Solvability Theorem for BVPs Suppose f in the BVP

$$y'' = f(x, y, y'),\ \text{for } a \leqslant x \leqslant b \text{ with } y(a) = \alpha \text{ and } y(b) = \beta$$

is continuous on the set

$$D = \{(x, y, y') \mid a \leqslant x \leqslant b, -\infty < y < \infty, -\infty < y' < \infty\}$$

and that the partials f_y and $f_{y'}$ are also continuous on D. If

- $f_y(x, y, y') > 0, \quad \forall (x, y, y') \in D$
- $\exists M > 0 \in \mathbb{R}$ s.t. $|f_{y'}(x, y, y')| \leqslant M \quad \forall (x, y, y') \in D$

Then the BVP has a unique solution.

Linear Shooting Method For general BVPs in the form

$$f(x, y, y') = y'' = p(x)y' + q(x)y + r(x)$$

for $a \leqslant x \leqslant b, \quad y(a) = \alpha, \quad y(b) = \beta$ which satisfies the simplified Solvability Theorem for BVPs:

- $p(x), q(x), r(x)$ are continuous on $[a, b]$
- $q(x) > 0$ on $[a, b]$

Then the BVP has a unique solution. We can approximate this solution by first solving the two IVPs for y_1 and y_2 of

$$\begin{aligned} y_1'' &= p(x)y_1' + q(x)y_1 + r(x), & y_1(a) &= \alpha, \quad y_1'(a) = 0 \\ y_2'' &= p(x)y_2' + q(x)y_2, & y_2(a) &= 0, \quad y_2'(a) = 1 \end{aligned}$$

using any IVP solver and then define the solution as

$$y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2(x)$$

Check: Plug y_1'' and y_2'' in above and check the BVP conditions.

Nonlinear Shooting Method Similar to the linear technique, except we approximate the solution to the boundary value problem using the solutions to a sequence of IVPs involving t in the form

$$y'' = p(x)y' + q(x)y + r(x), \qquad y(a) = \alpha, \quad y'(a) = t_k$$

ensuring that $\lim_{k\to\infty} y(b, t_k) = y(b) = \beta$. Use either Secant-Euler or Secant-RK4 to solve within each of M subintervals for the solution until $y^{(N)} - \beta < \epsilon$

Nonlinear Shooting with Secant-Euler Given number of iterations N, number of subintervals M, initial t_0, t_1 and $h = \frac{b-a}{M}$

$$F(t_k) = y_{t_k}^{(N)}(b) - \beta, \quad M_k = \frac{F(t_k) - F(t_{k-1})}{t_k - t_{k-1}}$$

$$\mathbf{w}(x) = \begin{pmatrix} y \\ y'' \end{pmatrix}, \quad V(x, y) = \begin{pmatrix} y' \\ y'' \end{pmatrix}, \quad \mathbf{w}' = V(x, \mathbf{w})$$

$k = 1, \ldots, N$ or $F(t_k) < \epsilon$

$$\begin{cases} t_{k>2} = t_{k-1} - \frac{F(t_{k-1})}{M_{k-1}} \\ \mathbf{w}^{(0)} = \left[\begin{smallmatrix} 0 \\ t_k \end{smallmatrix}\right] \\ \mathbf{w}^{(k)} = \mathbf{w}^{(k-1)} - hV(x_k, y^{(k-1)}) \end{cases}$$

Nonlinear Shooting with Newton Iteration $k = 1, \ldots, N$ or $F(t_k) < \epsilon$

$$\begin{cases} t_{k>1} = t_{k-1} - \frac{F(t_{k-1})}{\frac{\mathrm{d}}{\mathrm{d}t} y^{(k)}(b, t_{k-1})} \\ \mathbf{w}^{(0)} = \left[\begin{smallmatrix} 0 \\ t_k \end{smallmatrix}\right] \\ \mathbf{w}^{(k)} = \mathbf{w}^{(k-1)} - hV(x_k, y^{(k-1)}) \end{cases}$$

