PHYS624 Notes

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1 Motivation

Why do we need QFT? Consider a problem we already know, the Hydrogen atom. Consider an electron in the $2s$ state. If we wait long enough, it will decay to the $1s$ state. How long does it take for the electron to decay to decay from $2s \to 1s$? In QM, we spent time computing the energy splitting between the two states, which sets the frequency of the Lyman line. We never posed the question of how long it takes to transition between the two states. The reason for this is that in non-relativistic QM, the transition time is infinite, these are stationary states, they never decay. Since both mass and energy are conserved separately, the decay process

$$2s \to 1s + \text{photons}$$

can never occur, since particle number is conserved in non-relativistic QM. Usually in non-relativistic QM, we begin with a particle and we maintain that particle, the norm of the particle's wavefunction is conserved in time. In this case, we have photons that come into being, and we need some formalism that allows us to describe processes like this. Let us write down the Schrodinger equation for the Hydrogen atom:

$$\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi\left(\boldsymbol{r},t\right) - \frac{e^2}{r}\psi\left(\boldsymbol{r},t\right) = i\hbar\frac{\partial\psi}{\partial t}$$

We have a wavefunction that describes the electron, and the $\frac{e^2}{r}$ term comes from the electromagnetic field, and we are treating this field completely classically, we are using the classical Coulomb potential. The photons are quantum objects, and they are excitations in the EM field, which means we need to treat them quantum mechanically. Just like we quantized the motion of the electron into $\psi\left(\boldsymbol{r},t\right)$, we need to quantize the EM field in order to obtain a quantum mechanical formalism for the decay process.

QFT has many subtleties, but there is a central idea that we want to highlight. In QM, we discuss wave-particle duality: if we quantize the motion of particles, we observe wave behavior. What we will see in this course is that if we quantize the motion of waves, we get particles, the duality holds bidirectionally. Most of this course will be exploring the quantization of waves and how they generate particles.

What does this other direction of the duality mean? For every particle we think of in nature, we can start with a field description, and each particle will be an excitation of the field, i.e. an electron is an excitation of the electron field.

There are three ingredients that go into QFT:

1. Non-relativistic quantum mechanics
2. Special Relativity
3. Classical field theory

With these three things, we can produce relativistic quantum field theory. In fact, we can pick any of two of these, and we have a consistent subject. For example, if we put non-relativistic QM and special relativity together, we get relativistic QM. If we put classical field theory and special relativity together, we get relativistic classical field theory (such as E&M). Finally, if we put non-relativistic QM together with classical field theory, we will get non-relativistic QFT.

In this course, we will choose to discuss classical field theory and special relativity, to obtain relativistic classical field theory. We will then consider relativistic QM, then non-relativistic QFT, and then finally we will put them all together to look at relativistic QFT.

2 Classical Field Theory

This discussion is taken from the last chapter of Goldstein.

2.1 Discrete Systems

We can think of classical field theory as the mechanics of continuous media. The way we approach a continuous system is to take it as the limit of a discrete system with many degrees of freedom, we take the continuum limit to recover the continuous system.

Consider an infinitely long elastic rod that undergoes longitudinal vibrations, that is, compression waves. We will approximate this as an infinite chain of point masses spaced a distance a apart, connected by massless springs with spring constant k:

Suppose we have a vibration along this chain, we displace the masses from their equilibrium positions. We label the displacements by η, and we index each mass, that is, the displacement of the ith mass is η_i. At equilibrium, $\eta_i = 0$ for all i.

We can write down the kinetic energy in the chain:

$$T = \frac{1}{2} m \sum_i \dot{\eta}_i^2$$

And the potential energy:

$$V = \frac{1}{2} k \sum_i \left(\eta_{i+1} - \eta_i\right)^2$$

And then write down the Lagrangian:

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} \sum_i \left[m\dot{\eta}_i^2 - k\left(\eta_{i+1} - \eta_i\right)^2 \right] \end{aligned}$$

We can rewrite this to introduce the chain spacing:

$$L = \sum aL_i$$

Where

$$L_i = \frac{1}{2}\frac{m}{a}\dot{\eta}_i^2 - \frac{1}{2} ka \left(\frac{\eta_{i+1} - \eta_i}{a}\right)^2$$

We now want to relate the quantity ka to the Young's modulus of the material, Y. To do this, we first note that for an elastic rod, from Hooke's Law, the force is equal to the Young's modulus times ξ, the extension per unit length:

$$F = Y\xi$$

Let us apply this to our system. Consider a constant force being applied to one end of the rod. In this case, we have a uniform tension being applied to the springs, which is given by Hooke's Law for a spring:

$$F = k\left(\eta_{i+1} - \eta_i\right)$$

We can rewrite this:

$$F = ka\left(\frac{\eta_{i+1} - \eta_i}{a}\right)$$

Now recall that a is the chain spacing, and therefore ka is the Young's modulus, and the remaining term is exactly the displacement per unit length.

