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Asset Pricing, Financial Markets, and Linear Algebra

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The mathematics of financial markets involves some concepts already part of probability theory, linear algebra, and optimization, expressed in different terms. For that reason, these concepts provide classroom examples even for introductory courses in these subjects.

The fundamental theorems of asset pricing, for example, provide an opportunity to discuss arbitrage and complete markets. These are related here to concepts of discrete probability, expected value, linear dependence, and vector basis. Linear programming, i.e., the simplex method, is illustrated by a small-scale problem that can be worked out as a classroom example in optimization courses.

Though the discussion of some of these concepts might not be elementary, we show how important it is to translate properly, theoretical concepts into mathematical ones. The advanced reader may wish to consult the works of de Finetti [**2**] and Lad [**4**] to explore further the fruitful connections between the fundamental theorems of probability and asset pricing.

Basic concepts

The *state of the world* is a complete specification of all the relevant events over a specific time horizon. We work with a finite number of possible states, each denoted by some ω_i, and we refer to $\Omega = \{\omega_1, \ldots, \omega_k\}$ as the *state space*, the set of all states. Here we consider only one-period investments, during which period only one state occurs.

Another important concept is *contingent claim*, a claim for a monetary payment contingent on the specific state of the world. An example of contingent claim is a share of some company. The owner of the share has the right to receive, for instance, part of the profit of this company. Even a simple bet is a type of contingent claim. More formally, such a claim can be represented by a k-dimensional vector of payoffs, each element corresponding to the payoff received if the corresponding state occurs. Here

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we define *assets* as contingent claims, since they are contracts that pay off different amounts of money under different states of the world. The *price* of some asset is the amount of one specific good, typically money, that one willingly exchanges for one unit of this asset. Mathematically, price is a real-valued function on assets.

The *return rate* of asset A when state ω_j occurs is denoted A_j, and is given by the price of the asset after the period—a day, month, or year—divided by its price at the beginning of the period. A *risky asset* is an asset whose return is uncertain, i.e., a random variable, since it depends on the state of the world. For instance, suppose asset A pays a return rate of 3% if state 1 occurs, 40% if state 2 happens, -6% if state 3 happens, and 10% when state 4 is observed. In this case, $A_1 = 1.03$, $A_2 = 1.4$, $A_3 = 0.94$, and $A_4 = 1.1$. We use vector notation and write $A = (1.03, 1.4, 0.94, 1.1)$.

A *riskless asset* pays a constant return rate, no matter which state of the world happens, that is, there is no uncertainty about its return. All our examples assume the existence of a riskless asset O, whose return rate is denoted by $O_j \equiv r$. Usually, it is also assumed that one can borrow or lend money at this riskless rate, just as one can buy or sell risky assets.

A probability distribution on states, or probability *measure*, is denoted by $(p_1, p_2, \ldots, p_k)$, where p_j is the probability that state j happens. A probability measure is an element of $\mathcal{P} = \{(p_1, p_2, \ldots, p_k) : 0 \le p_j \le 1, \sum_{j=1}^{k} p_j = 1\}$, the $(k-1)$-dimensional unit simplex. Given such a measure, the expected return rate of asset A is:

$$E_p[A] = \sum_{j=1}^{k} A_j p_j.$$

Arbitrage

Suppose that a risky asset A has a return rate strictly bigger than the riskless rate r no matter which state occurs, that is, $A_j > r$ for $j = 1, \ldots, k$. Then any investor can, at the beginning of the period, borrow a dollar from the bank promising to pay a return r, and buy a dollar's worth of A. At the end of the period he would get A_j, and after paying r to the bank, obtain a sure profit of $(A_j - r) > 0$. This prospect is called *arbitrage* and means that one can earn something even with a net investment of zero dollars.

An opportunity like this will not go unnoticed! Other investors, attempting to exploit it, will drive up the demand for A, raising its price and lowering its return. Thus, attempts to profit from arbitrage opportunities tend to destroy them. Analogous reasoning applies if one knows of a risky asset that offers a return rate that is always *smaller* than r. Any *portfolio*, i.e., a bundle of assets, could be formed to exploit arbitrage opportunities trading in the riskless asset and some risky asset A whenever one knows that either $A_j > r$, or that $A_j < r$, for all j.

We conclude that in real, free markets the riskless rate and the random returns of the risky assets will not allow arbitrage opportunities. Suppose, however, that there is an arbitrage opportunity A; then either $E_p[A] > r$ for all p in $\mathcal{P}$, or $E_p[A] < r$ for all p. Conversely, if there is *one* probability measure q such that the expected returns of *all* assets given by this measure is r, then for a risky asset A, A_j is smaller than r for some states and greater than r for other states because $A \neq O$. Such a distribution q is called *risk neutral* because the expected return of all assets is the same as the return rate of the riskless asset.

