Lecture 20: The Divergence Theorem *RHB 9.7 9.8*

20. 1. Integral Definition of Divergence (RHB 9.7)

If $\underline{A}$ is a vector field in the region R, and P is a point in R, then the divergence of $\underline{A}$ at P may be **defined** by

$$\boxed{\operatorname{div} \underline{A} \;=\; \lim_{\mathrm{V}\to 0} \frac{1}{\mathrm{V}} \int_{\mathrm{S}} \underline{A}\cdot \underline{dS}}$$

where S is a **closed** surface in R which encloses the volume V. The limit must be taken so that the point P is within V.

This definition of $\operatorname{div} \underline{A}$ is **basis independent**.

We now prove that our original definition of *div* is recovered in Cartesian co-ordinates

Let P be a point with Cartesian coordinates (x_0, y_0, z_0) situated at the *centre* of a small rectangular block of size $\delta_1 \times \delta_2 \times \delta_3$, so its volume is $\delta V = \delta_1\, \delta_2\, \delta_3$.

- On the **front** face of the block, orthogonal to the x axis at $x = x_0 + \delta_1/2$ we have *outward* normal $\underline{n} = \underline{e}_1$ and so $\underline{dS} = \underline{e}_1\, dy\, dz$
- On the **back** face of the block orthogonal to the x axis at $x = x_0 - \delta_1/2$ we have *outward* normal $\underline{n} = -\underline{e}_1$ and so $\underline{dS} = -\underline{e}_1\, dy\, dz$

Hence $\underline{A} \cdot \underline{dS} = \pm\, A_1\, dy\, dz$ on these two faces. Let us denote the two surfaces orthogonal to the e_1 axis by S_1.

The contribution of these two surfaces to the integral $\displaystyle\int_S \underline{A} \cdot \underline{dS}$ is given by

$$\begin{aligned}
\int_{S_1} \underline{A} \cdot \underline{dS} \;&=\; \int_z\!\int_y \Big\{ A_1(x_0 + \delta_1/2, y, z) \;-\; A_1(x_0 - \delta_1/2, y, z) \Big\}\, dy\, dz \\
&=\; \int_z\!\int_y \Bigg\{ \left[A_1(x_0, y, z) \;+\; \frac{\delta_1}{2}\, \frac{\partial A_1(x_0, y, z)}{\partial x} \;+\; O(\delta_1^2) \right] \\
&\qquad\qquad -\; \left[A_1(x_0, y, z) \;-\; \frac{\delta_1}{2}\, \frac{\partial A_1(x_0, y, z)}{\partial x} \;+\; O(\delta_1^2) \right] \Bigg\}\, dy\, dz \\
&=\; \int_z\!\int_y \delta_1\, \frac{\partial A_1(x_0, y, z)}{\partial x}\, dy\, dz
\end{aligned}$$

where we have dropped terms of $O(\delta_1^2)$ in the Taylor expansion of A_1 about (x_0, y, z).

So

$$\frac{1}{\delta V}\int_{S_1} \underline{A}\cdot \underline{dS} = \frac{1}{\delta_2\,\delta_3}\int_z\int_y \frac{\partial A_1(x_0,y,z)}{\partial x}\,dy\,dz$$

As we take the limit $\delta_1, \delta_2, \delta_3 \to 0$ the integral tends to $\frac{\partial A_1(x_0,y_0,z_0)}{\partial x}\,\delta_2\,\delta_3$ and we obtain

$$\lim_{\delta V\to 0}\frac{1}{\delta V}\int_{S_1}\underline{A}\cdot\underline{dS} \;=\; \frac{\partial A_1(x_0,y_0,z_0)}{\partial x}$$

With similar contributions from the other 4 faces, we find

$$\operatorname{div}\underline{A} \;=\; \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \;=\; \underline{\nabla}\cdot\underline{A}$$

in agreement with our original definition in Cartesian co-ordinates.

Note that the integral definition gives an intuitive understanding of the divergence in terms of net flux leaving a small volume around a point $\underline{r}$. **In pictures:** for a small volume dV

20. 2. The Divergence Theorem (Gauss's Theorem)

If $\underline{A}$ is a vector field in a volume V, and S is the closed surface bounding V, then

$$\boxed{\int_V \underline{\nabla}\cdot\underline{A}\,dV \;=\; \int_S \underline{A}\cdot\underline{dS}}$$

The divergence theorem is the generalisation to 3-D of

$$\int_a^b \frac{df(x)}{dx}\,dx \;=\; f(b)\;-\;f(a)$$

Proof : We derive the divergence theorem by making use of the integral definition of div $\underline{A}$

$$\operatorname{div}\underline{A} \;=\; \lim_{\Delta\mathrm{V}\to 0}\frac{1}{\Delta\mathrm{V}}\int_{\mathrm{S}}\underline{A}\cdot\underline{dS}.$$

Since this **definition** of div $\underline{A}$, is valid for infinitesimal volumes of arbitrary shape (in the previous subsection we just considered rectangular blocks), we can build a smooth surface S from a large number of blocks of volume ΔV^i and surface ΔS^i. We have

$$\operatorname{div}\underline{A}(\underline{r}^i) \;=\; \frac{1}{\Delta V^i}\int_{\Delta S^i}\underline{A}\cdot\underline{dS} + (\epsilon^i)$$

where $\epsilon^i \to 0$ as $\Delta V^i \to 0$. Now multiply both sides by ΔV^i and sum over all i

$$\sum_{i=0}^{N-1} \operatorname{div} \underline{A}(\underline{r}^i)\, \Delta V^i \;=\; \sum_{i=0}^{N-1} \int_{\Delta S^i} \underline{A} \cdot \underline{dS}$$

On rhs the contributions from surface elements *interior* to S cancel. This is because where two blocks touch, the outward normals are in *opposite* directions, implying that the contributions to the respective integrals cancel.