Now let us take the continuum limit of our discrete system. To do this, we move from the discrete index i to a continuous index x. When we make this replacement, we have that η_i becomes $\eta\left(x\right)$, and η_{i+1} becomes $\eta\left(x+a\right)$. Previously, we labelled each mass by its counted number. Instead, we now label each mass by its location at equilibrium. $\eta\left(x\right)$ is the displacement of the mass that, when the system is in equilibrium, would be sitting at location x. Note that x is not a dynamical variable, it is just a constant that labels the equilibrium locations, $\eta\left(x\right)$ is the dynamical variable. This is essentially downgrading x to the level of t, instead of a dynamical variable, it is a parameter that the actual dynamical variables depend on, which is foreshadowing the introduction of relativity, but everything here is completely classical.

Now if we look at how our expressions change when we make this continuum limit:

$$\frac{\eta_{i+1} - \eta_i}{a} \to \frac{\eta\left(x+a\right) - \eta\left(x\right)}{a}$$

Now we note that in the continuum limit, this becomes $d\eta/dx$:

$$\frac{\eta\left(x+a\right) - \eta\left(x\right)}{a} = \frac{d\eta}{dx}$$

Now if we look at our summation in the continuum limit:

$$a\sum_i \to \int dx$$

And at our m/a, which now becomes the mass per unit length, μ:

$$\frac{m}{a} \to \mu$$

Putting all of these together, we find that the full continuum Lagrangian is given by

$$L = \frac{1}{2}\int \mathrm{d}x \left[\mu\dot{\eta}^2 - Y\left(\frac{\partial \eta}{\partial x}\right)^2\right]$$

Now using the Lagrangian, we can obtain the equation of motion. Let us first look at the discrete equations of motion for the ith mass:

$$m\ddot{\eta}_i - k\left(\eta_{i+1} - \eta_i\right) - k\left(\eta_i - \eta_{i-1}\right) = 0$$

Suppose we now take the continuum limit of this equation. In this case, we have that:

$$\eta_{i+1} - \eta_i \to a \left(\frac{\partial \eta}{\partial x}\right)\bigg|_x$$
$$\eta_i - \eta_{i-1} \to a \left(\frac{\partial \eta}{\partial x}\right)\bigg|_{x-a}$$

This gives us the continuum expression:

$$a\left[\mu \frac{\partial^2 \eta}{\partial t^2} - ka\frac{\partial^2 \eta}{\partial x^2}\right] = 0$$

Now recall that $Y = ka$, so we have the continuum equation of motion:

$$\mu \frac{\partial^2 \eta}{\partial t^2} - Y\frac{\partial^2 \eta}{\partial x^2} = 0$$

Which is the wave equation, and our wave velocity will be $v = \sqrt{Y/\mu}$. We obtained this by taking the continuum limit of the discrete equation of motion, but let us now recover this directly from the continuum Lagrangian that we derived earlier, rather than first discussing the discrete case.

2.2 Continuous Lagrangian Formalism

Usually, when we do particle mechanics, we write down an action, which is the time integral of a Lagrangian. In this case our Lagrangian is itself an integral over a variable. We denote the integrand as the Lagrangian density, $\mathcal{L}$:

$$L = \frac{1}{2}\int \mathrm{d}x \left[\mu\dot{\eta}^2 - Y\left(\frac{\partial \eta}{\partial x}\right)^2\right]$$
$$\mathcal{L} = \frac{1}{2}\left[\mu\dot{\eta}^2 - Y\left(\frac{\partial \eta}{\partial x}\right)^2\right]$$

Using this denotation, the action is the time integral and the spatial integral of $\mathcal{L}$.

We want to obtain the equation of motion directly from $\mathcal{L}$. In general, the Lagrangian density is a function of η and it its partials[1], along with the parameters that we have, x and t:

$$\mathcal{L} = \mathcal{L}\left(\eta, \frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial t}, x, t\right)$$

Starting from this, we define the action[2]:

$$S = \int_{t_1}^{t_2}\int_{x_1}^{x_2} \mathrm{d}x\,\mathrm{d}t\,\mathcal{L}$$

We want to extremize the action with respect to variations of the dynamical variable, η. Note that we fix the endpoints in t and x, at both ends of the trajectory. We thus fix the variation of η at the endpoints to be zero.

