yet.

Let's illustrate the two kinds of termination arguments with two examples. For the `bundle` function, it suffices to warn readers about chunk size `0`:

```
; [List-of 1String] N -> [List-of String]
; bundles sub-sequences of s into strings of length n
; termination (bundle s 0) loops unless s is '()
(define (bundle s n)
  ...)
```

In this case, it is possible to define a predicate that precisely describes when `bundle` terminates. Do so before you read on. For `quick-sort`, the key observation is that each recursive use of `quick-sort` receives a shorter list:

```
; [List-of Number] -> [List-of Number]
; creates a sorted variant of alon
; termination both recursive calls to quick-sort
; processes list that are guaranteed to miss the pivot
(define (quick-sort alon)
  ...)
```

You cannot define a predicate for this class; otherwise you could modify the function and ensure that it always terminates.

In one case, the list consists of the numbers that are strictly smaller than the pivot; the other one is for numbers strictly larger.

Exercise 352. Develop a checked version of `bundle` that is guaranteed to terminate for all inputs. It may signal an error for those cases where the original version loops. ∎

Exercise 353. Consider the following definition of `smaller-items`, one of the two "problem generators" for `quick-sort`:

```
; [List-of Number] Number -> [List-of Number]
; creates a list with all those numbers on alon
; that are smaller than or equal to n
(define (smaller-items alon n)
  (cond
    [(empty? alon) '()]
    [else (if (<= (first alon) threshold)
              (cons (first alon) (smaller-items (rest alon) threshold))
              (smaller-items (rest alon) threshold))]))
```

What can go wrong when this version is used with the `quick-sort` definition from Recursion that Ignores Structure? ∎

Exercise 354. When you worked on exercise 349 or exercise 347, you may have produced looping solutions. Similarly, exercise 353 actually reveals how brittle the

termination argument is for `quick-sort`. In all cases, the argument relies on the idea that `smaller-items` and `larger-items` produce lists that are maximally as long as the given list, and our understanding that neither includes the given pivot in the result.

Based on this explanation, modify the definition of `quick-sort` so that both functions already receive lists that are shorter than the given one. ∎

30.3 Structural versus Generative Recursion

The template for algorithms is so general that it includes structurally recursive functions. Consider the left side of figure 81. This template is specialized to deal with one trivial clause and one generative step. If we replace `trivial?` with `empty?` and `generate` with `rest`, we get a template for list-processing functions; see the right side of figure 81.

```
(define (general P)
  (cond
    [(trivial? P) (solve P)]
    [else
      (combine-solutions
        P
        (general (generate P)))]))
```

```
(define (special P)
  (cond
    [(empty? P) (solve P)]
    [else
      (combine-solutions
        P
        (special (rest P)))]))
```

Figure 81: From generative to structural recursion

Exercise 355. Define `solve` and `combine-solutions` so that

- `special` computes the length of its input,
- `special` negates each number on the given list of strings, and
- `special` uppercases the given list of strings.

What do you conclude from these exercises? ∎

Is there a real difference between structural recursive design and the one for generative recursion?

Our answer is “it depends.” Of course, we could say that all functions using structural recursion are just special cases of generative recursion. This “everything is equal” attitude, however, is of no help if we wish to understand the process of designing functions. It confuses two kinds of design that require different forms of knowledge and that have different consequences. One relies on a systematic data analysis and not much more; the other requires a deep, often mathematical, insight into the problem-solving process itself. One leads programmers to naturally terminating functions; the other requires a termination argument. Conflating these two approaches is simply unhelpful.

30.4 Making Choices

When you interact with a function `f` that consumes lists of numbers and produces sorted variants, it is impossible for you to know whether `f` is the `sort>` function or `quick-sort`. The two functions behave in an observably equivalent way. This raises the question of

which of the two a programming language should provide in its library. More generally, when we can design a function using structural recursion and generative recursion, we need to figure out which one to pick.

To understand this choice, let's discuss another classical example of generative recursion from mathematics: the problem of finding the greatest common divisor of two positive natural numbers. All such numbers have at least one divisor in common: 1. On occasion, this is also the only common divisor. For example, 2 and 3 have only 1 as common divisor because 2 and 3, respectively, are the only other divisors. Then again, 6 and 25 are both numbers with several divisors:

- 6 is evenly divisible by 1, 2, 3, and 6;
- 25 is evenly divisible by 1, 5, and 25.

Still, their greatest common divisor is 1. In contrast, 18 and 24 have many common divisors and their greatest common divisor is 6:

- 18 is evenly divisible by 1, 2, 3, 6, 9, and 18;
- 24 is evenly divisible by 1, 2, 3, 4, 6, 8, 12, and 24.

Completing the first three steps of the design recipe is straightforward:

```
; N[>= 1] N[>= 1] -> N
; finds the greatest common divisor of n and m


(check-expect (gcd 6 25) 1)
(check-expect (gcd 18 24) 6)

(define (gcd n m)
  1)
```

The signature specifies the precise inputs: natural numbers greater or equal to `1`.

From here we design both a structural and a generative recursive solution. Since this part of the book is about generative recursion, we merely present a structural solution in figure 82 and leave the design ideas to exercises. Just note that `(= (remainder n i) (remainder m i) 0)` encodes the idea that both `n` and `m` are "evenly divisible" by `i`.

```
(define (gcd-structural n m)
  (local (; N -> N
          ; determines the greatest divisor of n and m less than i
          (define (greatest-divisor-<= i)
            (cond
              [(= i 1) 1]
              [else (if (= (remainder n i) (remainder m i) 0)
                        i
                        (greatest-divisor-<= (- i 1)))])))
    (greatest-divisor-<= (min n m))))
```

Figure 82: Finding the greatest common divisor via structural recursion

Exercise 356. Explain in your words how `greatest-divisor-<=` works. Use the design recipe to find the right words. Why is `greatest-divisor-<=` applied to `(min n m)` in the body of the `local` definition? ∎

Observable equivalence plays a central role in Dr. John Stone suggested mming ges. the material on the greatest common divisor.

Although the design of `gcd-structural` is rather straightforward, it is also naive. It simply tests for every number between the smaller of `n` and `m` and `1` whether it divides both `n` and `m` evenly and returns the first such number. For small `n` and `m`, this works just fine. Consider the following example, however:

```
(gcd-structural 101135853 45014640)
```

The result is `177` and to get there, `gcd-structural` had to check the "evenly divisible" condition for 101135676, that is, 101135853 - 177, numbers. Checking that many `remainder`s—actually twice as many—is a large effort, and even reasonably fast computers spend many seconds on this task.

Exercise 357. Copy `gcd-structural` into the definitions area of DrRacket and evaluate

```
(time (gcd-structural 101135853 45014640))
```

in the interactions area. ∎

Since mathematicians recognized the inefficiency of this structural function a long time ago, they studied the problem of finding divisors in depth. The essential insight is that

> for two natural numbers `larger` and `smaller`, their greatest common divisor is equal to the greatest common divisor of `smaller` and the remainder of `larger` divided into `smaller`.

Here is how we can articulate this insight as an equation:

```
(gcd larger smaller) = (gcd smaller (remainder larger smaller))
```

Since `(remainder larger smaller)` is smaller than both `larger` and `smaller`, the right-hand side use of `gcd` consumes `smaller` first.

Here is how this insight applies to our small example:

- The given numbers are `18` and `24`.
- According to the insight, they have the same greatest common divisor as `18` and `6`.
- And these two have the same greatest common divisor as `6` and `0`.

And here we seem stuck because `0` is unexpected. But, `0` can be evenly divided by every number, meaning we have found our answer: `6`.

Working through the example not only validates the basic insight but also suggests how to turn the insight into an algorithm:

- when the smaller of the two given numbers is `0`, we are faced with a trivial case;
- the larger of the two numbers is the solution in the trivial case;
- generating a new problem requires a single `remainder` operation; and
- the above equation tells us that the answer to the newly generated problem is also the answer to the originally given problem.

In other words, the answers for all four questions from the design recipe just fall out.

```
(define (gcd-generative n m)
```

```
(local (; N[>= 1] N[>=1] -> N
        ; generative recursion
        ; (gcd larger smaller) = (gcd larger (remainder larger smaller))
        (define (clever-gcd larger smaller)

          (cond
            [(= smaller 0) larger]
            [else (clever-gcd smaller (remainder larger smaller))])))
  (clever-gcd (max m n) (min m n))))
```

Figure 83: Finding the greatest common divisor via generative recursion

Figure 83 presents the definition of the algorithm. The `local` definition introduces the workhorse of the function: `clever-gcd`. Its first `cond` line discovers the trivial case by comparing `smaller` to `0` and produces the matching solution. The generative step uses `smaller` as the new first argument and `(remainder larger smaller)` as the new second argument to `clever-gcd`.

If we now use `gcd-generative` with our complex example from above:

```
(gcd-generative 101135853 45014640)
```

we see that the response is nearly instantaneous. A hand-evaluation shows that `clever-gcd` recurs only nine times before it produces the solution:

```
...
= (clever-gcd 101135853 45014640)
= (clever-gcd 45014640 11106573)
= (clever-gcd 11106573 588348)
= (clever-gcd 588348 516309)
= (clever-gcd 516309 72039)
= (clever-gcd 72039 12036)
= (clever-gcd 12036 11859)
= (clever-gcd 11859 177)
= (clever-gcd 177 0)
```

This also means that it checks only nine `remainder` conditions, clearly a much smaller effort than `gcd-structural` expends.

Exercise 358. Copy `gcd-generative` into the definitions area of DrRacket and evaluate

```
(time (gcd-structural 101135853 45014640))
```

in the interactions area. ∎

You may now think that generative recursion design has discovered a much faster solution to the `gcd` problem, and you may conclude that generative recursion is always the right way to go. This judgment is too rash for three reasons. First, even a well-designed algorithm isn't always faster than an equivalent structurally recursive function. For example, `quick-sort` wins only for large lists; for small ones, the standard `sort>` function is faster. Worse, a badly designed algorithm can wreak havoc on the performance of a program. Second, it is typically easier to design a function using the recipe for structural recursion. Conversely, designing an algorithm requires an idea of how to generate new problems, a step that often requires some deep insight. Finally, people who read functions can easily understand structurally recursive functions, even without much documentation. To understand an algorithm, the generative step must be explained really well, but generating a really good explanation can be a lot of hard work.

Experience shows that most functions in a program employ structural design; only a few exploit generative recursion. When we encounter a situation where a design could use the recipe for either structural or generative recursion, the best approach is to start with a structural version. If it turns out to be too slow for the task at hand, explore the use of generative recursion.

Exercise 359. Evaluate

```
(quick-sort (list 10 6 8 9 14 12 3 11 14 16 2))
```

by hand. Show only those lines that introduce a new recursive call to `quick-sort`. How many recursive applications of `quick-sort` are required? How many recursive applications of the `append` function? Suggest a general rule for a list of length `n`.

Evaluate

```
(quick-sort (list 1 2 3 4 5 6 7 8 9 10 11 12 13 14))
```

by hand. How many recursive applications of `quick-sort` are required? How many recursive applications of `append`? Does this contradict the first part of the exercise? ∎

Exercise 360. Add `sort>` and `quick-sort` to the definitions area. Run tests on the functions to ensure that they work on basic examples. Also develop `create-tests`, a function that creates large test cases randomly. Then explore how fast each works on various lists.

Does the experiment confirm the claim that the plain `sort>` function wins in many comparisons over `quick-sort` for short lists and vice versa?

Determine the cross-over point. Use this information to build a `clever-sort` function that behaves like `quick-sort` for large lists and switches over to the plain `sort>` function for lists below this cross-over point. See exercise 346. ∎

Exercise 361. Given the header material for `gcd-structural`, a naive use of the design recipe might use the following template or some variant:

```
(define (gcd-structural n m)
  (cond
    [(and (= n 1) (= m 1)) ...]
    [(and (> n 1) (= m 1)) ...]
    [(and (= n 1) (> m 1)) ...]
    [else (... (gcd-structural (sub1 n) (sub1 m)) ...
               ... (gcd-structural (sub1 n) m) ...
               ... (gcd-structural n (sub1 m)) ...)]))
```

Why is it impossible to find a divisor with this strategy? ∎

Exercise 362. Exercise 361 means that the design for `gcd-structural` calls for some planning and a design by composition approach.