Secant/Forward Euler's Method Is a first-order version of Runge-Kutta. For a mesh size n and initial guess $\mathbf{x}^{(0)}$

$$b = -\frac{1}{n} F(x)$$

$$\begin{cases} A = J(\mathbf{x}^{(k-1)}) \\ \mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} + A^{-1} b \end{cases}$$

Shooting Method Convergence Theorem Assume:

- The conditions of the BVP Solvability Theorem are met
- $f(t), f'(t), f''(t)$ are continuous in the neighborhood of t
- $f'(t) \neq 0$

Then

- $t_k \to t_{*,N}$ as $k \to \infty$ provided that the initial guess was close enough ($|t_0 - t_*| < r, |t_1 - t_*| < r$) and the IVP grid is fine enough ($N \geqslant R$).
- $t_{*,N} \to t_*$ and $y_{t_{*,N}}^{(j)} \to y(x_j)$ as $N \to \infty$

$$|t_{*,N} - t_*| \leqslant Ch^p$$

$$\max_{0 \leqslant j \leqslant N} |y_{t_{*,N}} - y(x_j)| \leqslant Ch^p$$

where $p = 1$ for Secant-Euler or Newton-Euler and $p = 4$ for Runge-Kutta-4.

Runge-Kutta For n iterations and initial guess $\mathbf{x}^{(0)}$. Does not require a good guess of $\mathbf{x}^{(0)}$ and converges quickly, though for RK4, it requires for linear systems to be solved when computing the $\mathbf{k}$ values, so n steps requires solving $4n$ linear systems. One step may be enough to get an accurate solution.

The generalized $\mathbf{k}_i$ values, where α_i are weights and $h = \frac{1}{n}$

$$\mathbf{k}_i = -h[J(\mathbf{x}^{(k)}) + \alpha_{i-1}\mathbf{k}_{i-1})]^{-1} F(\mathbf{x}^{(0)})$$

For the fourth-order Runge-Kutta problem ($n = 4$), the $\mathbf{k}_i$ values are as below, where $\alpha = (0, \frac{1}{2}, \frac{1}{2}, 1)$.

$$h = 1/n \quad \mathbf{b} = -hF(\mathbf{x})$$

$$\begin{cases} \mathbf{k}_1 = (J(\mathbf{x}^{(i)}))^{-1}\mathbf{b} \\ \mathbf{k}_2 = (J(\mathbf{x}^{(i)} + \frac{1}{2}\mathbf{k}_1))^{-1}\mathbf{b} \\ \mathbf{k}_3 = (J(\mathbf{x}^{(i)} + \frac{1}{2}\mathbf{k}_2))^{-1}\mathbf{b} \\ \mathbf{k}_4 = (J(\mathbf{x}^{(i)} + \mathbf{k}_3))^{-1}\mathbf{b} \\ \mathbf{x}^{(i+1)} = \mathbf{x}^{(i)} + \frac{1}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4) \end{cases}$$

Centered-Difference Equations for Non/Linear Problems

$$y'(x_i) = \frac{1}{2h}[y(x_{i+1}) - y(x_{i-1})] - \frac{h^2}{6} y'''(\eta_i)$$

$$y''(x_i) = \frac{1}{h^2}[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})] - \frac{h^2}{12} y^{(4)}(\xi_i)$$

$$\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} = p\frac{y(x_{i+1}) - y(x_{i-1})}{2h} + qy(x_i) + r$$

Linear Finite Difference Method For general BVPs, where $a \leqslant x \leqslant b, \quad y(a) = \alpha, \quad y(b) = \beta$

$$f(x, y, y') = y'' = p(x)y' + q(x)y + r(x)$$

The Centered-Difference Equations are rewritten in the form

$$\left(\frac{-w_{i+1} + 2w_i - w_{i-1}}{h^2}\right) + p(x_i)\left(\frac{w_{i+1} - w_{i-1}}{2h}\right) + q(x_i)w_i = -r(x_i)$$

Define $w_0 = \alpha$, $w_{N+1} = \beta$, solve the tridiagonal $N \times N$ matrix $A\mathbf{w} = \mathbf{b}$ for the N interior nodes (subintervals).

