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Topics: Functions of Random Variables, Distribution of sum of RVs, Covariance and Correlation

Q. 1 X and Y are two independent exponential random variables with parameters λ_1 and λ_2 respectively. Let Z be a random variable defined as $Z = \max[X,Y]$ then find the distribution of Z.

Q. 2 Let X be a continuous random variable with CDF $F_X(x)$ which is strictly increasing and Y be a derived random variable defined as $Y = F_X(X)$. Find the distribution of Y.
Note that CDF of any given random variable is just another function.

Q. 3 Let X be a discrete random variable with PMF p_X and let Y be a continuous random variable, independent from X, with PDF f_Y . Derive a formula for the PDF of the random variable $X+Y$.

Q. 4 A random variable X has mean 0 and variance 1. A random variable Y has mean 1 and variance 2, and X and Y are independent. Suppose random variables U and V are given by $U = X+2Y$, $V = X-Y$. What is the covariance $\mathrm{Cov}(U,V)$? What is the correlation coefficient $\rho_{U,V}$? Can U and V be independent?

Q. 5 Consider n independent tosses of a k-sided fair die. Let X_i be the number of tosses that result in outcome i. Find Cov(X_1 , X_2) of X_1 and X_2.
Hint : Define a bernoulli random variable A_t that is equal to 1 if and only if the t^{th} toss resulted in 1. Note that $X_1 = A_1 + A_2 + \ldots + A_n$.

Q. 6 Understanding variance and risk. Let c be an integer where $c > 1$. We are given c *independent* random variables, denoted $X_1,\ldots,X_c$. and c *positively correlated* random variables, denoted by $Y_1,\ldots,Y_c$ such that correlation between any two random variables Y_i and Y_j is a constant value $\rho < 1$. All the given random variables have same mean and variance.

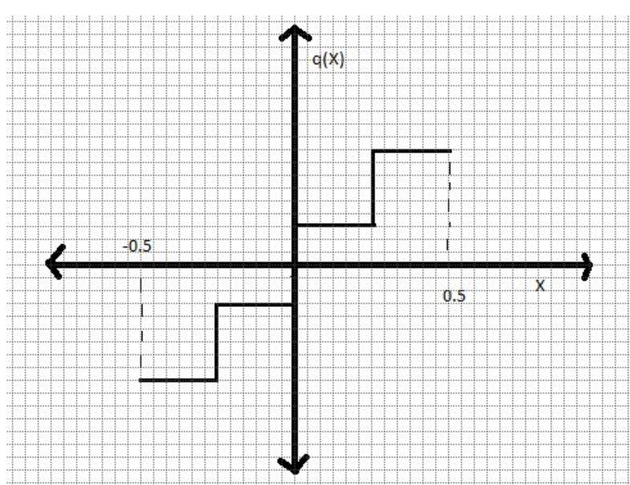
1. Which of the following has a lower variance :

$$\mathrm{Var}(X_1 + X_2 \cdots + X_c)\ ,\ \mathrm{Var}(Y_1 + Y_2 \cdots + Y_c) \text{ and } \mathrm{Var}(cX_1)?$$

   Compute each of these to relate them.

2. Suppose each random variable denotes an investment in a given stock. Consider the following cases, investing c times in a single stock, investing in c independent stocks and investing in c positively correlated stocks. Which investement is invovled with less risk? Explain your answer using the result to the previous question.

Q. 7 Let X be a random variable uniformly distributed between -0.5 and 0.5 and q be a uniform quantizer function with resolution of n bits. If $n = 2$, the quantizer function is shown below.



Let $Y = q(X)$ be a derived random variable. Random variable Y takes four values for $n = 2$, where each value will be associated with two bits. The quantized values random variable Y takes are the middle values of the uniform intervals over which the things are quantized i.e., all values between 0.5 and 0.25 for $n = 2$ are mapped to the middle value which is 0.375.

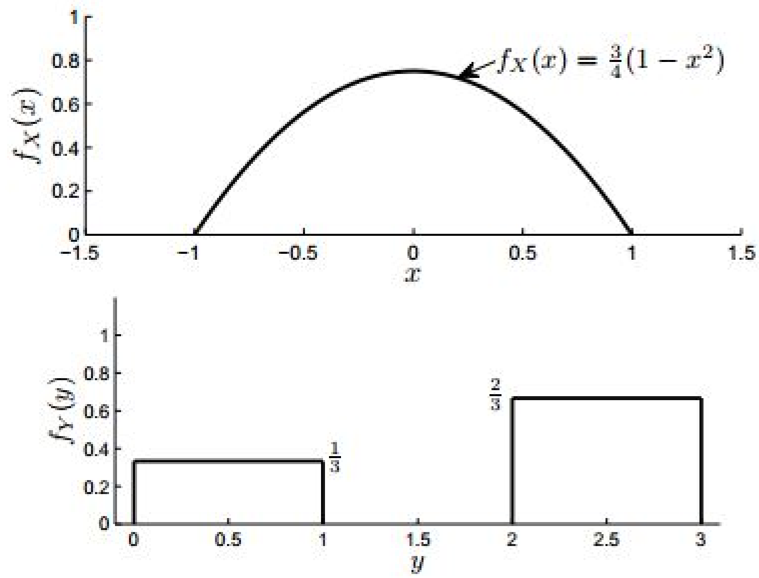
SNR is used as a measure of quality for the quantizer and is defined as

$$SNR = \frac{\sigma_X^2}{E[(X - q(X))^2]}$$

Find SNR for n=2. SNR in db is defined as $SNR_{db} = 10\log_{10} SNR$ .What happens to SNR_{db} as number of bits, n increases?

Extra Practice Problems

Q. 8 Let X and Y be two independent random variables. Their probability density functions are shown below.



Let $Z = X + Y$. Find $f_Z(z)$.

Q. 9 Romeo and Juliet have a date at a given time, and each, independently, will be late by amounts of time, X and Y , respectively, that are exponentially distributed with parameter λ. Find the PDF of $Z = X - Y$.