Complex Analysis Notes

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Contents

We defined the idea of a ring:

$$R = (R, +, \cdot)$$

We can adjoin an element to a ring:

$$R[x] = \{\sum_{i=0}^{n} r_i x^i | r_i \in R, n \in \mathbb{N}\}$$

We can also choose to start with a ring and adjoin an element:

$$\mathbb{Z}[x] \ s.t. \ x^2 = 2$$

This gives us elements of the form

$$a_o + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

We can remove the even powers because they are just equal to 2, and we can remove the odd powers except for $a_1 x$ because we can pull out even powers of x from them. This leaves us with:

$$\mathbb{Z}[x] = \{a + bx | x^2 = 2\}$$

We can define multiplication of ring elements:

$$(a + bx)(c + dx) = (ac + bxc + 2bdx^2 + adx)$$

This proves closure, since we the output we have is of the same form of an element of the ring. Note that we have defined $x = \sqrt{2}$, which in $\mathbb{Z}$ means nothing. This means that we can redefine the square root of 2 without ever changing the definition of a square root.
If we take the ring $\mathbb{Z}/3\mathbb{Z}$, we have defined a quotient ring, where we take elements of the larger set and groups them together, in this case setting everything of the form $\{0 + 3k\} = \bar{0}$ (equal to the same element). Everything of the form $\{1 + 3k\} = \bar{1}$, and this same logic is true for 2. Quotient rings and groups create sets of equivalence classes.
How can we use the ideas of adjoining elements, quotient rings, and ideals?
Take $R[x]/(x)$, the ring adjoining x modding out the ideal generated by x. In this case, the ideal (x) can be defined:

$$(x) = \{kx | k \in R\}$$

Elements of $R[x]$ look like

$$r_o + r_1 x + r_2 x^2 + r_3 x^3 + \cdots + r_n x^n$$

Since we have set all multiples of x to 0, we are left with just r_o, which leaves us with just R. This tells us that $R[x]/(x) = R$. What else can we mod out?
What if we start with $\mathbb{R}[x]/(x^2)$? This sets all elements of the form $kx^2 = 0$. This leaves us with elements of the form:

$$\mathbb{R}[x]/(x^2) = \{a + bx | x^2 = 0\}$$

Note that x is not a member of $\mathbb{R}$, so asking what x is equal to is a bad question, since x is a unique arbitrary value that is not in the reals. Let's now think about some objects in this space. Say we want to find the multiplicative inverse of $1 + x$:

$$(1 + x)(a + bx) = 1 + 0x$$

We can't divide yet, since division only makes sense if you have a multiplicative inverse in the first place. Instead, we can multiply it out:

$$a + bx + ax + bx^2 = 1 + 0x$$

This leaves us with $1 - 1x$, which is the inverse of $1 + x$. In order to make it a field, we have to have inverses for all elements, so let's find the inverse for arbitrary elements:

$$(a + bx)(c + dx) = 1 + 0x$$

$$ac + (bc + ad)x = 1 + 0x$$

This tells us that $ac = 1$, and that $bc + ad = 0$. For any given a,b, $c = \frac{1}{a}$ and $d = \frac{-b\frac{1}{a}}{a}$. This actually tells us that there is a category of elements that do not have inverses, as if $a = 0$ and $b \neq 0$, we have an issue. This tells us that $\mathbb{R}[x]/(x^2)$ is not a field. Instead, we can take $\mathbb{R}[x]/(x^2 + 1) = \{a + bx | x^2 = -1\}$. Note that we are simply saying that $x^2 + 1$ should be absorbed into $\bar{0}$, not that there exists an element that when squared is -1. In fact, we can also write the space as $\{a + bx | x^2 \in \overline{-1}\}$.

Lets go back to figuring out inverses in this new system:

$$(a + bx)(c + dx) = (ac + bdx^2) + (bc + ad)x = (ac - bd) + (bc + ad)x$$

Given a and b, we know that

$$ac - bd = 1$$

$$bc + ad = 0$$

Solving this system for c and d:

$$c = \frac{a}{a^2 + b^2} \quad d = \frac{-b}{a^2 + b^2}$$

In this case we find that we are okay as long as a and b are simultaneously 0, which is a case that we don't care about since the additive identity by definition does not require an identity. We have now confirmed that the space is a field!
Let's take elements of $\mathbb{R}^2$ x and y:

$$(x, y) \in \mathbb{R}^2$$

$$+ : (x, y), (w, z) \to (x + w, y + z)$$

$$* : a * (x, y) \to (ax, ay)$$

We want to check whether this is a vector space. We have to check the 8 (or 10 if you prefer) axioms that define a vector space. We can also add a new operation, multiplication:

$$\cdot : (x, y) \cdot (w, z) \to (xw - yz, yw + xz)$$

Or the norm operation:

$$| \ | : (x, y) \to \sqrt{x^2 + y^2}$$

Or the Arg operation:

$$Arg : (x, y) \to \arctan(\frac{y}{x})$$

The vector space does not care that we are adding new operations, it is staying a vector space. The question is whether or not these new operations provide structure worth studying. Had we used just looked at the first and third operations, we would have obtained a field. Note that both of these lead to the complex numbers, and yet we never talked about an element that when squared equals -1.

If we take $(a, 0) \cdot (1, 0)$, we see we get $(a, 0)$. If we take a $(a, b) \cdot (c, d)$, we know that we will find some combination that will get us $(1, 0)$ (we've already done this). If we take $(0, 1) \cdot (0, 1)$, we see that it just happens to give us $(-1, 0)$! If we ask whether or not $\mathbb{R}$ is a subspace or subfield of what we have, we would intuitively say yes, but technically speaking we can't compare scalars and vectors! So instead we say that the Reals are isomorphic to a subspace or subfield:

$$\{a | a \in \mathbb{R}\} \cong \{(a, 0) | a \in \mathbb{R}\}$$

This is the formulation that we will be looking at. We will immediately stop looking at the ordered pair format, as we can simply say:

$$(a, b) = (a, 0) + (0, b) = a * (1, 0) + b * (0, 1) \cong (a, b)$$

While this may look circular, note that the first one is in the vector space and the last one is in $\mathbb{R}$. This proves the isomorphism that we were looking at.
Finally, after 2 days of complex analysis, we are defining i:

$$\boxed{i := (0, 1)}$$

1 Section 1

We have defined elements of the complex numbers as oredered pairs (a, b), and we have defined componentwise addition and multiplication.

Theorem 1.1. *$\forall a, b \in \mathbb{R}$,*

$$(a, 0) + (b, 0) = (a + b, 0)$$
$$(a, 0) \cdot (b, 0) = (ab, 0)$$

Proof. It's trivial. $\square$

This theorem gives us the isomorphism that maps the real numbers to the complex numbers:

$$\mathbb{R} \cong S \subseteq \mathbb{C}$$

Note that there is no i, so let's define it:

$$i := (0, 1)$$

Note that we haven't said that $i^2 = -1$, we say that $i^2 = (-1, 0)$, which under the natural isomorphism to the reals becomes -1.

Theorem 1.2. $i^2 = -1$

Proof.

$$i^2 = (0,1)^2 = (-1,0) \cong -1$$

□

Theorem 1.3. *Under the* $\mathbb{R} \cong S$ *isomorphism,*

$$(a,b) = a + bi$$

Proof.

$$a + bi = a(1,0) + b(0,1) = (a,0) + (0,b) = (a,b)$$

□

There are additional operations that act on the vector space, such as the Re operation, which pulls out the component that is real, and the Im operation, which pulls out the i term:

$$Re(a+bi) := a$$

$$Im(a+bi) := b$$

$$Re : \mathbb{C} \to \mathbb{R}$$

2 Section 2

The complex numbers form a field.

3 Section 3

Here we are defining division, or multiplication by the inverse. In this case we are fine since multiplicative inverses exist in the complex field.

$$z_1 z_2 = 0 \to z_1 = 0 \ v \ z_2 = 0$$

$$\frac{z_1}{z_2} := z_1 z_2^{-1} \quad z_2 \neq 0$$

Here are some other properties that we should prove:

$$\frac{z_1 + z_2}{z_3} = \frac{z_1}{z_3} + \frac{z_2}{z_3}$$

$$(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}$$

$$\frac{z_1 z_2}{z_3 z_4} = \frac{z_1}{z_3}\frac{z_2}{z_4}$$

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k}$$

4 Section 4

We have the idea of $x + iy$ that can be represented as a directed segment (sort of a vector). The absolute value doesn't really have meaning anymore, but we can consider it the distance to the origin, just like in $\mathbb{R}$.

$$|z| = \sqrt{x^2 + y^2}$$

This is called the **modulus** of z.
If we look at $|z - z_o| = R$, we see that it forms a circle (constant distance from a point) centered at z_o. We can also rewrite the modulus of z squared:

$$|z|^2 = (Re\, z)^2 + (Im\, z)^2$$

This tells us that $Re\, z \leq |Re(z)| \leq |z|$ and $Im\, z \leq |Im(z)| \leq |z|$. We have another theorem, the Triangle Inequality:

Theorem 4.1.

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

We are interested in the statement that

$$|z_1 + z_2| = ||z_1| - |z_2||$$

Proof.

$$|z_1| = |(z_1 + z_2) + (-z_2)| \leq |z_1 + z_2| + |-z_2|$$

$$|z_1| \leq |z_1 + z_2| + |z_2|$$

Now reverse the 1's and 2's, and you will be able to prove the statement we want. $\square$

5 Section 5

We define the conjugate of z:

$$\bar{z} := a - bi$$

Lets talk about the properties of the conjugate:

$$\bar{\bar{z}} = z$$

$$|\bar{z}| = |z|$$

$$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$$

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

$$Re(z) = \frac{z + \bar{z}}{2}$$

$$Im(z) = \frac{z - \bar{z}}{2i}$$

$$z\bar{z} = |z|^2$$

$$|z| > 0$$

(unless $z = 0$)
The Cauchy-Schwarz inequality:

$$|\sum_{j=1}^{n} a_j b_j|^2 \leq \sum_{j=1}^{n} |a_j|^2 \cdot \sum_{j=1}^{n} |b_j|^2$$

Let's do some proofs:

$$|z_1 z_2| = |z_1||z_2|$$

$$|z_1 z_2|^2 = (z_1 z_2)\overline{z_1 z_2}$$

$$= z_1 z_2 \bar{z}_1 \bar{z}_2$$

$$= |z_1|^2 |z_2|^2$$

Lemma: $\overline{\bar{z}w} = z\bar{w}$, $z\bar{w} + \bar{z}w = 2Re(z\bar{w})$. Looking at

$$|z+w|^2 = (z+w)(\overline{z+w}) = (z+w)(\bar{z}+\bar{w})$$

$$z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w}$$

$$= |z|^2 + 2Re(z\bar{w}) + |w|^2$$

Note that we can see that we are trying to prove the triangle inequality, so we work backwards from what we want:

$$\leq (|z| + |w|)^2$$

Expanding out we see that we have to use the property that Re has, allowing us to do the following:

$$\leq |z|^2 + 2|z\bar{w}| + |w|^2$$

$$\leq |z|^2 + w|z||w| + |w|^2$$

Lets talk about the uniqueness of zero in the complex numbers. Suppose we have 2 unique additive identities:

$$\exists 0_a, 0_b \;\; s.t. 0_a \neq 0_b$$

$$0_a + 0_b = 0_a$$

$$0_a + 0_b = 0_b + 0_a$$

$$0_b + 0_a = 0_b$$

Therefore, there can be only 1 unique zero.

6 Section 6

Let's define polar. $z = x + iy$, $z := r\cos\theta + ir\sin\theta = re^{i\theta}$. Note that if $z = 0$, θ is undefined, so we see that this form doesn't work for all complex numbers. If we allow $r \geq 0$, $r = |z|$. We also define the argument of z:

$$arg(z) := \{\theta | \tan\theta = \frac{Im(z)}{Im(z)}\}$$

This sort of sucks because we want it to be a single object, but instead we have an infinite set. Thus, we make a new operation, Arg:

$$Arg(z) = \theta \; s.t. \; \tan\theta = \frac{Im(z)}{Re(z)} \; and \; -\pi < \theta \leq \pi$$

We choose the bounds on θ because we don't want to mess up the real axis. This also ruins limits as we approach certain points, making calculus difficult to do.