The very explanation of “greatest common denominator” suggests a two-stage approach. First design a function that can compute the list of divisors of a natural number. Second, design a function that picks the largest common number in the list of divisors of `n` and the list of divisors of `m`. The overall function would look like this:

```
(define (gcd-structural smaller larger)
  (largest-common (divisors smaller smaller) (divisors smaller larger)))
```

Ideally, you should use sets

not lists.

```
; N[>= 1] N[>= 1] -> N
; computes the list of divisors of l smaller or equal to k
(define (divisors k l)
  '())

; [List-of N] [List-of N] -> N
; finds the largest number common to both k and l
(define (largest-common k l)
  1)
```

Before you design `largest-common` explain why `divisors` consumes two numbers and why it consumes `smaller` in both cases. ∎

31 Variations on the Theme

The design of an algorithm starts with an informal description of a process of how to create a problem that is more easily solvable than the given one and whose solution contributes to the solution of the given problem. Coming up with this kind of idea requires inspiration, penetration of an area, and experience with many different kinds of examples.

This chapter presents several illustrative examples of algorithms. Some are directly drawn from mathematics, which is the source of many ideas; others come from computational settings. The first example is a graphical illustration of our principle: the Sierpinski triangle. The second one explains the divide-and-conquer principle with the simple mathematical example of finding the root of a function. It then shows how to turn this idea into a fast algorithm for searching sequences, a widely used application. The third section concerns “parsing” of sequences of 1Strings, also a common problem in real-world programming.

31.1 Fractals, a First Taste

Fractals play an important role in computational geometry. Flake writes in *The Computational Beauty of Nature* (The MIT Press, 1998) that “geometry can be extended to account for objects with a fractional dimension. Such objects, known as *fractals*, come very close to capturing the richness and variety of forms found in nature. Fractals possess structural self-similarity on multiple ... scales, meaning that a piece of a fractal will often look like the whole.”

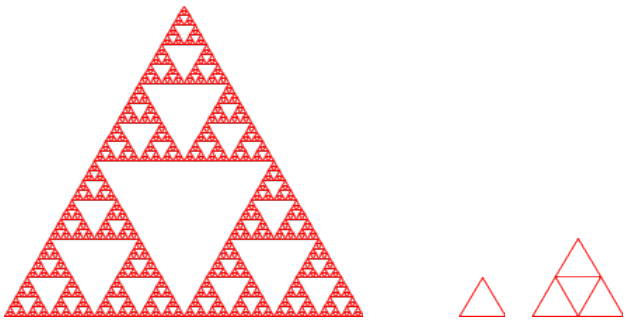


Figure 84: The Sierpinski triangle

Figure 84 displays an example of a fractal shape, known as the Sierpinski triangle. The basic shape is an (equilateral) triangle, like the one in the center. When this triangle is composed sufficiently many times in a triangular fashion, we get the left-most shape.

The right-most image in figure 84 explains the generative step. When taken by itself, it says that given a triangle, find the midpoint of each side and connect them to each other. This step yields four triangles; repeat the process for each of the outer of these three triangles unless these triangles are too small.

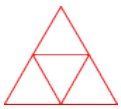
An alternative explanation, well suited for the shape composition functions in the `"image"` library, is based on the transition from the image in the center to the image on the right. By juxtaposing two of the center triangles and then placing one copy above these two, we also get the shape on the right:

We owe this solution to Dr. Marc Smith.

```
> (sf SMALL)
```

```
> (beside (sf SMALL) (sf SMALL))
```

```
> (above (sf SMALL) (beside (sf SMALL) (sf SMALL)))
```



This section uses the alternative description to design the Sierpinski algorithm; Accumulators as Results deals with the first description. Given that the goal is to generate the image of an equilateral triangle, we encode the problem with a (positive) number, the length of the triangle's side. This decision, yields a signature, purpose statement, and header:

```
; Number -> Image
; creates Sierpinski triangle of size side

(define (sierpinski side)
  (triangle side 'outline 'red))
```

Now it is time to address the four questions of generative recursion:

- When the given number is so small that drawing triangles inside of it is pointless, the problem is trivial.
- In that case, it suffices to generate a triangle.
- Otherwise, the algorithm must generate a Sierpinski triangle of size *side* / 2 because juxtaposing two such triangles in either direction yields one of size *side*.
- If `half-sized` is the Sierpinski triangle of size *side* / 2, then

  ```
  (above half-sized
         (beside half-sized half-sized))
  ```

  is a Sierpinski triangle of size *side*.

With these answers, it is straightforward to define the function.

```
(define SMALL 4)

(define small-triangle (triangle SMALL 'outline 'red))

; Number -> Image
; generative creates Sierpinski triangle of size side by generating
; one of size side/2 and placing one copy above two composed copies

(check-expect (sierpinski SMALL) small-triangle)
(check-expect (sierpinski (* 2 SMALL))
              (above small-triangle
                     (beside small-triangle small-triangle)))

(define (sierpinski side)
  (cond
    [(<= side SMALL) (triangle side 'outline 'red)]
    [else (local ((define half-sized (sierpinski (/ side 2))))
            (above half-sized
                   (beside half-sized half-sized)))]))
```

Figure 85: The Sierpinski algorithm

Figure 85 spells out the details. The “triviality condition” translates to `(<= side SMALL)` for some constant `SMALL`. For the trivial answer, the function returns a (red) triangle of the given size. In the recursive case, a `local` expression introduces the name `half-sized` for the Sierpinski triangle that is half as big as the specified size. Once the recursive call has generated the small Sierpinski triangle, the `above`-`beside` composition of three copies yields the desired triangle.

The figure highlights two other points. First, the purpose statement is articulated as an explanation of **what** the function accomplishes

```
; create Sierpinski triangle of size side by ...
```

and **how** it accomplishes this goal:

```
; generating one of size side/2 and
; placing one copy above two composed copies
```

Second, the examples illustrate the two possible cases: one if the given size is small enough, and one for a size that is too large still. In the latter case, the expression that computes the expected value explains exactly the meaning of the purpose statement.

Since `sierpinski` is based on generative recursion, defining the function and testing it is not the last step. We must also consider why the algorithm terminates for any given legal input. The input of `sierpinski` is a single positive number. If the number is smaller than `SMALL`, the algorithm terminates. Otherwise, the recursive call uses a number that is half as large as the given one. Hence, the algorithm must terminate for all positive `sides`, assuming `SMALL` is positive, too.

One view of the Sierpinski process is that it divides its problem in half until it is immediately solvable. With a little imagination, you can see that the process can be used to search for numbers with certain properties. The next section explains this idea in detail.

31.2 Binary Search

Applied mathematicians model the real-world with non-linear equations and then try to solve them. Specifically, they translate problems into a function *f* from numbers to numbers and look for some number *r* such that

$$f(r) = 0.$$

The value *r* is called the *root* of *f*.

Here is a problem from the physical domain:

> **Sample Problem:** A rocket is flying at the constant speed of *v* miles per hour on a straight line towards some target, d_0 miles away. It then accelerates at the rate of *a* miles per hour squared for *t* hours. When will it hit its target?

Physics tells us that the distance covered is the following function of time:

$$d(t) = (v*t + 1/2 * a * t^2)$$

The question of when it hits the target ask us to find the time t_0 such that the object reaches the desired goal:

$$d_0 = (v*t_0 + 1/2 * a * t_0^2)$$

From algebra we know that this is a quadratic equation and that it is possible to solve such equations given d_0, *a*, and *v* satisfy certain conditions.

Generally such problems call for more complex equations than quadratic ones. In response, mathematicians have spent the last few centuries developing root-finding methods for different types of functions. In this section, we study a solution that is based on the **Intermediate Value Theorem**, an early result of mathematical analysis. The resulting algorithm is a primary example of generative recursion based on a mathematical theorem. It has been adapted to other uses and has become known as the *binary search* algorithm in computer science.

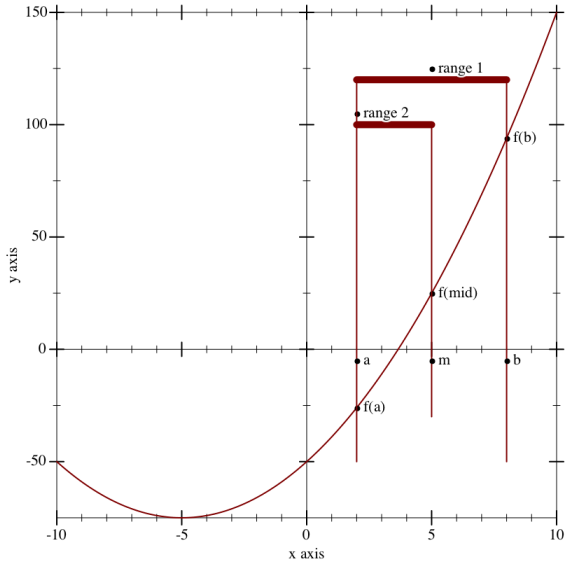


Figure 86: A numeric function *f* with root in interval [*a*,*b*] (stage 1)

The Intermediate Value Theorem says that a continuous function f has a root in an interval $[a,b]$ if $f(a)$ and $f(b)$ are on the opposite side of the x axis. By *continuous* we mean a function that doesn't "jump," that doesn't have gaps, and that proceeds on a "smooth" path.

Figure 86 illustrates the Intermediate Value Theorem in a graphical manner. The function f is below the x axis at a and above the x-axis at b. It is a continuous function, as suggested by the uninterrupted, smooth graph. And indeed, f intersects the x axis somewhere between a and b, labeled "range 1" in the figure.

We thank Dr. Neil Toronto for `plot`.

Now take a look at the midpoint between a and b:

$$m = (a+b) / 2 .$$

It partitions the interval $[a,b]$ into two smaller, equally large intervals. We can now compute the value of f at m and see whether it is below or above 0. Here $f(m) > 0$, so according to the Intermediate Value Theorem, the root is in the left interval: $[a,m]$. Our picture confirms this because the root is in the left half of the interval, labeled "range 2" in figure 86.

We now have a description of the key step in the root-finding process. The next step is to translate this description into a ISL+ algorithm. Our first task is to agree on the exact task of `find-root`. Clearly the algorithm consumes a function and the boundaries of the interval in which we expect to find a root:

```
; [Number -> Number] Number Number -> ...
(define (find-root f left right) ...)
```

The three parameters can't be just any function and numbers. For `find-root` to work, we must assume that

```
(or (<= (f left) 0 (f right))
    (<= (f right) 0 (f left)))
```

holds, that is, `f`'s values for `left` and `right` must be on opposite side's of the x axis.

Next we need to fix the function's result and formulate a purpose statement. Simply put, `find-root` finds an interval that contains a root. The search divides the interval until its size, `(- right left)`, is tolerably small, say, smaller than some constant `TOLERANCE`. At that point, the function could produce one of three results: the left boundary, the right one, or a representation of the interval. Any one of them completely identifies the interval, and since it is simpler to return numbers, we pick the left boundary. Here is the complete header material:

```
; [Number -> Number] Number Number -> Number
; determine R such that f has a root in [R,(+ R TOLERANCE)]
; assume f is continuous
; assume (or (<= (f left) 0 (f right)) (<= (f right) 0 (f left)))
; generative divide interval in half, the root is in one of the two
; halves, pick according to assumption
(define (find-root f left right)
  0)
```

Exercise 363. Consider the following function definition:

```
; Number -> Number
(define (poly x)
```

```
  (* (- x 2) (- x 4)))
```

It defines a binomial for which we can determine its roots by hand:

```
> (poly 2)
0
> (poly 4)
0
```

Use `poly` to formulate a test for `find-root`.

Also use `poly` to illustrate the root-finding process. Start with the interval [3,5] and tabulate the information as follows:

step	*left*	*f*(*left*)	*right*	*f*(*right*)	*mid*	*f*(*mid*)
n=1	3	-1	6.00	8.00	4.25	1.25
n=2	3	-1	4.25	1.25	?	?

Assume `TOLERANCE` is `0.5`.

Our next task is to address the four fundamental questions of algorithm design:

1. We need a condition that describes when the problem is solved and a matching answer. Given our discussion so far, this is straightforward:

   ```
   (<= (- right left) TOLERANCE)
   ```

2. The matching result in the trivial case is `left`.

3. For the generative case, we need an expression that generates new problems for `find-root`. According to our informal description, this step first requires determining the midpoint and its function value:

   ```
   (local ((define mid (/ (+ left right) 2))
           (define f@mid (f mid)))
     ...)
   ```

   The midpoint is then used to pick the next interval. Following Intermediate Value Theorem, the interval [*left,mid*] is the next candidate if

   ```
   (or (<= (f left) 0 f@mid) (<= f@mid 0 (f left)))
   ```

   while [*mid,right*] is used for the recursive call if

   ```
   (or (<= f@mid 0 (f right)) (<= (f right) 0 f@mid))
   ```

   Translated into code, the body of the `local` expression must be a conditional:

   ```
   (local ((define mid (/ (+ left right) 2))
           (define f@mid f@mid))
     (cond
       [(or (<= (f left) 0 f@mid) (<= f@mid 0 (f left)))
        (... (find-root f left mid) ...)]
       [(or (<= f@mid 0 (f right)) (<= (f right) 0 f@mid))
        (... (find-root f mid right) ...)]))
   ```

   In both clauses, we use `find-root` to continue the search.