$$\begin{cases} A = \begin{pmatrix} 2 + h^2 q(x_1) & -1 + \frac{h}{2}p(x_1) & & 0 \\ -1 - \frac{h}{2}p(x_2) & \ddots & \ddots & \\ & \ddots & \ddots & -1 + \frac{h}{2}p(x_{N-1}) \\ 0 & & -1 - \frac{h}{2}p(x_N) & 2 + h^2 q(x_N) \end{pmatrix}, \\ \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-1} \\ w_N \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -h^2 r(x_1) + \left(1 + \frac{h}{2}p(x_1)\right) w_0 \\ -h^2 r(x_2) \\ \vdots \\ -h^2 r(x_{N-1}) \\ -h^2 r(x_N) + \left(1 + \frac{h}{2}p(x_N)\right) w_{N+1} \end{pmatrix} \end{cases}$$

Linear Finite-Difference Convergence Theorem The tridiagonal linear system has a unique solution provided that $h < 2/L$ where $L = \max_{a \leqslant x \leqslant b} |p(x)|$. Establish $y^{(4)}$ is continuous for truncation error $O(h^2)$.

Nonlinear Finite Difference Method For general BVPs, where $a \leqslant x \leqslant b, \quad y(a) = \alpha, \quad y(b) = \beta$

$$f(x, y, y') = y'' = p(x)y' + q(x)y + r(x)$$

To guarantee a unique solution, assume that f satisfies:

- f and its partials f_y and $f_{y'}$ are continuous on
$$D = \{(x, y, y') \mid a \leqslant x \leqslant b, -\infty < y < \infty, -\infty < y' < \infty\}$$
- $f_y(x, y, y') \geqslant \delta$ on D, for some $\delta > 0$
- Constants k and L exist such that
$$k = \max_{(x,y,y') \in D} |f_y(x, y, y')|, \quad L = \max_{(x,y,y') \in D} |f_{y'}(x, y, y')|$$

The Centered Difference Method uses $w_0 = \alpha$, $w_{N+1} = \beta$:

$$-\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} + f\left(x_i, w_i, \frac{w_{i+1} - w_{i-1}}{2h}\right) = 0$$

for $i = 1, \ldots, N$. Multiply both sides by h^2 and we can use Newton's Method to approximate the solution, so the next iterations of the method will converge to the solution provided that the initial approximation of $w_i^{(0)}$ is close to the solution and that the Jacobian matrix is nonsingular. Solve for $\mathbf{v}$ and accumulate with $\mathbf{w}$ to obtain the solution. The Jacobian matrix is tridiagonal, and the method converges of order $O(h^2)$.

$$\begin{cases} J(w_1, \ldots, w_N)_{i,j}^{(k)} = \\ \begin{cases} -1 + \frac{h}{2} f_{y'}\left(x_i, w_i, \frac{w_{i+1} - wi-1}{2h}\right), & i = j - 1, \quad j = 2, \ldots, N \\ 2 + h^2 f_y\left(x_i, w_i, \frac{w_{i+1} - w_{i-1}}{2h}\right), & i = j \quad j = 1, \ldots, N \\ -1 - \frac{h}{2} f_{y'}\left(x_i, w_i, \frac{w_{i+1} - wi-1}{2h}\right), & i = j - 1, \quad j = 2, \ldots, N \end{cases} \\ J^{(k)}\mathbf{v}^{(k)} = \left((-\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} + f\left(x_i, w_i, \frac{w_{i+1} - w_{i-1}}{2h}\right)\right) \\ \mathbf{w}^{(k)} = \mathbf{w}^{(k-1)} + \mathbf{v}^{(k)} \end{cases}$$

