Lattice Field Theory Notes

Hersh Kumar

Contents

1 Overview of Continuum Theory

1.1 Path Integrals in the Continuum

Consider a general quantum mechanical system in 1D with Hamiltonian:

$$H = \frac{p^2}{2m} + V\left(x\right)$$

The quantity of interest is the partition function:

$$\begin{aligned} Z &= \mathrm{Tr}\left[e^{-iHt}\right] \\ &= \int dx\ \langle x|e^{-iHt}|x\rangle \end{aligned}$$

which will give us the generating functional for correlation functions. First, let us compute the free matrix element $\langle x|e^{-iH_0t}|y\rangle$:

$$\begin{aligned} \langle x|e^{-iH_0t}|y\rangle &= \langle x|e^{-ip^2t/2m}|y\rangle \\ &= \int dp\ \langle x|e^{-ip^2t/2m}|p\rangle \langle p|y\rangle \\ &= \int dp\ \langle x|p\rangle \langle p|y\rangle\, e^{-ip^2t/2m} \\ &= \int \frac{dp}{2\pi} e^{-ip^2t/2m+i(x-y)} \\ &= \sqrt{\frac{m}{2\pi it}} e^{(x-y)^2m/2t} \end{aligned}$$

This gives us the free matrix elements, but to introduce the potential we can Trotter decompose the full time evolution operator:

$$\begin{aligned} e^{-iH\delta t} &= e^{-iV(x)\delta t/2} e^{-iH_0\delta t} e^{-iV(x)\delta t/2} + \mathcal{O}\left(\delta t^2\right) \\ &= \int dx\, dy\, e^{-iV(x)\delta t/2} \left|x\right\rangle \left\langle x\right| e^{-iH_0\delta t} \left|y\right\rangle \left\langle y\right| e^{-iV(x)\delta t/2} + \mathcal{O}\left(\delta t^2\right) \end{aligned}$$

We can now reconstruct our time evolution operator:

$$e^{-iHt} = \underbrace{e^{-iHt/N} e^{-iHt/N} \dots e^{-iHt/N}}_{N \text{ times}}$$

Where each term uses the above decomposition, with $\delta t = t/N$.

$$e^{-iHt} = \int d\boldsymbol{x} d\boldsymbol{y}\, e^{-iV(x)\delta t/2} \left|x_1\right\rangle \left\langle x_1\right| e^{-iH_0t/2} \left|y_1\right\rangle \left\langle y_1\right| e^{-iV(x)\delta t/2} \left|x_2\right\rangle \dots \left\langle x_N\right| e^{iH_0\delta t} \left|y_N\right\rangle \left\langle y_N\right| e^{-iV(x)\delta t/2} + \mathcal{O}\left(\delta t^2\right)$$

We can continue the derivation to show that:

$$Z = \left(\frac{m}{2\pi i\delta t}\right)^{N/2} \int dx_1 \dots dx_{N-1} e^{iS(x, \dots x_{N-1})}$$

Where

$$S\left(x_{1},\ldots x_{N-1}\right)=\frac{1}{2m}\frac{1}{\delta t}\sum_{j=1}^{N}\left(x_{j+1}-x_{j}\right)^{2}-\delta t\sum_{j=1}^{N}\frac{V\left(x_{j}\right)+V\left(x_{j+1}\right)}{2}$$

We can see that we can take the limit of this:

$$\lim_{\delta t\to 0}\lim_{N\to\infty}S\left(x_{1},\ldots,x_{N-1}\right)=\int_{0}^{t}dt'\,\left(\frac{1}{2}m\dot{x}\left(t'\right)^{2}-V\left(x\left(t'\right)\right)\right)$$

Which is the continuum action!

Since we have a finite number of integration measures, we can compute this path integral numerically. However, the integrand is an oscillating phase, which leads to the Monte Carlo sign problem. To avoid this, we perform an analytic continuation to imaginary time, $t=-i\tau$:

$$Z=\int dx_{1}\ldots dx_{N-1}e^{-S_{E}\left(x_{1},\ldots x_{N-1}\right)}$$

Where

$$S_{E}\left(x_{1},\ldots x_{N-1}\right)=\int_{0}^{t}d\tau'\,\left(\frac{1}{2}m\dot{x}\left(\tau'\right)^{2}+V\left(x\left(\tau'\right)\right)\right)$$

This is very similar to the Boltzmann distribution from statistical mechanics. Thus we can interpret the Euclidean path integral with periodic boundary conditions as the canonical partition function of the corresponding thermal system.

1.2 Correlation Functions

Correlation functions are of great importance as they give us infromation about the mass spectrum of a particle, hadronic contributions to $\mu\ g-2$, weak decays, etc. Correlation functions follow an operator expectation value. Let us first consider a Euclidean two point correlator:

$$\begin{aligned}\left\langle x\left(t\right)x\left(0\right)\right\rangle &=\frac{1}{Z}\int dx_{1}\ldots dx_{N}\,x\left(t\right)x\left(0\right)e^{-S\left(x_{1},\ldots x_{N}\right)}\\ &=\frac{1}{Z}\mathrm{Tr}\left[xe^{-Ht}xe^{-H(T-t)}\right]\\ &=\frac{1}{Z}\sum_{m,n}\left\langle n|e^{-HT}x|m\right\rangle\left\langle m|e^{-H(T-t)}|n\right\rangle\\ &=\frac{1}{Z}\sum_{m,n}\left\langle n|x|m\right\rangle\left\langle m|x|n\right\rangle e^{-t\Delta E_{n}}e^{-(T-t)\Delta E_{m}}\\ &=\frac{\sum_{n,m}\left\langle n|x|m\right\rangle\left\langle m|x|n\right\rangle e^{-t\Delta E_{n}}e^{-(T-t)\Delta E_{m}}}{1+e^{-T\Delta E_{1}}+e^{-T\Delta E_{2}}+\cdots}\end{aligned}$$

In the limit of large T, we recover the two point correlator:

$$\left\langle x\left(t\right)x\left(0\right)\right\rangle=\sum_{n,m}\left\langle n|x|m\right\rangle\left\langle m|x|n\right\rangle e^{-t\Delta E_{n}}$$

From this, we can extract the mass/energy spectrum from a two-point function.