Analysis 2 Notes Compilation

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1.1 Proofs of the Schwarz Inequality

$$\boxed{|\mathbf{a}||\mathbf{b}| \geq |\mathbf{a} \cdot \mathbf{b}|}$$

1.1.1 Proof 1

let $\mathbf{z} = u - proj_{\mathbf{v}}\mathbf{u}$.
Lemma:

$$\begin{aligned} \mathbf{z} \cdot \mathbf{v} &= (\mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}) \cdot \mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{v} - (\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}})\mathbf{v} \cdot \mathbf{v} \\ &= 0 \end{aligned}$$

So $\mathbf{z} \perp \mathbf{v}$.
Theorem:

$$|\mathbf{z}|^2 + |proj_{\mathbf{v}}\mathbf{u}|^2 = |\mathbf{u}|^2$$
$$|\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \cdot \mathbf{v}|^2 + |\mathbf{z}|^2 = |\mathbf{u}|^2$$
$$|\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}|^2 \cdot |\mathbf{v}|^2 + |\mathbf{z}|^2 = |\mathbf{u}|^2$$
$$\frac{(\mathbf{u} \cdot \mathbf{v})^2}{\mathbf{v} \cdot \mathbf{v}} + |\mathbf{z}|^2 = |\mathbf{u}|^2$$
$$(\mathbf{u} \cdot \mathbf{v})^2 + |\mathbf{z}|^2|\mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2$$
$$\sqrt{(\mathbf{u} \cdot \mathbf{v})^2 + |\mathbf{z}|^2|\mathbf{v}|^2} = |\mathbf{u}||\mathbf{v}|$$

Since $|\mathbf{z}|^2|\mathbf{v}|^2$ is always positive, we know that:

$$\sqrt{(\mathbf{u} \cdot \mathbf{v})^2} \leq \sqrt{(\mathbf{u} \cdot \mathbf{v})^2 + |\mathbf{z}|^2|\mathbf{v}|^2} = |\mathbf{u}||\mathbf{v}|$$
$$|\mathbf{u}||\mathbf{v}| \geq |\mathbf{a} \cdot \mathbf{v}| \quad \square$$

1.1.2 Proof 2

Take

$$(a_1 z + b_1)^2 + (a_2 z + b_2)^2 + (a_3 z b_3)^2$$
$$= (a_1^2 + a_2^2 + a_3^2)^2 z^2 + 2z(a_1 b_1 + a_2 b_2 + a_3 b_3) + (b_1^2 + b_2^2 + b_3 2^2)$$

This is a quadratic that is always positive, meaning that it cannot have 2 real roots, so its discriminant is ≤ 0. We can rewrite this quadratic as:

$$= |\mathbf{a}|^2 z^2 + 2(\mathbf{a} \cdot \mathbf{b})z + |\mathbf{b}|^2$$

The discriminant of this quadratic is:

$$(2\mathbf{a} \cdot \mathbf{b})^2 - 4|\mathbf{a}|^2|\mathbf{b}|^2 \leq 0$$

This can be simplified down to the inequality we want:

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}| \cdot |\mathbf{b}| \quad \square$$

2 14.5

A line is defined by two points, or a point and a slope. Given the equations

$$x = 1 + 2t$$

$$y = 1 + 3t$$

We can generate a single equation for the line using vectors:

$$\begin{aligned}\langle x, y\rangle &= \langle 1+2t, 1+3t\rangle \\ &= \langle 1, 1\rangle + \langle 2t, 3t\rangle \\ \langle x, y\rangle &= \langle 1, 1\rangle + t\langle 2, 3\rangle\end{aligned}$$

Where $\langle x, y\rangle$ is written as $\mathbf{r}$, and the vector/point $\langle 1, 1\rangle = \mathbf{r_o}$, and $\langle 2, 3\rangle = \mathbf{a}$.
We have 2 methods of representing the equation of a line, parametric equations and the vector equation. However, a third form, symmetric form, can be generated by eliminating the parameter (solving the parametric equations for t and then setting them equal to each other):

$$t = \frac{x-1}{2}$$

$$t = \frac{y-1}{3}$$

$$\frac{x-1}{2} = \frac{y-1}{3}$$

We can generalize these forms for n dimensions easily. For parametric, we simply add the z equation, $z = z_o + ta_3$. For the vector form, the vectors in the equation simply increase in dimension, leaving the form of the equation the same. For the symmetric, we add a new equality, $\frac{z-z_0}{d_3}$. Consider the Earth at $(0, 5, 0)$, moving in the direction $\langle 0, -2, 0\rangle$. The line describing the Earth's motion is

$$\langle 0, 5, 0\rangle + t\langle 0, -2, 0\rangle$$

Consider an asteroid moving towards the Earth at location $(7, 0, 0)$, moving in direction $\langle -1, 0, 0\rangle$. The line describing the asteroids motion is

$$\langle 7, 0, 0\rangle + t\langle -1, 0, 0\rangle$$

When do they intersect?

