PHYS611 Notes

Hersh Kumar

Contents

1 Electrostatics

In the theory of electromagnetism, we define two three dimensional vector fields, $\boldsymbol{E}$ and $\boldsymbol{B}$.

We begin with the change in the magnetic field over time:

$$\frac{d\boldsymbol{B}}{dt} = e\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right)$$

We see that if we have no $\boldsymbol{B}$ field, the electric field causes acceleration. If instead we had pure magnetic field, we would have gyrations. In particular, this would do no work, since $\boldsymbol{F}_L \cdot \boldsymbol{v} = 0$ in the pure magnetic field case. Now let us consider the case where we have an $\boldsymbol{E}$ field perpendicular to a $\boldsymbol{B}$ field. A particle placed in this system would tend to go in the direction given by $\boldsymbol{E} \times \boldsymbol{B}$. In fact, given the special velocity:

$$\boldsymbol{v} = c\frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2}$$

One can work out that this particle would continue to move with this velocity, unaffected by the fields.

As we add more charges, we start to form currents. Consider an electric field pointing to the right, and a mixture of positive and negative charges placed in the field. The field will cause the particles to move to either side of the system, based on the sign of the charge. This polarization would create an electric field oriented against the original electric field. This is where things become more complicated, and Maxwell's equations come in.

The first two equations are the stationary equations:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{E} &= 4\pi\rho \\ \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 \end{aligned}$$

Which tell us that charges are the source of electric fields, and there are no magnetic field sources.

The other two are Faraday's Law, or the induction equation:

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}$$

And Ampere's Law:

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}$$

We can see that not all of these equations are equal, by taking the divergence of both sides of Faraday's Law:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \boldsymbol{B})$$

Now noting that the divergence of a curl is always zero, we see that the divergence of $\boldsymbol{B}$ must be constant as a function of time:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \text{const.}$$

We see that the equation $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ is a constraint equation,we are constraining the constant to be zero. In other words, it is an initial condition for Faraday's Law.

Similarly, we can show that $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\rho$ is an initial condition for Ampere's Law. Using the same process, we take the divergence of both sides:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) = \frac{4\pi}{c}\boldsymbol{\nabla} \cdot \boldsymbol{J} + \frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \boldsymbol{E})$$

Once again noting that the left side must be zero. Now in order to obtain Gauss's Law, we must apply the conservation of charge. The mathematical formulation to maintain the inflow and outflow of charge is known as the continuity equation:

$$\frac{\partial Q}{\partial t} = -\oint_V \boldsymbol{J} dS$$

The change in charge in a particular volume is dependent on the flux of the charge through the boundary, which is the current. Now we note that the definition of charge is the charge density integrated over the volume:

$$Q = \int_V \rho dV$$

On the right side of the equation, we can write the surface integral as the volume integral of the divergence of the current:

$$-\oint_V \boldsymbol{J} dS = -\int_V \boldsymbol{\nabla} \cdot \boldsymbol{J} dV$$

Thus we can put these all together:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$$

Now we see that we can express the divergence of $\boldsymbol{J}$ in terms of the time derivative of the charge density. We can look back at what we initially had:

$$0 = \frac{4\pi}{c}\left(-\frac{\partial \rho}{\partial t}\right) + \frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \boldsymbol{E})$$

Which leaves us with a general form of Gauss's Law:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\rho + C$$

Where we can see that the equation $\boldsymbol{\nabla} \cdot E = 4\pi\rho$ is a constraint equation for Ampere's Law.

The first form of mathematical massaging that we will employ is to consider things in terms of potentials. We have 6 field components, 3 from both the electric and magnetic fields. 4 of these are independent, since we have 2 constraint equations. These 4 components are split into the scalar potential and the vector potential, ϕ and $\boldsymbol{A}$ respectively. The vector potential describes the magnetic field:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \rightarrow \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

Inserting this definition into Faraday's Law:

$$-\frac{1}{c}\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\times\boldsymbol{A}\right)=\boldsymbol{\nabla}\times\boldsymbol{E}$$
$$\boldsymbol{\nabla}\times\left(\boldsymbol{E}+\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\right)=0$$

In order for the left side to be zero, the term that we are taking the curl of must be the gradient of a scalar function, $\boldsymbol{\nabla}\phi$:

$$-\boldsymbol{\nabla}\phi=\boldsymbol{E}+\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}$$

Where by convention we consider the negative gradient of a scalar function. This then gives us that:

$$\boldsymbol{E}=-\boldsymbol{\nabla}\phi-\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}$$

We can now rewrite the system of Maxwell's equations in terms of our potentials. The first thing we do is insert them into Gauss's Law:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=4\pi\rho$$
$$-\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\phi\right)-\frac{1}{c}\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A}\right)=4\pi\rho$$
$$-\boldsymbol{\nabla}^2\phi-\frac{1}{c}\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A}\right)=4\pi\rho$$

In the electrostatics case, we would drop the time derivative term, and we would have the expected Poisson's equation.

Inserting our potentials into Ampere's Law:

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{A}=\frac{4\pi}{c}\boldsymbol{J}+\frac{1}{c}\frac{\partial}{\partial t}\left(-\boldsymbol{\nabla}\phi-\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\right)$$

The double curl gives two terms, the gradient of the divergence of $\boldsymbol{A}$, and the Laplacian of $\boldsymbol{A}$:

$$\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A}\right)-\boldsymbol{\nabla}^2\boldsymbol{A}=\frac{4\pi}{c}\boldsymbol{J}+\frac{1}{c}\frac{\partial}{\partial t}\left(-\boldsymbol{\nabla}\phi-\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\right)$$

To prove the double curl identity, we can use the Levi-Civita symbol. We need to know two formulas. The first is for the ith component of a vector cross product:

$$\left(\boldsymbol{A}\times\boldsymbol{B}\right)_i=e_{ijk}A_jB_k$$

Where we assume Einstein summation notation, we are summing over j and k.

The other useful formula is:

$$e_{ijk}e_{imn}=\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{km}$$

Now going back to the curl of a curl:

$$\left(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{A}\right)_i=e_{ijk}\partial_j\left(e_{klm}\partial_lA_m\right)$$

Now noting that the symmetry of the Levi-Civita symbol means that each shift in indices picks up a minus sign, we can shift the second one into e_{lmk}:

$$\begin{aligned}
(\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A})_i &= e_{ijk}\partial_j \left(e_{lmk}\partial_l A_m\right) \\
&= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\partial_j\partial_l A_m \\
&= \partial_m\partial_i A_m - \partial_j\partial_j A_i \\
&= \partial_i\partial_m A_m - \partial_j^2 A_i
\end{aligned}$$

We see that the first term is a gradient of a divergence, and the second term is a Laplacian.

Rewriting our Ampere's Law expression:

$$-\boldsymbol{\nabla}^2\boldsymbol{A} + \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = \frac{4\pi}{c}\boldsymbol{J} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)$$

Thus we have rewritten Maxwell's equations as two equations, in terms of the scalar and vector potentials:

$$\begin{aligned}
-\boldsymbol{\nabla}^2\boldsymbol{A} + \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} &= \frac{4\pi}{c}\boldsymbol{J} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right) \\
\boldsymbol{\nabla}^2\phi &= -4\pi\rho - \frac{1}{c}\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A}\right)
\end{aligned}$$

Looking at the top equation, the left side is a wave equation, but to make sense of the right side of the equation, we make use of the gauge freedom of the vector potential, we can add the gradient of a scalar function to the vector potential and maintain the same equations:

$$\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\Lambda$$

Essentially, the vector potential is not unique.

Consequently, we have gauge freedom for the scalar potential:

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}$$

We now change ϕ to ϕ' and $\boldsymbol{A}$ to $\boldsymbol{A}'$, while keeping $\boldsymbol{E}$ the same:

$$\begin{aligned}
-\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t} &= -\boldsymbol{\nabla}\phi' - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t} - \frac{1}{c}\boldsymbol{\nabla}\left(\frac{\partial\Lambda}{\partial t}\right) \\
\phi' &= \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}
\end{aligned}$$

Which is the gauge freedom of the scalar potential.

Since the choice of Λ is arbitrary, we can "fix" the gauge.

One such choice is the Lorentz gauge, where $\boldsymbol{\nabla}\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0$. This gives us the equations:

$$\begin{aligned}
-\boldsymbol{\nabla}^2\boldsymbol{A} + \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} &= \frac{4\pi}{c}\boldsymbol{J} \\
\boldsymbol{\nabla}^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} &= -4\pi\rho
\end{aligned}$$

We see that this decouples the two equations, making it a useful choice of gauge.

Now let us prove that we can always fix the gauge in such a way. Consider the case where we have $\boldsymbol{A}$ and ϕ such that $\boldsymbol{\nabla}\cdot\boldsymbol{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}\neq 0$. Can we change the gauge such that the shifted potentials do satisfy the Lorentz gauge?

We must make the shift $\boldsymbol{A}'=\boldsymbol{A}+\boldsymbol{\nabla}\Lambda$, and $\phi'=\phi-\frac{1}{c}\frac{\partial\Lambda}{\partial t}$. Now computing $\boldsymbol{\nabla}\cdot\boldsymbol{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}$:

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}=\left(\boldsymbol{\nabla}\cdot\boldsymbol{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}\right)+\boldsymbol{\nabla}^2\Lambda-\frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}$$

We want to see if this can be made zero by picking Λ. This can be rewritten as a wave equation:

$$\boldsymbol{\nabla}^2\Lambda-\frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}=\text{known terms.}$$

This wave equation always has a solution, and thus we can always fix the Lorentz gauge.

Another choice of gauge is the Coulomb gauge, where we impose that $\boldsymbol{\nabla}\cdot\boldsymbol{A}=0$. This again is always guaranteed to be possible, since we have that $\boldsymbol{\nabla}\cdot(\boldsymbol{A}'+\boldsymbol{\nabla}\Lambda)=0$, which is equivalent to the Poisson equation $\boldsymbol{\nabla}^2\Lambda=-\boldsymbol{\nabla}\cdot\boldsymbol{A}$, which always has a solution. This gauge simplifies one of the equations, but the second equation remains coupled, so it is still difficult. In the time-independent case, the two gauges are equivalent.

1.1 Conservation of Energy

Now let us consider what conservation laws this system must obey. To begin, we define the energy density of the electromagnetic field:

$$\mathcal{W}=\frac{E^2+B^2}{8\pi}$$

Let us compute the time derivative of the energy density:

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{W}&=\frac{\partial}{\partial t}\left(\frac{E^2+B^2}{8\pi}\right)\\&=\frac{2}{8\pi}\left(\boldsymbol{E}\cdot\frac{\partial\boldsymbol{E}}{\partial t}\right)+\frac{2}{8\pi}\left(\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t}\right)\end{aligned}$$

Now using the Faraday and Ampere equations to replace time derivatives with spatial derivatives:

$$\begin{aligned}\frac{\partial\boldsymbol{B}}{\partial t}&=-c\boldsymbol{\nabla}\times\boldsymbol{E}\\\frac{\partial\boldsymbol{E}}{\partial t}&=c\boldsymbol{\nabla}\times\boldsymbol{B}-4\pi\boldsymbol{J}\end{aligned}$$

We can insert these into our expression:

$$\frac{\partial}{\partial t}\mathcal{W}=-\boldsymbol{J}\cdot\boldsymbol{E}+\frac{c}{4\pi}\underbrace{(\boldsymbol{E}\cdot\boldsymbol{\nabla}\times\boldsymbol{B}-\boldsymbol{B}\cdot\boldsymbol{\nabla}\times\boldsymbol{E})}_{\boldsymbol{\nabla}\cdot\left(\frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B}\right)}$$

This is beginning to look like a conservation law:

$$\frac{\partial}{\partial t}\mathcal{W}_{\text{EM}}+\boldsymbol{\nabla}\cdot\boldsymbol{S}=-\boldsymbol{J}\cdot\boldsymbol{E}$$

Where $\boldsymbol{S}$ is the Poynting vector:

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B}$$

If we drop the current term for now, we see that the Poynting flux tells us how much electromagnetic energy is moving through the volume. However, if we have currents and particles, we have the additional $\boldsymbol{J}\cdot\boldsymbol{E}$ term. This additional term is the rate of work done by the electric field on the ensemble of particles.

For a group of particles, we can define the current density:

$$\boldsymbol{J} = \frac{\sum_{\text{charges}} q_i \boldsymbol{v}_i}{V}$$

If we dot this against the electric field:

$$\boldsymbol{J}\cdot\boldsymbol{E} = \frac{\sum_{\text{charges}} q\boldsymbol{E}\cdot\boldsymbol{v}_i}{V}$$

Where we assume that the volume is very small, so the electric field is about the same for all of the particles. We see that this is just the Lorentz force on each particle, times the velocity of the particle:

$$\begin{aligned}\boldsymbol{J}\cdot\boldsymbol{E} &= \frac{\sum_{\text{charges}} \boldsymbol{F}_{L,i}\cdot\boldsymbol{v}_i}{V} \\ &= \frac{\sum_{\text{charges}} \frac{\partial\mathcal{E}_i^{\text{kin}}}{\partial t}}{V}\end{aligned}$$

If we integrate the version of this conservation law over a volume:

$$\frac{\partial}{\partial t}\left(\int_V \mathcal{W}_{EM}\,dV\right) = -\oint_V \boldsymbol{S}\,d\boldsymbol{A} - \frac{d}{dt}\left(\mathcal{E}_{\text{kin}}\right)$$

Which is our conservation law over a volume.

Now let us prove that radiation requires time dependence. Consider a circular ring of charges, with a constant current. Intuitively, this will not radiate, but each individual charge on the ring is accelerating, since it is moving in a non-linear orbit. We can show that this does not radiate by computing the divergence of the Poynting vector, and then integrating over the volume:

$$\begin{aligned}\int_V d^3x\,\boldsymbol{\nabla}\cdot(\boldsymbol{E}\times\boldsymbol{B}) &= -\int d^3x\,\boldsymbol{J}\cdot\boldsymbol{E} \\ &\quad - \int d^3x\,\phi\cdot\boldsymbol{\nabla}\boldsymbol{J} \\ &= 0\end{aligned}$$

1.2 Conservation of Momentum

We have derived the conservation law for energy in the electromagnetic theory:

$$\frac{\partial\mathcal{W}_{EM}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{S} = -\boldsymbol{j}\cdot\boldsymbol{E}$$

We now want to derive the conservation law for the momentum of fields and particles.

To do this, we begin with the time derivative of the momentum of every particle being equal to the Lorentz force at the location of the particle (using Newton's Second Law):

$$\frac{d}{dt}\boldsymbol{p}_i = e\left(\boldsymbol{E}\left(\boldsymbol{x}_i\right) + \frac{\boldsymbol{v}_i}{c} \times \boldsymbol{B}\left(\boldsymbol{x}_i\right)\right)$$

We can make a volume weighted momentum vector (momentum density of the charged particles in the system):

$$\boldsymbol{P} = \frac{1}{V}\sum_i \boldsymbol{p}_i$$

We want an evolution equation for this, so we compute the time derivative:

$$\begin{aligned}\frac{d\boldsymbol{P}}{dt} &= \frac{1}{V}\sum_i \frac{d\boldsymbol{p}_i}{dt} \\ &= \frac{1}{V}\sum_i e\left(\boldsymbol{E}\left(\boldsymbol{x}_i\right) + \frac{\boldsymbol{v}_i}{c} \times \boldsymbol{B}\left(\boldsymbol{x}_i\right)\right)\end{aligned}$$

In this case, the volume is the infinitesimal volume of particles very close together in space, $V = d^3\boldsymbol{x}$. In this case, we can state that we have some average field value over this small volume, so instead of evaluating the fields at each point, we have a constant field over the entire volume, for both the electric and magnetic fields. In this case, we note that the sum of the charges over the volume, $\frac{1}{V}\sum_i e$ is equivalent to the charge density, ρ, and we note that $\sum_i e\boldsymbol{v}_i$ is the total current $\boldsymbol{J}$[1], and we divide by the volume, so we have the current density $\boldsymbol{j}$:

$$\frac{d\boldsymbol{P}}{dt} = \rho\boldsymbol{E}\left(\boldsymbol{x}\right) + \frac{1}{c}\boldsymbol{j} \times \boldsymbol{B}\left(\boldsymbol{x}\right)$$

Now we want to use Maxwell's equations to massage the right side of this equation, to arrive at an analogous result to the conservation law for energy. We first note that, by rewriting Maxwell's equations:

$$\rho = \frac{1}{4\pi}\boldsymbol{\nabla} \cdot \boldsymbol{E}$$

And similarly for the current density:

$$\boldsymbol{j} = \frac{c}{4\pi}\left(\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t}\frac{1}{c}\right)$$

We can now insert these into the right hand side of our evolution equation:

$$\frac{d\boldsymbol{P}}{dt} = \frac{1}{4\pi}\left(\boldsymbol{E} \ \cdot \boldsymbol{\nabla} \cdot \boldsymbol{E} + \boldsymbol{\nabla} \times \boldsymbol{B} \times \boldsymbol{B}\right) - \frac{1}{4\pi c}\frac{\partial \boldsymbol{E}}{\partial t} \times \boldsymbol{B}$$

