

PHYS611 Notes

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1 Electrostatics

In the theory of electromagnetism, we define two three dimensional vector fields, \mathbf{E} and \mathbf{B} .

We begin with the change in the magnetic field over time:

$$\frac{d\mathbf{B}}{dt} = e \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right)$$

We see that if we have no \mathbf{B} field, the electric field causes acceleration. If instead we had pure magnetic field, we would have gyrations. In particular, this would do no work, since $\mathbf{F}_L \cdot \mathbf{v} = 0$ in the pure magnetic field case. Now let us consider the case where we have an \mathbf{E} field perpendicular to a \mathbf{B} field. A particle placed in this system would tend to go in the direction given by $\mathbf{E} \times \mathbf{B}$. In fact, given the special velocity:

$$\mathbf{v} = c \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

One can work out that this particle would continue to move with this velocity, unaffected by the fields.

As we add more charges, we start to form currents. Consider an electric field pointing to the right, and a mixture of positive and negative charges placed in the field. The field will cause the particles to move to either side of the system, based on the sign of the charge. This polarization would create an electric field oriented against the original electric field. This is where things become more complicated, and Maxwell's equations come in.

The first two equations are the stationary equations:

$$\begin{aligned}\nabla \cdot \mathbf{E} &= 4\pi\rho \\ \nabla \cdot \mathbf{B} &= 0\end{aligned}$$

Which tell us that charges are the source of electric fields, and there are no magnetic field sources.

The other two are Faraday's Law, or the induction equation:

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$$

And Ampere's Law:

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}$$

We can see that not all of these equations are equal, by taking the divergence of both sides of Faraday's Law:

$$\nabla \cdot (\nabla \times \mathbf{E}) = -\frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{B})$$

Now noting that the divergence of a curl is always zero, we see that the divergence of \mathbf{B} must be constant as a function of time:

$$\nabla \cdot \mathbf{B} = \text{const.}$$

We see that the equation $\nabla \cdot \mathbf{B} = 0$ is a constraint equation, we are constraining the constant to be zero. In other words, it is an initial condition for Faraday's Law.

Similarly, we can show that $\nabla \cdot \mathbf{E} = 4\pi\rho$ is an initial condition for Ampere's Law. Using the same process, we take the divergence of both sides:

$$\nabla \cdot (\nabla \times \mathbf{B}) = \frac{4\pi}{c} \nabla \cdot \mathbf{J} + \frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E})$$

Once again noting that the left side must be zero. Now in order to obtain Gauss's Law, we must apply the conservation of charge. The mathematical formulation to maintain the inflow and outflow of charge is known as the continuity equation:

$$\frac{\partial Q}{\partial t} = - \oint_V \mathbf{J} dS$$

The change in charge in a particular volume is dependent on the flux of the charge through the boundary, which is the current. Now we note that the definition of charge is the charge density integrated over the volume:

$$Q = \int_V \rho dV$$

On the right side of the equation, we can write the surface integral as the volume integral of the divergence of the current:

$$- \oint_V \mathbf{J} dS = - \int_V \nabla \cdot \mathbf{J} dV$$

Thus we can put these all together:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Now we see that we can express the divergence of \mathbf{J} in terms of the time derivative of the charge density. We can look back at what we initially had:

$$0 = \frac{4\pi}{c} \left(- \frac{\partial \rho}{\partial t} \right) + \frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E})$$

Which leaves us with a general form of Gauss's Law:

$$\nabla \cdot \mathbf{E} = 4\pi\rho + C$$

Where we can see that the equation $\nabla \cdot \mathbf{E} = 4\pi\rho$ is a constraint equation for Ampere's Law.

The first form of mathematical massaging that we will employ is to consider things in terms of potentials. We have 6 field components, 3 from both the electric and magnetic fields. 4 of these are independent, since we have 2 constraint equations. These 4 components are split into the scalar potential and the vector potential, ϕ and \mathbf{A} respectively. The vector potential describes the magnetic field:

$$\nabla \cdot \mathbf{B} = 0 \rightarrow \mathbf{B} = \nabla \times \mathbf{A}$$

Inserting this definition into Faraday's Law:

$$-\frac{1}{c} \frac{\partial}{\partial t} (\nabla \times \mathbf{A}) = \nabla \times \mathbf{E}$$

$$\nabla \times \left(\mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

In order for the left side to be zero, the term that we are taking the curl of must be the gradient of a scalar function, $\nabla\phi$:

$$-\nabla\phi = \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$$

Where by convention we consider the negative gradient of a scalar function. This then gives us that:

$$\mathbf{E} = -\nabla\phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$$

We can now rewrite the system of Maxwell's equations in terms of our potentials. The first thing we do is insert them into Gauss's Law:

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

$$-\nabla \cdot (\nabla\phi) - \frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = 4\pi\rho$$

$$-\nabla^2\phi - \frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = 4\pi\rho$$

In the electrostatics case, we would drop the time derivative term, and we would have the expected Poisson's equation.

Inserting our potentials into Ampere's Law:

$$\nabla \times \nabla \times \mathbf{A} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial}{\partial t} \left(-\nabla\phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right)$$

The double curl gives two terms, the gradient of the divergence of \mathbf{A} , and the Laplacian of \mathbf{A} :

$$\nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial}{\partial t} \left(-\nabla\phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right)$$

To prove the double curl identity, we can use the Levi-Civita symbol. We need to know two formulas. The first is for the i th component of a vector cross product:

$$(\mathbf{A} \times \mathbf{B})_i = \epsilon_{ijk} A_j B_k$$

Where we assume Einstein summation notation, we are summing over j and k .

The other useful formula is:

$$\epsilon_{ijk} \epsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}$$

Now going back to the curl of a curl:

$$(\nabla \times \nabla \times \mathbf{A})_i = \epsilon_{ijk} \partial_j (\epsilon_{klm} \partial_l A_m)$$

Now noting that the symmetry of the Levi-Civita symbol means that each shift in indices picks up a minus sign, we can shift the second one into e_{lmk} :

$$\begin{aligned} (\nabla \times \nabla \times \mathbf{A})_i &= e_{ijk} \partial_j (e_{lmk} \partial_l A_m) \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_j \partial_l A_m \\ &= \partial_m \partial_i A_m - \partial_j \partial_j A_i \\ &= \partial_i \partial_m A_m - \partial_j^2 A_i \end{aligned}$$

We see that the first term is a gradient of a divergence, and the second term is a Laplacian.

Rewriting our Ampere's Law expression:

$$-\nabla^2 \mathbf{A} + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{4\pi}{c} \mathbf{J} - \nabla \left(\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \right)$$

Thus we have rewritten Maxwell's equations as two equations, in terms of the scalar and vector potentials:

$$\begin{aligned} -\nabla^2 \mathbf{A} + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} &= \frac{4\pi}{c} \mathbf{J} - \nabla \left(\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \right) \\ \nabla^2 \phi &= -4\pi\rho - \frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) \end{aligned}$$

Looking at the top equation, the left side is a wave equation, but to make sense of the right side of the equation, we make use of the gauge freedom of the vector potential, we can add the gradient of a scalar function to the vector potential and maintain the same equations:

$$\mathbf{A}' = \mathbf{A} + \nabla \Lambda$$

Essentially, the vector potential is not unique.

Consequently, we have gauge freedom for the scalar potential:

$$\mathbf{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$$

We now change ϕ to ϕ' and \mathbf{A} to \mathbf{A}' , while keeping \mathbf{E} the same:

$$\begin{aligned} -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} &= -\nabla \phi' - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \frac{1}{c} \nabla \left(\frac{\partial \Lambda}{\partial t} \right) \\ \phi' &= \phi - \frac{1}{c} \frac{\partial \Lambda}{\partial t} \end{aligned}$$

Which is the gauge freedom of the scalar potential.

Since the choice of Λ is arbitrary, we can “fix” the gauge.

One such choice is the Lorentz gauge, where $\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0$. This gives us the equations:

$$\begin{aligned} -\nabla^2 \mathbf{A} + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} &= \frac{4\pi}{c} \mathbf{J} \\ \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} &= -4\pi\rho \end{aligned}$$

We see that this decouples the two equations, making it a useful choice of gauge.

Now let us prove that we can always fix the gauge in such a way. Consider the case where we have \mathbf{A} and ϕ such that $\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \neq 0$. Can we change the gauge such that the shifted potentials do satisfy the Lorentz gauge?

We must make the shift $\mathbf{A}' = \mathbf{A} + \nabla \Lambda$, and $\phi' = \phi - \frac{1}{c} \frac{\partial \Lambda}{\partial t}$. Now computing $\nabla \cdot \mathbf{A}' + \frac{1}{c} \frac{\partial \phi'}{\partial t}$:

$$\nabla \cdot \mathbf{A}' + \frac{1}{c} \frac{\partial \phi'}{\partial t} = \left(\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \right) + \nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2}$$

We want to see if this can be made zero by picking Λ . This can be rewritten as a wave equation:

$$\nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2} = \text{known terms.}$$

This wave equation always has a solution, and thus we can always fix the Lorentz gauge.

Another choice of gauge is the Coulomb gauge, where we impose that $\nabla \cdot \mathbf{A} = 0$. This again is always guaranteed to be possible, since we have that $\nabla \cdot (\mathbf{A}' + \nabla \Lambda) = 0$, which is equivalent to the Poisson equation $\nabla^2 \Lambda = -\nabla \cdot \mathbf{A}$, which always has a solution. This gauge simplifies one of the equations, but the second equation remains coupled, so it is still difficult. In the time-independent case, the two gauges are equivalent.

1.1 Conservation of Energy

Now let us consider what conservation laws this system must obey. To begin, we define the energy density of the electromagnetic field:

$$\mathcal{W} = \frac{E^2 + B^2}{8\pi}$$

Let us compute the time derivative of the energy density:

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{W} &= \frac{\partial}{\partial t} \left(\frac{E^2 + B^2}{8\pi} \right) \\ &= \frac{2}{8\pi} \left(\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \right) + \frac{2}{8\pi} \left(\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) \end{aligned}$$

Now using the Faraday and Ampere equations to replace time derivatives with spatial derivatives:

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= -c \nabla \times \mathbf{E} \\ \frac{\partial \mathbf{E}}{\partial t} &= c \nabla \times \mathbf{B} - 4\pi \mathbf{J} \end{aligned}$$

We can insert these into our expression:

$$\frac{\partial}{\partial t} \mathcal{W} = -\mathbf{J} \cdot \mathbf{E} + \frac{c}{4\pi} \underbrace{(\mathbf{E} \cdot \nabla \times \mathbf{B} - \mathbf{B} \cdot \nabla \times \mathbf{E})}_{\nabla \cdot \left(\frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \right)}$$

This is beginning to look like a conservation law:

$$\frac{\partial}{\partial t} \mathcal{W}_{\text{EM}} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}$$

Where \mathbf{S} is the Poynting vector:

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}$$

If we drop the current term for now, we see that the Poynting flux tells us how much electromagnetic energy is moving through the volume. However, if we have currents and particles, we have the additional $\mathbf{J} \cdot \mathbf{E}$ term. This additional term is the rate of work done by the electric field on the ensemble of particles.

For a group of particles, we can define the current density:

$$\mathbf{J} = \frac{\sum_{\text{charges}} q_i \mathbf{v}_i}{V}$$

If we dot this against the electric field:

$$\mathbf{J} \cdot \mathbf{E} = \frac{\sum_{\text{charges}} q \mathbf{E} \cdot \mathbf{v}_i}{V}$$

Where we assume that the volume is very small, so the electric field is about the same for all of the particles. We see that this is just the Lorentz force on each particle, times the velocity of the particle:

$$\begin{aligned} \mathbf{J} \cdot \mathbf{E} &= \frac{\sum_{\text{charges}} \mathbf{F}_{L,i} \cdot \mathbf{v}_i}{V} \\ &= \frac{\sum_{\text{charges}} \frac{\partial \mathcal{E}_i^{\text{kin}}}{\partial t}}{V} \end{aligned}$$

If we integrate the version of this conservation law over a volume:

$$\frac{\partial}{\partial t} \left(\int_V \mathcal{W}_{EM} dV \right) = - \oint_V \mathbf{S} d\mathbf{A} - \frac{d}{dt} (\mathcal{E}_{\text{kin}})$$

Which is our conservation law over a volume.

Now let us prove that radiation requires time dependence. Consider a circular ring of charges, with a constant current. Intuitively, this will not radiate, but each individual charge on the ring is accelerating, since it is moving in a non-linear orbit. We can show that this does not radiate by computing the divergence of the Poynting vector, and then integrating over the volume:

$$\begin{aligned} \int_V d^3x \nabla \cdot (\mathbf{E} \times \mathbf{B}) &= - \int d^3x \mathbf{J} \cdot \mathbf{E} \\ &\quad - \int d^3x \phi \cdot \nabla J \\ &= 0 \end{aligned}$$

1.2 Conservation of Momentum

We have derived the conservation law for energy in the electromagnetic theory:

$$\frac{\partial \mathcal{W}_{EM}}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{j} \cdot \mathbf{E}$$

We now want to derive the conservation law for the momentum of fields and particles.

To do this, we begin with the time derivative of the momentum of every particle being equal to the Lorentz force at the location of the particle (using Newton's Second Law):

$$\frac{d}{dt}\mathbf{p}_i = e \left(\mathbf{E}(\mathbf{x}_i) + \frac{\mathbf{v}_i}{c} \times \mathbf{B}(\mathbf{x}_i) \right)$$

We can make a volume weighted momentum vector (momentum density of the charged particles in the system):

$$\mathbf{P} = \frac{1}{V} \sum_i \mathbf{p}_i$$

We want an evolution equation for this, so we compute the time derivative:

$$\begin{aligned} \frac{d\mathbf{P}}{dt} &= \frac{1}{V} \sum_i \frac{d\mathbf{p}_i}{dt} \\ &= \frac{1}{V} \sum_i e \left(\mathbf{E}(\mathbf{x}_i) + \frac{\mathbf{v}_i}{c} \times \mathbf{B}(\mathbf{x}_i) \right) \end{aligned}$$

In this case, the volume is the infinitesimal volume of particles very close together in space, $V = d^3\mathbf{x}$. In this case, we can state that we have some average field value over this small volume, so instead of evaluating the fields at each point, we have a constant field over the entire volume, for both the electric and magnetic fields. In this case, we note that the sum of the charges over the volume, $\frac{1}{V} \sum_i e$ is equivalent to the charge density, ρ , and we note that $\sum_i e\mathbf{v}_i$ is the total current \mathbf{J}^1 , and we divide by the volume, so we have the current density \mathbf{j} :

$$\frac{d\mathbf{P}}{dt} = \rho \mathbf{E}(\mathbf{x}) + \frac{1}{c} \mathbf{j} \times \mathbf{B}(\mathbf{x})$$

Now we want to use Maxwell's equations to massage the right side of this equation, to arrive at an analogous result to the conservation law for energy. We first note that, by rewriting Maxwell's equations:

$$\rho = \frac{1}{4\pi} \nabla \cdot \mathbf{E}$$

And similarly for the current density:

$$\mathbf{j} = \frac{c}{4\pi} \left(\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} \right)$$

We can now insert these into the right hand side of our evolution equation:

$$\frac{d\mathbf{P}}{dt} = \frac{1}{4\pi} (\mathbf{E} \cdot \nabla \cdot \mathbf{E} + \nabla \times \mathbf{B} \times \mathbf{B}) - \frac{1}{4\pi c} \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}$$

Rewriting the second term as $\frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) - \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t}$:

$$\frac{d\mathbf{P}}{dt} = \frac{1}{4\pi} (\mathbf{E} \cdot \nabla \cdot \mathbf{E} + \nabla \times \mathbf{B} \times \mathbf{B}) - \frac{1}{4\pi c} \left[\frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) - \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} \right]$$

¹Consider the change in charge in the volume if we shift a charge e by some Δx , $\Delta q = e\Delta x$. Dividing by Δt , we have that $\Delta q/\Delta t = e\Delta x/\Delta t$, which gives us that $J = ev$

If we define $\mathbf{g} = \frac{1}{4\pi c} \mathbf{E} \times \mathbf{B}$:

$$\frac{d\mathbf{P}}{dt} + \frac{\partial \mathbf{g}}{\partial t} = \frac{1}{4\pi} (\nabla \cdot \mathbf{E} \cdot \mathbf{E} + \nabla \times \mathbf{B} \times \mathbf{B} - \mathbf{E} \times \nabla \times \mathbf{E})$$

We see that \mathbf{g} takes on the role of the momentum of the electromagnetic field, \mathbf{P}_{EM} , and is in fact equal to the Poynting vector divided by c^2 , $\mathbf{P}_{EM} = \frac{\mathbf{S}}{c^2}$. Now looking at the term $\mathbf{E} \times \nabla \times \mathbf{E}$:

$$\begin{aligned} (\mathbf{E} \times \nabla \times \mathbf{E})_i &= e_{ijk} E_j (e_{klm} \partial_l E_m) \\ &= e_{ijk} E_j e_{lmk} \partial_l E_m \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) E_j \partial_l E_m \\ &= E_m \partial_i E_m - E_l \partial_l E_i \end{aligned}$$

Writing this in vector notation:

$$\mathbf{E} \times \nabla \times \mathbf{E} = \frac{1}{2} \nabla E^2 - (\mathbf{E} \nabla) \mathbf{E}$$

Inserting this result into our equation:

$$\frac{d\mathbf{P}}{dt} + \frac{\partial \mathbf{g}}{\partial t} = \frac{1}{4\pi} \left[(\mathbf{E} \nabla) \mathbf{E} + (\mathbf{B} \nabla) \mathbf{B} - \nabla \left(\frac{E^2 + B^2}{2} \right) - \nabla \cdot \mathbf{E} \cdot \mathbf{E} \right]$$

Taking the i th component of the right side:

$$\begin{aligned} \left[\frac{d\mathbf{P}}{dt} + \frac{\partial \mathbf{g}}{\partial t} \right]_i &= \frac{1}{4\pi} \left(E_j \partial_j E_i + B_j \partial_j B_i - \delta_{ij} \partial_j \left(\frac{E^2 + B^2}{2} \right) + E_i \partial_j E_j \right) \\ &= \frac{1}{4\pi} [\partial_j (E_i E_j) + \partial_j (B_i B_j)] - \delta_{ij} \partial_j \mathcal{W}_{EM} \\ &= -\partial_j T_{EM}^{ij} \end{aligned}$$

Where we have added the term $B_i \partial_j B_j$, because by Maxwell's laws, the divergence of the magnetic field is zero, so we can add this term. We define spatial components of T_{EM} , the Maxwell stress tensor:

$$\sigma_{ij} = \frac{E_i E_j + B_i B_j}{4\pi} - \delta_{ij} \mathcal{W}_{EM}$$

Thus we have that the change in the total momentum in our system is equal to the derivative of the stress tensor:

$$\frac{d}{dt} (\mathbf{P}_{EM} + \mathbf{P})_i = -\partial_j T_{EM}^{ij}$$

We can integrate this over the volume:

$$\frac{d}{dt} \left[\int_V (\mathbf{P}_{EM} + \mathbf{P}) dV \right]_i = - \oint_{\mathbf{A}} n_j T_{EM}^{ij} d\mathbf{A}$$

Which gives us our conservation law.

We will later see that T_{EM} will have 4 indices, and the spatial components will be the σ_{ij} s.

We have now derived the conservation laws of the electromagnetic theory.

1.3 Laplace's Equation in Cylindrical Coordinates

In electrostatics, we have the case that $\mathbf{E} = -\nabla\varphi$. In the case with no contained charges, we then have that $\nabla \cdot \mathbf{E} = 0$, from which we have Laplace's equation:

$$\nabla^2\varphi = 0$$

In cylindrical coordinates, we have (ρ, ϕ, z) . Let us first consider the case where we have z symmetry, so the Laplacian is of the form:

$$\nabla^2\varphi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \varphi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \varphi}{\partial \phi^2}$$

We solve Laplace's equation via separation of variables:

$$\varphi = R(\rho) \Phi(\phi)$$

Which leaves us with

$$\frac{\Phi}{\rho} \frac{d}{d\rho} \left(\rho \frac{dR}{d\rho} \right) + \frac{1}{\rho^2} \frac{d^2\Phi}{d\phi^2} = 0$$

Which we can rewrite:

$$\frac{\rho}{R} \frac{d}{d\rho} \left(\rho \frac{dR}{d\rho} \right) + \frac{1}{\Phi} \frac{d^2\Phi}{d\phi^2} = 0$$

Now noting that the two terms are dependent on separate variables, they must both be equal to a constant²:

$$\begin{aligned} \frac{1}{\Phi} \frac{d^2\Phi}{d\phi^2} &= -m^2 \\ \frac{\rho}{R} \frac{d}{d\rho} \left(\rho \frac{dR}{d\rho} \right) &= m^2 \end{aligned}$$

We can now solve the two separately. We expect sines and cosines for the Φ equation, and by power counting, we expect a polynomial for the R equation, $R \sim \rho^\gamma$. By inserting this solution back into the radial equation, we find that $\gamma = \pm m$.

In the special case where $m = 0$, we have that $R \sim \text{const}$, as well as $R \sim \ln \rho$ as our two linearly independent solutions.

For a problem that contains the origin, we can drop the $\ln \rho$ and the ρ^{-m} terms, since they diverge at $\rho = 0$. Thus our potential is given by

$$\varphi = \sum_m (a_m \cos(m\phi) + b_m \sin(m\phi)) c_m \rho^m$$

From here, we have to utilize the boundary conditions of the problem in order to pin down the values of a_m , b_m , and c_m . For example, consider the case where φ is defined as:

$$\varphi = \begin{cases} V, & \phi \in (0, \pi) \\ -V, & \phi \in (\pi, 2\pi) \end{cases}$$

²We choose $-m^2$ for the Φ equation because we expect the solution for that equation to be periodic, so we want an equation that gives us an oscillator equation

First, we see that we need the sines, because the potential is odd:

$$\varphi = \sum_m a_m \sin(m\phi) \rho^m$$

The second thing to note is that we are restricted to odd m , because the even coefficients will be given by:

$$\begin{aligned} a_{\text{even}} &\sim \int \varphi(r=a, \phi) \sin(2n\phi) d\phi \\ &= V \int_0^\pi \sin(2n\phi) d\phi - V \int_\pi^{2\pi} \sin(2n\phi) d\phi \\ &= 0 \end{aligned}$$

We see that we can drop all of the even coefficients, and normalize ρ to the circle:

$$\varphi = \sum_{m \text{ odd}} a_m \sin(m\phi) \left(\frac{\rho}{a}\right)^m$$

Now we can see that

$$\varphi(\rho=a, \phi) = \sum_{m \text{ odd}} a_m \sin(m\phi)$$

And thus we can solve for a_m , by computing the projection onto the m th basis function:

$$\begin{aligned} a_m &= \frac{1}{\pi} \int_0^{2\pi} d\phi \sin(m\phi) \varphi(\rho=a, \phi) \\ &= \frac{1}{\pi} \left[\int_0^\pi V \sin(m\phi) d\phi - \int_\pi^{2\pi} V \sin(m\phi) d\phi \right] \\ &= \frac{2V}{\pi} \int_0^\pi d\phi \sin(m\phi) \\ &= \frac{2V}{\pi m} (1 - \cos(m\pi)) \\ &= \frac{4V}{\pi m} \end{aligned}$$

Thus we have that

$$\varphi = \frac{4V}{\pi} \sum_{m \text{ odd}} \frac{1}{m} \left(\frac{\rho}{a}\right)^m \sin(m\phi)$$

To write this in a closed form, we express the sine as a complex exponent, and then use the geometric series to rewrite the sum:

$$\begin{aligned} \varphi &= \frac{4V}{\pi} \sum_{m \text{ odd}} \frac{1}{m} \left(\frac{\rho}{a}\right)^m \sin(m\phi) \\ &= \frac{4V}{\pi} \text{Im} \left[\sum_{m \text{ odd}} \frac{1}{m} \left(\frac{\rho}{a} e^{i\phi}\right)^m \right] \end{aligned}$$

Now consider the series $\sum_{m \text{ odd}} \frac{z^m}{m}$. If we differentiate this with respect to z :

$$\frac{d}{dz} \left[\sum_{m \text{ odd}} \frac{z^m}{m} \right] = \sum_{m \text{ odd}} z^{m-1}$$

$$\begin{aligned}
&= \frac{1}{1-z^2} \\
&= \frac{1}{2} \left(\frac{1}{1-z} + \frac{1}{1+z} \right)
\end{aligned}$$

From this, we have that

$$\sum_{m \text{ odd}} \frac{z^m}{m} = \frac{1}{2} \ln \left[\frac{1+z}{1-z} \right]$$

Now noting that our original series matches this, with $z = \frac{\rho}{a} e^{i\phi}$, we now only have to deal with the imaginary part of this. First, we note that

$$\frac{1+z}{1-z} = \left| \frac{1+z}{1-z} \right| e^{i\theta}$$

If we now consider the imaginary part of the log of this:

$$\ln \left(\operatorname{Im} \left[\frac{1+z}{1-z} \right] \right) = \theta$$

Now finding what θ is in our case, we multiply both the numerator and denominator by $(1-z^*)$:

$$\frac{(1+z)(1-z^*)}{(1-z)(1-z^*)} = \frac{1-|z|^2 + 2i \operatorname{Im} z}{1+|z|^2 - 2 \operatorname{Re} z}$$

This must be equal to some $a + bi$, in which case $\theta = \arctan \left(\frac{b}{a} \right)$, so we have that

$$\theta = \arctan \left(\frac{2 \operatorname{Im} z}{1-|z|^2} \right)$$

Now inserting the expression for z :

$$\theta = \arctan \left(\frac{2 \frac{\rho}{a} \sin \phi}{1 - \left(\frac{\rho}{a} \right)^2 \sin^2 \phi} \right)$$

From this, we can put together the total closed form of the potential:

$$\varphi = \frac{4V}{\pi} \arctan \left(\frac{2 \frac{\rho}{a} \sin \phi}{1 - \left(\frac{\rho}{a} \right)^2 \sin^2 \phi} \right)$$

When doing this whole process in 3D, we will follow the same logic, we begin with Laplace's equations in cylindrical:

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \varphi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \varphi}{\partial \phi^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$$

We can separate this:

$$\varphi = R(\rho) \Phi(\phi) Z(z)$$

Inserting this, and then separating variables:

$$\frac{1}{R} \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dR}{d\rho} \right) + \frac{1}{\rho^2} \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0$$

Again noting that we expect periodicity in the ϕ direction, we will choose the constant to be $-\nu^2$ for the Φ equation. For the Z equation, we expect the solution to decay at infinity, so we expect something of the form $Z \sim e^{-kz}$ ³. The radial equation that we are left with is the Bessel equation⁴:

$$\rho \frac{d}{d\rho} \left(\rho \frac{dR}{d\rho} \right) + (k^2 \rho^2 - \nu^2) R = 0$$

Which has solutions $J_m(k\rho)$ and $J_{-m}(k\rho)$, which differ by their behavior at the origin, J_ν decays at the origin, and $J_{-\nu}$ diverges at the origin. In the case where we contain the origin, we discard the $J_{-\nu}$ solutions, and we pick only the J_ν . We can then write down the full solution:

$$\varphi(\rho, \phi, z) = \sum_{\nu, n} c J_\nu(k_n \rho) (d \sin(\nu \phi) + e \cos(\nu \phi)) (a e^{k_n z} + b e^{-k_n z})$$

Now let us discuss the orthogonality and the asymptotic behavior of the Bessel functions. For the asymptotic behavior, for small x :

$$J_{\pm \nu}(x) = \begin{cases} \frac{\left(\frac{x}{2}\right)}{\Gamma(\nu+1)} \\ \frac{\left(\frac{x}{2}\right)}{\Gamma(1-\nu)} \end{cases}$$

For large x :

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

We can also consider the zeroes of the Bessel function, where $x_{\nu n}$ is the n th zero of the Bessel function $J_\nu(x)$.

Consider the case where we have the boundary condition that $\varphi(\phi = a) = 0$. In order for this to be satisfied, we need the Bessel function $J_\nu(k_{\nu, n} a)$ to be zero. For this to be true, we need $k_{\nu, n} a = x_{\nu, n}$.

This is an example of Dirichlet boundary conditions, and the other type of boundary condition are the von Neuman boundary conditions, where:

$$\frac{\partial \varphi}{\partial n} = 0$$

In this case, the derivatives of the Bessel functions will be zero:

$$J'_\nu \big|_{\rho=a} = 0$$

Now let us consider the proof of orthogonality of the Bessel functions. Suppose we have $J_p(\alpha x)$ and $J_p(\beta x)$. We want to show that these are orthogonal unless $\alpha = \beta$, that is:

$$\int_0^1 x J_p(\alpha x) J_p(\beta x) dx = \delta_{\alpha\beta}$$

³If we restrict ourselves to finite z , then we would have to keep the positive exponent as well.