$$\langle 0, 5, 0\rangle + t\langle 0, -2, 0\rangle = \langle 7, 0, 0\rangle + t\langle -1, 0, 0\rangle$$

Solving for t in a certain direction gives us the following:

$$0 = 7 - t$$

$$t = 7$$

They intersect! The Earth is destroyed at time $t = 7$! (Hint: not actually). If we solve in the y direction:

$$5 - 2t = 0$$

$$t = \frac{5}{2}$$

What? They don't collide? But what's going on? The two lines should obviously intersect?
The issue occurs because we are using the same parameter for both equations. To fix this, we use s instead of t for the second equation:

$$\langle 7, 0, 0\rangle + s\langle -1, 0, 0\rangle$$

We set them equal again and solve in components:

$$0 + 0t = 7 - s$$
$$s = 7$$

In the y:

$$5 - 2t = 0 + 0s$$
$$t = \frac{5}{2}$$

We know now that at some point, the x and y coordinates will match for both equations. However, we are neglecting the z direction, but in this case we don't have to worry because both sides are always 0, so the Earth is doomed (yay!).
In general, when solving the system of equations, if the system is not solvable, the two lines never intersect. If both t and s exist, but are not equal, the lines **intersect** (they're in the same place at different times). If $t = s$, then the lines **collide** (same place same time).

3 14.6

When it comes to vectors, we have a new way of thinking about planes. If we have two nonparallel and nonzero vectors, we can create a plane. However, we can also make a plane with only a single vector, a normal vector. If given a plane known to be normal to the plane, and the point at which it is normal, we can define a unique plane.
Let there be a point denoted P_o, with coordinates (x_o, y_o, z_o). Given a normal vector, the plane is the locus of all points satisfying the equation of the plane. How do we get the equation of the plane? To take a dot product, we would need a second vector, but we only have a point. However, we have an infinite set of points on the plane. If we choose an arbitrary point (x, y, z), the point is on the plane if the vector formed by (x, y, z) and (z_o, y_o, z_o) is perpendicular to $\mathbf{r_o}$. Formally:
For any $\langle x, y, z\rangle$ s.t:

$$\langle x, y, z\rangle - \mathbf{r_o} \perp \mathbf{n}$$

where $\mathbf{n}$ is the normal vector. This can also be written as:

$$(\mathbf{r} - \mathbf{r_o}) \cdot \mathbf{n} = 0$$

This is known as vector form We can also write it in the planar version of point-slope form (known as standard form):

$$a(x - x_o) + b(y - y_o) + c(z - z_o) = 0$$

All planes have an equation of this form, but does every equation of ths form define a plane? If $\mathbf{n} = \mathbf{0}$, the equation is not a plane. Other than this special case, these steps are all reversible, so we can work backwards to the dot product definition of a plane from any of these equations. Therefore we can state that all equations of this form (except the weird case) defines planes in $\mathbb{R}^3$. We know that two planes are parallel if their normal vectors are parallel. Unless you go higher than 3-space, skew planes don't exist. To find the angle between two planes, simply find the angle between their normal vectors, so use the vectors in place of the planes themselves. By convention we use the smaller angle between the two planes (the angle less than $\frac{\pi}{2}$).
There are 6 distances that we are working with:

3.1 Point to Point

The distance between point P_o and P_1 can be given by the Pythagorean theorem, which can also be done via the magnitude of the vector $\mathbf{P_oP_1}$.

3.2 Plane to Plane

Lets assume we have 2 parallel planes (otherwise the distance is 0). What we do is pick a point on one plane, and travel along the normal vector until we strike the other plane. This will be the shortest distance from the point chosen and the other plane. This essentially reduces the problem to the distance from a point to a plane.

3.3 Point to Plane

3.3.1 Method 1

We pick a point on the plane that satisfies the following conditions:

1. It is on the plane
2. It is on the line $P_o + t \cdot \mathbf{n}$ (the equation of the line following the normal vector to the point not on the plane).

and then we use point to point to solve for the distance. However, this is a pretty algebra-heavy, so we have another method.

3.3.2 Method 2

Choose any point on the plane P_1, and project it onto the normal vector of $\mathbf{P_oP_1}$:

$$d = |proj_{\mathbf{n}}\mathbf{P_oP_1}|$$

3.4 Point to Line

We have a point P_o, and a line l. We want to find the distance from P_o to the closest point on l.

3.4.1 Method 1 (Don't do this)

The distance to an arbitrary point is defined by:

$$d = \sqrt{((x_1 + at) - x_o)^2 + (yx_1 + bt) - y_o)^2 + ((z_1 + ct) - z_o)^2}$$

We now need to minimize this distance, and we use a neat trick and simply minimize the argument to the square root, which is easier than messing with the whole function.

3.4.2 Method 2

We pick and arbitrary point on this line P_1, and we project it:

$$proj_{\mathbf{a}}\mathbf{P_oP_1}$$

where $\mathbf{a}$ is the direction vector of l. This forms a right triangle, and we can then use the Pythagorean Theorem to find the magnitude of the unknown leg, which is the one that we want.