4. The answer to the final question is obvious. Since the recursive call to `find-root` finds the root of `f`, there is no need to process its solution any further.

The completed function is displayed in figure 87. The following exercises suggest some tests and a termination argument.

```
; [Number -> Number] Number Number -> Number
; determines R such that f has a root in [R,(+ R TOLERANCE)]
; assume f is continuous
; assume (or (<= (f left) 0 (f right)) (<= (f right) 0 (f left)))
; generative divide interval in half, the root is in one of the two
; halves, pick according to assumption
(define (find-root f left right)
  (cond
    [(<= (- right left) TOLERANCE) left]
    [else
      (local ((define mid (/ (+ left right) 2))
              (define f@mid f@mid))
        (cond
          [(or (<= (f left) 0 f@mid) (<= f@mid 0 (f left)))
           (find-root f left mid)]
          [(or (<= f@mid 0 (f right)) (<= (f right) 0 f@mid))
           (find-root f mid right)]))]))
```

Figure 87: The *find-root* algorithm

Exercise 364. Use exercise 363 to test `find-root`. Experiment with different `TOLERANCE`s. ∎

Exercise 365. Develop a polynomial `f2` that has two roots. Then use `find-root` with `f2` and an interval that contains both roots. ∎

Exercise 366. The `find-root` algorithm terminates for all (continuous) `f`, `left`, and `right` for which the assumption holds. Why? Formulate a termination argument.

Hint Suppose the arguments of `find-root` describe an interval of size `S1`. How large is the distance between `left` and `right` for the first, second, third recursive call to `find-root`? After how many steps is `(- right left)` smaller than or equal to `TOLERANCE`? ∎

Exercise 367. As presented in figure 87, `find-root` computes the value of `f` for each boundary value twice to generate the next interval. Use `local` to avoid this re-computation.

In addition, `find-root` recomputes the value of a boundary across recursive calls. For example, `(find-root f left right)` computes `(f left)` and, if [*left,mid*] is chosen as the next interval, `find-root` computes `(f left)` again. Introduce a helper function that is like `find-root` but consumes not only `left` and `right` but also `(f left)` and `(f right)` at each recursive stage.

How many re-computations of `(f left)` does this design maximally avoid?

Note The two additional arguments to this helper function change at each recursive stage but the change is related to the change in the numeric arguments. These arguments are so-called *accumulators*, which are the topic of Accumulators. ∎

Exercise 368. A function `f` is *monotonically increasing* if `(<= (f a) (f b))` whenever `(< a b)`. Simplify `find-root` assuming the given function is not only continuous but also monotonically increasing. ∎

Exercise 369. A table is a structure of two fields: the natural number `VL` and a function

`array`, which consumes natural numbers and, for those between `0` and `VL` (exclusive), produces answers:

```
(define-struct table [length array])
; A Table is a
;   (make-table N [N -> Number])
```

Since this data structure is somewhat unusual, it is critical to illustrate it with examples:

```
(define table1 (make-table 3 (lambda (i) i)))

; N -> Number
(define (a2 i)
  (if (= i 0) pi (error "table2 is not defined for i =!= 0")))

(define table2 (make-table 1 a2))
```

Here `table1`'s array function is defined for more inputs that its length field allows; `table2` is defined for just one input, namely `0`. Finally, we also define a useful function for looking up values in tables:

```
; Table N -> Number
; looks up the ith value in array of t
(define (table-ref t i)
  ((table-array t) i))
```

The root of a table `t` is the number in `(table-array t)` that is closest to `0`. A *root index* is a natural number `i` such that `(table-ref t i)` is a root of table `t`.

A table `t` is monotonically increasing if `(table-ref t 0)` is less then `(table-ref t 1)`, `(table-ref t 1)` is less than `(table-ref t 2)`, and so on.

Design `find-linear`. The function consumes a monotonically increasing table and finds the smallest index for a root of the table. Use the structural recipe for N, proceeding from `0` through `1`, `2`, and so on to the `array-length` of the given table. This kind of root-finding process is often called a *linear search*.

Design `find-binary`, which also finds the smallest index for root of a monotonically increasing table but uses generative recursion to do so. Like ordinary binary search, the algorithm narrows an interval down to the smallest possible size and then chooses the index. Don't forget to formulate a termination argument.

Hint The key problem is that a table index is a **natural** number, not a plain number. Hence the interval boundary arguments for `find` must be natural numbers. Consider how this observation (1) the nature of trivially solvable problem instances, (2) the midpoint computation, (3) and the decision which interval to generate next?

Consider `(make-table 1024 a)` and assume `(= (a 1023) 0)`. How many recursive calls to `find` are needed in `find-linear` and `find-binary` respectively? ∎

ASL provides `vectors`, which are similar to tables; `arrays` is another name for the same idea.

31.3 A Glimpse at Parsing

As mentioned in Incremental Refinement, computers come with files, which provide a form of permanent memory. From our perspective a *file* is just a list of 1Strings, though interrupted by a special string:

```
; A File is one of:
; - '()
; - (cons "\n" File)
; - (cons 1String File)
; interpretation "\n" represents the newline character
```

The idea is that Files are broken into lines, where `"\n"` represents the so-called newline character, which indicates the end of a line. Let's also introduce lines before we move on:

```
; A Line is [List-of 1String]
```

Many programs need to process files as list of lines. For example, the file

```
(list "h" "o" "w" " " "a" "r" "e" " " "y" "o" "u" "\n"
      "d" "o" "i" "n" "g" "?" "\n"
      "a" "n" "y" " " "p" "r" "o" "g" "r" "e" "s" "s" "?")
```

might have to be processed as a list of three lines:

```
(list (list "h" "o" "w" " " "a" "r" "e" " " "y" "o" "u")
      (list "d" "o" "i" "n" "g" "?")
      (list "a" "n" "y" " " "p" "r" "o" "g" "r" "e" "s" "s" "?"))
```

Similarly, the file

```
(list "a" "b" "c" "\n"
      "d" "e" "\n"
      "f" "g" "h" "\n")
```

also corresponds to a list of three lines:

```
(list (list "a" "b" "c")
      (list "d" "e")
      (list "f" "g" "h"))
```

Stop! What are the list-of-lines representation for these three cases: `'()`, `(list "\n")`, and `(list "\n" "\n")`?. Why are these examples important test cases?

The problem of turning a sequence of 1Strings into a list of lines is called the *parsing* problem. Many programming languages provide functions that retrieve lines, words, numbers and other kinds of so-called tokens from files. But even if they do, it is common that programs need to parse these tokens even further. This section provides a glimpse at a parsing technique. Parsing is so complex and so central to the creation of full-fledged software applications, however, that most undergraduate curricula come with at least one course on parsing. So do not think you can tackle real parsing problems properly even after mastering this section.

We start by stating the obvious—a signature, a purpose statement, one of the above examples, and a header—for a function that turns a File into a list of Lines:

```
; File -> [List-of Line]
; converts a file into a list of lines

(check-expect (file->list-of-lines
                (list "a" "b" "c" "\n" "d" "e" "\n" "f" "g" "h" "\n"))
              (list (list "a" "b" "c")
                    (list "d" "e")
                    (list "f" "g" "h")))
```

The exact convention differs from one operating system to another, but for our purposes this is irrelevant.

```
(define (file->list-of-lines afile)
  '())
```

It is also easy to describe the parsing process, given our experience with Recursion without Structure:

1. The problem is trivially solvable if the file is `'()`.
2. In that case, the file doesn't contain a line.
3. Otherwise, the file contains at least `"\n"` or one 1String. These items—up to and including the first `"\n"`, if any—must be separated from the rest of the File. The remainder is a new problem of the same kind that `file->list-of-lines` can solve.
4. It then suffices to `cons` the initial segment as a single line to the list of Lines that result from the rest of the File.

The four questions suggest a straightforward instantiation of the template for generative recursive functions. Because the separation of the initial segment from the rest of the file requires a scan of an arbitrarily long list of 1Strings, we put two auxiliary functions on our wish list: `first-line`, which collects all 1Strings up to, but excluding, the first occurrence of `"\n"` or the end of the list; and `remove-first-line`, which removes the very same items that `first-line` collects.

```
; File -> [List-of Line]
; converts a file into a list of lines

(check-expect (file->list-of-lines '("\n" "\n"))
              '(() ()))

(check-expect (file->list-of-lines
                (list "a" "b" "c" "\n" "d" "e" "\n" "f" "g" "h" "\n"))
              (list (list "a" "b" "c")
                    (list "d" "e")
                    (list "f" "g" "h")))

(define (file->list-of-lines afile)
  (cond
    [(empty? afile) '()]
    [else
      (cons (first-line afile)
            (file->list-of-lines (remove-first-line afile)))]))

; File -> Line
; retrieves the prefix of afile up to the first occurrence of NEWLINE
(define (first-line afile)
  (cond
    [(empty? afile) '()]
    [(string=? (first afile) NEWLINE) '()]
    [else (cons (first afile) (first-line (rest afile)))]))

; File -> Line
; drops the suffix of afile behind the first occurrence of NEWLINE
(define (remove-first-line afile)
  (cond
    [(empty? afile) '()]
```

```
      [(string=? (first afile) NEWLINE) (rest afile)]
      [else (remove-first-line (rest afile))]))


(define NEWLINE "\n")
```

Figure 88: Translating a file into a list of lines

From here, it is easy to create the rest of the program. In `file->list-of-lines`, the answer in the first clause must be `'()` because an empty file does not contain any lines. The answer in the second clause must `cons` the value of `(first-line afile)` onto the value `(file->list-of-lines (remove-first-line afile))`, because the first expression computes the first line and the second one computes the rest of the lines. Finally, the auxiliary functions process their inputs in a structurally recursive manner; their development is a straightforward exercise. Figure 88 collects the three function definitions and the definition for `NEWLINE`.

Here is how `file->list-of-lines` processes the second test:

```
(file->list-of-lines (list "a" "b" "c" "\n" "d" "e" "\n" "f" "g" "h" "\n"))
=
(cons (list "a" "b" "c")
      (file->list-of-lines (list "d" "e" "\n" "f" "g" "h" "\n")))
=
(cons (list "a" "b" "c")
      (cons (list "d" "e")
            (file->list-of-lines (list "f" "g" "h" "\n"))))
=
(cons (list "a" "b" "c")
      (cons (list "d" "e")
            (cons (list "f" "g" "h")
                  (file->list-of-lines '()))))
=
(cons (list "a" "b" "c")
      (cons (list "d" "e")
            (cons (list "f" "g" "h")
                  '())))
=

(list (list "a" "b" "c")
      (list "d" "e")
      (list "f" "g" "h"))
```

This evaluation is another reminder that the argument of the recursive application of `file->list-of-lines` is almost never the rest of the given file. It also shows why this generative recursion is guaranteed to terminate for every given File. Every recursive application consumes a list that is shorter than the given one, meaning the recursive process stops when the process reaches `'()`.

Exercise 370. Design the function `tokenize`. It turns a Line into a list of tokens. Here a token is either a 1String or a String that consists of lower-case letters and nothing else. That is, all white-space 1Strings are dropped; all other non-letters remain as is; and all consecutive letters are bundled into "words." **Hint** Read up on the `string-whitespace?` function. ∎

Exercise 371. Design `create-matrix`. The function consumes a number n and a list of n^2 numbers. It produces a list of n lists of n numbers, for example:

```
(check-expect
  (create-matrix 2 (list 1 2 3 4))
  (list (list 1 2)
        (list 3 4)))
```

Make up a second example. ∎

32 Mathematical Examples

Many solutions to mathematical problems employ generative recursion. A future programmer must get to know such solutions for two reasons. On one hand, a fair number of programming tasks are essentially about turning these kinds of mathematical ideas into programs. On the other hand, practicing with such mathematical problems often proves inspirational for the design of algorithms. This chapter deals with three such problems.

32.1 Newton's Method

Binary Search introduces one method for finding the root of a mathematical function. As the exercises in the same section sketch, the method naturally generalizes to computational problems, such as finding certain values in tables, vectors, and arrays. In mathematical applications, programmers tend to employ methods that originate from analytical mathematics. A prominent one is due to Newton. Like binary search, the so-called *Newton method* repeatedly improves an approximation to the root until it is "close enough." Starting from a guess, say, $r1$, the essence of the process is to construct the tangent of f at $r1$ and to determine its root. While the tangent approximates the function, it is also straightforward to determine its root. By repeating this process sufficiently often, an algorithm can find a root r for which $(f\ r)$ is close enough to 0.