We described a circle of radius R centered at z_o:

$$|z - z_o| = R$$

We can also write this as:

$$z = z_o + Re^{i\theta}$$

7 Section 7

We have talked about expressing complex numbers via polar:

$$z = r\cos\theta + i\sin\theta$$

And in this odd shorthand that we haven't fully justified:

$$z = re^{i\theta}$$

What happens when we multiply them?

$$z_1 \cdot z_2 = (r_1\cos\theta_1 + ir_1\sin\theta_1)(r_1\cos\theta_2 + ir_2\sin\theta_2)$$

$$= r_1r_2(\cos\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2) + i(r_1\cos_1 r_2\sin\theta_2 + r_1\sin\theta_1 r_2\cos\theta_2)$$

$$= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

This is what we expected, and we see that our result is what we wanted:

$$z_1 \cdot z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$$

We can also equivalently do division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

And inversion:

$$z^{-1} = \frac{1}{r}e^{-i\theta}$$

We also claim that $arg(z_1z_2) = arg(z_1) + arg(z_2)$ but we can't do this for Arg, as the sum of Args might be larger than π, making it go out of the bounds of Arg. So we find that as long as the sum of the Args remain in the domain, the statement is also true for Arg. However, we notice that we don't know how to add sets! We define set addition as follows:

$$H + K := \{h + k | h \in H \wedge k \in K\}$$

With this definition of set addition, the statement makes sense and holds true. We also claim that

$$arg(\frac{z_1}{z_2}) = arg(z_1) - arg(z_2)$$

and that

$$arg(z^{-1}) = -arg(z)$$

We can also write down a statement about exponentiation:

$$z^n = r^n e^{in\theta}$$

We define this as repeated multiplication against itself (as long as n is a positive integer). We could prove this via induction (do this). We can use $(z^{-1})^n$ to conclude that this also holds true for $n \in \mathbb{Z}^-$. For $n = 0$, we know that the real portion goes to 1 and the cosines and sines go to 1 as well. We know that the left hand side is 1 because of convention.
We also claim that $(e^{i\theta})^n = e^{i\theta n}$, and we can prove this simply via the above statement. Note that we still have only proven this for integer n. This is an interesting result:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

This is deMoivre's theorem.

8 Section 8

We begin by claiming that if $z_1 = z_2$, we require $r_1 = r_2$ and $\theta_1 = \theta_2$, but we actually have to add in a $2\pi k$:

$$r_1 = r_2 \;\; for\; r > 0$$

$$\theta_1 = \theta_2 + 2\pi k$$

or $z_1 = 0$, $z_2 = 0$. This causes what is known as a symmetry break, as we could use either $r > 0$ or $r < 0$, as they do the same thing. However, we choose to use $r > 0$ because if r is the modulus of z, it must be positive.

9 Section 9

Let's now bring up the definition of the roots of unity. We know that $\sqrt[n]{1} = 1$, but we also have -1! So we have to account for this somehow. We define the $\sqrt[n]{1}$ to be different from the solutions to $z^n - 1 = 0$. We can define the roots of unity by thinking of complex multiplication as an operation, $z_1(z) = (r_1 r)e^{i(\theta_1+\theta)}$, which allows us to define repeated multiplication:

$$z^n = z(z(z\ldots(z))\cdots)$$

In order for a number to be a root of 1, the following must be true:

$$(e^{i\theta})^n = 1$$

We see that we are really just looking at the modulus, $|z|^n$, leading us back to the equation we had:

$$z^n - 1 = 0$$

$$|z|^n = 1 \to 1$$

$$n\theta = 0 + 2k\pi$$

$$\theta = \frac{2\pi k}{n}$$

This gives us a set of solutions:

$$\{e^{i(\frac{-2\lfloor\frac{n}{2}\rfloor\pi}{n})}, \ldots, e^{i(\frac{2\lfloor\frac{n}{2}\rfloor\pi}{n})}\}$$

In practice we don't write this, and we call $\omega_n = e^{i(\frac{2\pi}{n})}$ the first root of unity. We can find the other roots of unity:

$$\{\omega_n, \omega_n^2, \ldots, \omega_n^{n-1}, 1\}$$

To generalize this, we need to solve $z^n - c = 0$, which will give us

$$\{(\sqrt[n]{c})\omega_n^j\}$$

Note that the single root of unit generates a cyclic group, as multiplying the roots with each other gives more roots of unity.
Let's now define a **principal root**:
For some $z \in \mathbb{C}$, $z = r_o e^{i\theta_o}$, $\theta_o \int (-\pi, pi]$. The principal root is defined as

$$\sqrt[n]{z} := (\sqrt[n]{r_o})e^{i\frac{\theta_o}{n}}$$

Note that this is not the same thing as $z^{\frac{1}{n}}$.
Let's do a quick example, looking at the cube roots of -8. By definition, the principal root is

$$\sqrt[n]{-8} := 8^{\frac{1}{3}} \cdot e^{\frac{i\,Arg(-8)}{3}} = 2e^{i\frac{\pi}{3}}$$

We can also solve for all numbers that satisfy the condition, by solving:

$$r^3 e^{3i\theta} = 8e^{i(\pi + 2k\pi)}$$

to find that $r = 2$ and $\theta = \frac{\pi}{3} + \frac{2\pi k}{3}$. This gives us 3 unique solutions between 0 and 2π:

$$2e^{i\frac{\pi}{3}} \quad 2e^{i\pi} \quad 2e^{i\frac{5\pi}{3}}$$

10 Section 10

We need the idea of different neighborhoods (balls) in the complex space. We need the following concepts:

1. ϵ neighborhood
2. Deleted neighborhood
3. Interior point
4. Exterior point
5. Boundary Point
6. Boundary
7. Open
8. Closed
9. Closure
10. Neither
11. Connected
12. Domain

13. Bounded
14. Accumulation point

Definition 10.1. We can define ϵ neighborhoods as a sphere, which in complex we have defined to be $|z - z_o| < \epsilon$.

Definition 10.2. An interior point is a point z_o such that

$$\exists B_\epsilon(z_o) < S$$

where S is the region that we are looking at.

Definition 10.3. An exterior point is a point z_o such that

$$\exists B_\epsilon(z_o) \cap S = \emptyset$$

Note that points that are on the boundary do not count, as we can't make a ball of size 0, as ϵ is defined to be non-zero.

Definition 10.4. A boundary point is a point z_o such that

$$\forall B_\epsilon(z_o),\ B_\epsilon(z_o) \cap S \neq \emptyset,\ B_\epsilon(z_o) \nsubseteq S$$

By convention, boundary points do not need to be points that are in the set of the region.

Definition 10.5. The boundary of region S ∂S is the set of all boundary points.

Definition 10.6. The Closure of a region S $\bar{S} = S \cup \partial S$ (The region and the boundary points).

Definition 10.7. A set is **closed** if $S = \bar{S}$ or

$$\partial S \subseteq S$$

We can use a slightly larger definition:

$$S = int(S) \cup \partial S$$

Or using the complement:

$$S = (ext(S))^c$$

We can also use the fact that S^c is open to define that the set is closed.

Definition 10.8. We want an **open** set to have no boundary points in the set:

$$\forall x \in \partial S,\ x \notin S$$

or that the intersection between the set and the boundary is the null set:

$$S \cap \partial S = \varnothing$$

We can also avoid using the boundary entirely:

$$S = int(S)$$

This tests whether every point in S is in the interior of S.

Definition 10.9. A **clopen** set is a set that is both closed and open. This can only occur when your boundary set is empty, so examples include the real numbers and the null set.

Definition 10.10. Two points in a set $x, y \in S$ are **connected** means that there exists a finite set of segments in S s.t. $S_1 : x \to a_1 \ \ldots \ S_n : a_{n-1} \to y$.

Definition 10.11. A **bounded** is a set that can be contained inside of an epsilon ball:

$$\exists \epsilon > 0 \ s.t. S \subseteq B_\epsilon(z_o)$$

Definition 10.12. Given some $\{z_1, z_2, \ldots\} \subseteq S$, the **accumulation point** is given by $\lim_{n\to\infty} z_n$. Another definition is slightly esoteric:

$$\forall \epsilon > 0 \ B_\epsilon \cap S \neq \varnothing$$

Definition 10.13. In topology, connected means that it is not equivalent to a disjoin union of two or more nonempty sets.

Definition 10.14. Path-connected:

$$\forall a, b, \in S \ \ \exists \, continuous \, f : \ [0,1] \to S \ \ s.t. \, f(0) = a \, f(1) = b$$

There are a couple of basic topological functions, such as the Topologist's Sine Curve:

$$(x, \sin(\frac{1}{x})) \cup (0,0) \ \ x \in (0,1]$$

Definition 10.15. A curve is **Simply connected** is it is path connected and can be deformed into other paths. (Coffee cup = donut)

We can take a sphere and we realize that we can map every point to a point in $\mathbb{R}^2$, and paths away from the origin in $\mathbb{R}^2$ lead to the bottom of the sphere, and we call this map a stereographic projection, and considering that point infinity is called 1 point compactification.

11 Section 11

This chapter talks about functions in complex, where

$$f : S \to \mathbb{C}, S \subseteq \mathbb{C}$$

We can write these in a more recognizable way:

$$w = f(z)$$

$$f(x + iy) = u(x,y) + iv(x,y)$$

We can also express it in polar:

$$f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$$

Let's talk about the function z^n. When $n = 1$:

$$f(z) = z^1$$

$$u = x$$

$$v = y$$

When $n = 2$:

$$f(z) = z^2$$

$$u = x^2 - y^2$$

$$v = 2xy$$

When $n = 3$, we see that it gets really annoying to do, as we have to expand out some stuff. We could have also done it as $r^3 e^{i3\theta}$. With these we can now work all rational functions, and we can use radicals (if we ssume the principal root is the only output).

12 Section 12

We can represent a function as an ordered pair:

$$f(z) = u(x, y) + iv(x, y) = (u, v)$$

We technically need 4 dimensions to graph this, but we can use level curves instead.

13 Section 13

This section is what allows us to split exponentials like $e^{ax+iy} = e^{ax}e^{iy}$. Lets start by defining what it means to exponentiate complex numbers:

$$e^z = e^{x+iy} := e^x e^{iy}$$

We've done it! When in doubt, just say you're right. Note that what we have just agrees with what he have done in the real numbers, and notationally we can see that it we didn't split anything, which while not necessary is useful. We could have chosen any different definition, but we wanted something that kept the properties that real exponentiation had, like the basic algebraic rules as well as it being one to one, increasing, always positive, etc.

14 Section 14

Let's talk about limits, because we want to do calculus. If we have a limit:

$$\lim_{z \to z_0} f(z) = w_0$$

We want to define a limit the same way that we defined limits, using epsilon balls:

$$\forall \epsilon > 0 \, \exists \delta > 0 \; s.t. \; 0 < |z - z_0| < \delta \to |f(z) - w_o| < \epsilon$$

$$z \in B_\delta(z_o) \setminus \{z_o\} \to f(z) \in B_\epsilon(w_o)$$

Note that the fact that we are saying that $\epsilon > 0$ and $\delta > 0$ implies that they are members of $\mathbb{R}$, as the complex numbers have no defined ordering.

Theorem 14.1. *There is only one limit to a function at a point.*

Proof. Take the point z_0 and say it maps to two different points, w_0 and w_1. If we construct epsilon balls around both, with some ϵ, we will get two δ balls to the preimage. WLOG, we can just choose one of them:

$$\delta = min(\delta_0, \delta_1)$$

$$|f(z) - w_0| < \epsilon$$

$$|f(z) - w_1| < \epsilon$$

We can then define the distance between the two points:

$$|w_1 - w_0| = |(f(z) - w_0) + (-(f(z) - w_1))|$$

By the triangle inequality:

$$\leq |f(z) - w_0| + |f(z) - w_1| \leq 2\epsilon$$

We know now that

$$|w_1 - w_o| \leq 2\epsilon \quad \forall \epsilon > 0$$

Because the modulus is always positive, we know that $|w_1 - w_0| = 0 \to w_1 = w_0$. □

Let's do some limits:

$$\lim_{z \to 0} \frac{z}{z} = 1$$

$$\lim_{z \to 0} \frac{z}{\bar{z}}$$

For this one, if we plug in the real component as 0, we see that we get -1, and if we plug in the complex component as 0, we get 1, and since we just proved uniqueness, this limit doesn't exist.

15 Section 15

Let's talk about properties of limits.

Theorem 15.1.

$$\lim_{z \to z_o} f(z) = w_o \; iff:$$

$$\lim_{(x,y) \to (x, y_o)} u(x, y) = u_o$$

$$\lim_{(x,y) \to (x_o, y_o)} v(x, y) = v_o$$

This means that we can break up the limit into the real and imaginary components, which is great because it works the way we think, but sucks because multivariable limits really suck.