Rewriting the second term as $\frac{\partial}{\partial t}\left(\boldsymbol{E} \times \boldsymbol{B}\right) - \boldsymbol{E} \times \frac{\partial \boldsymbol{B}}{\partial t}$:

$$\frac{d\boldsymbol{P}}{dt} = \frac{1}{4\pi}\left(\boldsymbol{E} \ \cdot \boldsymbol{\nabla} \cdot \boldsymbol{E} + \boldsymbol{\nabla} \times \boldsymbol{B} \times \boldsymbol{B}\right) - \frac{1}{4\pi c}\left[\frac{\partial}{\partial t}\left(\boldsymbol{E} \times \boldsymbol{B}\right) - \boldsymbol{E} \times \frac{\partial \boldsymbol{B}}{\partial t}\right]$$

[1]Consider the change in charge in the volume if we shift a charge e by some Δx, $\Delta q = e\Delta x$. Dividing by Δt, we have that $\Delta q/\Delta t = e\Delta x/\Delta t$, which gives us that $J = ev$

If we define $\boldsymbol{g} = \frac{1}{4\pi c}\boldsymbol{E} \times \boldsymbol{B}$:

$$\frac{d\boldsymbol{P}}{dt} + \frac{\partial \boldsymbol{g}}{\partial t} = \frac{1}{4\pi}\left(\boldsymbol{\nabla} \cdot \boldsymbol{E} \cdot \boldsymbol{E} + \boldsymbol{\nabla} \times \boldsymbol{B} \times \boldsymbol{B} - \boldsymbol{E} \times \boldsymbol{\nabla} \times \boldsymbol{E}\right)$$

We see that $\boldsymbol{g}$ takes on the role of the momentum of the electromagnetic field, $\boldsymbol{P}_{EM}$, and is in fact equal to the Poynting vector divided by c^2, $\boldsymbol{P}_{EM} = \frac{\boldsymbol{S}}{c^2}$. Now looking at the term $\boldsymbol{E} \times \boldsymbol{\nabla} \times \boldsymbol{E}$:

$$\begin{aligned}
\left(\boldsymbol{E} \times \boldsymbol{\nabla} \times \boldsymbol{E}\right)_i &= e_{ijk}E_j\left(e_{klm}\partial_l E_m\right) \\
&= e_{ijk}E_j e_{lmk}\partial_l E_m \\
&= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)E_j\partial_l E_m \\
&= E_m\partial_i E_m - E_l\partial_l E_i
\end{aligned}$$

Writing this in vector notation:

$$\boldsymbol{E} \times \boldsymbol{\nabla} \times \boldsymbol{E} = \frac{1}{2}\boldsymbol{\nabla}\boldsymbol{E}^2 - \left(\boldsymbol{E}\boldsymbol{\nabla}\right)\boldsymbol{E}$$

Inserting this result into our equation:

$$\frac{d\boldsymbol{P}}{dt} + \frac{\partial \boldsymbol{g}}{\partial t} = \frac{1}{4\pi}\left[\left(\boldsymbol{E}\boldsymbol{\nabla}\right)\boldsymbol{E} + \left(\boldsymbol{B}\boldsymbol{\nabla}\right)\boldsymbol{B} - \boldsymbol{\nabla}\left(\frac{E^2 + B^2}{2}\right) - \boldsymbol{\nabla} \cdot \boldsymbol{E} \cdot \boldsymbol{E}\right]$$

Taking the ith component of the right side:

$$\begin{aligned}
\left[\frac{d\boldsymbol{P}}{dt} + \frac{\partial \boldsymbol{g}}{\partial t}\right]_i &= \frac{1}{4\pi}\left(E_j\partial_j E_i + B_j\partial_j B_i - \delta_{ij}\partial_j\left(\frac{E^2 + B^2}{2}\right) + E_i\partial_j E_j\right) \\
&= \frac{1}{4\pi}\left[\partial_j\left(E_iE_j\right) + \partial_j\left(B_iB_j\right)\right] - \delta_{ij}\partial_j\mathcal{W}_{EM} \\
&= -\partial_j T^{ij}_{EM}
\end{aligned}$$

Where we have added the term $B_i\partial_j B_j$, because by Maxwell's laws, the divergence of the magnetic field is zero, so we can add this term. We define spatial components of T_{EM}, the Maxwell stress tensor:

$$\sigma_{ij} = \frac{E_iE_j + B_iB_j}{4\pi} - \delta_{ij}\mathcal{W}_{EM}$$

Thus we have that the change in the total momentum in our system is equal to the derivative of the stress tensor:

$$\frac{d}{dt}\left(\boldsymbol{P}_{EM} + \boldsymbol{P}\right)_i = -\partial_j T^{ij}_{EM}$$

We can integrate this over the volume:

$$\frac{d}{dt}\left[\int_V \left(\boldsymbol{P}_{EM} + \boldsymbol{P}\right)dV\right]_i = -\oint_{\boldsymbol{A}} n_j T^{ij}\, d\boldsymbol{A}$$

Which gives us our conservation law.

We will later see that T_{EM} will have 4 indices, and the spatial components will be the σ_{ij}s.

We have now derived the conservation laws of the electromagnetic theory.

1.3 Laplace's Equation in Cylindrical Coordinates

In electrostatics, we have the case that $\boldsymbol{E} = -\boldsymbol{\nabla}\varphi$. In the case with no contained charges, we then have that $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$, from which we have Laplace's equation:

$$\boldsymbol{\nabla}^2 \varphi = 0$$

In cylindrical coordinates, we have (ρ, ϕ, z). Let us first consider the case where we have z symmetry, so the Laplacian is of the form:

$$\boldsymbol{\nabla}^2 \varphi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \varphi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \varphi}{\partial \phi^2}$$

We solve Laplace's equation via separation of variables:

$$\varphi = R(\rho)\,\Phi(\phi)$$

Which leaves us with

$$\frac{\Phi}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2}\frac{d^2\Phi}{d\phi^2} = 0$$

Which we can rewrite:

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

Now noting that the two terms are dependent on separate variables, they must both be equal to a constant[2]:

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2$$
$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) = m^2$$

We can now solve the two separately. We expect sines and cosines for the Φ equation, and by power counting, we expect a polynomial for the R equation, $R \sim \rho^\gamma$. By inserting this solution back into the radial equation, we find that $\gamma = \pm m$.

In the special case where $m = 0$, we have that $R \sim$ const, as well as $R \sim \ln\rho$ as our two linearly independent solutions.

For a problem that contains the origin, we can drop the $\ln\rho$ and the ρ^{-m} terms, since they diverge at $\rho = 0$. Thus our potential is given by

$$\varphi = \sum_m \left(a_m \cos(m\phi) + b_m \sin(m\phi)\right) c_m \rho^m$$

From here, we have to utilize the boundary conditions of the problem in order to pin down the values of a_m, b_m, and c_m. For example, consider the case where φ is defined as:

$$\varphi = \begin{cases} V, & \phi \in (0, \pi) \\ -V, & \phi \in (\pi, 2\pi) \end{cases}$$

[2]We choose $-m^2$ for the Φ equation becase we expect the solution for that equation to be periodic, so we want an equation that gives us an oscillator equation

First, we see that we need the sines, because the potential is odd:

$$\varphi = \sum_m a_m \sin(m\phi)\,\rho^m$$

The second thing to note is that we are restricted to odd m, because the even coefficients will be given by:

$$\begin{aligned} a_{\text{even}} &\sim \int \varphi(r=a,\phi)\sin(2n\phi)\;d\phi \\ &= V\int_0^{\pi}\sin(2n\phi)\;d\phi - V\int_{\pi}^{2\pi}\sin(2n\phi)\;d\phi \\ &= 0 \end{aligned}$$

We see that we can drop all of the even coefficients, and normalize ρ to the circle:

$$\varphi = \sum_{m\,\text{odd}} a_m \sin(m\phi)\left(\frac{\rho}{a}\right)^m$$

Now we can see that

$$\varphi(\rho=a,\phi) = \sum_{m\,\text{odd}} a_m \sin(m\phi)$$

And thus we can solve for a_m, by computing the projection onto the mth basis function:

$$\begin{aligned} a_m &= \frac{1}{\pi}\int_0^{2\pi} d\phi\;\sin(m\phi)\,\varphi(\rho=a,\phi) \\ &= \frac{1}{\pi}\left[\int_0^{\pi} V\sin(m\phi)\;d\phi - \int_{\pi}^{2\pi} V\sin(m\phi)\;d\phi\right] \\ &= \frac{2V}{\pi}\int_0^{\pi} d\phi\;\sin(m\phi) \\ &= \frac{2V}{\pi m}\left(1-\cos(m\pi)\right) \\ &= \frac{4V}{\pi m} \end{aligned}$$

Thus we have that

$$\varphi = \frac{4V}{\pi}\sum_{m\,\text{odd}}\frac{1}{m}\left(\frac{\rho}{a}\right)^m \sin(m\phi)$$

To write this in a closed form, we express the sine as a complex exponent, and then use the geometric series to rewrite the sum:

$$\begin{aligned} \varphi &= \frac{4V}{\pi}\sum_{m\,\text{odd}}\frac{1}{m}\left(\frac{\rho}{a}\right)^m \sin(m\phi) \\ &= \frac{4V}{m}\mathrm{Im}\left[\sum_{m\,\text{odd}}\frac{1}{m}\left(\frac{\rho}{a}e^{i\phi}\right)^m\right] \end{aligned}$$

Now consider the series $\sum_{m\,\text{odd}}\frac{z^m}{m}$. If we differentiate this with respect to z:

$$\frac{d}{dz}\left[\sum_{m\,\text{odd}}\frac{z^m}{m}\right] = \sum_{m\,\text{odd}} z^{m-1}$$

$$
\begin{aligned}
&= \frac{1}{1-z^2} \\
&= \frac{1}{2}\left(\frac{1}{1-z} + \frac{1}{1+z}\right)
\end{aligned}
$$

From this, we have that

$$
\sum_{m\,\text{odd}} \frac{z^m}{m} = \frac{1}{2}\ln\left[\frac{1+z}{1-z}\right]
$$

Now noting that our original series matches this, with $z = \frac{\rho}{a}e^{i\phi}$, we now only have to deal with the imaginary part of this. First, we note that

$$
\frac{1+z}{1-z} = \left|\frac{1+z}{1-z}\right| e^{i\theta}
$$

If we now consider the imaginary part of the log of this:

$$
\ln\left(\text{Im}\left[\frac{1+z}{1-z}\right]\right) = \theta
$$

Now finding what θ is in our case, we multiply both the numerator and denominator by $(1-z^*)$:

$$
\frac{(1+z)(1-z^*)}{(1-z)(1-z^*)} = \frac{1-|z|^2 + 2i\ \text{Im}\ z}{1+|z|^2 - 2\ \text{Re}\ z}
$$

This must be equal to some $a+bi$, in which case $\theta = \arctan\left(\frac{b}{a}\right)$, so we have that

$$
\theta = \arctan\left(\frac{2\,\text{Im}\ z}{1-|z|^2}\right)
$$

Now inserting the expression for z:

$$
\theta = \arctan\left(\frac{2\frac{\rho}{a}\sin\phi}{1-\left(\frac{\rho}{a}\right)^2\sin^2\phi}\right)
$$

From this, we can put together the total closed form of the potential:

$$
\varphi = \frac{4V}{\pi}\arctan\left(\frac{2\frac{\rho}{a}\sin\phi}{1-\left(\frac{\rho}{a}\right)^2\sin^2\phi}\right)
$$

When doing this whole process in 3D, we will follow the same logic, we begin with Laplace's equations in cylindrical:

$$
\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\phi^2} + \frac{\partial^2\varphi}{\partial z^2} = 0
$$

We can separate this:

$$
\varphi = R(\rho)\,\Phi(\phi)\,Z(z)
$$

Inserting this, and then separating variables:

$$\frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\frac{1}{\rho^2}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0$$

Again noting that we expect periodicity in the ϕ direction, we will choose the constant to be $-\nu^2$ for the Φ equation. For the Z equation, we expect the solution to decay at infinity, so we expect something of the form $Z \sim e^{-kz}$ [3]. The radial equation that we are left with is the Bessel equation[4]:

$$\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\left(k^2\rho^2-\nu^2\right)R$$

Which has solutions $J_m\left(k\rho\right)$ and $J_{-m}\left(k\rho\right)$, which differ by their behavior at the origin, J_ν decays at the origin, and $J_{-\nu}$ diverges at the origin. In the case where we contain the origin, we discard the $J_{-\nu}$ solutions, and we pick only the J_ν. We can then write down the full solution:

$$\varphi\left(\rho,\phi,z\right)=\sum_{\nu,n}cJ_\nu\left(k_n\rho\right)\left(d\sin\left(\nu\phi\right)+e\cos\left(\nu\phi\right)\right)\left(ae^{k_nz}+be^{-k_nz}\right)$$

Now let us discuss the orthogonality and the asymptotic behavior of the Bessel functions. For the asymptotic behavior, for small x:

$$J_{\pm\nu}\left(x\right)=\begin{cases}\frac{\left(\frac{x}{2}\right)}{\Gamma(\nu+1)}\\ \frac{\frac{x}{2}}{\Gamma(1-\nu)}\end{cases}$$

For large x:

$$J_\nu\left(x\right)\approx\sqrt{\frac{2}{\pi x}}\sin\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)$$

We can also consider the zeroes of the Bessel function, where $x_{\nu n}$ is the nth zero of the Bessel function $J_\nu\left(x\right)$.

Consider the case where we have the boundary condition that $\varphi\left(\phi=a\right)=0$. In order for this to be satisfied, we need the Bessel function $J_\nu\left(k_{\nu,n}\right)$ to be zero. For this to be true, we need $k_{\nu,n}a=x_{\nu,n}$.

This is an example of Dirichlet boundary conditions, and the other type of boundary condition are the von Neuman boundary conditions, where:

$$\frac{\partial\varphi}{\partial n}=0$$

In this case, the derivatives of the Bessel functions will be zero:

$$J'_\nu\Big|_{\rho=a}=0$$

Now let us consider the proof of orthogonality of the Bessel functions. Suppose we have $J_p\left(\alpha x\right)$ and $J_p\left(\beta x\right)$. We want to show that these are orthogonal unless $\alpha=\beta$, that is:

$$\int_0^1 xJ_p\left(\alpha x\right)J_p\left(\beta x\right)\,dx=\delta_{\alpha\beta}$$

[3] If we restrict ourselves to finite z, then we would have to keep the positive exponent as well.
[4] In general, the equation of the form $x\left(xu'\right)'+\left(\alpha^2x^2-p^2\right)u=0$ has solution $u=J_p\left(\alpha x\right)$

Where the factor of x comes from the Jacobian in cylindrical.

The equations that generate these solutions are (respectively):

$$x\left(xu'\right)' + \left(\alpha^2 x^2 - p^2\right)u = 0$$
$$x\left(xv'\right)' + \left(\beta^2 x^2 - p^2\right)v = 0$$

If we subtract these two equations from each other:

$$v\left(xu'\right)' - u\left(xv'\right)' + \left(\alpha^2 - \beta^2\right)xuv = 0$$

Which can be rewritten as a derivative:

$$\frac{d}{dx}\left(vxu' - uxv'\right) + \left(\alpha^2 - \beta^2\right)xuv = 0$$

Integrating this from 0 to 1:

$$\int_0^1 \frac{d}{dx}\left(vxu' - uxv'\right) + \left(\alpha^2 - \beta^2\right)xuv\,dx = 0$$
$$v\left(1\right)u'\left(1\right) - u\left(1\right)v'\left(1\right) + \left(\alpha^2 - \beta^2\right)\int_0^1 xuv\,dx = 0$$

Now we see that we have two additional terms, that are not necessarily zero. In order to make the first two terms zero, we need $J_p\left(\beta\right)$ and $J_p\left(\alpha\right)$ to both be equal to zero, meaning that α and β are zeroes of the Bessel function. After this, we see that the remaining integral is the orthogonality integral that we care about, and we see that this must be zero unless $\alpha = \beta$.

Now let us consider the case where we have the orthogonality relation:

$$\int_0^a d\rho\,\rho J_\nu\left(x_{\nu,n}\frac{\rho}{a}\right)J_\nu\left(x_{\nu,n'}\frac{\rho}{a}\right) = \delta_{nn'}\frac{a^2}{2}J_{\nu+1}^2\left(x_{\nu,n}\right)$$

We also have recursion relations between different orders of the Bessel functions:

$$\frac{d}{dx}\left(x^p J_p\right) = x^p J_{p-1}$$
$$J_{p-1} + J_{p+1} = \frac{2p}{x}J_p$$

Now that we have developed this machinery, let us consider a problem with a cylinder subdivided into two sections. At $z = 0$, we have a disk of radius a, with height L. Suppose we have a boundary condition that the potential on the base and top sides is $\varphi = V$, and on the surface of the side of the cylinder, $\varphi = 0$. We want to find the potential inside the cylinder, given these boundary conditions.