Newton proved this fact.

Clearly this process relies on two pieces of domain knowledge concerning tangents: their slopes and their roots. Informally, a tangent of f at some point $r1$ is the line that goes through the point $(r1, f(r1))$ and has the same slope as f. One mathematical way to obtain the tangent's slope is to pick two close points on the x axis that are equidistant from $r1$ and to use the slope of the line determined by f at those two points. The convention is to choose a small number ε and to work with $r1 + \varepsilon$ and $r1 - \varepsilon$. That is, the points are $(r1 - \varepsilon, f(r1 - \varepsilon))$ and $(r1 + \varepsilon, f(r1 + \varepsilon))$, which determines a line and a slope:

$$slope = \frac{f(r1+\epsilon) - f(r1-\epsilon)}{(r1+\epsilon) - (r1-\epsilon)} = \frac{f(r1+\epsilon) - f(r1-\epsilon)}{2 \cdot \epsilon}$$

Stop! Solve the next exercise.

Exercise 372. Translate this mathematical formula into the ISL+ function `slope`, which maps function `f` and a number `x` to the slope of `f` at `x`. Assume that `EPSILON` is a global constant. For your examples, use functions whose exact slope you can figure out, say, horizontal lines, linear functions, and perhaps polynomials if you know some calculus. ∎

The second piece of domain knowledge concerns the root of a tangent, which is just a line or a linear function. The tangent goes through $(r1, f(r1))$ and has the above *slope*. Mathematically, it is defined as

$$tangent(x) = slope \cdot x + f(r1)$$

Finding the root of this tangent also means finding a value *root-of-tangent* so that the

function value equals 0:

$$0 = slope \cdot root\text{-}of\text{-}tangent + f(r1)$$

In contrast to arbitrary functions, though, we can solve this equation in a straightforward manner:

$$root\text{-}of\text{-}tangent = r1 - \frac{f(r1)}{slope}$$

Stop! Solve the next exercise.

Exercise 373. Translate this mathematical formula into the function `root-of-tangent`, which maps function `f` and a number `x` to the root of the tangent through (`x`,`(f x)`). You need to solve exercise 372 first and reuse the solution here. ∎

Now we can use the design recipe to translate the description of Newton's process into an ISL+ program. The function—let's call it `newton` in honor of its inventor—consumes a function `f` and a number `r1`, the current guess:

```
; [Number -> Number] Number -> Number
; finds a number r such that (f r) is small
; generative repeatedly generate improved guesses using f and r

(define (newton f r1)
  1.0)
```

For the template of `newton`, we turn to the central four question of the design recipe for generative recursion:

1. If `(f r1)` is close enough to `0`, the problem is solved. Close to `0` could be mean `(f r1)` is a small positive number or a small negative number. Hence we translate this idea into

   ```
   (<= (abs (f r1)) EPSILON)
   ```

   That is, we determine whether the absolute value is small.

2. The solution is `r1`.

3. The generative step of the algorithm consists of finding the root of the tangent of `f` at `r1`, which generates the next guess. By applying `newton` to `f` and this new guess, we resume the process.

4. The answer of the recursion is also the answer of the original problem.

With these in place, putting together the function is a matter of formulating formulas as ISL+ code.

```
; [Number -> Number] Number -> Number
; finds a number r such that (<= (abs (f r)) EPSILON)

(check-within (newton poly 1) 2 EPSILON)

(check-within (newton poly 3.5) 4 EPSILON)

(define (newton f r1)
  (cond
    [(<= (abs (f r1)) EPSILON) r1]
```

```
    [else (newton f (root-of-tangent f r1))]))

; see exercise 372
(define (slope f r)
  1.0)

; see exercise 373
(define (root-of-tangent f r)
  1.0)
```

Figure 89: The Newton process

Figure 89 displays the definition of `newton`. It includes two tests that are derived from the tests in Binary Search for the `find-root` function. After all, both functions search for the root of a function and `poly` has two known roots.

We are not finished with the design of `newton`. The new, seventh step of the design recipe calls for an investigation into the termination behavior of the function. For `newton`, the problem shows up with `poly` already. Recall its definition:

```
; Number -> Number
(define (poly x)
  (* (- x 2) (- x 4)))
```

As mentioned and as figure 90 shows, its roots are `2` and `4`. The figure also shows that between the two roots the function flattens out. For a mathematically inclined person, this shape almost immediately suggests the question what `newton` computes for an initial guess of `3`:

```
> (poly 3)
-1
> (newton poly 3 #f)
/: division by zero
```

The explanation is that `slope` produces a “bad” value and `root-of-tangent` reacts to it in its own bad way:

```
> (slope poly 3)
0
> (root-of-tangent poly 3)
/: division by zero
```

The `newton` function just propagates this error.

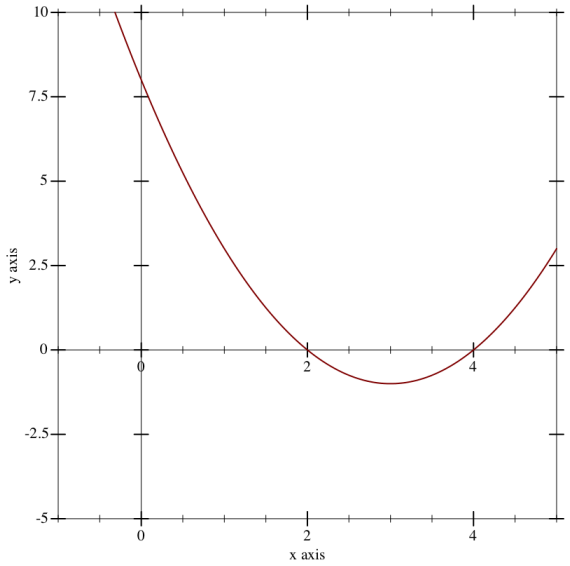


Figure 90: A plot of `poly` on the interval [`-1`,`5`]

In addition to this run-time error, `newton` exhibits two other problems with respect to termination. Fortunately, we can demonstrate both with `poly`. The first one concerns the nature of numbers, which we briefly touched on in The Arithmetic of Numbers. It is safe to ignore the distinction between exact and inexact numbers for many beginner exercises in programming but when it comes to translating mathematics into programs, you need to proceed with extreme caution. Consider the following:

```
> (newton poly 2.9999)
```

An ISL+ program treats `2.9999` as an exact number and the computations in `newton` process it as such, though because the numbers aren't integers, the computation uses exact rational fractions. Since the arithmetic for fractions can get much slower than the arithmetic for inexact numbers, the above function call takes a significant amount of time in DrRacket. Depending on your computer, it may take between a few second and a minute or more. If you happen to choose other numbers that trigger this form of computation, it may seem as if the call to `newton` does not terminate at all.

The second problem concerns true non-termination. Here is the example:

```
> (newton poly 3.0)
```

It uses the inexact number `3.0` as the initial guess, which unlike `3` causes a different kind of problem. Specifically, the `slope` function now produces an inexact `0` for poly while `root-of-tangent` jumps to infinity:

```
> (slope poly 3.0)
#i0.0
> (root-of-tangent poly 3.0)
#i+inf.0
```

As a result, the evaluation immediately falls into an infinite loop.

In short, `newton` exhibits the full range of complex termination behavior. For some inputs, the function produces a correct result. For some others, it signals errors. And for yet others, it goes into infinite loop or appears to go into one. The header for `newton`—or

The calculation in `newton`

some other piece of writing—must warn others who wish to use the function and future readers of these complexities, and good math libraries in common programming languages do so.

Exercise 374. Design the function `double-amount`, which computes how many months it takes to double a given amount of money when a savings account pays interest at a fixed rate on a monthly basis.

Hint The function must know the current amount and the initially given one; see ... Add Expressive Power.

Domain Knowledge With a minor algebraic manipulation, you can show that the given amount is irrelevant. Only the interest rate matters. Also domain experts know that doubling occurs after roughly $72/r$ as long as the interest rate r is within a certain range.

turns `+inf.0` into `+nan.0`, a piece of

This exercise is due to Adrian German.

are

r."

then continues to use this guess.

32.2 Numeric Integration

Many problems in physics boil down to determining the area under a curve:

> **Sample Problem:** A car drives at constant speed of v meters per second. How far does it travel in `5`, `10`, `15` seconds?
>
> A rocket lifts off at the constant rate of acceleration of $12m/s^2$. What height does it reach after `5`, `10`, `15` seconds?

Physics tells us that a vehicle travels $d = v \cdot t$ meters if it moves at a constant speed v. For vehicles that accelerate, the distance traveled depends on the square of the time t passed:

$$\frac{1}{2} \cdot a \cdot t^2$$

In general, the law tells us that the distance corresponds to the area under the graph of speed $v(t)$ over time t.

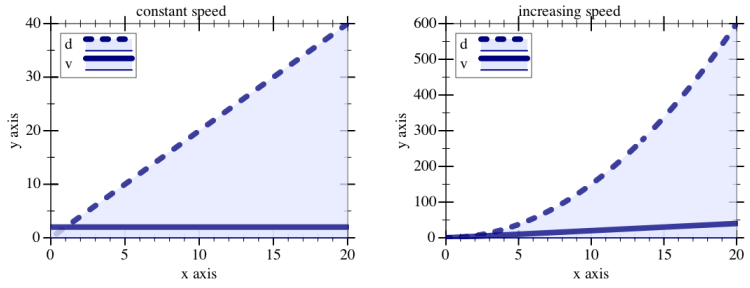


Figure 91: Distance traveled with constant vs accelerating speed

Figure 91 illustrates the idea in a graphical manner. On the left, we see an overlay of two graphs: the solid flat line is the speed of the vehicle and the rising dashed line is the distance traveled. A quick check shows that the latter is indeed the area determine by the former and the x axis at **every point in time**. Similarly, the graphs on the right show the relationship between a rocket moving at constantly increasing speed and the height it reaches. Determining this area under the graph of a function for some specific interval is called (function) *integration*.

While mathematicians know formulas for the two sample problems that give precise answers, the general problem calls for computational solutions. The problem is that curves often come with complex shapes, more like those in figure 92, which suggests that someone needs to know the area between the *x* axis, the vertical lines labeled *a* and *b*, and the graph of *f*. Applied mathematicians determine such areas in an approximate manner, summing the areas of many small geometric shapes. It is therefore natural to develop algorithms that deal with these calculations.

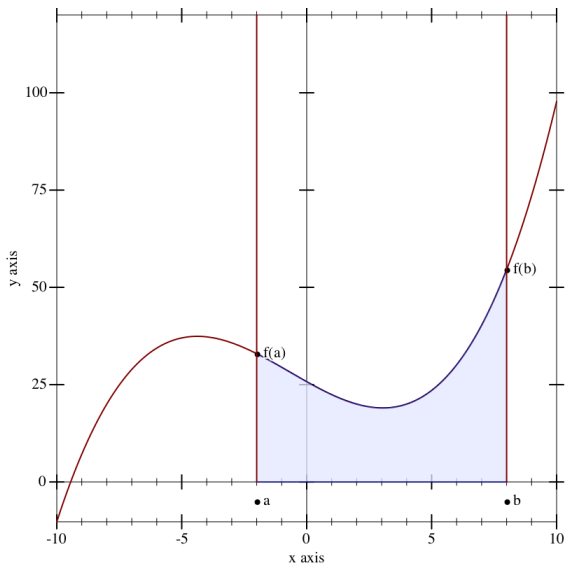


Figure 92: Integrating a function *f* between *a* and *b*

An integration algorithm consumes three inputs: the function `f` and two borders, `a` and `b`. The fourth part, the *x* axis, is implied. This suggests the following signature:

```
; [Number -> Number] Number Number -> Number
```

In order to understand the idea behind integration, it is best to study simple examples such as a constant function or a linear one. Thus, consider

```
(define (constant x) 20)
```

Passing `constant` to `integrate`, together with `12` and `22`, describes a rectangle of width `10` and height `20`. The area of this rectangle is `200`, meaning we get this test:

```
(check-expect (integrate constant 12 22) 200)
```

Similarly, let's use linear to create a second test:

```
(define (linear x) (* 2 x))
```

If we use `linear`, `0`, and `10` with `integrate`, the area is a triangle with a base width of `10` and a height of `20`. Here is the example reformulated as a test:

```
(check-expect (integrate linear 0 10) 100)
```

After all, a triangle's area is half of the product of its base width and height.