Proof. Assume that $\lim_{(x,y) \to (x_o, y_o)} u = v_o$, and respectively $v \to v_o$. Using the definition of a limit, this tells us that

$$0 < \sqrt{(x - x_o)^2 + (y - y_o)^2} < \delta_1 \to |u - u_o| < \epsilon$$

$$0 < \sqrt{(x - x_o)^2 + (y - y_o)^2} < \delta_2 \to |v - v_o| < \epsilon$$

We want to show that $|(u + iv) - (u_o + iv_o)|$. Rearranging:

$$|(u + iv) - (u_o + iv_o)| = |(u - u_o) + i(v - v_o)| \leq |u - u_o| + |i(v - v_o)| \leq |u - u_o| + |v - v_o|$$

In order to make everything work out when we add them, we set $\epsilon = \epsilon/2$, and make $\delta = min(\delta_1, \delta_2)$. We can then rewrite the above expression via the assumption:

$$|u - u_o| + |v - v_o| = \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$\leq \epsilon$$

We now need to prove it the other way (since its an if and only if). We assume that

$$\forall \epsilon > 0 \; \exists \delta > 0 \; s.t.$$
$$0 < |(x + iy) - (x_o - iy_o)| < \delta \to |(u + iv) - (u_o + iv_o)| < \delta$$

We want to show that $|u - v_o| < \epsilon$. From an earlier proof, we know that $|Re(z)| \leq |z|$, and we see that $|u - u_o|$ is the real component of $w - w_o$.

$$|u - u_o| = |Re(w - w_o)| \leq |w - w_o| = |(u + iv) - (u_o + iv_o)| < \epsilon$$

We can then do this same thing for the imaginary component of z. □

Theorem 15.2. *This theorem tells us that sums, products, and quotients work the way we think they should.*

Proof. By using the above theorem, we can just split the limit into two limits and use the fact that multivariable limits work the way they do. □

Theorem 15.3.

$$\lim_{z \to z_o} P(z) = P(z_o)$$

16 Section 16

Here is where we'll talk about limits involving infinity. We can define an epsilon ball around infinity:

$$B_\epsilon(\infty) = \{z | \, |z| > \frac{1}{\epsilon}\}$$

From this we can now make some definitions:

$$\lim_{z \to z_o} f(z) = \infty$$

For this one, we see that $z \in B_\delta(z_o) \setminus \{z_o\} \to f(z) \in B_\epsilon(\infty)$. From here we can see that $|f(z)| > \frac{1}{\epsilon} \to \frac{1}{|f(z)|} < \epsilon$. This means that $\frac{1}{f(z)}$ is in the epsilon ball around 0, $B_\epsilon(0)$. This gives us the definition:

$$\lim_{z \to z_o} f(z) = \infty \;\; iff \;\; \lim_{z \to z_o} \frac{1}{f(z)} = 0$$

Let's try to get the definition for

$$\lim_{z \to infty} f(z) = w_o$$

We take $z \in B_\delta(\infty) \setminus \{\infty\} \to |f(z) - w_o| < \epsilon$, which tells us that $|z| > \frac{1}{\delta}$. From here we can rearrange to tell us that $|\frac{1}{z} - 0| < \delta$, which tells us that $\frac{1}{z} \in B_\epsilon(0)$. We can exclude 0 because since z is a complex number we can't achieve 0. This gives us a good definition:

$$\lim_{z \to \infty} f(z) = w_o \;\; iff \;\; \lim_{z \to 0} f(\frac{1}{z}) = w_o$$

In the case that we want to look at

$$\lim_{z \to \infty} f(z) = \infty \;\; iff \;\; \lim_{z \to 0} \frac{1}{f(\frac{1}{z})} = 0$$

17 Section 17

Continuity at z_o means $\lim_{z\to z_o} f(z) = f(z_o)$. This means that the limit operation commutes with the function f.

Let's do another limit to infinity:

$$\lim_{z\to 1} \frac{(z+2)}{(z-1)(z+3)} = \lim_{z\to 1} \frac{z+2}{z^2+2z+3}$$

We know that a limit goes to infinity if:

$$\lim_{z\to z_o} f(z) = \infty \to \lim_{z\to z_o} \frac{1}{f} = 0$$

We see that this is true, so the limit does go to infinity.
This section is about continuity.

Definition 17.1. A function f is **continuous** at z_o if

$$\lim_{z\to z_o} f(z) = f(z_o)$$

Definition 17.2. We want to define function composition, $f(g)$:

$$\exists \delta \ s.t. \ |f(g(z)) - f(g(z_o))| < \epsilon$$

$$if \ \ |g(z) - g(z_o)| < \delta$$

Theorem 17.1. *If $f(z)$ is continuous and $f(z) \neq 0$ at z_o, then $\exists \epsilon$ s.t. $\forall z \in B_\epsilon(z_o)$, $f(z) \neq 0$.*

Proof. We know that we can make an epsilon ball such that we are within δ:

$$|f(z) - f(z_o)| < \delta$$

We can force a maximal distance away from z_o, and we can just make the distance smaller than the distance to 0, such as $|f(z_o)|/2$. □

Theorem 17.2. *For $f(z) = u(x,y) + iv(x,y)$, f is continuous at z_o iff u and v is continuous at (x_o, y_o).*

Proof. Since we know we can just split the limit apart, we can just split the limit into two limits and in order for the sum to exist, both limits must exist, so both components must be continous. □

18 Section 18

Definition 18.1. For $f(z)$ defined on all $z \in B_\epsilon(z_o)$,

$$f'(z_o) := \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{z\to z_o} \frac{f(z) - f(z_o)}{z - z_o}$$

Definition 18.2. We define the increment of $f(z)$:

$$\Delta f := f(z + \Delta z) - f(z)$$

and we can say:

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f(z)}{\Delta z}$$

Prove that the derivative of $5z + 3$ is 5.

$$f'(z) = \lim_{\Delta z \to 0} \frac{5z + 5\Delta z + 3 - 5z - 3}{\Delta z} = 5$$

Find the derivative of $f(z) = |z|^2$. We find that the limit doesn't exist other than at 0, so things are not good.

Theorem 18.1. *Differentiability implies continuity.*

Proof.

$$\lim_{z \to z_o} f(z) - f(z_o) = \lim_{z \to z_o} \frac{f(z) - f(z)}{z - z_o} \cdot \lim_{z \to z_o} z - z_o = f'(z_o) \cdot 0 = 0$$

□

19 Section 19

Theorem 19.1. *Derivatives of constants, exponents, sums, products, quotients and chain rule (sorta) all work the same way as they do in real numbers.*

20 Section 20

$$w = f(z) = u(x, y) + iv(x, y)$$
$$\Delta w = f(z + \Delta z) - f(z)$$
$$= (u(x + \Delta x, y + \Delta y) - u(x, y)) + i(v(x + \Delta x, y + \Delta y) - v(x, y))$$
$$f'(z) = \lim_{(\Delta x, \Delta y) \to (0,0)} Re(\frac{\Delta w}{\Delta z}) + i \lim_{(\Delta x, \Delta y) \to (0,0)} Im(\frac{\Delta w}{\Delta z})$$

Taking the first limit and going along the path $\Delta y = 0$, we see that Δz becomes Δx:

$$Re(f'(z)) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x}$$

We see that this is simply a partial derivative, U_x. If we look at the imaginary portion on this path:

$$Im(f'(z)) = \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} = V_x$$

We can now consider the other path, where $\Delta x = 0$:

$$Re(f'(z)) = \lim_{\Delta y \to 0} Re(\frac{u(x, y + \Delta y) + iv(x, y + \Delta y) - u(x, y) - iv(x, y)}{i\Delta y}) = V_y$$

If we look at the imaginary component of that, we see that we're dividing by i in the denominator, so we have to negate it in order to get it in the right form:

$$= -U_y$$

These are the Cauchy-Riemann Equations:

$$\exists f'(z) \to u_x = v_y \quad v_x = -u_y$$

If we have some function f, the derivative is:

$$f'(z) = u_x + iv_x = u_x - iu_y = v_y - iu_y = v_y + iv_x$$

21 Section 21

Theorem 21.1. *Take some function $f(z) = u(x,y) + iv(x,y)$ defined on some $B_\epsilon(z_o)$ and the first partials of u and v exist $\forall z \in B_\epsilon(z_o)$, if u_x, u_y, v_x, and v_y satisfy the Cauchy-Riemann equations and are continuous at z_o, then $f'(z_o)$ exists and it is equal to $u_x + iv_x$ or equivalent.*

Proof. We know that $\Delta z = \Delta x + i\Delta y$, and that $\Delta w = f(z_o + \Delta z) - f(z_o)$, which is equivalent to $\Delta u + i\Delta v$ where $\Delta u = u(x_o + \Delta x, y_o + \Delta y) - u(x_o, y_o)$ and $\Delta v = v(x_o + \Delta x, y_o + \Delta y) - v(x_o, y_o)$. By continuity, we claim that

$$\Delta u = u_x(x_o, y_o)\Delta x + u_y(x_o, y_o)\Delta y + \epsilon_1\sqrt{(\Delta x)^2 + (\Delta y)^2}$$

for $\epsilon_1 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. The same thing is true for Δv with some ϵ_2. If we now sub these definitions for Δu and Δv into our rewriting of Δw:

$$u_x\Delta x + u_y\Delta y + \epsilon_1|\Delta z| + iv_x\Delta x + iv_y\Delta y + \epsilon|\Delta z|$$

Applying Cauchy-Riemann Equations:

$$\Delta w = u_x(\Delta x + i\Delta y) + u_y(\Delta y - i\Delta x) + (\epsilon_1 + i\epsilon_2)|\Delta z|$$

$$\Delta w = u_x\Delta z - u_y i\Delta z + (\epsilon_1 + i\epsilon_2)|\Delta z|$$

$$= u_x\Delta z + iv_x\Delta z + (\epsilon_1 + \epsilon_2)|z|$$

If we now consider $\Delta w/\Delta z$:

$$\frac{\Delta w}{\Delta z} = u_x + iv_x + (\epsilon_1 + i\epsilon_2)\frac{|\Delta z|}{\Delta z}$$

We now apply a limit as $\Delta z \to 0$, and we see that the entire third term will go to 0, since the epsilons go to 0, and the right part is bounded, so we know that the entire term will go to 0. We are then left with:

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = u_x + iv_x$$

which exists. $\square$

22 Section 22

We know that $z = x + iy = re^{i\theta}$, and that $w = u + iv = f(z)$. If we want to find

$$\frac{\partial u}{\partial r} =$$

We can use the conversion $x = r\cos\theta$ and $y = r\sin\theta$, along with the chain rule:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

Computing the 4 different derivatives:

$$u_r = u_x \cos\theta + u_y \sin\theta$$

$$u_\theta = -u_x r\sin\theta + u_y r\cos\theta$$

$$v_r = v_x \cos\theta + v_y \sin\theta$$

$$v_\theta = -v_x r\sin\theta + v_y r\cos\theta$$

If we assume that the CR hold true, we know that $u_x = v_y$, which gets us the fact that $ru_r = v_\theta$, and we can apply the other part of CR to get that $rv_r = -u_\theta$. This provides polar Cauchy-Riemann:

$$ru_r = v_\theta, \quad rv_r = -u_\theta$$

If we take a function like $f(z) = \sqrt[n]{r}e^{i\theta/n}$, $r > 0$, with some bounds on θ, we can compute the regular partials, and then we can compute the new ones, and we find that they indeed hold true. From the fact that the CR holds true and the results are continuous, we know that the derivative f' exists:

$$f' = e^{-i\theta}(u_r + iv_r)$$

$$= e^{-i\theta}(\frac{1}{n}r^{1/n-1}\cos\frac{\theta}{n} + \frac{i}{n}r^{1/n-1}\sin\frac{\theta}{n})$$

$$= \frac{1}{n}r^{1/n-1}e^{-i\theta}e^{i(1/n-1)\theta}$$

$$= \frac{1}{n}r^{1/n-1}e^{i(1/n-1)\theta}$$

$$= \frac{1}{n}r^{1/n}\frac{1}{r}e^{i\theta/n}$$

$$\frac{1}{n}f(z)\frac{1}{re^{i\theta}}$$

$$= \frac{1}{n}z^{1/n-1}$$

We see that things work the same way that we think they should work.

23 Section 23

Definition 23.1. $f(z)$ is holomorphic on an open set if $f'(z)$ exists everywhere in the set.

Implicitly, this tells us that $f(z)$ is holomorphic over a closed set if it is holomorphic on some open set containing the closed set. In particular, holomorphic at z_0 means that $\exists B_\epsilon(z_0)$ on which f is holomorphic.

Definition 23.2. $f(z)$ is **entire** if it's holomorphic on $\mathbb{C}$

Definition 23.3. If $f(z_0)$ is not holomorphic and

$$\forall \epsilon \ \exists z_1 | z_1 \in B_\epsilon(z_0),\ z_1 \textit{ is holomorphic}$$

z_0 is a **singular point**.

Definition 23.4. f is **analytic** at z_o means $\exists$ a convergent power series that converges to $f(z)$ $\forall z \in B_\epsilon(z_o)$.

If $f'(z) = 0 \forall z \in D \to f(z) = w_0 \forall z \in D$.

Proof. We can't do an integral because we don't know how to do those yet, but we can say that the existence of the derivative implies that CR holds. This tells us that $u_x = u_y = v_x = v_y = 0$. Consider a segment in D from P to P'. We know that $\vec{r} \parallel PP'$, and we can compute a directional derivative:

$$D_{\vec{r}}(u(x,y)) = \nabla u \cdot \hat{r} = 0$$

This tells us that $f(P) = f(P')$. For arbitrary $z_1,\ z_2 \in D$, there exists some finite path of such segments $z_1 = P_1 \to P_2 \to \cdots \to P_n = z_2$, $f(z_1) = f(P_1) = \cdots = f(P_n) = f(z_2)$. $\square$

24 Section 24

If f and $\bar{f}$ are both analytic, $f(z)$ must be constant. Let's write out f:

$$f = u + iv$$
$$Re(f) = u$$
$$Im(f) = v$$
$$\bar{f} = u - iv$$
$$Re(\bar{f}) = u$$
$$Im(\bar{f}) = -v$$

Using CR on f:

$$y_x = v_y$$
$$u_y = -v_x$$

And using it on $\bar{f}$:

$$u_x = -v_y$$
$$u_y = v_x$$

From this we know that $v_y = -v_y \to v_y = 0 \to u_x = 0$. This then tells us that $u_y = v_x = 0$. You can now do exactly what we did in the proof above to generate paths and prove that the entire function is constant.