⁴In general, the equation of the form $x(xu')' + (\alpha^2 x^2 - p^2)u = 0$ has solution $u = J_p(\alpha x)$

Where the factor of x comes from the Jacobian in cylindrical.

The equations that generate these solutions are (respectively):

$$\begin{aligned} x (xu')' + (\alpha^2 x^2 - p^2) u &= 0 \\ x (xv')' + (\beta^2 x^2 - p^2) v &= 0 \end{aligned}$$

If we subtract these two equations from each other:

$$v (xu')' - u (xv')' + (\alpha^2 - \beta^2) xuv = 0$$

Which can be rewritten as a derivative:

$$\frac{d}{dx} (vxu' - uxv') + (\alpha^2 - \beta^2) xuv = 0$$

Integrating this from 0 to 1:

$$\begin{aligned} \int_0^1 \frac{d}{dx} (vxu' - uxv') + (\alpha^2 - \beta^2) xuv \, dx &= 0 \\ v(1)u'(1) - u(1)v'(1) + (\alpha^2 - \beta^2) \int_0^1 xuv \, dx &= 0 \end{aligned}$$

Now we see that we have two additional terms, that are not necessarily zero. In order to make the first two terms zero, we need $J_p(\beta)$ and $J_p(\alpha)$ to both be equal to zero, meaning that α and β are zeroes of the Bessel function. After this, we see that the remaining integral is the orthogonality integral that we care about, and we see that this must be zero unless $\alpha = \beta$.

Now let us consider the case where we have the orthogonality relation:

$$\int_0^a d\rho \, \rho J_\nu \left(x_{\nu,n} \frac{\rho}{a} \right) J_\nu \left(x_{\nu,n'} \frac{\rho}{a} \right) = \delta_{nn'} \frac{a^2}{2} J_{\nu+1}^2(x_{\nu,n})$$

We also have recursion relations between different orders of the Bessel functions:

$$\begin{aligned} \frac{d}{dx} (x^p J_p) &= x^p J_{p-1} \\ J_{p-1} + J_{p+1} &= \frac{2p}{x} J_p \end{aligned}$$

Now that we have developed this machinery, let us consider a problem with a cylinder subdivided into two sections. At $z = 0$, we have a disk of radius a , with height L . Suppose we have a boundary condition that the potential on the base and top sides is $\varphi = V$, and on the surface of the side of the cylinder, $\varphi = 0$. We want to find the potential inside the cylinder, given these boundary conditions.

Since we have no ϕ dependence in the boundary conditions, we only have to deal with one Bessel function, J_0 , since $\Phi = e^{\pm i\nu\phi} = 1$, so $\nu = 0$:

$$\varphi = \sum_n C_n J_0(k_{0,n}\rho) \cosh\left(k_{0,n}\left(z - \frac{L}{2}\right)\right)$$

Where we have replaced the sum of positive and negative exponentials with a hyperbolic cosine centered around $L/2$, since our problem is symmetric in the z direction around $L/2$, and \sinh is not

symmetric around the centerpoint. Using the fact that $\varphi|_{\rho=a} = 0$, we have that $k_{0,n}a = x_{0,n}$, and we can start using the upper and lower boundary conditions (we actually get the same result for $z = L$ and $z = 0$):

$$V = \sum C_n J_0 \left(\frac{x_{0,n} \rho}{a} \right) \cosh \left(\frac{x_{0,n} L}{a} \right)$$

Now applying the orthogonality of the Bessel functions, we multiply both sides by a Bessel function and integrate, using the orthogonality relation on the right side to get a delta function:

$$\int_0^a V \rho J_0 \left(\frac{x_{0,n'} \rho}{a} \right) d\rho = C_n \delta_{n,n'} \cosh \left(\frac{x_{0,n} L}{2a} \right) J_1^2(x_{0,n})$$

To compute the integral on the left side, we can use one of the recursion relations. First, letting $\xi = \frac{x_{0,n'} \rho}{a}$, the left side becomes:

$$\int V J_0(\xi) \left(\frac{a}{x_{0,n'}} \right)^2 \xi d\xi$$

Which, by the recursion relation previously stated, which relates the derivative of a Bessel function of one order to the Bessel function of the order below:

$$\int V J_0(\xi) \left(\frac{a}{x_{0,n'}} \right)^2 \xi d\xi = V \left(\frac{a}{x_{0,n'}} \right)^2 x_{0,n'} J_1(x_{0,n'})$$

From this, we are left with

$$\varphi = 2V \sum_{n=1}^{\infty} \frac{\cosh \left(\frac{x_{0,n}}{a} \left(z - \frac{L}{2} \right) \right) J_0 \left(x_{0,n} \frac{\rho}{a} \right)}{x_{0,n} \cosh \left(\frac{x_{0,n} L}{2a} \right) J_1(x_{0,n})}$$

If we look at this in the limit where $L \rightarrow \infty$, we see that we are left with something of the form of $e^{\frac{x_{0,n}}{a}(z-L)}$, which indicates that the potential becomes concentrate that the bottom and top faces of the cylinder, and is exponentially vanishing in between, which matches the physical intuition, the two faces don't interact with each other for a very large cylinder.

Consider the case where the potential on a disk in a plane is given by $\varphi = V$, and everywhere else on the plane $\varphi = 0$. Once again, we have the general solution, but we can discard the ϕ dependence once more, setting $\nu = 0$. In this case, we have terms of the form:

$$J_0(k\rho) e^{-kz}$$

Where k can take on any value. Thus, we have an infinite superposition of solutions with varying k :

$$\varphi = \int_0^{\infty} J_0(k\rho) e^{-kz} g(k) dk$$

We want to solve for the coefficients $g(k)$. At $z = 0$:

$$\int_0^{\infty} J_0(k\rho) g(k) dk = M(\rho)$$

Where $M(\rho)$ is defined:

$$M(\rho) = \begin{cases} V, & \rho < a \\ 0, & \rho \geq a \end{cases}$$

Multiplying both sides by $\rho J_0(k'\rho)$, and then integrating:

$$\int_0^\infty \int_0^a \rho J_0(k'\rho) J_0(k\rho) g(k) d\rho dk = \int_0^a V \rho J_0(k'\rho) d\rho$$

The left side gives us a delta function, $\frac{1}{k} \delta(k - k')$. We can solve for $g(k)$, and then write out φ :

$$\varphi = Va \int_0^\infty dk e^{-kz} J_0(k\rho) J_1(ka)$$

1.4 Conformal Mapping

Consider a complex function $f(z) = u + iv$, where $z = x + iy = re^{i\phi}$.

We define the derivative of a complex function at some point z_0 ⁵:

$$f'(z) \big|_{z=z_0} = \lim_{\Delta z \rightarrow 0} \frac{\Delta f}{\Delta z}$$

Consider the derivative of $f(z) = |z|^2$:

$$\lim_{\Delta z \rightarrow 0} \left(\frac{|z + \Delta z|^2 - |z|^2}{\Delta z} \right)$$

We can see that along the real axis, this limit is real, and along the complex axis, the limit is complex, and therefore the limit does not exist, and the function is not differentiable.

If a function's derivatives exist at every point, the function is analytic, and satisfies the following conditions:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}$$

Which are the Cauchy-Riemann conditions.

Proof. We want to show that $\frac{df}{dz}$ is defined and unique for all x and y .

Starting from the definition:

$$\frac{df}{dz} = \lim_{\delta z \rightarrow 0} \frac{\delta f}{\delta z}$$

⁵Note that the condition for this derivative to exist is more stringent than the real version, the limit along *any* trajectory to the point must exist for the derivative to exist. In fact, the existence of the first derivative guarantees the existence of all higher orders of derivatives.

Now we note that $\delta z = \delta x + i\delta y$, and therefore $\delta f = \delta u + i\delta v$. Thus:

$$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}$$

We now consider two approaches to the point (x, y) , one that is parallel to the x -axis, and the other that is parallel to the y -axis. We will show that the requirement that these two approaches are equivalent will imply the Cauchy-Riemann conditions.

Approaching along $\delta y = 0$, We have that

$$\begin{aligned}\lim_{\delta z \rightarrow 0} \frac{\delta f}{\delta z} &= \lim_{\delta x \rightarrow 0} \frac{\delta u + i\delta v}{\delta x} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}\end{aligned}$$

Approaching along $\delta x = 0$:

$$\begin{aligned}\lim_{\delta z \rightarrow 0} \frac{\delta f}{\delta z} &= \lim_{\delta y \rightarrow 0} \frac{\delta u + i\delta v}{i\delta y} \\ &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\end{aligned}$$

Now we enforce the condition that these two are equal:

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Now separating the real and imaginary components:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}$$

Which are the Cauchy-Riemann conditions. □

We want to be able to map the solutions of Laplace's equation in one domain to another domain. To do this, let us define a mapping. A mapping is a function $f(z)$ that creates a correspondence between one point in the complex plane to a different complex plane, $\omega = f(z)$. ω is known as the image of z , under the mapping f .

Consider the mapping $\omega = i + ze^{i\pi/4}$. Looking at this, if we insert $z = re^{i\phi}$, we see that we rotate the angle, $re^{i\phi} \rightarrow re^{i(\phi + \frac{\pi}{4})}$. What happens if we look at an area of the original complex plane? We can see that a square in the original plane is rotated by $\pi/4$, and shifted upwards by i .

Consider a point in the complex plane, $z = x + iy$. We can consider an infinitesimal displacement, $dz = dx + i dy$. We can compute the magnitude:

$$|dz|^2 = (dx)^2 + (dy)^2$$

This is how we can compute lengths in the regular complex plane. We can consider lengths in the ω plane:

$$|d\omega|^2 = (du)^2 + (dv)^2$$

We can note that $|dz|^2 = |d\omega|^2 \left(\left| \frac{dz}{d\omega} \right| \right)^2$:

$$|dz|^2 = \left| \frac{dz}{d\omega} \right|^2 (du^2 + dv^2)$$

We see that lengths in the ω plane are scaled by the Jacobian term, $\left| \frac{dz}{d\omega} \right|^2$. Things that started out orthogonal to each other in the z plane will remain orthogonal in the ω plane, just scaled.

Now let us discuss the underlying theorem that we will be using.

Theorem 1.1. Consider a function $\phi(u, v)$, which is a solution of Laplace's equation in two dimensions:

$$\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} = 0$$

We can define a conformal mapping $\omega = f(z) = u(x) + iv(y)$, which generates a new function $\psi(x, y)$:

$$\psi(x, y) = \phi(u(x, y), v(x, y))$$

Which is a solution to Laplace's equation in the new plane:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Proof. Consider $\frac{\partial \psi}{\partial x}$:

$$\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x}$$

We can consider the second derivative:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial \phi}{\partial u} \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial u} \right) \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial v} \right) \frac{\partial v}{\partial x}$$

Applying the chain rule to the second and fourth term:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial \phi}{\partial u} \frac{\partial^2 u}{\partial x^2} + \left[\frac{\partial^2 \phi}{\partial u \partial v} \frac{\partial v}{\partial x} + \frac{\partial^2 \phi}{\partial u^2} \frac{\partial u}{\partial x} \right] \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \left[\frac{\partial^2 \phi}{\partial v \partial u} \frac{\partial u}{\partial x} + \frac{\partial^2 \phi}{\partial v^2} \frac{\partial v}{\partial x} \right] + \frac{\partial \phi}{\partial v} \frac{\partial^2 v}{\partial x^2}$$

We can do the same exact thing for the second derivative of ψ with respect to y :

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial \phi}{\partial u} \frac{\partial^2 u}{\partial y^2} + \left[\frac{\partial^2 \phi}{\partial u \partial v} \frac{\partial v}{\partial y} + \frac{\partial^2 \phi}{\partial u^2} \frac{\partial u}{\partial y} \right] \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \left[\frac{\partial^2 \phi}{\partial v \partial u} \frac{\partial u}{\partial y} + \frac{\partial^2 \phi}{\partial v^2} \frac{\partial v}{\partial y} \right] + \frac{\partial \phi}{\partial v} \frac{\partial^2 v}{\partial y^2}$$

Now we can apply the Cauchy-Riemann conditions, which relate the partials of u and v with respect to x and y :

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}$$

Now we can add the two second derivatives, which should give us zero (by Laplace's equation). When we do this, the first terms of both expressions cancel out, and the last terms of both expressions cancel out. Then looking at the mixed derivative terms, we see that after adding them together, and applying the Cauchy-Riemann conditions, they cancel out. We are then left with just the terms with second derivatives of ϕ , which when factored out, by our definition of ϕ , satisfy Laplace's equation, meaning that they sum to zero. Thus, we have that

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Meaning that ψ satisfies Laplace's equation in the new plane. \square

Let us do an example. Consider the Laplace equation in the $u - v$ plane (u along the horizontal direction, v along the vertical), defined on a rectangle. Suppose that $\phi = 0$ on the lower edge of the rectangle, and $\phi = V$ on the top edge. We then have von Neumann boundary conditions on the sides, $\frac{\partial \phi}{\partial u} = 0$. Suppose that the height of the rectangle is L . The solution to this system is:

$$\phi = V \left(\frac{v}{L} \right)$$

Where $L = \pi$. We now define the conformal mapping $\omega = f(z) = \ln z$. Now we want to find out what the rectangle looks like in the image plane. To do this, we consider $z = re^{i\varphi}$. In this case, $u = \ln r$, and $v = \varphi$, where we are choosing the branch where φ ranges from 0 to π . If we now look at the transformation of each of the 4 boundaries, we see that we have a large semicircle in the upper half plane, a smaller semicircle in the upper half plane, and then two lines connecting the two along the x axis⁶. Along the two semicircles, we still have that $\frac{\partial \psi}{\partial \bar{n}} = 0$, since the relationships between the derivatives of ψ and ϕ remain the same under the conformal mapping. On the left line segment, $\psi = V$, and on the right segment we have $\psi = 0$.

Now we can look at the transformation of ϕ into ψ :

$$\begin{aligned}\psi(x, y) &= \frac{V}{\pi} v(x, y) \\ &= \frac{V}{\pi} \varphi(x, y) \\ &= \frac{V}{\pi} \arctan \left(\frac{y}{x} \right)\end{aligned}$$

Let us do a (mindblowing) example. Consider two lines in the $u - v$ plane, $u = \pi$ and $v = -\pi$. This can be thought of as the setup for an infinite parallel plate capacitor. The lines of constant

⁶Essentially a half-donut shape.

electric field are where we have constant u , and the lines of constant potential are when we have constant v . Now consider the mapping $z = w + e^w$. We can first consider where the $v = 0$ and $v = \pm\pi$ transform to. Expanding our transformation:

$$x + iy = u + iv + e^{u+iv}$$

Which gives us that

$$\begin{aligned} x &= u + e^u \cos v \\ y &= v + e^u \sin v \end{aligned}$$

Looking at the case where $v = 0$:

$$\begin{aligned} x &= u + e^u \\ y &= 0 \end{aligned}$$

We see that for $u = -\infty$, $x = -\infty$ as well. For $u = 0$, we have $x = 1$. For $u = \infty$, this maps to $x = \infty$.

Now looking at $v = \pm\pi$:

$$\begin{aligned} x &= u - e^u \\ y &= \pm\pi \end{aligned}$$

Looking at the function $x = u - e^u$, we see that for very negative u , we asymptote to $x = u$, and for very positive u , we have $-e^u$. This is extremized at $u = 0$, so $x = -1$. By looking at the second derivative, we see that this extrema must be an maximum. Thus, we have mapped all values of u onto a range of x values from $-\infty$ to -1 . Similarly, the same behavior occurs for the $-\pi$ line, we “fold” the infinite line into an infinite half-line. Thus we have mapped an infinite capacitor into a capacitor with an edge, at $x = -1$. We know the solution to the infinite parallel plate capacitor case, which we can then conformally map to solve for the capacitor with an edge. If we find the solution (using Mathematica or Python), we can see that the field lines at the edge of the capacitor bulge outwards.

1.5 Multipole Expansion

Suppose we have some charge density $\rho(\mathbf{x}')$, with arbitrary shape. We want to find the electric potential at a point \mathbf{x} that is very far away, a distance r from the origin. We want to expand the potential in powers of r :

$$\varphi = \frac{Q}{r} + \mathcal{O}\left(\frac{1}{r^n}\right)$$

We can solve this system exactly. First, we look at a charge element in the charge distribution:

$$dq = \rho(\mathbf{x}') d^3\mathbf{x}'$$

We can then find the potential at \mathbf{x} due to this charge element at \mathbf{x}' :

$$\frac{dq}{|\mathbf{x} - \mathbf{x}'|} = \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

We can now integrate over all of the charges to get the total potential:

$$\begin{aligned}\varphi &= \int_V \frac{dq}{|\mathbf{x} - \mathbf{x}'|} \\ &= \int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}'\end{aligned}$$

We want to find out the behavior of the potential that is universal at long distances, regardless of ρ . We can look at the denominator:

$$\begin{aligned}|\mathbf{x} - \mathbf{x}'| &= (x^2 + (x')^2 + 2\mathbf{x} \cdot \mathbf{x}')^{1/2} \\ &= x \left(1 + \left(\frac{x'}{x} \right)^2 + 2 \frac{\mathbf{x} \cdot \mathbf{x}'}{x^2} \right)^{1/2}\end{aligned}$$

Now if we assume that the second two terms are small, ε , and using the expansion of $1/\sqrt{1+\varepsilon}$:

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{x} \left[1 + \frac{\mathbf{x} \cdot \mathbf{x}'}{x^2} - \left(\frac{x'}{x} \right)^2 \frac{1}{2} + \frac{3}{8} 4 \frac{(\mathbf{x} \cdot \mathbf{x}')^2}{x^4} + \mathcal{O}(x^3) \right]$$

Now looking at the potential:

$$\varphi = \frac{\int \rho(\mathbf{x}') d^3\mathbf{x}'}{x} + \frac{\int \rho(\mathbf{x}') (\mathbf{x} \cdot \mathbf{x}') d^3\mathbf{x}'}{x^3} + \frac{1}{2} \int \frac{\rho(\mathbf{x}')}{x} \left[\frac{3(\mathbf{x} \cdot \mathbf{x}')^2}{x^4} - \frac{(x')^2}{x^2} \right] + \dots$$

We have found an expansion of the potential in terms of inverse powers of x . We can look at the first term, we see that it is the total charge over the distance, as expected of the first term. Looking at the second term, we have the scalar product of the dipole moment and \mathbf{x} , where $\mathbf{P} = \int \rho(\mathbf{x}') d^3\mathbf{x}' = \int dq \mathbf{x}'$. The third term is the quadrupole moment, $Q_{ij} = \int d\mathbf{x}' \rho(\mathbf{x}') (3x_i x_j - (x')^2 \delta_{ij})$:

$$\varphi = \frac{Q}{x} + \frac{\mathbf{P} \cdot \mathbf{x}}{x^3} + \frac{1}{2x^5} Q_{ij} x_i x_j + \dots$$

This is the multipole expansion of the potential. A similar expansion can be done for gravity, to find gravitational potentials. However, in that case, we do not have negative masses. In this expansion, we have no radiation from the dipole term, since

$$\mathbf{P} = \int m \mathbf{r}' d^3\mathbf{x}'$$

Looking at the radiation, which is proportional to $\ddot{\mathbf{P}}$:

$$\ddot{\mathbf{P}} = \int m \mathbf{a}' d^3\mathbf{x}'$$

Which must be equal to zero, and thus there is no dipole gravitational radiation. The fact that the radiation only appears at the quadrupole term is why gravitational radiation is harder to detect than electromagnetic radiation.

The potential due to a dipole is given by

$$\varphi = \frac{p \cos \theta}{r^2}$$

Where $p = ed$, where e is the charge difference and d is the distance between the two charges.

We have the potential due to a dipole, using the multipole expansion, at a point \mathbf{x} :

$$\varphi = \frac{\mathbf{p} \cdot \mathbf{x}}{x^3}$$

We can find the electric field, using the fact that $\mathbf{E} = -\nabla\varphi$:

$$\begin{aligned} E_i &= -\partial_i \left[\frac{p_j x_j}{x^3} \right] \\ &= -p_j \partial_i \left[\frac{x_j}{x^3} \right] \\ &= -p_j \frac{\delta_{ij}}{x^3} + \frac{3p_j}{x^4} x_j \partial_i (\sqrt{x^2 + y^2 + z^2}) \\ &= -\frac{p_i}{x^3} + \frac{3p_j}{x^4} x_j \frac{x_i}{x} \end{aligned}$$

From this, we have that

$$\mathbf{E} = -\frac{\mathbf{p}}{x^3} + \frac{3(\mathbf{p} \cdot \mathbf{x})}{x^4} \hat{n}$$

Where $\hat{n} = \frac{\mathbf{x}}{x}$. This can be rewritten:

$$\mathbf{E} = -\frac{\mathbf{p}}{x^3} + \frac{3(\mathbf{p} \cdot \hat{n})}{x^3} \hat{n}$$

This is the electric field due to a single dipole. Note that the electric field falls off as $\sim \frac{1}{x^3}$, faster than that of a single charge.

If we write down the electric field of the dipole taking into account the quadrupole moment, we would see that the quadrupole terms would fall off as $\sim \frac{1}{x^4}$.

Consider some charge distribution function $\rho(\mathbf{r})$. The exact potential at some point \mathbf{r} is given by:

$$\varphi = \int \frac{\rho(\mathbf{r}') d^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

Looking at the denominator:

$$\begin{aligned} \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta}} &= \frac{1}{r} \frac{1}{\sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r} \cos \theta}} \\ &= \sum_{l=0}^{\infty} \frac{(r')^l}{r^{l+1}} P_l(\cos \theta) \end{aligned}$$

Where θ is the angle between \mathbf{r} and \mathbf{r}' . If we want to write this in terms of the angles of the two points (lets call them $\mathbf{r} = (r, \Theta, \Phi)$ and $\mathbf{r}' = (r', \theta, \phi)$), we can use the addition relation for Legendre polynomials:

$$P_l(\cos \theta') = \sum_{m=-l}^l P_l^{|m|}(\cos \Theta) P_l^{|m|}(\cos \theta) \frac{(l - |m|)!}{(l + |m|)!} e^{-im(\Phi - \phi)}$$

Now let us use the spherical harmonics:

$$Y_{l,m \geq 0} = (-1)^m i^l \left(\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!} \right)^{1/2} P_l^m(\cos \theta) e^{im\phi}$$

$$Y_{l,-|m|} = (-1)^{l-m} Y_{l,|m|}^*$$

We can now write down the multipole expansion in terms of the spherical harmonics:

$$\begin{aligned} \varphi(\mathbf{x}) &= \sum_{l=0}^{\infty} \sum_{m=-l}^l \int \frac{(r')^l}{r^{l+1}} \left(\frac{4\pi}{2l+1} \right) Y_{lm}^*(\Theta, \Phi) Y_{lm}(\theta, \phi) \rho(\mathbf{r}') d^3\mathbf{r}' \\ &= \sum_{l=0}^{\infty} \frac{1}{r^{2l+1}} \sum_{m=-l}^l \left(\frac{4\pi}{2l+1} \right)^{1/2} Q_m^{(l)}(\theta, \phi) Y_{lm}^*(\Theta, \Phi) \end{aligned}$$

Where

$$Q_m^{(l)} = \int d^3\mathbf{r}' \rho(\mathbf{r}') (r')^l \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta, \phi)$$

This is the general form of the multipole expansion.

1.6 Potential Energy

Suppose we have some external electric field, with potential $\varphi(\mathbf{x})$. If we place a charge distribution $\rho(\mathbf{x})$ into this potential, what is the potential energy? For a single charge, we have $q\varphi(\mathbf{x})$, and we can sum this over all charges:

$$\begin{aligned} U &= \sum_a q_a \varphi(\mathbf{x}_a) \\ &= \int \frac{\rho(\mathbf{x}') \varphi(\mathbf{x}') d^3\mathbf{x}'}{dq} \end{aligned}$$

Suppose we want to generate a multipole-like expansion of the potential energy. We can insert the Taylor expansion of the potential:

$$\varphi(\mathbf{x}') = \varphi(0) + (\mathbf{x}' \cdot \nabla) \varphi + \frac{1}{2} x'_i x'_j \frac{\partial^2 \varphi}{\partial x'_i \partial x'_j}$$

Now using this in the definition of U , we can go order by order:

$$U = U^{(0)} + U^{(1)} + U^{(2)} + \dots$$

and then compute each order:

$$\begin{aligned} U^{(0)} &= \varphi(0) Q_{\text{tot}} \\ U^{(1)} &= \int \rho(\mathbf{x}') x'_j \underbrace{(\partial_j \varphi)}_{-E_j(0)} d^3\mathbf{x}' \\ &= -\mathbf{P} \cdot \mathbf{E}(0) \end{aligned}$$

$$\begin{aligned}
U^{(2)} &= \frac{1}{2} \left(\frac{\partial^2 \varphi}{\partial x'_i \partial x'_j} \right) \Big|_0 \int \rho(\mathbf{x}') \left[x'_i x'_j - \frac{1}{3} \delta_{ij} x'^2 \right] d^3 \mathbf{x}' \\
&= \frac{Q_{ij}}{6} \frac{\partial^2 \varphi}{\partial x'_i \partial x'_j} \Big|_0
\end{aligned}$$

Using this, let us consider the situation with two dipoles, very far away from each other, both aligned in the same direction, and with strengths \mathbf{d}_1 and \mathbf{d}_2 . We want to know what the mutual force between them is. Note that, just by intuition, we know that the force must be repulsive, since the two dipoles are aligned.

To begin, we can compute the $U^{(1)}$ term:

$$U^{(1)} = -\mathbf{d}_2 \cdot \mathbf{E}_1$$

We have previously derived the electric field due to a dipole:

$$\mathbf{E}_1 = \frac{3\hat{n}(\hat{n} \cdot \mathbf{d}_1)}{R^3} - \frac{\mathbf{d}_1}{R^3}$$

From this, we have that

$$\begin{aligned}
U^{(1)} &= -\mathbf{d}_2 \cdot \mathbf{E}_1 \\
&= \frac{(\mathbf{d}_1 \cdot \mathbf{d}_2) R^2 - 3(\mathbf{d}_1 \cdot \mathbf{R})(\mathbf{d}_2 \cdot \mathbf{R})}{R^5}
\end{aligned}$$

To compute the force, we can compute the gradient of this:

$$\mathbf{F} = -\nabla U$$

This gradient contains many terms, which we can compute beforehand:

$$\begin{aligned}
\nabla(R) &= \frac{\mathbf{R}}{R} \\
\nabla(\mathbf{d} \cdot \mathbf{R}) &= \partial_i (d_j R_j) \\
&= \mathbf{d} \\
\nabla\left(\frac{1}{R^5}\right) &= -\frac{5}{R^6} \left(\frac{\mathbf{R}}{R}\right) \\
\nabla\left(\frac{1}{R^3}\right) &= -\frac{3}{R^4} \frac{\mathbf{R}}{R}
\end{aligned}$$

Putting these together, we can compute the force:

$$\begin{aligned}
\mathbf{F} &= -\nabla U \\
&= \frac{3}{R^5} \left[(\mathbf{d}_1 \cdot \mathbf{d}_2) \mathbf{R} + \mathbf{d}_1 (\mathbf{d}_2 \cdot \mathbf{R}) + \mathbf{d}_2 (\mathbf{d}_1 \cdot \mathbf{R}) - \frac{5}{R^2} (\mathbf{d}_1 \cdot \mathbf{R})(\mathbf{d}_2 \cdot \mathbf{R}) \mathbf{R} \right]
\end{aligned}$$

We now note that this scales as $\sim \frac{1}{R^4}$. We can compute the component of the force along the vector between the two dipoles, \hat{R} . Discarding the terms that are perpendicular to \mathbf{R} , we find that

$$\mathbf{F}_{\hat{R}} = \frac{3}{R^5} \mathbf{R} (\mathbf{d}_1 \cdot \mathbf{d}_2)$$

We see that the repulsion or attraction of the force depends on the relative orientation of the dipoles, $\mathbf{d}_1 \cdot \mathbf{d}_2$, as expected.

