M368K
Homework #12

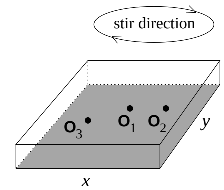
Burden and Faires.

Section 12.2 (#6b[1], 8b[1], 10b[1]).

[1]Use $\Delta x = \frac{1}{4}$ and $\Delta t = \frac{1}{10}$ instead of the given spacings. Explicitly write the discrete equations. Compute up to $t = \frac{2}{10}$; compare approx and exact solns at each node at final time. There is a typo: $\cos \pi(x - \frac{1}{2})$ should be $\cos[\pi(x - \frac{1}{2})]$.

Programming mini-project.

A research group is studying how the distribution of nutrients changes with time in a thin, aquatic incubation chamber. The nutrients are supplied at location O_1, transported by diffusion and advection in slowly stirred water, and consumed at locations O_2 and O_3. The concentration $u(x, y, t)$ of the nutrients is modeled by



$$
\begin{array}{llll}
u_t = \alpha \Delta u - v \cdot \nabla u + \eta, & & a \le x \le b, \quad c \le y \le d, & 0 \le t \le T \\
u(a, y, t) = 1, & u(b, y, t) = 1, & c \le y \le d, \quad 0 \le t \le T & \\
u(x, c, t) = 1, & u(x, d, t) = 1, & a \le x \le b, \quad 0 \le t \le T & \\
u(x, y, 0) = 1, & & a \le x \le b, \quad c \le y \le d &
\end{array}
$$

where $\Delta u = u_{xx} + u_{yy}$ is the Laplacian of u, $\nabla u = (u_x, u_y)$ is the gradient of u, α is a diffusion coefficient, $v = (v_1, v_2)$ is the water velocity and η is the load function which represents the nutrient source and sinks. Uniform boundary and initial conditions are assumed. For concreteness, we take $[a, b] = [c, d] = [-1, 1]$, $\alpha \equiv 0.03$, $v_1 = y$, $v_2 = -x$ and $\eta = \sum_{i=1}^{3} \gamma_i e^{-30(x-\alpha_i)^2 - 30(y-\beta_i)^2}$ where $\{\alpha_1, \beta_1, \gamma_1\} = \{0.2, 0.2, 0.6\}$, $\{\alpha_2, \beta_2, \gamma_2\} = \{0.6, 0.2, -0.7\}$ and $\{\alpha_3, \beta_3, \gamma_3\} = \{-0.5, 0, -0.6\}$. Below we use Δx, Δy and Δt to denote grid parameters.

(a) Download the C++ files `program12.cpp` and `fwddiff2D.cpp` from the course webpage. Complete the file `fwddiff2D.cpp` so that it implements the forward-difference method for initial-boundary-value problems of the type $u_t = P u_{xx} + Q u_{yy} + p u_x + q u_y + r u + \eta$ on rectangular domains. As a check, your code should be able to reproduce the constant solution $u \equiv 1$ in the case when $\eta \equiv 0$ in the above problem.

(b) Solve the IBVP on the time interval $[0, T] = [0, 5]$ using the following sets of decreasing grid parameters. Record the value of u at $(x, y, t) = (0.6, 0.6, 5)$ for each set. [As a check, the first set should give $u = 0.95370$.] What happens as the grid is refined? Does the approximation at $(0.6, 0.6, 5)$ appear to be converging to a definite value? Briefly explain why or why not.

Δx	Δy	Δt
$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{20}$
$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{25}$
$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{30}$

(c) Using a grid with $\Delta x = \Delta y = \frac{1}{15}$ and a value of Δt at least 50% below the stability threshold, solve the IBVP on the intervals $[0, T] = [0, 0.5]$, $[0, 3]$ $[0, 10]$, $[0, 20]$ and $[0, 40]$. At each final time, compute the average concentration u_{avg} over the entire xy-grid. Does u_{avg} appear to approach a steady-state value as time increases? Is the nutrient supply sufficient to maintain $u_{\text{avg}} \ge 0.90$ for all times considered? Use the supplementary file `matlab12.m` to produce contour maps of the nutrient distribution at times $t = 0.5$, 3 and 40. Submit these maps along with your findings about u_{avg}.

Turn in: versions of `fwddiff2D.cpp` and `program12.cpp` used for this problem and responses to (b),(c).