25 Section 25

Definition 25.1. For real valued functions $\mathbb{R}^2 \to \mathbb{R}$ with continuous second partials, if

$$H_{xx} + H_{yy} = 0$$

the function is **harmonic**. This is Laplace's equation.

Theorem 25.1. *If $f = u + iv$ is analytic on domain D, then u,v are harmonic on D.*

Proof. Since f is analytic, we can use CR:

$$u_x = v_y \;\; u_y = -v_x$$

$$u_{xx} = v_{yx} \;\; u_{yy} = -v_{xy}$$

By Clairaut's,

$$u_{xx} = -u_{yy}$$

and likewise for u_{xy}. □

Definition 25.2. v is a **harmonic conjugate** of u means that u and v are harmonic and satisfy CR. Note that the relationship is not symmetric.

Theorem 25.2. *$f = u + iv$ is holomorphic iff v is a harmonic conjugate of u.*

Proof. The proof is trivial and is left as an exercise that nobody will do. □

Given that $u = x^2 - y^2$, how would we find a harmonic conjugate? First we write out the CR:

$$u_x = 2x = v_y$$

$$v_x = 2y$$

$$v_y = 2x$$

To solve this, we can do what we did to find potentials in MV:

$$v = 2xy + D(y) \;\; v = 2xy + E(x)$$

$$v = 2xy$$

26 Section 26

We talk about the "Blue Car Lemma":

Lemma 26.1. *If $f(z) = 0$ along a segment of $B_\epsilon(z_o)$, then $f(z) = 0 \;\forall z \in B_\epsilon(z_o)$*

Lemma 26.2. *If f is analytic on D and $f(z) = 0$ on a segment or domain in D, then $f(z) = 0 \;\forall z \in D$.*

Proof. Assume Lemma 26.1. We can draw a segment in the domain that is "blue" ($f(z) = 0$). We can then draw epsilon balls around the segment, and by the first lemma they are all also "blue". From here, we can "chain" these epsilon balls, making sure that they contain a part of an epsilon ball that is 0. Since D is connected, we are sure that we can draw line segments to any point in our domain. □

Lemma 26.3. *If f and g are analytic on D, and $f = g$ on a segment or domain of D, then $f = g\ \forall z \in D$.*

Proof. Assume Lemma 26.1. We can make a new function $h = f - g$, which is 0 on the segment. By Lemma 26.1, $h = 0$ everywhere on D, meaning that $f = g$. □

Sadly, we cannot prove the Blue Car Lemma until section 68.

Corollary 26.3.1. *A real valued $f : \mathbb{R} \to \mathbb{R}$ has a **unique** analytic continuation $\tilde{f} : D \to \mathbb{C}$, $(a, b) \leq D \leq \mathbb{C}$.*

27 Section 27

We skip this section.

28 Section 28

We haved defined the idea of e^{iy}, but we've been using it as shorthand notation. We can actual define exponentiation by a complex number:

Definition 28.1.

$$e^z := e^x e^{iy} := e^x(\cos y + i \sin y)$$

We are defining the function $\widetilde{exp} : \mathbb{C} \to \mathbb{C}$, s.t. $\widetilde{exp}|_{\mathbb{R}} = exp$, and $\widetilde{exp}$ is analytic. This function is the analytic continuation of the exponential from real numbers. We can check these properties. If we set $y = 0$:

$$e^z = e^x(\cos 0) = e^x$$

And we can see that CR holds, and that the partials are continuous. This does prove that the analytic continuation theorem holds.
What if we wanted to define $e^{\frac{1}{n}}$, $n \in \mathbb{N}$. We see that this is just the root operation, and we'd have to look at the principal root of the function, not the family of functions. Note that all of the other properties that we have proved with the other definition still hold, since the two definitions are equal.
How can we prove that $e^z \neq 0$ at any point? We can look at the norm squared:

$$|e^z|^2 = e^x e^{iy} e^x e^{-iy} = e^{zx} e^0 = e^{2x} = 0$$

Let's talk about some cool properties of this function. Property number 1:

$$\exists z |\, e^z < 0$$

To prove this, we can write it out and see that if $y = \pi$, we end with $e^z = -1$.
Property number 2: We know that $e^{z+k(2\pi i)} = e^z s$, which means that our function is periodic with period $2\pi i$.

29 Section 29

We like having inverses, and so we want to find something that agrees with the real version of logarithms. We can look at $e^w = z$, and write $w = u + iv$ and $z = r3^{i\theta}$. We can then write the original statement as $e^u e^{iv} = re^{i\theta}$. This tells use that $e^u = r$ and $e^{iv} = e^{i\theta}$. This first one tells us that $u = \ln r$ (real logarithms) and $v = \Theta + k2\pi$, because the exponential function is periodic. This tells us an expression for w:

$$w = \ln r + i(\Theta + 2k\pi)$$

This is a problem because this gives us a family of answers, but we want a singular answer.

$$\log z = \ln r + i(\Theta + 2k\pi)$$

for $\alpha < \Theta < \alpha + 2\pi$. We can look at $e^{\log z} = e^{\ln r + i\theta + 2ik\pi} = re^{i\Theta}$. Also note that using log in complex implies the use of base e for the logarithms. We have now defined a function that is the inverse of e^z! Right?.

We can look at the $\log(e^1)$, which we want to be 1, but sadly we get $1 + 2\pi ki$, so we know that the log function is not actually the inverse. We then define $Log(z) = \log z$ with $k = 0$ (given a choice of α). Note that for the sake of keeping the real axis in the domain, we actually define α to go from $-\pi$ to π in most cases. However, this actually gets rid of the negative real axis, so why not include that one too, $\alpha < \Theta \leq \alpha + 2\pi$?

30 Section 30

Turns out that including that axis actually causes a discontinuity, due to the jump. This means that we can't eliminate the effect of α, technically creating another infinite class of *Log* functions. This is known as a **Branch** of the *Log* function. A branch of a function is a function in which we have chosen restrictions such that we have analyticity. A **Branch Cut** is the collection of restricted points. A **Branch Point** is any point in the intersection of all branch cuts. In the case of *Log*, the origin is a branch point, as all values of α cause us to have a branch cut that intersects with the origin.

We can prove that what we got is analytic by showing that the polar form of CR holds true, and we have continuity, so the function is analytic. We can now compute the derivative directly:

$$\frac{d}{dz}(Log\, z) = e^{i\theta}(u_r + iv_r)$$

$$= e^{-i\theta}(\frac{1}{r} + 0) = \frac{1}{z}$$

Let's get formal definitions for a branch:

Definition 30.1. for f, which is many valued, F is a **branch** if:

1. F is single-valued
2. $F(z_o) = f(z_o)$ for some choice of $f(z_o), z_o$ in the domain
3. $F(z)$ is analytic

31 Section 31

$$\log(z_1 z_2) = \log z_1 + \log z_2$$

This means nothing to us, because we don't know what set addition is. If we choose one value for the left hand side, and pick one value for $\log z_1$, we want it to be true that there is a value for $\log z_2$ such that the equation holds. This same idea holds for anything permutation of what we choose. We can switch to *Log*s, but then there are conditions that will wrap over and put us on the branch cut, and we see that the equation is no longer true.

32 Section 32

Assume that $z \neq 0$ and $c \in \mathbb{C}$. We want to define z^c:

$$z^c := e^{c \log z}$$

We know that this works if $c \in \mathbb{R}$, but we want to show that this works for $c \in \mathbb{C}$. The issue is that this is multi-valued, so we have an issue. Let's move on for a second, and instead look at i^i:

$$i^i = e^{i \log i} = e^{i(\ln 1 + \pi/2 i + 2k\pi i)}$$

$$= e^{-\pi/2 + 2k\pi}$$

This is also multi-valued, but notice that all the values are real. In order to make this single-valued, why don't we just pick one!
Moving back to arbitrary z^c, why don't we just restrict what the argument should be?

Definition 32.1. Using $\log z = \ln r + i\theta$ for $r > 0$ and $\alpha < \theta < \alpha + 2\pi$ ($\alpha \in \mathbb{R}$), z^c is single-valued and analytic.

Let's now take the derivative of this:

$$\frac{d}{dz}(z^c) = \frac{d}{dz} e^{c \log z}$$

$$= e^{c \log z} \frac{d}{dz}(c \log z) = z^c c \frac{1}{z} = c z^{z-1}$$

The principal value of z^c is $e^{cLog\, z}$ for $-\pi < Arg(z) < \pi$. We see that the function is analytic over it's domain, but not over all complex numbers.
What if we want to define c^z?

$$c^z = e^{z \log c}$$

This is not a problem, and in fact this is entire, provided that we are single-valued (using a branch). Taking the derivative:

$$\frac{d}{dz}(c^z) = \frac{d}{dz}(e^{z \log c})$$

$$= c^z \log c$$

33 Section 33

We know that $e^{ix} = \cos x + i \sin x$, and that $e^{-ix} = \cos x - i \sin x$, so we can add them in order to eliminate the sin, giving us the fact that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

And we can also define $\sin x$:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

We can also think of this as getting the imaginary component of e^{ix} via manipulating the conjugate. From these ideas that we actually already knew, we can replace x with z and actually define the analytic continuation of sin and cos. We know that they are analytic because everything that we used to construct them are analytic!

$$\sin z := \frac{e^{ix} + e^{-ix}}{2i}$$

$$\cos z := \frac{e^{ix} + e^{-ix}}{2}$$

We can now prove identities, such as $\sin^2 + \cos^2 = 1$:

Proof.

$$\frac{(e^{ix} + e^{-ix}}{2i})^2 + (\frac{e^{ix} + e^{-ix}}{2})^2$$

Doing this out and cancelling terms, we see that we do indeed get 1. $\square$

34 Section 34

$$\sinh(z) := \frac{e^z - e^{-z}}{2}$$

$$\cosh(z) := \frac{e^z + e^{-z}}{2}$$

Some identities:

$$\cos(iz) = \cosh(z)$$

$$i \sin(iz) = -\sinh(z)$$

$$\sin(iz) = i \sinh(z)$$

We can also prove that $|\sinh(x)|^2 = \sinh^2 x + \sinh^2 y$, and that $|\cosh z|^2 = \sinh^2 x + \cos^2 y$. We can use these to find a pretty interesting identity:

$$|\sinh y| \leq |\cosh z| \leq \cosh y$$

If we let $w = iz = ix - y$:

$$|\sinh(Im\, w)| \leq |\cosh w| \leq \cosh(Im\, w)$$

We can derive other identities from here, such as

$$|\sinh x| \leq |\cos x| \leq \cosh x$$

35 Section 35

We want to find the inverses of trig and hyperbolic trig. We know that

$$\sin z = w = \frac{e^{iz} - e^{-iz}}{2i}$$

We use the classic trick of switching the variables and solving:

$$z = \frac{e^{iw} - e^{-iw}}{2i}$$

We multiply both sides by e^{iw}:

$$ze^{iw} = \frac{e^{2iw} - 1}{2i}$$

$$2ize^{iw} = e^{2iw} - 1$$

$$e^{2iw} - 2ize^{iw} - 1 = 0$$

$$e^{iw} = \frac{2iz \pm \sqrt{-4z^2 + 4}}{2}$$

$$e^{iw} = iz \pm \sqrt{1 - z^2}$$

Taking the log to isolate w:

$$w = -i \log(iz \pm \sqrt{1 - z^2})$$

This is multivalued, so we have to use Log, and pick either the plus or minus case.
We cant take the derivative of this:

$$\frac{d}{dz} \arcsin z = -i \frac{1}{iz \pm \sqrt{1 - z^2}} (i \pm \frac{1}{2}(1 - z^2)^{-\frac{1}{2}}(-2z))$$

$$= \frac{1 \pm iz(1 - z^2)^{-1/2}}{1z \pm \sqrt{1 - z^2}}$$

Using the conjugate:

$$= \frac{iz \mp \sqrt{1 - z^2} \pm i^2 z^2 (1 - z^2)^{-1/2} - iz}{-z^2 - (1 - z^2)}$$

$$= \mp(\sqrt{1 - z^2} + \frac{z^2}{\sqrt{1 - z^2}})$$

$$= \mp \frac{1}{\sqrt{1 - z^2}}$$

36 Section 36

We will first conduct calculus on functions mapping $\mathbb{R}$ to $\mathbb{C}$.

$$w(t) = u(t) + iv(t)$$

$u, v : \mathbb{R} \to \mathbb{R}$ and $t \in \mathbb{R}$.

Definition 36.1.

$$\frac{d}{dt}w(t) = u'(t) + iv'(t)$$

We already know that we can take the derivatives of real functions, so we can just differentiate them. Note that the definition does not actually directly rely on the definition of a limit.