3.4.3 Method 3

We can also determine a point P_1 such that the point is on the line and the vector formed between the point and P_o is orthogonal to the line.

$$\mathbf{a} \cdot \mathbf{P_1 P_o} = 0$$

We then know that this is the shortest vector possible, and we cna now just find the magnitude.

3.5 Line to Plane

We assume that the line is parallel to the plane (can be checked by dotting the normal to the plane and the direction vector of the line). We can pick a point on the line, and then the distance boils down to point to plane.

3.6 Line to Line

If they intersect, the distance is 0. If they are parallel, it boils down to point to line. If they are skew however, we have an issue.
One nice fact is that given any 2 skew lines, we can generate a plane along each line such that the two planes are parallel.
Suppose we have two lines:

$$l_1 = P_o + \mathbf{a}t$$
$$l_2 = P_1 + \mathbf{b}t$$

Let $\mathbf{n} = \mathbf{a} \times \mathbf{b}$ (orthogonal to both $\mathbf{a}$ and $\mathbf{b}$). This allows to make two planes with the same normal vector. These planes are then guaranteed to be parallel to each other. This now becomes plane to plane, which is itself just point to plane.

3.7 Some Plane Stuff

When two planes intersect, they form a line, whose equation we wish to find. We can easily find the point $(x_o, y_o, 0)$ at which this line strikes the xy plane, or any other of the planes we know (it has to be on both planes). If this point exists, we can define a line:

$$L = (x_0, y_0, 0) + \mathbf{a}t$$

where $\mathbf{a} = \mathbf{n_1} \times \mathbf{n_2}$.
Here's a formula method for plane to plane distance: let $(x_1, y_1, z_1) \in P_1$, $(x_2, y_2, z_2) \in P_2$, and $\mathbf{n}$ be the normal vector for plane P_1.

$$\text{comp}_{\mathbf{n_1}} \mathbf{P_1 P_2} = |\frac{\mathbf{P_1 P_2} \cdot \mathbf{n}}{|\mathbf{n_1}|}| = |\frac{a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1)}{\sqrt{a^2 + b^2 + c^2}}| = |\frac{ax_2 + by_2 + cz_2 - d}{\sqrt{a^2 + b^2 + c^2}}|$$

4 14.7

What is a cylinder?
A cylinder is defined as the extrusion of a planar curve c in some direction $\mathbf{a}$.
This is a very stupid definition, since it implies that planes and lines are cylinders, and the curve does not need to be closed. This is the formal definition of a cylinder:

$$\{c + t\mathbf{a} | c \in C\}$$

This tells us something interesting, cylinders are just infinite collections of lines.

5 15.1

$$\mathbb{R} \to \mathbb{R}^3$$

This function has an input t, and output $\mathbf{r}(t)$

$$t \mapsto \mathbf{r}(t)$$

This can be written as:

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

Looking at $\mathbf{r}(1)$, it is both a point in 3-space, and is also a vector.
We define a curve $\mathbf{r}(t)$:

$$\mathbf{r}(t) = \{\langle x, y, z \rangle | x = f(t), y = g(t), z = h(t)\}$$

We can restrict the domain:

$$t \in [a, b]$$

Doing this, we can look at the endpoints of the function:

$$\mathbf{r}(a), \mathbf{r}(b)$$

A curve is defined as **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$. Geometrically there is an issue with this sometimes, as a shape can look closed but algebraically the two endpoints don't match (ex. a circle is visually closed but the enpoints don't always match up).
A curve is **simple** if:

$$\forall\, t_1, t_2 \in (a, b), \mathbf{r}(t_1) \neq \mathbf{r}(t_2)$$

In English, the curve doesn't cross itself.

5.1 Arclength

To find the arclength, we take a very small section of the curve, from $\mathbf{r}(t_i)$ to $\mathbf{r}(t_{i+1})$. From the first point to the second point, there is some difference in the x,y, and z directions, called Δx_i, Δy_i, and Δz_i. To approximate the arclength of this section, we assume that the curve is locally linear and that we can approximate it with a line. This makes the arclength

$$\Delta S_i \approx \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}$$

We know that the actual arclength will end up as an integral, meaning that there has to be a dt somewhere. So what we do is multiply $\frac{\Delta t_i}{\Delta t_i}$ onto the approximation we have.

$$\Delta S_i \approx \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}\frac{\Delta t_i}{\Delta t_i}$$

The top Δt_i turns into the dt that we want, and the bottom Δt_i we stick into the square root and distribute it to all of the terms:

$$\Delta S_i \approx \sqrt{(\frac{\Delta x_i}{\Delta t_i})^2 + (\frac{\Delta y_i}{\Delta t_i})^2 + (\frac{\Delta z_i}{\Delta t_i})^2}\Delta t_i$$

As we take smaller and smaller timesteps, we end up with:

$$\Delta S_i \approx \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} \Delta t_i$$