Since we have no ϕ dependence in the boundary conditions, we only have to deal with one Bessel function, J_0, since $\Phi = e^{\pm i\nu\phi} = 1$, so $\nu = 0$:

$$\varphi = \sum_n C_n J_0\left(k_{0,n}\rho\right)\cosh\left(k_{0,n}\left(z - \frac{L}{2}\right)\right)$$

Where we have replaced the sum of positive and negative exponentials with a hyperbolic cosine centered around $L/2$, since our problem is symmetric in the z direction around $L/2$, and sinh is not symmetric around the centerpoint. Using the fact that $\varphi|_{\rho=a} = 0$, we have that $k_{0,n}a = x_{0,n}$, and we can start using the upper and lower boundary conditions (we actually get the same result for $z = L$ and $z = 0$):

$$V = \sum C_n J_0\left(\frac{x_{0,n}\rho}{a}\right)\cosh\left(\frac{x_{0,n}}{a}\frac{L}{2}\right)$$

Now applying the orthogonality of the Bessel functions, we multiply both sides by a Bessel function and integrate, using the orthogonality relation on the right side to get a delta function:

$$\int_0^a V\rho J_0\left(\frac{x_{0,n'}\rho}{a}\right)\, d\rho = C_n\delta_{n,n'}\cosh\left(\frac{x_{0,n}L}{2a}\right)J_1^2\left(x_{0,n}\right)$$

To compute the integral on the left side, we can use one of the recursion relations. First, letting $\xi = \frac{x_{0,n'}\rho}{a}$, the left side becomes:

$$\int VJ_0\left(\xi\right)\left(\frac{a}{x_{0,n'}}\right)^2\xi\, d\xi$$

Which, by the recursion relation previously stated, which relates the derivative of a Bessel function of one order to the Bessel function of the order below:

$$\int VJ_0\left(\xi\right)\left(\frac{a}{x_{0,n'}}\right)^2\xi\, d\xi = V\left(\frac{a}{x_{0,n'}}\right)^2 x_{0,n'}J_1\left(x_{0,n'}\right)$$

From this, we are left with

$$\varphi = 2V\sum_{n=1}^{\infty}\frac{\cosh\left(\frac{x_{0,n}}{a}\left(z-\frac{L}{2}\right)\right)}{x_{0,n}\cosh\left(\frac{x_{0,n}L}{2a}\right)}\frac{J_0\left(x_{0,n}\frac{\rho}{a}\right)}{J_1\left(x_{0,n}\right)}$$

If we look at this in the limit where $L\to\infty$, we see that we are left with something of the form of $e^{\frac{x_{0,n}}{a}(z-L)}$, which indicates that the potential becomes concentrate that the bottom and top faces of the cylinder, and is exponentially vanishing in between, which matches the physical intution, the two faces don't interact with each other for a very large cylinder.

Consider the case where the potential on a disk in a plane is given by $\varphi = V$, and everywhere else on the plane $\varphi = 0$. Once again, we have the general solution, but we can discard the ϕ dependence once more, setting $\nu = 0$. In this case, we have terms of the form:

$$J_0\left(k\rho\right)e^{-kz}$$

Where k can take on any value. Thus, we have an infinite superposition of solutions with varying k:

$$\varphi = \int_0^\infty J_0\left(k\rho\right)e^{-kz}g\left(k\right)\, dl$$

We want to solve for the coefficients $g\left(k\right)$. At $z = 0$:

$$\int_0^\infty J_0\left(k\rho\right)g\left(k\right)\, dk = M\left(\rho\right)$$

Where $M(\rho)$ is defined:

$$M(\rho) = \begin{cases} V, & \rho < a \\ 0, & \rho \geq a \end{cases}$$

Multiplying both sides by $\rho J_0(k'\rho)$, and then integrating:

$$\int_0^\infty \int_0^a \rho J_0\left(k'\rho\right) J_0(k\rho)\, g(k)\; d\rho\, dk = \int_0^a V \rho J_0\left(k'\rho\right)\; d\rho$$

The left side gives us a delta function, $\frac{1}{k}\delta(k-k')$. We can solve for $g(k)$, and then write out φ:

$$\varphi = Va \int_0^\infty dk\, e^{-kz} J_0(k\rho)\, J_1(ka)$$

1.4 Conformal Mapping

Consider a complex function $f(z) = u + iv$, where $z = x + iy = re^{i\phi}$.

We define the derivative of a complex function at some point z_0[5]:

$$f'(z)\,\big|_{z=z_0} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$$

Consider the derivative of $f(z) = |z|^2$:

$$\lim_{\Delta z \to 0} \left(\frac{|z + \Delta z|^2 - |z|^2}{\Delta z} \right)$$

We can see that along the real axis, this limit is real, and along the complex axis, the limit is complex, and therefore the limit does not exist, and the function is not differentiable.

If a function's derivatives exist at every point, the function is analytic, and satisfies the following conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Which are the Cauchy-Riemann conditions.

Proof. We want to show that $\frac{df}{dz}$ is defined and unique for all x and y.

Starting from the definition:

$$\frac{df}{dz} = \lim_{\delta z \to 0} \frac{\delta f}{\delta z}$$

[5] Note that the condition for this derivative to exist is more stringent than the real version, the limit along *any* trajectory to the point must exist for the derivative to exist. In fact, the existence of the first derivative guarantees the existence of all higher orders of derivatives.

Now we note that $\delta z = \delta x + i\delta y$, and therefore $\delta f = \delta u + i\delta v$. Thus:

$$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}$$

We now consider two approaches to the point (x, y), one that is parallel to the x-axis, and the other that is parallel to the y-axis. We will show that the requirement that these two approaches are equivalent will imply the Cauchy-Riemann conditions.

Approaching along $\delta y = 0$, We have that

$$\begin{aligned}\lim_{\delta z \to 0} \frac{\delta f}{\delta z} &= \lim_{\delta x \to 0} \frac{\delta u + i\delta v}{\delta x} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\end{aligned}$$

Approaching along $\delta x = 0$:

$$\begin{aligned}\lim_{\delta z \to} \frac{\delta f}{\delta z} &= \lim_{\delta y \to 0} \frac{\delta u + i\delta v}{i\delta y} \\ &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\end{aligned}$$

Now we enforce the condition that these two are equal:

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Now separating the real and imaginary components:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}$$

Which are the Cauchy-Riemann conditions. □

We want to be able to map the solutions of Laplace's equation in one domain to another domain. To do this, let us define a mapping. A mapping is a function $f(z)$ that creates a correspondence between one point in the complex plane to a different complex plane, $\omega = f(z)$. ω is known as the image of z, under the mapping f.

Consider the mapping $\omega = i + ze^{i\pi/4}$. Looking at this, if we insert $z = re^{i\phi}$, we see that we rotate the angle, $re^{i\phi} \to re^{i\left(\phi + \frac{\pi}{4}\right)}$. What happens if we look at an area of the original complex plane? We can see that a square in the original plane is rotated by $\pi/4$, and shifted upwards by i.

Consider a point in the complex plane, $z = x + iy$. We can consider an infinitesimal displacement, $dz = dx + i\,dy$. We can compute the magnitude:

$$|dz|^2 = (dx)^2 + (dy)^2$$

This is how we can compute lengths in the regular complex plane. We can consider lengths in the ω plane:

$$|d\omega|^2 = (du)^2 + (dv)^2$$

We can note that $|dz|^2 = |d\omega|^2 \left(\left|\frac{dz}{d\omega}\right|\right)^2$:

$$|dz|^2 = \left|\frac{dz}{d\omega}\right|^2 \left(du^2 + dv^2\right)$$

We see that lengths in the ω plane are scaled by the Jacobian term, $\left|\frac{dz}{d\omega}\right|^2$. Things that started out orthogonal to each other in the z plane will remain orthogonal in the ω plane, just scaled.

Now let us discuss the underlying theorem that we will be using.

Theorem 1.1. *Consider a function $\phi(u, v)$, which is a solution of Laplace's equation in two dimensions:*

$$\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2} = 0$$

We can define a conformal mapping $\omega = f(z) = u(x) + iv(y)$, which generates a new function $\psi(x, y)$:

$$\psi(x, y) = \phi(u(x, y), v(x, y))$$

Which is a solution to Laplace's equation in the new plane:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

Proof. Consider $\frac{\partial\psi}{\partial x}$:

$$\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}$$

We can consider the second derivative:

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial\phi}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial v}\frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial v}\right)\frac{\partial v}{\partial x}$$

Applying the chain rule to the second and fourth term:

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial\phi}{\partial u}\frac{\partial^2 u}{\partial x^2} + \left[\frac{\partial^2\phi}{\partial u\partial v}\frac{\partial v}{\partial x} + \frac{\partial^2\phi}{\partial u^2}\frac{\partial u}{\partial x}\right]\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\left[\frac{\partial^2\phi}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial^2\phi}{\partial u\partial v}\frac{\partial v}{\partial x}\right] + \frac{\partial\phi}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

We can do the same exact thing for the second deriative of ψ with respect to y:

$$\frac{\partial^2\psi}{\partial y^2} = \frac{\partial\phi}{\partial u}\frac{\partial^2 u}{\partial y^2} + \left[\frac{\partial^2\phi}{\partial u\partial v}\frac{\partial v}{\partial y} + \frac{\partial^2\phi}{\partial u^2}\frac{\partial u}{\partial y}\right]\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\left[\frac{\partial^2\phi}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial^2\phi}{\partial u\partial v}\frac{\partial v}{\partial y}\right] + \frac{\partial\phi}{\partial v}\frac{\partial^2 v}{\partial y^2}$$

Now we can apply the Cauchy-Riemann conditions, which relate the partials of u and v with respect to x and y:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}$$

Now we can add the two second derivatives, which should give us zero (by Laplace's equation). When we do this, the first terms of both expressions cancel out, and the last terms of both expressions cancel out. Then looking at the mixed derivative terms, we see that after adding them together, and applying the Cauchy-Riemann conditions, they cancel out. We are then left with just the terms with second derivatives of ϕ, which when factored out, by our definition of ϕ, satisfy Laplace's equation, meaning that they sum to zero. Thus, we have that

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

Meaning that ψ satisfies Laplace's equation in the new plane. $\square$

Let us do an example. Consider the Laplace equation in the $u-v$ plane (u along the horizontal direction, v along the vertical), defined on a rectangle. Suppose that $\phi = 0$ on the lower edge of the rectangle, and $\phi = V$ on the top edge. We then have von Neumann boundary conditions on the sides, $\frac{\partial\phi}{\partial u} = 0$. Suppose that the height of the rectangle is L. The solution to this system is:

$$\phi = V\left(\frac{v}{L}\right)$$

Where $L = \pi$. We now define the conformal mapping $\omega = f(z) = \ln z$. Now we want to find out what the rectangle looks like in the image plane. To do this, we consider $z = re^{i\varphi}$. In this case, $u = \ln r$, and $v = \varphi$, where we are choosing the branch where φ ranges from 0 to π. If we now look at the transformation of each of the 4 boundaries, we see that we have a large semicircle in the upper half plane, a smaller semicircle in the upper half plane, and then two lines connecting the two along the x axis[6]. Along the two semicircles, we still have that $\frac{\partial\psi}{\partial\hat{n}} = 0$, since the relationships between the derivatives of ψ and ϕ remain the same under the conformal mapping. On the left line segment, $\psi = V$, and on the right segment we have $\psi = 0$.

Now we can look at the transformation of ϕ into ψ:

$$\begin{aligned}\psi(x, y) &= \frac{V}{\pi}v(x, y) \\ &= \frac{V}{\pi}\varphi(x, y) \\ &= \frac{V}{\pi}\arctan\left(\frac{y}{x}\right)\end{aligned}$$

Let us do a (mindblowing) example. Consider two lines in the $u-v$ plane, $u = \pi$ and $v = -\pi$. This can be thought of as the setup for an infinite parallel plate capacitor. The lines of constant

[6] Essentially a half-donut shape.

electric field are where we have constant u, and the lines of constant potential are when we have constant v. Now consider the mapping $z = w + e^w$. We can first consider where the $v = 0$ and $v = \pm\pi$ transform to. Expanding our transformation:

$$x + iy = u + iv + e^{u+iv}$$

Which gives us that

$$\begin{aligned} x &= u + e^u \cos v \\ y &= v + e^u \sin v \end{aligned}$$

Looking at the case where $v = 0$:

$$\begin{aligned} x &= u + e^u \\ y &= 0 \end{aligned}$$

We see that for $u = -\infty$, $x = -\infty$ as well. For $u = 0$, we have $x = 1$. For $u = \infty$, this maps to $x = \infty$.

Now looking at $v = \pm\pi$:

$$\begin{aligned} x &= u - e^u \\ y &= \pm\pi \end{aligned}$$

Looking at the function $x = u - e^u$, we see that for very negative u, we asymptote to $x = u$, and for very positive u, we have $-e^u$. This is extremized at $u = 0$, so $x = -1$. By looking at the second derivative, we see that this extrema must be an maximum. Thus, we have mapped all values of u onto a range of x values from $-\infty$ to -1. Similarly, the same behavior occurs for the $-\pi$ line, we "fold" the infinite line into an infinite half-line. Thus we have mapped an infinite capacitor into a capacitor with an edge, at $x = -1$. We know the solution to the infinite parallel plate capacitor case, which we can then conformally map to solve for the capacitor with an edge. If we find the solution (using Mathematica or Python), we can see that the field lines at the edge of the capacitor bulge outwards.

1.5 Multipole Expansion

Suppose we have some charge density $\rho(\boldsymbol{x}')$, with arbitrary shape. We want to find the electric potential at a point $\boldsymbol{x}$ that is very far away, a distance r from the origin. We want to expand the potential in powers of r:

$$\varphi = \frac{Q}{r} + \mathcal{O}\left(\frac{1}{r^n}\right)$$

We can solve this system exactly. First, we look at a charge element in the charge distribution:

$$dq = \rho\left(\boldsymbol{x}'\right) d^3\boldsymbol{x}'$$

We can then find the potential at $\boldsymbol{x}$ due to this charge element at $\boldsymbol{x}'$:

$$\frac{dq}{|\boldsymbol{x} - \boldsymbol{x}'|} = \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}$$

We can now integrate over all of the charges to get the total potential:

$$\begin{aligned}\varphi &= \int_V \frac{dq}{|\boldsymbol{x}-\boldsymbol{x}'|} \\ &= \int_V \frac{\rho\left(\boldsymbol{x}'\right)}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3\boldsymbol{x}'\end{aligned}$$

We want to find out the behavior of the potential that is universal at long distances, regardless of ρ. We can look at the denominator:

$$\begin{aligned}\left|\boldsymbol{x}-\boldsymbol{x}'\right| &= \left(x^2 + \left(x'\right)^2 + 2\boldsymbol{x}\cdot\boldsymbol{x}'\right)^{1/2} \\ &= x\left(1+\left(\frac{x'}{x}\right)^2 - 2\frac{\boldsymbol{x}\cdot\boldsymbol{x}'}{x^2}\right)^2\end{aligned}$$

Now if we assume that the second two terms are small, ε, and using the expansion of $1/\sqrt{1+\varepsilon}$:

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{x}\left[1+\frac{\boldsymbol{x}\cdot\boldsymbol{x}'}{x^2} - \left(\frac{x'}{x}\right)^2\frac{1}{2} + \frac{3}{8}4\frac{\left(\boldsymbol{x}\cdot\boldsymbol{x}'\right)^2}{x^4} + \mathcal{O}\left(x^3\right)\right]$$

Now looking at the potential:

$$\varphi = \frac{\int \rho\left(\boldsymbol{x}'\right)\, d^3\boldsymbol{x}'}{x} + \frac{\int \rho\left(\boldsymbol{x}'\right)\left(\boldsymbol{x}\cdot\boldsymbol{x}'\right) d^3\boldsymbol{x}'}{x^3} + \frac{1}{2}\int \frac{\rho\left(\boldsymbol{x}'\right)}{x}\left[\frac{3\left(\boldsymbol{x}\cdot\boldsymbol{x}'\right)^2}{x^4} - \frac{\left(x'\right)^2}{x^2}\right] + \dots$$

We have found an expansion of the potential in terms of inverse powers of x. We can look at the first term, we see that it is the total charge over the distance, as expected of the first term. Looking at the second term, we have the scalar product of the dipole moment and $\boldsymbol{x}$, where $\boldsymbol{P} = \int \rho\left(\boldsymbol{x}'\right)\, d^3\boldsymbol{x}' = \int dq\, \boldsymbol{x}'$. The third term is the quadrupole moment, $Q_{ij} = \int d\boldsymbol{x}'\, \rho\left(\boldsymbol{x}'\right)\left(3\boldsymbol{x}_i\boldsymbol{x}_j - \left(x'\right)^2\delta_{ij}\right)$:

$$\varphi = \frac{Q}{x} + \frac{\boldsymbol{P}\cdot\boldsymbol{x}}{x^3} + \frac{1}{2x^5}Q_{ij}x_ix_j + \dots$$

This is the multipole expansion of the potential. A similar expansion can be done for gravity, to find gravitational potentials. However, in that case, we do not have negative masses. In this expansion, we have no radiation from the dipole term, since

$$\boldsymbol{P} = \int m\boldsymbol{r}'\, d^3\boldsymbol{x}'$$

Looking at the radiation, which is proportional to $\ddot{\boldsymbol{P}}$:

$$\ddot{\boldsymbol{P}} = \int m\boldsymbol{a}'\, d^3\boldsymbol{x}'$$

Which must be equal to zero, and thus there is no dipole gravitational radiation. The fact that the radiation only appears at the quadrupole term is why gravitational radiation is harder to detect than electromagnetic radiation.

