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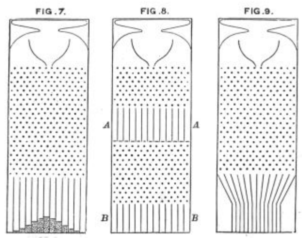
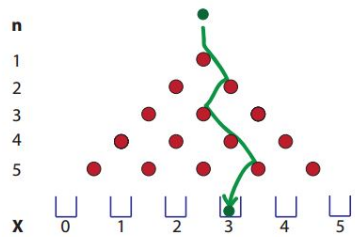
Topics: Multiple Discrete Random Variables, Conditioning, Independent Random Variables

Q. 1 Galton Box Problem

The basic design of the Galton box (also called bean machine or quincunx) is shown in figures below. On the left are the actual drawings/design of Sir Francis Galton on the right my idealization model for the board.





Balls are released at the top of a triangle of pegs, say with n layers. The pegs are aligned such that roughly with probability p the ball goes to the right, otherwise it moves left, where typically $p = 1/2$. The ball eventually lands in one of the bins at the bottom.

1. Under the above idealized model, what is the PMF for the random variable X denoting the number of bin a typical ball would end up in?
2. Watch the any simulated demonstation or youtube video of the physical system to see a Galton box where a bunch of balls are released. As the balls accumulate you start to see that the fraction in each bin, i.e., the empirical distribution associated with the experiment is quite close (but not always) to the true distribution computed earlier. How does the number of balls in the experiment affect the proximity of the empirical distribution to our model?

Q. 2 The joint PMF of two random variables X and Y is given by

$$P_{X,Y}(x,y) = \begin{cases} c(x^2+y^2), & \text{if } x = \{1,2,4\}\ ; y = \{1,3\} \\ 0 & \text{otherwise} \end{cases}$$

1. Determine the constant c.
2. What is $P(Y < X)$?
3. What is $P(Y = X)$?
4. Find the marginal PMF's of X and Y. Are X ,Y independent?
5. Find the expectations $E[X], E[Y]$ and $E[XY]$.
6. Find the variances $\text{var}(X)$ and $\text{var}(X+Y)$.
7. Let A denote the event $X \geq Y$. Find $E[X|A]$ and $\text{var}(X|A)$.

Q. 3 The joint PMF of two random variables R and S is given by

$$P_{RS}(r,s) = \begin{cases} \frac{4}{45} & r = 1\ , s = 1 \\ \frac{6}{45} & r = 1\ , s = 2 \\ \frac{6}{45} & r = 1\ , s = 3 \\ \frac{6}{45} & r = 2\ , s = 1 \\ \frac{9}{45} & r = 2\ , s = 2 \\ \frac{9}{45} & r = 2\ , s = 3 \\ \frac{2}{45} & r = 3\ , s = 1 \\ \frac{3}{45} & r = 3\ , s = 2 \\ 0 & r = 3\ s = 3 \end{cases}$$

Let $A = \{S \neq 3\}, X = R+S, Y = R-S$.

1. Find $p_S(s)$ and $p_{S|A}(s)$.
2. Find $p_{R,Y}(r,y)$.
3. Find $p_{X|A}(x)$.

Q. 4 The joint probability mass function of the random variables X,Y,Z is

$$p_{X,Y,Z}(0,0,0)=\frac{1}{9},\ p_{X,Y,Z}(0,0,1)=\frac{1}{9},\ p_{X,Y,Z}(0,1,0)=\frac{1}{18},\ p_{X,Y,Z}(0,1,1)=\frac{2}{9},$$
$$p_{X,Y,Z}(1,0,0)=\frac{2}{9},\ p_{X,Y,Z}(1,0,1)=\frac{1}{18},\ p_{X,Y,Z}(1,1,0)=\frac{1}{9},\ p_{X,Y,Z}(1,1,1)=\frac{1}{9}\ .$$

1. Find $E[XY\sqrt{Z}]$.
2. Argue that $E[XY+XZ^2+YZ]=E[XY]+E[XZ^2]+E[YZ]$. and also compute its value.
3. Are X , Y independent given Z =1 ? Compute $E[XY|Z=1]$.

Q. 5 Choose a number X at random from the set of numbers $\{1,2,3,4\}$. Now choose a number at random from the subset no larger than X, that is, from $\{1,2,\ldots X\}$. Call this second number Y.

1. Find the joint mass function of X and Y.
2. Find the conditional mass function of X given that $Y=i$. Do it for $i=1,2,\ldots,4$. Also, find $E[Y]$.
3. Are X and Y independent? Why?

Q. 6 The PMF for the result of any one roll of a three sided die with faces numbered 1,2 and 3 is

$$p_X(x)=\begin{cases}\frac{1}{2}, & x=1\\ \frac{1}{4}, & x=2\\ \frac{1}{4}, & x=3\\ 0, & \text{otherwise}\end{cases}$$

Consider a sequence of six independent rolls of this die, and let X_i be the random variable corresponding to the i^{th} roll.

1. What is the probability that exactly three of the rolls have result equal to 3?
2. What is the probability that the first roll is 1, given that exactly two of the six rolls have result of 1?
3. We are told that exactly three of the rolls resulted in 1 and exactly three resulted in 2. Given this information, what is the probability that the sequence of rolls is 121212?
4. Conditioned on the event that at least one roll resulted in 3, find the conditional PMF of the number of 3s.