This gives us the arclength formula:

$$S = \int_a^b \sqrt{(x')^2 + (y')^2 + (z')^2}\, dt$$

If we look at this, this looks similar to the formula for 2d arclength, and is also the magnitude of a vector, in this case the $\mathbf{r}'$ vector. The derivative of $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ Take the curve:

$$\mathbf{r}(t) = \langle \sin^2(\frac{\pi t}{2}), \sin^2(\frac{\pi t}{2}), \sin^2(\frac{\pi t}{2}) \rangle$$

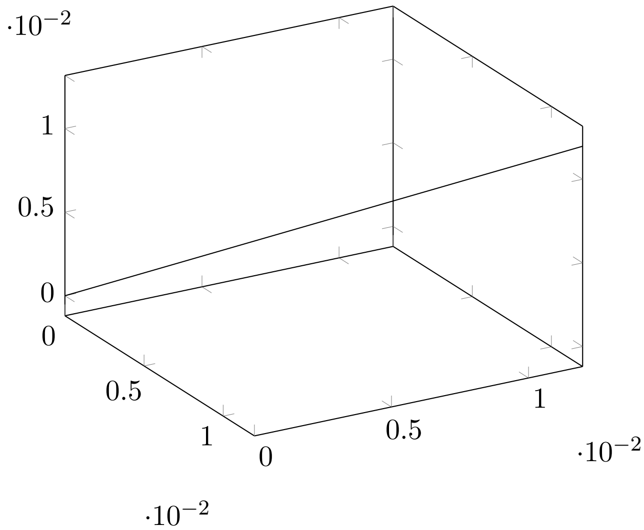
from $t : 0 \to 4$. Calculate the arclength.

$$S = \int_0^4 \sqrt{3(\frac{d}{dt}(\sin^2(\frac{\pi t}{2})))^2}\, dt$$

$$S = \sqrt{3} \int_0^4 \sqrt{(2 \sin \frac{\pi t}{2} (\cos(\frac{\pi t}{2})) \frac{\pi}{2})^2}\, dt$$

$$S = \sqrt{3} \int_0^4 \pi \sin(\frac{\pi t}{2}) \cos(\frac{\pi t}{2})\, dt$$

If we keep going from here, we find that the integral equals 0, which makes no sense, since it means that the curve goes nowhere, which we know is not true. This problem occurred because we took the square root of a square, which means that we should have split into cases. There are a couple methods of fixing this, one of which is to simply use absolute values, which leads to splitting the integral into multiple integrals. The other method is to notice symmetry in the original function itself. Plotting the function, we find that graph goes back and forth between two points 4 times, meaning that we can compute one integral (from 0 to 1) and then multiply the result by 4.


$\cdot 10^{-2}$
1
0.5
0
0
0.5
1
0
0.5
1
$\cdot 10^{-2}$
$\cdot 10^{-2}$

This brings us to the 4th property of a vector-valued function, the "doubling back" of the function, which caused the issue in our integral.

A curve is **smooth** iff for $\mathbf{r} = \langle f, g, h \rangle$, f', g', and h' all exist, and $\neg\exists t \in (a, b)$ s.t. $f'(t) = g'(t) = h'(t) = 0$.

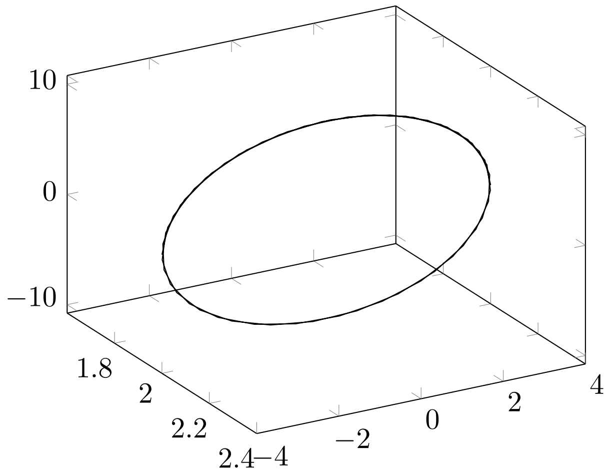
Essentially, if all 3 derivatives are never 0 at the same time. However, this is a very strong condition, as if the motion stops and then begins moving again in the same direction with doubling back, the integral we use would still work, even though the curve is not smooth. We could define being smooth at a point, but its more interesting to talk about smoothness on an interval.
Problem Not 7:
Sketch

$$\langle 2, 4\cos t, 9\sin t \rangle$$

for $t \geq 0$. We know the x won't change, so we shift the origin to $(2, 0, 0)$. We see a sin and a cos, so we know we'll get something sorta like a circle:

$$(\frac{y}{4})^2 + (\frac{z}{9})^2 = 1$$

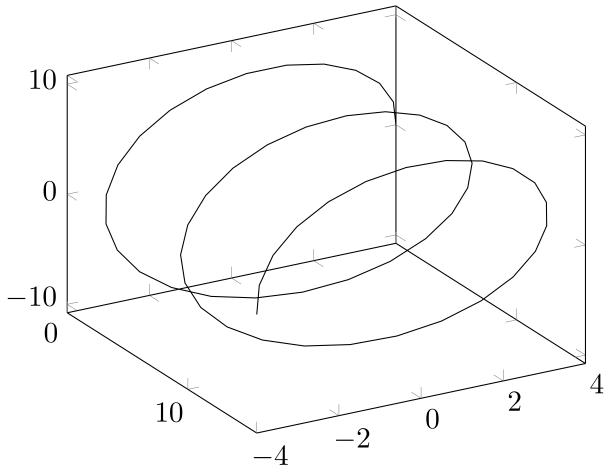
This is an ellipse, so we can graph it:


10
0
−10
1.8
2
2.2
2.4
−4
−2
0
2
4

Problem 7:
Sketch

$$\langle t, 4\cos t, 9\sin t\rangle$$

We know that looking at this head on will be an ellipse, but the fact that the x value is t means that we'll have something sorta like a helix:


10
0
−10
0
10
−4
−2
0
2
4

6 15.2

Lets look at a limit:

$$\lim_{t\to c}\mathbf{r}(t)$$

If we look at a regular limit:

$$\lim_{t\to c}x(t)=L$$

This means that as t goes to c, $x(t)$ approaches L. In epsilon-delta form,

$$\forall\epsilon>0\,\exists\delta\, s.t.\, 0<|t-c|<\delta\to|x(t)-L|<\epsilon$$

Consider $B_\epsilon(L)$ (a ball of radius ϵ around L). Remember that balls are like spheres, but filled in. In 2 dimensions, a ball is simply a disk, a filled in circle.

$$\forall\epsilon>0\,\exists\delta\ \text{ s.t. } t\in B_\delta(c)\setminus\{c\}\to x(t)\in B_\epsilon(L)$$

Going back to the original vector function:

$$\lim_{t\to c}\mathbf{r}(t)=\mathbf{L}$$

$$t\in B_\delta(c)\setminus\{c\}\to\mathbf{r},(t)\in B_\delta(\mathbf{L})$$

$$0<|t-c|<\delta\to|\mathbf{r}(t)-\mathbf{L}|<\epsilon$$

Limits work exactly the same! We could also have just defined this:

$$\lim_{t\to c}\mathbf{r}(t):=\langle\lim_{t\to c}x,\lim_{t\to c}y,\lim_{t\to c}z\rangle$$

This is essentially componentwise limits. This is great! since if limits work pretty much the same way, it means that derivatives should also work the same way!

$$\frac{d}{dt}\mathbf{r}(t):=\lim_{\Delta t\to 0}\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}$$

If we split this componentwise:

$$\langle x',y',z'\rangle$$

This means that we are doing Analysis 1 stuff 3 times. This componentwise stuff also spreads to continuity, meaning that if all 3 subfunctions are continuous, the whole thing is continuous.

Integrals are just limits of Riemann Sums, and sums are componentwise, meaning that:

$$\int\mathbf{r}(t)\,dt:=\lim\sum\mathbf{r}(t_i*)\Delta t_i$$

$$=\langle\int x\,dt,\int y\,dt,\int z\,dt\rangle$$

If we look at the derivative of a sum:

$$\frac{d}{dt}(\mathbf{u}(t)+\mathbf{v}(t))=\langle\frac{d}{dt}(x_1(t)+x_2(t)),\frac{d}{dt}(y_1(t)+y_2(t)),\frac{d}{dt}(z_1(t)+z_2(t))\rangle$$

This is just the sum of the derivatives, showing that derivative rules are the same.

$$\frac{d}{dt}(\mathbf{u}(t)+\mathbf{v}(t))=\frac{d}{dt}\mathbf{u}(t)+\frac{d}{dt}\mathbf{v}(t)$$

7 15.4

7.1 Unit Tangent Vector

We know that for any curve, we can define a unit tangent vector $\mathbf{T}$, which we know is parallel to the derivative of the curve at the point.

$$\mathbf{T} \parallel \mathbf{r}'(t)$$

$$\boxed{\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}}$$

If the vector is always parallel to the derivative, this means that if we reparameterize the function $\mathbf{r}(t)$, we get a tangent vector that points in the direction opposite to the original tangent vector.

The unit tangent vector is invariant to parameterization, meaning that the unit tangent will only differ by up to a $\pm$ issue if we use different parameters.

When we transfer between different parameters, we are just substituting in one parameter for another:

$$\mathbf{r_1}(u) = \langle 2u, 4u \rangle,\ \mathbf{r_2}(t) = \langle t, 2t \rangle$$

We can rewrite $\mathbf{r_2}$ in terms of u by just setting $t = 2u$:

$$\mathbf{r_1}(u) = \mathbf{r_2}(2u)$$

This is essentially function composition.