The potential due to a dipole is given by

$$\varphi = \frac{p\cos\theta}{r^2}$$

Where $p = ed$, where e is the charge difference and d is the distance between the two charges.

We have the potential due to a dipole, using the multipole expansion, at a point $\boldsymbol{x}$:

$$\varphi = \frac{\boldsymbol{p} \cdot \boldsymbol{x}}{x^3}$$

We can find the electric field, using the fact that $\boldsymbol{E} = -\boldsymbol{\nabla}\varphi$:

$$\begin{aligned}
E_i &= -\partial_i \left[\frac{p_j x_j}{x^3}\right] \\
&= -p_j \partial_i \left[\frac{x_j}{x^3}\right] \\
&= -p_j \frac{\delta_{ij}}{x^3} + \frac{3p_j}{x^4} x_j \partial_i \left(\sqrt{x^2 + y^2 + z^2}\right) \\
&= -\frac{p_i}{x^3} + \frac{3p_j}{x^4} x_j \frac{x_i}{x}
\end{aligned}$$

From this, we have that

$$\boldsymbol{E} = -\frac{\boldsymbol{p}}{x^3} + \frac{3\left(\boldsymbol{p} \cdot \boldsymbol{x}\right)}{x^4} \hat{n}$$

Where $\hat{n} = \frac{\boldsymbol{x}}{x}$. This can be rewritten:

$$\boldsymbol{E} = -\frac{\boldsymbol{p}}{x^3} + \frac{3\left(\boldsymbol{p} \cdot \hat{n}\right)\hat{n}}{x^3}$$

This is the electric field due to a single dipole. Note that the electric field falls off as $\sim \frac{1}{x^3}$, faster than that of a single charge.

If we write down the electic field of the dipole taking into account the quadrupole moment, we would see that the quadrupole terms would fall off as $\sim \frac{1}{x^4}$.

Consider some charge distribution function $\rho\left(\boldsymbol{r}\right)$. The exact potential at some point $\boldsymbol{r}$ is given by:

$$\varphi = \int \frac{\rho\left(\boldsymbol{r}\right)' \, d^3\boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|}$$

Looking at the denominator:

$$\begin{aligned}
\frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} &= \frac{1}{r} \frac{1}{\sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\theta}} \\
&= \sum_{l=0}^{\infty} \frac{\left(r'\right)^l}{r^{l+1}} P_l\left(\cos\theta\right)
\end{aligned}$$

Where θ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$. If we want to write this in terms of the angles of the two points (lets call them $\boldsymbol{r} = (r, \Theta, \Phi)$ and $\boldsymbol{r}' = (r', \theta, \phi)$), we can use the addition relation for Legendre polynomials:

$$P_l\left(\cos\theta'\right) = \sum_{m=-l}^{l} P_l^{|m|}\left(\cos\Theta\right) P_l^{|m|}\left(\cos\theta\right) \frac{(l - |m|)!}{(l + |m|)!} e^{-im(\Phi - \phi)}$$

Now let us use the spherical harmonics:

$$
\begin{aligned}
Y_{l,m\geq 0} &= (-1)^m \, i^l \left(\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!} \right)^{1/2} P_l^m \left(\cos\theta\right) e^{im\phi} \\
Y_{l,-|m|} &= (-1)^{l-m} \, Y^*_{l,|m|}
\end{aligned}
$$

We can now write down the multipole expansion in terms of the spherical harmonics:

$$
\begin{aligned}
\varphi\left(\boldsymbol{x}\right) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int \frac{(r')^l}{r^{l+1}} \left(\frac{4\pi}{2l+1} \right) Y^*_{lm}\left(\Theta, \Phi\right) Y_{lm}\left(\theta, \phi\right) \rho\left(\boldsymbol{r}'\right) \, d^3\boldsymbol{r}' \\
&= \sum_{l=0}^{\infty} \frac{1}{r^{2l+1}} \sum_{m=-l}^{m=l} \left(\frac{4\pi}{2l+1} \right)^{1/2} Q_m^{(l)}\left(\theta, \phi\right) Y^*_{lm}\left(\Theta, \Phi\right)
\end{aligned}
$$

Where

$$
Q_m^{(l)} = \int d^3\boldsymbol{r}' \, \rho\left(\boldsymbol{r}'\right) \left(r'\right)^l \sqrt{\frac{4\pi}{2l+1}} Y_{lm}\left(\theta, \phi\right)
$$

This is the general form of the multipole expansion.

1.6 Potential Energy

Suppose we have some external electric field, with potential $\varphi\left(\boldsymbol{x}\right)$. If we place a charge distribution $\rho\left(\boldsymbol{x}\right)$ into this potential, what is the potential energy? For a single charge, we have $q\varphi\left(\boldsymbol{x}\right)$, and we can sum this over all charges:

$$
\begin{aligned}
U &= \sum_a q_a \varphi\left(\boldsymbol{x}_a\right) \\
&= \int \frac{\rho\left(\boldsymbol{x}'\right) \varphi\left(\boldsymbol{x}'\right) \, d^3\boldsymbol{x}'}{dq}
\end{aligned}
$$

Suppose we want to generate a multipole-like expansion of the potential energy. We can insert the Taylor expansion of the potential:

$$
\varphi\left(\boldsymbol{x}'\right) = \varphi\left(0\right) + \left(\boldsymbol{x}'\boldsymbol{\nabla}\right)\varphi + \frac{1}{2} x'_i x'_j \frac{\partial^2 \varphi}{\partial x'_i \partial x'_j}
$$

Now using this in the definition of U, we can go order by order:

$$
U = U^{(0)} + U^{(1)} + U^{(2)} + \dots
$$

and then compute each order:

$$
\begin{aligned}
U^{(0)} &= \varphi\left(0\right) Q_{\text{tot}} \\
U^{(1)} &= \int \rho\left(\boldsymbol{x}'\right) x'_j \underbrace{\left(\partial_j \varphi\right)}_{-E_j(0)} \, d^3\boldsymbol{x}' \\
&= -\boldsymbol{P} \cdot \boldsymbol{E}\left(0\right)
\end{aligned}
$$

$$\begin{aligned}
U^{(2)} &= \frac{1}{2}\left(\frac{\partial^2\varphi}{\partial x_i'\partial x_j'}\right)\Big|_0 \int \rho\left(\boldsymbol{x}'\right)\left[x_i'x_j' - \frac{1}{3}\delta_{ij}x'^2\right] d^3\boldsymbol{x}' \\
&= \frac{Q_{ij}}{6}\frac{\partial^2\varphi}{\partial x_i'\partial x_j'}\Big|_0
\end{aligned}$$

Using this, let us consider the situation with two dipoles, very far away from each other, both aligned in the same direction, and with strengths $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$. We want to know what the mutual force between them is. Note that, just by intuition, we know that the force must be repulsive, since the two dipoles are aligned.

To begin, we can compute the $U^{(1)}$ term:

$$U^{(1)} = -\boldsymbol{d}_2 \cdot \boldsymbol{E}_1$$

We have previously derived the electric field due to a dipole:

$$\boldsymbol{E}_1 = \frac{3\hat{n}\left(\hat{n}\cdot\boldsymbol{d}_1\right)}{R^3} - \frac{\boldsymbol{d}_1}{R^3}$$

From this, we have that

$$\begin{aligned}
U^{(1)} &= -\boldsymbol{d}_2 \cdot \boldsymbol{E}_1 \\
&= \frac{\left(\boldsymbol{d}_1\cdot\boldsymbol{d}_2\right)R^2 - 3\left(\boldsymbol{d}_1\cdot\boldsymbol{R}\right)\left(\boldsymbol{d}_2\cdot\boldsymbol{R}\right)}{R^5}
\end{aligned}$$

To compute the force, we can compute the gradient of this:

$$\boldsymbol{F} = -\boldsymbol{\nabla} U$$

This gradient contains many terms, which we can compute beforehand:

$$\begin{aligned}
\boldsymbol{\nabla}\left(R\right) &= \frac{\boldsymbol{R}}{R} \\
\boldsymbol{\nabla}\left(\boldsymbol{d}\cdot\boldsymbol{R}\right) &= \partial_i\left(d_jR_j\right) \\
&= \boldsymbol{d} \\
\boldsymbol{\nabla}\left(\frac{1}{R^5}\right) &= -\frac{5}{R^6}\left(\frac{\boldsymbol{R}}{R}\right) \\
\boldsymbol{\nabla}\left(\frac{1}{R^3}\right) &= -\frac{3}{R^4}\frac{\boldsymbol{R}}{R}
\end{aligned}$$

Putting these together, we can compute the force:

$$\begin{aligned}
\boldsymbol{F} &= -\boldsymbol{\nabla} U \\
&= \frac{3}{R^5}\left[\left(\boldsymbol{d}_1\cdot\boldsymbol{d}_2\right)\boldsymbol{R} + \boldsymbol{d}_1\left(\boldsymbol{d}_2\cdot\boldsymbol{R}\right) + \boldsymbol{d}_2\left(\boldsymbol{d}_1\cdot\boldsymbol{R}\right) - \frac{5}{R^2}\left(\boldsymbol{d}_1\cdot\boldsymbol{R}\right)\left(\boldsymbol{d}_2\cdot\boldsymbol{R}\right)\boldsymbol{R}\right]
\end{aligned}$$

We now note that this scales as $\sim \frac{1}{R^4}$. We can compute the component of the force along the vector between the two dipoles, $\hat{R}$. Discarding the terms that are perpendicular to $\boldsymbol{R}$, we find that

$$\boldsymbol{F}_{\hat{R}} = \frac{3}{R^5}\boldsymbol{R}\left(\boldsymbol{d}_1\cdot\boldsymbol{d}_2\right)$$

We see that the repulsion or attraction of the force depends on the relative orientation of the dipoles, $\boldsymbol{d}_1\cdot\boldsymbol{d}_2$, as expected.

2 Magnetostatics

2.1 Biot-Savart Law

From Ampere's Law, in the static case, the magnetic field is generated by electric current:

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J}$$

Using the definition of the vector potential:

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

We can rewrite Ampere's Law:

$$\boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \boldsymbol{A}\right) - \boldsymbol{\nabla}^2 \boldsymbol{A} = \frac{4\pi}{c}\boldsymbol{J}$$

In the Coulomb gauge (recalling that we can let $\boldsymbol{A'} = \boldsymbol{A} + \boldsymbol{\nabla}\psi$ for any scalar function ψ, where $\boldsymbol{\nabla}\psi = -\boldsymbol{\nabla} \cdot \boldsymbol{A}$), $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$, which means that the first term is zero, so we are left with:

$$\boldsymbol{\nabla}^2 \boldsymbol{A} = -\frac{4\pi}{c}\boldsymbol{J}$$

This has the solution[7]:

$$\boldsymbol{A} = \frac{1}{c}\int \frac{\boldsymbol{J}\left(\boldsymbol{x'}\right)\, d^3\boldsymbol{x'}}{\left|\boldsymbol{x} - \boldsymbol{x'}\right|}$$

We can now compute the magnetic field:

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

This has the form $\boldsymbol{\nabla} \times (f\boldsymbol{a})$, which we can write out:

$$\begin{aligned}\left(\boldsymbol{\nabla} \times (f\boldsymbol{a})\right)_i &= e_{ijk}\partial_j\left(f a_k\right)\\ &= e_{ijk}\left(\partial_j f\right) a_k + e_{ijk} f \partial_j a_k\end{aligned}$$

From this, we have that

$$\boldsymbol{\nabla} \times (f\boldsymbol{a}) = f\boldsymbol{\nabla} \times \boldsymbol{a} + \boldsymbol{\nabla} f \times \boldsymbol{a}$$

Applying this to the magnetic field, we can see that we will need to compute:

$$\begin{aligned}&\boldsymbol{\nabla}_{\boldsymbol{x}}\left(\frac{1}{\left|\boldsymbol{x} - \boldsymbol{x'}\right|}\right)\\ &\qquad = \boldsymbol{\nabla}_{\boldsymbol{x}}\frac{1}{\left(\boldsymbol{x} - \boldsymbol{x'}\right)^{1/2}}\\ &\qquad = -\frac{\left(\boldsymbol{x} - \boldsymbol{x'}\right)}{\left(\boldsymbol{x} - \boldsymbol{x'}\right)^{3/2}}\end{aligned}$$

[7] Note that this is almost the exact same case as the scalar potential and the charge density, just with an extra factor of c.

Inserting this into our expression for $\boldsymbol{B}$:

$$\begin{aligned}\boldsymbol{B} &= \boldsymbol{\nabla} \times \boldsymbol{A} \\ &= \frac{1}{c} \int \frac{\boldsymbol{J}\left(\boldsymbol{x}'\right) \times \boldsymbol{R}}{R^3} \, d^3\boldsymbol{x}'\end{aligned}$$

This is the expression for the magnetic field generated by a static current, and is known as the Biot-Savart Law.

Let us now return to Ampere's Law in the static case:

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c} \boldsymbol{J}$$

Now let us take the surface integral of both sides:

$$\begin{aligned}\oint_S \left(\boldsymbol{\nabla} \times \boldsymbol{B}\right) \cdot d\boldsymbol{S} &= \frac{4\pi}{c} I \\ \oint_l \boldsymbol{B} \cdot d\boldsymbol{l} &= \frac{4\pi}{c} I\end{aligned}$$

Where I is the total current through the region, and in the second line, we have applied Stokes' Theorem to the left side. This is the integral form of Ampere's Law.

Ampere's Law in this form is most useful for cases with symmetries, such as when we are computing the magnetic field around a wire. If we have an infinite wire with current $\boldsymbol{J}$, and we want to find the magnetic field at some distance r away from the axis, we can use Ampere's Law:

$$\begin{aligned}\oint_l \boldsymbol{B} \cdot d\boldsymbol{l} &= \frac{4\pi}{c} I \\ 2\pi r B_r &= \frac{4\pi}{c} I\end{aligned}$$

From which we find that $B_r = \frac{2I}{cr}$.

2.2 Loop of Wire

Let us consider another common situation, the case of a circular loop of wire, with radius a, that has current $\boldsymbol{J}$ running through it. We can define the current:

$$j_\phi = \frac{I}{a} \delta\left(\theta - \frac{\pi}{2}\right) \delta\left(r - a\right)$$

We can compute the total current:

$$I = \int_{a+\varepsilon}^{a-\varepsilon} j_\phi \, d\phi \, r dr$$

We want to calculate the vector potential in Cartesian, so we can write the current in terms of the Cartesian basis vectors:

$$\boldsymbol{j} = -\hat{x} j_\phi \sin\phi + \hat{y} j_\phi \cos\phi$$

Prior to beginning our computation, let us also note that

$$\delta\left(f\left(x\right)\right)=\frac{\delta\left(x\right)}{\left|f'\left(x\right)\right|}$$

To prove this, recall that:

$$\int_{-\infty}^{\infty}\delta\left(x\right)g\left(x\right)\,dx=g\left(0\right)$$

Now we can consider:

$$\begin{aligned}\int_{-\infty}^{\infty}\delta\left(\alpha x\right)g\left(x\right)\,dx&=\int_{-\infty}^{\infty}\delta\left(\xi\right)g\left(\frac{\xi}{\alpha}\right)\frac{d\xi}{\alpha}\\&=\frac{g\left(0\right)}{\alpha}\end{aligned}$$

From this, we see that $\delta\left(\alpha x\right)=\frac{1}{\alpha}\delta\left(x\right)$. If we look at the case where α is negative, we see that we get a $-1/\alpha$, meaning that we take an absolute value of α for the general relation. We can generalize this to any function $f\left(x\right)$ inside the delta function, and we have the previously stated result.

Using this, we can change the delta function in terms of θ:

$$\delta\left(\theta-\frac{\pi}{2}\right)=\delta\left(\cos\theta\right)\sin\theta$$

We can now compute $A_y\left(r,\theta\right)$:

$$A_y\left(r,\theta\right)=\frac{I}{c}\int\frac{d\left(\cos\theta'\right)\,r'^2\,dr'\,d\phi'}{\left|\boldsymbol{x}-\boldsymbol{x}'\right|}\left(\cos\phi'\right)\frac{\delta\left(\cos\theta'\right)\sin\left(\theta'\right)\delta\left(r-a\right)}{a}$$

We can rewrite the denominator:

$$\left|\boldsymbol{x}-\boldsymbol{x}'\right|=\left(r^2+r'^2-2rr'\cos\gamma\right)^{1/2}$$

Where γ is the angle between the two vectors, $\boldsymbol{r}$ and $\boldsymbol{r}'$. We can compute the angle between them by writing them in Cartesian:

$$\boldsymbol{r}=r\begin{bmatrix}\sin\theta\\0\\\cos\theta\end{bmatrix}$$
$$\boldsymbol{r}'=\begin{bmatrix}\sin\theta'\cos\phi'\\\sin\theta'\sin\phi'\\\cos\theta'\end{bmatrix}$$

From this, we have that:

$$\begin{aligned}\cos\gamma&=\frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{rr'}\\&=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\left(\phi-\phi'\right)\end{aligned}$$

Now we can replace everything in our integral with angular coordinates, rather than Cartesian.

