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Topics: discrete random variables, functions of, expectation and variance

Q. 1 The discrete random variable K has the following PMF:

$$p_K(k) = \begin{cases} b & k=0 \\ 2b & k=1 \\ 3b & k=2 \\ 0 & \text{otherwise} \end{cases}$$

1. Find b.
2. Determine the values of $P(K<2)$, $P(K \leq 2)$, and $P(0<K<2)$
3. Compute $E[K]$ and $\mathrm{Var}(K)$.
4. Let $Y = \frac{1}{1+K}$, find the PMF of Y.
5. Compute $E[Y]$ and $E[Y^2]$.

Sol:

1. $b+2b+3b=1$, $b=1/6$
2. $P(K<2)=\frac{1}{2}$, $P(K\leq 2)=1$, and $P(0<K<2)=\frac{1}{3}$
3. $E[K]=\frac{1}{6}(0)+\frac{1}{3}(1)+\frac{1}{2}(2)=\frac{4}{3}$

$$\mathrm{Var}(K)=\frac{1}{6}(0-\frac{4}{3})^2+\frac{1}{3}(1-\frac{4}{3})^2+\frac{1}{2}(2-\frac{4}{3})^2=\frac{4}{3}$$

Q. 2 The number of N calls arriving at a switchboard during a period of one hour has the PMF

$$p_N(n)=\frac{10^n e^{-10}}{n!} \qquad n=0,1,\ldots$$

1. What is the probability that at least two calls arrive within one hour?
2. What is the probability that at most three calls arrive within one hour?
3. What is the probability that the number of calls that arrive within one hour is greater than three but less than or equal to six?

Sol:

1. $P(n>2)=1-(\frac{10^0e^{-10}}{0!}+\frac{10^1e^{-10}}{1!})=.9995$
2. $P(n\leq 3)=\frac{10^0e^{-10}}{0!}+\frac{10^1e^{-10}}{1!}+\frac{10^2e^{-10}}{2!}+\frac{10^3e^{-10}}{3!}=.0103$
3. $P(3<n\leq 6)=\frac{10^4e^{-10}}{4!}+\frac{10^5e^{-10}}{5!}+\frac{10^6e^{-10}}{6!}=.1198$

Q. 3 A man claims to have extrasensory perception. As a test, a fair coin is flipped 10 times, and the man is asked to predict the outcome in advance. He gets 7 out of the 10 correct. What is the probability that he would have done at least this well if he had no ESP?

Sol:

Let X be the number of correct predictions that are made. If the man had no ESP, the probability of a success in each trial would be 0.5 . Since the trials are independent, $X \sim$ Binomial(10,0.5) .

$$P(X\geq 7)=\sum_{i=7}^{10}\binom{10}{i}(0.5)^{10}=0.172$$

Q. 4 St. Petersburg paradox. You toss independently a fair coin and you count the number of tosses until the first tail appears. If this number is n, you receive 2^n dollars. What is the expected amount that you will receive? How much would you be willing to pay to play this game?

Sol: The expected value of the gain for a single game is infinite since if X is your gain, then

$$E[X] = \sum_{k=1}^{\infty} 2^k * 2^{-k} = \sum_{k=1}^{\infty} 1 = \infty.$$

Thus if you are faced with the choice of playing for given fee f or not playing at all, and your objective is to make the choice that maximizes your expected net gain, you would be willing to pay any value of f. However, this is in strong disagreement with the behavior of individuals. In fact experiments have shown that most people are willing to pay only about 20 dollars to 30 dollars to play the game. The discrepancy is due to a presumption that the amount one is willing to pay is determined by the expected gain. However, expected gain does not take into account a persons attitude towards risk taking.

Q. 5 (Resource dimensioning under uncertainty) An Internet service provider has m modems with which it serves n customers. Customers are not always active, so m need not be greater than or equal to n. In fact, customers are only active with probability $p = 0.05$, and their activity is mutually independent. Let X be a random variable denoting the total load on the system at a typical time, i.e., the total number of active customers in the system. Let q denote the probability that the system is able to meet the current load, i.e., q is a measure of the quality of service. If it is high users are more likely to be blocked. If it is low most users are likely to be able to access a modem. In this problem you will consider three types of problems associated with this types of scenario:

1. Suppose the Internet service provider has $n = 10$ customers and $m = 4$ modems. what is the quality of service q it can deliver to its customers?

2. Suppose the Internet service provider has $n = 20$ customers, how many modems m does it need to ensure that the quality of service is at least $q = 0.9$.

3. Suppose the Internet service provider has $m = 2$ modems, what is the maximum number of customers n that it can support while ensuring a quality of service of at least $q = 0.2$.