2 Magnetostatics

2.1 Biot-Savart Law

From Ampere's Law, in the static case, the magnetic field is generated by electric current:

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}$$

Using the definition of the vector potential:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

We can rewrite Ampere's Law:

$$\nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \frac{4\pi}{c} \mathbf{J}$$

In the Coulomb gauge (recalling that we can let $\mathbf{A}' = \mathbf{A} + \nabla\psi$ for any scalar function ψ , where $\nabla\psi = -\nabla \cdot \mathbf{A}$), $\nabla \cdot \mathbf{A} = 0$, which means that the first term is zero, so we are left with:

$$\nabla^2 \mathbf{A} = -\frac{4\pi}{c} \mathbf{J}$$

This has the solution⁷:

$$\mathbf{A} = \frac{1}{c} \int \frac{\mathbf{J}(\mathbf{x}') d^3\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|}$$

We can now compute the magnetic field:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

This has the form $\nabla \times (f\mathbf{a})$, which we can write out:

$$\begin{aligned} (\nabla \times (f\mathbf{a}))_i &= e_{ijk} \partial_j (f a_k) \\ &= e_{ijk} (\partial_j f) a_k + e_{ijk} f \partial_j a_k \end{aligned}$$

From this, we have that

$$\nabla \times (f\mathbf{a}) = f \nabla \times \mathbf{a} + \nabla f \times \mathbf{a}$$

Applying this to the magnetic field, we can see that we will need to compute:

$$\begin{aligned} \nabla_x \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) &= \nabla_x \frac{1}{(\mathbf{x} - \mathbf{x}')^{1/2}} \\ &= -\frac{(\mathbf{x} - \mathbf{x}')}{(\mathbf{x} - \mathbf{x}')^{3/2}} \end{aligned}$$

⁷Note that this is almost the exact same case as the scalar potential and the charge density, just with an extra factor of c .

Inserting this into our expression for \mathbf{B} :

$$\begin{aligned}\mathbf{B} &= \nabla \times \mathbf{A} \\ &= \frac{1}{c} \int \frac{\mathbf{J}(\mathbf{x}') \times \mathbf{R}}{R^3} d^3\mathbf{x}'\end{aligned}$$

This is the expression for the magnetic field generated by a static current, and is known as the Biot-Savart Law.

Let us now return to Ampere's Law in the static case:

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}$$

Now let us take the surface integral of both sides:

$$\begin{aligned}\oint_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} &= \frac{4\pi}{c} I \\ \oint_l \mathbf{B} \cdot d\mathbf{l} &= \frac{4\pi}{c} I\end{aligned}$$

Where I is the total current through the region, and in the second line, we have applied Stokes' Theorem to the left side. This is the integral form of Ampere's Law.

Ampere's Law in this form is most useful for cases with symmetries, such as when we are computing the magnetic field around a wire. If we have an infinite wire with current \mathbf{J} , and we want to find the magnetic field at some distance r away from the axis, we can use Ampere's Law:

$$\begin{aligned}\oint_l \mathbf{B} \cdot d\mathbf{l} &= \frac{4\pi}{c} I \\ 2\pi r B_r &= \frac{4\pi}{c} I\end{aligned}$$

From which we find that $B_r = \frac{2I}{cr}$.

2.2 Loop of Wire

Let us consider another common situation, the case of a circular loop of wire, with radius a , that has current \mathbf{J} running through it. We can define the current:

$$j_\phi = \frac{I}{a} \delta\left(\theta - \frac{\pi}{2}\right) \delta(r - a)$$

We can compute the total current:

$$I = \int_{a+\varepsilon}^{a-\varepsilon} j_\phi d\phi r dr$$

We want to calculate the vector potential in Cartesian, so we can write the current in terms of the Cartesian basis vectors:

$$\mathbf{j} = -\hat{x} j_\phi \sin \phi + \hat{y} j_\phi \cos \phi$$

Prior to beginning our computation, let us also note that

$$\delta(f(x)) = \frac{\delta(x)}{|f'(x)|}$$

To prove this, recall that:

$$\int_{-\infty}^{\infty} \delta(x) g(x) dx = g(0)$$

Now we can consider:

$$\begin{aligned} \int_{-\infty}^{\infty} \delta(\alpha x) g(x) dx &= \int_{-\infty}^{\infty} \delta(\xi) g\left(\frac{\xi}{\alpha}\right) \frac{d\xi}{\alpha} \\ &= \frac{g(0)}{\alpha} \end{aligned}$$

From this, we see that $\delta(\alpha x) = \frac{1}{\alpha} \delta(x)$. If we look at the case where α is negative, we see that we get a $-1/\alpha$, meaning that we take an absolute value of α for the general relation. We can generalize this to any function $f(x)$ inside the delta function, and we have the previously stated result.

Using this, we can change the delta function in terms of θ :

$$\delta\left(\theta - \frac{\pi}{2}\right) = \delta(\cos \theta) \sin \theta$$

We can now compute $A_y(r, \theta)$:

$$A_y(r, \theta) = \frac{I}{c} \int \frac{d(\cos \theta') r'^2 dr' d\phi'}{|\mathbf{x} - \mathbf{x}'|} (\cos \phi') \frac{\delta(\cos \theta') \sin(\theta') \delta(r - a)}{a}$$

We can rewrite the denominator:

$$|\mathbf{x} - \mathbf{x}'| = (r^2 + r'^2 - 2rr' \cos \gamma)^{1/2}$$

Where γ is the angle between the two vectors, \mathbf{r} and \mathbf{r}' . We can compute the angle between them by writing them in Cartesian:

$$\begin{aligned} \mathbf{r} &= r \begin{bmatrix} \sin \theta \\ 0 \\ \cos \theta \end{bmatrix} \\ \mathbf{r}' &= \begin{bmatrix} \sin \theta' \cos \phi' \\ \sin \theta' \sin \phi' \\ \cos \theta' \end{bmatrix} \end{aligned}$$

From this, we have that:

$$\begin{aligned} \cos \gamma &= \frac{\mathbf{r} \cdot \mathbf{r}'}{rr'} \\ &= \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi') \end{aligned}$$

Now we can replace everything in our integral with angular coordinates, rather than Cartesian.

When we are working in the $x - z$ plane, we have that $\phi = 0$, so $\cos \gamma = \sin \theta \cos \phi'$. We can then write out our integral:

$$A_y(r, \theta) = \frac{Ia}{c} \int \frac{d\phi' \cos \phi'}{(r^2 + a^2 - 2ar \sin \theta \cos \phi')^{1/2}}$$

This is exactly solvable in particular cases. In the case where we are along the z axis, by symmetry, the field must be exclusively along the z direction. The electric field in this case is then purely radial:

$$\begin{aligned} B_r &= \left[\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_r) \right] \Big|_{\theta=0} \\ &= \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left[\int \frac{d\phi' \cos \phi' \sin \theta}{(r^2 + a^2 - 2ar \sin \theta \cos \phi')^{1/2}} \right]_{\theta=0} \\ &= \frac{Ia}{rc \sin \theta} \left[\underbrace{\int_0^{2\pi} \frac{d\phi' \cos \phi'}{(r^2 + a^2 - 0)^{1/2}}}_0 + \frac{Ia}{c} \sin \theta \int \frac{ar \cos \theta \cos^2 \phi' d\phi'}{(r^2 + a^2 - 2ar \sin \theta \cos \phi')^{3/2}} \right] \\ &= \frac{Ia^2}{c} \frac{\int_0^{2\pi} \cos^2 \phi' d\phi'}{(r^2 + a^2)^{3/2}} \\ &= \frac{I\pi a^2}{c} \frac{1}{(r^2 + a^2)^{3/2}} \\ &= \frac{\mu}{(r^2 + a^2)^{3/2}} \end{aligned}$$

Where we have defined the dipole moment:

$$\mu = \frac{I\pi a^2}{c}$$

Now moving away from exact results, we can do a long distance expansion, considering the case where $r \gg a$. In this case, the square root in the denominator becomes $r \left(1 - \frac{a}{r} \sin \theta \cos \phi'\right)$, so we can rewrite the integral:

$$\begin{aligned} A_y &= \frac{Ia}{c} \left[\int \frac{d\phi' \cos \phi' \sin \theta}{(r^2 + a^2 - 2ar \sin \theta \cos \phi')^{1/2}} \right]_{\theta=0} \\ &= \frac{Ia}{cr^2} \sin \theta \int_0^{2\pi} d\phi' \cos^2 \phi' \\ &= \frac{\mu \sin \theta}{r^2} \end{aligned}$$

From this, we find that:

$$\begin{aligned} B_r &= \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_y \sin \theta) \\ &= \frac{2\mu}{r^3} \cos \theta \end{aligned}$$

$$\begin{aligned}
 B_\theta &= -\frac{1}{r \sin \theta} \frac{\partial}{\partial r} (r \sin \theta A_\phi) \\
 &= \frac{\mu \sin \theta}{r^3}
 \end{aligned}$$

We see that a loop of current, at a large distances, takes the form of a magnetic dipole.

2.3 Finite Wire

Let us consider the current generated by a finite length of wire. Suppose we have a wire from $-L$ to L along the z axis, with current \mathbf{J} . We can write down the vector potential:

$$\mathbf{A} = \frac{1}{c} \int \frac{\mathbf{J} d^3 \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

If we are a distance ρ away from the z axis, the vector potential will be exactly along the z axis, we will only have A_z . We can replace $\mathbf{J} d^3 \mathbf{r}'$ with $I dz'$:

$$A_z(\rho) = \frac{I}{c} \int_{-L}^L \frac{dz'}{\sqrt{z'^2 + \rho^2}}$$

Integrals of the form:

$$\int \frac{dx}{\sqrt{x^2 + a^2}}$$

Can be solved with the trig sub $x = a \tan \theta$:

$$\begin{aligned}
 \int \frac{dx}{\sqrt{x^2 + a^2}} &= \int \frac{a \frac{d\theta}{\cos^2 \theta}}{a \sqrt{1 + \tan^2 \theta}} \\
 &= \int \frac{d\theta}{\cos \theta}
 \end{aligned}$$

Multiplying by $\cos \theta / \cos \theta$:

$$\begin{aligned}
 \int \frac{d\theta}{\cos \theta} &= \int \frac{d(\sin \theta)}{1 - \sin^2 \theta} \\
 &= \int \frac{dt}{1 - t^2} \\
 &= \int \frac{1}{2} \left[\frac{1}{1 - t} + \frac{1}{1 + t} \right] dt \\
 &= \frac{1}{2} \ln \frac{1 + \sin \theta}{1 - \sin \theta}
 \end{aligned}$$

Now looking back at our original integral, we have $\sin \theta = \frac{\tan \theta}{\sqrt{\tan^2 \theta + 1}}$. We also know that $\tan \theta = x/a$, so $\sin \theta = \frac{x}{\sqrt{x^2 + a^2}}$:

$$\begin{aligned}
 A_z(\rho) &= \frac{I}{2c} \ln \left[\frac{\sqrt{z'^2 + \rho^2} + z'}{\sqrt{z'^2 + \rho^2} - z'} \right]_{-L}^L \\
 &= \frac{I}{2c} \left[\ln \left(\frac{\sqrt{L^2 + \rho^2} + L}{\sqrt{L^2 + \rho^2} - L} \right) - \ln \left(\frac{\sqrt{L^2 + \rho^2} - L}{\sqrt{L^2 + \rho^2} + L} \right) \right]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{I}{c} \ln \left[\frac{\sqrt{L^2 + \rho^2} + L}{\sqrt{L^2 + \rho^2} - L} \right] \\
&= \frac{I}{c} \ln \left[\frac{\sqrt{1 + \frac{\rho^2}{L^2}} + 1}{\sqrt{1 + \frac{\rho^2}{L^2}} - 1} \right]
\end{aligned}$$

For large L , this becomes:

$$\begin{aligned}
A_z(\rho) &= \frac{I}{c} \ln \left[\frac{2 + \frac{1}{2} \frac{\rho^2}{L^2}}{\frac{1}{2} \frac{\rho^2}{L^2}} \right] \\
&= \frac{I}{c} \ln \left[1 + \frac{4L^2}{\rho^2} \right] \\
&= \frac{I}{c} [\ln 4 + 2 \ln L - 2 \ln \rho]
\end{aligned}$$

From this, we can find the magnetic field, which we expect to be ϕ dependent, since we expect a toroidal field:

$$\begin{aligned}
B_\phi &= -\partial_\rho A_z \\
&= -\frac{\partial}{\partial \rho} \left[\frac{I}{c} [\ln 4 + 2 \ln L - 2 \ln \rho] \right] \\
&= \frac{2I}{c\rho}
\end{aligned}$$

Which matches the result from Ampere's Law for an infinite wire.

2.4 Boundary Conditions

If we have a surface, and we look at the normal and tangential components of the electric and magnetic field at a particular point on the surface:

$$\begin{aligned}
\Delta B_n &= \\
\Delta B_t &= \\
\Delta E_n &= \\
\Delta E_t &=
\end{aligned}$$

For the electric field, we can make a Gaussian pillbox, and we find that the normal component must be related to the surface charge density:

$$\Delta E_n = 4\pi\sigma_s$$

The tangential component can be shown to be zero, by using the fact that $\nabla \times \mathbf{E} = 0$, we can show that there cannot be a discontinuity in the electric field:

$$\Delta E_t = 0$$

For the magnetic field, we have that

$$\begin{aligned}\nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= 0\end{aligned}$$

Using the same argument as the normal component of the electric field, we have that

$$\Delta B_n = 0$$

Since there is no magnetic charge, the right side is just zero.

For the tangential magnetic field, we can apply similar logic as the tangential electric field case:

$$\Delta \mathbf{B} = \frac{4\pi}{c} \mathbf{j}_s$$

2.5 Finite Thickness Wire

Consider an infinite wire along the z axis, with radius a , and current density \mathbf{J} . We can compute the vector potential:

$$\begin{aligned}\nabla^2 \mathbf{A} &= -\frac{4\pi}{c} \mathbf{J} \\ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial A_z}{\partial \rho} \right) &= -\frac{4\pi}{c} J_z\end{aligned}$$

Where we have discarded the other two directions by symmetry.

$$A_z = -\frac{\pi}{c} J \rho^2 + C \ln \rho + D$$

At $\rho = 0$, we want $C = 0$, since the $\ln 0$ diverges. Thus, inside the wire:

$$A_z = \begin{cases} D - \frac{\pi}{c} J \rho^2 & \rho < a \\ D' + C \ln \rho & \rho > a \end{cases}$$

Now computing the magnetic field:

$$B_\phi = \begin{cases} \frac{2\pi J \rho}{c} & \rho < a \\ -\frac{C}{\rho} & \rho > a \end{cases}$$

Now stitching the two of these together at $\rho = a$:

$$\begin{aligned}\frac{2\pi J a}{c} &= -\frac{C}{a} \\ C &= -\frac{2\pi a^2 J}{c}\end{aligned}$$

Now noting that $\pi a^2 J = I$, we have that

$$B_\phi = \begin{cases} \frac{2\pi J \rho}{c} & \rho < a \\ \frac{2I}{c\rho} & \rho > a \end{cases}$$

We see that the result outside of the wire matches the infinitely thin wire case.

2.6 Hydrogen Atom

Consider the current density:

$$J_\phi = J_0 r e^{-r/a} \sin \theta$$

This is the current generated by the $|n, l, m\rangle = |2, 1, m\rangle$ state of the hydrogen atom. We can look at the vector potential in the ϕ direction, which will generate our B_r and B_θ . We can look at the Laplacian:

$$\frac{1}{r} \frac{d^2}{dr^2} (r A_\phi) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\phi) \right) = -\frac{4\pi}{c} J_\phi$$

We posit a solution to this equation of the form $A_\phi = \frac{\sin \theta}{r^2} f(r)$. We are essentially saying that we expect this to behave like a dipole from far away, and the short-range behavior is encoded in $f(r)$. Inserting this back into the diffeq:

$$\begin{aligned} \frac{1}{r} \frac{d^2}{dr^2} \left(\frac{f}{r} \right) + \frac{f}{r^3} \frac{1}{r} (-2) &= -\frac{4\pi}{c} J_0 r e^{-r/a} \\ \frac{f''}{r^2} - \frac{2f'}{r^3} &= -\frac{4\pi}{c} J_0 r e^{-r/a} \\ \frac{d}{dr} \left(\frac{f'}{r^2} \right) &= -\frac{4\pi}{c} J_0 r e^{-r/a} \end{aligned}$$

Integrating both sides and solving for $f(r)$:

$$f(r) = -\frac{4\pi}{c} J_0 a^2 e^{-r/a} [r^3 + 4ar^2 + 8a^2r + 8a^3] + K_1 r^3 + K_2$$

Discarding the K_1 term, since that generates a constant field at ∞ , which we don't want, we are left with:

$$A_\phi = -\frac{4\pi}{c} J_0 \sin \theta a^2 \left[e^{-r/a} \left[\frac{r^3 + 4ar^2 + 8a^2r^2 + 8a^3}{r^2} \right] + \frac{K_2}{r^2} \right]$$

Now noting that at $r = 0$, we see that we scale as⁸:

$$\frac{8a^3}{r^2} + \frac{K_2}{r^2}$$

In order to stop this from diverging at $r = 0$, we need $K_2 = -8a^3$. We see that we have used two boundary conditions, $A_\phi \rightarrow 0$ as $r \rightarrow \infty$, and $A_\phi \rightarrow \text{const.}$ as $r \rightarrow 0$. We then have the solution:

$$A_\phi = -\frac{4\pi}{c} J_0 \sin \theta a^2 \left[e^{-r/a} \left[\frac{r^3 + 4ar^2 + 8a^2r^2 + 8a^3}{r^2} \right] - \frac{8a^3}{r^2} \right]$$

At $r \rightarrow \infty$, the exponential term drops out, so the scaling is as:

$$A_\phi \sim \frac{32\pi J_0 a^5 \sin \theta}{cr^2}$$

⁸This is done by Taylor expanding the exponential, $e^{-r/a} = 1 - r/a$, and then multiplying through and noting that the $1/r$ terms drop out, and we can just match the $1/r^2$ terms.

Derivation of Hydrogen Current

In classical physics, when we discuss current, we think of $\mathbf{J} = e\mathbf{v}$. However, in quantum mechanics, we have the analogue:

$$\mathbf{J} = \frac{e\mathbf{p}}{m}$$

Where $\mathbf{p} = -i\hbar\nabla$. We can then consider expectation values:

$$\langle\psi|\mathbf{J}|\psi\rangle = -\frac{ie\hbar}{m}(\psi^*\partial_i\psi - \psi\partial_i\psi^*)$$

If we insert the eigenstates $|\psi_{nlm}\rangle$ of Hydrogen, we will get the aforementioned current for Hydrogen, based on the particular choice of n, l , and m .

2.7 Multipole Expansion of the Vector Potential

We have the solution to the governing equation for magnetostatics, $\nabla^2\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}$:

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c} \int \frac{\mathbf{J}(\mathbf{r}') d^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

We can look at this in the $\mathbf{r} \rightarrow \infty$ limit:

$$\frac{1}{c} \frac{1}{r} \left[\int \mathbf{J}(\mathbf{r}') d^3\mathbf{r}' \right] + \frac{1}{c} \frac{1}{r^3} \left[\int \mathbf{J}(\mathbf{r}') (\mathbf{r}, \mathbf{r}') d^3\mathbf{r}' \right] + \dots$$

This second term is the magnetic dipole term. Note that the first term is zero, there are no magnetic monopoles. To prove this, note that we can express a component of the current in the k direction as:

$$\begin{aligned} J_k &= \nabla \cdot (r_k \mathbf{J}) \\ &= \partial_i (r_k J_i) \\ &= \delta_{ki} J_i + r_k \partial_i J_i \\ &= J_k \end{aligned}$$

Where we note that the second term is the divergence of \mathbf{J} , which, in a time-independent system, is zero. Inserting this into our expression for the monopole term:

$$\int \nabla \cdot (r_k \mathbf{J}) d^3\mathbf{r} = \oint_S (r_k \mathbf{J}) dS$$

As we take $r \rightarrow \infty$, this right side must be zero since we assume a localized current, and we can push the bounds out to infinity, making the integral zero, and thus the overall magnetic monopole term is zero.

For the second term, we can write it out in components:

$$\frac{1}{c} \int \frac{J_k r_l r'_l}{r^3} d^3\mathbf{r}' = \frac{r_l}{c r^3} \underbrace{\int J_k r'_l d^3\mathbf{r}'}_{T_{kl}}$$

We define this integral as T_{kl} . Now let us state two relations:

$$\begin{aligned}\varepsilon_{lki} (\mathbf{r}' \times \mathbf{J})_i &= r'_l J_k - r'_k J_l \\ \nabla' (r'_l r'_k \mathbf{J}) &= r'_l J_k + r'_k J_l\end{aligned}$$

Essentially, we will decompose the tensor into a symmetric and antisymmetric terms⁹, which are a gradient and a vector product respectively. Writing out the decomposition:

$$J_k r'_l = \frac{1}{2} \underbrace{(r'_l J_k + r'_k J_l)}_{\nabla' (r'_l r'_k \mathbf{J})} + \frac{1}{2} \underbrace{(r'_l J_k - r'_k J_l)}_{\varepsilon_{lki} (\mathbf{r}' \times \mathbf{J})_i}$$

From this, we have that:

$$T_{kl} = \frac{1}{2} \int d^3 \mathbf{r}' [\nabla' (r'_l r'_k \mathbf{J}) + \varepsilon_{lki} (\mathbf{r}' \times \mathbf{J})_i]$$

Now taking a step back and proving these relations¹⁰, let us take the components of the gradient relation:

$$\begin{aligned}\partial_i (r'_l r'_k J_i) &= \delta_{il} r'_k J_i + r'_l \delta_{ki} J_i \\ &= r'_k J_l + r'_l J_k\end{aligned}$$

Thus we can rewrite our tensor T_{kl} , and write out A_k :

$$A_k = \frac{1}{2} \int \varepsilon_{lki} (\mathbf{r}' \times \mathbf{J})_i \frac{d^3 \mathbf{r}'}{c} \frac{r_l}{r^3}$$

Note that the gradient term must go to zero, by the same argument we used to rule out magnetic monopoles. Now noting that we can rewrite this as a vector product:

$$A_k = \frac{(\mathbf{m} \times \mathbf{r})_k}{r^3}$$

Where we have defined \mathbf{m} as:

$$\mathbf{m} = \frac{1}{2c} \int d^3 \mathbf{r}' (\mathbf{r}' \times \mathbf{J})$$

This is the magnetic moment of a dipole.

Consider a cloud of charges, each of which has the same ratio of charge to mass:

$$\frac{e_a}{m_a} = \frac{e}{m}$$

We can compute the magnetic dipole moment of the ensemble:

$$\mathbf{m} = \frac{1}{2c} \sum_a e_a \mathbf{r}_a \times \mathbf{v}_a$$

⁹Helmholtz decomposition.

¹⁰He only proved one of them.

Where we have implicitly used the fact that $\mathbf{J}_a = e_a \mathbf{v}_a$. Now noting that $\mathbf{r} \times \mathbf{v}$ takes the form of angular momentum:

$$\begin{aligned}\mathbf{m} &= \frac{1}{2c} \sum_a e_a \mathbf{r} \times \mathbf{v}_a \\ &= \frac{e}{2mc} \sum_a \mathbf{r}_a \times \mathbf{p}_a \\ &= \frac{e}{2mc} \mathbf{L}\end{aligned}$$

Thus we see a fundamental relation between the total angular momentum and the magnetic dipole moment. In quantum mechanics, the relation between the spin and magnetic moment of a single electron is given by the Bohr magneton:

$$\mu_B = \frac{e\hbar}{2m_e c}$$

Now let us consider the magnetic field of a magnetic dipole. We need to compute the curl:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Inserting the dipole vector potential, and looking at a particular component:

$$\begin{aligned}B_i &= \varepsilon_{ijk} \partial_j \left(\frac{\varepsilon_{klm} m_l r_m}{r^3} \right) \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_j \left(\frac{m_l r_m}{r^3} \right) \\ &= \partial_m \left(\frac{m_i r_m}{r^3} \right) - \partial_l \left(\frac{m_l r_i}{r^3} \right) \\ &= m_i \partial_m \left(\frac{r_m}{r^3} \right) - m_l \partial_l \left(\frac{r_i}{r^3} \right) \\ &= 4\pi m_i \delta^3(\mathbf{r}) - m_l \left(\frac{\delta_{il}}{r^3} - \frac{3r_i r_l}{r^5} + \frac{4\pi}{3} \delta^3(\mathbf{r}) \right)\end{aligned}$$

Where we have used the fact that

$$\begin{aligned}\partial_\alpha \left(\frac{r_\beta}{r^3} \right) &= -\partial_\alpha \partial_\beta \left(\frac{1}{r} \right) \\ &= \frac{\delta_{\alpha\beta}}{r^3} - \frac{3r_\alpha r_\beta}{r^5} + \frac{4\pi}{3} \delta_{\alpha\beta} \delta^3(\mathbf{r})\end{aligned}$$

The third term is required in order to account for the case where $\alpha = \beta$, where we know that the Laplacian of $1/r$ is $-4\pi\delta^3(\mathbf{r})$.

From our component of \mathbf{B} , we have that

$$\mathbf{B} = -\frac{\mathbf{m}}{r^3} + \frac{3\mathbf{r}(\mathbf{m}, \mathbf{r})}{r^5} + \frac{8\pi}{3} \mathbf{m} \delta^3(\mathbf{r})$$

Which is the magnetic field generated by a magnetic dipole.

Now consider a sphere of radius R that is uniformly magnetized, there is a uniform magnetic field inside the sphere. We want to find the magnetic field outside of the sphere. Our first guess is to say

that the magnetic field outside is dipolar. To check this, let us consider the boundary conditions on the magnetic field inside and outside. The normal component of the magnetic field, B_r , must be continuous. Inside the sphere, we have $B_r = B_0 \cos \theta$, and outside we expect it to match the dipole radial component:

$$B_r = \frac{2m \cos \theta}{r^3}$$

Now matching these components at the surface of the sphere:

$$\begin{aligned} B_0 \cos \theta &= \frac{2m \cos \theta}{R^3} \\ B_0 &= \frac{2m}{R^3} \\ m &= \frac{B_0 R^3}{2} \end{aligned}$$

We see that we can relate the strength of the interior magnetic field to the exterior magnetic field. We can compute the integral of the field over the volume over a sphere centered around the origin:

$$\begin{aligned} \int \mathbf{B} dV &= \frac{2\mathbf{m}}{R^3} \frac{4\pi}{3} R^3 \\ &= \frac{8\pi}{3} \mathbf{m} \end{aligned}$$

We see that there is no R dependence for the total magnetic field, if we fix \mathbf{m} , then shrinking R increases the strength of the magnetic field:

$$B_0 \sim \frac{1}{R^3}$$

Now if we look at the dipole field integrated over a volume:

$$\int \left[\mathbf{B}_d(\mathbf{r}) + \frac{8\pi}{3} \mathbf{m} \delta(\mathbf{r}) \right] d^3\mathbf{r} = 0 + \frac{8\pi}{3} \mathbf{m}$$

We can think of the delta function term as being generated by shrinking the radius of the sphere to be smaller and smaller, while holding the total magnetic field integrated over the volume to be finite.

2.8

In the electrostatics case, we compute the electric force via:

$$\int \rho(\mathbf{r}') \mathbf{E}(\mathbf{r}') d^3\mathbf{r}'$$

Where we consider the force qE , for all charges.

By analogy, we define the magnetic force:

$$\mathbf{F}_B = \frac{1}{c} \int \mathbf{J}(\mathbf{r}') \times \mathbf{B}(\mathbf{r}') d^3\mathbf{r}'$$

We can think of this as looking at the Lorentz force due to every charge:

$$\sum_a \frac{e}{c} \mathbf{v}_a \times \mathbf{B}$$

Which, when we look at infinitesimal charges, gives us the integral of the cross product $\mathbf{J} \times \mathbf{B}$.

Consider two wires, with currents \mathbf{J}_1 and \mathbf{J}_2 . Each generates its own magnetic field, which affects the other wire. We want to find the force of the magnetic field caused by \mathbf{J}_1 acting on the current \mathbf{J}_2 .