[1]Note that we are not technically restricted to just the first order partials, but for higher order partials, we end up with differential equations that are harder to solve and produce spurious solutions.

[2]Chacko uses I to denote the action.

Suppose the variation is of the form:

$$\eta\left(x,t\right)=\eta_0\left(x,t\right)+\delta\eta\left(x,t\right)$$

In this form, our previous fixing of the variation is written as:

$$\begin{aligned}\delta\eta\left(x_1,t\right)=\delta\eta\left(x_2,t\right)=0\\ \delta\eta\left(x,t_1\right)=\delta\eta\left(x,t_2\right)=0\end{aligned}$$

We can now write out the variation in the action:

$$\delta S=\iint \mathrm{d}x\,\mathrm{d}t\,\left[\frac{\partial\mathcal{L}}{\partial\eta}\delta\eta+\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\eta}{\partial x}\right)}\delta\left(\frac{\partial\eta}{\partial x}\right)+\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\eta}{\partial t}\right)}\delta\left(\frac{\partial\eta}{\partial t}\right)\right]$$

Now thinking back to Lagrangian dynamics, we note that

$$\begin{aligned}\delta\left(\frac{\partial\eta}{\partial x}\right)&=\frac{\partial\left(\eta_0+\delta\eta\right)}{\partial x}-\frac{\partial\eta_0}{\partial x}\\ &=\frac{\partial}{\partial x}\delta\eta\end{aligned}$$

And similarly for $\frac{\partial\eta}{\partial t}$. This allows us to rewrite the change in our action:

$$\delta S=\iint \mathrm{d}x\,\mathrm{d}t\,\left[\frac{\partial\mathcal{L}}{\partial\eta}\delta\eta+\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\eta}{\partial x}\right)}\underbrace{\delta\left(\frac{\partial\eta}{\partial x}\right)}_{\frac{\partial}{\partial x}\delta\eta}+\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\eta}{\partial t}\right)}\underbrace{\delta\left(\frac{\partial\eta}{\partial t}\right)}_{\frac{\partial}{\partial t}\delta\eta}\right]$$

Now integrating by parts, and noting that the boundary terms vanish because of the fixed boundary conditions:

$$\delta S=\iint \mathrm{d}x\,\mathrm{d}t\,\left[\frac{\partial\mathcal{L}}{\partial\eta}-\frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\eta}{\partial x}\right)}\right)-\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\eta}{\partial t}\right)}\right)\right]\delta\eta$$

If we set $\delta S=0$, then we see that the only way for this to be true is if everything in the square brackets is zero, which is the same as the usual Lagrangian argument. This leaves us with the Euler-Lagrange equation for the continuous case:

$$\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\eta}{\partial t}\right)}\right)+\frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\eta}{\partial x}\right)}\right)-\frac{\partial\mathcal{L}}{\partial\eta}=0$$

This is the equation that must be satisfied on the classical trajectory, the one that extremizes the action. Note that in the discrete case, we had a set of coupled ODEs, but in the continuous case we have a single PDE.

Let us apply this to our elastic rod, and see if this recovers the previously obtained equation of motion. We have our Lagrangian density:

$$\mathcal{L}=\frac{1}{2}\mu\left(\frac{\partial\eta}{\partial t}\right)^2-\frac{1}{2}Y\left(\frac{\partial\eta}{\partial x}\right)^2$$

Now computing our partials:

$$\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial x}\right)} = -Y\frac{\partial \eta}{\partial x}$$
$$\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial t}\right)} = \mu\frac{\partial \eta}{\partial t}$$

Similarly, we can look at $\frac{\partial \mathcal{L}}{\partial \eta}$:

$$\frac{\partial \mathcal{L}}{\partial \eta} = 0$$

Which, when inserted into our equation, gives us:

$$\mu\frac{\partial^2 \eta}{\partial t^2} - Y\frac{\partial^2 \eta}{\partial x^2} = 0$$

Which is exactly what we obtained from taking the discrete system to the continuum limit.

In this case, we have only used a single dynamical field, η. How do we generalize this to multiple fields?

