Lattice Field Theory Notes

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Contents

1 Overview of Continuum Theory

Consider a general quantum mechanical system in 1D with Hamiltonian:

$$H = \frac{p^2}{2m} + V(x)$$

The quantum of interest is the partition function:

$$Z = \mathrm{Tr}\left[e^{-iHt}\right]$$

Which will give us the generating functional for correlation functions. We can write the free matrix element, with just the kinetic energy:

$$\langle x|e^{-iHt}|y\rangle = \langle x|e^{-ip^2t/2m}|y\rangle$$

And then insert a complete set of momentum states:

$$\begin{aligned}\langle x|e^{-iHt}|y\rangle &= \int dp\ \langle x|e^{-ip^2t/2m}|p\rangle\langle p|y\rangle \\ &= \int dp\ \langle x|p\rangle\langle p|y\rangle\, e^{-ip^2t/2m} \\ &= \int \frac{dp}{2\pi} e^{-ip^2t/2m+ip(x-y)} \\ &= \sqrt{\frac{m}{2\pi i t}} e^{(x-y)^2/2t}\end{aligned}$$

Unfortunately, momentum eigenstates are not eigenstates of the Hamiltonian. We can Trotter decompose this:

$$e^{-iH\delta t} = e$$

$$e^{-iHt} = \underbrace{e^{-iHt/N}e^{-iHt/N}\cdots e^{-iHt/N}}_{N\text{ times}}$$

From which we have

$$\int dx_1\, dx_2, \ldots, dx_N\, dy_1\, dy_2\ \ldots dy_N\, e^{-iV(x)\delta t/2}\left|x_1\right\rangle\left\langle x_1\right| e^{-iH_0\delta t}\left|y_1\right\rangle\left\langle y_1\right| e^{-iV(x)\delta t/2}\left|x_2\right\rangle \ldots \left\langle x_N\right| e^{iH_0\delta t}\left|y_N\right\rangle\left\langle y_N\right| e^{-iV(x$$

Now inserting completeness relations between x and y, we have

$$Z = \left(\frac{m}{2\pi i\delta t}\right)^{N/2}\int dx_1\ \ldots\ dx_{N-1}$$