We can apply the formula for the magnetic field of current \mathbf{J}_1 :

$$\mathbf{B}_1 = \frac{1}{c} \int \frac{d^3 \mathbf{r}' \mathbf{J}_1(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

The force due to this field on the second current is:

$$\begin{aligned} \mathbf{F}_B^{(2)} &= \frac{1}{c^2} \iint \mathbf{J}_2(\mathbf{r}) \times \frac{(\mathbf{J}_1(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}'))}{|\mathbf{r} - \mathbf{r}'|^3} d^3 \mathbf{r}' d^3 \mathbf{r} \\ &= \frac{1}{c^2} \iint \frac{\mathbf{J}_1(\mathbf{r}') (\mathbf{J}_2 \cdot (\mathbf{r} - \mathbf{r}')) - (\mathbf{r} - \mathbf{r}') (\mathbf{J}_2 \cdot \mathbf{J}_1)}{|\mathbf{r} - \mathbf{r}'|^3} d^3 \mathbf{r}' d^3 \mathbf{r} \end{aligned}$$

Where we have used the identity for the vector product $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$:

$$\mathbf{a} \times \mathbf{b} \times \mathbf{c} = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

Now let us consider the integral:

$$\int \frac{(\mathbf{J}_2(\mathbf{r}) \cdot (\mathbf{r} - \mathbf{r}'))}{|\mathbf{r} - \mathbf{r}'|} d^3 \mathbf{r}$$

Looking at $\text{div} \left(\frac{\mathbf{J}_2}{|\mathbf{r} - \mathbf{r}'|} \right)$:

$$\nabla \cdot \left(\frac{\mathbf{J}_2}{|\mathbf{r} - \mathbf{r}'|} \right) = \mathbf{J}_2 \cdot \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) + \underbrace{\frac{\nabla \cdot \mathbf{J}_2}{|\mathbf{r} - \mathbf{r}'|}}_0$$

Using this to rewrite our integral:

$$\begin{aligned} \int \frac{(\mathbf{J}_2(\mathbf{r}) \cdot (\mathbf{r} - \mathbf{r}'))}{|\mathbf{r} - \mathbf{r}'|} d^3 \mathbf{r} &= - \int \nabla \cdot \left(\frac{\mathbf{J}_2}{|\mathbf{r} - \mathbf{r}'|} \right) d^3 \mathbf{r}' \\ &= 0 \end{aligned}$$

Where we have used the fact that the integral of a divergence is zero. This leaves our force integral as:

$$\mathbf{F}_B^{(2)} = - \frac{1}{c^2} \iint \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} (\mathbf{J}_2 \cdot \mathbf{J}_1) d^3 \mathbf{r}' d^3 \mathbf{r}$$

Note that if the two currents are in the same direction, this is an attractive force, parallel currents attract. We can also note the similarity of the electric force between two charge densities:

$$\mathbf{F}_E = \iint \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \rho_1 \rho_2 d^3\mathbf{r} d^3\mathbf{r}'$$

Except that the magnetic force is scaled by a factor of the ratio of $(v/c)^2$:

$$|F_B| = \left(\frac{v}{c}\right)^2 |F_E|$$

Let us consider the force due to a uniform magnetic field. In this case, the force is given by:

$$\mathbf{F} = I \int d\mathbf{l} \times \mathbf{B}_0$$

Let us expand our magnetic field around a center point \mathbf{r} :

$$\mathbf{B}(\mathbf{r}') = \mathbf{B}(\mathbf{r}) + (\mathbf{r}' \cdot \nabla) \mathbf{B} + \dots$$

We are saying that our magnetic field is roughly uniform for the length scale of our current distribution. Writing out the force:

$$\mathbf{F} = \frac{1}{c} \int d^3\mathbf{r}' \varepsilon_{ipk} J_p (r'_l \partial_l) B_k$$

We can look at a subsection of the integrand:

$$\begin{aligned} \int d^3\mathbf{r}' J_k r'_l &= -\frac{1}{2} \varepsilon_{kli} \int d^3\mathbf{r}' (\mathbf{r}' \times \mathbf{J})_i \\ &= -\frac{1}{2} \varepsilon_{kli} m_i c \end{aligned}$$

Inserting this into our original integral:

$$\begin{aligned} \mathbf{F}_i &= \frac{1}{c} \int d^3\mathbf{r}' \varepsilon_{ipk} J_p (r'_l \partial_l) B_k \\ &= -\varepsilon_{ipk} \varepsilon_{pls} m_s \partial_l B_k \\ &= (\delta_{il} \delta_{ks} - \delta_{is} \delta_{kl}) m_s \partial_l B_k \\ &= m_l \partial_i B_k - m_i \partial_k B_l \end{aligned}$$

From this, we note that the second term has a $\partial_k B_k$, which is the divergence of \mathbf{B} , which is zero, thus we have that

$$\mathbf{F} = m_k \nabla B_k$$

In the case where \mathbf{m} is a constant, one can show that this reduces to

$$\mathbf{F} = \nabla (\mathbf{m} \cdot \mathbf{B})$$

This is the force acting on a current distribution \mathbf{J} , generated by a uniform magnetic field \mathbf{B} .

If we apply a vector calculus identity:

$$\begin{aligned} \mathbf{F} &= \nabla (\mathbf{m} \cdot \mathbf{B}) \\ &= \mathbf{m} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{m}) + (\mathbf{m} \nabla) \mathbf{B} + (\mathbf{B} \nabla) \mathbf{m} \end{aligned}$$

If we apply Ampere's Law, $\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}$, which, in this region, means that $\nabla \times \mathbf{B} = 0$, and noting that $\nabla \mathbf{m} = 0$ since we assumed that \mathbf{m} is constant, so we are left with

$$\mathbf{F} = (\mathbf{m} \nabla) \mathbf{B}$$

Note that if we are working in the region where $\nabla \times \mathbf{B} = 0$, we can use an analogous definition to the electrostatic scalar potential:

$$\mathbf{B} = -\nabla \psi$$

Which we can then plug into the definition of \mathbf{F} and we have another way of deriving $\mathbf{F} = (\mathbf{m} \nabla) \mathbf{B}$. We can also define a potential:

$$\mathbf{F} = -\nabla V$$

Where $V = -\mathbf{m} \cdot \mathbf{B}$.

We can also define the work done by the magnetic force:

$$\begin{aligned} -dW &= \mathbf{F} \cdot d\mathbf{r} \\ &= m_k \underbrace{\partial_i B_k dx_i}_{dB_k} \\ &= m_k dB_k \end{aligned}$$

From this, we can determine the total work:

$$W = - \int \mathbf{m} d\mathbf{B}$$

Consider a magnetic field sourced close to the z -axis in cylindrical, with no ϕ dependence. If we are far from the source current, again $\nabla \times \mathbf{B} = 0$, so we have that $\mathbf{B} = -\nabla \psi$. We can look at Laplace's equation:

$$\nabla^2 \psi = 0$$

Which we can write out in cylindrical coordinates:

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right) + \frac{\partial^2 \psi}{\partial z^2} = 0$$

If we are close to the axis (small ρ), we can assume we have a power expansion:

$$\psi = \sum_n a_n(z) \rho^n$$

Inserting this into Laplace's equation:

$$\sum_n a_n n^2 \rho^{n-2} + \sum_n a_n''(z) \rho^n = 0$$

We can expand out the first few terms in the first summation:

$$\sum_n a_n(z) n^2 \rho^{n-2} = \frac{a_1}{\rho} + 4a_2 + 9a_3\rho + \dots$$

And we can expand out the second summation:

$$\sum_n a_n''(z) \rho^n = a_0'' + a_1''\rho + a_2''\rho^2 + \dots$$

Thus we have:

$$\frac{a_1}{\rho} + 4a_2 + 9a_3\rho + \dots + a_0'' + a_1''\rho + a_2''\rho^2 + \dots = 0$$

At the axis, the potential will be $\psi = a_0(z)$, which means that $B_z(z, \rho = 0) = -a_0'(z)$. If we are not at the axis, we have to cancel out the $1/\rho$ term, so $a_1 = 0$. We will see that all even coefficients, a_{2k} , will be dependent only on a_0 , and we have the general relation:

$$[a_{k+2}(k+2)^2 + a_k'']\rho^k = 0$$

From the ψ expansion, we can determine the field \mathbf{B} to any power of ρ , close to the axis, by taking the negative gradient of the potential. For the lowest order expansion:

$$B_z(z) = -a_0'(z)$$

If we look at higher orders:

$$B_z(\rho, z) = B_z(z) - \frac{\rho^2}{4} B_z'' + \dots$$

2.9 Magnetic Torque Equation

Consider the quantity \mathbf{M} :

$$\mathbf{M} = \frac{1}{c} \int d^3\mathbf{r} \, \mathbf{r} \times \mathbf{J} \times \mathbf{B}$$

Now noting the relation between the magnetic moment and the angular momentum, we have $\mathbf{m} = \gamma \mathbf{L}$, we see that

$$\dot{\mathbf{L}} = \gamma \mathbf{L} \times \mathbf{B}$$

We can see that the magnitude of \mathbf{L} does not change, but the direction changes. This gives us the Larmor precession:

$$\dot{\mathbf{L}} = \boldsymbol{\Omega} \times \mathbf{L}$$

Where $\boldsymbol{\Omega} = \gamma \mathbf{B}$.

What does the $\mathbf{J} \times \mathbf{B}$ force do to distributions of current inside magnetic fields? We can look at the form of the force:

$$\begin{aligned} \frac{1}{c} \mathbf{J} \times \mathbf{B} &= \frac{1}{4\pi} (\nabla \times \mathbf{B} \times \mathbf{B}) \\ &= \frac{1}{4\pi} \left(\underbrace{(\mathbf{B} \nabla) \mathbf{B}}_{\text{mag. tension}} - \underbrace{\nabla \left(\frac{B^2}{2} \right)}_{\text{mag. pressure}} \right) \end{aligned}$$

The magnetic tension term wants to “straighten” out the magnetic field lines, and this is counteracted by the magnetic pressure term.

Looking at a specific case, recall the thick wire, which generated a toroidal magnetic field:

$$\mathbf{B}_\phi = \begin{cases} \frac{2I\rho}{a^2c}, & \rho < a \\ \frac{2I}{c\rho}, & \rho > a \end{cases}$$

From this (since $\mathbf{J} = 0$ outside of the wire), we see that the tension and pressure effects cancel each other out exactly outside of current sources. Inside the wire, we can look at the two terms of the force:

$$\begin{aligned} \mathbf{F}_{\text{tension}} &= \frac{1}{4\pi} (\mathbf{B} \nabla) \mathbf{B} \\ &= -\frac{1}{4\pi} \frac{B_\rho^2}{\rho} \hat{\rho} \\ \mathbf{F}_{\text{pressure}} &= -\frac{1}{8\pi} \nabla (B_\phi^2) \\ &= -\frac{\partial}{\partial \rho} \left(\frac{B_\phi^2}{8\pi} \right) \hat{\rho} \\ &= -\frac{B_\phi^2}{4\pi\rho} \hat{\rho} \end{aligned}$$

We see that both forces work together, to squeeze the wire. If we have some resisting pressure P , which provides force $\mathbf{F} = -\nabla P$, we can generate the equilibrium condition:

$$\frac{\partial}{\partial \rho} \left(P + \frac{B^2}{8\pi} \right) = \text{const.}$$

If we are in the special case where $\mathbf{J} = \alpha \mathbf{B}$, we have what is known as a force-free state, where $\mathbf{J} \times \mathbf{B} = 0$. An example of a force-free state is the corona of the sun. These are also known as Taylor states¹¹.

¹¹Not the Taylor of Taylor series fame.

3 Electromagnetic Waves

3.1 Wave Equations

We now move to electromagnetic waves, for which we must rewrite Maxwell's equations in their time-dependent forms, with no sources:

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}\end{aligned}$$

And in a vacuum, we have the initial conditions of the other two Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0$$

Now let us take the curl of both sides of the first equation:

$$\begin{aligned}\nabla \times \nabla \times \mathbf{E} &= -\frac{1}{c} \frac{\partial}{\partial t} (\nabla \times \mathbf{B}) \\ \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}\end{aligned}$$

From this, we have the wave equation for the electric field:

$$\left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \mathbf{E} = 0$$

Working in the Lorenz gauge¹², where $\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0$, we can rewrite Maxwell's equations in terms of the potentials:

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{A}) &= \frac{1}{c} \frac{\partial}{\partial t} \left(-\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) \\ \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} &= \nabla \left(-\frac{1}{c} \frac{\partial \phi}{\partial t} \right) - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} \\ \underbrace{\nabla \left(\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \right)}_0 - \nabla^2 \mathbf{A} &= -\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2}\end{aligned}$$

This leads to two uncoupled wave equations:

$$\begin{aligned}\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} &= 0 \\ \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} &= 0\end{aligned}$$

¹²This choice of gauge is made more clear when working in 4-dimensional notation, where we are stating that the 4-derivative of A_μ is zero.

Looking at the \mathbf{A} wave equation, consider solutions of the form:

$$\mathbf{A}(\mathbf{r}, t) = u(\mathbf{r}, t) \mathbf{S}$$

Where \mathbf{S} is some constant vector. Inserting this, we find an equation in terms of u :

$$\nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0$$

Once we have \mathbf{A} , we can use the gauge condition to find ϕ :

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= -c \nabla \cdot \mathbf{A} \\ &= -c \partial_i (u S'_i) \\ &= -c S'_i \partial_i u \end{aligned}$$

Which we can integrate to find ϕ :

$$\phi = -c \mathbf{S} \cdot \int_{-\infty}^t dt' \nabla u(\mathbf{r}, t')$$

If u satisfies its own wave equation, then we generate a ϕ that solves a wave equation.

3.2 Method of Characteristics

Consider waves propagating in one direction (which is allowed in the vacuum, since no directions are considered special). In this formalism, the wave equation takes the form:

$$\frac{\partial^2 \mathbf{w}}{\partial z^2} - \frac{\partial^2 \mathbf{w}}{\partial t^2} \frac{1}{c^2} = 0$$

Note that the second order differential operators can be decomposed into two first order differentials:

$$\frac{\partial^2 \mathbf{w}}{\partial z^2} - \frac{\partial^2 \mathbf{w}}{\partial t^2} \frac{1}{c^2} = \left(\frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial z} - \frac{1}{c} \frac{\partial}{\partial t} \right) \mathbf{w}$$

Using this, we can rewrite our wave equation as:

$$\left(\frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial z} - \frac{1}{c} \frac{\partial}{\partial t} \right) \mathbf{w} = 0$$

Now consider a change of variables, $\xi = z + ct$, and $\eta = z - ct$. Applying the chain rule:

$$\begin{aligned} \frac{\partial}{\partial \xi} &= \frac{\partial}{\partial z} \frac{\partial z}{\partial \xi} + \frac{\partial}{\partial t} \frac{\partial t}{\partial \xi} \\ \frac{\partial}{\partial \eta} &= \frac{\partial}{\partial z} \frac{\partial z}{\partial \eta} + \frac{\partial}{\partial t} \frac{\partial t}{\partial \eta} \end{aligned}$$

To find these derivatives, we note that $z = \frac{1}{2}(\xi + \eta)$, and $ct = \frac{1}{2}(\xi - \eta)$. From this, we can rewrite our chain rules:

$$\frac{\partial}{\partial \xi} = \frac{1}{2} \left(\frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t} \right)$$

$$\frac{\partial}{\partial \eta} = \frac{1}{2} \left(\frac{\partial}{\partial t} - \frac{1}{c} \frac{\partial}{\partial t} \right)$$

Now noting that our original decomposition of the wave equation reduces to:

$$\frac{\partial^2 \mathbf{w}}{\partial \xi \partial \eta} = 0$$

From this, we see that any function of only η , or a function of only ξ , are solutions to this equation:

$$\mathbf{w} = \mathbf{f}(\xi) + \mathbf{g}(\eta)$$

Intuitively, this makes sense, ξ and η describe waves moving to a particular direction, either left or right. An example of a solution would be $\sin \eta$, or rather $\sin(z - ct)$, which is a rightwards travelling wave. Another solution would be $\sin \xi = \sin(z + ct)$, a leftwards travelling wave. Note that the waves are travelling at the speed of light. The functions \mathbf{f} and \mathbf{g} can be anything, they are dependent on the initial conditions for the wave equation. Also note that this means that the *form* of the wave remains constant, it simply moves in a direction as a function of time. The lines $\xi = z + ct$ and $\eta = z - ct$ are known as the characteristics of our wave equation.

3.3 Transverse EM Waves

Suppose we have a wave travelling in the z direction. We want to find $\mathbf{E}(z, t)$ and $\mathbf{B}(z, t)$. To begin, we first convince ourselves that $\mathbf{E}_z = 0$. To derive this, we begin with the divergence condition:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \frac{\partial E_z}{\partial z} &= 0 \end{aligned}$$

We also have Ampere's Law:

$$\begin{aligned} \left(\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \right)_z &= (\nabla \times \mathbf{B})_z \\ &= \partial_x B_y - \partial_y B_x \\ &= 0 \end{aligned}$$

Where we have used the fact that the partials in directions other than z of the magnetic field must be zero, the wave is propagating only along the z . Thus we have that

$$\frac{\partial E_z}{\partial t} = 0$$

The electric field in the z direction must be constant in space and constant in time, meaning that it is a global constant. If we assume that the electric field at ∞ is zero, then we can set $E_z = 0$. From this, we have two solutions:

$$\begin{aligned} \mathbf{E}_+ &= \mathbf{f}_\perp(z - ct) \\ \mathbf{E}_- &= \mathbf{g}_\perp(z + ct) \end{aligned}$$

Where the perpendicular denotes that the vector functions \mathbf{f}_\perp and \mathbf{g}_\perp do not have z components, $\mathbf{k} \cdot \mathbf{E} = 0$. This is the definition of a transverse electromagnetic mode (TEM).

Plane waves take the form of $\mathbf{A}e^{i\mathbf{k}\cdot\mathbf{r}}$, and if we take the divergence of this:

$$\begin{aligned}\left(\nabla \cdot (\mathbf{A}e^{i\mathbf{k}\cdot\mathbf{r}})\right)_i &= \partial_i (A_i e^{i\mathbf{k}\cdot\mathbf{r}}) \\ &= A_i e^{i\mathbf{k}\cdot\mathbf{r}} i k_i \delta_{ij}\end{aligned}$$

From this, we have that

$$\nabla \cdot (\mathbf{A}e^{i\mathbf{k}\cdot\mathbf{r}}) = i (\mathbf{k} \cdot \mathbf{A}e^{i\mathbf{k}\cdot\mathbf{r}})$$

By Gauss's Law, this must be equal to $4\pi\rho_{\mathbf{k}}$:

$$i (\mathbf{k} \cdot \mathbf{E}_{\mathbf{k}}) = 4\pi\rho_{\mathbf{k}}$$

Where $\rho_{\mathbf{k}}$ is the Fourier component of the charge density in the \mathbf{k} direction.

We have the two possible electric field propagating solutions, now what does the \mathbf{B} field solution look like? Looking at the relation between \mathbf{B} and \mathbf{E} :

$$-\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}$$

If we assume that \mathbf{B} takes the form $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$:

$$\begin{aligned}-\frac{i\omega}{c} B_{\mathbf{k}} &= i (\mathbf{k} \times \mathbf{E}_{\mathbf{k}}) \\ B_{\mathbf{k}} &= \hat{\mathbf{e}}_{\mathbf{k}} \times \mathbf{E}_{\mathbf{k}}\end{aligned}$$

Looking at this in components:

$$\begin{aligned}e_{ijl} \partial_j (B_l e^{i(k_m r_m - \omega t)}) &= e_{ijl} B_l \underbrace{\partial_j (e^{i(k_m r_m - \omega t)})}_{k_m i \delta_{jm} e^{i(k_m r_m - \omega t)}} \\ &= e_{ijl} B_l i k_j e^{i(k_m r_m - \omega t)} \\ &= i (\mathbf{k} \times \mathbf{B})\end{aligned}$$

Not quite sure what he was doing here.

From this, we have that \mathbf{B} is perpendicular to \mathbf{E} , as well as the direction of propagation. Another consequence is that $|\mathbf{B}| = |\mathbf{E}|$.

We can also derive the perpendicularity of the \mathbf{B} field by starting with the electric field solutions, in the general case, not assuming that they are plane waves. Once again applying the induction equation:

$$\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

We suppose that we have two solutions, \mathbf{B}_+ and \mathbf{B}_- :

$$\begin{aligned}\frac{1}{c} \frac{\partial}{\partial t} \mathbf{B}_+ &= - \begin{bmatrix} 0 \\ 0 \\ \frac{\partial}{\partial z} \end{bmatrix} \times \mathbf{f}_{\perp}(z, t) \\ &= -\hat{\mathbf{e}}_z \times \frac{\partial \mathbf{f}_{\perp}}{\partial z}\end{aligned}$$

$$\begin{aligned}
&= -\hat{e}_z \times \frac{1}{c} \frac{\partial}{\partial t} (-\mathbf{f}_\perp) \\
\frac{1}{c} \frac{\partial}{\partial t} \mathbf{B}_- &= - \begin{bmatrix} 0 \\ 0 \\ \frac{\partial}{\partial z} \end{bmatrix} \times \mathbf{g}_\perp(z, t) \\
&= -\hat{e}_z \times \frac{\partial \mathbf{g}_\perp}{\partial z} \\
&= -\hat{e}_z \times \frac{1}{c} \frac{\partial}{\partial t} (\mathbf{g}_\perp)
\end{aligned}$$

From these, we have that

$$\begin{aligned}
\mathbf{B}_+ &= \hat{e}_z \times \mathbf{f}_\perp \\
\mathbf{B}_- &= -\hat{e}_z \times \mathbf{g}_\perp
\end{aligned}$$

This entire process is direction agnostic in the vacuum case, we can always define a direction of propagation:

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}_\perp (\mathbf{k} \cdot \mathbf{r} - ickt) \\
\mathbf{B} &= \mathbf{k} \times \mathbf{E}
\end{aligned}$$

We can also define the phase and group speeds. Phase speed is defined as:

$$\phi = (\mathbf{k} \cdot \mathbf{r} - kct)$$

Using this, we can define how surfaces of constant phase are moving. In this case, these surfaces are planes, since ϕ is defined by a linear equation in multiple coordinates. The plane is orthogonal to the vector \mathbf{k} , and if we look at how it is moving, it will remain perpendicular to \mathbf{k} and moves at the phase speed:

$$\mathbf{v}_{\text{phase}} = \hat{e}_k \frac{\omega}{k}$$

In the vacuum case:

$$\mathbf{v}_{\text{phase}} = c$$

The phase speed is not a physical speed, it is the velocity of a plane of constant phase, and it can in fact be faster than the speed of light, since no physical objects are actually moving faster than the speed of light.

How does the energy in the plane waves depend on the amplitude of the wave itself? We can define the energy density:

$$\begin{aligned}
U_{\text{EM}} &= \frac{1}{8\pi} (\mathbf{E} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{B}) \\
&= \frac{1}{4\pi} |\mathbf{E}_\perp|^2
\end{aligned}$$

We can define a Poynting flux \mathbf{S} , which defines how the electromagnetic energy is propagating through space:

$$\begin{aligned}\mathbf{S} &= \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \\ &= \frac{c}{4\pi} |\mathbf{E}_\perp|^2 \hat{e}_k \\ &= cU_{\text{EM}} \hat{e}_k\end{aligned}$$

And this satisfies the continuity equation:

$$\frac{\partial U_{\text{EM}}}{\partial t} + \nabla \cdot \mathbf{S} = 0$$

We can also define \mathbf{g} , the momentum of the wave:

$$\begin{aligned}\mathbf{g} &= \frac{\mathbf{S}}{c^2} \\ &= \frac{U_{\text{EM}}}{c} \hat{e}_k\end{aligned}$$

We see that we recover the expected relation, $\mathcal{E} = pc$.

3.3.1 Polarization

Now let us consider the polarization of this system. To recap, we have monochromatic plane waves:

$$\mathbf{E}(\mathbf{r}, t) = \mathcal{E}_\perp e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$$

Where $\omega = kc$. We discussed the energy density of the wave:

$$U_{\text{EM}} = \frac{1}{8\pi} \left[[\text{Re} \mathbf{E}]^2 + [\text{Re} \mathbf{B}]^2 \right]$$

If we average the energy over a wave period:

$$\begin{aligned}\overline{U_{\text{EM}}} &= \langle U_{\text{EM}} \rangle_t \\ &= \frac{1}{8\pi} \left\langle \left[\frac{1}{4} (\mathbf{E} + \mathbf{E}^*)^2 + \frac{1}{4} [\mathbf{B} + \mathbf{B}^*]^2 \right] \right\rangle\end{aligned}$$

In these units, the two terms are the same:

$$\begin{aligned}\overline{U_{\text{EM}}} &= \frac{2}{8\pi} \frac{1}{4} \langle (\mathbf{E} + \mathbf{E}^*)^2 \rangle \\ &= \frac{1}{16\pi} \langle 2\mathbf{E} \cdot \mathbf{E}^* \rangle \\ &= \frac{1}{8\pi} |\mathcal{E}_\perp|^2\end{aligned}$$

Where we have used the fact that the average of $(\mathbf{E} + \mathbf{E}^*)^2$ over a period vanishes except for the cross term, where the exponentials vanish.

Now let us attempt to look at the geometry of the field in the plane orthogonal to the direction of propagation (\mathbf{k}). We can decompose vectors in this plane in terms of \hat{e}_1 and \hat{e}_2 :

$$\mathbf{E} = (\mathcal{E}_1 \hat{e}_1 + \mathcal{E}_2 \hat{e}_2) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$$

We can consider \mathcal{E}_1 and \mathcal{E}_2 as complex numbers:

$$\mathcal{E}_1 = Ae^{i\delta_1} \quad \mathcal{E}_2 = Be^{i\delta_2}$$

From these, we can write down the real part of the wave:

$$\begin{aligned} \text{Re}\mathbf{E} &= A \cos(\phi + \delta_1) \hat{e}_1 + B \cos(\phi + \delta_2) \hat{e}_2 \\ &= E_1 \hat{e}_1 + E_2 \hat{e}_2 \end{aligned}$$

Where we have denoted $\phi = \mathbf{k} \cdot \mathbf{r} - \omega t$.

As we evolve in t , the field remains in this plane, but this vector will move around. What we want to do is look at the shape that this vector traces out over time.

Consider the expressions:

$$\begin{aligned} \frac{E_1}{A} \sin \delta_2 - \frac{E_2}{B} \sin \delta_1 &= \cos(\phi + \delta_1) \sin \delta_2 - \cos(\phi + \delta_2) \sin \delta_1 \\ &= \sin(\delta_2 - \delta_1) \cos \phi \\ \frac{E_1}{A} \cos \delta_2 - \frac{E_2}{B} \cos \delta_1 &= \sin(\delta_2 - \delta_1) \sin \phi \end{aligned}$$

If we square these two and add them together, we find the equation:

$$\left(\frac{E_1}{A}\right)^2 + \left(\frac{E_2}{B}\right)^2 - 2\frac{E_1}{A}\frac{E_2}{B} \cos \delta = \sin^2 \delta$$

Where we have defined a relative phase $\delta = \delta_2 - \delta_1$. We can now see that if $\cos \delta = 0$, then we have exactly the equation of an ellipse. If $\delta \neq \frac{\pi}{2}$, we can just shift our axes and we will have a rotated ellipse.

From this, we see that we can classify the polarizations based on δ . If $\delta = m\pi$ for $m \in \mathbb{Z}$, then the $\sin^2 \delta = 0$, and the $\cos \delta = \pm 1$, so we have the ellipse reducing to a line:

$$\text{Re}\mathbf{E} = (A\hat{e}_1 + B\hat{e}_2) \cos(\phi + \delta_1)$$

The trajectory of the field in the plane will oscillate back and forth along the same direction, which we denote as **linear polarization**.

We can also define **circular polarization**, where $\delta = \frac{m\pi}{2}$ where $m = 1, 3, \dots$, where $A = B = \frac{A}{\sqrt{2}}$, we have the electric field moving in a circle around the plane. We can define two subpolarizations in this case, where we are rotating either clockwise or counterclockwise in the plane. These denote Left-handed Circular Polarization (LCP), and Right-Handed Circular Polarization (RHCP)¹³. The choice of the label depends on the perspective, whether we are looking at the tail of the wave as it propagates or if the wave is moving directly towards us.

We can generalize the polarization using 4 parameters, generally denoted I, Q, U and V . I is the total intensity, and V denotes the amount of circular polarization (the sign of V denotes the handed-ness of the circular polarization). U and V denote the two independent linear modes of polarization. We can relate the intensity to the other 3 parameters:

$$I^2 \geq Q^2 + U^2 + V^2$$

¹³Red Hot Chili Peppers?!

In the cases that we have previously considered (perfectly polarized emission):

$$\begin{aligned} I &= A^2 + B^2 \\ Q &= A^2 - B^2 \\ U &= 2AB \cos \delta \\ V &= 2AB \sin \delta \end{aligned}$$

In the linearly polarized case we considered, $\delta = m\pi$, we have that

$$\begin{aligned} V &= 0 \\ Q &\neq 0 \\ U &\neq 0 \end{aligned}$$

For the circular case, where $\delta = \frac{m\pi}{2}$, $m = 1, 3, \dots$

$$\begin{aligned} V &= \pm AB \\ U &= 0 \\ Q &= 0 \end{aligned}$$

These are the Stokes parameters for the different cases of perfectly polarized emission in a plane.