Let $w(t) = e^{it}$. The biggest consequence of derivatives in scrub calculus was the MVT, so lets jump to that. Let $t \in [0, 2\pi]$. We know that

$$w'(t) = ie^{it}$$

Note that $w(0) = 1$ and $w(2\pi) = 1$, so we can say that

$$w(2\pi) - w(0) = 2\pi w(t_o)$$

We are saying that there exists a point such that $w'(t) = 0$, by the MVT. However, we realize that this can't be true, since we can see that $|w'(t)| = 1$. This is a massive problem, as if the MVT fails, then all of calculus is sort of broken, since we base a lot of other theorems on this.
Let's try the following derivative:

$$\frac{d}{dt}(e^{z_o t}) = (e^{x_o t}\cos(y_o t))' + i(e^{x_o t}\sin(y_o t))$$

$$= x_o e^{x_o t}\cos(y_o t) - y_o e^{x_o t}\sin(y_o t) + ix_o e^{x_o t}\sin(y_o t) + iy_o e^{x_o t}\cos(y_o t)$$

We can do this out, and we see that we are left with exactly what we expected.
Take the function $f(z) = z^3$. Take the curve $\mathcal{C} : t + (1-t)i$. The function is not one of the functions that we have been doing, but if we plug it into the path, we have the function as a function of a single real parameter t. For now though, let's stick with z. We can compute

$$\frac{f(i) - f(1)}{i - 1} = \frac{-i - 1}{i - 1}$$

We see that the modulus of the output is 1. We can now take the derivative. Technically we would need to convert to the t space and then convert back, but we can just say that power rule holds because we know it's analytic:

$$f' = 3z^2$$

The minimal modulus of the derivative on the curve is $3(\frac{\sqrt{2}}{2})^2 = \frac{3}{2}$. Once again, we see that the minimum modulus is too high to satisfy the MVT's requirement of a derivative with a modulus of 1.

37 Seciton 37

For $w(t) = u(t) + iv(t)$,

$$\int_a^b w(t)\,dt := \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

We can write that $u(t) = Re(w)$, so we can just take the left integral:

$$Re\int_a^b w(t)\,dt = \int_a^b u(t)\,dt = \int_a^b Re(w(t))\,dt$$

This tells us that the Real operator and the integral commute. (Resbawt-Sbarewt Theorem) Notice that this is a linear operator, which can be defined as follows:

$$T : X \to Z \ s.t. \ T(ax + y) = aT(x) + T(y)$$

We claim that the integral exists provided that u and t are finitely piecewise continuous over the region. We also maintain the rule that constants pull out of the integral (whether they be real or complex).
If $W' = w$, this means that $U' = u$ and $V' = u$, which means that we can use the FTC on the smaller pieces, which means that integration retains most of the properties that we remember from scrub calculus. Lets prove something interesting:

$$|\int_a^b w(t)\,dt| \leq \int_a^b |w(t)|\,dt$$

Proof. We split this into cases, the first of which is when $\int_a^b w(t)\,dt = 0$. This one solves itself, since we know that the right hand side is always positive. Case 2 is when

$$\int_a^b w(t)\,dt \in \mathbb{C} - \{0\}$$

Then, $\exists r_o, \theta_o$ such that $\int_a^b w(t)\,dt = r_o e^{i\theta_o}$ for $r_o \neq 0$. This tells us that

$$r_o = e^{-i\theta_o} \int_a^b w(t)\,dt$$

$$r_o = \int_a^b e^{i-\theta_o} w(t)\,dt$$

We know that the right hand side has to be real since the left hand side is real, so we can do some cool things:

$$\int_a^b e^{-i\theta_o} w(t)\,dt = Re \int_a^b e^{-i\theta_o} w(t)\,dt = \int_a^b Re(e^{-i\theta_o} w(t))\,dt$$

We know that that entire term must be less than or equal to

$$\leq \int_a^b |e^{-i\theta_o} w(t)|\,dt$$

$$\leq \int_a^b |w(t)|\,dt$$

This technique is a proof by reversion to real values, as we can take advantage of real calculus when doing a lot of complex proofs. □

38 Section 38

We have been restricting the complex numbers to the real axis, so we want to extend this to full complex. This means that when we do integration, we have two choices, either treat it as a double integral where z can vary in two dimensions, or treat integration as path integration. We choose to stick to path integration, as it leads to more interesting math.
In complex, we don't call them path integrals, we call them Countour integrals. A contour is a finite union of smooth arcs. We can discuss simple and closed contours, which have similar definitions as they did before.

38.1 Jordan Curve Theorem

Theorem 38.1. *Every simple closed contour is the set of boundary points for two distinct domains, the interior domain and the exterior domain, where the interior domain is bounded and the exterior domain is unbounded.*

This seems intuitive, but this is actually incredibly difficult to prove.
We have defined already that $z'(t) = x'(t) + iy'(t)$. Assume that $z(t)$ represents a differentiable arc. We can find the arclength the same way we normally do:

$$L = \int_a^b |z'(t)|\, dt$$

This is invariant under reparameterization, so we can say that $t = \phi(\tau)$, then $\frac{d}{d\tau} z(\phi(\tau)) = \frac{dz}{dt} \cdot \phi'$. Taking the modulus:

$$|z' \cdot \phi'| = |z'| \cdot |\phi'|$$

We see that we assume that $\phi'(\tau) \geq 0$, which is useful because it tells us that the curve always maintains the same orientation.

$$\int_{t=a}^{t=b} |z'|\, dt = \int_{\tau=\alpha}^{\tau=\beta} |z'|\phi' dt$$

We can talk about the unit tangent vector of a function:

$$\vec{T} = \frac{z'}{|z'|}$$

The angle from the horizontal is also easy to get, its simply $\theta = Arg(z')$.
We can reverse orientation:

$$\int_{-a}^{-b} w(-t)\, dt = \int_a^b w(\tau)\, d\tau$$

where $t = -\tau$.

39 Section 39

What we would rather want to do is path integrals:

$$\int_{\mathcal{C}} f(z)\, dz$$

If we are independent of contour, we can rewrite the integral:

$$\int_{z_1}^{z_2} f(z)\, dz$$

We can define a path integral:

$$\int_{\mathcal{C}} f(z)\, dz := \int_a^b f(z(t)) \cdot z'(t)\, dt$$

40 Section 40

40.1 Example 1

Consider the integral

$$\int_{\mathcal{C}} \bar{z}\, dz$$

Over the circular curve from $-2i \to 2i$ with radius 2. We can parameterize the path in terms of θ:

$$z = 2e^{i\theta},\ \theta : -\frac{\pi}{2} \to \frac{\pi}{2}$$

$$z' = 2ie^{i\theta}$$

$$\bar{z} = 2e^{-i\theta}$$

Plugging it all in:

$$\int_{-\pi/2}^{\pi/2} (2e^{-i\theta})2ie^{i\theta}\, d\theta = 4\pi i$$

However, there might be an easier way of doing it, by looking at the fact that $|z| = 2$ and that $\bar{z} = \frac{4}{z}$.

40.2 Example 2

Integrate

$$\int_{OA\cup AB} f(z)\, dz$$

where $f(z) = y - x - i3x^2$, with $A(0,1)$, $B(1,1)$ and $O(0,0)$.
For the line OA, we know that $x = 0$, and that $y : 0 \to 1$. We also know that $z = x + iy$, which tells us that $dz = dx + idy$:

$$\int_0^1 (y - x - i3x^2)_{x=0} \cdot (dx + idy)_{dx=0}$$

$$\int_0^1 y \cdot idy$$

$$\frac{iy^2}{2}|_0^1 = \frac{i}{2}$$

For the line AB, we know that $y = 1$, and that $x : 0 \to 1$. We know that $dy = 0$, so we can plug into the integral:

$$\int_0^1 (1 - x - i3x^2) \cdot (dx) = \frac{1}{2} - i$$

40.3 Example 3

$\mathcal{C} : z = z(t)$, $t : a \to b$, $z : z_1 \to z_2$. We want to compute

$$\int_{\mathcal{C}} z\,dz = \int_a^b z \cdot z'\,dt$$

We can notice that the derivative of z^2 with respect to t is $2zz'$, so we can rewrite the integrand

$$\int_a^b \frac{d}{dt}(\frac{z^2}{2})\,dt$$

$$= \frac{z^2}{2}\bigg|_a^b$$

$$= \frac{z_2^2 - z_1^2}{2}$$

40.4 Example 4

$f(z) = 1$ for $y < 0$, and $f(z) = y$ for $y > 0$. The path $\mathcal{C} : z : -1 - i \to 1 + i$ along $y = x^3$. So we can say that $z = x + ix^3$ for $x : -1 \to 1$. We can integrate around the contour:

$$\int_{\mathcal{C}} f(z)\,dz = \int_{-1}^0 1 + 3ix^2\,dx + \int_0^1 4(x^3) \cdot (1 + 3ix^2)\,dx$$

$$= (x + ix^3)_{-1}^0 + (x^4 + i2x^6)_0^1$$

$$= 2 + 3i$$

40.5 Example 5

Take the integral

$$\int_{\mathcal{C}} \bar{z}\,dz$$

for the curve $\mathcal{C} : z = \sqrt{4 - y^2} + iy$, $y : -2 \to 2$.

$$\int_{-2}^2 (\sqrt{4 - y^2}(\frac{1}{2}(-2y)(4 - y^2)^{-1/2}) + i)\,dy$$

Writing this all out:

$$\int_{-2}^2 -y + \frac{iy^2}{\sqrt{4 - y^2}} + y + i\sqrt{4 - y^2}\,dy$$

$$= \int_{-2}^2 \frac{4i}{\sqrt{4 - y^2}}$$

This is a pain, but we already know the answer! It's $4\pi i$! We did this integral in polar, and it was a lot nicer than this.

Theorem 40.1. *If we take the modulus of a path integral:*

$$|\int_{\mathcal{C}} f(z)\,dz| = |\int_a^b f(z(t))z'(t)\,dt|$$

$$\leq \int_a^b |f(z(t))z'(t)\,dt|$$

$$\leq \int_a^b |f(z(t))| \cdot |z'(t)|\,dt$$

Suppose we have some M such that for all z on the path $\mathcal{C}$, $|f(z)| \leq M$,

$$|\int_{\mathcal{C}} f(z)\,dz| \leq \int_a^b M|z'(t)|\,dt$$

$$|\int_{\mathcal{C}} f(z)\,dz| \leq ML$$

This is the ML theorem (because we use an M and an L).

41 Section 41

41.1 Example

We have a curve from i to 1 along the line segment connecting them. We have some $f(x) = \frac{1}{z^4}$. We know that the maximum of the function:

$$Max|f(z)| = \frac{1}{Min|z^4|}$$

$$= \frac{1}{(Min|z|)^4}$$

$$= \frac{1}{(\frac{\sqrt{2}}{2})^4}$$

The length of the line segment is $L = \sqrt{2}$, so we can conclude that

$$|\int_{\mathcal{C}} \frac{1}{z^4}\,dz| \leq 4\sqrt{2}$$

This isn't really helpful, but this is just a small example.

41.2 Example 2

If we have the rational function

$$f(z) = \frac{2z^2 - 1}{z^4 + 5z^2 + 4}$$

And we want to find the maximal modulus, we know that it is less than or equal to the maximum modulus of the numerator divided by the minimal modulus of the denominator:

$$Max|f(z)| \leq \frac{Max|2z^2 - 1|}{Min|z^4 + 5z^2 + 4|}$$

Taking the numerator, we know that we can use the triangle inequality:

$$|2z^2 + 1| \leq 2|z|^2 + 1$$

Looking at the denominator, we can split it into a product:

$$|z^2 + 5z + 4| = |z^4 + 4| \cdot |z^2 + 1| \geq ||z^2| - 4| \cdot ||z^2| - 1|$$

This tells us that

$$Max(f(z)) \leq \frac{2max|z^2| + 1}{|min|z|^2 - 4| \cdot |min|z|^2 - 1|}$$

Notice that the function in example 2 actually has 2 discontinuities, one at i and one at $2i$. If we took the curve to be a semicircle with some radius R and the real axis, we can create circles around the two discontinuities, and we can see that we want to do something similar to Green's theorem. We know that the value of the integral inside the outside boundary is 0 (in the limit case where $R \to \infty$). We can then set something up that looks like 4 integrals added together, one of them being the one that we just did, and 2 being the integrals of the circles, and the last one being some integral along the real axis. This means that if we can solve the complex integrals around the loops, we can solve the real integral, which is cool.

41.3 Example 3

$$\int_{\mathcal{C}} \frac{dz}{z^2 \sin z} \, dz$$

where the curve is a square gvoerned by segments:

$$y = (N + \frac{1}{2})\pi$$

$$x = (N + \frac{1}{2})\pi$$

$$y = -(N + \frac{1}{2})\pi$$

$$x = -(N + \frac{1}{2})\pi$$

We know that to bound this, we want to look at the min of the denominator:

$$min|z^2 \sin z| = min(|z^2| \cdot |\sin z|) \geq (min|z|)^2 \cdot min|\sin z|$$

This one we get easily (call $min|\sin z|$ A):

$$\geq ((N + \frac{1}{2})\pi)^2 \cdot A$$

$$\geq A(\frac{1}{2}(2N + 1)\pi)^2$$

Solving for L now:

$$L = 4 \cdot 2 \cdot (N + \frac{1}{2})\pi$$

$$= 4\pi(2N + 1)$$

$$ML = \frac{16}{A\pi(2N + 1)}$$

Once again, we see that in a limit case this goes to 0.

42 Section 42

The FTC with gradient (FToPI) was only useful iff we had path independence, which meant that we had the TFAE.

