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Topics: Central Limit Theorem, Pollster Problem, Law of Large Numbers, MAP estimators.

Q. 1 A modem transmits one million bits. Each bit is 0 or 1 independently with equal probability. Estimate the probability that at least 502,000 ones have transmitted.

Q. 2 On any given flight, an airline's goal is to have the plane be as full as possible. For this reason, they sometimes choose to overbook.

1. If on average 10% of customers cancel their tickets, and they do so independently of each other, what is the probability that a particular flight will be overbooked if the airline sells 320 tickets for a plane that has a maximum capacity of 300 people.
2. What is the probability that a plane with maximum capacity of 150 people will be overbooked if the airline sells 160 tickets.
3. Given your answers on the previous questions, offer a comment on the relationship between the ability to overbook and the capacity of the airplane.

Q. 3 A certain town has a Saturday night movie audience of 600 who must choose between two comparable movie theaters. Assume that the movie-going public is composed of 300 couples, and each couple independently flips a fair coin to decide which theatre to patronize.

1. Using the central limit theorem approximation, determine the minimum number of seats each theater must have so that the proability of exactly one theater running out of seats is less than 0.1.
2. Repeat, assuming that each of the 600 customers makes an independent decision (instead of acting in pairs).

Q. 4 To maintain a speed of v miles/hour in the presence of a headwind of speed w mi/hr, a cyclist must generate a power output $y = 50 + (v + w - 15)^3$ Watts. During each mile of road, the wind speed W is a continuous uniform (0,10) random variable that is independent of the wind speed in any other mile.

1. Lance the cyclist rides at constant velocity v = 15 mi/hr mile after mile. Let Y denote Lance's power output over a randomly chosen mile. What is Lance's average power output $E[Y]$?
2. What is the PDF $f_Y(y)$?
3. Mile after mile, another cyclist Ashwin rides at a constant power $\hat{y}$ Watts in the presence of the same variable headwinds. Let $\hat{V}$ denote Ashwin's velocity over a randomly chosen mile. Ashwin chooses $\hat{y}$ so that $E[\hat{V}]$ = 16 mi/hr. What is $\hat{y}$?
4. Ashwin and Lance race across America (a 3000 mile race). Use the central limit theorem to estimate the probability $P[A]$ that Ashwin wins.

Q. 5 In a market survey, a company wishes to estimate the fraction of respondents who favor its product A. The poll is to be conducted with confidence level of 0.95, and the margin of error of 4 percent. Assuming that surveying a respondent costs \$0.5, how large should the company's budget be in order for the survey to meet the above criterion?
Hint:

$$\text{Consider: } P(|M_n - a| < e) > c$$

e is the margin of error, and c is the confidence level.

Q. 6 An airline burns X gallons of jet fuel for each mile it travels. X is random, as it depends on atmospheric conditions, flight altitude, and wind in the upper atmosphere. We would like to define the efficiency of the plane in terms miles per gallon (MPG). Since the plane travels 1 mile using X gallons of fuel, we can measures MPG as

$$\eta = E[\frac{1}{X}] \text{ or } \eta' = \frac{1}{E[X]}.$$

Since both η and η' have the units of miles per gallon, which is the better measure of efficiency? Hint: Suppose $X_1, \ldots, X_m$ are iid where X_i is the fuel consumed for mile i of an m mile trip. What is your trip MPG? What does the law of large numbers say?

Q. 7 We have two boxes, each containing three balls: one black and two white in box 1; two black and one white in box 2. We choose one of the boxes at random, where the probability of choosing box 1 is equal to some given p, and then draw a ball.

1. Describe the MAP rule for deciding the identity of the box based on whether the drawn ball is black or white.
2. Assuming that $p = \frac{1}{2}$, find the probability of an incorrect decision and compare it with the probability of error if no ball had been drawn.