3.4 Radiation

We want to find the general formula for the electromagnetic field generated by a moving charge. To do this, we introduce the concept of retarded potentials. We are introducing a source to Maxwell's equations:

$$\begin{aligned} \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\frac{4\pi}{c} \mathbf{J} \\ \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} &= -4\pi \rho \end{aligned}$$

Where we are again working in the Lorenz gauge:

$$\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0$$

In the case of a single charge, our charge density is given by

$$\rho = e(t) \delta(\mathbf{R})$$

Where \mathbf{R} is the location of the charge. Away from the point where the charge is, we have the same scalar potential, with charge density being zero:

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0$$

If we think about the frame in which we are moving with the charge, we expect that we should have spherical symmetry with respect to \mathbf{R} . We can thus replace $\nabla^2 \phi$ with the spherically symmetric Laplacian:

$$\frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial \phi}{\partial R} \right) - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0$$

Now we can show that we can remove the R dependence by substituting in a new function $\phi = \frac{\chi(R,t)}{R}$. We can first compute the derivatives of ϕ :

$$\phi' = \frac{\chi'}{R} - \frac{1}{R^2}\chi$$

From which we can find

$$\begin{aligned}(R^2\phi')' &= (R\chi' - \chi)' \\ &= R\chi''\end{aligned}$$

Thus we have that our equation turns into:

$$\frac{\partial^2\chi}{\partial R^2} - \frac{1}{c^2} \frac{\partial^2\chi}{\partial t^2} = 0$$

This is something that we have solved previously:

$$\chi = f_1\left(t - \frac{R}{c}\right) + f_2\left(t + \frac{R}{c}\right)$$

From which we can divide by R to obtain the solution for ϕ .

Which of these two solutions should we keep? First, note that the function f_1 depends on the state of charge that existed at time $t - \frac{R}{c}$, which is in the past. This is causal, at some time t , we have to wait for the information to propagate (at the speed of light) to us. Thus we pick the $f_1\left(t - \frac{R}{c}\right)$ potential, it describes something that has happened at a previous t affecting us at a later t . This is known as the retarded potential.

Also note that in the asymptotic limit where $R \rightarrow 0$, we know that $\chi = e(t)$, since we need to recover the charge distribution at the point charge itself. From this, we have that

$$\phi = \frac{e\left(t - \frac{R}{c}\right)}{R}$$

Just like we did when moving from point charge potentials to continuous charge distributions, we can move from a moving point charge to a moving charge distribution ρ :

$$\phi = \int \underbrace{\frac{1}{|\mathbf{r} - \mathbf{r}'|}}_R \rho\left(\mathbf{r}', t - \frac{R}{c}\right) d^3\mathbf{r}' + \phi_0$$

This gives us the potential for the electromagnetic wave generated by any time-dependent charge density.

Similarly, we can compute the vector potential:

$$\mathbf{A} = \frac{1}{c} \int \frac{1}{|\mathbf{r} - \mathbf{r}'|} \mathbf{J}\left(\mathbf{r}', t - \frac{R}{c}\right) d^3\mathbf{r}' + \mathbf{A}_0$$

While this may seem simple, these integrals are very computationally difficult. We will first look at the case where

$$\rho(\mathbf{r}, t) = q\delta(\mathbf{r} - \mathbf{r}_0(t))$$

Where we have a point charge moving on the trajectory $\mathbf{r}_0(t)$.

In practice, we find that $t - \frac{R}{c}$ is a difficult quantity to work with. Instead, we introduce a second integral, over t' , and then insert a delta function to select the correct value:

$$\phi(\mathbf{r}, t) = \int d^3\mathbf{r}' \int dt' \frac{\rho(\mathbf{r}, t')}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - t + \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)$$

Where, as usual, \mathbf{r}' are the locations of the charges in the charge distribution. Inserting our definition of ρ :

$$\phi(\mathbf{r}, t) = q \int dt' \int d^3\mathbf{r}' \frac{\delta(\mathbf{r} - \mathbf{r}_0(t'))}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - t + \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)$$

If we integrate over the \mathbf{r}' first, which gets rid of the delta function on \mathbf{r}' :

$$\phi(\mathbf{r}, t) = q \int dt' \frac{1}{|\mathbf{r} - \mathbf{r}_0(t')|} \delta\left(t' - t + \frac{|\mathbf{r} - \mathbf{r}_0(t')|}{c}\right)$$

Now applying a property of the delta function:

$$\int \delta(f(x)) dx = \frac{1}{|f'(x)|} \Big|_{\text{zeros}}$$

From this, we can compute the zeros of the argument to the delta function:

$$t_{\text{zero}} = t - \frac{|\mathbf{r} - \mathbf{r}_0(t_{\text{zero}})|}{c}$$

To do this, let us denote the argument as the function $g(t')$:

$$g(t') = t' - t + \frac{|\mathbf{r} - \mathbf{r}_0(t')|}{c}$$

Computing the derivative with respect to t' :

$$\begin{aligned} \frac{\partial g(t')}{\partial t'} &= 1 + \frac{1}{c} \frac{d}{dt'} \left(\sqrt{\mathbf{R}(t') \cdot \mathbf{R}(t')} \right) \\ &= 1 - \frac{1}{2c} \frac{2\mathbf{v}(t') \cdot \mathbf{R}(t')}{R(t')} \\ &= 1 - \beta(t') \end{aligned}$$

Where we have used the fact that

$$\begin{aligned} \frac{d}{dt'} (\mathbf{R} \cdot \mathbf{R}) &= 2 \frac{d\mathbf{R}}{dt'} \cdot \mathbf{R} \\ &= -2\mathbf{v}(t') \cdot \mathbf{R}(t') \end{aligned}$$

And we have defined $\beta(t') = \frac{v}{c}$. If we also define a direction vector $\hat{n}(t') = \frac{\mathbf{R}(t')}{R(t')}$, we can write our potential as:

$$\phi(\mathbf{r}, t) = \left[\frac{q}{|\mathbf{r} - \mathbf{r}_0(t')|} \underbrace{\frac{1}{|1 - \beta(t') \cdot \hat{n}(t')|}}_{\mathcal{D}} \right]_{t'=t_{\text{zero}}}$$

This second term \mathcal{D} is known as the Doppler factor. Consider a frequency ω' that is the result of a Doppler shift given by \mathbf{k} and \mathbf{v} :

$$\begin{aligned}\omega' &= \gamma (\omega \pm \mathbf{k} \cdot \mathbf{v}) \\ &= \gamma \omega (1 \pm \boldsymbol{\beta} \cdot \hat{n})\end{aligned}$$

We see that we have the same functional form, hence the term Doppler factor.

Also note that if we have a velocity less than c , there is only one root t_{zero} , which makes sense by causality.

We can follow a similar process for the vector potential, and we will find that:

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{c} \left[\frac{q\mathbf{v}(t')}{|\mathbf{r} - \mathbf{r}_0(t')|} \mathcal{D} \right]_{t'=t_{\text{zero}}}$$

These are the Liénard-Wiechert potentials.

Now from these potentials, we can find the electric and magnetic fields:

$$\mathbf{E} = -\nabla\phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$$

To begin, let us do some preliminary computations. To begin, we want to compute (noting that for ease of computation, we use t' and t_{zero} interchangeably):

$$\begin{aligned}\frac{\partial R}{\partial t} &= \frac{\partial R}{\partial t'} \frac{\partial t'}{\partial t} \\ &= \frac{\partial}{\partial t'} (\sqrt{\mathbf{R} \cdot \mathbf{R}}) \frac{\partial t'}{\partial t} \\ &= -\frac{\mathbf{R} \cdot \mathbf{v}}{R} \frac{\partial t'}{\partial t}\end{aligned}$$

If we differentiate the definition of t_{zero} with respect to t :

$$\frac{\partial t'}{\partial t} = 1 - \frac{\partial R}{\partial t} \frac{1}{c}$$

Solving this for $\frac{\partial R}{\partial t}$ and inserting it into the expression that we had before:

$$\begin{aligned}-\frac{\mathbf{R} \cdot \mathbf{v}}{R} \frac{\partial t'}{\partial t} &= c \left(1 - \frac{\partial t'}{\partial t} \right) \\ \frac{\partial t'}{\partial t} &= \frac{1}{1 - \boldsymbol{\beta} \cdot \hat{n}} \\ &= \mathcal{D}\end{aligned}$$

We can then write out $\frac{\partial R}{\partial t}$:

$$\frac{\partial R}{\partial t} = -(\hat{n} \cdot \boldsymbol{\beta}) \mathcal{D}$$

and also compute the gradient of R :

$$\nabla R(\mathbf{r}, t') = \hat{n} + \frac{\partial R}{\partial t'} \nabla t'$$

Now looking at the gradient of t' :

$$\nabla t' = -\frac{1}{c} \nabla R$$

Inserting this into our previous expression:

$$\nabla R = \hat{n} + \frac{\partial R}{\partial t'} \left(-\frac{1}{c} \nabla R \right)$$

Inserting the previously computed $\frac{\partial R}{\partial t'}$, we find that

$$\begin{aligned} \nabla t' &= -\frac{\mathbf{R}}{c \left(R - \frac{\mathbf{R} \cdot \mathbf{v}}{c} \right)} \\ &= -\frac{\hat{n}}{c (1 - \hat{n} \cdot \boldsymbol{\beta})} \\ \nabla R &= \frac{\hat{n}}{1 - \hat{n} \cdot \boldsymbol{\beta}} \end{aligned}$$

We can also find that

$$\frac{\partial t'}{\partial t} = \frac{1}{1 - \hat{n} \cdot \boldsymbol{\beta}}$$

This represents the effect of special relativity, the clock of the moving particle varies when compared to the clock of an observer that is far away.

Now that we have the required derivatives, we can write out the electric field:

$$\mathbf{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$$

Looking at $\nabla \phi$:

$$\begin{aligned} \nabla \phi &= q \nabla \left(\frac{1}{R - \boldsymbol{\beta} \cdot \mathbf{R}} \right) \\ &= -\frac{q}{(1 - \boldsymbol{\beta} \cdot \hat{n}) R^2} \nabla (R - \boldsymbol{\beta} \cdot \mathbf{R}) \\ &= -\frac{q}{R^2 (1 - \boldsymbol{\beta} \cdot \hat{n})^2} \left[\frac{\hat{n}}{1 - \hat{n} \cdot \boldsymbol{\beta}} - \nabla (\boldsymbol{\beta} \cdot \mathbf{R}) \right] \end{aligned}$$

Looking at this last term:

$$\partial_i (\boldsymbol{\beta}_j R_j) = \beta_j \left[\frac{\partial R_j}{\partial t'} \frac{\partial t'}{\partial x_i} + \frac{\partial R_j}{\partial x_i} \right] + R_j \frac{\partial \beta_j}{\partial t'} \frac{\partial t'}{\partial x_i}$$

Looking at $\frac{\partial R_j}{\partial x_i}$:

$$\frac{\partial R_j}{\partial x_i} = \frac{\partial (x_j - r_{0,j}(t'))}{\partial x_i}$$

$$= \delta_{ij}$$

We also note that $\frac{\partial R_j}{\partial t'} = -c(\hat{n} \cdot \boldsymbol{\beta})$. Thus we have that

$$\partial_i (\beta_j R_j) = \beta_i + \frac{n_i}{1 - \hat{n} \cdot \boldsymbol{\beta}} \beta^2 + (\mathbf{R} \cdot \boldsymbol{\beta}) \left(-\frac{\hat{n}}{c} \right) \frac{1}{1 - \hat{n} \cdot \boldsymbol{\beta}}$$

If we now insert this back into the expression for $\nabla \phi$:

$$\nabla \phi = -\frac{q}{R^2 (1 - \hat{n} \cdot \boldsymbol{\beta})^3} \left[\hat{n} \left(1 - \beta^2 + \frac{\dot{\boldsymbol{\beta}} \cdot \mathbf{R}}{c} \right) - \boldsymbol{\beta} (1 - \hat{n} \cdot \boldsymbol{\beta}) \right]$$

Now we can compute the derivative of the vector potential:

$$\begin{aligned} -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} &= -\frac{q}{c} \frac{\partial}{\partial t} \left(\frac{\boldsymbol{\beta}}{R(1 - \hat{n} \cdot \boldsymbol{\beta})} \right) \\ &= -\frac{q}{c} \frac{\partial}{\partial t} \left(\frac{\boldsymbol{\beta}}{R - \mathbf{R} \cdot \boldsymbol{\beta}} \right) \\ &= \frac{q}{c} \frac{\boldsymbol{\beta}}{R^2 (1 - \hat{n} \cdot \boldsymbol{\beta})^2} \frac{d}{dt} (\mathbf{R} - \mathbf{R} \cdot \boldsymbol{\beta}) - \frac{1}{c} \frac{\dot{\boldsymbol{\beta}}}{R(1 - \hat{n} \cdot \boldsymbol{\beta})} \end{aligned}$$

Doing out this time derivative:

$$\frac{d}{dt'} (\mathbf{R} - \mathbf{R} \cdot \boldsymbol{\beta}) \frac{\partial t'}{\partial t} = \frac{1}{1 - \hat{n} \cdot \boldsymbol{\beta}} [-c\hat{n} \cdot \boldsymbol{\beta} + c\beta^2 - \dot{\boldsymbol{\beta}} \cdot \mathbf{R}]$$

Which leaves us with

$$-\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = \frac{q\boldsymbol{\beta} (c\beta^2 - c\hat{n} \cdot \boldsymbol{\beta} - \dot{\boldsymbol{\beta}} \cdot \mathbf{R})}{cR^2 (1 - \hat{n} \cdot \boldsymbol{\beta})^3} - \frac{q\dot{\boldsymbol{\beta}}}{cR(1 - \hat{n} \cdot \boldsymbol{\beta})^2}$$

Now putting this all together:

$$\begin{aligned} \mathbf{E} &= -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \\ &= \frac{q}{R^2 (1 - \hat{n} \cdot \boldsymbol{\beta})^3} \left[(\hat{n} \cdot \boldsymbol{\beta}) \left(1 - \beta^2 + \frac{\dot{\boldsymbol{\beta}} \cdot \mathbf{R}}{c} \right) \right] - \frac{q}{cR} \frac{\dot{\boldsymbol{\beta}}}{(1 - \hat{n} \cdot \boldsymbol{\beta})^2} \\ &= \frac{q}{R^2} \frac{(1 - \beta^2)(\hat{n} \cdot \boldsymbol{\beta})}{(1 - \hat{n} \cdot \boldsymbol{\beta})^3} + \frac{q}{cR} \left[\frac{(\dot{\boldsymbol{\beta}} \cdot \hat{n})(\hat{n} - \boldsymbol{\beta})}{(1 - \hat{n} \cdot \boldsymbol{\beta})^3} - \frac{\dot{\boldsymbol{\beta}}}{(1 - \hat{n} \cdot \boldsymbol{\beta})^2} \right] \end{aligned}$$

The first term is a static field, and is enhanced in the direction of motion. If $\boldsymbol{\beta}$ is constant, the field scales as $1/R^2$, with just the static field. However, if $\boldsymbol{\beta}$ is not constant, the second term will give us a non-zero Poynting flux, and we will have radiation. This matches our intuition, an accelerating charge produces radiation.

If we look at the second term, which we will denote \mathbf{E}_{wave} :

$$\mathbf{E}_{\text{wave}} = \frac{q}{cR(1 - \hat{n} \cdot \boldsymbol{\beta})^3} \left[(\dot{\boldsymbol{\beta}} \cdot \hat{n})(\hat{n} - \boldsymbol{\beta}) - \dot{\boldsymbol{\beta}}(1 - \hat{n} \cdot \boldsymbol{\beta}) \right]$$

$$= \frac{q}{cR(1 - \hat{n} \cdot \boldsymbol{\beta})^3} [\hat{n} \times (\hat{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]$$

The magnetic field will be given by:

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} \\ &= \hat{n}_{\text{ret}} \cdot \mathbf{E} \end{aligned}$$

Where $\hat{n} = \frac{\mathbf{R}}{R}$, evaluated at t_{ret} .

Also note that we have done no approximations here, everything is exact.

Could we think of an easier way of deriving the static electric field? We can compute $\mathbf{E}_{\text{static}}$ by Lorentz transforming the electric field of a static particle.

If we want to find the total radiation, we need to integrate the Poynting flux. The Poynting vector is given by:

$$\begin{aligned} \mathbf{S} &= \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \\ &= \frac{c}{4\pi} E_{\text{wave}}^2 \hat{n} \\ &= \frac{c}{4\pi} \frac{q^2}{c^2 R^2} \left[\frac{\hat{n} \times (\hat{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}}{(1 - \hat{n} \cdot \boldsymbol{\beta})^3} \right]^2 \hat{n} \end{aligned}$$

If we want the flux through a sphere of radius R :

$$\begin{aligned} \frac{dU}{dt d\Omega} &= R^2 (\mathbf{S} \cdot \hat{n}) \\ &= \frac{q^2}{4\pi c} \frac{[\hat{n} \times (\hat{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]^2}{(1 - \hat{n} \cdot \boldsymbol{\beta})^6} \end{aligned}$$

This is in the frame of an observer far away. To get the radiation in a frame that is close to the particle (but not the particle itself, it's off by a factor of γ), we need

$$\begin{aligned} \frac{dU}{dt' d\Omega} &= \frac{dt}{dt'} \frac{dU}{dt d\Omega} \\ &= \frac{q^2}{4\pi c} \frac{[\hat{n} \times (\hat{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]^2}{(1 - \hat{n} \cdot \boldsymbol{\beta})^5} \end{aligned}$$

Which is what is traditionally denoted the power, P . Note that U here is the integral of the energy density U_{EM} , the total energy.

Now let us consider the case where the velocity is very small, $\beta \ll 1$. In this case, we can drop β , but not $\dot{\boldsymbol{\beta}}$, so we have that

$$\frac{dU}{dt d\Omega} = \frac{q^2}{4\pi c^3} |\hat{n} \times \hat{n} \times \mathbf{a}|^2$$

Where $\mathbf{a} = \frac{d\mathbf{v}}{dt} = c \frac{d\boldsymbol{\beta}}{dt}$. Note that $\frac{dU}{dt' d\Omega} = \frac{dU}{dt d\Omega}$, since we are in the non-relativistic regime.

Suppose we are accelerating in the z direction, and our \hat{n} makes some angle θ from the z axis. In this case, we have that

$$\frac{dU}{dt d\Omega} = \frac{q^2}{4\pi c^3} a^2 \sin^2 \theta$$

Doing out a solid angle integral:

$$\begin{aligned} \frac{dU}{dt} &= \frac{q^2 a^2}{4\pi c^3} \iint \sin^2 \theta d(\cos \theta) d\phi \\ &= \frac{q^2 a^2}{4\pi c^3} 2\pi \left(\frac{4}{3} \right) \\ &= \frac{2}{3} \frac{q^2 a^2}{c^3} \end{aligned}$$

This is the non-relativistic total power formula. If we note that we have something that looks like a dipole, $qa = q|\dot{\mathbf{r}}| = \ddot{\mathbf{d}}$, we have that

$$\frac{dU}{dt} = \frac{2}{3} \frac{|\ddot{\mathbf{d}}|^2}{c^3}$$

Which is known as the Larmor formula.

In the relativistic case, the Doppler factor makes the radiation sensitive to the relative direction of the velocity and the acceleration. Let us first consider the case where \mathbf{a} is along β . This is known as Bremsstrahlung or braking radiation. In this case, we have that

$$\frac{dU}{dt' d\Omega} = \frac{q^2}{4\pi c^3} \frac{\phi^2 \sin^2 \theta}{(1 - \beta \cos \theta)^5}$$

For low velocities, the radiation is perpendicular to the direction of motion, but as the particle gets faster and faster, we see two lobes start to point in the direction of the motion. We can Taylor expand the Doppler factor:

$$(1 - \beta \cos \theta) \approx \frac{1 + \gamma^2 \theta^2}{2\gamma^2}$$

Which gives us the asymptotic expression:

$$\frac{dU}{dt' d\Omega} = \frac{8q^2}{\pi c^3} \frac{a^2 \sin^2 \theta}{(1 + \gamma^2 \theta^2)^5} \gamma^{10}$$

Which we can relate to a constant C :

$$\begin{aligned} \frac{dU}{dt' d\Omega} &= \frac{8q^2}{\pi c^3} \frac{a^2 \sin^2 \theta}{(1 + \gamma^2 \theta^2)^5} \gamma^{10} \\ &\approx C \frac{\theta^2}{(1 + \theta^2 \gamma^2)^5} \end{aligned}$$

We see that the width between the two lobes is comparable to $\frac{1}{\gamma}$, and the locations of the beams are at $\theta_{\max} = \pm \frac{1}{2\gamma}$.

Consider the frame of the moving charge, where it sees itself to be at rest. In this case, the beam is perpendicular, $\theta = \frac{\pi}{2}$. We can Lorentz transform into the frame of an observer. In the frame of the particle, we have a wave given by (\mathbf{k}', ω') , and in the lab frame we have (\mathbf{k}, ω) .

In the case where the acceleration \mathbf{a} is parallel to the velocity β , we see that for low velocity, the Doppler factor in the expression for the power is less important, and the two lobes of radiation are perpendicular to the direction of motion. If we start to approach $\beta \sim 1$, our Doppler factor becomes more important, and the lobes tilt to point in the direction of the velocity and acceleration, and the emission is concentrated within the region subtended by the angle $\theta_{\max} \sim \frac{1}{2\gamma}$.

We can also think having a lab frame, with wavevector \mathbf{k} and frequency ω , and then the frame of a moving charge (\mathbf{k}', ω') . In the moving charge frame, the interesting behavior is when we are perpendicular to the motion of the charged particle. We can think the Lorentz boost transformation between these two frames.

The motion perpendicular to the velocity does not change:

$$\mathbf{k}_{\perp} = \mathbf{k}'_{\perp}$$

However, the parallel components are transformed:

$$k_{\parallel} = \gamma \left(k'_{\parallel} + \beta \frac{\omega'}{c} \right)$$

We are interested in the angle of the photon with respect to the direction of the velocity of the charged particle in the lab frame, which we denote θ . This is defined in the lab frame:

$$\begin{aligned} \cot \theta &= \frac{k_{\parallel}}{k_{\perp}} \\ &= \frac{\gamma \left(k'_{\parallel} + \beta \frac{\omega'}{c} \right)}{k'_{\perp}} \end{aligned}$$

Now noting that $k'_{\parallel} = k' \cos \theta'$, and $k'_{\perp} = k' \sin \theta'$, and $\omega' = ck'$, we have that

$$\cot \theta = \gamma \left[\cot \theta' + \frac{\beta}{\sin \theta'} \right]$$

This is what is known as relativistic aberration, and describes the change in the angle of the photon propagation based on the frame. In particular, we are interested in the case where $\theta' \sim \frac{\pi}{2}$, so the $\cot \theta'$ is very small, so we have that $\cot \theta \sim \gamma\beta$, which inverted gives us:

$$\begin{aligned} \tan \theta &\approx \frac{1}{\beta\gamma} \\ &\approx \frac{1}{\gamma} \end{aligned}$$

which is the same result we derived previously.

Now let us consider the case where the particle was moving along the z axis, and then experienced acceleration in the x direction. In this case, the acceleration is perpendicular to the velocity. A simple example of this is an electron in a magnetic field.

We can use the general formula for the angular distribution of the power:

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\mathbf{n} \times (\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}|}{(1 - \mathbf{n} \cdot \boldsymbol{\beta})^5}$$

We can now apply this to the case where the acceleration is perpendicular to the velocity. We can decompose the vector \mathbf{n} in spherical:

$$\mathbf{n} = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix}$$

We also have $\boldsymbol{\beta}$ and \mathbf{a} :

$$\boldsymbol{\beta} = \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix} \quad \mathbf{a} = c\dot{\boldsymbol{\beta}} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$$

Now we can compute the angular power, where we denote the double vector product as \mathbf{A} . If we do the product out:

$$|\mathbf{A}|^2 = (1 - \beta \cos \theta)^2 + \sin^2 \theta \cos^2 \phi (\beta^2 - 1)$$

Inserting this into the angular power expression:

$$\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi^2 c^3} \frac{1}{(1 - \beta \cos \theta)^3} \left[1 - \frac{\sin^2 \theta \cos^2 \phi}{\gamma^2 (1 - \beta \cos \theta)^2} \right]$$

Note that at $\theta = 0$, along the direction of the motion, we have maximized power being radiated, and again, most of the power is radiated in a $\theta \sim \frac{1}{\gamma}$ cone, with respect to the particle motion. This means that if the observer is stationary, they will see pulses of radiation, when the beam cone is directed towards the observer. If $\gamma \gg 1$, this is known as synchrotron radiation, and if $\gamma \sim 1$, it is known as cyclotron radiation.

If we integrate the power over the sphere, we get the total power output:

$$P = \int_0^\pi \int_0^{2\pi} \frac{dP}{d\Omega} \sin \theta d\theta d\phi$$

This is difficult to compute, so instead, we can transform into a frame where the motion is subrelativistic, in which case the difference between the acceleration being in the direction of motion or across the direction of motion becomes negligible. We can then boost back to the original, relativistic frame, and it turns out that the power is invariant across frames.

We again have two frames. In the frame with wavevector k' , the particle is at rest, and in the frame with wavevector k , the particle is moving relativistically. If we want the power in the relativistic frame, and then we transform the time and energy into the non-relativistic frame:

$$\begin{aligned} P &= \frac{\Delta E}{\Delta t} \\ &= \frac{\gamma (\Delta E' + \mathbf{v} \cdot d\mathbf{p}')}{\gamma (dt' + \mathbf{v} \cdot \frac{d\mathbf{r}}{c})} \end{aligned}$$

Where we have defined the 4-vector that transforms¹⁴ for the energy as (E, \mathbf{p}) . We also note that the particle is at rest in the primed frame, so $dr' = 0$:

$$P = \frac{\gamma(dE' + \mathbf{v} \cdot d\mathbf{p}')}{\gamma dt'}$$

We can also note that the particle is not losing momentum, the power radiation is symmetric, we are only losing energy. In that case, $d\mathbf{p}' = 0$, so we have that

$$\begin{aligned} P &= \frac{\gamma dE'}{\gamma dt'} \\ &= P' \end{aligned}$$

We see that the power is invariant under the Lorentz frame change. Note that this is only true if we are considering the power over the entire sphere.

Thus we can look at the non-relativistic power formula:

$$P = \frac{2}{3c^3} |\ddot{\mathbf{d}}|^2$$

Now we only need to find the acceleration in the non-relativistic frame.

In the lab frame, we have a Lorentz force generating acceleration:

$$e \left(\frac{\mathbf{v}}{c} \times \mathbf{B} \right)$$

But in the shifted frame, we have

$$e(\mathbf{E}' + \boldsymbol{\beta} \times \mathbf{B}) = e\mathbf{E}'$$

Now applying Newton's Second Law, since we are in a non-relativistic frame:

$$m\ddot{\mathbf{r}} = e\mathbf{E}'$$

From this, we can compute our dipole moment:

$$\begin{aligned} |\ddot{\mathbf{d}}|^2 &= e^2 |\ddot{\mathbf{r}}|^2 \\ &= e^2 \frac{e^2 |\mathbf{E}'|^2}{m^2} \end{aligned}$$

Now using the Lorentz transformation of the electric field to find \mathbf{E}' :

$$\begin{aligned} \mathbf{E}' &= \gamma \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \\ &= \gamma(\boldsymbol{\beta} \times \mathbf{B}) \end{aligned}$$

We can put this all together to get the power, which is the same in both frames:

$$\mathbf{P} = P'$$

¹⁴We will be using the $(+ - - -)$ metric.

$$= \frac{2}{3c^3} \frac{e^4}{m^2} \gamma^2 |\boldsymbol{\beta} \times \mathbf{B}|^2$$

This is the total power radiated by a charge accelerating perpendicular to the direction of motion.

Note that formally, we did not subtract out the momentum and energy generated by the radiation itself. This is known as the radiation reaction force. If we increase the force of the field, we can generate a radiation reaction force that is comparable to the force of the field itself. This occurs when the field strength reaches some critical value, B_{cr} :

$$B_{\text{cr}} \sim \frac{e}{r_{\text{cl}}^2}$$

Where $r_{\text{cl}} \sim \frac{e^2}{m_e c^2} \sim 10^{-13}$ cm. This is where classical electrodynamics begins to break down. Another way of thinking about this is that quantum effects begin to take effect when the energy of the emitted photon is comparable to the rest mass:

$$\hbar\omega_B \sim m_e c^2$$

This is where we start to see the effects of quantum electrodynamics¹⁵.

3.4.1 Fourier Harmonics

Now let us look at the Fourier harmonics of our Liénard-Wiechert potentials, which will allow us to compute the spectrum of our radiation. What we will find is that we need the Fourier transform of our current.

Recall that our potential is given by

$$\phi = \int \frac{1}{|\mathbf{r} - \mathbf{r}'|} \rho\left(\mathbf{r}', t - \frac{R}{c}\right) d^3\mathbf{r}'$$

Where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, and the observer is at \mathbf{r} . In Fourier space:

$$\phi_\omega e^{-i\omega t} = \int \frac{1}{R} \rho_\omega e^{-i\omega(t - \frac{R}{c})} d^3\mathbf{r}'$$

From this, we can find that:

$$\phi_\omega = \int \frac{1}{R} \rho_\omega e^{ikR} d^3\mathbf{r}'$$

Where we have used the fact that $\omega/c = k$. Inserting the Fourier transform of the charge density:

$$\rho_\omega = \int_{-\infty}^{\infty} \rho e^{i\omega t} dt$$

We find that we have a double integral:

$$\phi_\omega = \int_{-\infty}^{\infty} dt \int \frac{\rho}{R} e^{i(\omega t + kR)} d^3\mathbf{r}'$$

This is the definition of the Fourier harmonic for a retarded potential.