Thus if there is at least one risk-neutral probability measure, there are no arbitrage opportunities. We state this as a theorem. See Duffie [**1**, p. 4] for a formal proof.

The first fundamental theorem of asset pricing. There are no arbitrage opportunities if and only if there is a *risk neutral* probability measure, i.e., a measure q satisfying

$$E_q[A] = \sum_{j=1}^{k} A_j q_j = r$$

for all assets A.

Is this risk neutral measure unique? To answer this question, we need another concept from financial markets theory. As mentioned above, an asset may be represented by a vector of dimension equal to the number of states of the world. A *market* is a set of available assets. The set of all portfolios based on the assets in a market is a vector subspace of $\Re^k$.

A market is *complete* if we can arrange a portfolio with any conceivable payoff vector. In terms of linear algebra, a complete market is one in which the set of available assets spans the k-dimensional space of all possible payoff vectors.

A claim is *redundant* if its payoff in all states can be obtained by buying or selling a fixed portfolio of other claims. In linear algebra terms, a claim is redundant if it is dependent on the other claims available on the market. A market is *incomplete* if the number of states is greater than the number of non-redundant claims. To establish the incompleteness of a market system, it is not sufficient merely to count equations involving payoff vectors and unknowns: One must find the *largest* set of non-redundant claims, i.e., a basis for the set of portfolios.

Now we can state an extended version of the First Fundamental Theorem using risk neutrality and complete markets (see Duffie [**1**, pp. 29–30]).

The second fundamental theorem of asset pricing. Assuming that one can trade a given number of risky assets and one riskless asset, there is a unique risk-neutral probability measure if and only if the market is complete and there are no arbitrage opportunities.

Completeness of a market implies that there are $N = k$ assets with linear independent returns. If there are $N < k$ linearly independent assets, we can find not only one risk neutral measure but also a whole convex polytope of them. The Second Theorem is regarded as fundamental because one can set a price for any asset in a complete market by calculating the expected value of the asset with respect to the risk neutral probability measure.

A two-asset example

Consider a market $\mathcal{M}$ with just two assets, one of which is riskless. Suppose also there are three states of the world: $\Omega = \{\omega_1, \omega_2, \omega_3\}$. The risky asset, denoted A, pays 10% if state 1 occurs, 2% if state 2 occurs, and -5% if state 3 happens. The riskless asset, denoted O, pays 6% no matter what state occurs. In vector notation $A = (1.1, 1.02, 0.95)$, $O = (1.06, 1.06, 1.06)$, and $\mathcal{M} = \{A, O\}$.

According to the no-arbitrage principle, the expected return of asset A must be 6% per period. To find a risk-neutral probability measure, let $q = (q_1, q_2, q_3)$. Then we must solve the system:

$$\begin{cases} 1.06q_1 + 1.06q_2 + 1.06q_3 = 1.06 \\ 1.1q_1 + 1.02q_2 + 0.95q_3 = 1.06 \end{cases}$$

in a way that ensures q constitutes a probability measure; that is, we seek a positive solution. Note that the first equation is just the condition that the probabilities sum to one. Using the fact that $q_3 = 1 - q_2 - q_1$, we rewrite the second equation as $0.15q_1 + 0.07q_2 = 0.11$. However, we cannot identify a unique value of q_1 and q_2 with just one equation.

We can represent graphically the set of possible probability measures—see Figure 1. Since $q_1 + q_2 \leq 1$, the shaded triangle represents all probability measures. The risk-neutral probability measure must also satisfy $0.15q_1 + 0.07q_2 = 0.11$. The points on this line that are also inside the triangle, represent risk-neutral probability measures. The solution to the market restriction equation is not unique, but is composed of the 1-dimensional convex polytope of solutions represented by the thick line segment in Figure 1.