If $\mathbf{r_1}(t)$ and $t = f(u)$, we can say that $\mathbf{r_1}(f(u)) = \mathbf{r_2}(u)$. If we want to take the derivative:

$$\mathbf{r_2}'(u) = \frac{d}{du}(\mathbf{r_2}(u)) = \frac{d}{du}(\mathbf{r_1}(f(u))) = \mathbf{r_1}'|_{f(u)} \cdot f'(u)$$

Notice that the $\mathbf{r_1}'$ is with respect to t, and the $f'(u)$ is with respect to u. The substitution $t = f(u)$ is known as a reparameterization. We also know that $\mathbf{r_2}'(u)$ is parallel to $\mathbf{T}(u)$, and $\mathbf{r_1}'|_{f(u)}$ is parallel to $\mathbf{T}(t)$. This means that the difference between tangent vectors occurs only if $f'(u)$ is negative. Parameterization is essentially a particle moving in the same path in different ways, and if the parameterization is negative, the particle is moving backwards.

$$\boxed{\mathbf{T}(t_1)|_p = \mathbf{T}(t_2)|_p}$$

For $t_1 = f(t_2),\ f' > 0$.

Lets look at vectors orthogonal to $\mathbf{T}$. We want to find a unit vector orthogonal to $\mathbf{T}$. We can't use $\mathbf{N} = \frac{\mathbf{r}''}{|\mathbf{r}''|}$, since it is not necessarily orthogonal to the tangent vector. We know the nice fact that a vector is orthogonal to its derivative if the vector is of constant magnitude. This means that we can define the unit normal vector:

$$\boxed{\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}}$$

This vector is guaranteed to be orthogonal to $\mathbf{T}$.

7.2 2D Curvature

Take a curve on a plane. The curvature of a line should indicate how "curvy" the curve is. We know that something that is definitely not curvy is a line, so we need something that is 0 for a straight line, but not otherwise. We take a look at the derivative of the curve at 2 different points. If we are on a line, we expect the derivatives to be the same, but on a curve, they will differ. We can measure how curvy something is by finding the difference in the angle between the tangent vectors and $\hat{i}$. We define curvature as a scalar K:

$$K = |\frac{d}{d?}\theta|$$

What are we taking the derivative in relation to? We can't use t because the curvature shouldn't care about the amount of time it takes for the angle to change (looking at a rollercoaster, two parts could have the same curvature, but one will be weighted higher if the Δt was only 1 second, versus 20 seconds), and it should be about the shape itself. We need a parameter that is uniform throughout the entire shape. Thus, we use arclength.

$$\boxed{K = |\frac{d}{dS}\theta|}$$

Bad News: This is a terrifying reparameterization.
Good News: There's other ways to do it.
Pushing ahead:

$$S(t) = \int_{t_0}^{t} \sqrt{(f')^2 + (g')^2}\, dt$$

Taking the derivative:

$$\frac{dS}{dt} = \sqrt{(f')^2 + (g')^2} = |\mathbf{r}'(t)|$$

We now do something weird. We reparameterize arclength to be a function of itself.

$$S(S) = \int_{S_0}^{S} \sqrt{(f'(S))^2 + (g'(S))^2}\, dS$$

taking the derivative:

$$\frac{dS}{dS} = \sqrt{(f'(S))^2 + (g'(S))^2}$$

$$1 = |\mathbf{r}'(S)|$$

This is a vector that is tangent to curve and has magnitude 1. There is another vector that we know that follows these conditions, $\mathbf{T}$. This tells us:

$$\mathbf{r}'(S) = \mathbf{T}$$

Note that the left side is using S as its parameter, while $\mathbf{T}$ can use any parameter we want. The positive/negative issue drops out due to the absolute value on the outside of the formula.

What we hope is that there is a relationship between the angle θ and $\mathbf{T}$. We can generate a right triangle wth angle θ and hypotenuse 1. This gives the other sides as $\cos\theta$ and $\sin\theta$. We can generate another triangle, with hypotenuse $\mathbf{T}$. The two other legs of the triangle are just the components of $\mathbf{T}$. Since these two triangles are the same, we know that $\mathbf{T}(\theta) = \langle \cos\theta, \sin\theta \rangle$. This sorta sucks, since we don't want to use θ. However, this isn't an issue, since we can reparameterize $\mathbf{T}$ in terms of whatever we want. Since we want to find $\frac{d\theta}{dS}$, we just take the derivative!

$$\frac{d}{dS}\mathbf{T}(S) = \mathbf{T}'(S) = \frac{d}{dS}\langle \cos\theta, \sin\theta\rangle = \langle -\sin\theta\frac{d\theta}{dS}, \cos\theta\frac{d\theta}{dS}\rangle$$

This leaves us with

$$\mathbf{T}'(S) = \frac{d\theta}{dS}\langle -\sin\theta, \cos\theta\rangle$$

How do we get rid of that vector? We take the magnitude!

$$|\mathbf{T}'(S)| = |\frac{d\theta}{dS}| \cdot \sqrt{(-\sin\theta)^2 + (\cos\theta)^2}$$

$$\boxed{|\mathbf{T}'(S)| = |\frac{d\theta}{dS}|}$$

This gives us another formula:

$$\boxed{K = |\frac{d}{dS}\mathbf{T}|}$$

Good News: This is better
Bad News: We don't have the parameterization in terms of S!