Newton's Method Exhibits quadratic convergence; requires continuous g derivatives and that $A(\mathbf{x}) = J(\mathbf{x})$ is nonsingular around the radius of the solution. Given an initial guess $\mathbf{x}^{(0)}$:

$$\begin{aligned} \mathbf{x}^{(k)} = G(\mathbf{x}^{(k-1)}) &= \mathbf{x}^{(k-1)} - A(\mathbf{x}^{(k-1)})^{-1} F(\mathbf{x}^{(k-1)}) \\ &= \mathbf{x}^{(k-1)} - J(\mathbf{x}^{(k-1)})^{-1} F(\mathbf{x}^{(k-1)}) \end{aligned}$$

A weakness in Newton's is the requirement to compute and invert $J(\mathbf{x})$, avoided by finding a $\mathbf{y}$ such that

$$J(\mathbf{x}^{(k-1)})\mathbf{y} = -F(\mathbf{x}^{(k-1)})$$

Then $\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} + \mathbf{y}$. Newton's Method requires $n^2 + n$ functional evaluations, n^2 for the Jacobian matrix, n for the evaluation of F, and $O(n^3)$ arithmetic operations to solve the linear system.

Piecewise Basis Functions We must first partition the domain such that $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$ for n subintervals. Pay attention to the value of x; remember these equations are piecewise. Remember this especially when integrating.

$$\phi_i(x) = \begin{cases} 0, & a \leqslant x \leqslant x_{i-1} \\ \frac{1}{h_{i-1}}(x - x_{i-1}) & x_{i-1} < x \leqslant x_i \\ \frac{1}{h_i}(x - x_{i+1}) & x_i < x \leqslant x_{i+1} \\ 0, & x_{i+1} < x \leqslant b \end{cases}$$

$$\phi_i'(x) = \begin{cases} 0, & a \leqslant x \leqslant x_{i-1} \\ \frac{1}{h_{i-1}} & x_{i-1} < x \leqslant x_i \\ -\frac{1}{h_i} & x_i < x \leqslant x_{i+1} \\ 0, & x_{i+1} < x \leqslant b \end{cases}$$

Piecewise Linear Finite Element Method The matrix A is symmetric, tridiagonal, and positive-definite (given $p(x) \geqslant p_{min} > 0$, so the linear system is stable with respect to roundoff error. We have $|\phi(x) - y(x)| = O(h^2)$ convergence for each x in $[a, b]$ due to the first-degree interpolating polynomial $y^*(x)$.

$$\begin{cases} a_{ij} = \int_a^b [p(x)\phi_i'(x)\phi_j'(x) + q(x)\phi_i'(x)\phi_j'(x)]\mathrm{d}x \\ b_i = \int_a^b f(x)\phi_i(x)\mathrm{d}x \\ A\mathbf{c} = \mathbf{b} \\ y^*(x) = \sum_{i=1}^n c_i\phi_i(x) \end{cases}$$

or

$$\begin{cases} Q_{1,i} = \left(\frac{1}{h_i}\right)^2 \int_{x_i}^{x_{i+1}} (x_{i+1} - x)(x - x_i)q(x)\mathrm{d}x, & i = 1, \ldots, n-1 \\ Q_{2,i} = \left(\frac{1}{h_{i-1}}\right)^2 \int_{x_{i-1}}^{x_i} (x - x_{i-1})^2 q(x)\mathrm{d}x, & i = 1, \ldots, n \\ Q_{3,i} = \left(\frac{1}{h_i}\right)^2 \int_{x_i}^{x_{i+1}} (x_{i+1} - x)^2 q(x)\mathrm{d}x, & i = 1, \ldots, n \\ Q_{4,i} = \left(\frac{1}{h_{i-1}}\right)^2 \int_{x_{i-1}}^{x_i} p(x)\mathrm{d}x, & i = 1, \ldots, n+1 \\ Q_{5,i} = \frac{1}{h_{i-1}} \int_{x_{i-1}}^{x_i} (x - x_{i-1})f(x)\mathrm{d}x, & i = 1, \ldots, n \\ Q_{6,i} = \frac{1}{h_i} \int_{x_i}^{x_{i+1}} (x_{i+1} - x)f(x)\mathrm{d}x, & i = 1, \ldots, n \\ a_{i,i} = Q_{4,i} + Q_{4,i+1} + Q_{2,i} + Q_{3,i}, & i = 1, \ldots, n \\ a_{i,i+1} = -Q_{4,i+1} + Q_{1,i}, & i = 1, \ldots, n-1 \\ a_{i,i-1} = -Q_{4,i} + Q_{1,i-1}, & i = 2, \ldots, n \\ b_i = Q_{5,i} + Q_{6,i}, & i = 1, \ldots, n \\ A\mathbf{c} = \mathbf{b} \\ y^*(x) = \sum_{i=1}^n c_i\phi_i(x) \end{cases}$$