For a third example, we exploit some domain-specific knowledge. As mentioned, mathematicians know how to determine the area under some functions in a precise

manner. For example, the area under the function

$$square(x) = 3 \cdot x^2$$

on the interval $[a,b]$ can be calculated with the following formula

$$b^3 - a^3 .$$

Here is how to turn this idea into a concrete test:

```
(define (square x) (* 3 (sqr x)))

(check-expect (integrate square 0 10) (- (expt 10 3) (expt 0 3)))
```

```
(define EPSILON 0.1)

; [Number -> Number] Number Number -> Number
; computes the area under the graph of f between a and b
; assume (< a b) holds

(check-within (integrate (lambda (x) 20) 12 22) 200 EPSILON)
(check-within (integrate (lambda (x) (* 2 x)) 0 10) 100 EPSILON)
(check-within (integrate (lambda (x) (* 3 (sqr x))) 0 10) 1000 EPSILON)

(define (integrate f a b)
  0.0)
```

Figure 93: A generic integration function

Figure 93 collects the result of the first three steps of the design recipe. The figure adds a purpose statement and an obvious assumption concerning the two interval boundaries. Instead of `check-expect` it uses `check-within`, which anticipates the numerical inaccuracies that comes with computational approximations in such calculations. Analogously, the header of `integrate` specifies `0.0` as the return result, signaling that the function is expected to return an inexact number.

The following two exercises show how to turn domain knowledge into integration functions. Both functions compute rather crude approximations. While the design of the first uses only mathematical formulas, the second also exploits a bit of structural design ideas. Solving these exercises creates the necessary appreciation for the core of this section, which presents a generative-recursive integration algorithm.

Exercise 375. Kepler suggested a simple integration method. To compute a rough estimate of the area under f between a and b, proceed as follows:

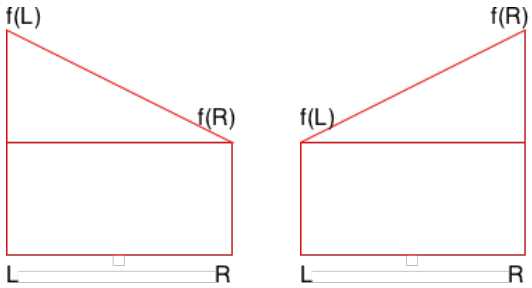
1. divide the interval into half at $mid = (a + b) / 2$;
2. compute the areas of the two *trapezoids*, each determined by four points:
   - $[(a,0),(a,f(a)),(mid,0),(mid,f(mid))]$
   - $[(mid,0),(mid,f(mid)),(b,0),(b,f(b))]$;
3. then add the two areas.

The method is known as *Kepler's rule*.

Design the function `integrate-kepler`. That is, turn the mathematical knowledge into a

ISL+ function. Make sure to adapt the test cases from figure 93 to this use. Which of the three tests fails and by how much?

Domain knowledge Let us take a close look at the kind of trapezoids whose area you need to compute. Here are the two basic shapes, without a coordinate system to reduce clutter:



The left assumes $f(L) > f(R)$ while the right one shows the case where $f(L) < f(R)$. Surprisingly, it is possible to calculate the area of these trapezoids with a single formula:

$$[(R - L) \cdot f(R)] + [\frac{1}{2} \cdot (R - L) \cdot (f(L) - f(R))]$$

Stop! Convince yourself that this formula **adds** the area of the triangle to the area of the lower rectangle for the left trapezoid above, while it **subtracts** the area of the triangle from the area of the large rectangle for the right one.

Also show that the above formula is equal to

$$\frac{(R - L) \cdot (f(L) + f(R))}{2}$$

This is a mathematical way to convince yourself of the asymmetry of the formula. ∎

Exercise 376. Another simple integration method divides the area into many small rectangles. Each rectangle has a fixed width and is as tall as the function graph in the middle of the rectangle. Adding up the areas of the rectangles produces an estimate of the area under the function's graph.

Let's use

$$R = 10$$

to stand for the number of rectangles to be considered. Hence the width of each rectangle is

$$W = \frac{(b - a)}{R} .$$

For the height of one of these rectangles, we determine the value of f at its midpoint. The first midpoint is clearly at a plus half of the width of the rectangle,

$$S = \frac{width}{2} ,$$

which means its area is

$$W \cdot f(a + S) .$$

To compute the area of the second rectangle, we must add the width of one rectangle to the first midpoint:

$$W \cdot f(a + W + S) \ ,$$

For the third one, we get

$$W \cdot f(a + 2 \cdot W + S) \ .$$

In general, we can use the following formula for the *i*th rectangle:

$$W \cdot f(a + i \cdot W + S) \ .$$

The first rectangle has index 0, the last one *R* - 1.

Using this sequence of rectangles, we can now determine the area under the graph as a series:

$$\sum_{i=0}^{i=R-1} W \cdot f(a+i \cdot W+S) = W \cdot f(a+0 \cdot W+S) + \ldots + \ldots W \cdot f(a+(R-1) \cdot W+S) \ .$$

Design the function `integrate-rectangles`. That is, turn the description of the rectangle process an ISL+ function. Make sure to adapt the test cases from figure 93 to this use.

The more rectangles the algorithms uses, the closer its estimate is to the actual area. Make `R` a top-level constant and increase it by factors of `10` until the algorithm's accuracy eliminates problems with `EPSILON` value of `0.1`.

Decrease `EPSILON` to `0.01` and increase `R` enough to eliminate any failing test cases again. Compare the result to exercise 375. ∎

The Kepler method of exercise 375 immediately suggests a divide-and-conquer strategy like binary search introduced in Binary Search. Roughly speaking, the algorithm would split the interval into two pieces, recursively compute the area of each piece, and add the two results.

Exercise 377. Develop the algorithm `integrate-dc`, which integrates a function `f` between the boundaries `a` and `b` using a divide-and-conquer strategy. Use Kepler's method when the interval is sufficiently small. ∎

The plain divide-and-conquer approach of exercise 377 is wasteful. Consider a function whose graph is level in one part and rapidly changes in another; see figure 94 for a concrete example. For the level part on the graph, it is pointless to keep splitting the interval. It is just as easy to compute the trapezoid for the complete interval as for the two halves. For the wavy part, however, the algorithm must continue dividing the interval until the irregularities of the graph are reasonably small.

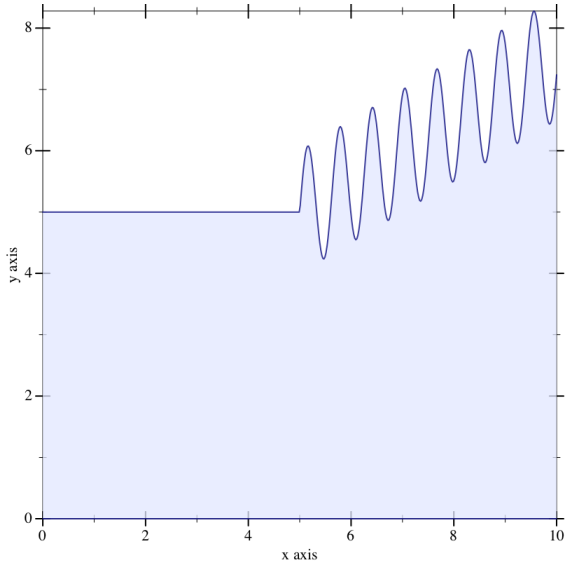


Figure 94: A candidate for adaptive integration

To discover when *f* is level, we can change the algorithm as follows. Instead of just testing how large the interval is, the new algorithm computes the area of three trapezoids: the given one, and the two halves. If the difference between the two is less than the area of a small rectangle of height ε and width *b* - *a*,

$$\cdot (b - a)$$

it is safe to assume that the overall area is a good approximation. In other words, the algorithm determines whether *f* changes so much that it affects the error margin. If so, it continues with the divide-and-conquer approach; otherwise it stops and uses the Kepler approximation.

Exercise 378. Design `integrate-adaptive`. That is, turn the recursive process description into an ISL+ algorithm. Make sure to adapt the test cases from figure 93 to this use.

Do not discuss the termination of `integrate-adaptive`

Does `integrate-adaptive` necessarily compute a better answer than either `integrate-kepler` or `integrate-rectangles`? Which aspect is `integrate-adaptive` guaranteed to improve? ∎

Terminology The algorithm is called *adaptive integration* because it automatically allocates time to those parts of the graph that needs it and spends little time on the others. Specifically, for those parts of *f* that are level, it performs just a few calculations; for the other parts, it inspects small intervals to decrease the error margin. Computer science knows many adaptive algorithms, and `integrate-adaptive` is just one of them.

32.3 Extended Exercise: Gaussian Elimination

Mathematicians not only search for solutions of equations in one variable; they also study whole systems of linear equations:

Sample Problem: In a bartering world, the values of coal (*x*), oil (*y*), and

natural gas (z) are determined by the following exchange equations:

$$\begin{array}{rcrcrcl} 2\cdot x & + & 2\cdot y & + & 3\cdot z & = & 10 \\ 2\cdot x & + & 5\cdot y & + & 12\cdot z & = & 31 \\ 4\cdot x & + & 1\cdot y & - & 2\cdot z & = & 1 \end{array} \qquad (\dagger)$$

A solution to such a system of equations consists of a collection of numbers, one per variable, such that if we replace the variable with its corresponding number, the two sides of each equation evaluate to the same number. In our running example, the solution is

$$x = 1, y = 1, \text{ and } z = 2.$$

We can easily check this claim:

$$\begin{array}{rcrcrcl} 2\cdot 1 & + & 2\cdot 1 & + & 3\cdot 2 & = & 10 \\ 2\cdot 1 & + & 5\cdot 1 & + & 12\cdot 2 & = & 31 \\ 4\cdot 1 & + & 1\cdot 1 & - & 2\cdot 2 & = & 1 \end{array}$$

The three equations reduce to

$$10 = 10, 31 = 31, \text{ and } 1 = 1.$$

```
; An SOE is a non-empty Matrix
; constraint if its length is n (in N), each item has length (+ n 1)
; interpretation an SOE represents a system of linear equations

; An Equation is [List-of Number]
; constraint an Equation contains at least two numbers.
; interpretation if (list a_1 ... a_n b) is an Equation, a_1, ..., a_n are
; the left-hand side variable coefficients and b is the right-hand side

; A Solution is [List-of Number]

; examples:
(define M
  (list (list 2 2  3 10)
        (list 2 5 12 31)
        (list 4 1 -2  1)))

(define S '(1 1 2))
```

Figure 95: A data representation for systems of equations

Figure 95 introduces a data representation for our problem domain. It also illustrates how to represent the sample system of equations and its solution with this data representation. This representation captures the essence of a system of equations, namely, the numeric coefficients of the variables on the left-hand side and the right-hand side values. The names of the variables don't play any role because they are like parameters of functions; meaning, as long as they are consistently renamed the equations have the same solutions.

For the rest of this section. and especially the exercises, it is convenient to use the following functions:

```
; Equation -> [List-of Number]
; extracts the left-hand side from a row in a matrix
(check-expect (lhs (first M)) '(2 2 3))
(define (lhs e)
```

```
  (reverse (rest (reverse e))))

; Equation -> Number
; extracts the left-hand side from a row in a matrix
(check-expect (rhs (first M)) 10)
(define (rhs e)
  (first (reverse e)))
```

Exercise 379. Design the function `check-solution`. It consumes an SOE and a Solution. Its result is `#true` if plugging in the numbers from the Solution for the variables in the Equations of the SOE produces equal left-hand side values and right-hand side values; otherwise the function produces `#false`.

Hint Design the function `plug-in` first. It consumes the left-hand side of an Equation and a Solution and calculates out the value of the left-hand side when the numbers from the solution are plugged in for the variables. ∎

Gaussian elimination is a standard method for finding solutions to systems of linear equations. It consists of two steps. The first step is to transform the system of equations into a system of different shape but with the same solution. The second step is to find solutions to one equation at a time. Here we focus on the first step because it is another interesting instance of generative recursion.

The first step of the Gaussian elimination algorithm is called "triangulation" because the result is a system of equations in the shape of a triangle. In contrast, the original system is a rectangle. To understand this terminology, take a look at this list, which represents the original system:

```
(list (list 2 2  3 10)
      (list 2 5 12 31)
      (list 4 1 -2 1))
```

Triangulation transforms this matrix into the following:

```
(list (list 2 2  3 10)
      (list   3  9 21)
      (list      1  2))
```

As promised, the shape of this system of equations is (roughly) a triangle. Stop! Solve the next exercise.