Lets try a different method. Suppose the curve is in terms of x and y.

$$S(x) = \int_{x_0}^{x} \sqrt{1 + (f')^2}\, dx$$

We also know the following:

$$\frac{d\theta}{dS} \cdot \frac{dS}{dx} = \frac{d\theta}{dx}$$

We can find the second term via FTC, and we want the first term. We also need to find the last term, so we need a way to relate x and θ.

$$\frac{dS}{dx} = \sqrt{1 + (f')^2}$$

This gives us the first of the quantities that we want.

To find $\frac{d\theta}{dx}$, we can generate a triangle with the hypotenuse being a tangent vector at any point on the curve, giving us the following relationship:

$$\tan\theta = y'$$

$$\theta = \arctan(y')$$

$$\frac{d\theta}{dx} = \frac{1}{1 + (y')^2}y''$$

Plugging in the pieces that we have just found:

$$\frac{d\theta}{dS} = \frac{y''}{(1 + (y')^2)^{\frac{3}{2}}}$$

Since curvature is given by the absolute value of $\frac{d\theta}{dS}$:

$$\boxed{K := \frac{|y''|}{(1 + (y')^2)^{\frac{3}{2}}}}$$

This definition looks really bad, but it is much easier than the others.

We can also use a parameterized version of it:

$$\frac{d\theta}{dS}\frac{dS}{dt}$$

Generating a similar triangle as before, we find a slightly more complicated formula for θ:

$$\tan\theta = \frac{g'}{f'}$$

$$\theta = \arctan(\frac{g'}{f'})$$

$$\frac{d\theta}{dt} = \frac{1}{1 + (\frac{g'}{f'})^2}\frac{f'f'' - f'g''}{(f'^2)}$$

$$\frac{d\theta}{dt} = \frac{f'g'' - g'f''}{f'^2 + g'^2}$$

Solving for the other bit:

$$S = \int_{t_o}^{t}\sqrt{f'^2 + g'^2}$$

$$\frac{dS}{dt} = \sqrt{f'^2 + g'^2}$$

Plugging these values back in:

$$\boxed{K = |\frac{f'g'' - g'f''}{(f'^2 + g'^2)^{\frac{3}{2}}}|}$$

We see that if we have y as a function of x, this new equation agrees with the previous equation, which is nice.

Doing some dimensional analysis on this new equation:

$$\frac{\frac{m}{s}\frac{m}{s^2} - \frac{m}{s}\frac{m}{s^2}}{(\frac{m^2}{s^2})^{\frac{3}{2}}}$$

Cancelling things out:

$$\frac{1}{m}$$

This tells us that curvature has units of inverse length, which agrees with $\frac{d\theta}{dS}$.

Consider a circle of the form $\mathbf{r}(t) = \langle r\cos t, r\sin t\rangle$.

$$K = \frac{|r\cos t(-r\cos t) - (-r\sin t)(-r\sin t)|}{((-r\sin t)^2 + (r\cos t)^2)^{\frac{3}{2}}}$$

$$K = \frac{|-r^2|}{(r^2)^{\frac{3}{2}}}$$

$$K = \frac{1}{r}$$

This makes sense, as circles have constant curviness, and larger circles are less curvy than smaller circles (Think about the Earth being huge and locally flat). We use this to define a **circle of curvature**, which is an alternative to using the tangent to the curve to model the curve. We instead use a circle with the same curvature as the curve at that point. This also defines the **center of the circle of curvature**, which is, as the name implies, the center of the approximation circle. The tangent direction and the path towards the center of the circle of curvature are orthogonal. Think of circles of curvature to be "circles of best fit".

8 15.5

Now we want to discuss 3-space. Most of it can be generalized, but the definition of θ doesn't really exist anymore, as a curve could be very curvy and yet be at an angle of 0 with $\hat{i}$. Not having a θ also breaks the parameterized version, as it is derived from θ. However, one form still works:

$$K := |\frac{d}{dS}\mathbf{T}|$$

We have moved the colon, as we find that if we are in 2D, our previous statement is indeed correct. However, the above equation holds true for all dimensions, not just 2D. This method is still a really painful method of computing curvature, so 15.5 is about finding a better way. However this involves some stuff that seems sort of random.
Note that the scalar v denotes the speed, the magnitude of $\mathbf{v}$. This is also equivalent to $\frac{dS}{dt}$. We will use these interchangeably.
We start with the unit tangent vector:

$$\mathbf{T} = \frac{\mathbf{r}(t)}{|\mathbf{r}(t)|}$$

$$\mathbf{r}'(t) = |\mathbf{r}'(t)|\mathbf{T}(S)$$

Taking the derivative to get acceleration:

$$\mathbf{r}''(t) = \frac{dv}{dt}\mathbf{T}(S) + \frac{dS}{dt}\frac{d}{dt}(\mathbf{T}(S))$$