Theorem 42.1. *For $f(z)$ continuous on domain D, TFAE:*

1. $\exists F(z) | F'(z) = f(z) \, \forall z \in D$
2. *For any two contours $\mathcal{C}_1, \mathcal{C}_2 \in D$, such that $\mathcal{C}_1, \mathcal{C}_2 : z_1 \to z_2$,*

$$\int_{\mathcal{C}_1} f(z)\, dz = \int_{\mathcal{C}_2} f(z)\, dz$$

3. *For any closed contour $\mathcal{C} \in D$,*

$$\oint_{\mathcal{C}} f(z)\, dz = 0$$

Proof. Statements 2 and 3 are proved the same way that we did in scrub calculus, so we want to prove 1 implies 2:
We have some curve $\mathcal{C} : z = z(t)\, t \in [a, b]$. We also have the fact that

$$\frac{d}{dt} F(z(t)) = F'(z(t)) \cdot z'(t) = f(z)_{z(t)} \cdot z'(t)$$

If we take the path integral

$$\begin{aligned} \int_{\mathcal{C}} f(z)\, dz &= \int_a^b f(z)_{z(t)} z'(t)\, dt \\ &= F(z(t))\big|_a^b \\ &= F(z_2) - F(z_1) \end{aligned}$$

This has proved it, as the curve is no longer in play here, so the statement does not rely on the parameterization that we used.
We now want to prove that 2 implies 1:
We are assuming that the integration is path independent, and we want to show that there exists an antiderivative, so we just make one.

$$F(z) := \int_{z_o}^{z} f(s)\, ds$$

We don't need to specify a path because we have path independence, so it doesn't matter as long as we have the start and endpoints. We now compute the derivative:

$$F(z + \Delta z) - F(z) = \int_{z_o}^{z+\Delta z} f(s)\, ds - \int_{z_o}^{z} f(s)\, ds$$

$$= \int_{z}^{z+\Delta z} f(s)\, ds$$

We choose the path to be the segment from $z \to z + \Delta z$, and we know that $L = |\Delta z|$, and the integral becomes:

$$\int_{z}^{z+\Delta z} 1\, ds = \Delta z$$

We now do a weird step:

$$f(z) = \frac{1}{\Delta z}(f(z)\Delta z) = \frac{1}{\Delta z}(f(z)\int_z^{z+\Delta z} 1\, ds) = \frac{1}{\Delta z}\int_z^{z+\Delta z} f(z)\, ds$$

f is continuous in D, so we know it's continuous at z. This tells us that

$$\lim_{s \to z} f(s) = f(z)$$

In another form:

$$\forall \epsilon\, \exists \delta \mid |s - z| < \delta \to |f(s) - f(z)| < \epsilon$$

That last term is interesting, because it's bounding a modulus, and we have a length, so we see that we are building to ML.
s on $\mathcal{C}$ has $|s - z| \leq |\Delta z|$. Let $|\Delta z| < \delta$.

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_z^{z+\Delta z} f(s)\, ds - \frac{1}{\Delta z}\int_z^{z+\Delta z} f(z) ds$$

$$= \frac{1}{\Delta z}\int_z^{z+\Delta z} (f(s) - f(z))\, ds$$

Looking at the modulus of this statement:

$$|\frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z)| = \frac{1}{|\Delta z|}|\int_z^{z+\Delta z} f(s) - f(z)\, ds|$$

For $|s - z| < \delta$, $|f(s) - f(z)| < \epsilon$. Consider $|\Delta z| < \delta$. On $\mathcal{C}$, $|s - z| < |\Delta z|$, so $|s - z| < \delta$. Therefore, on $\mathcal{C}$, $RHS \leq \frac{1}{|\Delta z|} \cdot \epsilon \cdot |\Delta z|$ by ML.

$$\forall \epsilon\, \exists \delta \text{ s.t. } |\Delta z| < \delta \to |\frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z)| < \epsilon$$

This is by definition:

$$\lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) = f(z)$$

This is the definition of an antiderivative. □

43 Section 43

43.1 Example 1

We are interested in the contour integral

$$\int_{\mathcal{C}_1} z^{1/2}\, dz$$

Where $\mathcal{C}_1$ is any contour moving from $z = -3$ to $z = 3$ lying above the real axis (except for endpoints), on the branch of $z^{1/2}$ where $z^{1/2} = \sqrt{r}e^{i\theta/2}$, where $r > 0$ and $0 < \theta < \frac{\pi}{2}$.
Notice that the problem as given is unsolvable because the branch gets rid of the positive x axis, so we have to change the branch to $r > 0$ $\frac{-\pi}{2} < \theta < \frac{3\pi}{2}$ in order to remove another axis that we don't care about.

44 Section 44

For a simple closed contour, $z = z(t)$, $t \in [a, b]$, clockwise, and f is analytic $\forall z \in \mathcal{C} \cup int(\mathcal{C})$:

$$\int_{\mathcal{C}} f(z)\,dz = (\int_{\mathcal{C}} u\,dx - v\,dy) + i(\int_{\mathcal{C}} v\,dx + u\,dy)$$

Proof. The left hand side can be rewritten as

$$\int_{\mathcal{C}} f(z(t))z'(t)\,dt$$

$$z'(t) = x' + iy'$$

$$= (dx + idy)\frac{1}{dt}$$

□

If f is analytic, f must be continuous, which tells us that u and v are continuous, and the first partials are also continuous by the fact that f' is continuous. From this, by Green's (and CR) on the right hand side:

$$\int_{\mathcal{C}} f(z)\,dz = 0$$

This is the Cauchy part of Cauchy-Goursat theorem, and the Goursat part gets rid of the reliance on the continuity of f'.

45 Section 46

Remember that a simply connected domain is a region such that every simple closed contour in D has $int(\mathcal{C}) \subseteq D$. Extended Cauchy-Goursat states that if f is analytic on all z in a simply connected domain D implies that $\oint_{\mathcal{C}} f(z)\,dz = 0$.

Corollary 45.0.1. *f being analytic for all z in simply connected domain D implies that there exists an F such that $F' = f$.*

Proof. via TFAE. □

We can use this to get a pokeball theorem, where we can draw boundaries that allow us to break up non simply-connected regions into regions that are simply connected and then add them all up.

Theorem 45.1. *if f is analytic on $\mathcal{C}$,$\mathcal{C}_k$, $int\mathcal{C} \setminus U int(\mathcal{C}_k)$, then*

$$\oint_{\mathcal{C}} f\,dz + \sum_{k=1}^{n} \oint_{\mathcal{C}_k} f(z)\,dz = 0$$

46 Section 47

46.1 The Cauchy Integral Formula

Theorem 46.1. *If f is analytic on $\mathcal{C} \cup int(\mathcal{C})$, for simple, closed $\mathcal{C}$, oriented positively, with $z_o \in int(\mathcal{C})$, then*

$$f(z_o) = \frac{1}{\tau i} \oint_{\mathcal{C}} \frac{f(z)\,dz}{z - z_o}$$

where $\tau = 2\pi$ (because its the day after π day).

The real point of the integral isn't to solve for the value at z_o, but its to rexpress the integral in terms of f:

$$\oint_{\mathcal{C}} \frac{f(z)\,dz}{z - z_o} = 2if(z_o)$$

This allows us to compute the integral without actually needing to do integrals. cool.

Proof. Take some contour $\mathcal{C}$ with a point z_o inside it, and a second contour $\mathcal{C}_\rho$ being a circle of radius ρ around z_o such that $\mathcal{C}_\rho \subseteq int(\mathcal{C})$. We claim that $\frac{f(z)}{z-z_o}$ is analytic everywhere except at z_o. We know that $\mathcal{C}_\rho := |z - z_o| = \rho$. From this, we know that

$$0 = \oint_{\mathcal{C}} \frac{f(z)}{z - z_o}\,dz - \oint_{\mathcal{C}_\rho} \frac{d(z)\,dz}{z - z_o}$$

We know that these two integrals are equal to each other. We can take the integral

$$\oint_{\mathcal{C}} \frac{f(z) - f(z_o)}{z - z_o}\,dz = \oint_{\mathcal{C}} \frac{f(z)\,dz}{z - z_o} - f(z_o) \oint_{\mathcal{C}} \frac{dz}{z - z_o}$$

We also know that the integral is equal to

$$\oint_{\mathcal{C}} \frac{f(z) - f(z_o)}{z - z_o}\,dz$$

We now apply ML, and say that we let $\rho < \delta$. On $\mathcal{C}_\rho$, $|z - z_o| = \rho$. ML tells us that

$$|\oint_{\mathcal{C}} \frac{f(z) - f(z_o)}{z - z_o}\,dz| < |2\pi\rho\frac{\epsilon}{\rho}| = 2\pi\epsilon$$

ϵ is free, so we can just set it to be 0, getting us 0 for the entire thing. □

47 Section 48

47.1 Morera's Theorem

Theorem 47.1. *For f continuous on domain D, if for all closed contours in D:*

$$\oint_{\mathcal{C}} f\,dz = 0$$

means that f is analytic on D. This theorem

Proof. By TFAE, we know that $\exists F|F' = f$ on D. Since F is analytic, we know that f is. $\square$

Note that this theorem is the converse of Cauchy-Goursat.

Corollary 47.1.1. *If f is some $u+iv$ defined and analytic at $z = x+iy$, u and v have continuous partials of all orders at that point*

Lemma 47.2. *If f is analytic on a contour $\mathcal{C}$ and its interior, and $z \in int(\mathcal{C})$:*

$$f^n(z) = \frac{n!}{2\pi i}\oint_{\mathcal{C}} \frac{f(s)\,ds}{(s-z)^{n+1}}$$

This is huge, as we are doing integrals via doing derivatives, which is really cool.

Proof. By the CIF, we know that

$$f(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(s)\,ds}{s-z}$$

Our sketchy proof says that we should just differentiate both sides:

$$f'(z) = \frac{1}{2\pi i}\frac{d}{dz}\oint_{\mathcal{C}} \frac{f(s)\,ds}{s-z}$$

Commuting the derivative to the inside of the integral:

$$= \frac{1}{2\pi i}\frac{d}{dz}(\frac{1}{s-z})f(s)\,ds$$

$$= \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(s)\,ds}{(s-z)^2}$$

We iterate this procedure, and we see that we get what we wanted. Note that we haven't really justified that you can commute the derivative into the inside of the integral. $\square$

Theorem 47.3. *If f is analytic at a point, derivatives of all order exist and are analytic at that point.*

48 Section 49

48.1 Liouville's Theorem

Theorem 48.1. *If f is entire and bounded, then f must be constant.*

Proof. Apply the lemma that says:

$$|f'(z_o)| \leq \frac{M_r}{R}$$

for M_r being $max|f(z)|$ on $\mathcal{C}_R$. If f is bounded, then $\exists M|M \geq |f(z)| \forall z$. The max on the curve is less than or equal to the overall max ($M_R \leq M$):

$$|f'(z_o)| \leq \frac{M}{R}$$

This may not seem like a huge difference, but we can choose arbitrary M, but not M_R, so we can actually make the right side go to 0:

$$|f'(z_o)| = 0$$

We can repeat this for all $z_o \in \mathbb{C}$. $\square$

Corollary 48.1.1. $\sin z$ *is unbounded!*

48.2 Fundamental Theorem of Algebra

The proof relies on the fact that if $f(z)$ has no root, then $\frac{1}{f(z)}$ is entire. From Liouville, we can create a proof by contradiction.

Theorem 48.2. *Given some $P(z)$:*

$$P(z) = \sum_{i=0}^{n} a_i z^i$$

$a_n \neq 0, n \in \mathbb{N}$, yields at least one zero.

Proof. Say not. Then $\frac{1}{P(z)}$ entire, as well as bounded. Consider

$$w = \frac{a_o}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z}$$
$$P(z) = (a_n + w)z^n$$

For any set $\{a_i\}$, $\exists R|\forall|z| > R$,

$$|\frac{a_i}{z^{n-i}}| < |\frac{a_n}{2n}|$$

By triangle inequality, we can conclude $|w| < \frac{|a_n|}{2}$. Let's look at $|P(z)|$:

$$\begin{aligned} |P(z)| &= |a_n + w| \cdot |z|^n \\ &\geq ||a_n| - |w|| \cdot |z|^n \\ &> \frac{|a_n|}{2} R^n \end{aligned}$$

for $|z| \geq R$.

$$|f| = |\frac{1}{p}| < \frac{2}{|a_n|R^n}$$

for all z outside the circle. Inside and on the region, f is continuous, which means that it is bounded because of it being a closed region. This implies that f is entire and bounded, and thus constant. This causes a contradiction, and as such $\exists z_o|P(z_o) = 0$. □

49 Section 50

49.1 Maximum Modulus Principle

Theorem 49.1. *If f is analytic, not constant on D, then $|f(z)|$ has no maximum in D.*

Corollary 49.1.1. *If f is constant on a closed, bounded region R, and not constant on $intR$, then the maximum modulus exists and is on ∂R.*

50 Section 51

This section is about sequences. We see that they work very much the same way as they did before. We can talk about the limit of series:

$$\lim_{n\to\infty} z_n = z \to \exists N|\forall n > N, |z - z_n| < \epsilon$$

A limit is unique if it exists.

Theorem 50.1. *If $z_n = x_n + iy_n$, $z = x + iy$, then z_n converges to z if x_n converges to x and y_n converges to y.*

51 Section 52

Series also work the way they should:

$$S_n = \sum_{n=1}^{N} z_n$$

$$S = \lim S_n$$

if it exists, in which case it converges.