Suppose we are now working with more spatial dimensions. In this case, we move from t, x to x^μ, where μ is an index, $\mu = 0, 1, 2, 3$, where $x^0 = t$, $x^1 = x$, $x^2 = y$, and $x^3 = z$. Suppose we have a general number of fields, so η becomes η^ρ, where ρ indexes over some arbitrary number of indices, it may be a Lorentz index, or it could be any number of scalar fields. We keep this arbitrary so that we can derive all cases at once.

We can write down the general Lagrangian density:

$$\mathcal{L} = \mathcal{L}\left(\eta^\rho, \partial_\nu \eta^\rho, x^\nu\right)$$

We want to extremize the action, which is now an integral over all spacetime:

$$S = \int \mathrm{d}^4x\, \mathcal{L}$$

Note that in this formalism, space and time are on equal footing, so it will be easy to generalize to relativity.

Now looking at variations in η^ρ:

$$\eta^\rho = {\eta_0}^\rho + \delta\eta^\rho$$

We again fix the endpoints, $\delta\eta^\rho = 0$ at the endpoints in spacetime.

We can look at variations in the action, where we use Einstein notation, summation over repeated indices is implied:

$$\delta S = \int \mathrm{d}^4x\, \left[\frac{\partial \mathcal{L}}{\partial \eta^\rho}\delta\eta^\rho + \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \eta^\rho\right)}\delta\left(\partial_\nu \eta^\rho\right)\right]$$

Again noting that $\delta\left(\partial_n u\eta^\rho\right) = \partial_\nu\left(\delta\eta^\rho\right)$, and integrating by parts, we have that

$$\delta S = \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial\eta^\rho}\delta\eta^\rho - \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\eta^\rho\right)}\right)\right]$$

Setting this equal to zero, and using the same argument as the single field case, we have the general Euler-Lagrange equation in the continuous formalism:

$$\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\eta^\rho\right)}\right) - \frac{\partial\mathcal{L}}{\partial\eta^\rho} = 0$$

With this, we can take a very general Lagrangian density, and then obtain the equation of motion.

Recall the classical dynamics of a single point particle. In this case, we have the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

We see that this is very similar to our continuous result, we just make space take the same footing as time, and we recover the same form.

2.3 Energy-Momentum Tensor

Recall from point particle mechanics, that we have energy conservation if the Lagrangian does not explicitly depend on time. In our continuous formalism, we will show that if the Lagrangian density does not depend on x^0, we have energy conservation, and if the density does not depend on x^i then p^i is conserved.

Let us first recall the classical proof of this, which we will then generalize to the field formalism.

If we have no explicit dependence of L on t, then we have that

$$L = L\left(q_i, \dot{q}_i\right)$$

If this is the case, then

$$\begin{aligned}\frac{dL}{dt} &= \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\frac{d}{dt}\left(\dot{q}_i\right) + \frac{\partial L}{\partial q_i}\frac{dq_i}{dt} \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\dot{q}_i\right) - \dot{q}_i\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) + \frac{\partial L}{\partial q_i}\frac{dq_i}{dt}\end{aligned}$$

Where we have rewritten the first term. Now applying the equation of motion, we know that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

Inserting this, we see that the second and third terms cancel:

$$\frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\dot{q}_i\right)$$

Which can be rewritten:

$$\frac{d}{dt}\left(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i} - L\right) = 0$$

Now nothing that this is just the time derivative of the Hamiltonian:

$$\frac{d}{dt}H = 0$$

We see that the total energy (the Hamiltonian) is a constant of the motion.

Now let us generalize this to the field formalism. We have a Lagrangian density, which is a function of our fields η^ρ, their partials, $\partial_\nu \eta^\rho$, but explicitly not a function of x^μ:

$$\mathcal{L} = \mathcal{L}\left(\eta^\rho, \partial_\nu \eta^\rho\right)$$

We can look at $\frac{\partial L}{\partial x^\nu}$:

$$\frac{\partial \mathcal{L}}{\partial x^\nu} = \frac{\partial \mathcal{L}}{\partial \eta^\rho}\partial_\nu \eta^\rho + \frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha \eta^\rho\right)}\partial_\nu\partial_\alpha\eta^\rho$$

Now rewriting the second term, just as we did in the classical derivation:

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha \eta^\rho\right)}\partial_\nu\partial_\alpha\eta^\rho = \partial_\alpha\left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha \eta^\rho\right)}\partial_\nu\eta^\rho\right] - \partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha \eta^\rho\right)}\right)\partial_\nu\eta^\rho$$

By the equation of motion, we see that the second term here cancels with the first term in the equation above.