Exercise 380. Check that the following system of equations

$$\begin{array}{rcrcrcrl} 2\cdot x & + & 2\cdot y & + & 3\cdot z & = & 10 & \\ & & 3\cdot y & + & 9\cdot z & = & 21 & (*) \\ & & & & 1\cdot z & = & 2 & \end{array}$$

has the same solution as the one labeled with (†). Do so by hand and with `check-solution` from exercise 379. ∎

The key idea of triangulation is to subtract the first Equation from the remaining ones. To subtract one Equation from another means to subtract the corresponding coefficients in the two Equations. With our running example, subtracting the first from the second equation yields the following matrix:

```
(list (list 2 2  3 10)
      (list 0 3  9 21)
      (list 4 1 -2  1))
```

The goal of these subtractions is to put a `0` into the first column of all but the first equation. For the third equation, getting a `0` into the first position means subtracting the first equation **twice** from the third one:

```
(list (list 2  2  3  10)
      (list 0  3  9  21)
      (list 0 -3 -8 -19))
```

Following convention, we drop the leading `0`'s from the last two equations:

```
(list (list 2  2  3   10)
      (list    3  9   21)
      (list   -3 -8  -19))
```

Generally speaking, we first multiply each item in the first row with `2` and then subtract the result from the last row. As mentioned, these subtractions do not change the solution; that is, the solution of the original system is also the solution of the transformed one.

Mathematics teaches how to prove such facts. We just use them.

Exercise 381. Check that the following system of equations

$$\begin{array}{rcrcrcr} 2\cdot x & + & 2\cdot y & + & 3\cdot z & = & 10 \\ & & 3\cdot y & + & 9\cdot z & = & 21 \\ - & & 3\cdot y & - & 8\cdot z & = & -19 \end{array} \qquad (\ddagger)$$

has the same solution as the one labeled with (†). Again do so by hand and with `check-solution` from exercise 379. ∎

Exercise 382. Design `subtract`. The function consumes two Equations of equal length. It subtracts the second from the first, item by item, as many times as necessary to obtain an Equation with a `0` in the first position. Since the leading coefficient is known to be `0`, `subtract` returns the rest of the list that results from the subtractions. ∎

Now consider the rest of the SOE:

```
(list (list  3  9   21)
      (list -3 -8  -19))
```

It is also an SOE and we can thus apply the same algorithm again. For our running example, this next subtraction step calls for subtracting the first Equation `-1` times from the second one—that is equivalent to adding the two equations. Doing so yields

```
(list (list 3  9 21)
      (list    1  2))
```

And the remainder of this list is a single equation and obviously cannot be simplified any further.

Exercise 383. Here is the data definition for triangular systems of equations:

```
; A TM is [List-of Equation]
; such that the Equations are of decreasing length:
;   n + 1, n, n - 1, ..., 2.
; interpretation represents a triangular matrix
```

Design the `triangulate` algorithm:

```
; SOE -> TM
; triangulates the given system of equations
(define (triangulate M)
```

```
   '(1 2))
```

Turn the above example into a test and spell out explicit answers for the four questions based on our loose description.

Do not yet deal with the termination step of the design recipe. ∎

Unfortunately, the solution to exercise 383 occasionally fails to produce the desired triangular system. Consider the following representation of a system of equations:

```
(list (list 2  3  3 8)
      (list 2  3 -2 3)
      (list 4 -2  2 4))
```

Its solution is $x = 1$, $y = 1$, and $z = 1$.

The first step is to subtract the first row from the second and to subtract it twice from the last one, which yields the following matrix:

```
(list (list  2  3  3   8)
      (list     0 -5  -5)
      (list    -8 -4 -12))
```

Next triangulation would focus on the rest of the matrix:

```
(list (list   0 -5  -5)
      (list  -8 -4 -12))
```

but the first item of this matrix is `0`. Since it is impossible to divide by `0`, the algorithm signals an error via `subtract`.

To overcome this problem, we need to use another piece of knowledge from our problem domain. Mathematics tells us that switching equations in a system of equations does not affect the solution. Of course, as we switch equations, we must eventually find an equation whose leading coefficient is not `0`. Here we can simply swap the first two:

```
(list (list  -8 -4 -12)
      (list   0 -5  -5))
```

From here we may continue as before, subtracting the first equation from the remaining one `0` times. The final triangular matrix is:

```
(list (list 2  3  3   8)
      (list   -8 -4 -12)
      (list      -5  -5))
```

Stop! Show that $x = 1$, $y = 1$, and $z = 1$ is still a solution to this system of equations.

Exercise 384. Revise the algorithm `triangulate` from exercise 383 so that it rotates the equations first to find one with a leading coefficient that is not `0` before it subtracts the first equation from the remaining ones.

Does this algorithm terminate for all possible system of equations?

Hint The following expression rotates a non-empty list `L`:

```
(append (rest L) (list (first L)))
```

Explain why. ∎

Some systems of equations don't have a solution. Consider the following example:

$$\begin{array}{rcrcrcl} 2 \cdot x & + & 2 \cdot y & + & 2 \cdot z & = & 6 \\ 2 \cdot x & + & 2 \cdot y & + & 4 \cdot z & = & 8 \\ 2 \cdot x & + & 2 \cdot y & + & 1 \cdot z & = & 2 \end{array}$$

If you try to triangulate this system—by hand or with your solution from exercise 384—you discover that it yields an intermediate matrix all of whose equations start with `0`:

$$\begin{array}{rcrcrcr} 2 \cdot x & + & 2 \cdot y & + & 2 \cdot z & = & 6 \\ 0 \cdot x & + & 0 \cdot y & + & 2 \cdot z & = & 2 \\ 0 \cdot x & + & 0 \cdot y & - & 1 \cdot z & = & -4 \end{array}$$

Exercise 385. Modify the `triangulate` function from exercise 384 so that it signals an error if it encounters an SOE whose leading coefficients are all `0`. ∎

After we obtain a triangular system of equations such as (*) in exercise 380, we can solve the equations, one at a time. In our specific example, the last equation says that z is 2. Equipped with this knowledge, we can eliminate z from the second equation through a substitution:

$$3 \cdot y + 9 \cdot 2 = 21 \ .$$

Doing so, in turn, determines the value for y:

$$y = \frac{21 - 9 \cdot 3}{3}$$

In other words, we now know that $z = 2$ and $y = 1$, knowledge that we can plug into in the first equation:

$$2 \cdot x + 2 \cdot 1 + 3 \cdot 2 = 10$$

And again, this yields an equation in a single variable, which we know how to solve:

$$x = \frac{10 - (2 \cdot 1 + 3 \cdot 2)}{2}$$

This finally yields a value for x and thus the complete solution for the entire system of equations.

Exercise 386. Design the `solve` function. It consumes triangular systems of equations and produces a solution.

Hint Use structural recursion for the design. Start with the design of a function that solves a single linear equation in $n+1$ variables, given a solution for the last n variables. In general, this function plugs in the values for the rest of the left-hand side, subtracts the result from the right-hand side, and divides by the first coefficient. Experiment with this suggestion and the above examples.

Challenge Use an existing abstraction and `lambda` to define `solve` as a one-liner. ∎

Exercise 387. Define the function `gauss`. It combines the `triangulate` function from exercise 385 and the `solve` function from exercise 386. ∎

33 Algorithms that Backtrack

Problem solving isn't always straightforward. Sometimes we make progress by pursuing one approach only to discover that we are stuck because we took a wrong turn. One obvious option is to backtrack to the place where we took the wrong turn and to take a

different turn. Some algorithms work just like that. This chapter presents two instances. The first section deals with an algorithm for traversing graphs. The second one is an extended exercise that uses backtracking in the context of a chess puzzle.

33.1 Traversing Graphs

Graphs are ubiquitous in our world and the world of computing. Imagine a group of people, say, the students in your school. Write down all the names and connect the names of those people that know each other. You have just created your first undirected graph.

Now take a look at figure 96, which displays a small directed graph. It consists of seven nodes—the circled letters—and nine edges—the arrows. The graph may represent a small version of a so-called email network. Imagine a company and all the emails that go back and forth. Write down the email addresses of all employees. Then, address by address, draw an arrow from the address to all those addresses to whom the owner sends emails during a week. This is how you would create the directed graph in figure 96, though it might end up looking much more complex, almost impenetrable.

In general, a *graph* consists of a collection of *nodes* and a collection of *edges*, which connect nodes. In a *directed graph*, the edges represent one-way connections between the nodes; in an *undirected graph*, the edges represent two way connections between the nodes. In this context, the following is a common type of problem:

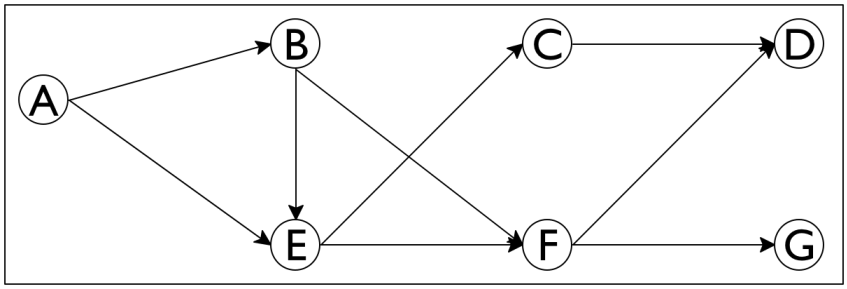
> **Sample Problem:** Design an algorithm that proposes a way to introduce one person to another in a directed email graph for a large company. The program consumes a directed graph representing established email connections and two email addresses. It returns a sequence of email addresses that connect the first email with the second.

Mathematical scientists call the desired sequence a *path*.

Figure 96 provides a basis for making the sample problem concrete. For example, you may wish to test whether the program can find a path from *C* to *D*. This particular path consists of the origination node *C* and the destination node *D*. In contrast, if you wish to connect *E* with *D*, there are two paths:

- send email from *E* to *F* and then to *D*.
- send it from *E* to *C* and then to *D*.

Sometimes it is impossible to connect two nodes with a path. In the graph of figure 96, you cannot move from *C* to *G* by following the arrows.



Social scientists use such algorithms to figure out the power structure in a company from such email graphs. Similarly they can use such graphs to predict the probable activities of

participants, even without knowledge of the content of the emails.

Figure 96: A directed graph

Looking at figure 96 you can easily figure out how to get from one node to another without thinking much about how you did it. So imagine for a moment that the graph in figure 96 is a large park. Also imagine someone says you are located at *E* and you need to get to *G*. You can clearly see two paths, one leading to *C* and another one leading to *F*. Follow the first one and make sure to remember that it is also possible to get from *E* to *F*. Now you have a new problem, namely, how to get from *C* to *G*. The key insight is that this new problem is just like the original problem; it asks you to find a path from one node to another. Furthermore, if you can solve the problem, you know how to get from *E* to *G*—just add the step from *E* to *C*. But there is no path from *C* to *G*. Fortunately, you remember that it is also possible to go from *E* to *F*, meaning you can *backtrack* to some point where you have a choice to make and re-start the search from there.

Now let's design this algorithm in a systematic manner. Following the general design recipe, we start with a data analysis. Here are two compact list-based representations of the graph in figure 96:

```
(define sample-graph
  '((A (B E))
    (B (E F))

    (C (D))
    (D ())
    (E (C F))
    (F (D G))
    (G ())))
```

```
(define sample-graph
  '((A B E)
    (B E F)

    (C D)
    (D)
    (E C F)
    (F D G)
    (G)))
```

Both contain one list per node. Each of these lists starts with the name of a node followed by its (immediate) *neighbors*, that is, nodes reachable by following a single arrow. The two differ in how they connect the (name of the) node and its neighbors: the left one uses `list` while the right one uses `cons`. For example, the second list represents node *B* with its two outgoing edges to *E* and *F* in figure 96. On the left `'B` is the first name on a two-element list; on the right it is the first name on a three-element list.

Exercise 388. Translate one of the above definitions into proper list form using `list` and proper symbols.

The data representation for nodes is straightforward:

```
; A Node is a Symbol
```

Formulate a data definition to describe the class of all *Graph* representations, allowing an arbitrary number of nodes and edges. Only one of the above representations has to belong to Graph.

Design the function `neighbors`. It consumes a Node `n` and a Graph `g` and produces the list of immediate neighbors of `n` in `g`.

Using your data definitions for Node and Graph—regardless of which one you chose, as long as you also designed `neighbors`—we can now formulate a signature and a purpose statement for `find-path`, the function that searches a path in a graph:

```
; Node Node Graph -> [List-of Node]
; finds a path from origination to destination in G
```

```
(define (find-path origination destination G)
  '())
```

What this header leaves open is the exact shape of the result. It implies that the result is a list of nodes, but it does not say exactly which nodes the list contains.