Sol:

1. $P(X > 4) = \sum_{i=5}^{10} \binom{10}{i} (0.05)^i (0.95)^{10-i} \leq q$

2. $Solve : P(X > m) = \sum_{i=m+1}^{20} \binom{20}{i} (0.05)^i (0.95)^{20-i} =\leq .9$

3. $Solve : P(X > 2) = \sum_{i=3}^{n} \binom{n}{i} (0.05)^i (0.95)^{n-i} =\leq .2$

Q. 6 (Zipf's Law) Read the brief article on Zipf's Law which is posted in the course materials section in the Discrete RVs folder. Zipf's Law can be "intuitively" stated as follows: Suppose we have a countable collection of items and we label them $x = 1, 2, \ldots.$ based on their rank in terms of their popularity, i.e., item x is the xth most popular item. Under Zipf's Law with parameter $\alpha > 1$ the probability that the ith item is chosen/appears/is requested is **inversely proportional to its rank to the power** α. In other words if X denotes the rank of a random item following Zipf's law with parameter α, then

$$p_X(x) \propto \frac{1}{x^\alpha} \quad \text{for} \quad x = 1, 2, 3 \ldots.$$

This can be used to model the popularity of web sites, movies, songs on iTunes, uses of letters in the alphabet etc.

1. Find an expression in terms of α for the proportionality constant making p_X a pmf when $\alpha = 2$. Note: In general can show that for $\alpha \in [0, 1]$ the proportionality constant does not exist and thus there is no pmf.

2. What is the expected value of X when $\alpha = 2$.

3. Suppose there are a finite number of items, so $x = 1, 2, \ldots, n$ and now let $\alpha = 1$. Find an expression, in terms of n for the proportionality constant for the pmf.

4. How many times more likely is the most popular item versus the xth most popular item. For some large but finite n compare this to the case where the popularity profile is given by a geometric distribution. Which distribution concentrates more probability on the higher ranked, i.e., lower x, values.

Sol:

1. Suppose $p_X(x) = \frac{k}{x^\alpha}$ for $x = 1, 2, 3....$, then to make p_X a pmf, we need $\sum_{x=1}^{\infty} p_X(x) = 1$. This gives us, $k \cdot \sum_{x=1}^{\infty} \frac{1}{x^\alpha} = 1$. If $\sum_{x=1}^{\infty} \frac{1}{x^\alpha}$ is bounded, then $k = \frac{1}{\sum_{x=1}^{\infty} \frac{1}{x^\alpha}}$. For $\alpha = 2$, we have that $\sum_{x=1}^{\infty} \frac{1}{x^\alpha} = \frac{\pi^2}{6}$, and $k = \frac{6}{\pi^2}$.

   For general case, $\frac{1}{x^\alpha} > \int_x^{x+1} \frac{1}{x^\alpha}$ for $\alpha > 0$, and $x = 1, 2, 3.....$ Using this inequity, we have $\sum_{x=1}^{\infty} \frac{1}{x^\alpha} > \int_1^{\infty} \frac{1}{x^\alpha}$. Using this, you could prove that for $\alpha \in [0, 1]$, $\sum_{x=1}^{\infty} \frac{1}{x^\alpha}$ goes to infinity thus the proportional constant does not exist.

   For $\alpha > 1$, $\frac{1}{x^\alpha} < \int_{x-1}^{x} \frac{1}{x^\alpha}$, thus $\sum_{x=1}^{\infty} \frac{1}{x^\alpha} < \int_0^{\infty} \frac{1}{x^\alpha}$.

   $\int_0^{\infty} \frac{1}{x^\alpha}$ is bounded for $\alpha > 1$, thus the proportionality constant exists and this could be a pmf.

2. $E[X] = \sum_{x=1}^{\infty} x \cdot \frac{1}{x^2} = \sum_{x=1}^{\infty} \frac{1}{x}$ This value would goes to infinity as n goes to infinity.

3. Similar to question (1), $k = \frac{1}{\sum_{x=1}^{n} \frac{1}{x}}$.

4. The ratio between the most popular item and the xth most popular item is $\frac{\frac{k}{1^\alpha}}{\frac{k}{x^\alpha}} = x^\alpha$. For a geometric distribution with parameter p, this ratio becomes $\frac{p}{(1-p)^{x-1}} = (\frac{1}{1-p})^{x-1}$.

   To compare which one of the two ratios is larger, we could take log of the two ratios first and compare them. The log of the ratio for Zipf's law is $\alpha \cdot \log x$, and the log of ratio for geometric distribution is $(x-1) \cdot \log(\frac{1}{1-p})$. We know that x grows faster than $\log x$, so geometry distribution concentrates more probability on the higher ranked.