Poisson Equation Finite-Difference Method In general form:

$$\nabla^2 u(x, y) \equiv \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$
$$R = \{(x, y) \mid a < x < b, \ c < y < d\}$$
$$u(x, y) = g(x, y), \quad (x, y) \in S$$

S is a boundary of R. If f and g are continuous then there is a unique solution. We first choose a grid of step sizes $h = (b-a)/n$ and $k = (d - c)/m$; n and m are the number of steps for the x and y axis. $x_i = a + ih$ and $y_i = c + ik$ are gridlines whose intersections form mesh points. The Centered-Difference Formulas:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(x_i, y_j) &= \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, y_j) \\ \frac{\partial^2 u}{\partial y^2}(x_i, y_j) &= \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{k^2} - \frac{k^2}{12}\frac{\partial^4 u}{\partial y^4}(x_i, \eta_j) \end{aligned}$$

$\xi_i \in (x_{i-1}, x_{i+1})$; $\eta_j \in (y_{j-1}, y_{j+1})$. Difference-Equation form:

$$2\left[\left(\frac{h}{k}\right)^2 + 1\right] w_{i,j} - (w_{i+1,j} + w_{i-1,j}) - \left(\frac{h}{k}\right)^2 (w_{i,j+1} + w_{i,j-1}) = -h^2 f(x_i, y_j)$$

$\lambda = \left(\frac{h}{k}\right)^2$. Label the grid points $P_l = (x_i, y_j)$ and $w_l = w_{i,j}$, where $l = i + (m - 1 - j)(n - 1)$ for $i = 1, \ldots, n-1$ and $j = 1, \ldots, m-1$ to obtain equations at each point P_i. Set the RHS of those equations to the adjacent boundary conditions. Put the P_i equations in a matrix and solve for w_i. Gauss-Seidel is used to solve the matrix: it is Symmetric Positive-Definite and diagonally dominant (Symmetric Block Tridiagonal).

Poisson Equation Solving Methods Iterative methods should be used for large systems, specifically SOR. Otherwise direct techniques usually work better (less roundoff).

Forward Difference Method For Parabolic (heat/diffusion) PDEs:

$$\frac{\partial u}{\partial t}(x,t) = \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t), \qquad 0 < x < l, \quad t > 0$$
$$u(0,t) = u(l,t) = 0, \quad t > 0; \quad u(x,0) = f(x), \quad 0 \leqslant x \leqslant l$$

The discrete equation is

$$w_{i,j+1} = (1-2\lambda)w_{i,j} + \lambda(w_{i+1,j} + w_{i-1,j})$$

Where $w_{i,0} = f(x_i)$. The system can be written in matrix form:

$$A = \begin{pmatrix} 1-2\lambda & \lambda & & 0 \\ \lambda & \ddots & \ddots & \\ & \ddots & \ddots & \lambda \\ 0 & & \lambda & 1-2\lambda \end{pmatrix}$$

where $\lambda = k\left(\frac{\alpha}{h}\right)^2$, $h = \Delta x$, $k = \Delta t$. Using the matrix form, further approximations of $\mathbf{w}$ can be obtained by multiplying:

$$\mathbf{w}^{(j)} = A\mathbf{w}^{(j-1)}$$

The Forward Difference method is conditionally stable: for staibilty, $\rho(A) \leqslant 1$ because of initial error propogation in the explicit nature of the difference method:

$$\mathbf{w}^{(1)} = A(\mathbf{w}^{(0)} + \mathbf{e}^{(0)}) = A\mathbf{w}^{(0)} + A\mathbf{e}^{(0)}$$

Choose h, k such that $\lambda \leqslant \frac{1}{2}$ to ensure $O(k + h^2)$ convergence.