When we are working in the $x-z$ plane, we have that $\phi = 0$, so $\cos\gamma = \sin\theta\cos\phi'$. We can then write out our integral:

$$A_y\left(r,\theta\right) = \frac{Ia}{c}\int\frac{d\phi'\ \cos\phi'}{\left(r^2+a^2-2ar\sin\theta\cos\phi'\right)^{1/2}}$$

This is exactly solvable in particular cases. In the case where we are along the z axis, by symmetry, the field must be exclusively along the z direction. The electric field in this case is then purely radial:

$$\begin{aligned}
B_r &= \left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_r\right)\right]\Big|_{\theta=0} \\
&= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\int\frac{d\phi'\cos\phi'\sin\theta}{\left(r^2+a^2-2ar\sin\theta\cos\phi'\right)^{1/2}}\right]_{\theta=0} \\
&= \frac{Ia}{rc\sin\theta}\left[\underbrace{\int_0^{2\pi}\frac{d\phi'\cos\phi'}{\left(r^2+a^2-0\right)^{1/2}}}_{0}+\frac{Ia}{c}\sin\theta\int\frac{ar\cos\theta\cos^2\phi' d\phi'}{\left(r^2+a^2-2ar\sin\theta\cos\phi'\right)^{3/2}}\right] \\
&= \frac{Ia^2}{c}\frac{\int_0^{2\pi}\cos^2\phi'\,d\phi'}{\left(r^2+a^2\right)^{3/2}} \\
&= \frac{I\pi a^2}{c}\frac{1}{\left(r^2+a^2\right)^{3/2}} \\
&= \frac{\mu}{\left(r^2+a^2\right)^{3/2}}
\end{aligned}$$

Where we have defined the dipole moment:

$$\mu = \frac{I\pi a^2}{c}$$

Now moving away from exact results, we can do a long distance expansion, considering the case where $r \gg a$. In this case, the square root in the denominator becomes $r\left(1-\frac{a}{r}\sin\theta\cos\phi'\right)$, so we can rewrite the integral:

$$\begin{aligned}
A_y &= \frac{Ia}{c}\left[\int\frac{d\phi'\cos\phi'\sin\theta}{\left(r^2+a^2-2ar\sin\theta\cos\phi'\right)^{1/2}}\right]_{\theta=0} \\
&= \frac{Ia}{cr^2}\sin\theta\int_0^{2\pi}d\phi'\cos^2\phi' \\
&= \frac{\mu\sin\theta}{r^2}
\end{aligned}$$

From this, we find that:

$$\begin{aligned}
B_r &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_y\sin\theta\right) \\
&= \frac{2\mu}{r^3}\cos\theta
\end{aligned}$$

$$\begin{aligned} B_\theta &= -\frac{1}{r\sin\theta}\frac{\partial}{\partial r}\left(r\sin\theta A_\phi\right) \\ &= \frac{\mu\sin\theta}{r^3} \end{aligned}$$

We see that a loop of current, at a large distances, takes the form of a magnetic dipole.

2.3 Finite Wire

Let us consider the current generated by a finite length of wire. Suppose we have a wire from $-L$ to L along the z axis, with current $\boldsymbol{J}$. We can write down the vector potential:

$$\boldsymbol{A} = \frac{1}{c}\int \frac{\boldsymbol{J}\, d^3\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

If we are a distance ρ away from the z axis, the vector potential will be exactly along the z axis, we will only have A_z. We can replace $\boldsymbol{J}\, d^3\boldsymbol{r}'$ with $Id\boldsymbol{z}$:

$$A_z\left(\rho\right) = \frac{I}{c}\int_{-L}^{L}\frac{dz'}{\sqrt{z'^2+\rho^2}}$$

Integrals of the form:

$$\int\frac{dx}{\sqrt{x^2+a^2}}$$

Can be solved with the trig sub $x = a\tan\theta$:

$$\begin{aligned} \int\frac{dx}{\sqrt{x^2+a^2}} &= \int\frac{a\frac{d\theta}{d\cos^2\theta}}{a\sqrt{1+\tan^2\theta}} \\ &= \int\frac{d\theta}{\cos\theta} \end{aligned}$$

Multiplying by $\cos\theta/\cos\theta$:

$$\begin{aligned} \int\frac{d\theta}{\cos\theta} &= \int\frac{d\left(\sin\theta\right)}{1-\sin^2\theta} \\ &= \int\frac{dt}{1-t^2} \\ &= \int\frac{1}{2}\left[\frac{1}{1-t}+\frac{1}{1+t}\right]dt \\ &= \frac{1}{2}\ln\frac{1+\sin\theta}{1-\sin\theta} \end{aligned}$$

Now looking back at our original integral, we have $\sin\theta = \frac{\tan\theta}{\sqrt{\tan^2\theta+1}}$. We also know that $\tan\theta = x/a$, so $\sin\theta = \frac{x}{\sqrt{x^2+a^2}}$:

$$\begin{aligned} A_z\left(\rho\right) &= \frac{I}{2c}\ln\left[\frac{\sqrt{z'^2+\rho^2}+z'}{\sqrt{z'^2+\rho^2}-z'}\right]_{-L}^{L} \\ &= \frac{I}{2c}\left[\ln\left(\frac{\sqrt{L^2+\rho^2}+L}{\sqrt{L^2+\rho^2}-L}\right)-\ln\left(\frac{\sqrt{L^2+\rho^2}-L}{\sqrt{L^2+\rho^2}+L}\right)\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{I}{c} \ln \left[\frac{\sqrt{L^2+\rho^2}+L}{\sqrt{L^2+\rho^2}-L} \right] \\
&= \frac{I}{c} \ln \left[\frac{\sqrt{1+\frac{\rho^2}{L^2}}+1}{\sqrt{1+\frac{\rho^2}{L^2}}-1} \right]
\end{aligned}
$$

For large L, this becomes:

$$
\begin{aligned}
A_z(\rho) = \frac{I}{c} \ln \left[\frac{2+\frac{1}{2}\frac{\rho^2}{L^2}}{\frac{1}{2}\frac{\rho^2}{L^2}} \right] \\
&= \frac{I}{c} \ln \left[1 + \frac{4L^2}{\rho^2} \right] \\
&= \frac{I}{c} \left[\ln 4 + 2\ln L - 2\ln \rho \right]
\end{aligned}
$$

From this, we can find the magnetic field, which we expect to be ϕ dependent, since we expect a toroidal field:

$$
\begin{aligned}
B_\phi &= -\partial_\rho A_z \\
&= -\frac{\partial}{\partial \rho} \left[\frac{I}{c} \left[\ln 4 + 2\ln L - 2\ln \rho \right] \right] \\
&= \frac{2I}{c\rho}
\end{aligned}
$$

Which matches the result from Ampere's Law for an infinite wire.

2.4 Boundary Conditions

If we have a surface, and we look at the normal and tangential components of the electric and magnetic field at a particular point on the surface:

$$
\begin{aligned}
\Delta B_n &= \\
\Delta B_t &= \\
\Delta E_n &= \\
\Delta E_t &=
\end{aligned}
$$

For the electric field, we can make a Gaussian pillbox, and we find that the normal component must be related to the surface charge density:

$$
\Delta E_n = 4\pi \sigma_s
$$

The tangential component can be shown to be zero, by using the fact that $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$, we can show that there cannot be a discontinuity in the electric field:

$$
\Delta E_t = 0
$$

For the magnetic field, we have that

$$\nabla \cdot \boldsymbol{B} = 0$$
$$\nabla \times \boldsymbol{B} = 0$$

Using the same argument as the normal component of the electric field, we have that

$$\Delta B_n = 0$$

Since there is no magnetic charge, the right side is just zero.

For the tangential magnetic field, we can apply similar logic as the tangential electric field case:

$$\Delta \boldsymbol{B} = \frac{4\pi}{c} j_s$$

2.5 Finite Thickness Wire

Consider an infinite wire along the z axis, with radius a, and current density $\boldsymbol{J}$. We can compute the vector potential:

$$\nabla^2 \boldsymbol{A} = -\frac{4\pi}{c}\boldsymbol{J}$$
$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_z}{\partial\rho}\right) = -\frac{4\pi}{c}J_z$$

Where we have discarded the other two directions by symmetry.

$$A_z = -\frac{\pi}{c}J\rho^2 + C\ln\rho + D$$

At $\rho = 0$, we want $C = 0$, since the $\ln 0$ diverges. Thus, inside the wire:

$$A_z = \begin{cases} D - \frac{\pi}{c}J\rho^2 & \rho < a \\ D' + C\ln\rho & \rho > a \end{cases}$$

Now computing the magnetic field:

$$B_\phi = \begin{cases} \frac{2\pi J\rho}{c} & \rho < a \\ -\frac{C}{\rho} & \rho > a \end{cases}$$

Now stitching the two of these together at $\rho = a$:

$$\frac{2\pi Ja}{c} = -\frac{C}{a}$$
$$C = -\frac{2\pi a^2 J}{c}$$

Now noting that $\pi a^2 J = I$, we have that

$$B_\phi = \begin{cases} \frac{2\pi J\rho}{c} & \rho < a \\ \frac{2I}{c\rho} & \rho > a \end{cases}$$

We see that the result outside of the wire matches the infinitely thin wire case.

2.6 Hydrogen Atom

Consider the current density:

$$J_\phi = J_0 r e^{-r/a} \sin\theta$$

This is the current generated by the $|n, l, m\rangle = |2, 1, m\rangle$ state of the hydrogen atom. We can look at the vector potential in the ϕ direction, which will generate our B_r and B_θ. We can look at the Laplacian:

$$\frac{1}{r}\frac{d^2}{dr^2}\left(rA_\phi\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right)\right) = -\frac{4\pi}{c}J_\phi$$

We posit a solution to this equation of the form $A_\phi = \frac{\sin\theta}{r^2} f\left(r\right)$. We are essentially saying that we expect this to behave like a dipole from far away, and the short-range behavior is encoded in $f\left(r\right)$. Inserting this back into the diffeq:

$$\begin{aligned}
\frac{1}{r}\frac{d^2}{dr^2}\left(\frac{f}{r}\right) + \frac{f}{r^3}\frac{1}{r}\left(-2\right) &= -\frac{4\pi}{c}J_0 r e^{-r/a} \\
\frac{f''}{r^2} - \frac{2f'}{r^3} &= -\frac{4\pi}{c}J_0 r e^{-r/a} \\
\frac{d}{dr}\left(\frac{f'}{r^2}\right) &= -\frac{4\pi}{c}J_0 r e^{-r/a}
\end{aligned}$$

Integrating both sides and solving for $f\left(r\right)$:

$$f\left(r\right) = -\frac{4\pi}{c}J_0 a^2 e^{-r/a}\left[r^3 + 4ar^2 + 8a^2 r + 8a^3\right] + K_1 r^3 + K_2$$

Discarding the K_1 term, since that generates a constant field at ∞, which we don't want, we are left with:

$$A_\phi = -\frac{4\pi}{c}J_0 \sin\theta a^2\left[e^{-r/a}\left[\frac{r^3 + 4ar^2 + 8a^2r^2 + 8a^3}{r^2}\right] + \frac{K_2}{r^2}\right]$$

Now noting that at $r = 0$, we see that we scale as[8]:

$$\frac{8a^3}{r^2} + \frac{K_2}{r^2}$$

In order to stop this from diverging at $r = 0$, we need $K_2 = -8a^3$. We see that we have used two boundary conditions, $A_\phi \to 0$ as $r \to \infty$, and $A_\phi \to$ const. as $r \to 0$. We then have the solution:

$$A_\phi = -\frac{4\pi}{c}J_0 \sin\theta a^2\left[e^{-r/a}\left[\frac{r^3 + 4ar^2 + 8a^2r^2 + 8a^3}{r^2}\right] - \frac{8a^3}{r^2}\right]$$

At $r \to \infty$, the exponential term drops out, so the scaling is as:

$$A_\phi \sim \frac{32\pi J_0 a^5 \sin\theta}{cr^2}$$

[8] This is done by Taylor expanding the exponential, $e^{-r/a} = 1 - r/a$, and then multiplying through and noting that the $1/r$ terms drop out, and we can just match the $1/r^2$ terms.

Derivation of Hydrogen Current

In classical physics, when we discuss current, we think of $\boldsymbol{J} = e\boldsymbol{v}$. However, in quantum mechanics, we have the analogue:

$$\boldsymbol{J} = \frac{e\boldsymbol{p}}{m}$$

Where $\boldsymbol{p} = -i\hbar\boldsymbol{\nabla}$. We can then consider expectation values:

$$\langle\psi|\boldsymbol{J}|\psi\rangle = -\frac{ie\hbar}{m}\left(\psi^*\partial_i\psi - \psi\partial_i\psi^*\right)$$

If we insert the eigenstates $|\psi_{nlm}\rangle$ of Hydrogen, we will get the aforementioned current for Hydrogen, based on the particular choice of n, l, and m.

2.7 Multipole Expansion of the Vector Potential

We have the solution to the governing equation for magnetostatics, $\boldsymbol{\nabla}^2\boldsymbol{A} = -\frac{4\pi}{c}\boldsymbol{J}$:

$$\boldsymbol{A}\left(\boldsymbol{r}\right) = \frac{1}{c}\int\frac{\boldsymbol{J}\left(\boldsymbol{r}'\right)\, d^3\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

We can look at this in the $\boldsymbol{r}\to\infty$ limit:

$$\frac{1}{c}\frac{1}{r}\left[\int\boldsymbol{J}\left(\boldsymbol{r}'\right)\, d^3\boldsymbol{r}'\right] + \frac{1}{c}\frac{1}{r^3}\left[\int\boldsymbol{J}\left(\boldsymbol{r}'\right)\left(\boldsymbol{r},\boldsymbol{r}'\right)\, d^3\boldsymbol{r}'\right] + \dots$$

This second term is the magnetic dipole term. Note that the first term is zero, there are no magnetic monopoles. To prove this, note that we can express a component of the current in the k direction as:

$$\begin{aligned}
J_k &= \boldsymbol{\nabla}\left(r_k\boldsymbol{J}\right)\\
&= \partial_i\left(r_kJ_i\right)\\
&= \delta_{ki}J_i + r_k\partial_iJ_i\\
&= J_k
\end{aligned}$$

Where we note that the second term is the divergence of J, which, in a time-independent system, is zero. Inserting this into our expression for the monopole term:

$$\int\boldsymbol{\nabla}\left(r_k\boldsymbol{J}\right)\, d^3\boldsymbol{r} = \oint_S\left(r_k\boldsymbol{J}\right)d\boldsymbol{S}$$

As we take $r\to\infty$, this right side must be zero since we assume a localized current, and we can push the bounds out to infinity, making the integral zero, and thus the overall magnetic monopole term is zero.

For the second term, we can write it out in components:

$$\frac{1}{c}\int\frac{J_kr_lr_l'}{r^3}\, d^3\boldsymbol{r}' = \frac{r_l}{cr^3}\underbrace{\int J_kr_l'\, d^3\boldsymbol{r}'}_{T_{kl}}$$

We define this integral as T_{kl}. Now let us state two relations:

$$\begin{aligned}\varepsilon_{lki}\left(\boldsymbol{r}' \times \boldsymbol{J}\right)_i &= r'_l J_k - r'_k J_l \\ \boldsymbol{\nabla}'\left(r'_l r'_k \boldsymbol{J}\right) &= r'_l J_k + r'_k J_l\end{aligned}$$

Essentially, we will decompose the tensor into a symmetric and antisymmetric terms[9], which are a gradient and a vector product respectively. Writing out the decomposition:

$$J_k r'_l = \frac{1}{2}\underbrace{\left(r'_l J_k + r'_k J_l\right)}_{\boldsymbol{\nabla}'\left(r'_l r'_k \boldsymbol{J}\right)} + \frac{1}{2}\underbrace{\left(r'_l J_k - r'_k J_l\right)}_{\varepsilon_{lki}(\boldsymbol{r}\times\boldsymbol{J})_i}$$

From this, we have that:

$$T_{kl} = \frac{1}{2}\int d^3\boldsymbol{r}' \left[\boldsymbol{\nabla}'\left(r'_l r'_k \boldsymbol{J}\right) + \varepsilon_{lki}\left(\boldsymbol{r} \times \boldsymbol{J}\right)_i\right]$$

Now taking a step back and proving these relations[10], let us take the components of the gradient relation:

$$\begin{aligned}\partial_i\left(r'_l r'_k J_i\right) &= \delta_{il} r'_k J_i + r'_l \delta_{ki} J_i \\ &= r'_k J_l + r'_l J_k\end{aligned}$$

Thus we can rewrite our tensor T_{kl}, and write out A_k:

$$A_k = \frac{1}{2}\int \varepsilon_{lki}\left(\boldsymbol{r}' \times \boldsymbol{J}\right)_i \frac{d^3\boldsymbol{r}'}{c}\frac{r_l}{r^3}$$

Note that the gradient term must go to zero, by the same argument we used to rule out magnetic monopoles. Now noting that we can rewrite this as a vector product:

$$A_k = \frac{(\boldsymbol{m} \times \boldsymbol{r})_k}{r^3}$$

Where we have defined $\boldsymbol{m}$ as:

$$\boldsymbol{m} = \frac{1}{2c}\int d^3\boldsymbol{r}' \left(\boldsymbol{r}' \times \boldsymbol{J}\right)$$

This is the magnetic moment of a dipole.