Q. 7 (Jensen's Inequality) I will show in class that, for a function $g : \mathbb{R} \to \mathbb{R}$, in general $E[g(X)] \neq g(E[X])$ for a random variable X. We did note that if g was a linear function, however, this is indeed true. In this problem we consider this further in the case of convex functions. A function is **convex** if for all $x, y \in \mathbb{R}$ and for any $\lambda \in [0, 1]$ we have that

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y).$$

If $-g$ is convex, then we say that g is **concave**.

1. Use the above definition to draw a representative convex and concave functions.

2. Suppose X is a discrete random variable that takes two values only and g is convex. Show that $E[g(X)] \geq g(E[X])$. It follows that if g is concave then $E[g(X)] \leq g(E[X])$.

3. This result is true in general, i.e., for any kind of random variable. To convince yourself, can you extend this result to the case X a discrete random variables that takes three values? Hint: you could perhaps condition...

Sol:

1. Typical picture for convex function: Picture for concave function: The intuition behind a convex function is that, we can think of $(x, g(x))$ and $(y, g(y))$ as two points. The line segment between $(x, g(x))$ and $(y, g(y))$ lies above the figure of $g(\cdot)$.

2. Suppose X could take two values, x_1 with probability p and x_2 with probability $1-p$. (I'll use w/p for "with probability").

$$E[g(X)] = p \cdot g(x_1) + (1-p) \cdot g(x_2) \geq g(p \cdot x_1 + (1-p) \cdot x_2) = g(E[X])$$

   For $g(\cdot)$ is concave, we just need to change $\geq$ to $\leq$.

3. It's similar to the question above. Suppose the three values X can take are, x_1 w/p p_1, x_2 w/p p_2 and x_3 w/p p_3. $p_1 + p_2 + p_3 = 1$. Notice that, we make no assumption of which one of x_1, x_2, x_3 is larger here.

$$g(\frac{p_1}{p_1+p_2}x_1 + \frac{p_2}{p_1+p_2}x_2) \leq \frac{p_1}{p_1+p_2}g(x_1) + \frac{p_2}{p_1+p_2}g(x_2)$$

$$
\begin{aligned}
g(E[X]) &= g(p_1 \cdot x_1 + p_2 \cdot x_2 + p_3 \cdot x_3) \\
&= g((p_1+p_2) \cdot (\frac{p_1}{p_1+p_2} x_1 + \frac{p_2}{p_1+p_2} x_2) + p_3 \cdot x_3) \\
&\leq (p_1+p_2) \cdot g(\frac{p_1}{p_1+p_2} x_1 + \frac{p_2}{p_1+p_2} x_2)) + p_3 \cdot g(x_3) \\
&\leq p_1 \cdot g(x_1) + p_2 \cdot g(x_2) + p_2 \cdot g(x_2) \\
&= E[g(X)]
\end{aligned}
$$

Q. 8 (Utility functions and risk averse versus seeking decision-making) A utility function u of some some quantity say x is sometimes used to model how happy a person is with getting x. In the case where the allocation to is a random variable can interpret $E[u(X)]$ as the average utility the person will see for an allocation which is modeled by X. Given two random variables X and Y if $E[u(X)] \geq E[u(Y)]$ we say that the person prefers the uncertainty associated with X to Y. If a persons utility function is concave we say the person is *risk averse*. Indeed recall that if u is a concave function then

$$E[u(X)] \leq u(E[X])$$

i.e., the person prefers to receive the average value vs the expected value of the utility. If it is convex then we call the person risk seeking, i.e.

$$E[u(X)] \geq u(E[X])$$

they prefer a scenario with uncertainty versus receiving the expected value.

1. Suppose you are given a choice between two 'bets':
   - You win \$100 with probability 0.9 or \$ 0 with probability 0.1.
   - You win \$90 for sure.

   Which one do you prefer? Are you risk seeking or risk averse?

2. Suppose you are given a choice between two 'bets':
   - You lose \$100 with probability 0.9 or \$ 0 with probability 0.1.
   - You lose \$90 for sure.

   Now which one do you prefer? Are you risk seeking or risk averse?

3. **Something to think about.** Most people will give different answers to Parts 1 and 2 above. Which means that they "view" uncertainty in earnings differently than in losses. Think about some time in your life where you had to make such decisions, this could be associated with insurance, buying lottery tickets, etc..

Sol: This is a open question. We want to give you a brief understanding of risk seeking and risk averse.
Generally, one is said to be risk seeking if he has a "convex function" $g(\cdot)$ to evaluate his gain. For example, in the first sub-question of Q5, if you have a convex function for your gain, the expectation of gain for \$100 and \$0 case is higher than the gain of \$90 case, and you would choose the first bet. You tend to earn more while taking more risks, thus you are risk seeking. If you have a "concave function", you may choose the second bet and avoid risk.
Similar argument works for the "losing money" case.