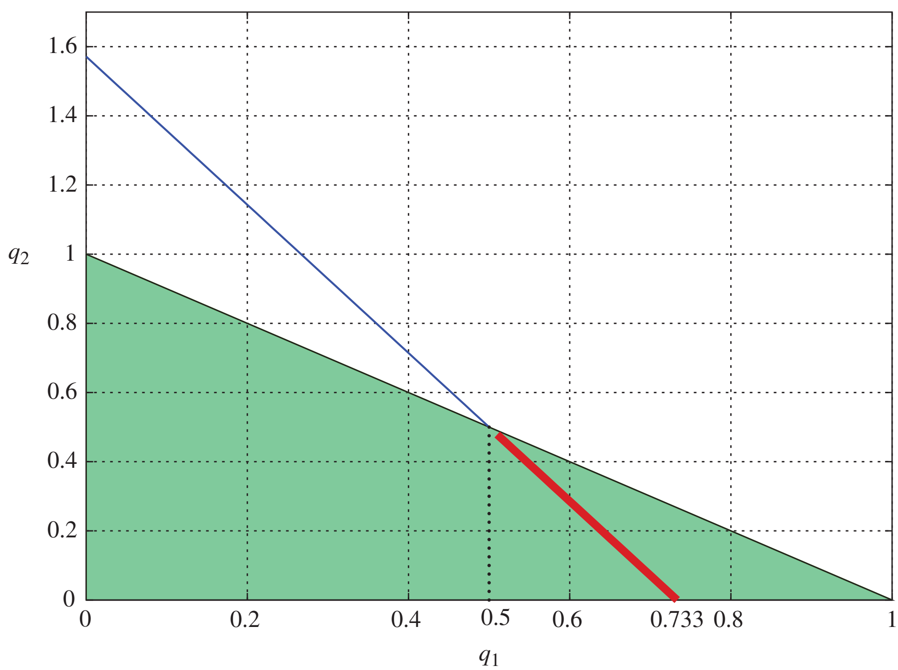

Figure 1. A convex set of risk-neutral measures.

This market is not complete: We have two non-redundant claims and three states. Another way to say this is that, although A and O are linearly independent, they do not form a basis for $\Re^3$. Therefore one cannot arrange any conceivable payoff vector using only A and O.

To "complete the market," we must extend the set $\mathcal{M} = \{A, O\}$ to a basis. This is possible because the vectors in $\mathcal{M}$ are linearly independent.

Completing the market

Suppose we have a second risky asset, B, that pays nothing if state 1 occurs, 30% if state 2 happens, or -22% if state 3 happens. Now, the no-arbitrage assumption implies

$$\begin{cases} 1.06q_1 + 1.06q_2 + 1.06q_3 = 1.06 \\ 1.1q_1 + 1.02q_2 + 0.95q_3 = 1.06 \\ q_1 + 1.3q_2 + 0.78q_3 = 1.06, \end{cases}$$

a system in three unknowns that can be solved easily. Adopting the same approach as above, we can write it as

$$\begin{cases} 0.15q_1 + 0.07q_2 = 0.11 \\ 0.22q_1 + 0.52q_2 = 0.28, \end{cases}$$

and represent it as in Figure 2. Now we have a single point where the lines cross, representing the risk-neutral probability measure $q = (0.601, 0.284, 0.115)$. We have completed $\mathcal{M}$ to form a basis $\mathcal{M}' = \{A, B, O\}$ of $\Re^3$. This particular system is solvable. Other cases, where there is a redundant third asset, can be used to illustrate linear systems of equations without a unique solution.

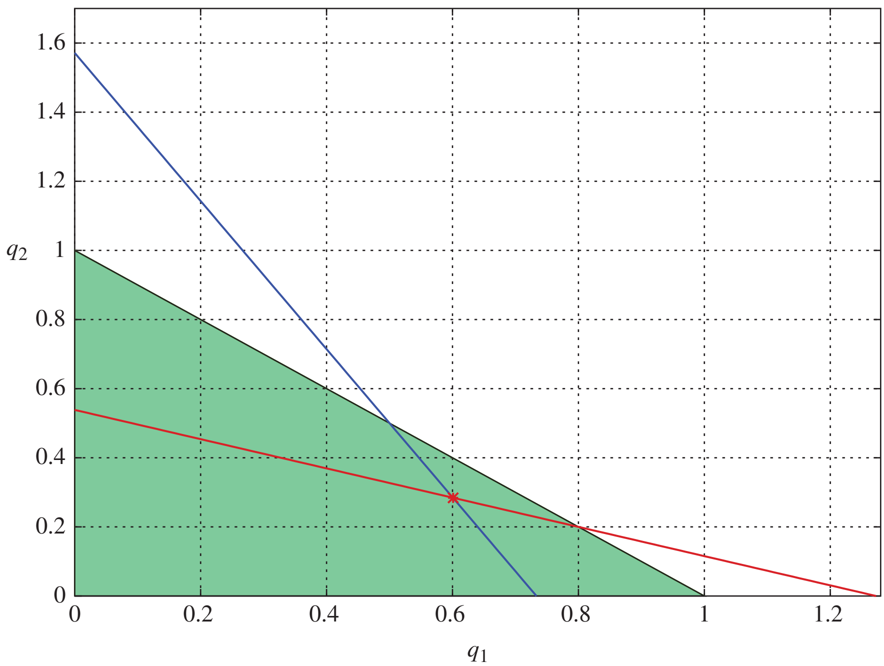

Figure 2. A unique risk-neutral measure.