¹⁵In this case, the Schwinger field.

Looking at the case of a single charge, $\rho = q\delta^3(\mathbf{r} - \mathbf{r}_0(t))$, so we have:

$$\phi_\omega = \int_{-\infty}^{\infty} dt \frac{q}{R(t)} e^{i\omega(t + \frac{R(t)}{c})}$$

Where $R = |\mathbf{r} - \mathbf{r}_0(t)|$. Similarly, we can look at the Fourier transform of the vector potential:

$$\mathbf{A}_\omega = \frac{q}{c} \int_{-\infty}^{\infty} \frac{\mathbf{v}(t)}{R(t)} e^{i(t + \frac{R(t)}{c})} dt$$

Now (returning to the general charge distribution case) let us consider the large distance expansion of these formulas. Let \mathbf{r}' be the distance from the origin to a piece of charge, and \mathbf{R}_0 be the distance from the origin to the observer. We define $\mathbf{R} = \mathbf{R}_0 - \mathbf{r}'$, and we state that we are in the limit where R is large:

$$\begin{aligned} R &= \sqrt{(\mathbf{R}_0 - \mathbf{r}') \cdot (\mathbf{R}_0 - \mathbf{r}')} \\ &= \sqrt{R_0^2 - 2(\mathbf{R}_0 \cdot \mathbf{r}')} \\ &\approx R_0 \left[1 - \frac{\mathbf{n} \cdot \mathbf{r}'}{R_0} \right] \\ &= R_0 - \mathbf{n} \cdot \mathbf{r}' \end{aligned}$$

In this limit our potential becomes:

$$\phi = \frac{1}{R_0} \int \rho \left(t - \frac{R_0}{c} + \frac{\mathbf{r}' \cdot \mathbf{n}}{c} \right) d^3\mathbf{r}'$$

We know that

$$\mathbf{E} = \mathbf{B} \times \mathbf{n}$$

We can find \mathbf{B} from the curl of the vector potential, $\mathbf{B} = \nabla \times \mathbf{A}$. We know that we should be able to have everything in terms of a delayed time variable, $\xi = t - \frac{x}{c}$. Thus, we replace the curl with derivatives with respect to ξ . We can write out the magnetic field in components:

$$\begin{aligned} B_i &= e_{ijk} \partial_j A_k \\ &= e_{ijk} \frac{\partial A_k}{\partial \xi} \frac{\partial \xi}{\partial x_j} \\ &= E_{ijk} \frac{\partial A_k}{\partial \xi} \left(-\frac{n_j}{c} \right) \\ &= -\frac{1}{c} (\mathbf{n} \times \mathbf{A})_i \end{aligned}$$

We can then construct the vector potential:

$$\mathbf{A} = \frac{1}{cR_0} \int \mathbf{J} \left(t - \frac{R_0}{c} + \frac{\mathbf{r}' \cdot \mathbf{n}}{c} \right) d^3\mathbf{r}'$$

We can also look at the Fourier harmonic of this:

$$\mathbf{A}_\omega = \frac{e^{ikR_0}}{cR_0} \int \mathbf{J}_\omega e^{-ik\mathbf{r}'} d^3\mathbf{r}'$$

We see that we have the leading order $\frac{1}{R_0}$ term in the long distance expansion.

Now if we drop the $e^{-i\mathbf{k}\mathbf{r}'}$, we see that things become easier to compute. This can be dropped in the case where $\mathbf{k}\mathbf{r}'$ is small. If we let T be the characteristic time scale at which the currents are changing, we can say that this is the same time scale at which the radiation will occur. If a is the size of the system, then we can say that \mathbf{r}' is of order a , since outside a there are no currents. The $|\mathbf{k}\mathbf{r}'|$ is then comparable to $\frac{2\pi}{\lambda}a$. We can then relate λ to T , $\lambda = cT$, so we have that

$$|\mathbf{k}\mathbf{r}'| \sim \frac{2\pi a}{cT}$$

We can thus ignore this factor if this is much less than 1, so the size of the system a is much less than the characteristic wavelength λ , or more transparently:

$$\frac{2\pi a}{cT} \ll 1$$

Or in other notation, denoting $\frac{a}{T}$ as the characteristic velocity, this must be much less than the speed of light:

$$\frac{a}{T} \ll c$$

In this case, we can obtain nicer formulae, which are known as dipole radiation. Our vector potential is given by:

$$\mathbf{A} = \frac{1}{cR_0} \int \mathbf{J} \left(t - \frac{R_0}{c} \right) d^3\mathbf{r}'$$

The current is defined as the sum over local charge densities times local velocities:

$$\mathbf{J}(t) = \sum_{\text{charges}} \rho(\mathbf{r}') \mathbf{v}$$

From which we have

$$\begin{aligned} \mathbf{A} &= \frac{\sum e\mathbf{v}}{cR_0} \Big|_{t-\frac{R_0}{c}} \\ &= \frac{\dot{\mathbf{d}}}{cR_0} \end{aligned}$$

Now that we have the vector potential, we can use our formula for the magnetic field:

$$\mathbf{B} = \frac{1}{c^2 R_0} \ddot{\mathbf{d}} \times \mathbf{n}$$

We can find the electric field:

$$\mathbf{E} = \mathbf{B} \times \mathbf{n}$$

And we can compute the angular power distribution:

$$\frac{dP}{d\Omega} = \frac{|\ddot{\mathbf{d}} \times \mathbf{n}|^2}{4\pi c^3}$$

Which in this case is true for a distribution of charges in the far field limit, not just a single charge. This is similar to the multipole expansion for the static fields, just for the case of moving charge distributions.

Missed a lecture here.

4 Particle Motion

We have thus far considered two different classes of problems, those where we have the ρ and \mathbf{J} distributions, and we used them to solve electrostatic and magnetostatic problems, using these two equations:

$$\begin{aligned}\nabla \cdot \mathbf{E} &= 4\pi\rho \\ \nabla \times \mathbf{B} &= \frac{4\pi}{c}\mathbf{J}\end{aligned}$$

The second class of problems is those where we have time dependent ρ and \mathbf{J} , and we solve the time-dependent versions of our two equations:

$$\begin{aligned}\nabla \times \mathbf{B} &= \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \\ \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\end{aligned}$$

We denote the first class of problems to be “statics”, and the second to be “radiation”.

The third class that we can consider is “particle motion”, when we know \mathbf{E} and \mathbf{B} as functions of position and time, and we want to find things like the velocity of a particle in these fields.

Let us begin with the Lorentz equation:

$$\frac{d\mathbf{p}}{dt} = e \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right)$$

If we assume that we know \mathbf{E} and \mathbf{B} , we want to find how \mathbf{v} and \mathbf{x} change. We will also consider the relativistic case, in which case we need to use the relativistic relations:

$$\begin{aligned}\mathbf{p}_{\text{rel}} &= m\mathbf{v}\gamma \\ \mathbf{v} &= \frac{\mathbf{p}c^2}{E_{\text{rel}}} \\ E_{\text{rel}} &= mc^2\gamma \\ \frac{\partial E_{\text{rel}}}{\partial t} &= \mathbf{v} \frac{d\mathbf{p}_{\text{rel}}}{dt}\end{aligned}$$

Where γ is the Lorentz factor :

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Using these, we can write down an expression for the acceleration:

$$\begin{aligned}\frac{d}{dt}(m\mathbf{v}\gamma) &= m\gamma \frac{d\mathbf{v}}{dt} + m\mathbf{v} \frac{d\gamma}{dt} \\ &= m\gamma \frac{d\mathbf{v}}{dt} + \frac{\mathbf{v}}{c^2} \frac{dE}{dt} \\ &= m\gamma \frac{d\mathbf{v}}{dt} + \frac{\mathbf{v}}{c^2} (q\mathbf{v} \cdot \mathbf{E})\end{aligned}$$

Where we note that only the electric field does work. Now equating this to the Lorentz force, by Newton's Second Law:

$$\begin{aligned}\frac{d}{dt}(m\mathbf{v}\gamma) &= q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \\ m\frac{d\mathbf{v}}{dt} &= \frac{q}{\gamma}\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} - \frac{\mathbf{v}}{c^2}(\mathbf{v} \cdot \mathbf{E})\right)\end{aligned}$$

We can see that the second term only comes into play if the particle is relativistic. In the ultra-relativistic limit, the electric field terms cancel, and accelerations are generated mostly by magnetic fields.

4.1 Constant Electric Field

Let us consider the case where the electric field is aligned with the z axis, and the electric field is constant:

$$\dot{p}_x = \dot{p}_y = 0 \quad \dot{p}_z = eE_z$$

Suppose we have some initial momentum, in the x direction:

$$p_x = p_0 \quad p_y = 0$$

Integrating our expression for \dot{p}_z :

$$p_z = eE_z t$$

From this, we can find the energy¹⁶:

$$\mathcal{E} = \sqrt{\mathcal{E}_0^2 + (eE_z t)^2 c^2}$$

Where \mathcal{E}_0 encompasses the rest mass and the initial momentum:

$$\mathcal{E}_0^2 = m^2 c^4 + p_0^2 c^2$$

We can see that the scaling of \mathcal{E} in this relativistic case is linear in t , rather than scaling as t^2 as we would expect in the non-relativistic case, $\mathcal{E} = \frac{p^2}{2m} \sim t^2$.

Now we want to find the velocities:

$$\begin{aligned}\frac{dx}{dt} &= v_x \\ &= \frac{p_x c^2}{\mathcal{E}} \\ &= \frac{p_0 c^2}{\sqrt{\mathcal{E}_0^2 + (eE_z t)^2 c^2}} \\ \frac{dz}{dt} &= v_z\end{aligned}$$

¹⁶Now using \mathcal{E} to distinguish the energy from the electric field

$$\begin{aligned}
&= \frac{p_z c^2}{\mathcal{E}} \\
&= \frac{e E_z t c^2}{\sqrt{\mathcal{E}_0^2 + (e E_z t)^2} c^2}
\end{aligned}$$

We can now integrate these two get the trajectories:

$$z = \frac{\mathcal{E}}{qE} \sqrt{\mathcal{E}_0^2 + (qEct)^2}$$

The x integral is slightly more difficult, we have an integral of the form:

$$\int \frac{dt}{\sqrt{1+t^2}}$$

Which gives a hyperbolic arcsine:

$$x = \frac{p_0 c}{qE} \operatorname{arcsinh} \left(\frac{qEct}{\mathcal{E}_0} \right)$$

Using this, we can write z in terms of x to get a parameterized trajectory:

$$z = \frac{\mathcal{E}_0}{qE} \cosh \left(\frac{qEx}{p_0 c} \right)$$

We can see that if the particle is accelerated to small energies, then $qEx \ll mc^2$, the change in energy is less than the rest mass. If we then Taylor expand the hyperbolic cosine, we would get quadratic terms, which give us the classical parabolic trajectory.

4.2 Constant Magnetic Field

In the case of the constant magnetic field, we have that

$$\frac{d\mathbf{v}}{dt} = \frac{q}{\gamma m} \frac{\mathbf{v}}{c} \times \mathbf{B}$$

Where γ is constant, since the magnetic field does no work. Thus we have that

$$\frac{d\mathbf{v}}{dt} = \mathbf{v} \times \boldsymbol{\omega}_L$$

Where $\boldsymbol{\omega}_L$ is the Larmor frequency:

$$\boldsymbol{\omega}_L = \frac{e\mathbf{B}}{mc\gamma}$$

In this case, we see that the trajectory we get is a spiral that moves along the direction of \mathbf{B} , and in the plane orthogonal to that direction, we have harmonic oscillator motion, with frequency $\boldsymbol{\omega}_L$. The perpendicular momentum is given by:

$$p_{\perp} = \frac{\mathcal{E} v_{\perp}}{c^2}$$

$$\begin{aligned}
&= \frac{\mathcal{E}\omega r_c}{c^2} \\
&= \frac{eB}{c} r_c
\end{aligned}$$

Where r_c is the Larmor radius. This gives a relation between the momentum and the radius of the spiral. Note that this assumes that the particle is not affecting the fields, they are fixed. If we did allow for this, the spiral would become tighter and tighter over time, instead of having a fixed radius.

4.3 Constant Electric and Magnetic Fields

There are several different setups for having constant electric and magnetic fields. The first is if $\mathbf{E} \perp \mathbf{B}$, and the second is when $\mathbf{E} \cdot \mathbf{B} \neq 0$, there is some overlap in the directions. The reason that these two are qualitatively different is that $B^2 - E^2$ is a relativistic invariant. In the case where they are perpendicular, we can move to a frame where one of the fields is zero. However, in the case where $\mathbf{E} \cdot \mathbf{B} \neq 0$, we can note that this dot product is a relativistic invariant, and we can move into a frame where the fields are perpendicular. Thus we have 4 cases:

1. $\mathbf{E} \cdot \mathbf{B} = 0$, the fields are parallel.
2. $\mathbf{E} \perp \mathbf{B}, E < B$, the fields are perpendicular, and we can remove the electric field.
3. $\mathbf{E} \perp \mathbf{B}, E > B$, the fields are perpendicular, and we can remove the magnetic field.
4. $\mathbf{E} \perp \mathbf{B}, E = B$, the fields are perpendicular, but we cannot remove either fields.

The case where $\mathbf{E} = \mathbf{B}$ and $\mathbf{E} \perp \mathbf{B}$ is relevant for particles inside electromagnetic waves. The case where $E > B$ is uncommon, since electric fields are generally smaller than the magnetic fields.

Let us consider the case where the fields are perpendicular and $E < B$. We will do the non-relativistic case. Consider a magnetic field along the z axis, and the electric field along the y axis. We can write down the non-relativistic equation of motion:

$$\begin{aligned}
m\dot{v}_x &= \frac{qB}{c} v_y \\
m\dot{v}_y &= qE - \frac{qB}{c} v_x \\
m\dot{v}_z &= 0
\end{aligned}$$

Thus v_z is a constant, so for simplicity we will consider the case where $v_z = 0$. We then have coupled equations for v_x and v_y , which reduce to the harmonic oscillator equation:

$$\begin{aligned}
m\ddot{v}_y &= -\frac{qB}{c} \dot{v}_x \\
&= -\left(\frac{qB}{mc}\right)^2 v_y
\end{aligned}$$

We see that we have the harmonic oscillator solution in the y direction, with some coefficient u , determined by initial conditions:

$$v_y = -u \sin(\omega t)$$

Where $\omega = \frac{qB}{mc}$. Inserting this solution back in and finding v_x :

$$v_x = v_d + u \cos(\omega t)$$

Where v_d is the drift velocity, $v_d = c \frac{E}{B}$. Integrating these for the positions:

$$x = \frac{u}{\omega} \sin(\omega t) + v_d t$$

$$y = \frac{u}{\omega} \cos(\omega t)$$

We can take a look at the total Lorentz force, and try to find the velocity that causes this to be zero:

$$\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} = 0$$

Applying a $\times \mathbf{B}$ to both sides:

$$\mathbf{E} \times \mathbf{B} = -\frac{\mathbf{v}}{c} \times \mathbf{B} \times \mathbf{B}$$

Since \mathbf{B} , \mathbf{E} , and \mathbf{v} are perpendicular to each other, we have that

$$\mathbf{v} = c \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

If the particle is moving with this velocity, it doesn't feel any force. This can be shown to be the frame in which the electric field vanishes, and the drift term drops out.

Let us consider our solution in different limits. If $u = 0$, the particle will move in a straight line, with constant y . If there is no drift, it will move in a circle in the plane, but if $u > v_d$, the particle moves in a drifting circle.

We have now see that the case where $\mathbf{E} < \mathbf{B}$ gives rise to Larmor motion with a drift.

4.4 Trajectories in Adiabatic Magnetic Fields

What happens to particles when the magnetic field is varying very slowly? By very slowly, we mean that the change over a Larmor period is much less than the field itself:

$$\dot{B}T \ll B$$

We can also consider slow variations in space:

$$r_c \nabla B \ll B$$

In the case where the field varies slowly in space, we have “guiding center motion”, where the particle undergoes Larmor motion with a drift, but the trajectory of the drift changes. We will find that the motion is given by the Larmor motion $\boldsymbol{\xi}(t)$, which varies quickly over time, and the motion of the center, $\mathbf{R}_g(t)$:

$$\mathbf{r} = \boldsymbol{\xi}(t) + \mathbf{R}_g(t)$$

Before we do this, let us consider the classical case where the magnetic field changes over time, the Betatron. We have a particle with a circular orbit, and the magnetic field is changing in time. In

this case, the radius of the orbit will change, but we can tune the B field to keep the radius the same. The changing magnetic field will also induce an electric field:

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$$

Which can be written as

$$\begin{aligned} \oint \nabla \times \mathbf{E} d\mathbf{S} &= \oint \mathbf{E} d\mathbf{l} \\ &= -\frac{1}{c} \frac{\partial}{\partial t} \Phi_B \\ E_\phi \cdot 2\pi r &= -\frac{1}{c} \pi r^2 \dot{B}_{\text{avg}} \end{aligned}$$

Where Φ_B is the magnetic flux through the circular orbit, and $B_{\text{avg}} = \frac{\Phi_B}{\pi r^2}$. The field that the particle experiences as it goes along the orbit is \mathbf{B}_{orb} , and we can tune B_{avg} and \mathbf{B}_{orb} such that the radius of the orbit remains the same, and the particle will continuously accelerate. We can write down the induced electric field:

$$E_\phi = -\frac{1}{2c} r \dot{B}_{\text{avg}}$$

The change in the momentum is given by just the electric force:

$$\begin{aligned} \frac{dE}{dt} &= qE_\phi \\ &= -\frac{qr}{2c} \dot{B}_{\text{avg}} \end{aligned}$$

We have circular motion, so the change in momentum in the radial direction is:

$$\begin{aligned} \hat{r} \frac{d\mathbf{p}}{dt} &= -\omega \mathbf{p} \\ &= \frac{qv}{c} \mathbf{B}_{\text{orb}} \end{aligned}$$

From this, we have that

$$\dot{p} = -\frac{rq}{c} \mathbf{B}_{\text{orb}}$$

Taking a time derivative of this, and then setting the two relations equal to each other, we have that:

$$\dot{B}_{\text{orb}} = \frac{1}{2} \dot{B}_{\text{avg}}$$

Thus we can maintain that the radius is constant, and the particle continues to accelerate over time, creating an electron accelerator (although not a very good one).

Now let us return to the motion of a charged particle with guiding center motion. As the particle travels, it feels a drift velocity that generates a centrifugal force towards the center of curvature, which is known as “curvature drift”, a particle moving along a curved field line in a plane will feel a

drift out of the plane to keep it on the field line. The particle also experiences what is known as “grad-B drift”, along the direction of the magnetic field.

The motion is governed by two variables, the “fast” variable $\zeta(t)$, and the “slow” variable, $R_{gc}(t)$. This is true if r_c , the gyromagnetic radius is much less than the scale at which the magnetic field changes in space. We are essentially looking at the case where we can neglect the gyromagnetic orbit, and focus on just the overall guiding center motion. We will consider two separate drifts, one that occurs when the particle moves transverse to the field line, and then another effect that occurs even without transverse motion, caused by the fact that the field lines are curving. Thus we have motion caused by v_\perp and motion caused by v_\parallel , and we will consider these two separately, before putting them together.

We need to consider the effective force that the particle feels as the magnetic field changes, which modifies the orbit to generate the guiding center motion. To do this, we need to consider how the magnetic field changes as we move through space. Suppose that at some point O , the magnetic field is exactly along z , $\mathbf{B}(O) \parallel \hat{z}$. We can calculate how the field strength changes, by looking at the magnitude at some nearby point:

$$\begin{aligned} B &= \sqrt{B_z^2 + B_x^2} \\ &\approx B_z \sqrt{1 + \frac{B_x^2}{B_z^2}} \\ &= B_z + \frac{B_x^2}{2B_z} + \dots \end{aligned}$$

Where we have, by noting that at O , the field is all along the z direction, assumed that $B_x \ll B_z$, since the field varies slowly. This leads to a gradient of the field strength:

$$\nabla B = \nabla B_z + \frac{B_x \nabla B_x}{B_z} + \dots$$

Now we note that $\frac{\partial B_z}{\partial x} \sim \frac{\partial B_x}{\partial z}$, since $(\nabla \times \mathbf{B})_y = 0$. Now relating the gradients to the curvature of the field:

$$\begin{aligned} \nabla_\perp B &= \frac{\partial B_z}{\partial x} \hat{x} \\ &= \frac{\partial B_x}{\partial z} \hat{x} \end{aligned}$$

Now estimating $\frac{\partial B_x}{\partial z}$, if we have some point that is slightly away from O , where we have some small $B_x(P)$:

$$\frac{\partial B_x}{\partial z} \approx \frac{B_x(P) - \overbrace{B_x(O)}^0}{\Delta z}$$

If R is the local radius of curvature of the field line at P , and $\Delta\theta$ is the change in angle between O and P :

$$\frac{\partial B_x}{\partial z} \approx \frac{B_x(P)}{\Delta z}$$

$$\begin{aligned}
&= \frac{B(P) \Delta\theta}{\Delta z} \\
&= \frac{B(P) \Delta\theta}{R_c \Delta\theta} \\
&= \frac{B(P)}{R_c}
\end{aligned}$$

If we use this, we find the relation:

$$\nabla_{\perp} B = -\frac{B(P)}{R_c^2} \mathbf{R}_c$$

Where the minus sign comes from having the direction of the radius of curvature vector point from the center of curvature to the point along the field line.

As the particle moves in its circular orbit in the $x - y$ plane, the centripetal force does not match up at all points along the orbit because the changing field leads to an effective force. If the force is in the x direction :

$$\begin{aligned}
0 &= \mathbf{f}_x + \frac{e}{c} \mathbf{v} \times \mathbf{B} \\
\mathbf{v}_{\text{dr}} &= \frac{c}{e} \frac{\mathbf{f}_x \times \mathbf{B}}{B^2}
\end{aligned}$$

We can average the force along the orbit to get the effective force:

$$\mathbf{f}_x = \int_0^{2\pi} \mathbf{F}_x d\theta$$

To obtain \mathbf{f}_x , we can look at the instantaneous Lorentz force:

$$\mathbf{F} = \frac{q}{c} \mathbf{v} \times \mathbf{B}$$

Where $\mathbf{B} = B_z \hat{z}$, and the velocity is perpendicular and directed tangentially:

$$\mathbf{v} = \begin{pmatrix} v_{\perp} \sin \theta \\ -v_{\perp} \cos \theta \\ 0 \end{pmatrix}$$

Now computing the vector product to get the Lorentz force explicitly:

$$\begin{aligned}
\mathbf{F} &= \frac{q}{c} \mathbf{v} \times \mathbf{B} \\
&= -\frac{q}{c} v_{\perp} B_z (\cos \theta \hat{x} + \sin \theta \hat{y})
\end{aligned}$$

Now we have to insert the varying magnetic field, and then average along the trajectory. Inserting the Taylor expansion:

$$B_z(\theta) \approx B_z + \frac{v_{\perp}}{\omega_c} \frac{\partial B_z}{\partial x} \cos \theta$$

Where $\frac{v_{\perp}}{\omega_c}$ is the Larmor radius¹⁷. Now averaging over the orbit:

$$\begin{aligned} \mathbf{f}_x &= \int_0^{2\pi} \mathbf{F}_x d\theta \\ &= -\frac{1}{2} \frac{q}{c} \frac{v_{\perp}^2}{\omega_c} \left(\frac{\partial B_z}{\partial x} \right) \end{aligned}$$

Now inserting this effective force into the expression for the drift velocity:

$$\mathbf{v}_{\text{dr}} = \frac{\mathcal{E} v_{\perp}^2}{2qcB^2} (\hat{B} \times \nabla_{\perp} B)$$

Where \mathcal{E} is the energy of the particle, and we have used the fact that $\omega_c = \frac{ecB}{\mathcal{E}} = \frac{ecB}{mc^2\gamma}$. Note that the particle drifts in the direction perpendicular to the magnetic field and the force, which in this case is the y direction. Note that the drift disappears if we take $B \rightarrow \infty$.

Another way that we can derive this is to classify the trajectory into two parts:

$$\mathbf{R}(t) = \mathbf{R}_x(t) + \boldsymbol{\zeta}(t)$$

And we can Taylor expand the magnetic field:

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= \mathbf{B}(\mathbf{R}(t)) \\ &= \mathbf{B}(\mathbf{R}_x(t)) + (\boldsymbol{\zeta} \cdot \nabla) \mathbf{B}(\mathbf{R}_x(t)) \end{aligned}$$

We can then compute the average force:

$$\langle \mathbf{F} \rangle = \frac{e}{c} \langle \dot{\boldsymbol{\zeta}} \times (\boldsymbol{\zeta} \cdot \nabla) \mathbf{B}(\mathbf{R}_x) \rangle$$

Now writing down $\dot{\boldsymbol{\zeta}}$:

$$\dot{\boldsymbol{\zeta}} = \omega_c \boldsymbol{\zeta} \times \hat{B}$$

We also know the average of $\boldsymbol{\zeta}$ in two directions:

$$\langle \zeta_{\alpha} \zeta_{\beta} \rangle = \frac{1}{2} \zeta^2 \delta_{\alpha\beta}$$

Which we can see by noting that ζ_{α} and ζ_{β} will be sines and cosines, which when multiplied together, will average to zero unless we have a \cos^2 or \sin^2 , giving us the delta function. Now we can write down the average force explicitly, which we can do using Levi-Civita symbols:

$$\begin{aligned} \frac{c}{e} \langle \mathbf{F} \rangle_i &= \langle \varepsilon_{ijk} \dot{\zeta}_j (\zeta_l \partial_l) B_k \rangle \\ &= \langle \varepsilon_{ijk} \omega_c \varepsilon_{jpr} \zeta_p b_r \zeta_l \partial_l B_k \rangle \\ &= \frac{1}{2} \zeta^2 \omega \left(\underbrace{b_i \partial_k B_k - b_k \partial_i B_k}_{\nabla \cdot \mathbf{B} = 0} \right) \\ &= -\frac{1}{2} \zeta^2 \omega_c b_k \partial_k (B b_i) \end{aligned}$$

¹⁷Which is identical to the gyromagnetic radius.

Now using the identity:

$$(\mathbf{n} \nabla) \mathbf{n} = -\frac{\mathbf{n}}{R_c}$$

We find the same result as in our geometric derivation¹⁸.

Now let us consider the second type of drift. Suppose that the particle is attempting to move along the curved field line. There will be a Lorentz force towards the center of curvature. The Lorentz force must be able to turn the particle, and thus the parallel component of the velocity will not be able to cause this, we need a component of \mathbf{v} that provides a Lorentz force. To do this, the particle must drift out of the plane.

To compute the amplitude of this effect, we can write down the centrifugal force and set it equal to the Lorentz force:

$$\frac{e}{c} \mathbf{v}_{\text{dr}} \times \mathbf{B} = -\frac{mv_{\parallel}^2}{R_c} \hat{R}_c$$

From this, we find that

$$\mathbf{v}_{\text{dr}} = \frac{c\mathcal{E}}{qc^2 R_c B} v_{\parallel}^2 (\hat{R}_c \times \hat{B})$$

If we compute the ratio between these two velocities:

$$\frac{v_{\text{dr}}}{v_{\parallel}} \sim \frac{r_c(v_{\parallel})}{R_c}$$

Where $r_c(v_{\parallel})$ is:

$$r_c(v_{\parallel}) = \frac{m_e c^2}{e B_0} v_{\parallel}$$

We can now put our drift velocities together:

$$\mathbf{v}_{\text{dr}} = \frac{\mathcal{E}}{qcB^2} \left(v_{\parallel}^2 + \frac{1}{2} v_{\perp}^2 \right) (\hat{B} \times \nabla_{\perp} B)$$

Where we have used the fact that

$$\nabla_{\perp} B = -\frac{B}{R_c^2} \mathbf{R}_c$$

We have seen that as the particle tries to gyrate in the slowly changing magnetic field, it will have to drift with this velocity (for first order drift). Now we need to look at how v_{\parallel} and v_{\perp} as the particle moves. To determine this, we will compute an adiabatic invariant. First, we note that the energy is conserved, since the magnetic field does no work:

$$\mathcal{E} = \left[(p_{\parallel}^2 + p_{\perp}^2) c^2 + m^2 c^4 \right]^{1/2}$$

The other conserved quantity will be the adiabatic invariant. There are several different ways to think about this quantity.

¹⁸TODO: finish this up?