To appreciate this ambiguity and why it matters, let's study the examples from above. In ISL+, we can now formulate them like this:

```
(find-path 'C 'D Graph)
(find-path 'E 'D Graph)
(find-path 'C 'G Graph)
```

The first call to `find-path` must return a unique path, the second one must choose one from two, and the third one must signal that there is no path from `'C` to `'G` in `sample-graph`. Here are two possibilities then on how to construct the return value:

- The result of the function consists of all nodes leading from the `origination` node to the `destination` node, including those two. In this case, an empty path could be used to express the lack of a path between two nodes.
- Alternatively, since the call itself already lists two of the nodes, the output could mention only the "interior" nodes of the path. Now the answer for the first call would be `'()` because `'D` is an immediate neighbor of `'C`. Of course, this also means that `'()` no longer signals failure.

It is easy to imagine others, such as skipping either of the two given nodes.

Concerning the lack-of-a-path issue, we must choose a distinct value for signaling this notion. Because `#false` is distinct, meaningful, and works in either case, we opt for it. As for the multiple-paths issue, we postpone making a choice for now and list both possibilities in the example section:

```
; A Path is [List-of Node]
; interpretation The list of nodes specifies a sequence of
; immediate neighbors that leads from the first Node on the
; list to the last one.

; Node Node Graph -> [Maybe Path]
; finds a path from origination to destination in G
; if there is no path, the function produces #false

(check-expect (find-path 'C 'D Graph) '(C D))
(check-member-of (find-path 'E 'D Graph) '(E F D) '(E C D))
(check-expect (find-path 'C 'G Graph) #false)

(define (find-path origination destination G)
  #false)
```

Our next design step is to understand the four essential pieces of the function: the "trivial problem" condition, a matching solution, the generation of a new problem, and the combination step. The above discussion of the search process and the analysis of the three examples suggest answers:

1. If the two given nodes are directly connected with an arrow in the given graph, the path consists of just these two nodes. But there is an even simpler case, namely, when the `origination` argument of `find-path` is equal to its `destination`.
2. In that second case, the problem is truly trivial and the matching answer is `(list`

destination).

3. If the arguments are different, the algorithm must inspect all immediate neighbors of origination and determine whether there is a path from any one of those to destination. In other words, picking one of those neighbors generates a new instance of the "find a path" problem.

4. Finally, once the algorithm has a path from a neighbor of origination to destination, it is easy to construct a complete path from the former to the latter—just add the origination node to the list.

From a programming perspective, the third point is critical. Since a node can have an arbitrary number of neighbors, the "inspect all neighbors" task is too complex for a single primitive. We need an auxiliary function whose task it is to consume a list of nodes and to generate a new path problem for each of them. Put differently, the function is a list-oriented version of find-path.

Let's call this auxiliary function find-path/list and let's formulate a wish:

```
; [List-of Node] Node Graph -> [Maybe Path]
; finds a path from some node on lo-originations to destination
; if there is no path, the function produces #false
(define (find-path/list lo-originations destination G)
  #false)
```

Using this wish, we can fill in the generic template for generative-recursive functions to get a first draft of find-path:

```
(define (find-path origination destination G)
  (cond
    [(symbol=? origination destination) (list destination)]
    [else (... origination ...
           (find-path/list (neighbors origination G) destination G)
           ...)]))
```

It uses the neighbors from exercise 388 and the wish list function find-path/list and otherwise uses the answers to the four questions about generative recursive functions.

The rest of the design process is about details of composing these functions properly. Consider the signature of find-path/list. Like find-path, it produces [Maybe Path]. That is, if it finds a path from any of the neighbors, it produces this path; otherwise, if none of the neighbors is connected to destination, the function produces #false. Hence the answer of find-path depends on the kind of result find-path/list produces, meaning the code must distinguish the two possible answers with a cond expression:

```
(define (find-path origination destination G)
  (cond
    [(symbol=? origination destination) (list destination)]
    [else (local ((define next (neighbors origination G))
                  (define candidate (find-path/list next destination G)))
            (cond
              [(boolean? candidate) ...]
              [else ; (cons? candidate)
                ...]))]))
```

The two cases reflect the two kinds of answers we might receive: a boolean or a list. In the first case, find-path/list cannot find a path from any neighbor to destination,

meaning `find-path` itself cannot construct such a path either. In the second case, the auxiliary function found a path, but `find-path` must still add `origination` to the front of this path because `candidate` starts with one of `origination`'s neighbors not `origination` itself as agreed upon above.

```
; Node Node Graph -> [Maybe Path]
; finds a path from origination to destination in G
; if there is no path, the function produces #false
(define (find-path origination destination G)
  (cond
    [(symbol=? origination destination) (list destination)]
    [else (local ((define next (neighbors origination G))
                  (define candidate (find-path/list next destination G)))
            (cond
              [(boolean? candidate) #false]
              [else (cons origination candidate)]))]))


; [List-of Node] Node Graph -> [Maybe Path]
; finds a path from some node on lo-Os to D
; if there is no path, the function produces #false
(define (find-path/list lo-Os D G)
  (cond
    [(empty? lo-Os) #false]
    [else (local ((define candidate (find-path (first lo-Os) D G)))
            (cond
              [(boolean? candidate) (find-path/list (rest lo-Os) D G)]
              [else candidate]))]))
```

Figure 97: Finding a path in a graph

Figure 97 contains the complete definition of `find-path`. It also contains a definition of `find-path/list`, which processes its first argument via structural recursion. For each node in the list, `find-path/list` uses `find-path` to check for a path. If `find-path` indeed produces a path, that path is the answer. Otherwise, `find-path` produces `#false` and the function backtracks.

Note Trees discusses backtracking in the structural world. A particularly good example is the function that searches blue-eyed ancestors in a family tree. When the function encounters node, it first searches one branch of the family tree, say the father's, and if this search produces `#false`, it searches the other half. Since graphs generalize trees, comparing this function with `find-path` is an instructive exercise. **End**

Lastly, we need to check whether `find-path` produces an answer for all possible inputs. It is relatively easy to check that when given the graph in figure 96 and any two nodes in this graph, `find-path` always produces some answer. Stop! Solve the next exercise before you read on.

Exercise 389. Test `find-path`. Use the function to find a path from `'A` to `'G` in `sample-graph`. Which one does it find? Why?

Design the function `test-on-all-nodes`, which consumes a graph `g` and tries to find a path `find-path` between all pairs of nodes in `g`. If it succeeds, it produces `#true`. Test the function on `sample-graph`. ∎

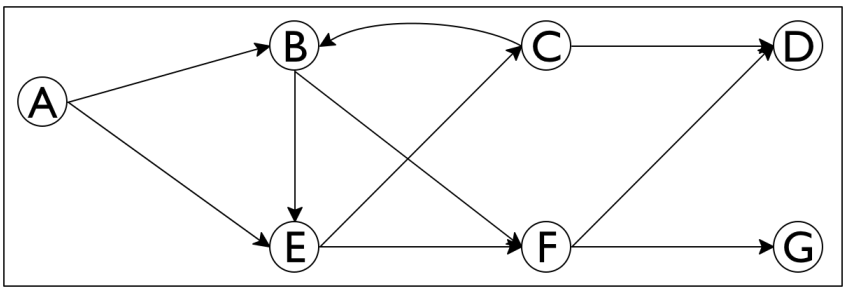


Figure 98: A directed graph with cycle

For other graphs, however, `find-path` may not terminate for certain pairs of nodes. Consider the graph in figure 98.

Stop! Define `cyclic-graph` to represent the graph in this figure.

Compared to figure 96, this new graph contains only one extra edge, a connection from *C* to *B* though this seemingly small addition allows to start a search in a node and to return to the same node by just following the arrows. Specifically, it is possible to move from *B* to *E* to *C* and back to *B*. Indeed, when `find-path` is applied to `'B`, `'D`, and `cyclic-graph`, it fails to stop as the following hand-evaluation confirms:

```
(find-path 'B 'D cyclic-graph)
= ... (find-path 'B 'D cyclic-graph) ...
= ... (find-path/list (list 'E 'F) 'D cyclic-graph) ...
= ... (find-path 'E 'D cyclic-graph) ...
= ... (find-path/list (list 'C 'F) 'D cyclic-graph) ...

= ... (find-path 'C 'D cyclic-graph) ...
= ... (find-path/list (list 'B 'D) 'D cyclic-graph) ...
= ... (find-path 'B 'D cyclic-graph) ...
= ...
```

The hand-evaluation shows that after seven applications of `find-path` and `find-path/list`, ISL+ must evaluate the exact same expression from where it started. Since the same input triggers the same evaluation for any defined function, we now know that `find-path` does not terminate for these inputs.

In summary, the **termination** argument goes like this. If some given graph is free of cycles, `find-path` produces some output for any given inputs. After all, every path can only contain a finite number of nodes and the number of paths is finite, too. The function therefore either exhaustively inspects all solutions starting from some given node or finds a path from the origination to the destination node. If, however, a graph contains a cycle, that is, a path from some node back to itself, `find-path` may not produce a result for some inputs.

You know only one exception to this rule: `random`.

The next part presents a program design technique that addresses just this kind of problem. In particular, it presents a variant of `find-path` that can deal with cycles in a graph.

Exercise 390. Test `find-path` on `'B`, `'C`, and the graph in figure 98. Also use `test-on-all-nodes` from exercise 389 on this graph. ∎

Exercise 391. Re-design the `find-path` program as a single function definition. Remove parameters from the locally defined functions. ∎

Exercise 392. Re-design `find-path/list` so that it uses an existing list abstraction from figure 55 instead of explicit structural recursion.

Once you have this version tested, merge the two functions.

Note Read the documentation for Racket's `ormap`. How does it differ from ISL+'s `ormap` function? Would the former be helpful here? ∎

Note on Data Abstraction You may have noticed that the `find-path` function does not need to know how Graph is defined. As long as you provide a correct `neighbors` function for Graph, `find-path` works perfectly fine. In short, the `find-path` program uses **data abstraction**.

In a way, data abstraction works just like function abstraction, as discussed in Abstraction. Here you could create a function `abstract-find-path`, which would consume one more parameter than `find-path`: `neighbors`. As long as you always handed `abstract-find-path` a graph `G` from `Graph` and the matching `neighbors` function, it would process the graph properly. While the extra parameter suggests abstraction in the conventional sense, the required relationship between two of the parameters—`G` and `neighbors`—really means that `abstract-find-path` is also abstracted over the definition of Graph. Since the latter is a data definition, the idea is dubbed data abstraction.

When programs grow large, data abstraction becomes a critical tool for the construction of a program's components. The next volume in the "How to Design" series addresses this idea in depth; the next section illustrates the idea with another example. **End**

Exercise 393. Finite State Machines poses a problem concerning finite state machines and strings but immediately defers to this chapter because the solution calls for generative recursion. You have now acquired the design knowledge needed to tackle the problem.

Design the function `fsm-match`. It consumes the data representation of a finite state machine and a string. It produces `#true` if the sequence of characters in the string causes the finite state machine to transition from an initial state to a final state.

Since this problem is about the design of generative recursive functions, we provide the essential data definition and a data example:

```
(define-struct transition [current key next])
(define-struct fsm [initial transitions final])

; A FSM is (make-fsm FSM-State [List-of 1Transition] FSM-State)
; A 1Transition is
;   (make-transition FSM-State 1String FSM-State)
; A FSM-State is String

; data example: see exercise 98
(define fsm-a-bc*-d
  (make-fsm
   "AA"
   (list (make-transition "AA" "a" "BC")
         (make-transition "BC" "b" "BC")
         (make-transition "BC" "c" "BC")
         (make-transition "BC" "d" "DD"))
   "DD"))
```

The data example corresponds to the regular expression `a (b|c)* d`. As mentioned in

exercise 98, `"acbd"` is one example of an acceptable string; `"ad"` and `"abcd"` are two others. Of course, `"da"`, `"aa"`, or `"d"` do not match.

In this context, you are designing the following function:

```
; FSM String -> Boolean
; does the given string match the regular expression expressed as fsm
(define (fsm-match? a-fsm a-string)
  #false)
```

Hint Design the necessary auxiliary function locally to `fsm-match?`. In this context, represent the problem as a pair of parameters: the current state of the finite state machine and the remaining list of 1Strings. ∎

33.2 Extended Exercise: Checking (on) Queens

We thank Mr. Mark Engelberg for his suggestions on how to reformulate this section for young students.

The *n* queens puzzle is a famous problem from the world of chess that also illustrates the applicability of backtracking in a natural way. For our purposes, a chessboard is a grid of *n* by *n* squares. The queen is a game piece that can move in a horizontal, vertical, or diagonal direction arbitrarily far without "jumping" over another piece. We say that a queen *threatens* a square if it is on the square or can move to it. Figure 99 illustrates the notion in a graphical manner. The queen is in the second column and sixth row. The solid lines radiating out from the queen go through all those squares that are threatened by the queen.