Consider a cloud of charges, each of which has the same ratio of charge to mass:

$$\frac{e_a}{m_a} = \frac{e}{m}$$

We can compute the magnetic dipole moment of the ensemble:

$$\boldsymbol{m} = \frac{1}{2c}\sum_a e_a \boldsymbol{r}_\times \boldsymbol{v}_a$$

[9]Helmholtz decomposition.
[10]He only proved one of them.

Where we have implicitly used the fact that $\boldsymbol{J}_a = e_a \boldsymbol{v}_a$. Now noting that $\boldsymbol{r} \times \boldsymbol{v}$ takes the form of angular momentum:

$$\begin{aligned}
\boldsymbol{m} &= \frac{1}{2c} \sum_a e_a \boldsymbol{r}_\times \boldsymbol{v}_a \\
&= \frac{e}{2mc} \sum_a \boldsymbol{r}_a \times \boldsymbol{p}_a \\
&= \frac{e}{2mc} \boldsymbol{L}
\end{aligned}$$

Thus we see a fundamental relation between the total angular momentum and the magnetic dipole moment. In quantum mechanics, the relation between the spin and magnetic moment of a single electron is given by the Bohr magneton:

$$\mu_B = \frac{e\hbar}{2m_e c}$$

Now let us consider the magnetic field of a magnetic dipole. We need to compute the curl:

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

Inserting the dipole vector potential, and looking at a particular component:

$$\begin{aligned}
B_i &= \varepsilon_{ijk} \partial_j \left(\frac{\varepsilon_{klm} m_l r_m}{r^3} \right) \\
&= \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right) \partial_j \left(\frac{m_l r_m}{r^3} \right) \\
&= \partial_m \left(\frac{m_i r_m}{r^3} \right) - \partial_l \left(\frac{m_l r_i}{r^3} \right) \\
&= m_i \partial_m \left(\frac{r_m}{r^3} \right) - m_l \partial_l \left(\frac{r_i}{r^3} \right) \\
&= 4\pi m_i \delta^3 \left(\boldsymbol{r} \right) - m_l \left(\frac{\delta_{il}}{r^3} - \frac{3 r_i r_l}{r^5} + \frac{4\pi}{3} \delta^3 \left(\boldsymbol{r} \right) \right)
\end{aligned}$$

Where we have used the fact that

$$\begin{aligned}
\partial_\alpha \left(\frac{r_\beta}{r^3} \right) &= -\partial_\alpha \partial_\beta \left(\frac{1}{r} \right) \\
&= \frac{\delta_{\alpha\beta}}{r^3} - \frac{3 r_\alpha r_\beta}{r^5} + \frac{4\pi}{3} \delta_{\alpha\beta} \delta^3 \left(\boldsymbol{r} \right)
\end{aligned}$$

The third term is required in order to account for the case where $\alpha = \beta$, where we know that the Laplacian of $1/r$ is $-4\pi\delta^3 \left(\boldsymbol{r} \right)$.

From our component of $\boldsymbol{B}$, we have that

$$\boldsymbol{B} = -\frac{\boldsymbol{m}}{r^3} + \frac{3\boldsymbol{r} \left(\boldsymbol{m}, \boldsymbol{r} \right)}{r^5} + \frac{8\pi}{3} \boldsymbol{m} \delta^3 \left(\boldsymbol{r} \right)$$

Which is the magnetic field generated by a magnetic dipole.

Now consider a sphere of radius R that is uniformly magnetized, there is a uniform magnetic field inside the sphere. We want to find the magnetic field outside of the sphere. Our first guess is to say that the magnetic field outside is dipolar. To check this, let us consider the boundary conditions on the magnetic field inside and outside. The normal component of the magnetic field, B_r, must be continuous. Inside the sphere, we have $B_r = B_0 \cos\theta$, and outside we expect it to match the dipole radial component:

$$B_r = \frac{2m\cos\theta}{r^3}$$

Now matching these components at the surface of the sphere:

$$\begin{aligned}
B_0 \cos\theta &= \frac{2m\cos\theta}{R^3} \\
B_0 &= \frac{2m}{R^3} \\
m &= \frac{B_0 R^3}{2}
\end{aligned}$$

We see that we can relate the strength of the interior magnetic field to the exterior magnetic field. We can compute the integral of the field over the volume over a sphere centered around the origin:

$$\begin{aligned}
\int \boldsymbol{B}\, dV &= \frac{2\boldsymbol{m}}{R^3}\frac{4\pi}{3}R^3 \\
&= \frac{8\pi}{3}\boldsymbol{m}
\end{aligned}$$

We see that there is no R dependence for the total magnetic field, if we fix $\boldsymbol{m}$, then shrinking R increases the strength of the magnetic field:

$$B_0 \sim \frac{1}{R^3}$$

Now if we look at the dipole field integrated over a volume:

$$\int \left[\boldsymbol{B}_d\left(\boldsymbol{r}\right) + \frac{8\pi}{3}\boldsymbol{m}\delta\left(\boldsymbol{r}\right)\right]\, d^3\boldsymbol{r} = 0 + \frac{8\pi}{3}\boldsymbol{m}$$

We can think of the delta function term as being generated by shrinking the radius of the sphere to be smaller and smaller, while holding the total magnetic field integrated over the volume to be finite.

2.8

In the electrostatics case, we compute the electric force via:

$$\int \rho\left(\boldsymbol{r}'\right)\boldsymbol{E}\left(\boldsymbol{r}'\right)\, d^3\boldsymbol{r}'$$

Where we consider the force qE, for all charges.

By analogy, we define the magnetic force:

$$\boldsymbol{F}_B = \frac{1}{c}\int \boldsymbol{J}\left(\boldsymbol{r}'\right) \times \boldsymbol{B}\left(\boldsymbol{r}'\right)\, d^3\boldsymbol{r}'$$

We can think of this as looking at the Lorentz force due to every charge:

$$\sum_a \frac{e}{c}\boldsymbol{v}_a \times \boldsymbol{B}$$

Which, when we look at infinitesimal charges, gives us the integral of the cross product $\boldsymbol{J} \times \boldsymbol{B}$.

Consider two wires, with currents $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$. Each generates its own magnetic field, which affects the other wire. We want to find the force of the magnetic field caused by $\boldsymbol{J}_1$ acting on the current $\boldsymbol{J}_2$.

We can apply the formula for the magnetic field of current $\boldsymbol{J}_1$:

$$\boldsymbol{B}_1 = \frac{1}{c}\int \frac{d^3\boldsymbol{r}'\,\boldsymbol{J}_1(\boldsymbol{r}') \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

The force due to this field on the second current is:

$$\begin{aligned}
\boldsymbol{F}_B^{(2)} &= \frac{1}{c^2}\iint \boldsymbol{J}_2(\boldsymbol{r}) \times \frac{(\boldsymbol{J}_1(\boldsymbol{r}') \times (\boldsymbol{r}-\boldsymbol{r}'))}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\, d^3\boldsymbol{r}'\, d^3\boldsymbol{r} \\
&= \frac{1}{c^2}\iint \frac{\boldsymbol{J}_1(\boldsymbol{r}')(\boldsymbol{J}_2 \cdot (\boldsymbol{r}-\boldsymbol{r}')) - (\boldsymbol{r}-\boldsymbol{r}')(\boldsymbol{J}_2 \cdot \boldsymbol{J}_1)}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\, d^3\boldsymbol{r}\, d^3\boldsymbol{r}'
\end{aligned}$$

Where we have used the identity for the vector product $\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c}$:

$$\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{b}(\boldsymbol{a}, \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}, \boldsymbol{b})$$

Now let us consider the integral:

$$\int \frac{(\boldsymbol{J}_2(\boldsymbol{r}) \cdot (\boldsymbol{r}-\boldsymbol{r}'))}{|\boldsymbol{r}-\boldsymbol{r}'|}\, d^3\boldsymbol{r}$$

Looking at div $\left(\frac{\boldsymbol{J}_2}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)$:

$$\boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{J}_2}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) = \boldsymbol{J}_2 \cdot \boldsymbol{\nabla}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) + \underbrace{\frac{\boldsymbol{\nabla} \cdot \boldsymbol{J}_2}{|\boldsymbol{r}-\boldsymbol{r}'|}}_{0}$$

Using this to rewrite our integral:

$$\begin{aligned}
\int \frac{(\boldsymbol{J}_2(\boldsymbol{r}) \cdot (\boldsymbol{r}-\boldsymbol{r}'))}{|\boldsymbol{r}-\boldsymbol{r}'|}\, d^3\boldsymbol{r} &= -\int \boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{J}_2}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)\, d^3\boldsymbol{r}' \\
&= 0
\end{aligned}$$

Where we have used the fact that the integral of a divergence is zero. This leaves our force integral as:

$$\boldsymbol{F}_B^{(2)} = -\frac{1}{c^2}\iint \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}(\boldsymbol{J}_2 \cdot \boldsymbol{J}_1)\, d^3\boldsymbol{r}\, d^3\boldsymbol{r}'$$

Note that if the two currents are in the same direction, this is an attractive force, parallel currents attract. We can also note the similarity of the electric force between two charge densities:

$$\boldsymbol{F}_E = \iint \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\rho_1\rho_2\, d^3\boldsymbol{r}\, d^3\boldsymbol{r}'$$

Except that the magnetic force is scaled by a factor of the ratio of $(v/c)^2$:

$$|F_B| = \left(\frac{v}{c}\right)^2 |F_E|$$

Let us consider the force due to a uniform magnetic field. In this case, the force is given by:

$$\boldsymbol{F} = I\int d\boldsymbol{l}\times\boldsymbol{B}_0$$

Let us expand our magnetic field around a center point $\boldsymbol{r}$:

$$\boldsymbol{B}\left(\boldsymbol{r}'\right) = \boldsymbol{B}\left(\boldsymbol{r}\right) + \left(\boldsymbol{r}'\boldsymbol{\nabla}\right)\boldsymbol{B} + \dots$$

We are saying that our magnetic field is roughly uniform for the length scale of our current distribution. Writing out the force:

$$\boldsymbol{F} = \frac{1}{c}\int d^3\boldsymbol{r}'\,\varepsilon_{ipk}J_p\left(r_l'\partial_l\right)B_k$$

We can look at a subsection of the integrand:

$$\begin{aligned}\int d^3\boldsymbol{r}'\,J_k r_l' &= -\frac{1}{2}\varepsilon_{kli}\int d^3\boldsymbol{r}'\left(\boldsymbol{r}'\times\boldsymbol{J}\right)_i \\ &= -\frac{1}{2}\varepsilon_{kli}m_i c\end{aligned}$$

Inserting this into our original integral:

$$\begin{aligned}\boldsymbol{F}_i &= \frac{1}{c}\int d^3\boldsymbol{r}'\,\varepsilon_{ipk}J_p\left(r_l'\partial_l\right)B_k \\ &= -\varepsilon_{ipk}\varepsilon_{pls}m_s\partial_l B_k \\ &= \left(\delta_{il}\delta_{ks}-\delta_{is}\delta_{kl}\right)m_s\partial_l B_k \\ &= m_l\partial_i B_k - m_i\partial_k B_k\end{aligned}$$

From this, we note that the second term has a $\partial_k B_k$, which is the divergence of $\boldsymbol{B}$, which is zero, thus we have that

$$\boldsymbol{F} = m_k\boldsymbol{\nabla}B_k$$

In the case where $\boldsymbol{m}$ is a constant, one can show that this reduces to

$$\boldsymbol{F} = \boldsymbol{\nabla}\left(\boldsymbol{m}\cdot\boldsymbol{B}\right)$$

This is the force acting on a current distribution $\boldsymbol{J}$, generated by a uniform magnetic field $\boldsymbol{B}$.

If we apply a vector calculus identity:

$$\begin{aligned}\boldsymbol{F} &= \boldsymbol{\nabla}\left(\boldsymbol{m}\cdot\boldsymbol{B}\right)\\ &= \boldsymbol{m}\times\left(\boldsymbol{\nabla}\times\boldsymbol{B}\right) + \boldsymbol{B}\times\left(\boldsymbol{\nabla}\times\boldsymbol{m}\right) + \left(\boldsymbol{m}\boldsymbol{\nabla}\right)\boldsymbol{B} + \left(\boldsymbol{B}\boldsymbol{\nabla}\right)\boldsymbol{m}\end{aligned}$$

If we apply Ampere's Law, $\boldsymbol{\nabla}\times\boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J}$, which, in this region, means that $\boldsymbol{\nabla}\times\boldsymbol{B} = 0$, and noting that $\boldsymbol{\nabla}\boldsymbol{m} = 0$ since we assumed that $\boldsymbol{m}$ is constant, so we are left with

$$\boldsymbol{F} = \left(\boldsymbol{m}\boldsymbol{\nabla}\right)\boldsymbol{B}$$

Note that if we are working in the region where $\boldsymbol{\nabla}\times\boldsymbol{B} = 0$, we can use an analagous definition to the electrostatic scalar potential:

$$\boldsymbol{B} = -\boldsymbol{\nabla}\psi$$

Which we can then plug into the definition of $\boldsymbol{F}$ and we have another way of deriving $\boldsymbol{F} = \left(\boldsymbol{m}\boldsymbol{\nabla}\right)\boldsymbol{B}$. We can also define a potential:

$$\boldsymbol{F} = -\boldsymbol{\nabla}V$$

Where $V = -\boldsymbol{m}\cdot\boldsymbol{B}$.

We can also define the work done by the magnetic force:

$$\begin{aligned}-dW &= \boldsymbol{F}\cdot d\boldsymbol{r}\\ &= m_k\underbrace{\partial_i B_k dx_i}_{dB_k}\\ &= m_k dB_k\end{aligned}$$

From this, we can determine the total work:

$$W = -\int \boldsymbol{m}d\boldsymbol{B}$$

Consider a magnetic field sourced close to the z-axis in cylindrical, with no ϕ dependence. If we are far from the source current, again $\boldsymbol{\nabla}\times\boldsymbol{B} = 0$, so we have that $\boldsymbol{B} = -\boldsymbol{\nabla}\psi$. We can look at Laplace's equation:

$$\boldsymbol{\nabla}^2\psi = 0$$

Which we can write out in cylindrical coordinates:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{\partial^2\psi}{\partial z^2} = 0$$

If we are close to the axis (small ρ), we can assume we have a power expansion:

$$\psi = \sum_n a_n(z)\,\rho^n$$

Inserting this into Laplace's equation:

$$\sum_n a_n n^2 \rho^{n-2} + \sum_n a_n''(z)\,\rho^n = 0$$

We can expand out the first few terms in the first summation:

$$\sum_n a_n(z)\,n^2\rho^{n-2} = \frac{a_1}{\rho} + 4a_2 + 9a_3\rho + \dots$$

And we can expand out the second summation:

$$\sum_n a_n''(z)\,\rho^n = a_0'' + a_1''\rho + a_2''\rho^2 + \dots$$

Thus we have:

$$\frac{a_1}{\rho} + 4a_2 + 9a_3\rho + \cdots + a_0'' + a_1''\rho + a_2''\rho^2 + \dots = 0$$

At the axis, the potential will be $\psi = a_0(z)$, which means that $B_z(z, \rho = 0) = -a_0'(z)$. If we are not at the axis, we have to cancel out the $1/\rho$ term, so $a_1 = 0$. We will see that all even coefficients, a_{2k}, will be dependent only on a_0, and we have the general relation:

$$\left[a_{k+2}(k+2)^2 + a_k''\right]\rho^k = 0$$

From the ψ expansion, we can determine the field $\boldsymbol{B}$ to any power of ρ, close to the axis, by taking the negative gradient of the potential. For the lowest order expansion:

$$B_z(z) = -a_0'(z)$$

If we look at higher orders:

$$B_z(\rho, z) = B_z(z) - \frac{\rho^2}{4}B_z'' + \dots$$

2.9 Magnetic Torque Equation

Consider the quantity $\boldsymbol{M}$:

$$\boldsymbol{M} = \frac{1}{c}\int d^3\boldsymbol{r}\,\boldsymbol{r}\times\boldsymbol{J}\times\boldsymbol{B}$$

Now noting the relation between the magnetic moment and the angular momentum, we have $\boldsymbol{m} = \gamma\boldsymbol{L}$, we see that

$$\dot{\boldsymbol{L}} = \gamma\boldsymbol{L}\times\boldsymbol{B}$$

We can see that the magnitude of $\boldsymbol{L}$ does not change, but the direction changes. This gives us the Larmor precession:

$$\dot{\boldsymbol{L}} = \boldsymbol{\Omega} \times \boldsymbol{L}$$

Where $\boldsymbol{\Omega} = \gamma \boldsymbol{B}$.