The simplest way to look at this is to consider the energy of a particle in a slowly changing magnetic field. In this case, the particle will experience a changing electric field, and it will gain energy:

$$\begin{aligned}\Delta W &= -q \oint \mathbf{E} \cdot d\mathbf{l} \\ &= -q \frac{1}{c} \frac{\partial}{\partial t} \Phi_B \\ &= \frac{q}{c} \pi r_c^2 \dot{B}\end{aligned}$$

Where Φ_B is the magnetic flux, which is $\Phi_B = \pi r_c^2 B$. We can compute the time derivative of the energy:

$$\begin{aligned}\frac{\partial \mathcal{E}}{\partial t} &= \Delta W \frac{\omega_c}{2\pi} \\ &= \frac{q}{c} \frac{\omega_c}{2\pi} \pi r_c^2 \dot{B} \\ &= \frac{q}{2c} r_c^2 \omega_c \dot{B} \\ &= \frac{c^2}{2\mathcal{E}} \frac{dp_{\perp}^2}{dt}\end{aligned}$$

Where we have used the relation between energy and momentum, and the fact that $T = \frac{2\pi}{\omega_c}$. Now writing all of this down as a single conservation law:

$$\begin{aligned}\frac{dp_{\perp}^2}{dt} &= \frac{2\mathcal{E}}{c^2} \frac{\partial \mathcal{E}}{\partial t} \\ &= \frac{q\mathcal{E}}{c^3} r_c^2 \omega_c \dot{B}\end{aligned}$$

Now inserting the relations between the cyclotron frequency and radius to the momentum and energy:

$$\begin{aligned}r_c &= \frac{p_{\perp}^2 c^2}{e^2 B^2} \\ \omega_c &= \frac{ecB}{\mathcal{E}}\end{aligned}$$

We are left with:

$$\frac{dp_{\perp}^2}{dt} = \frac{p_{\perp}^2}{B} \dot{B}$$

Which, when rewritten, gives us:

$$\begin{aligned}\frac{1}{B} \frac{d}{dt} (p_{\perp}^2) - \frac{p_{\perp}^2}{B^2} \dot{B} &= 0 \\ \frac{d}{dt} \left(\frac{p_{\perp}^2}{B} \right) &= 0\end{aligned}$$

Which gives us an invariant quantity, p_{\perp}^2/B . This is known as an adiabatic invariant, as B changes, so does p_{\perp}^2 , in a way such that this quantity is kept constant.

Now let us consider another method to determine this adiabatic invariant. In classical mechanics, if we have a closed orbit, and some generalized coordinates and momenta \mathbf{P} and \mathbf{Q} , then

$$\frac{1}{2\pi} \oint \mathbf{P} \cdot d\mathbf{Q} = I$$

will be the first adiabatic invariant. Applying this to our case, taking into account the magnetic vector potential and the tangential momentum:

$$\begin{aligned} I &= \frac{1}{2\pi} \oint \left(\mathbf{p}_t + \frac{e}{c} \mathbf{A} \right) \cdot d\mathbf{r} \\ &= \frac{1}{2\pi} (p_t \cdot 2\pi r_c) + \frac{e}{2\pi c} \oint \mathbf{A} \cdot d\mathbf{r} \\ &= p_t r_c - \frac{e}{2\pi c} \int \mathbf{B} \cdot d\mathbf{S} \\ &= p_t r_c - \frac{e}{2\pi c} B \pi r_c^2 \\ &\sim \frac{p_t^2}{B} \end{aligned}$$

We see that we have derived the same adiabatic invariant. Another way to do this would be to consider the ϕ coordinate, in which case P_ϕ would be the angular momentum.

We have two invariant quantities:

$$\begin{aligned} p_\parallel^2 + p_\perp^2 &= \text{const.} \\ \frac{p_\perp^2}{B} &= \text{const.} \end{aligned}$$

With these two, we can find the trajectory of the guiding center:

$$\frac{d\mathbf{R}_{\text{gc}}}{dt} = \mathbf{v}_{\text{dr}} + \mathbf{v}_\parallel$$

Where \mathbf{v}_{dr} and \mathbf{v}_\parallel are determined by our two invariant quantities.

Consider the case of a particle in the Earth's magnetic field, initially with some p_\perp and p_\parallel . As the particle moves towards the Earth, the field strength increases, since the distance is decreasing. Based on our conservation laws, we find that the perpendicular momentum will also increase, since it scales as $p_\perp \sim C\sqrt{B}$. Because of this, the parallel momentum will be decreasing. For some particles, which have sufficiently large p_\perp initially, this will lead to the particle reflecting, and then being trapped in an oscillatory motion between symmetric points in the orbit around the Earth. This is known as the van Allen radiation belt.

If the particle has initial angle to the field line α_0 , then the constant in the scaling is $C = \frac{p_0^2 \sin^2 \alpha_0}{B_0}$. We would then find that

$$p_\parallel = p_0 \left[1 - \frac{B \sin^2 \alpha_0}{B_0} \right]^{1/2}$$

If B varies between B_{min} and B_{max} , then if $\sin^2 \alpha_0 < \frac{B_{\text{min}}}{B_{\text{max}}}$, the particles will never be reflected, and they will “escape” from the system. This is known as the loss cone¹⁹.

¹⁹Since it selects a cone in velocity space of particles that escape.

5 Relativistic Electromagnetism

The basic principles of relativity are the two postulates:

1. The speed of light in a vacuum is constant, and is the speed limit of the universe.
2. The laws of physics look the same in any inertial reference frame.

From the first postulate, we can see that time is not absolute, if this was the case, velocities would satisfy the Galilean principle, which is that velocities add:

$$v = v_1 + v_2$$

If this is the case, we would be able to exceed the speed of light, and thus time is not absolute. Consider the following thought experiment. Suppose we have two reference frames, one which is at rest, and the second moves with some \mathbf{v} relative to the first frame. If we have a light emitter in the moving frame, we can define two events, each a distance L away from the emitter. In the moving frame, the light travelled the same distance, and arrived at the same time, and thus the two events are simultaneous. However, in the reference frame, the light moves at the same speed, but one of the points is closer to the emitter, and the other is further away. Thus the events are no longer simultaneous.

We denote a point in spacetime (t, \mathbf{x}) as an event, and trajectories in spacetime are denoted as worldlines.

If we have two events in spacetime, the conserved quantity relating these two in any reference frame is the invariant spacetime interval. Suppose we have two points (t_1, \mathbf{x}_1) and (t_2, \mathbf{x}_2) . If we connect these with a light signal travelling between them, the distance travelled is related to the time taken for the light to travel between them:

$$\begin{aligned} (\mathbf{x}_1 - \mathbf{x}_2)^2 &= c^2 (t_2 - t_1)^2 \\ (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 &= c^2 (t_2 - t_1)^2 \end{aligned}$$

This is true no matter what frame we are working in:

$$(\mathbf{x}'_1 - \mathbf{x}'_2)^2 = c^2 (t'_2 - t'_1)^2$$

In this case where we have connected them with a light source, this value is zero:

$$ds^2 = 0$$

In general, we define the spacetime interval as:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

To prove that it is invariant, we note that if it wasn't then there must be some constant scaling that is frame-dependent:

$$ds = a(v_R) ds'$$

Where v_R is the relative velocity between the frames. Now consider another thought experiment. We have 3 frames, K , K_1 , and K_2 . K is at rest, K_1 has velocity v_1 , and K_2 has velocity v_2 .

Working in the frame K , we can construct the spacetime interval scaling for K_1 :

$$ds^2 = a(v_1) (ds_1)^2$$

And similarly for the interval scaling for K to K_2 :

$$ds^2 = a(v_2) (ds_2)^2$$

Now we can look at the scaling from K_1 to K_2 :

$$(ds_1)^2 = a(v_{12}) (ds_2)^2$$

We now have 3 scaling relations for the spacetime intervals. Now if we insert them into each other, we find that

$$\frac{a(v_2)}{a(v_1)} = a(v_{12})$$

This is the property that the function a has to satisfy. Note that v_{12} depends on the relative angle between K_1 and K_2 , while the left side is only dependent on the velocities. Because of this, the only way that this can be satisfied is if $a(v)$ is a constant, and in fact, the constant must be 1. This means that the spacetime interval will always be invariant under any frame changes, it is a conserved quantity.

Can we have two events happen at the same position in space? What condition do we have on the spacetime interval between them?

If we have two events, 1 and 2, and we know that the interval is an invariant quantity, we can write down the interval between them, and note that it must be the same in any frame:

$$c^2 t_{12}^2 - l_{12}^2 = c^2 t_{12}'^2 - l_{12}'^2$$

We can shift into a frame in which l_{12}' is zero, they occur in the same position in the frame. In this case, we must then have that the events are causally connected:

$$l_{12} < ct_{12}$$

The distance between the two events is less than the distance that light can travel in the same amount of time.

Now let us introduce the idea of the proper time. Consider a frame K at rest, and a moving frame K' with velocity \mathbf{v} . Writing down the interval transformation:

$$\begin{aligned} c^2 (dt')^2 &= (c dt)^2 - (dl)^2 \\ &= (c dt)^2 - (v dt)^2 \end{aligned}$$

Solving this for $(dt')^2$:

$$(dt')^2 = (dt)^2 \left(1 - \frac{v^2}{c^2}\right)$$

And we denote $1 - \frac{v^2}{c^2}$ as $1/\gamma^2$:

$$dt' = \frac{dt}{\gamma}$$

A clock that runs in a moving frame K' runs slower than the clock at rest, by a factor of the Lorentz factor γ . This is what is known as time dilation, and dt' is known as the proper time, the time measured in a frame in which the object is at rest.

This effect is notable because it means that particles with very small lifetimes survive for longer in our measurements, such as muons being able to penetrate the atmosphere despite having a lifetime of a microsecond. The same holds true for any relativistic particle with a low lifetime (ex. neutrinos).

We can also define transformations under which these things remain true, which are known as Lorentz transformations. We want transformations that keep the interval invariant:

$$(ds)^2 = c^2 dt^2 - (dx)^2$$

One way to think about this is to note that rotations do not affect distances, so rotations in our space give us invariance. In this case, since we have a minus sign, we have hyperbolic sines and cosines:

$$\begin{aligned} x &= x' \cosh(\psi) + ct' \sinh(\psi) \\ ct &= x' \sinh(\psi) + ct' \cosh(\psi) \end{aligned}$$

Written in matrix form:

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh \psi & \sinh \psi \\ \sinh \psi & \cosh \psi \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}$$

To find what ψ needs to be, let us consider a frame at rest, K , and a moving frame K' . The origin in the frame K' has $x' = 0$, which in the frame K , must be given by:

$$\begin{aligned} x &= ct' \sinh \psi \\ ct &= ct' \cosh \psi \end{aligned}$$

From this, we can derive that:

$$\frac{x}{ct} = \tanh(\psi)$$

Thus the origin of the frame K' moves as $x = (\tanh \psi) ct$. We also know that the frame moves with velocity v relative to the frame K , and thus we have that:

$$\begin{aligned} x &= \tanh(\psi) ct \\ &= vt \end{aligned}$$

From this, we have that

$$\begin{aligned} \tanh \psi &= \frac{v}{c} \\ &= \beta \end{aligned}$$

Now using the identities for the hyperbolic trig functions, we find that

$$\sinh \psi = \frac{\beta}{\sqrt{1 - \beta^2}}$$

$$\cosh \psi = \frac{1}{\sqrt{1 - \beta^2}}$$

Thus we can relate x and ct in different frames:

$$\begin{aligned} x &= \gamma (x' + \beta ct') \\ &= \gamma (x' + vt') \\ ct &= \gamma (ct' + x'\beta) \\ &= \gamma \left(t' + \frac{vx'}{c^2} \right) \end{aligned}$$

These are the Lorentz transformations.

We can determine the relativistic velocity transformations:

$$\begin{aligned} dx &= \gamma_f (dx' + v_f dt') \\ dy &= dy' \\ dt &= \gamma_f \left(dt' + \frac{v_f dx'}{c^2} \right) \end{aligned}$$

Where the subscript f denotes the frame velocity. From these, we find that

$$v = \frac{v' + v_f}{1 + \frac{v'v_f}{c^2}}$$

Which is the relativistic velocity transformation. We can also look at the velocity in the y direction:

$$\begin{aligned} v_y &= \frac{dy}{dt'} \\ &= \frac{dy'}{dt} \\ &= \frac{dy}{\gamma_f dt' \left(1 + \frac{v_f}{c^2} v_x \right)} \\ &= \frac{v_y}{\gamma_f \left(1 + \frac{v_f}{c^2} v_x \right)} \end{aligned}$$

We see that even though the frame was moving in the x direction, the velocity in the y direction is changed.

This leads to what is known as relativistic aberration. If we have an object moving in a frame K , with velocity at some angle relative to the axes:

$$\begin{aligned} v_x &= v \cos \theta \\ v_y &= v \sin \theta \end{aligned}$$

In the shifted frame, we have

$$\begin{aligned} v'_x &= v' \cos \theta' \\ v'_y &= v' \sin \theta' \end{aligned}$$

Looking at the tangent of the angle in the original frame:

$$\begin{aligned}\tan \theta &= \frac{v_y}{v_x} \\ &= \frac{v' \sin \theta' \sqrt{1 - \beta^2}}{v' \cos \theta' + v}\end{aligned}$$

If we consider the case of light, which moves at speed c in any frame:

$$\tan \theta_{\text{photon}} = \frac{\sin \theta' \sqrt{1 - \beta^2}}{\cos \theta' + \frac{v}{c}}$$

This is the relativistic aberration of light. Note that we also looked at relativistic aberration in the case of the radiation of a charged relativistic particle.

5.1 4-Vectors

We can define spacetime vectors with 4 components, (ct, \mathbf{x}) , and the spacetime interval is invariant under Lorentz transformations:

$$(x^0)^2 - ((x^1)^2 + (x^2)^2 + (x^3)^2) = \text{invariant}$$

Any set of four components that satisfy these relations form a 4-vector. Other examples include light waves:

$$(\omega, c\mathbf{k})$$

Since $\omega^2 = c^2 k^2$, these are “null” 4-vectors.

Another example is the 4-momentum of a massive particle:

$$(\mathcal{E}, \mathbf{p}c)$$

We define an object known as the Minkowski metric tensor $\eta_{\mu\nu}$:

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

Using this, we can define the scalar product between two vectors A^i and B^i :

$$\mathbf{A} \cdot \mathbf{B} = \eta_{ij} A^i B^j$$

We can also define covariant and contravariant vectors. One can say that covariant vectors define functions of contravariant vectors. Contravariant vectors are written as A^i (similar to position vectors, x^i), and covariant vectors have lower indices, A_i , and are basically functions²⁰. Consider $\hat{A} = A_i \hat{e}_i$, applied to a vector \mathbf{B} :

$$\hat{A}\mathbf{B} = A_i B^i$$

²⁰Sasha said he had a way of remembering which vector was which name, but he forgot how.

= scalar

This is why we can consider covariant vectors as functions. We can also transform between covariant and contravariant vectors via the metric tensor:

$$A_i = \eta_{ij} A^j$$

One example of a covariant vector is the gradient, it takes a vector, and returns a scalar.

$$\frac{\partial \phi}{\partial x^\mu}$$

Applying this to a contravariant vector:

$$\frac{\partial \phi}{\partial x^\mu} dx^\mu = d\phi$$

And thus this is a covariant vector, denoted ∇_μ .

We can also define timelike and spacelike vectors, much like we did for intervals. We do this by looking at the sign of the magnitude:

$$A^2 < 0 \rightarrow \text{spacelike}$$

$$A^2 = 0 \rightarrow \text{null}$$

$$A^2 > 0 \rightarrow \text{timelike}$$

In the case of the 4-momentum of a massive particle:

$$\mathcal{E}^2 - (pc)^2 > 0$$

We see that the 4-momentum is timelike. An example of a spacelike 4-vector is $(\rho c, \mathbf{J})$, the 4-vector composed of current and charge density, which can take either sign.

Let us now discuss 4-tensors of rank 2. These objects have 2 indices:

$$A^{ik}$$

Each of these has 16 components, each index has 4 possibilities, so this is essentially a 4×4 matrix. How do these transform under Lorentz transformations? Let us consider the simplest rank-2 4-tensor, the product of two vectors:

$$A^i B^k$$

We can derive the transformation for each of these individually, and then get the transformation for the product of the two. To do this, we need to look at how products of the form $A^0 B^0$ transform:

$$\begin{aligned} A^0 B^0 &\rightarrow \gamma \left(A^{0'} + \frac{v}{c} A^{1'} \right) \gamma \left(B^{0'} + \frac{v}{c} B^{1'} \right) \\ &= \gamma^2 \left[A^{0'} B^{0'} + \frac{v}{c} (A^{0'} B^{1'} + A^{1'} B^{0'}) + \frac{v^2}{c^2} (A^{1'} B^{1'}) \right] \end{aligned}$$

If we let this be the 00 component of the tensor A^{ij} :

$$A^{00} \rightarrow \gamma^2 \left(A^{0'0'} + \frac{v}{c} (A^{0'1'} + A^{1'0'}) + \frac{v^2}{c^2} A^{1'1'} \right)$$

Repeating this for all of the other components:

$$A^{11} \rightarrow \gamma^2 \left(A^{1'1'} + \frac{v}{c} (A^{1'0'} + A^{0'1'}) + \frac{v^2}{c^2} A^{0'0'} \right)$$

We will deal with two classes of these tensors, symmetric tensors:

$$A^{ik} = A^{ki}$$

And anti-symmetric tensors:

$$B^{ik} = -B^{ki}$$

An example of an anti-symmetric tensor is the field tensor $F^{\mu\nu}$:

$$F^{\mu\nu} = \partial_\nu A^\mu - \partial_\mu A^\nu$$

And an example of a symmetric tensor is the Maxwell stress tensor, $T^{\mu\nu}$.

Note that for anti-symmetric tensors, the diagonal elements must be zero, and define two vectors \mathbf{a} and \mathbf{b} :

$$B^{ik} = \begin{pmatrix} 0 & p_x & p_y & p_z \\ -p_x & 0 & -a_z & a_y \\ -p_y & a_z & 0 & -a_x \\ -p_z & -a_y & a_x & 0 \end{pmatrix}$$

We see that there are no longer 16 independent components, there are 6 independent components.

5.2 Relativistic Free Particle

In the context of relativity, the action is the change in the interval as the particle moves between two points a and b :

$$\begin{aligned} S &= -\alpha \int_a^b ds \\ &= -\alpha \int_{t_1}^{t_2} c \sqrt{1 - \frac{v^2}{c^2}} dt \end{aligned}$$

Where we have used the fact that $(ds)^2 = c^2 (dt)^2 - v^2 (dt)^2$. The quantity under the integral is denoted the Lagrangian:

$$S = \alpha \int_{t_1}^{t_2} \mathcal{L} dt$$

Which gives us that the Lagrangian for a relativistic particle is given by:

$$\mathcal{L} = -\alpha c \sqrt{1 - \frac{v^2}{c^2}}$$

If we look at the non-relativistic limit, by Taylor expanding for small velocities:

$$\lim_{v \ll c} \mathcal{L} = -\alpha c + \frac{\alpha v^2}{2c} + \dots$$

From this, we see that $\alpha = mc$, in order for this to reduce to the classical $\mathcal{L} = \frac{1}{2}mv^2$. Thus we have that the relativistic Lagrangian for a particle is:

$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{v^2}{c^2}}$$

From the Lagrangian, we can obtain the momentum:

$$\begin{aligned} \mathbf{p} &= \frac{\partial \mathcal{L}}{\partial \mathbf{v}} \\ &= -\frac{mc^2}{c^2} \frac{(-2\mathbf{v})}{2\sqrt{1 - \frac{v^2}{c^2}}} \\ &= m\mathbf{v}\gamma \end{aligned}$$

We can relate this to the force:

$$\begin{aligned} \mathbf{F} &= \frac{\partial \mathbf{p}}{\partial t} \\ &= \frac{d}{dt}(\gamma m\mathbf{v}) \end{aligned}$$

When the force is perpendicular to the velocity:

$$\mathbf{F} = \gamma m \frac{d\mathbf{v}}{dt}$$

And when the force is parallel to the velocity:

$$\mathbf{F} = \gamma^3 m \frac{d\mathbf{v}}{dt}$$

where we have computed the time derivative of $\gamma m\mathbf{v}$.

We can compute the energy:

$$\begin{aligned} \mathcal{E} &= \mathbf{p} \cdot \mathbf{v} - \mathcal{L} \\ &= mc^2\gamma \\ &= \mathcal{E}_{\text{rest}} + \mathcal{E}_{\text{kin}} \end{aligned}$$

Which is what we expect, we have a rest mass energy and a kinetic energy.

Now let us apply the action principle to derive the equation of motion. To begin, we must derive the variation of the action. The variation of the action must be zero, by the action principle:

$$\delta S = 0$$

We fix the start and endpoints at a and b , and vary the trajectory between them, to find the trajectory that minimizes the action. We can write out the change in the action:

$$\delta S = -mc \int_a^b \delta(ds)$$

Now noting that $ds = \sqrt{dx_i dx^i}$:

$$\begin{aligned}\delta S &= -mc \int_a^b \delta \left(\sqrt{dx_i dx^i} \right) \\ &= -mc \int_a^b \frac{\delta (dx_i dx^i)}{2\sqrt{dx_i dx^i}} \\ &= -mc \int_a^b \frac{2dx_i \delta (dx^i)}{2ds}\end{aligned}$$

Where we have noted that $\delta (dx_i) = \delta (dx^i)$. Now we can cancel the 2s, and rewrite this a little:

$$\begin{aligned}\delta S &= -mc \int_a^b \frac{dx_i}{ds} \delta (dx^i) \\ &= -mc \int_a^b U_i \delta (dx^i)\end{aligned}$$

Where we denote U_i as the covariant 4-velocity $\frac{dx_i}{ds}$, since x_i is covariant and ds is a relativistic scalar. Now integrating by parts:

$$\begin{aligned}\delta S &= -mc \int_a^b U_i d(\delta x^i) \\ &= -mc U_i \delta (x^i) \Big|_a^b + mc \int_a^b \frac{dU_i}{ds} (\delta x^i ds)\end{aligned}$$

We now note that the first term is zero, since the value at a and b is fixed, so we have that

$$\delta S = mc \int_a^b \frac{dU_i}{ds} (\delta (x^i) ds)$$

Looking at the components of U^i , the **contravariant** 4-velocity:

$$\begin{aligned}U^0 &= \frac{dx^0}{ds} \\ &= \frac{c dt}{c\sqrt{(dt)^2 - \frac{1}{c^2} d\mathbf{x}^2}} \\ &= \gamma \\ U^{i \in \{1,2,3\}} &= \frac{d\mathbf{x}}{c dt \sqrt{1 - \beta^2}} \\ &= \frac{\boldsymbol{\beta}}{\sqrt{1 - \beta^2}}\end{aligned}$$

Thus we have that

$$U^i = (\gamma, \gamma \boldsymbol{\beta})$$

And we can write the 4-momentum in terms of U^i :

$$p^i = mc U^i$$

$$= (mc\gamma, \mathbf{p})$$

To obtain the covariant 4-velocity, we can apply the metric:

$$U_i = \eta_{ij} U^j$$

Looking at the 4-momentum, we can rewrite it in terms of the energy:

$$\begin{aligned} p^i &= (mc\gamma, \mathbf{p}) \\ &= \left(\frac{\mathcal{E}}{c}, \mathbf{p} \right) \end{aligned}$$

Looking at the magnitude:

$$\begin{aligned} p_i p^i &= \frac{\mathcal{E}^2}{c^2} - \mathbf{p}^2 \\ &= m^2 c^2 \end{aligned}$$

If we do the same for the 4-velocity:

$$\begin{aligned} U_i U^i &= \gamma^2 - \gamma^2 \beta^2 \\ &= 1 \end{aligned}$$

We can also write down Lorentz transformation rules for energy and momentum:

$$\begin{aligned} \mathcal{E} &= \gamma (\mathcal{E}' + p'_x v_x) \\ p_x &= \gamma \left(p'_x + \frac{v_x \mathcal{E}'}{c^2} \right) \end{aligned}$$

5.3 Relativistic Interaction with EM Fields

We have thus far covered the kinematics of a single relativistic particle, so let us now consider a relativistic particle interacting with a fixed electromagnetic field.

We can introduce a term to the action, which we guess is proportional to the charge, and since we know that we will have a vector interaction, we know that we have to have the vector potential $A^i = (\phi, \mathbf{A})$, and we need a dx^j to keep it as a scalar:

$$S = -mc \int_a^b ds + \frac{q}{c} \int_a^b A_j dx^j$$

This is not the only choice that we can make. If we have powers of the vector potential, we have a non-linear theory, which does not agree with the experimental reality. The only other relativistic scalar that we can write is:

$$\int A ds$$

Which is Lorentz invariant. However, the results of this choice do not match what we witness in reality.

We can write out the action, noting that $A_i = (\phi, -\mathbf{A})$:

$$\begin{aligned} S &= \int_a^b \left(-mc ds + \frac{q}{c} \mathbf{A} \cdot d\mathbf{r} - q\phi dt \right) \\ &= \int_a^b \mathcal{L} dt \end{aligned}$$

From this, we have the Lagrangian:

$$\mathcal{L} = -mc\sqrt{1 - \frac{v^2}{c^2}} + \frac{q}{c} \mathbf{A} \cdot \mathbf{v} - q\phi$$

We can compute the generalized momentum:

$$\begin{aligned} \mathcal{P} &= \frac{\partial \mathcal{L}}{\partial \mathbf{v}} \\ &= \mathbf{p} + \frac{q}{c} \mathbf{A} \end{aligned}$$

We can construct the Hamiltonian:

$$\begin{aligned} \mathcal{H} &= \mathcal{P} \cdot \mathbf{v} - \mathcal{L} \\ &= mc^2\gamma + q\phi \end{aligned}$$

We could obtain the equation of motion via direct variation of the action, but we can instead use the Euler-Lagrange equations:

$$\frac{d}{dt} \mathcal{P} = \frac{\partial \mathcal{L}}{\partial \mathbf{r}}$$

Computing the time derivatives:

$$\begin{aligned} \frac{d}{dt} \left(\mathbf{p} + \frac{q}{c} \mathbf{A} \right) &= \frac{q}{c} \nabla_r (\mathbf{A} \cdot \mathbf{v}) - q \nabla \phi \\ &= \frac{q}{c} \mathbf{v} \times \mathbf{B} - e \nabla \phi + \frac{q}{c} (\mathbf{v} \nabla) \mathbf{A} \end{aligned}$$

Expanding out the left side:

$$\frac{d}{dt} \left(\mathbf{p} + \frac{q}{c} \mathbf{A} \right) = \frac{q}{c} \frac{\partial \mathbf{A}}{\partial t} + \frac{q}{c} (\mathbf{v} \nabla) \mathbf{A}$$

Thus we have that

$$\begin{aligned} \frac{d\mathbf{p}}{dt} &= \frac{q}{c} (\mathbf{v} \times \mathbf{B}) + q \underbrace{\left(-\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right)}_{\mathbf{E}} \\ \frac{d\mathbf{p}}{dt} &= q\mathbf{E} + \frac{q}{c} \mathbf{v} \times \mathbf{B} \end{aligned}$$

Which is the expected result, we have the Lorentz force equation, derived from the Lagrangian we wrote.

Let us redo this derivation using 4 dimensional notation. First, let us note that we have gauge invariance in the potential:

$$A'_i = A_i - \frac{df}{dx^i}$$

If we insert this into the Lagrangian, and look at the extra term in the action:

$$\begin{aligned}\Delta S &= \frac{q}{c} \int_a^b \frac{df}{dx^i} dx^i \\ &= \frac{q}{c} \int_a^b df \\ &= \frac{q}{c} (f(b) - f(a))\end{aligned}$$

We see that this is an extra constant term in the Lagrangian, which does nothing in our equations of motion, thus we have gauge invariance.

Let us find the equations of motion in the 4-dimensional notation. We want to vary our action:

$$\delta \int_a^b \left[-mc ds - \frac{q}{c} A_i dx^i \right] = \int_a^b mc dU_i \delta x^i + \text{boundary term} - \int_a^b \frac{q}{c} [A_i d(\delta x^i) + \delta A_i dx^i]$$

We can split the second term into two terms, and look at the first of those:

$$\begin{aligned}-\frac{q}{c} \int A_i d(\delta x^i) &= -\underbrace{\frac{q}{c} A_i \delta x^i \Big|_a^b}_{\text{boundary term}} + \frac{q}{c} \int_a^b dA_i \delta x^i \\ &= \frac{q}{c} \int_a^b \frac{\partial A_i}{\partial x^j} dx^j \delta x^i\end{aligned}$$

Looking at the second of the two terms:

$$-\frac{q}{c} \int_a^b \delta A_i dx^i = -\frac{q}{c} \int_a^b \frac{\partial A_i}{\partial x^j} \delta x^j dx^i$$

All together, we are left with

$$\delta S = \int_a^b \left[mc dU_i + \frac{q}{c} \left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} \right) dx^j \right] \delta x^i$$

Dividing everything by ds :

$$\delta S = \int_a^b \left[mc \frac{dU_i}{ds} + \frac{q}{c} \left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} \right) \frac{dx^j}{ds} \right] \delta x^i ds$$

Setting this equal to zero, we find that we have the 4D equation of motion:

$$mc \frac{dU_i}{ds} = \frac{q}{c} F_{ik} U^k$$

Where F_{ik} is an antisymmetric tensor ($F_{ki} = -F_{ik}$):

$$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}$$

Note that F , the electromagnetic field tensor, has 6 independent components, since the diagonals are zero, and the bottom triangle are just the inverse of the top triangle terms.