Theorem 51.1.

$$\sum x_n + iy_n = (\sum x_n) + i(\sum y_n)$$

Proof. Just use limit addition identities to split them. □

The NTT still applies (on the modulus), and absolute convergence is the same as it was before, we just replace absolute values with moduli. The first notably different thing deals with remainders. Let $\rho_N = S - S_N$ (the remainder):

$$|S_N - S| = |\rho_N - 0|, \ \ \sum z_n = S \to \lim \rho_N = 0$$

Power series also work the way they should.

52 Section 53

We want to prove that if the error in a Taylor series goes to 0, then the series converges (Taylor's theorem):

Proof. Let $|z| = r$, $\mathcal{C}_o$ a circle $|z| = r_o$, $r < r_o < R_o$. Because f is analytic on all $z \int \mathcal{C}_o \cup int(\mathcal{C}_o)$, so

$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}_o} \frac{1}{s} (\frac{1}{1 - z/s}) f(s)\, ds$$

$$\frac{1}{1-z} = \sum_{n=0}^{N-1} z^n + \frac{z^N}{1-z}$$

for $z \neq 1$. We can plug in z/s and factor out the s:

$$\frac{1}{s-z} = \frac{1}{s}(\sum_{n=0}^{N-1} (\frac{z}{s})^n + \frac{(z/s)^N}{1-(z/s)})$$

$$= \sum_{n=0}^{N-1} \frac{1}{s^{n+1}} z^n + z^N (\frac{1}{(s-z)s^N})$$

$$\oint_{\mathcal{C}_o} \frac{f(s)\, ds}{s-z} = \sum_{n=0}^{N-1} (\oint_{\mathcal{C}_o} \frac{f(s)\, ds}{s^{n+1}}) z^n + z^N \oint_{\mathcal{C}_o} \frac{f(s)\, ds}{(s-z)s^N}$$

We can sub in the extended Cauchy-Integral formula:

$$f(z) = \sum_{n=0}^{N-1} (\frac{f^{(n)}(0)}{n!}) z^n + \frac{z^N}{2\pi i} \oint_{\mathcal{C}} \frac{f(s)\, ds}{(s-z)s^N}$$

We want to show that the remainder term, the second term, goes to 0, so we use ML:

$$\lim_{N\to\infty} \rho_N = 0$$

$\mathcal{C}_o$ has radius $r_o > r = |z|$, so $|s - z| \geq ||s| - |z|| = |r_o - r|$.

$$|\rho_N| \leq \frac{r^N}{2\pi} \cdot \frac{max|f(s)|}{(r_o - r)r_o^N} \cdot 2\pi r_o$$

Lets say that the maximum in the numerator is M:

$$\leq \frac{Mr_o}{r_o - r}(\frac{r}{r_o})^N$$

As we take the limit, we see that only the portion in parentheses matters, and we know that $r < r_o$, so the limit wil go to 0. $\square$

53 Section 54

This section is showing that the Taylor series' of e^z, $\sin z$, $\cos z$, $\frac{1}{1-z}$, etc. work the same way as they did before.

54 Section 55

Lets do something like the partial fraction decomposition of

$$\frac{1}{z(4-z)} = \frac{A}{z} + \frac{B}{4-z}$$
$$= \frac{1}{4z} + \frac{1}{16}\frac{1}{1-(z/4)}$$
$$= \frac{1}{4z} + \frac{1}{16}\sum(\frac{z}{4})^n$$

This second term converges for $|z| < 4$, and the first term can't have $z = 0$.

$$\frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+z}}$$

This isn't a Taylor series because it converges on a "punctured" neighborhood around a point, since $z \neq 0$. This is a trivial annulus (annular region (washer)), in that it converges on a disk with a point missing. This is a Laurent series.

Theorem 54.1. *f analytic on annulus $R_1 < |z - z_o| < R_2$, $\mathcal{C}$ is a simple, closed contour in that domain, then*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_o)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_o)^n}$$

$$a_n = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{(z - z_o)^{n+1}}\, dz$$
$$b_n = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{(z - z_o)^{-n+1}}\, dz$$

If f is analytic on $B_{R_2}(z_o)$, $b_n = 0$ by Cauchy-Goursat, and we see that we have reduced to a Taylor series.

55 Section 56

Take $\oint_{\mathcal{C}} e^{\frac{1}{z}}\, dz = 2\pi i$, for $\mathcal{C}$ being a circle around $z = 0$. Also remember that $e^{1/z}$:

$$= \sum_{n=0}^{\infty} \frac{(\frac{1}{z}^n)}{n!}$$

And that

$$b_1 = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{e^{1/z}}{(z-0)^{-n+1}}\, dz$$

$$2\pi i b_1 = \oint_{\mathcal{C}} e^{1/z}\, dz$$

We can see that we can compute integrals without needing to do integrals. This is where things get really nice in complex.

55.1 Example 1

$$f(z) = \frac{z+1}{z-1}$$

We can set up a MacLaurin series:

$$1 - 2\sum_{n=0}^{\infty} z^n$$

for $|z| < 1$. We can instead try to find a series that converges when $|z| > 1$:

$$1 + \frac{2}{z-1} = 1 + \frac{2}{z}\left(\frac{1}{1-1/z}\right)$$

$$= 1 + \frac{2}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

when $|\frac{1}{z}| < 1$. We can then rewrite this:

$$= 1 + 2\sum_{n=0}^{\infty} z^{-(n+1)}$$

for $|z| > 1$. We now have 2 series for the same thing that converge in different areas.
We can fix our old definition of absolute convergence now.

Theorem 55.1. *If*

$$\sum_{n=0}^{\infty} a_n (z - z_o)^n$$

conerges at $z_1 \neq z_o$, then the series absolutely converges if $\forall |z - z_o| < R_1$ where $R_1 = |z_1 - z_o|$.

Theorem 55.2. *If z_1 lies inside the circle of convergence of a Taylor series, then we get uniform convergence $\forall |z - z_o| < R_1$.*

Definition 55.1. A series **uniformly converges** if within a circle of convergence,

$$\exists N_\epsilon | N > N_\epsilon \to |\rho_N(z)| < \epsilon$$

56 Section 58

Theorem 56.1. *If we have a convergent Taylor series:*

$$\sum_{n=0}^{\infty} a_n(z-z_o)^n = S(z)$$

is continuous $\forall z$ in its circle of convergence $z - z_o = R$.

Section 59

Theorem 56.2. *We have that $S(z) = \sum_{n=0}^{\infty} a_n(z-z_o)^n$ is continuous on a disk of convergence, as well as being analytic on the disk. This ties up the difference between analytic and holomorphic, as the definition of an analytic function is that it can be represented via a convergent Taylor series.*

Theorem 56.3. *If we have a contour $\mathcal{C}$ in $B_R(z_o)$, $g(z)$ continuous on $\mathcal{C}$, then*

$$\int_{\mathcal{C}} g(z)S(z)\,dz = \sum a_n \int_{\mathcal{C}} g(z)(s-z_o)\,dz$$

This just talks about the commutativity of integrals and series. We can think of the case when $g(z) = 1$, then the right side is exactly 0 (integral over a closed contour). We can then use Morera's theorem, which tells us that $S(z)$ is analytic, allowing us to prove the previous theorem.

If we have a Taylor series converging to $S(z)$ on $|z - z_o| = R$, then we have absolute convergence $\forall |z| < R$, uniform convergence on the same region, $S(z)$ is continuous on the interior of the ball around z_o, and $S(z)$ is analytic on the interior of the ball. We can turn this around, and say that at z_1 where S is not analytic, the series cannot converge beyond $|z_1|$. The info at a single point has given us a ton of info about the series around the point.

Theorem 56.4. *Differentiation also commutes with series:*

$$S(z) = \sum a_n(z-z_o) \to \forall z \in int(B_R(z_o))$$

$$S' = \sum_{n=1}^{\infty} na_n(z-z_o)^{n-1}$$

This seems the same as in singlevar, but in complex land, we have that integrals and derivatives are just sorta the same thing, so we can go back to the theorem that proves that integrals commute with series, and we can choose $g(z)$ such that we can prove it via integrals and CIF. Pretty cool.

57 Section 61

The one big result from this section is that we are allowed to multiply and divide series as we expected from singlevar.

58 Section 62

Definition 58.1. A **singular** point is a point at which you are not analytic at z_o but $\forall \epsilon \exists z \in B_\epsilon(z_o)$ analytic.

Definition 58.2. An **isolated singular** point are singular points where $\exists \epsilon$ s.t analytic on $B_\epsilon(z_o) \setminus \{z_o\}$

For isolated singular points, $\exists R$ such that there exists a Laurent series converging to f on $0 < |z - z_o| < R$.

$$b_n = \frac{2}{2\pi i} \int_{\mathcal{C}} \frac{f(z)\,dz}{(z - z_o)^{-n+1}}$$

$$b_1 = \frac{1}{2\pi i} \int_{\mathcal{C}} f(z)\,dz$$

So we define the residue:

$$Res_{z=z_o} f(z) = b_1$$

and note that for closed $\mathcal{C}$ in $0 < |z - z_o| < R$,

$$\oint_{\mathcal{C}} f(z)\,dz = 2\pi i Res_{z=z_o} f(z)$$

Note that the series must be centered around z_o. Note that this does not require analyticity at a point inside the region, and for more than one, we can use pokeball.

59 Section 63

59.1 Cauchy Residue Theorem

Theorem 59.1. *For a simple, closed, positively oriented contour $\mathcal{C}$, with f analytic on and inside $\mathcal{C}$ except at a finite number of singular points (therefore isolated) $\{z_k\}$, then*

$$\oint_{\mathcal{C}} f\,dz = 2\pi i \sum_k Res_{z=z_k}(f(z))$$

All that is really happening is a pokeball theorem.

Proof. We know that $\int_{\mathcal{C}} f - \sum_k \int_{\mathcal{C}_k} f = 0$, by Cauchy-Goursat. Taking the second term and applying residues, and moving them to the other side, we see that we can reapply this for all points k, and we have our proof. □

59.2 Example 1

Take the function

$$f(z) = \frac{1}{z + z^2}$$

On a region surrounding two points at which are singular, $z = 0$ and $z = -1$. We know that $Res_{z=0} f = 1$, via a Laurent series centered at 0, and we need a series that is centered at $z = -1$:

$$\frac{1}{z+1} \frac{1}{-1 + (1 + z)}$$

$$= \frac{-1}{z+1} \frac{1}{1-(z+1)}$$

$$= \sum_{n=0}^{\infty} (-(z+1)^{n-1})$$

Looking at this series, all we care about is the coefficient of the $\frac{1}{z+1}$ term, which is -1, telling us that

$$Res_{z=-1} f = -1$$

Adding the two residues, we get 0. This may seem to break the claim that analyticity is necessary to get contour integrals that equal 0, but the key is that every closed contour in a region implies analyticity, not that just a single one can prove analyticity.

59.3 Example 2

Consider

$$\frac{z+1}{z^2-2z}$$

on $\mathcal{C} = \{z||z| = 3\}$. We can easily see that we have singular points at 0 and 2, so we need to make Laurent series centered at both points. Starting with $z = 0$:

$$\frac{1}{z} \frac{z+1}{z-2}$$

We can then use PFD or other techniques to get rid of the z in the numerator:

$$\frac{1}{z}(1 + \frac{3}{z-2})$$

Each of these expressions has a Taylor expansion:

$$\frac{1}{z} + (\frac{-3}{2}) \frac{1}{z} \frac{1}{1-z/2}$$

$$\frac{1}{z} + \frac{-3}{2} \frac{1}{z}(1 + \frac{z}{2} + \dots)$$

We only care about when we have exactly one z^{-1},

$$1 - \frac{3}{2} = \frac{-1}{2} = Res_{z=0} f$$

Centering at $z = 2$:

$$\frac{z+1}{z^2-2z} = \frac{(z-2)+3}{z(z-2)}$$

$$= \frac{1}{z}(1 + \frac{3}{z-2})(\frac{1}{2+(z-2)})$$

$$f = (1 + \frac{3}{z-2})(\frac{1}{z})(\frac{1}{1+(\frac{z-2}{2})})$$

$$Res_{z=2} f = \frac{3}{2}$$

Adding the two, we find that the computation ends with $2\pi i$.

60 Section 64

Theorem 60.1. *If f is analytic on and in $\mathcal{C}$ except at a finite set of singular points, and f is analytic outside $\mathcal{C}$ and $\mathcal{C}$ is positively oriented, simple, and closed, then*

$$\oint_{\mathcal{C}} f\,dz = 2\pi i\,Res_{z=0}(\frac{1}{z^2}f(\frac{1}{z}))$$

If we have a bunch of isolated singular points, and we use a contour that contains some of them, we could work with the residues of the inner ones and add them up, or we could work with the residues on the outside and at ∞, and use those instead. This means we worry about at most half of the singular points that we are using. Going back to the problem we did before:

$$f = \frac{z+1}{z^2-2z}$$

We look at $f(\frac{1}{z})$:

$$\begin{aligned} f(1/z) &= \frac{z^2+z}{1-2z} \\ &= (z^2+z)(\frac{1}{1-2z}) \\ &= (z^2+z)(1+2z+4z^2+\dots) \end{aligned}$$

Multiplying the $f(\frac{1}{z})$ by $\frac{1}{z^2}$, we see that the residue of the expression is indeed $2\pi i$, the same answer that we got last time.