What does the $\boldsymbol{J} \times \boldsymbol{B}$ force do to distributions of current inside magnetic fields? We can look at the form of the force:

$$\begin{aligned}
\frac{1}{c}\boldsymbol{J} \times \boldsymbol{B} &= \frac{1}{4\pi}\left(\boldsymbol{\nabla} \times \boldsymbol{B} \times \boldsymbol{B}\right) \\
&= \frac{1}{4\pi}\left(\underbrace{\left(\boldsymbol{B}\boldsymbol{\nabla}\right)\boldsymbol{B}}_{\text{mag. tension}} - \underbrace{\boldsymbol{\nabla}\left(\frac{\boldsymbol{B}^2}{2}\right)}_{\text{mag. pressure}}\right)
\end{aligned}$$

The magnetic tension term wants to "straighten" out the magnetic field lines, and this is counteracted by the magnetic pressure term.

Looking at a specific case, recall the thick wire, which generated a toroidal magnetic field:

$$\boldsymbol{B}_\phi = \begin{cases} \frac{2I\rho}{a^2 c}, & \rho < a \\ \frac{2I}{c\rho}, & \rho > a \end{cases}$$

From this (since $\boldsymbol{J} = 0$ outside of the wire), we see that the tension and pressure effects cancel each other out exactly outside of current sources. Inside the wire, we can look at the two terms of the force:

$$\begin{aligned}
\boldsymbol{F}_{\text{tension}} &= \frac{1}{4\pi}\left(\boldsymbol{B}\boldsymbol{\nabla}\right)\boldsymbol{B} \\
&= -\frac{1}{4\pi}\frac{B_\rho^2}{\rho}\hat{\rho} \\
\boldsymbol{F}_{\text{pressure}} &= -\frac{1}{8\pi}\boldsymbol{\nabla}\left(B_\phi^2\right) \\
&= -\frac{\partial}{\partial\rho}\left(\frac{B_\phi^2}{8\pi}\right)\hat{\rho} \\
&= -\frac{B_\phi^2}{4\pi\rho}\hat{\rho}
\end{aligned}$$

We see that both forces work together, to squeeze the wire. If we have some resisting pressure P, which provides force $\boldsymbol{F} = -\boldsymbol{\nabla}P$, we can generate the equilibrium condition:

$$\frac{\partial}{\partial\rho}\left(P + \frac{B^2}{8\pi}\right) = \text{const.}$$

If we are in the special case where $\boldsymbol{J} = \alpha\boldsymbol{B}$, we have what is known as a force-free state, where $\boldsymbol{J} \times \boldsymbol{B} = 0$. An example of a force-free state is the corona of the sun. These are also known as Taylor states[11].

[11] Not the Taylor of Taylor series fame.

3 Electromagnetic Waves

3.1 Wave Equations

We now move to electromagnetic waves, for which we must rewrite Maxwell's equations in their time-dependent forms, with no sources:

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{E} &= -\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t}\\ \boldsymbol{\nabla}\times\boldsymbol{B} &= \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}\end{aligned}$$

And in a vacuum, we have the initial conditions of the other two Maxwell's equations:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$$

Now let us take the curl of both sides of the first equation:

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{E} &= -\frac{1}{c}\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\times\boldsymbol{B}\right)\\ \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{E}\right)-\boldsymbol{\nabla}^2\boldsymbol{E} &= -\frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2}\end{aligned}$$

From this, we have the wave equation for the electric field:

$$\left[\boldsymbol{\nabla}^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\boldsymbol{E} = 0$$

Working in the Lorenz gauge[12], where $\boldsymbol{\nabla}\cdot\boldsymbol{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}=0$, we can rewrite Maxwell's equations in terms of the potentials:

$$\begin{aligned}\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\boldsymbol{A}\right) &= \frac{1}{c}\frac{\partial}{\partial t}\left(-\boldsymbol{\nabla}\phi-\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\right)\\ \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A}\right)-\boldsymbol{\nabla}^2\boldsymbol{A} &= \boldsymbol{\nabla}\left(-\frac{1}{c}\frac{\partial\phi}{\partial t}\right)-\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2}\\ \underbrace{\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}\right)}_{0}-\boldsymbol{\nabla}^2\boldsymbol{A} &= -\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2}\end{aligned}$$

This leads to two uncoupled wave equations:

$$\begin{aligned}\boldsymbol{\nabla}^2\boldsymbol{A}-\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} &= 0\\ \boldsymbol{\nabla}^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} &= 0\end{aligned}$$

[12] This choice of gauge is made more clear when working in 4-dimensional notation, where we are stating that the 4-derivative of A_μ is zero.

Looking at the $\boldsymbol{A}$ wave equation, consider solutions of the form:

$$\boldsymbol{A}\left(\boldsymbol{r},t\right)=u\left(\boldsymbol{r},t\right)\boldsymbol{S}$$

Where $\boldsymbol{S}$ is some constant vector. Inserting this, we find an equation in terms of u:

$$\boldsymbol{\nabla}^2 u-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=0$$

Once we have $\boldsymbol{A}$, we can use the gauge condition to find ϕ:

$$\begin{aligned}\frac{\partial\phi}{\partial t}&=-c\boldsymbol{\nabla}\cdot\boldsymbol{A}\\&=-c\partial_i\left(uS'\right)\\&=-cS'\partial_i u\end{aligned}$$

Which we can integrate to find ϕ:

$$\phi=-c\boldsymbol{S}\cdot\int_{-\infty}^{t}dt'\,\boldsymbol{\nabla}u\left(\boldsymbol{r},t'\right)$$

If u satisfies its own wave equation, then we generate a ϕ that solves a wave equation.

3.2 Method of Characteristics

Consider waves propagating in one direction (which is allowed in the vacuum, since no directions are consider special). In this formalism, the wave equation takes the form:

$$\frac{\partial^2\boldsymbol{w}}{\partial z^2}-\frac{\partial^2\boldsymbol{w}}{\partial t^2}\frac{1}{c^2}=0$$

Note that the second order differential operators can be decomposed into two first order differentials:

$$\frac{\partial^2\boldsymbol{w}}{\partial z^2}-\frac{\partial^2\boldsymbol{w}}{\partial t^2}\frac{1}{c^2}=\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial z}-\frac{1}{c}\frac{\partial}{\partial t}\right)\boldsymbol{w}$$

Using this, we can rewrite our wave equation as:

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial z}-\frac{1}{c}\frac{\partial}{\partial t}\right)\boldsymbol{w}=0$$

Now consider a change of variables, $\xi=z+ct$, and $\eta=z-ct$. Applying the chain rule:

$$\begin{aligned}\frac{\partial}{\partial\xi}&=\frac{\partial}{\partial z}\frac{\partial z}{\partial\xi}+\frac{\partial}{\partial t}\frac{\partial t}{\partial\xi}\\\frac{\partial}{\partial\eta}&=\frac{\partial}{\partial z}\frac{\partial z}{\partial\eta}+\frac{\partial}{\partial t}\frac{\partial t}{\partial\eta}\end{aligned}$$

To find these derivatives, we note that $z=\frac{1}{2}\left(\xi+\eta\right)$, and $ct=\frac{1}{2}\left(\xi-\eta\right)$. From this, we can rewrite our chain rules:

$$\frac{\partial}{\partial\xi}=\frac{1}{2}\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)$$

$$\frac{\partial}{\partial\eta} = \frac{1}{2}\left(\frac{\partial}{\partial t} - \frac{1}{c}\frac{\partial}{\partial t}\right)$$

Now noting that our original decomposition of the wave equation reduces to:

$$\frac{\partial^2 \boldsymbol{w}}{\partial\xi\partial\eta} = 0$$

From this, we see that any function of only η, or a function of only ξ, are solutions to this equation:

$$\boldsymbol{w} = \boldsymbol{f}(\xi) + \boldsymbol{g}(\eta)$$

Intuitively, this makes sense, ξ and η describe waves moving to a particular direction, either left or right. An example of a solution would be $\sin\eta$, or rather $\sin(z - ct)$, which is a rightwards travelling wave. Another solution would be $\sin\xi = \sin(z + ct)$, a leftwards travelling wave. Note that the waves are travelling at the speed of light. The functions $\boldsymbol{f}$ and $\boldsymbol{g}$ can be anything, they are dependent on the initial conditions for the wave equation. Also note that this means that the *form* of the wave remains constant, it simply moves in a direction as a function of time. The lines $\xi = z + ct$ and $\eta = z - ct$ are known as the characteristics of our wave equation.

3.3 Transverse EM Waves

Suppose we have a wave travelling in the z direction. We want to find $\boldsymbol{E}(z,t)$ and $\boldsymbol{B}(z,t)$. To begin, we first convince ourselves that $\boldsymbol{E}_z = 0$. To derive this, we begin with the divergence condition:

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{E} &= 0 \\ \frac{\partial E_z}{\partial z} &= 0 \end{aligned}$$

We also have Ampere's Law:

$$\begin{aligned} \left(\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}\right)_z &= (\boldsymbol{\nabla}\times\boldsymbol{B})_z \\ &= \partial_x B_y - \partial_y B_x \\ &= 0 \end{aligned}$$

Where we have used the fact that the partials in directions other than z of the magnetic field must be zero, the wave is propagating only along the z. Thus we have that

$$\frac{\partial E_z}{\partial t} = 0$$

The electric field in the z direction must be constant in space and constant in time, meaning that it is a global constant. If we assume that the electric field at ∞ is zero, then we can set $E_z = 0$. From this, we have two solutions:

$$\begin{aligned} \boldsymbol{E}_+ &= \boldsymbol{f}_\perp(z - ct) \\ \boldsymbol{E}_- &= \boldsymbol{g}_\perp(z + ct) \end{aligned}$$

Where the perpendicular denotes that the vector functions $\boldsymbol{f}_\perp$ and $\boldsymbol{g}_\perp$ do not have z components, $\boldsymbol{k}\cdot\boldsymbol{E} = 0$. This is the definition of a transverse electromagnetic mode (TEM).

Plane waves take the form of $\boldsymbol{A}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$, and if we take the divergence of this:

$$\begin{aligned}\left(\boldsymbol{\nabla}\cdot\left(\boldsymbol{A}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right)\right)_i &= \partial_i\left(A_i e^P ik_i r_i\right)\\ &= A_i e^{ik_i r_i} ik_i\delta_{ij}\end{aligned}$$

From this, we have that

$$\boldsymbol{\nabla}\cdot\left(\boldsymbol{A}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right) = i\left(\boldsymbol{k}\cdot\boldsymbol{A}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right)$$

By Gauss's Law, this must be equal to $4\pi\rho_{\boldsymbol{k}}$:

$$i\left(\boldsymbol{k}\cdot\boldsymbol{E}_k\right) = 4\pi\rho_{\boldsymbol{k}}$$

Where $\rho_{\boldsymbol{k}}$ is the Fourier component of the charge density in the $\boldsymbol{k}$ direction.

We have the two possible electric field propagating solutions, now what does the $\boldsymbol{B}$ field solution look like? Looking at the relation between $\boldsymbol{B}$ and $\boldsymbol{E}$:

$$-\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times\boldsymbol{E}$$

If we assume that $\boldsymbol{B}$ takes the form $e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$:

$$\begin{aligned}-\frac{i\omega}{c}B_k &= i\left(\boldsymbol{k}\times\boldsymbol{E}_{\boldsymbol{k}}\right)\\ B_k &= \hat{e}_k\times\boldsymbol{E}_k\end{aligned}$$

Looking at this in components:

Not quite s
what he wa
doing here.

$$\begin{aligned}e_{ijl}\partial_j\left(B_l e^{i(k_m r_m-\omega t)}\right) &= e_{ijl}B_l\underbrace{\partial_j\left(e^{i(k_m r_m-\omega t)}\right)}_{k_m i\delta_{jm}e^{-(k_m r_m-\omega t)}}\\ &= e_{ijl}B_l ik_j e^{i(k_m r_m-\omega t)}\\ &= i\left(\boldsymbol{k}\times\boldsymbol{B}\right)\end{aligned}$$

From this, we have that $\boldsymbol{B}$ is perpendicular to $\boldsymbol{E}$, as well as the direction of propagation. Another consequence is that $|\boldsymbol{B}| = |\boldsymbol{E}|$.

We can also derive the perpendicularity of the $\boldsymbol{B}$ field by starting with the electric field solutions, in the general case, not assuming that they are plane waves. Once again applying the induction equation:

$$\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla}\times\boldsymbol{E}$$

We suppose that we have two solutions, $\boldsymbol{B}_+$ and $\boldsymbol{B}_-$:

$$\begin{aligned}\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B}_+ &= -\begin{bmatrix}0\\0\\\frac{\partial}{\partial z}\end{bmatrix}\times\boldsymbol{f}_\perp(z,t)\\ &= -\hat{e}_z\times\frac{\partial\boldsymbol{f}_\perp}{\partial z}\end{aligned}$$

$$
\begin{aligned}
&= -\hat{e}_z \times \frac{1}{c}\frac{\partial}{\partial t}\left(-\boldsymbol{f}_\perp\right) \\
\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B}_- &= -\begin{bmatrix} 0 \\ 0 \\ \frac{\partial}{\partial z} \end{bmatrix} \times \boldsymbol{g}_\perp\left(z, t\right) \\
&= -\hat{e}_z \times \frac{\partial \boldsymbol{g}_\perp}{\partial z} \\
&= -\hat{e}_z \times \frac{1}{c}\frac{\partial}{\partial t}\left(\boldsymbol{g}_\perp\right)
\end{aligned}
$$

From these, we have that

$$
\begin{aligned}
\boldsymbol{B}_+ &= \hat{e}_z \times \boldsymbol{f}_\perp \\
\boldsymbol{B}_- &= -\hat{e}_z \times \boldsymbol{g}_\perp
\end{aligned}
$$

This entire process is direction agnostic in the vacuum case, we can always define a direction of propagation:

$$
\begin{aligned}
\boldsymbol{E} &= \boldsymbol{E}_\perp\left(\boldsymbol{k}\cdot\boldsymbol{r} - ickt\right) \\
\boldsymbol{B} &= \boldsymbol{k} \times \boldsymbol{E}
\end{aligned}
$$

We can also define the phase and group speeds. Phase speed is defined as:

$$
\phi = \left(\boldsymbol{k}\cdot\boldsymbol{r} - kct\right)
$$

Using this, we can define how surfaces of constant phase are moving. In this case, these surfaces are planes, since ϕ is defined by a linear equation in multiple coordinates. The plane is orthogonal to the vector $\boldsymbol{k}$, and if we look at how it is moving, it will remain perpendicular to $\boldsymbol{k}$ and moves at the phase speed:

$$
\boldsymbol{v}_{\text{phase}} = \hat{e}_k \frac{\omega}{k}
$$

In the vacuum case:

$$
\boldsymbol{v}_{\text{phase}} = c
$$

The phase speed is not a physical speed, it is the velocity of a plane of constant phase, and it can in fact be faster than the speed of light, since no physical objects are actually moving faster than the speed of light.

How does the energy in the plane waves depend on the amplitude of the wave itself? We can define the energy density:

$$
\begin{aligned}
U_{\text{EM}} &= \frac{1}{8\pi}\left(\boldsymbol{E}\cdot\boldsymbol{E} + \boldsymbol{B}\cdot\boldsymbol{B}\right) \\
&= \frac{1}{4\pi}\left|\boldsymbol{E}_\perp\right|^2
\end{aligned}
$$

We can define a Poynting flux $\boldsymbol{S}$, which defines how the electromagnetic energy is propagating through space:

$$\begin{aligned}\boldsymbol{S} &= \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B} \\ &= \frac{c}{4\pi}\left|\boldsymbol{E}_\perp\right|^2\hat{e}_k \\ &= cU_{\text{EM}}\hat{e}_k\end{aligned}$$

And this satisfies the continuity equation:

$$\frac{\partial U_{\text{EM}}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{S} = 0$$

We can also define $\boldsymbol{g}$, the momentum of the wave:

$$\begin{aligned}\boldsymbol{g} &= \frac{\boldsymbol{S}}{c^2} \\ &= \frac{U_{\text{EM}}}{c}\hat{e}_k\end{aligned}$$

We see that we recover the expected relation, $\mathcal{E} = pc$.