Backward Difference Method Same Parabolic (heat/diffusion) PDE. The discrete equation is

$$w_{i,j-1} = (1+2\lambda)w_{i,j} - \lambda(w_{i+1,j} + w_{i-1,j})$$

Where $w_{i,0} = f(x_i)$. The system can be written in matrix form:

$$A = \begin{pmatrix} 1+2\lambda & -\lambda & & 0 \\ -\lambda & \ddots & \ddots & \\ & \ddots & \ddots & -\lambda \\ 0 & & -\lambda & 1+2\lambda \end{pmatrix}$$

where $\lambda = k\left(\frac{\alpha}{h}\right)^2$ $h = \Delta x$, $k = \Delta t$. Using the matrix form, further approximations of $\mathbf{w}$ can be obtained by solving:

$$\mathbf{w}^{(j-1)} = A\mathbf{w}^{(j)}$$

The Backward Difference method is unconditionally stable: The implicit-difference nature of the Backward Difference method. Since $\lambda > 0$, A is positive-definite, strictly diagonally dominant, and tridiagonal. Since $\rho(A^{-1}) < 1$ (since the eigenvalues of A^{-1} are reciprocals of those from A), $\lim_{n\to\infty}(A^{-1})^n\mathbf{e}^0 = 0$. The rate of convergence and local truncation error is $O(k + h^2)$.

Crank-Nicolson Method The discrete equation, using an averaged-difference method:

$$\frac{w_{i,j+1} - w_{i,j}}{k} = \frac{\alpha^2}{2}\left[\frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} + \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2}\right]$$

In matrix form this is written as

$$A = \begin{pmatrix} (1+\lambda) & -\frac{\lambda}{2} & & 0 \\ -\frac{\lambda}{2} & \ddots & \ddots & \\ & \ddots & \ddots & -\frac{\lambda}{2} \\ 0 & & -\frac{\lambda}{2} & (1+\lambda) \end{pmatrix}$$

$$B = \begin{pmatrix} (1-\lambda) & \frac{\lambda}{2} & & 0 \\ \frac{\lambda}{2} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{\lambda}{2} \\ 0 & & \frac{\lambda}{2} & (1-\lambda) \end{pmatrix}$$

where $\lambda = k\left(\frac{\alpha}{h}\right)^2$, $h = \Delta x$, $k = \Delta t$. Using the matrix form, further approximations of $\mathbf{w}$ can be obtained by solving:

$$A\mathbf{w}^{(j+1)} = B\mathbf{w}^{(j)}$$

Central-Difference Method For Hyperbolic (wave) PDEs:

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t), \qquad 0 < x < l, \qquad t > 0;$$
$$u(0,t) = u(l,t) = 0, \qquad t > 0;$$
$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 \leqslant x \leqslant l$$

The discrete equation is

$$w_{i,j+1} = 2(1-\lambda^2)wi,j + \lambda^2(w_{i+1,j} + w_{i-1,j}) - w_{i,j} - 1$$

where $w_{0,j} = w_{m,j} = 0$, $w_{i,0} = f(x_i)$, and

$$\mathbf{w}_{i,0} = f(x_i)$$
$$\mathbf{w}_{i,1} = (1-\lambda^2)f(x_i) + \frac{\lambda^2}{2}f(x_{i+1}) + \frac{\lambda^2}{2}f(x_{i-1}) + kg(x_i)$$

In matrix form this is written as

$$A = \begin{pmatrix} 2(1-\lambda^2) & \lambda^2 & & 0 \\ -\lambda & \ddots & \ddots & \\ & \ddots & \ddots & -\lambda \\ 0 & & \lambda^2 & 2(1-\lambda^2) \end{pmatrix}$$

where $\lambda = \alpha\frac{k}{h}$, $h = \Delta x$, $k = \Delta t$. Further approximations of $\mathbf{w}$ can be obtained by computing:

$$\mathbf{w}^{(j+1)} = A\mathbf{w}^{(j)} - \mathbf{w}^{(j-1)}$$

Since this method is explicit, it is conditionally stable: we must choose h, k such that $\lambda \leqslant 1$ to ensure $O(h^2 + k^2)$ convergence.