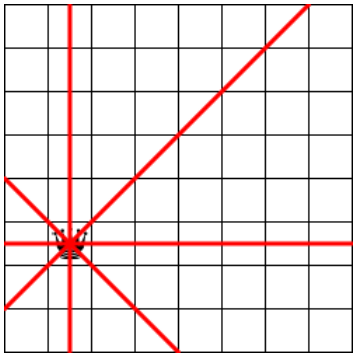


Figure 99: A chessboard with a single queen and the positions it threatens

The classical queen-placement problem is to place 8 queens on a 8 by 8 chessboard such that the queens on the board don't threaten each other. Computer scientists generalize the problem of course and ask whether it is possible to place *n* queens on a *n* by *n* chessboard such that the queens don't pose a threat to each other.

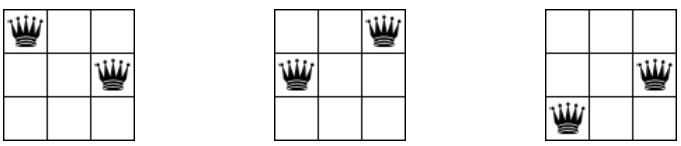


Figure 100: Three queen configurations for a 3 by 3 chess board

For $n = 2$, the puzzle obviously has no solution. A queen placed on any of the four squares threatens all remaining squares.

It turns out that there is also no solution for $n = 3$. Figure 100 presents three different placements of two queens. In each case, the left-most queen occupies a square in the left-most column while a second queen is placed in one of two squares that the first one does not threaten. The placement of a second queen ensures that all seven unoccupied squares are threatened by a queen, meaning it is impossible to place a third queen. Together, the three placements explore all possibilities of placing the first queen in a square of the first column.

Exercise 394. It is also possible to place the first queen in all squares of the top-most row, the right-most column, and the bottom-most row. Explain why all of these solutions are just like the three scenarios depicted in figure 100?

This leaves the central square. Is it possible to place even a second queen after you place one on the central square of a 3 by 3 board? ∎

Figure 101: Solutions for the n queens puzzle for 4 by 4 and 5 by 5 boards

Figure 101 displays two solutions for the n queens puzzle: the left one is for $n = 4$, while the right one solves the $n = 5$ version. The figure shows how in each case, a solution places one queen in each row and column of the board, which makes sense because a queen threatens the entire row and column that radiate out from its square.

Now that we have conducted a sufficiently detailed analysis of the problem, we can proceed to the solution phase. The analysis suggests several ideas:

1. The problem is about placing one queen at a time. When we place a queen on a board, we can mark the corresponding rows, columns, and diagonals as unusable for other queens.

2. By implication, the problem is really about placing n queens on a k by k chessboard in a non-threatening manner.

3. When we place another queen on a board, we consider only those spots that are not threatened.

4. But, in case the first choice prevents us from placing the remaining queens, we need to remember that there are other squares that we should still try.

5. If we are supposed to place a queen on a board but not safe squares are left, we need to backtrack to a point in the process where we chose one square over another.

In short, the solution process for the n queens puzzle has the same flavor as the “find a path” algorithm from Traversing Graphs.

```
; N -> [Maybe [List-of QP]]
; find a solution to the n queens problem
```

```
(check-member-of
 (n-queens 4)
 (list (make-posn 0 1) (make-posn 1 3) (make-posn 2 0) (make-posn 3 2))
 (list (make-posn 0 1) (make-posn 1 3) (make-posn 3 2) (make-posn 2 0))
 (list (make-posn 0 1) (make-posn 2 0) (make-posn 1 3) (make-posn 3 2))
 (list (make-posn 0 1) (make-posn 2 0) (make-posn 3 2) (make-posn 1 3))
 (list (make-posn 0 1) (make-posn 3 2) (make-posn 1 3) (make-posn 2 0))
 (list (make-posn 0 1) (make-posn 3 2) (make-posn 2 0) (make-posn 1 3))
 (list (make-posn 0 2) (make-posn 1 0) (make-posn 2 3) (make-posn 3 1))
 ...
 (list (make-posn 1 0) (make-posn 0 2) (make-posn 2 3) (make-posn 3 1))
 ...
 (list (make-posn 1 0) (make-posn 3 1) (make-posn 2 3) (make-posn 0 2))
 (list (make-posn 1 3) (make-posn 0 1) (make-posn 2 0) (make-posn 3 2))
 ...
 (list (make-posn 1 3) (make-posn 3 2) (make-posn 2 0) (make-posn 0 1))
 (list (make-posn 2 0) (make-posn 0 1) (make-posn 1 3) (make-posn 3 2))
 ...
 (list (make-posn 2 0) (make-posn 3 2) (make-posn 1 3) (make-posn 0 1))
 (list (make-posn 2 3) (make-posn 0 2) (make-posn 1 0) (make-posn 3 1))
 ...
 (list (make-posn 2 3) (make-posn 3 1) (make-posn 1 0) (make-posn 0 2))
 (list (make-posn 3 1) (make-posn 0 2) (make-posn 1 0) (make-posn 2 3))
 ...
 (list (make-posn 3 1) (make-posn 2 3) (make-posn 1 0) (make-posn 0 2))
 (list (make-posn 3 2) (make-posn 0 1) (make-posn 1 3) (make-posn 2 0))
 ...
 (list (make-posn 3 2) (make-posn 2 0) (make-posn 1 3) (make-posn 0 1)))

(define (n-queens n)
  (place-queens (board0 n) n))
```

Figure 102: Solutions for the 4 queens puzzle

Moving from the problem statement and the process description to a designed algorithm clearly calls for two data representations: one for the chess boards and one for positions on the board. Let's start with the latter because is basically dictated by the nature of the chess board:

```
; QP is (make-posn CI CI)
; CI is a natural number in [0,8)
; interpretation a CI denotes a row or column index for a chess board,
; (make-posn r c) specifies the square in the r-th row and the c-th column
```

The definition for CI could use [1,8] instead of [0, 8), but the two definitions are basically equivalent and counting up from 0 is what programmers do. Similarly, the so-called algebraic notation for chess positions uses the letters 'a through 'h for one of the board's dimension, meaning QP could have used CIs and such letters. Again, the two are roughly equivalent and with natural numbers it is easier in ISL+ to create many positions than with letters.

Exercise 395. Design the `threatening?` function. It consumes two QPs and determines whether queens placed on the two respective squares would threaten each other.

Domain Knowledge (1) Study figure 99. The queen in this figure threatens all squares on the horizontal, the vertical, and the diagonal lines. Conversely, a queen on any square on these lines threatens the queen.

(2) Translate your insights into mathematical conditions that relate the squares' coordinates to each other. For example, all squares on a horizontal have the same *y* coordinate. Similarly, all squares on one diagonal have coordinates whose sums are the same. Which diagonal is that? For the other diagonal, the differences between the two coordinates remains the same. Which diagonal does this idea describe?

Hint Once you have figured out the domain knowledge, formulate a test suite that covers horizontals, verticals, and both diagonals. Don't forget to include a pair of arguments for which `threatening?` must produce `#false`. ∎

Exercise 396. Design `render-queens`. The function consumes a natural number `n`, a list `l` of QPs, and an Image `i` representing a queen. It produces an image of an `n` by `n` chess board with images `i` placed according to `l`.

You may wish to look for an image for a chess queen on-line or create a simplistic one with the available image functions. ∎

As for a data representation for *Boards*, we postpone this step until we know how the algorithm processes them. Doing so in another exercise in data abstraction—see note at end of preceding chapter. Indeed, a data definition for Board isn't even necessary to state the signature for the algorithm proper:

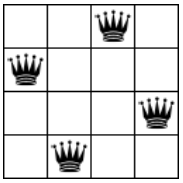
```
; N -> [Maybe [List-of QP]]
; find a solution to the n queens problem

; data example:
(define 4QUEEN-SOLUTION-2
  (list  (make-posn 0 2) (make-posn 1 0) (make-posn 2 3) (make-posn 3 1)))

(define (n-queens n)
  #false)
```

The puzzle proper is about finding a placement for *n* queens on an *n* by *n* chess board. So clearly, the algorithm consumes nothing else but a natural number, and it produces a representation for the *n* queen placements—if a solution exists. Since the latter can be represented with, for example, a list of QPs, one natural result signature is [List-of QP] or `#false`, where the latter represents the failure to find one.

Exercise 397. The next step is to develop examples and to formulate them as tests. We know that `n-queens` must fail when given `2` or `3`. For `4`, there are two solutions with real boards and four identical queens. Figure 101 shows one of them, on the left, and the other one is this:



In terms of data representations, however, there are many different ways to describe these two images via lists of QPs. Figure 102 sketches some. Your task is to fill in the dots. ∎

Exercise 398. The tests in figure 101 are awful. No real-world programmer will ever spell out all these possible outcomes.

One solution is to understand the lists of QPs as sets. After all, all lists that contain the same QPs in different order are equivalent as exercise 397 already suggests. Hence you could formulate the test for `n-queens` as

```
(check-expect (is-4-queens-result? (n-queens 4)) #true)

; [List-of QP] -> Boolean
; is the result equal [as a set] to either of two lists
(define (is-queens-result? x)
  (or (set=? 4QUEEN-SOLUTION-1 x) (set=? 4QUEEN-SOLUTION-2 x)))
```

Design the function `set=?`. It consumes two lists and determines whether they contain the same items—regardless of order.

Property testing is a second, even more general way to solve this problem. It calls for the design of a predicate that recognizes when a given list represents a solution to an `n` queens puzzle:

- A solution for an `n` queens puzzle must have length `n`.
- A QP on such a list may not threaten any other, distinct QP.

Design the `n-queens-solution?` function, which consumes a natural number `n` and a list of QPs and determines whether the latter is a solution to an `n` queens puzzle. Once you have tested this predicate, use it to formulate the tests for `n-queens`. ∎

Exercise 399. As bullet 2 above says, you really want to design a function that places *n* queens on a *k* by *k* chessboard in a non-threatening manner:

```
; Board N -> [Maybe [List-of QP]]
; places n queens on board. if possible; otherwise, returns #false

(define (place-queens a-board n)
  #false)
```

Figure 102 already refers to this function in the definition of `n-queens`.

Design the `place-queens` algorithm. Assume you have the following functions to deal with Boards:

```
; N -> Board
; creates the initial n by n board
(define (board0 n)
  ...)

; Board QP -> Board
; places a queen at qp on a-board
(define (add-queen a-board qp)
  a-board)

; Board -> [List-of QP]
; finds spots where it is still safe to place a queen
(define (find-open-spots a-board)
  '())
```

The first function is used in figure 102 to create the initial board representation for `place-queens`. You will need the other two to describe the generative steps for the algorithm. ∎

You cannot confirm that your solution to the preceding exercise works for the test cases of exercise 397 because it relies on an extensive wish list. Technically, it calls for a data representation of Boards that supports three specific functions. The remaining problem is then to formulate a definition for Board and to design the functions on the wish list.

Exercise 400. Develop a data definition for Board and design the three functions specified in exercise 399. Consider the following ideas:

- a Board collects those positions where a queen can still be placed;
- a Board contains the list of positions where a queen has been placed;
- a Board is a grid of `n` by `n` squares, each possibly occupied by a queen. **Hint** For this representation, consider using a structure to represent a square with one field for the *x* index, another one for *y*, and a third one saying whether the square is threatened.

Use one of the above ideas to solve this exercise.

Challenge Use all three ideas to come up with three different data representations of Board. Abstract your solution to exercise 399 and confirm that it works with any of your data representations of Board. ∎

34 Summary

This fifth part of the book introduces the idea of *eureka!* into program design. Unlike the structural design of the first four parts, *eureka!* design starts from an idea of how the program should solve a problem or process data that represents a problem. Designing here means coming up with a clever way to call a recursive function on a new kind of problem that is like the given one but simpler.

Keep in mind that while we have dubbed it **generative recursion**, most computer scientists refer to these functions as **algorithms**.

Once you have completed this part of the book, you understand the following about the design of generative recursion:

1. The standard outline of the design recipe remains valid.
2. The major change concerns the coding step. It introduces four new questions on going from the completely generic template for generative recursion to a complete function. With two of these questions, you work out the “trivial” parts of the solution process; and with the other two you work out the generative solution step.
3. The minor change is about the termination behavior of generative recursive functions. Unlike structurally designed functions, algorithms may not terminate for some inputs. This problem might be due to inherent limitations in the idea or the translation of the idea into code. Regardless, the future reader of your program deserves a warning about potentially “bad” inputs.

You will encounter some simple or well-known algorithms in your real-world programming tasks. And you will be expected to cope. For truly clever algorithms, software companies employ highly paid computer scientists, domain experts, and mathematicians to work out the conceptual details before they ask programmers to turn the concepts into programs. You must also be prepared for this kind of task, and the best

preparation is practice.