Linear programming

Returning to the two-assets example, let us find bounds for the risk-neutral probability, q_1, using the linear programming approach of Hailperin [**3**] and Lad [**4**]. If ω_1 occurs, the return rates of assets A and O equal 1.1 and 1.06, respectively; if state 2 happens, they are 1.02 and 1.06, etc. In matrix notation, we want the probability measure q that maximizes $(1, 0, 0)q^T$, subject to

$$\begin{bmatrix} 1.1 & 1.02 & 0.95 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1.06 \\ 1 \end{bmatrix}$$

and $q_i \geq 0$ for $i = 1, 2, 3$. The solution is $(0.73\overline{3}, 0, 0.26\overline{6})$, which, multiplied by $(1, 0, 0)^T$, gives $0.73\overline{3}$. If we solve the minimization problem, we find 0.5. These bounds are the same found by the previous approach, as can be shown in Figure 1.

Linear programming does not give the risk-neutral probability measures explicitly, but only bounds the probabilities for each state.

With the complete market $\mathcal{M}'$, we can state precisely the risk-neutral probability of each state and, therefore, we do not find bounds. Maximization and minimization of the state probabilities find the same answer, since the risk-neutral probability measure is now unique.

Linear programming is more useful when the states of the world have a great number of possibilities. The following example adapted from Lad [**4**, pp. 71–72] illustrates this.

An investor living in a certain country frames their view of the world in terms of the following potential events: U: unemployment rises above 13%; I: inflation rises 5%; and C: the exchange rate at the end of this year is greater than \$1.64 in the local currency.

For this investor there are $2^3 = 8$ states of the world. Each state is a three-tuple recording whether each of the above events happens. Using a self-evident notation, we have:

$$\omega_1 = UIC; \omega_2 = \overline{U}IC; \omega_3 = U\overline{I}C; \omega_4 = UI\overline{C};$$
$$\omega_5 = \overline{UI}C; \omega_6 = \overline{U}I\overline{C}; \omega_7 = U\overline{IC}; \omega_8 = \overline{UIC}.$$

We again assume that there are two assets in this economy—one, riskless, paying 6.7%, whatever state occurs. For this example, we let the risky asset be: $A = (1.06, 1.01, 1, 1.07, 1.06, 1.08, 0.98, 1.1)$.

By the no-arbitrage principle, the expected return of the risky asset is 6.7%, but this is clearly not enough to determine the risk-neutral probability measure because the market is not complete. An investment bank gives the probabilities for some specific events: $P(\overline{U}) = 0.75$, $P(I) = 0.1$, $P(C) = 0.2$, $P(U\overline{I}) = 0.2$, $P(\overline{UC}) = 0.6$, and $P(\overline{C}I) = 0.05$. The investor will be interested in finding the probability, given by the risk-neutral measure and consistent with the probabilities given by the bank, that the risky asset performs worse than the riskless asset.

The key to using linear programming is to note that we can write probabilities as disjunctions of the eight basic, mutually exclusive states. The event $\overline{U}$, for instance, equals $\omega_2 \vee \omega_5 \vee \omega_6 \vee \omega_8$, and, since these are exclusive, writing $q_j = P(\omega_j)$, $P(\overline{U}) = q_2 + q_5 + q_6 + q_8 = 0.75$. Therefore, the event "the risky asset performs worse than the riskless asset" is given by: $\omega_1 \vee \omega_2 \vee \omega_3 \vee \omega_5 \vee \omega_7$, whose probability is $q_1 + q_2 + q_3 + q_5 + q_7$.

Now we can set up our linear program.

We want to maximize $q_1 + q_2 + q_3 + q_5 + q_7 = (1, 1, 1, 0, 1, 0, 1, 0)q^T$ subject to

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1.06 & 1.01 & 1 & 1.07 & 1.06 & 1.08 & .98 & 1.1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} q = \begin{bmatrix} .75 \\ .1 \\ .2 \\ .2 \\ .6 \\ .05 \\ 1.067 \\ 1 \end{bmatrix}$$

and $q_i \geq 0$ for every $j = 1, \ldots, 8$. Computer software supplies the answer equal to 0.40. The minimum is 0.35. These bound the desired probability, consistent with the expected return rate of the risky asset, the no-arbitrage assumption, and the given probabilities. Thus, when dealing with a great number of states, linear programming is a useful tool for bounding the risk-neutral measure attached to each state.

Summary. Concepts from asset pricing and financial markets theory are used to illustrate concepts of linear algebra and linear programming.

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