Finite Element Method For the general PDE

$$\frac{\partial}{\partial x}\left(p(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + r(x,y)u(x,y) = f(x,y)$$

with $(x,y) \in D$, where D is a plane region with boundary S and boundary conditions $u(x,y) = g(x,y)$ on a portion of the boundary S_1. On S_2, $u(x,y)$ must satisfy

$$p(x,y)\frac{\partial u}{\partial x}(x,y)\cos\theta_1 + q(x,y)\frac{\partial u}{\partial y}(x,y)\cos\theta_2 + g_1(x,y)u(x,y) = g_2(x,y)$$

First divide the region into triangles $T_1, \ldots, T_M$ such that:

- $T_1, \ldots, T_k$ are triangles with no edges on S_1 or S_2
- $T_{k+1}, \ldots, T_N$ are triangles with at leasst one edge on S_2.
- $T_{N+1}, \ldots, T_M$ are the remaining triangles.
- Label the vertices of the triangle T_i by $(x_1^{(i)}, y_1^{(i)}), (x_2^{(i)}, y_2^{(i)}), (x_3^{(i)}, y_3^{(i)})$.
- Label the nodes (vertices) $E_1, \ldots, E_m$ where $E_1 \ldots, E_n$ are in $D \cup S_2$ and $E_{n+1}, \ldots, E_m$ are on S_1.

We must now find the elements of the matrix $A = (\alpha_{i,j})$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, is in the form

$$\alpha_{i,j} = \int\int_D \left[p\frac{\partial\phi_i}{\partial x}\frac{\partial\phi_j}{\partial x} + q\frac{\partial\phi_i}{\partial y}\frac{\partial\phi_j}{\partial y} - r\phi_i\phi_i\right]\mathrm{d}x\,\mathrm{d}y + \int_{\mathcal{S}_2} g_1\phi_i\phi_j\,\mathrm{d}\mathcal{S}_2$$

and β_i, for $i = 1, \ldots, n$, in the form

$$\beta_i = -\int\int_{\mathcal{D}} f\phi_i\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathcal{S}_2} g_2\phi_i\,\mathrm{d}\mathcal{S} - \sum_{k=n+1}^{m}\alpha_{ik}\gamma_k$$

where $\phi(x,y) = a + bx + cy$ for triangles. The $\phi_j^{(i)}$ equation for triangle i corresponds to the vertex E_j and produces systems

$$\begin{bmatrix} 1 & x_1^{(i)} & y_1^{(i)} \\ 1 & x_2^{(i)} & y_2^{(i)} \\ 1 & x_3^{(i)} & y_3^{(i)} \end{bmatrix}\begin{bmatrix} a_j^{(i)} \\ b_j^{(i)} \\ c_j^{(i)} \end{bmatrix} = \begin{bmatrix} j \stackrel{?}{=} i \\ j \stackrel{?}{=} i \\ j \stackrel{?}{=} i \end{bmatrix}$$

for whichever three $\phi_j^{(i)}$ equation are nonzero. Solving each system produces the a, b, and c values for the $\phi_j^{(i)}$ corresponding to the vertex for that triangle made from $x_j^{(i)}$ and $y_j^{(i)}$. Invoking the boundary conditions, we can consider the γ_i value corresponding to vertex E_i on the S_1 boundary to be the value at boundary. Now evaluate the big integrals over each triangle. Parametrization and projection over each triangle might be required to evaluate the line integral in β. Solve for γ in $A\gamma = b$. The solution is $\phi(x,y) = \sum_{i=1}^m \gamma_i\phi_i(x,y)$ for each triangle. Err. for elliptic 2nd-ord. probs. w/ smooth coef. fs. $O(h^2)$.