3.3.1 Polarization

Now let us consider the polarization of this system. To recap, we have monochromatic plane waves:

$$\boldsymbol{E}\left(\boldsymbol{r},t\right) = \boldsymbol{\mathcal{E}}_\perp e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$$

Where $\omega = kc$. We discussed the energy density of the wave:

$$U_{\text{EM}} = \frac{1}{8\pi}\left[[\text{Re}\boldsymbol{E}]^2 + [\text{Re}\boldsymbol{B}]^2\right]$$

If we average the energy over a wave period:

$$\begin{aligned}\overline{U_{\text{EM}}} &= \langle U_{\text{EM}}\rangle_t \\ &= \frac{1}{8\pi}\left\langle\left[\frac{1}{4}\left(\boldsymbol{E}+\boldsymbol{E}^*\right)^2 + \frac{1}{4}\left[\boldsymbol{B}+\boldsymbol{B}^*\right]^2\right]\right\rangle\end{aligned}$$

In these units, the two terms are the same:

$$\begin{aligned}\overline{U_{\text{EM}}} &= \frac{2}{8\pi}\frac{1}{4}\left\langle\left(\boldsymbol{E}+\boldsymbol{E}^*\right)^2\right\rangle \\ &= \frac{1}{16\pi}\left(2\boldsymbol{E}\boldsymbol{E}^*\right) \\ &= \frac{1}{8\pi}\left|\boldsymbol{\mathcal{E}}_\perp\right|^2\end{aligned}$$

Where we have used the fact that the average of $\left(\boldsymbol{E}+\boldsymbol{E}^*\right)^2$ over a period vanishes except for the cross term, where the exponentials vanish.

Now let us attempt to look at the geometry of the field in the plane orthogonal to the direction of propagation ($\boldsymbol{k}$). We can decompose vectors in this plane in terms of $\hat{e}_1$ and $\hat{e}_2$:

$$\boldsymbol{E} = \left(\mathcal{E}_1\hat{e}_1 + \mathcal{E}_2\hat{e}_2\right)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$$

We can consider $\mathcal{E}_1$ and $\mathcal{E}_2$ as complex numbers:

$$\mathcal{E}_1 = Ae^{i\delta_1} \qquad \mathcal{E}_2 = Be^{i\delta_2}$$

From these, we can write down the real part of the wave:

$$\begin{aligned}\mathrm{Re}\boldsymbol{E} &= A\cos(\phi+\delta_1)\,\hat{e}_1 + B\cos(\phi+\delta_2)\,\hat{e}_2 \\ &= E_1\hat{e}_1 + E_2\hat{e}_2\end{aligned}$$

Where we have denoted $\phi = \boldsymbol{k}\cdot\boldsymbol{r} - \omega t$.

As we evolve in t, the field remains in this plane, but this vector will move around. What we want to do is look at the shape that this vector traces out over time.

Consider the expressions:

$$\begin{aligned}\frac{E_1}{A}\sin\delta_2 - \frac{E_2}{B}\sin\delta_1 &= \cos(\phi+\delta_1)\sin\delta_2 - \cos(\phi+\delta_2)\sin\delta_1 \\ &= \sin(\delta_2-\delta_1)\cos\phi \\ \frac{E_1}{A}\cos\delta_2 - \frac{E_2}{B}\cos\delta_1 &= \sin(\delta_2-\delta_1)\sin\phi\end{aligned}$$

If we square these two and add them together, we find the equation:

$$\left(\frac{E_1}{A}\right)^2 + \left(\frac{E_2}{B}\right)^2 - 2\frac{E_1}{A}\frac{E_2}{B}\cos\delta = \sin^2\delta$$

Where we have defined a relative phase $\delta = \delta_2 - \delta_1$. We can now see that if $\cos\delta = 0$, then we have exactly the equation of an ellipse. If $\delta \neq \frac{\pi}{2}$, we can just shift our axes and we will have a rotated ellipse.

From this, we see that we can classify the polarizations based on δ. If $\delta = m\pi$ for $m \in \mathbb{Z}$, then the $\sin^2\delta = 0$, and the $\cos\delta = \pm 1$, so we have the ellipse reducing to a line:

$$\mathrm{Re}\boldsymbol{E} = (A\hat{e}_1 + B\hat{e}_2)\cos(\phi+\delta_1)$$

The trajectory of the field in the plane will oscillate back and forth along the same direction, which we denote as **linear polarization**.

We can also define **circular polarization**, where $\delta = \frac{m\pi}{2}$ where $m = 1, 3, \ldots$, where $A = B = \frac{\mathcal{A}}{\sqrt{2}}$, we have the electric field moving in a circle around the plane. We can define two subpolarizations in this case, where we are rotating either clockwise or counterclockwise in the plane. These denote Left-handed Circular Polarization (LCP), and Right-Handed Circular Polarization (RHCP)[13]. The choice of the label depends on the perspective, whether we are looking at the tail of the wave as it propagates or if the wave is moving directly towards us.

We can generalize the polarization using 4 parameters, generally denoted I, Q, U and V. I is the total intensity, and V denotes the amount of circular polarization (the sign of V denotes the handed-ness of the circular polarization). U and V denote the two independent linear modes of polarization. We can relate the intensity to the other 3 parameters:

$$I^2 \geq Q^2 + U^2 + V^2$$

[13]Red Hot Chili Peppers?!

In the cases that we have previously considered (perfectly polarizated emission):

$$
\begin{aligned}
I &= A^2 + B^2 \\
Q &= A^2 - B^2 \\
U &= 2AB\cos\delta \\
V &= 2AB\sin\delta
\end{aligned}
$$

In the linearly polarized case we considered, $\delta = m\pi$, we have that

$$
\begin{aligned}
V &= 0 \\
Q &\neq 0 \\
U &\neq 0
\end{aligned}
$$

For the circular case, where $\delta = \frac{m\pi}{2}, m = 1, 3, \ldots$

$$
\begin{aligned}
V &= \pm AB \\
U &= 0 \\
Q &= 0
\end{aligned}
$$

These are the Stokes parameters for the different cases of perfectly polarized emission in a plane.

3.4 Waves Generated By Moving Charges

We want to find the general formula for the electromagnetic field generated by a moving charge. To do this, we introduce the concept of retarded potentials. We are introducing a source to Maxwell's equations:

$$
\begin{aligned}
\boldsymbol{\nabla}^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} &= -\frac{4\pi}{c}\boldsymbol{J} \\
\boldsymbol{\nabla}^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} &= -4\pi\rho
\end{aligned}
$$

Where we are again working in the Lorenz gauge:

$$
\boldsymbol{\nabla} \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0
$$

In the case of a single charge, our charge density is given by

$$
\rho = e\left(t\right)\delta\left(\boldsymbol{R}\right)
$$

Where $\boldsymbol{R}$ is the location of the charge. Away from the point where the charge is, we have the same scalar potential, with charge density being zero:

$$
\boldsymbol{\nabla}^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0
$$

If we think about the frame in which we are moving with the charge, we expect that we should have spherical symmetry with respect to $\boldsymbol{R}$. We can thus replace $\boldsymbol{\nabla}^2\phi$ with the spherically symmetric Laplacian:

$$
\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial \phi}{\partial R}\right) - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0
$$

Now we can show that we can remove the R dependence by substituting in a new function $\phi = \frac{\chi(R,t)}{R}$. We can first compute the derivatives of ϕ:

$$\phi' = \frac{\chi'}{R} - \frac{1}{R^2}\chi$$

From which we can find

$$\begin{aligned}\left(R^2\phi'\right)' &= \left(R\chi' - \chi\right)' \\ &= R\chi''\end{aligned}$$

Thus we have that our equation turns into:

$$\frac{\partial^2 \chi}{\partial R^2} - \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t^2} = 0$$

This is something that we have solved previously:

$$\chi = f_1\left(t - \frac{R}{c}\right) + f_2\left(t + \frac{R}{c}\right)$$

From which we can divide by R to obtain the solution for ϕ.

Which of these two solutions should we keep? First, note that the function f_1 depends on the state of charge that existed at time $t - \frac{R}{c}$, which is in the past. This is causal, at some time t, we have to wait for the information to propagate (at the speed of light) to us. Thus we pick the $f_1\left(t - \frac{R}{c}\right)$ potential, it describes something that has happened at a previous t affecting us at a later t. This is known as the retarded potential.

Also note that in the asymptotic limit where $R \to 0$, we know that $\chi = e\left(t\right)$, since we need to recover the charge distribution at the point charge itself. From this, we have that

$$\phi = \frac{e\left(t - \frac{R}{c}\right)}{R}$$

Just like we did when moving from point charge potentials to continuous charge distributions, we can move from a moving point charge to a moving charge distribution ρ:

$$\phi = \int \underbrace{\frac{1}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|}}_{R}\rho\left(\boldsymbol{r}', t - \frac{R}{c}\right)\, d^3\boldsymbol{r}' + \phi_0$$

This gives us the potential for the electromagnetic wave generated by any time-dependent charge density.

Similarly, we can compute the vector potential:

$$\boldsymbol{A} = \frac{1}{c}\int \frac{1}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|}\boldsymbol{J}\left(\boldsymbol{r}', t - \frac{R}{c}\right)\, d^3\boldsymbol{r}' + \boldsymbol{A}_0$$

While this may seem simple, these integrals are very computationally difficult. We will first look at the case where

$$\rho\left(\boldsymbol{r}, t\right) = q\delta\left(\boldsymbol{r} - \boldsymbol{r}_0\left(t\right)\right)$$

Where we have a point charge moving on the trajectory $\boldsymbol{r}_0(t)$.

In practice, we find that $t - \frac{R}{c}$ is a difficult quantity to work with. Instead, we introduce a second integral, over t', and then insert a delta function to select the correct value:

$$\phi(\boldsymbol{r}, t) = \int d^3\boldsymbol{r}' \int dt' \, \frac{\rho(\boldsymbol{r}, t')}{|\boldsymbol{r} - \boldsymbol{r}'|} \delta\left(t' - t + \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{c}\right)$$

Where, as usual, $\boldsymbol{r'}$ are the locations of the charges in the charge distribution. Inserting our definition of ρ:

$$\phi(\boldsymbol{r}, t) = q \int dt' \int d^3\boldsymbol{r}' \, \frac{\delta(\boldsymbol{r} - \boldsymbol{r}_0(t'))}{|\boldsymbol{r} - \boldsymbol{r}'|} \delta\left(t' - t + \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{c}\right)$$

If we integrate over the $\boldsymbol{r}'$ first, which gets rid of the delta function on $\boldsymbol{r'}$:

$$\phi(\boldsymbol{r}, t) = q \int dt' \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_0(t')|} \delta\left(t' - t + \frac{|\boldsymbol{r} - \boldsymbol{r}_0(t')|}{c}\right)$$

Now applying a property of the delta function:

$$\int \delta(f(x)) \, dx = \frac{1}{|f'(x)|} \Big|_{\text{zeros}}$$

From this, we can compute the zeros of the argument to the delta function:

$$t_{\text{zero}} = t - \frac{|\boldsymbol{r} - \boldsymbol{r}_0(t_{\text{zero}})|}{c}$$

To do this, let us denote the argument as the function $g(t')$:

$$g\left(t'\right) = t' - t + \frac{|\boldsymbol{r} - \boldsymbol{r}_0(t')|}{c}$$

Computing the derivative with respect to t':

$$\begin{aligned}
\frac{\partial g(t')}{\partial t'} &= 1 + \frac{1}{c}\frac{d}{dt'}\left(\sqrt{\boldsymbol{R}(t') \cdot \boldsymbol{R}(t')}\right) \\
&= 1 - \frac{1}{2c}\frac{2\boldsymbol{v}(t') \cdot \boldsymbol{R}(t')}{R(t')} \\
&= 1 - \boldsymbol{\beta}\left(t'\right)
\end{aligned}$$

Where we have used the fact that

$$\begin{aligned}
\frac{d}{dt'}(\boldsymbol{R} \cdot \boldsymbol{R}) &= 2\frac{d\boldsymbol{R}}{dt'} \cdot \boldsymbol{R} \\
&= -2\boldsymbol{v}\left(t'\right) \cdot \boldsymbol{R}\left(t'\right)
\end{aligned}$$

And we have defined $\boldsymbol{\beta}(t') = \frac{\boldsymbol{v}}{c}$. If we also define a direction vector $\hat{n}(t') = \frac{\boldsymbol{R}(t')}{R(t')}$, we can write our potential as:

$$\phi(\boldsymbol{r}, t) = \left[\frac{q}{|\boldsymbol{r} - \boldsymbol{r}_0(t')|} \underbrace{\frac{1}{|1 - \boldsymbol{\beta}(t') \cdot \hat{n}(t')|}}_{\mathcal{D}}\right]_{t' = t_{\text{zero}}}$$

This second term $\mathcal{D}$ is known as the Doppler factor. Consider a frequency ω' that is the result of a Doppler shift given by $\boldsymbol{k}$ and $\boldsymbol{v}$:

$$\begin{aligned}\omega' &= \gamma\left(\omega \pm \boldsymbol{k}\cdot\boldsymbol{v}\right)\\ &= \gamma\omega\left(1 \pm \boldsymbol{\beta}\cdot\hat{n}\right)\end{aligned}$$

We see that we have the same functional form, hence the term Doppler factor.

Also note that if we have a velocity less than c, there is only one root t_{zero}, which makes sense by causality.

We can follow a similar process for the vector potential, and we will find that:

$$\boldsymbol{A}\left(\boldsymbol{r},t\right) = \frac{1}{c}\left[\frac{q\boldsymbol{v}\left(t'\right)}{\left|\boldsymbol{r}-\boldsymbol{r}_0\left(t'\right)\right|}\mathcal{D}\right]_{t'=t_{\text{zero}}}$$

These are the Liénard-Wiechert potentials.

Now from these potentials, we can find the electric and magnetic fields:

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}$$

To begin, let us do some preliminary computations. To begin, we want to compute (noting that for ease of computation, we use t' and t_{zero} interchangeably):

$$\begin{aligned}\frac{\partial R}{\partial t} &= \frac{\partial R}{\partial t'}\frac{\partial t'}{\partial t}\\ &= \frac{\partial}{\partial t'}\left(\sqrt{\boldsymbol{R}\cdot\boldsymbol{R}}\right)\frac{\partial t'}{\partial t}\\ &= -\frac{\boldsymbol{R}\cdot\boldsymbol{v}}{R}\frac{\partial t'}{\partial t}\end{aligned}$$

If we differentiate the definition of t_{zero} with respect to t:

$$\frac{\partial t'}{\partial t} = 1 - \frac{\partial R}{\partial t}\frac{1}{c}$$

Solving this for $\frac{\partial R}{\partial t}$ and inserting it into the expression that we had before:

$$\begin{aligned}-\frac{\boldsymbol{R}\cdot\boldsymbol{v}}{R}\frac{\partial t'}{\partial t} &= c\left(1-\frac{\partial t'}{\partial t}\right)\\ \frac{\partial t'}{\partial t} &= \frac{1}{1-\boldsymbol{\beta}\cdot\hat{n}}\\ &= \mathcal{D}\end{aligned}$$

We can then write out $\frac{\partial R}{\partial t}$:

$$\frac{\partial R}{\partial t} = -\left(\hat{n}\cdot\boldsymbol{\beta}\right)\mathcal{D}$$

and also compute the gradient of R:

$$\boldsymbol{\nabla} R\left(\boldsymbol{r}, t'\right) = \hat{n} + \frac{\partial R}{\partial t'} \boldsymbol{\nabla} t'$$

Now looking at the gradient of t':

$$\boldsymbol{\nabla} t' = -\frac{1}{c} \boldsymbol{\nabla} R$$

Inserting this into our previous expression:

$$\boldsymbol{\nabla} R = \hat{n} + \frac{\partial R}{\partial t'} \left(-\frac{1}{c} \boldsymbol{\nabla} R\right)$$

Inserting the previously computed $\frac{\partial R}{\partial t'}$, we find that

$$\begin{aligned}
\boldsymbol{\nabla} t' &= -\frac{\boldsymbol{R}}{c\left(R - \frac{\boldsymbol{R}\cdot\boldsymbol{v}}{c}\right)} \\
&= -\frac{\hat{n}}{c\left(1 - \hat{n}\cdot\boldsymbol{\beta}\right)} \\
\boldsymbol{\nabla} R &= \frac{\hat{n}}{1 - \hat{n}\cdot\boldsymbol{\beta}}
\end{aligned}$$

Once we have computed all of these gradients and derivatives, we find that the electric field is of the form:

$$\boldsymbol{E} = \frac{q\left(\hat{n} - \boldsymbol{\beta}\right)\left(1 - \beta^2\right)}{R^2\left(1 - \hat{n}\cdot\boldsymbol{\beta}\right)^3} + \frac{q\left(\hat{n} \times \left(\hat{n} - \boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right)}{cR\left(1 - \hat{n}\cdot\boldsymbol{\beta}\right)^3}$$

The first term is a static field, and is enhanced in the direction of motion. If $\boldsymbol{\beta}$ is constant, the field scales as $1/R^2$, with just the static field. However, if $\boldsymbol{\beta}$ is not constant, the second term will give us a non-zero Poynting flux, and we will have radiation. This matches our intuition, an accelerating charge produces radiation. The magnetic field will be given by:

$$\boldsymbol{B} = \hat{n} \times \boldsymbol{E}$$