60.1 Residue Example

Take the function $\frac{3-z}{z(z+2)}$, on the contour $\mathcal{C} : |z| = 3$. Let's do this the formal way first, finding the residue at 0 and the residue at -2, and add them together, starting with $z = 0$:

$$\begin{aligned} f = \frac{3-z}{z}\frac{1}{z+2} &= \frac{3-z}{z}\frac{1}{2+z} \\ &= (\frac{3}{z}-1)\frac{1}{z}\sum_{n=0}^{\infty}(\frac{-z}{2})^n \\ &= \frac{3}{2}z^{-1}+\dots \end{aligned}$$

This gives us the residue of $\frac{3}{2}$. Doing it with $z = -2$:

$$\begin{aligned} \frac{3-z}{z+2}\frac{1}{z} &= \frac{3-z}{z+2}\frac{-1}{2}\frac{1}{1-\frac{z+2}{2}} \\ &= (\frac{5}{z+2}-1)(\frac{-1}{2})(1-\dots) \end{aligned}$$

Which gives us a residue of $\frac{-5}{2}$. This tells us that

$$\int_{\mathcal{C}} f\,dz = 2\pi i(\frac{3}{2}-\frac{5}{2}) = -2\pi i$$

61 Section 65

Definition 61.1. Given the Laurent series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_o)^n + \sum_{n=1}^{\infty} b_n(z - z_o)^{-n}$, where the second term is known as the principal part, if $\exists m \,|\, b_m \neq 0 \wedge \forall p > m, b_p = 0$, then z_o is a pole of order m for $f(z)$. A simple pole is a pole of order 1. If $b_n = 0 \forall n$, z_o is a removable singular point (residue is 0). If there is no m such that $\forall p > m, b_p = 0$, then z_o is an essential singular point.

61.1 Picard's Theorem

Theorem 61.1. *For essential singular point z_o, $\forall\, w \in \mathbb{C}$ except possibly one value, and $\forall\, \epsilon > 0$, $\exists\, z_1, z_2, \ldots \in B_\epsilon(z_o)$ such that $f(z_1) = f(z_2) = \cdots = w$.*

This is a pretty insane theorem. Some examples of this include $e^{1/z}$, which apparently attains every single complex number as a value, infinitely, except for 0. Woah. Continuing on with poles:

Theorem 61.2. *for z_o being an isolated singular point of f is a pole of order m for f iff*

$$f(z) = \frac{\phi(z)}{(z - z_o)^m}$$

for some analytic $\phi(z)$ such that $\phi(z_o) \neq 0$. Further, if so,

$$Res_{z=z_o} f = \frac{\phi^{m-1}(z_o)}{(m-1)!}$$

Proof. Say that $f = \frac{\phi(z)}{(z-z_o)^m}$. We can rewrite ϕ:

$$\phi = \phi(z_o) + \phi'(z_o)(z - z_o) + \cdots + \frac{\phi^{(n)}(z_o)}{n!}(z - z_o)^n + \ldots$$

On the punctured ball,

$$f = \frac{\phi}{(z - z_o)^m} = \sum_{n=m}^{\infty} \frac{\phi^{(n)}(z_o)}{n!}(z - z_o)^{n-m}$$

This is a Laurent expansion, where the coefficient of the term with order m is nonzero. This is the definition of being a pole of order m. Going the other way, a pole of order m implies that we have some positive component of a Taylor series plus the b terms, culminating in a b_m term. We can define ϕ,

$$\phi(z) = (z - z_o)^m f \quad z \neq z_o$$

Defining it at z_o, we just fill it in with what we want it to be, b_m. This is analytic, and $\phi(z_o) \neq 0$. $\square$

This means that we can get rid of having to find expansions, or compute residues at infinity, and instead we can do some simple differentiation.

61.2 Example

$$f = \frac{\sinh z}{z^4}$$

This is singular at $z = 0$, and we can say that $\phi = \sinh z$:

$$\frac{\phi(z)}{(z-0)^4}$$

Everything looks fine, but we actually forgot to account for a condition. However, when we look at $\sinh 0$, we see that it equals 0, which means that we can't apply the theorem for that ϕ function.

61.3 Example 2

Compute the residue at i of

$$\frac{Log z}{(z^2+1)^2}$$

$$\frac{Log z}{(z^2+1)^2} = \frac{Log z}{(z-i)^2(z+i)^2} = \frac{\frac{Log z}{(z+i)^2}}{(z-1)^2}$$

Does this function work as ϕ? Well we know that this is indeed analytic, so we just need to look at $\phi(i)$, which via computation we can see is not 0, so we have a pole of order 2. From this, we can use the theorem:

$$Res_{z=i} f = \frac{\phi'(i)}{(2-1)!} = \phi'(i)$$

We can compute this, via the quotient rule:

$$= \frac{\pi/2 + i}{4}$$

61.4 Example 3

$$\oint_{\mathcal{C}} \frac{\cosh \pi z}{z(z^2+1)}\, dz$$

on $\mathcal{C} : |z| = 2$. At 0, we can rewrite it as:

$$\frac{\frac{\cosh \pi z}{z^2+1}}{z}$$

so the residue at 0 is $\phi(0) = 1$. At i:

$$\frac{\frac{\cosh \pi i}{z(z+i)}}{z-i}$$

This tells us that the residue is equal to $\phi(i)$:

$$\frac{\frac{e^{\pi i}+e^{-\pi i}}{2}}{2i^2} = \frac{1}{2}$$

Finding the residue at $-i$:

$$\frac{\frac{\cosh \pi z}{z(z-i)}}{z+i} \to \frac{1}{2}$$

Summing up the residues:

$$\oint_{\mathcal{C}} f\, dz = 4\pi i$$

62 Section 68

Definition 62.1. A zero of order m means that $\forall n \in [0, m-1] \leq \mathbb{Z}$, $f^{(n)}(z_o) = 0$, and $f^{(m)}(z_o) \neq 0$.

Theorem 62.1. *f, analytic at z_o has a zero of order m at z_o iff $\exists g$, analytic and $g(z_o) \neq 0$ such that $f = (z - z_o)^m g$.*

Proof. f being analytic means that we can make a Taylor series, in which $\forall\, n < m$, $f^{(n)}(z_o) = 0$, so we can start the series with a different counter:

$$\sum_{k=0}^{\infty} \frac{f^{(k+m)}(z_o)}{(k+m)!}(z - z_o)^{(k+m)}$$

Factoring, we see that we just pulled out a new Taylor series, which we can call g. In the other direction, we have an analytic g, which we can Taylor expand and multiply this by $(z - z_o)^m$:

$$f(z) = f(z_o)(z - z_o)^m + \frac{g'(z_o)}{1}(z - z_o)^{m+1} + \dots$$

This is just a Taylor series, which are unique, so we are done. $\square$

Theorem 62.2. *Given some f and z_o where f is analytic at zero z_o, then $\exists \epsilon \,|\, f(z) \neq 0 \,\forall B_\epsilon(z_o) \setminus \{z_o\}$.*

Proof. Because $f \neq 0$, not all derivatives of f are 0. $\exists m$ such that z_o is a zero of order m.

$$f = f(z - z_o)^m$$

such that $g(z_o) \neq 0$. g being analytic means that g is continuous. By the limit definition of continuity:

$$g(z_o) \neq 0 \;\exists \epsilon \; s.t.\; \forall z \in B_\epsilon(z_o) \to g(z) = 0$$

$\square$

Theorem 62.3. *f, z_o where f is analytic on $B_\epsilon(z_o)$ and $f(z_o) = 0$, $f(z) = 0$ on a domain or segment containing z_o, then $f(z) = 0$ for $B_\epsilon(z_o)$.*

Proof. Using contrapositive of the previous tells us that segment implies domain anyways, and we're analytic so we can make a Taylor expansion. On $B_\epsilon(z_o)$,

$$f = \sum_{n=0}^{\infty} \frac{0}{n!}(z - z_o)^n$$

By the uniqueness of Taylor series, we know that this will still converge on the entire ϵ ball. $\square$

That theorem was Blue Car theorem!

63 Section 69

This section talks about how zeroes are the same things as poles.

Theorem 63.1. *p, q analytic at z_o, $p(z_o) \neq 0$, and q has a zero of order m at z_o, then $\frac{p}{q}$ has a pole of order m at z_o.*

Proof.

$$q = (z - z_o)^m g$$

$$\frac{p}{q} = \frac{p/g}{(z - z_o)^m}$$

We know that this satisfies a theorem condition, so we know it has a pole. □

Theorem 63.2. *p, q are analytic at z_o. z_o is a simple pole of p/q, then $Res_{z=z_o} p/q = \frac{p(z_o)}{q'(z_o)}$.*

63.1 Example

Lets go back to the same old problem:

$$\frac{z-3}{z(z+2)} = \frac{p}{q}$$

We know that $q' = z + 2 + z = 2z + 2$, and $p(0) \neq 0$, $q(0) = 0$, and $q'(0) = 0$. We also know that $p(-2) \neq 0$, $q(-2) = 0$, and $q'(-2) \neq 0$. From here, we can compute the two residues via the theorem, and we see that we get the same residues, and that they sum up to the same answer that we've been getting for this problem.

64 Section 70

Theorem 64.1. *z_o is a pole of f, $\lim_{z \to z_o} f(z) = \infty$.*

Proof. Assume order m. (Note: a removable singularity is not a pole by the official definition. Also, essential singularities are also not poles.) Then $f(z) = \frac{\phi(z)}{(z-z_o)^m}$ where ϕ is analytic and $\phi(z_o) \neq 0$. Consider

$$\lim_{z \to z_o} \frac{1}{f(z)} = \lim_{z \to z_o} \frac{(z - z_o)^m}{\phi(z_o)} = \frac{0}{\phi(z_o)} = 0$$

□

Theorem 64.2. *For z_o a removable singularity of f, then f is analytic and bounded on some $B_\epsilon(z_o) \setminus \{z_o\}$.*

Proof. Define a new function g that is f except it is equal to a_o at z_o. By definition of a removable singularity:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_o)^n$$

on $B_\delta(z_o) \setminus \{z_o\}$. Then g is continuous on $\overline{B_\epsilon(z_o)}$ for $\epsilon < \delta$. □

Lemma 64.3. *If f is analytic and bounded on $B_\epsilon(z_o) \setminus \{z_o\}$, and if f is not analytic at z_o, then z_o is removable.*

Proof. Assume f is not analytic at z_o.

$$f(z) = \sum_{-\infty}^{\infty} c_n (z - z_o)^n$$

Let $\mathcal{C}$ be positively oriented and $|z - z_o| = \rho < \epsilon$. By construction of a Laurent series:

$$_n = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(z)\, dz}{(z - z_o)^{-n+1}}$$

But boundedness implies $\exists M, ||f(z)| \leq M \, on \, \mathcal{C}$. Meanwhile, $|z - z_o| = \rho$

$$C_n \leq \frac{1}{2\pi} (\frac{M}{\rho^{-n+1}}) 2\pi\rho = M\rho^n$$

by ML. As $\rho \to 0$, $C_n \to 0$. ☐

Theorem 64.4. *z_o is an essential singularity of f. ω_o is any complex number.*

$$\forall \epsilon > 0, \ |f(z) - \omega_o| < \epsilon$$

is satisfied at some $z \in B_\delta(z_o) \setminus \{z_o\}$.

Proof. Suppose not. $\exists \delta | f$ analytic on $B_\delta \setminus \{z_o\}$. Assume $\neg \exists z || f(z) - \omega_o| < \epsilon$.

$$0 < |z - z_o| < \delta \to |f(z) - \omega_o| \geq \epsilon$$

Let $g(z) = \frac{1}{f(z) - \omega_o}$. This function never divides by 0, and is bounded and analytic on the punctured delta ball. By the previous lemma, z_o is a removable singularity of g. Plugging the hole, let h be g everywhere except at z_o, where it is a_o. If $a_o \neq 0$, let $f = \frac{1}{h(z)} + \omega_o$, for $0 < |z - z_o| < \delta$. Looking at the lift of this function, we see that it is analytic, so z_o is removable, which causes a contradiction. If $a_o = 0$, then g has a zero of order m at z_o. Then f has a pole of order m, therefore not essential, therefore contradiction. ☐

65 Section 71

Principal value integration can be defined:

$$\int_{-\infty}^{\infty} f \, dx := \lim_{R \to \infty} \int_{-R}^{R} f \, dx$$

66 Section 78

Consider the real valued integral

$$\int_0^{2\pi} F(\sin\theta, \cos\theta) \, d\theta$$

If we let $z = e^{i\theta}$, then $dz = ie^{i\theta} d\theta = iz d\theta$, so $d\theta = \frac{dz}{iz}$. We also know thinks about $\sin\theta$ and $\cos\theta$:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

This can be used to replace crazy tan:

$$\int_0^{2\pi} \frac{f\theta}{5 + 4\sin\theta} = \int_{\mathcal{C}} \frac{dz}{iz(5 + 4(\frac{z - z^{-1}}{2i}))}$$

where $\mathcal{C} : |z| = 1$. Rewriting:

$$\int_{\mathcal{C}} \frac{dz}{2z^2 + 5iz - 2}$$

We can use residues to compute this integral without using integrations.