Taking the limit $N \to \infty$ we have, as claimed,

$$\int_V \underline{\nabla} \cdot \underline{A}\, dV \;=\; \int_S \underline{A} \cdot \underline{dS}\,.$$

For an elegant alternative proof see *Bourne & Kendall 6.2*

20. 3. The Continuity Equation (RHB 9.8.3, Dawber 5.4)

Consider a fluid with density field $\rho(\underline{r})$ and velocity field $\underline{v}(\underline{r})$. We have seen previously that the flux (volume per unit time) flowing across a surface is given by $\int_S \underline{v} \cdot \underline{dS}$.

Now consider a volume V bounded by the *closed* surface S. Since the flux measures flow *out* of V, the rate of change of the fluid mass M contained in V obeys

$$\frac{\partial M}{\partial t} = -\int_S \underline{J} \cdot \underline{dS}\,.$$

where $\underline{J} = \rho \underline{v}$ is the mass current. (We have assumed no sources or sinks of the fluid—taps or plugholes—see below).

The mass in V may be written as $M = \int_V \rho\, dV$ therefore we have

$$\frac{\partial}{\partial t}\int_V \rho\, dV + \int_S \underline{J} \cdot \underline{dS} = 0\,.$$

We now use the divergence theorem to rewrite the second term as a volume integral and we obtain

$$\int_V \left[\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J}\right] dV = 0$$

Now since this holds for arbitrary V we must have that

$$\boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0\,.}$$

This equation, known as the **continuity equation**, appears in many different contexts since it holds for any *conserved* quantity. Here we considered mass density ρ and mass current $\underline{J}$ of a fluid; but equally it could have been number density of molecules in a gas and current of molecules; electron density and electric current vector; thermal energy density and heat current vector; or even more abstract conserved quantities such as probability density!

To understand better the divergence of a vector field consider the divergence of the current in the continuity equation

$$\text{if} \quad \underline{\nabla} \cdot \underline{J}(\underline{r}) > 0 \quad \text{then} \quad \frac{\partial \rho}{\partial t} < 0 \quad \text{and the mass density at } \underline{r} \text{ decreases}$$

$$\text{if} \quad \underline{\nabla} \cdot \underline{J}(\underline{r}) < 0 \quad \text{then} \quad \frac{\partial \rho}{\partial t} > 0 \quad \text{and the mass density at } \underline{r} \text{ increases}$$

20. 4. Sources and Sinks (Dawber 5.3

Consider *time independent* behaviour where $\frac{\partial \rho}{\partial t} = 0$. The continuity equation tells us that for the density to be constant in time we must have $\underline{\nabla} \cdot \underline{J} = 0$ –flux into a point equals flux out.

However if we have a **source** or a **sink** of the field, the divergence is not zero at that point. In general the quantity

$$\frac{1}{V} \int_S \underline{A} \cdot \underline{dS}$$

tells us whether there are sources or sinks of the vector field $\underline{A}$ within V: if V contains

- a **source**, then $\int_S \underline{A} \cdot \underline{dS} = \int_V \underline{\nabla} \cdot \underline{A} > 0$
- a **sink**, then $\int_S \underline{A} \cdot \underline{dS} = \int_V \underline{\nabla} \cdot \underline{A} < 0$

If S contains neither sources nor sinks, then $\int_S \underline{A} \cdot \underline{dS} = 0$.

As an example consider **electrostatics**. You will have learned that electric field lines are conserved and can only start and stop at charges. A positive charge is a source of electric field (i.e. creates a positive flux) and a negative charge is a sink (i.e. absorbs flux or creates a negative flux).

The electric field due to a charge q at the origin is

$$\underline{E} = \frac{q}{4\pi\epsilon_0} \frac{\hat{\underline{r}}}{r^2}.$$

It is easy to verify that $\underline{\nabla} \cdot \underline{E} = 0$ except at the origin where the field is singular.

The flux integral for this type of field across a sphere (of any radius) around the origin was evaluated in the last lecture and we find the flux out of the sphere as:

$$\int_S \underline{E} \cdot \underline{dS} = \frac{q}{\epsilon_0}$$

Now since $\underline{\nabla} \cdot \underline{E} = 0$ away from the origin the results holds for any surface enclosing the origin. Moreover if we have several charges enclosed by S then

$$\int_S \underline{E} \cdot \underline{dS} = \sum_i \frac{q_i}{\epsilon_0}$$

This recovers *Gauss' Law* of electrostatics.

We can go further and consider a *charge density* of $\rho(\underline{r})$ per unit volume. Then

$$\int_S \underline{E} \cdot \underline{dS} = \int_V \frac{\rho(\underline{r})}{\epsilon_0} dV \ .$$

We can rewrite the lhs using the divergence theorem

$$\int_V \underline{\nabla} \cdot \underline{E} dV = \int_V \frac{\rho(\underline{r})}{\epsilon_0} dV$$

Since this must hold for arbitrary V we see

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho(\underline{r})}{\epsilon_0} \ .$$

which holds for all $\underline{r}$ and is one of Maxwell's equations of Electromagnetism.