How can we relate the components of F to the electric and magnetic fields? To do this, we can compute them directly:

$$\begin{aligned} F_{01} &= \frac{\partial A_1}{\partial x^0} - \frac{\partial A_0}{\partial x^1} \\ &= -\frac{1}{c} \frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x^1} \\ &= E_x \end{aligned}$$

Looking at other components:

$$\begin{aligned} F_{02} &= E_y \\ F_{03} &= E_z \end{aligned}$$

Thus we have that $F_{0i} = E_i$, and similarly $F_{i0} = -E_i$:

$$F_{ij} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & & \\ -E_y & & 0 & \\ -E_z & & & 0 \end{bmatrix}$$

Now we can look at the unfilled elements in this tensor:

$$\begin{aligned} F_{12} &= \frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2} \\ &= -\frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y} \\ &= -(\nabla \times \mathbf{A})_z \\ &= -B_z \end{aligned}$$

Which updates our tensor:

$$F_{ij} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & \\ -E_y & B_z & 0 & \\ -E_z & & & 0 \end{bmatrix}$$

If we look at F_{13} :

$$\begin{aligned} F_{13} &= \frac{\partial A_3}{\partial x^1} - \frac{\partial A_1}{\partial x^3} \\ &= (\nabla \times \mathbf{A})_y \\ &= B_y \end{aligned}$$

And thus we find that the field strength tensor is given by:

$$F_{ij} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

We can also write down the contravariant form of this:

$$F^{ik} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

To find this, we can use the transformation rule:

$$F^{ik} = \eta^{il} \eta^{km} F_{lm}$$

We then note that each application of the metric on a spatial index introduces a minus sign.

Now let us consider the Lorentz transformation of the field strength tensor. Suppose we do a Lorentz transformation along $x^1 = x$. How do the components of F^{ik} transform? For the electric field, we can look at F^{01} , F^{02} and F^{03} . Looking at F^{02} :

$$\begin{aligned} F^{02} &= (F^{02})' \\ &= \gamma \left(F^{0'2'} + \frac{v}{c} F^{1'2'} \right) \\ &= \gamma \left(-E'_y + \frac{v}{c} (-B'_z) \right) \\ &= \gamma \left(E'_y + \frac{v}{c} B'_z \right) \end{aligned}$$

Where we have used the fact that

$$t = \gamma \left(t' + \frac{v}{c^2} x' \right)$$

And then noted that $t \rightarrow F^{02}$, and $x \rightarrow F^{12}$. Thus we have that

$$E_y = \gamma \left(E'_y + \frac{v}{c} B'_z \right)$$

Similarly for E_z :

$$E_z = \gamma \left(E'_z + \frac{v}{c} B'_y \right)$$

Finally, looking at the transformation of F^{01} , which is more complicated since we have to transform both indices:

$$\begin{aligned} F^{01} &= \gamma \left(F^{0'1} + \frac{v}{c} F^{1'1} \right) \\ &= \gamma \left[\gamma \left(F^{0'1'} + \frac{v}{c} F^{0'0'} \right) + \gamma \frac{v}{c} \left(F^{1'1'} + \frac{v}{c} F^{1'0'} \right) \right] \\ &= \gamma \left(\gamma F^{0'1'} + \left(\frac{v}{c} \right)^2 \gamma F^{0'1'} \right) \\ &= F^{0'1'} \end{aligned}$$

Where we note that the diagonal elements must be zero, and we can flip indices with a minus sign. We see that the electric field along the Lorentz boost does not transform. This is how we can derive the Lorentz transformation of the electric field:

$$E_x = E'_x$$

$$E_y = \gamma \left(E'_y + \frac{v}{c} B'_z \right)$$

$$E_z = \gamma \left(E'_z - \frac{v}{c} B'_y \right)$$

Now suppose we want to see how F^{23} transforms if we do the same x direction Lorentz transformation:

$$F^{23} \rightarrow F^{2'3'}$$

Which implies that $B'_x = B_x$. If we look at F^{12} and F^{13} , transform those, and write them in terms of the magnetic and electric fields, we have the magnetic field transformation rules for an x boost:

$$B_x = B'_x$$

$$B_y = \gamma \left(B'_y - \frac{v}{c} E'_z \right)$$

$$B_z = \gamma \left(B'_z + \frac{v}{c} E'_y \right)$$

What happens if we boost into a frame where the electric field is zero? If $E' = 0$, we have that

$$B_x = B'_x$$

$$B_y = \gamma B'_y$$

$$B_z = \gamma B'_z$$

If we are in the case where $\mathbf{E} \perp \mathbf{B}$, then we have that

$$E_x = 0$$

$$E_y = \gamma \frac{v}{c} B'_z$$

$$= \frac{v}{c} B_z$$

$$E_z = -\frac{v}{c} \gamma B'_y$$

$$= -\frac{v}{c} B_y$$

Since the velocity is in the x direction, this seems to look like a vector product:

$$\mathbf{E} = -\frac{\mathbf{v}}{c} \times \mathbf{B}$$

We started with a frame where the electric field is zero, and we see that we can boost to a frame in which the electric and magnetic fields are perpendicular to each other and related to each other by the above equation.

Inverting this relationship, if we have perpendicular electric and magnetic fields, we can boost into a frame (perpendicular to both of them) moving with velocity $v = \frac{cE}{B}$, and the electric field will vanish. Note that in order for this argument to work, $E < B$, otherwise the boosted frame would have to move faster than the speed of light.

If instead we are in the situation where $E > B$ in a frame, we can shift into a frame where $B' = 0$:

$$B'_x = 0$$

$$\begin{aligned}
B_y &= -\gamma \frac{v}{c} E'_z \\
B_z &= \gamma \frac{v}{c} E'_y \\
E_x &= 0 \\
E_y &= \gamma E'_y \\
E_z &= \gamma E'_z
\end{aligned}$$

From this, we find that

$$\mathbf{B} = \frac{\mathbf{v}}{c} \times \mathbf{E}$$

And the velocity that the boosted frame travels with is

$$v = c \frac{B}{E}$$

We can generate a relativistic invariant from the field tensor, by contracting it with itself, to get a relativistic scalar, which will not change if we boost:

$$\begin{aligned}
F_{ik}F^{ik} &= \sum_{i,k} F_{ik}F^{ik} \\
&= -2\mathbf{E}^2 + 2\mathbf{B}^2 \\
&= 2(\mathbf{B}^2 - \mathbf{E}^2)
\end{aligned}$$

This is what is known as the first invariant of the electromagnetic field.

We can construct another invariant by constructing a dual tensor using the 4 dimensional Levi-Civita symbol:

$$F^{*lm} = \frac{1}{2} \varepsilon_{iklm} F_{ik}$$

Let us compute an element of this dual tensor:

$$\begin{aligned}
F^{*01} &= \frac{1}{2} \varepsilon_{ik01} F_{ik} \\
&= \frac{1}{2} [\varepsilon_{2301} F_{32} + \varepsilon_{3201} F_{23}] \\
&= -B_x
\end{aligned}$$

Where we have noted that $F_{32} = -F_{23}$. If we write down the tensor:

$$F^{*lm} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix}$$

We can now construct another relativistic scalar, by contracting the dual tensor with the field tensor:

$$F_{ik}F^{*ik} = -4(\mathbf{E} \cdot \mathbf{B})$$

$$= -4E_i B_i$$

This is our second relativistic invariant. This means that if we have an electric field parallel to a magnetic field, we can never move to a frame in which they are perpendicular.

Suppose we have an electric field and magnetic field with some shared component, $\mathbf{E} \cdot \mathbf{B} \neq 0$. Let us construct a special frame with velocity $\mathbf{v} \perp \mathbf{E}, \mathbf{B}$, in which $\mathbf{E}' \parallel \mathbf{B}'$. We denote the direction of \mathbf{v} to be the x direction.

We know that components along the boost are unchanged, and are zero because the velocity is perpendicular to the fields:

$$\begin{aligned} E_x &= E'_x = 0 \\ B_x &= B'_x = 0 \end{aligned}$$

We want $\mathbf{E}' \times \mathbf{B}' = 0$, the fields in the new frame should be parallel:

$$\begin{aligned} \mathbf{E}' \times \mathbf{B}' &= E'_y B'_z - E'_z B'_y \\ &= 0 \end{aligned}$$

Now we want to find the condition that relates \mathbf{v} , \mathbf{B} , and \mathbf{E} . Rewriting this with the boosted relations:

$$\mathbf{E}' \times \mathbf{B}' = \gamma^2 \left(E_y - \frac{v}{c} B_z \right) \left(B_z - \frac{v}{c} E_y \right) - \gamma^2 \left(E_z + \frac{v}{c} E_y \right) \left(E_z + \frac{v}{c} B_y \right) = 0$$

Now grouping terms by their dependence on v :

$$\frac{v^2}{c^2} (B_z E_y - E_z B_y) - \frac{v}{c} \underbrace{(B_z^2 + B_y^2 + E_z^2 + E_y^2)}_{E^2 + B^2} + \underbrace{(E_y B_z - E_z B_y)}_{(\mathbf{E} \times \mathbf{B})_x} = 0$$

From which we find that

$$\frac{\mathbf{v}}{c} = \frac{\mathbf{E} \times \mathbf{B}}{E^2 + B^2} \left[1 + \frac{v^2}{c^2} \right]$$

Note that in the non-relativistic limit, we see that

$$\frac{EB}{E^2 + B^2} \ll 1$$

5.3.1 4D Maxwell's Equations

Now that we have defined the electromagnetic field tensor (as well as its dual):

$$F_{ik} = \frac{\partial A_x}{\partial x^i} - \frac{\partial A_i}{\partial x^k}$$

And we know that $\mathbf{B} = \nabla \times \mathbf{A}$, and $\mathbf{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$, so we have

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \end{aligned}$$

We can see that these are straightforward to construct using the dual tensor, the derivative of the first row is the divergence of the \mathbf{B} field. We can express the derivative of a given row with respect to a particular coordinate:

$$\frac{\partial F^{*lm}}{\partial x^m}$$

If we choose $l = 0$:

$$\begin{aligned}\frac{\partial F^{*0m}}{\partial x^m} &= \frac{\partial F^{*01}}{\partial x^1} + \frac{\partial F^{*02}}{\partial x^2} + \frac{\partial F^{*03}}{\partial x^3} \\ &= \nabla \cdot \mathbf{B} \\ &= 0\end{aligned}$$

If we look at other values of l :

$$\begin{aligned}\frac{\partial F^{*1m}}{\partial x^m} &= \frac{1}{c} \frac{\partial B_x}{\partial t} + \underbrace{\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}}_{(\nabla \times \mathbf{E})_x} \\ &= 0\end{aligned}$$

We find that for all l :

$$\boxed{\frac{\partial F^{*lm}}{\partial x^m} = 0}$$

This gives us 4 equations, which are equivalent to the two Maxwell equations that we had above:

$$\begin{aligned}\partial_m F^{*lm} &= 0 \\ \downarrow \\ \nabla \cdot \mathbf{B} &= 0 \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\end{aligned}$$

To find the other two equations, we return to the action that relates the fields and particles:

$$S_{pf} = - \int \frac{q}{c} A_i dx^i$$

We define a 4-vector relating the currents:

$$\begin{aligned}j^i &= \rho \frac{dx^i}{dt} \\ &= (c\rho, \mathbf{J})\end{aligned}$$

We can prove that this construction is a 4-vector:

$$\rho = \sum_a q_a \delta(\mathbf{x} - \mathbf{x}_a) \rightarrow dq = \rho dV$$

dq is a relativistic invariant, since the charge of a single electron is a relativistic invariant. Therefore, the quantity $dq dx^i$ is a 4-vector:

$$dq dx^i = \rho dV dx^i$$

$$\begin{aligned}
&= \rho dV dt \frac{dx^i}{ddt} \\
&= \rho d^4x \frac{dx^i}{dt} \\
&= d^4x \left(\rho \frac{dx^i}{dt} \right)
\end{aligned}$$

Where $d^4x = dV dt$ is a Lorentz invariant. Since d^4x is a relativistic invariant, and so is dq , and dx^i is a 4-vector, $\rho \frac{dx^i}{dt}$ must also be a 4-vector. Thus j^i is a valid 4-vector.

We can rewrite our action to use j^i :

$$\begin{aligned}
S_{pf} &= -\frac{1}{c} \int \rho A_i dV \frac{dx^i}{dt} dt \\
&= -\frac{1}{c} \int j^i A_i d^4x
\end{aligned}$$

Now let us look at the action of just the fields. Since we will vary this action by taking derivatives, we need this to be quadratic in F , since when we take the derivative, we will recover equations of motions that are linear in the field strengths:

$$S_f = \frac{1}{16\pi c} \int F_{ik} F^{ik} d^4x$$

However, we can also insert something that is quadratic in the dual, which turns out to be a divergence:

$$F^{*lm} F_{lm} = 4 \frac{\partial}{\partial x^i} \left(\varepsilon_{iklm} A_k \frac{\partial A_m}{\partial x^l} \right)$$

If we inserted this into the action, we would have a 4D integral of a divergence, which would turn into a surface integral at infinity, which would give us zero. Thus, adding the second relativistic invariant is not needed. If we want our action to be relativistically invariant, and quadratic in F , these are the only two options that we can write in our action, and one of them is zero.

Thus our combined action is given by:

$$S = S_f + S_{pf}$$

Which we can vary:

$$\begin{aligned}
\delta S &= \delta (S_f + S_{pf}) \\
&= -\frac{1}{c} \int \left(\frac{1}{c} j^i \delta A_i + \frac{1}{8\pi} F^{ik} \delta F_{ik} \right) d^4x
\end{aligned}$$

Looking at δF_{ik} :

$$\delta F_{ik} = \delta \left[\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} \right]$$

So the second term looks like:

$$\begin{aligned} F^{ik} \delta F_{ik} &= F^{ik} \frac{\partial}{\partial x^i} (\delta A_k) - F^{ik} \frac{\partial}{\partial x^k} (\delta A_i) \\ &= -2F^{ik} \frac{\partial}{\partial x^k} (\delta A_i) \end{aligned}$$

Inserting this back into our change in the action, where we integrate by parts and drop the boundary term, we can set the variance of the action to zero:

$$\begin{aligned} \delta S &= \frac{1}{c} \int \left(\frac{1}{c} j^i + \frac{\partial F^{ik}}{\partial x^k} \frac{1}{4\pi} \right) \delta A_i d^4x \\ &= 0 \end{aligned}$$

This gets us that

$$\partial_k F^{ik} = -\frac{4\pi}{c} j^i$$

If we look at the first row of this tensor, $i = 0$, we find that this is equivalent to:

$$-\nabla \cdot \mathbf{E} = -4\pi\rho$$

Which gives us Gauss's Law. If we take the spatial components, we obtain Ampere's Law.

If we take a derivative of both sides of our expression with respect to i :

$$\partial_i (\partial_k F^{ik}) = -\frac{4\pi}{c} \partial_i j^i$$

This left side is zero²¹, and we have the charge continuity equation:

$$\frac{\partial c\rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

6 Waves in Continuous Media

We want to look at how electromagnetic waves propagate through a medium. We have an incoming wave, with some ω and \mathbf{k} , and it is modified to some ω' and \mathbf{k}' based on the properties of the medium that the wave enters.

The wave that is propagating through the medium still obeys Maxwell's equations:

$$\begin{aligned} \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} \\ \nabla \times \mathbf{B} &= \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

Where we have plane wave solutions:

$$\mathbf{E}, \mathbf{B} \sim e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$$

²¹By some tensor magic?

Inserting our solutions into the first of Maxwell's Equations:

$$-\frac{i\omega}{c}\mathbf{B} = -i\mathbf{k} \times \mathbf{E}$$

And the second:

$$i\mathbf{k} \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} - \frac{i\omega}{c}\mathbf{E}$$

From these, we find that

$$\frac{\omega}{c}\mathbf{B} = \mathbf{k} \times \mathbf{E}$$

Now let us suppose that our medium responds linearly to the wave, through the conductivity tensor ς :

$$J_i = \varsigma_{ij}E_j$$

We will find that when we insert our plane waves, we find something of the form:

$$\Lambda_{ij}E_j = 0$$

To find this, let us begin with:

$$\mathbf{B} = \frac{c}{\omega}\mathbf{k} \times \mathbf{E}$$

If we insert this into the second of our equations:

$$(i\mathbf{k} \times \mathbf{k} \times \mathbf{E}) \frac{c}{\omega} = \frac{4\pi}{c}\varsigma_{ij}E_j - \frac{i\omega}{c}\mathbf{E}$$

Multiplying by $\frac{i\omega}{c}$, we are left with

$$\begin{aligned} -\mathbf{k} \times \mathbf{k} \times \mathbf{E} &= \frac{4\pi i\omega}{c^2}\varsigma_{ij}E_j + \frac{\omega^2}{c^2}\mathbf{E} \\ &= \frac{\omega^2}{c^2}E_j \left[\delta_{ij} + \frac{4\pi i\varsigma_{ij}}{\omega} \right] \end{aligned}$$

We denote this the dielectric tensor:

$$\varepsilon_{ij} = \delta_{ij} + \frac{4\pi i\varsigma_{ij}}{\omega}$$

Expanding the left side:

$$-[\mathbf{k}(\mathbf{k} \cdot \mathbf{E}) - \mathbf{E}k^2] = \frac{\omega^2}{c^2}\varepsilon_{ij}E_j$$

Writing the left side in index notation, and then collecting all of the E_j terms:

$$\left[k^2\delta_{ij} - k_ik_j - \frac{\omega^2}{c^2}\varepsilon_{ij} \right] E_j = 0$$

Thus, we see that we have our Λ_{ij} :

$$\Lambda_{ij} = k^2 \delta_{ij} - k_i k_j - \frac{\omega^2}{c^2} \varepsilon_{ij}$$

The eigenvalues of this matrix provide the dispersion relation for the medium $\omega(\mathbf{k})$, and the eigenvectors provide the polarization directions.

The easiest case to work with is when the medium is isotropic and frequency independent:

$$\varepsilon_{ij} = \varepsilon \delta_{ij}$$

In which case we have something that looks like the vacuum case:

$$k^2 = \frac{\omega^2}{c^2} \varepsilon$$

We can also relate ε to the index of refraction:

$$\varepsilon = n^2$$

And thus we can relate the wavenumber to the index of refraction:

$$k^2 = \frac{\omega^2}{c^2} n^2$$

We can also define the phase velocity:

$$\begin{aligned} v_{ph} &= \frac{\omega}{k} \\ &= \frac{c}{n} \end{aligned}$$

And the group velocity:

$$v_{gr} = \frac{\partial \omega}{\partial k}$$

Let us look at different cases. Suppose we have a system of electrons and ions, and we want to construct the ς_{ij} tensor, so we can relate the current to the incoming wave.

Since we have no magnetic field, and we have no preferred direction, we know that the dielectric tensor has the form:

$$\varepsilon_{ij} = \varepsilon(\omega) \delta_{ij}$$

The electrons will feel the non-relativistic Lorentz force²²:

$$m_e \frac{\partial \mathbf{v}_e}{\partial t} = e \mathbf{E}_w + \frac{e}{c} \mathbf{v} \times \mathbf{B}_w$$

If the particle is non-relativistic, then the magnetic term drops out, since $v \ll c$. Thus, the current generated is given by:

$$\mathbf{J}_e = n_e e \mathbf{v}_e$$

²²In the cold approximation, where we assume that the electrons are essentially at rest initially, prior to the arrival of the wave.

Where n_e is the number density of electrons in the system.

The electrons behave like a harmonic oscillator coupled to the wave:

$$-i\omega m_e \mathbf{v}_e = e \mathbf{E}_w$$

From which we find the velocity:

$$\mathbf{v}_e = \frac{ie \mathbf{E}_w}{\omega m_e}$$

Which we can insert into the current relation:

$$\mathbf{J}_e = \frac{in_e e^2}{m_e \omega^2} \mathbf{E}_w$$

Which lets us see that

$$\zeta = \frac{in_e e^2}{m_e \omega}$$

From which we can find the dielectric tensor:

$$\begin{aligned} \varepsilon &= 1 + \frac{4\pi i \zeta}{\omega} \\ &= 1 - \underbrace{\frac{4\pi n_e e^2}{m_e}}_{\omega_p^2} \frac{1}{\omega^2} \\ &= 1 - \frac{\omega_p^2}{\omega^2} \end{aligned}$$

Where we denote this set of constants the plasma frequency, ω_p^2 .

If we now apply this to the case where $\mathbf{k} = k\hat{x}$, we find that our Λ matrix is:

$$\Lambda = \begin{bmatrix} -\frac{\omega^2}{c^2} \varepsilon & & \\ & k^2 - \frac{\omega^2}{c^2} \varepsilon & \\ & & k^2 - \frac{\omega^2}{c^2} \varepsilon \end{bmatrix}$$

If we have the matrix equation

$$\Lambda_{ij} E_j = 0$$

We have 3 equations, the first of which is:

$$\frac{\omega^2}{c^2} \varepsilon E_x = 0$$

This gives us a longitudinal ($\mathbf{E} \parallel \mathbf{k}$) mode, with $\omega = \omega_p$. The second and third of these equations gives us a transverse mode, $\mathbf{E} \perp \mathbf{k}$, with $k^2 = \frac{\omega^2}{c^2} \varepsilon$.

In the first case, electrons are pulled sideways, and the medium generates a restoring electric field, and the electrons are pulled back. In principle, with no damping, the system will continue to oscillate, and the oscillation occurs at the plasma frequency.

The transverse modes are very similar to the vacuum case, except we have a different relation between the frequency and the wavenumber:

$$k^2 = \frac{\omega^2}{c^2} \left(1 - \frac{\omega_p^2}{\omega^2} \right)$$

From which we find that

$$\omega^2 = \omega_p^2 + c^2 k^2$$

Note that we have a minimum frequency, the plasma frequency. If we consider a wave entering with a lower frequency, currents on the surface will cause the incoming wave to be fully reflected, only waves with frequencies higher than the plasma frequency can propagate through the medium.

Let us now consider the group velocity of an ensemble of plane waves. Suppose we have a signal, expressed in Fourier space:

$$\mathbf{E}(z, t) = \int_0^\infty A(\omega) e^{i[k(\omega)z - \omega t]} d\omega$$

We want to find the equation of motion for the electric field as a function of z and t .

We will either have an initial condition for the electric field outside the medium as a function of time, $E(0, t)$, or the electric field at a certain point in the medium as a function of time, $E(x, 0)$. Suppose that the frequency envelope is peaked around ω_0 . In this case, we can Taylor expand $k(\omega)$ around the peak frequency:

$$k(\omega) = k(\omega_0) + \frac{\partial k}{\partial \omega} \Big|_{\omega_0} \delta\Omega + \frac{1}{2} (\delta\Omega)^2 \frac{\partial^2 k}{\partial \omega^2} \Big|_{\omega_0} + \dots$$

Inserting this into our expression for \mathbf{E} :

$$\mathbf{E}(z, t) = \underbrace{e^{ik(\omega_0)z - \omega_0 t}}_{\text{carrier}} \int_0^\infty A(\omega_0 + \Omega) \exp \left[i \left(\frac{\partial k}{\partial \omega} \Omega + \frac{\Omega^2}{2} \frac{\partial^2 k}{\partial \omega^2} \right) z - i\Omega t \right] d\Omega$$

The integral is some function of z and t , which we can call $\tilde{A}(z, t)$, and this describes the modulation of the carrier frequency.

Now we can look at derivatives of \tilde{A} :

$$\begin{aligned} i \frac{\partial}{\partial t} \tilde{A} &= \Omega \tilde{A} \\ i \frac{\partial}{\partial z} \tilde{A} &= - \left(\Omega \frac{\partial k}{\partial \omega} + \frac{\omega^2}{2} \frac{\partial^2 k}{\partial \omega^2} \right) \tilde{A} \end{aligned}$$

Putting these two together, by multiplying the top equation by $\frac{\partial k}{\partial \omega}$:

$$i \left[\frac{\partial}{\partial z} + \frac{\partial k}{\partial \omega} \frac{\partial}{\partial t} \right] \tilde{A} = - \frac{\Omega^2}{2} \frac{\partial^2 k}{\partial \omega^2} \tilde{A}$$

From this, we can note that the right side is two time derivatives of \tilde{A} :

$$i \left[\frac{\partial}{\partial z} + \frac{\partial k}{\partial \omega} \frac{\partial}{\partial t} \right] \tilde{A} = \frac{\partial^2 k}{\partial \omega^2} \frac{1}{2} \frac{\partial^2 \tilde{A}}{\partial t^2}$$

Looking at this differential equation, we can see that if $k'_0 \gg k''_0 \delta \Omega$ ²³, where k_0 is the wavenumber related to ω_0 , we can eliminate the right side, and we are left with

$$\left(\frac{\partial}{\partial t} + v_{gr} \frac{\partial}{\partial z} \right) \tilde{A} = 0$$

Where v_{gr} is a constant. This is not a wave equation for \tilde{A} , instead, we have an advection equation, which has solutions of the form $\tilde{A}(z - v_{gr}t, t)$. Over time, the enveloping $\tilde{A}(t)$ will move over time, with velocity given by the group velocity. Thus the electric field takes the form:

$$\mathbf{E}(z, t) = \tilde{A}(z - v_{gr}t, t) e^{i(k_0 z - \omega_0 t)}$$

Looking back at the case of the electrons and ions, we find that the phase velocity is given by:

$$\begin{aligned} v_{ph} &= \frac{\omega}{k} \\ &= \frac{c}{\sqrt{\epsilon}} \\ &= \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} \end{aligned}$$

This seems to be moving faster than the speed of light! However, if we look at the group velocity:

$$\begin{aligned} v_{gr} &= \frac{\partial \omega}{\partial k} \\ &= \frac{c^2 k}{\sqrt{\omega_p^2 + c^2 k^2}} \\ &= \frac{c^2 k}{\omega} \\ &= \frac{c^2}{v_{ph}} \end{aligned}$$

From this, we find that

$$v_{gr} v_{ph} = c^2$$

The group velocity is fundamentally slower than the speed of light, and thus causality is preserved.

What are some other applications of these calculations? Suppose we have a faraway source of waves, with some distribution of frequencies, given by some maximum and some minimum frequency, ω_{max} and ω_{min} respectively. Suppose these waves are travelling through some region of space with electrons and ions, with no magnetic field. If the frequencies were all emitted at the same time, we will see that the waves don't arrive all at the same time, different frequencies arrive at different times because the group velocity depends on the frequency. We will see that the high frequencies will arrive first, and then we will see the low frequencies. If we measure the delay between the arrival times, this will actually tell us how much of the medium the waves travelled through before it reached the observer.

²³Essentially, the width of the signal wavepacket in frequency space is very small, otherwise the Taylor expansion around the carrier frequency would not be accurate.

Our group velocity is given by:

$$v_{gr} = c \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

If we say that the object is emitting frequencies in the radio spectrum, on the order of a gigahertz, and we are emitting through the interstellar medium (ISM), which has $n_e \sim 0.03$ electrons per cubic centimeter, we can make the assumption that $\omega \gg \omega_p$, and we can Taylor expand:

$$v_{gr} \approx c \left(1 - \frac{\omega_p^2}{2\omega^2} \right)$$

If we want to compute the difference in arrival times:

$$\begin{aligned} \Delta t &= \int \frac{dl}{v_{gr}(l)} \\ &= \frac{1}{2\pi m_e c \nu^2} \int_0^L n_e dl e^2 \end{aligned}$$

If we want the delay between two frequencies with ν_1 and ν_2 , we find that

$$\Delta t(\nu_1, \nu_2) = \frac{\int_0^L n_e dl e^2}{2\pi m_e c \nu_1^2}$$

If $\nu_2 \gg \nu_1$. If we measure the time delay, and compute the denominator, we can estimate the numerator, which is known as the dispersive measure:

$$\mathcal{DM} = \int_0^L n_e dl$$

Which is something that radio astronomers did for neutron stars in the 1970's.

Another use of this is to note that radar from the Earth's surface can be reflected off of the ionosphere, because the plasma frequency of the ionosphere is suddenly much higher than the plasma frequency of the lower atmosphere, and at some point we have total internal reflection.