To compute the second term, we use the chain rule:

$$\mathbf{r}''(t) = \frac{dv}{dt}\mathbf{T}(S) + v^2\mathbf{T}'(S)$$

Using the definition of $\mathbf{N}$:

$$\mathbf{r}''(t) = \frac{dv}{dt}\mathbf{T}(S) + v^2|\mathbf{T}'(S)|\mathbf{N}(S)$$

We see that part of the second term is simply the definition of curvature:

$$\mathbf{r}''(t) = \frac{dv}{dt}\mathbf{T}(S) + v^2K\mathbf{N}(S)$$

Looking at the curvature of a circle, we know it is $\frac{1}{r}$, but we define the circle's radius to be ρ:

$$\boxed{\mathbf{r}''(t) = \frac{dv}{dt}\mathbf{T}(S) + \frac{v^2}{\rho}\mathbf{N}(S)}$$

Think of a car on an exit ramp on the highway. You are both slowing down and turning, two types of acceleration. Slowing down is a change in speed, $\frac{dv}{dt}$, and turning is given by $\frac{v^2}{r}$. We see both of these in the above equation (the second one is $\frac{v^2}{\rho}$ because the radius is ρ). We have this expression for the acceleration of a particle:

$$\mathbf{r}''(t) = \frac{dv}{dt}\mathbf{T}(S) + \frac{v^2}{\rho}\mathbf{N}(S)$$

The two terms in this expression are $a_t\mathbf{T}$ and $a_n\mathbf{N}$, respectively. We want to find ways to get the value of these two terms. If we take the magnitude of this equation, we get the scalar acceleration a, and we cna find the magnitude of the right hand side by dotting it with itself and taking the square root:

$$a = \sqrt{(a_T\mathbf{T} + a_N\mathbf{N}) \cdot (a_T\mathbf{T} + a_N\mathbf{N})}$$

Squaring both sides and evaluating:

$$a^2 = a_T^2\mathbf{T} \cdot \mathbf{T} + 2a_T a_N\mathbf{T}\mathbf{N} + a_N^2\mathbf{N}\mathbf{N}$$

The second term is 0 due to $\mathbf{T} \cdot \mathbf{N} = 0$ (orthogonality). $\mathbf{T} \cdot \mathbf{T} = 1$, and same for $\mathbf{N}$, leaving us with:

$$\boxed{a^2 = a_T^2 + a_N^2}$$

This makes sense, since we can get this via vector addition of $\mathbf{T}$ and $\mathbf{N}$.

We would now like to get rid of one of these terms using a dot product, and we also want to dot it with something that is easy to get. $\mathbf{N}$ and $\mathbf{T}$ are sort of hard to get, but $\mathbf{r}'$ is in the same direction as $\mathbf{T}$ and is much easier to get.

$$\begin{aligned}\mathbf{r}' \cdot \mathbf{r}'' &= \mathbf{r}' \cdot (a_T\mathbf{T} + a_N\mathbf{N}) \\ &= a_T\mathbf{r}' \cdot \mathbf{T} + a_N\mathbf{r}' \cdot \mathbf{N} \\ &= a_T|\mathbf{r}'||\mathbf{T}|\cos\theta\end{aligned}$$

We know that θ is 0, so we are left with:

$$\mathbf{r}' \cdot \mathbf{r}'' = a_T|\mathbf{r}'||\mathbf{T}|$$

$$\boxed{a_T = \frac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|}}$$

This is the part of the magnitude in the tangential direction. This is also essentially just a projection onto $\mathbf{T}$. We now have a method of getting the tangential term, and if we really wanted, we could just solve for the normal term, but instead we want to derive a better expression.

We use a cross product (note that this removes arbitrary dimensions):

$$\mathbf{r}' \times \mathbf{r}'' = \mathbf{r}' \times (a_T\mathbf{T} + a_N\mathbf{N})$$

$$= a_T(\mathbf{r}' \times \mathbf{T}) + a_N(\mathbf{r}' \times \mathbf{N})$$

This first term goes to 0 because $\mathbf{r}' \parallel \mathbf{T}$.

$$\mathbf{r}' \times \mathbf{r}'' = a_N(\mathbf{r}' \times \mathbf{N})$$

To make the left side a scalar, we take the magnitude:

$$|\mathbf{r}' \times \mathbf{r}''| = a_N|\mathbf{r}'|$$

$$\boxed{a_N = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|}}$$

We already know what a_N is:

$$a_N = \frac{v^2}{\rho} = v^2 K = |\mathbf{r}'|^2 K$$

$$\boxed{K = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}}$$

9 16.1

Multivariable functions!
Definition:

$$f : \mathbb{R}^n \mapsto \mathbb{R}$$

or:

$$f : D \mapsto \mathbb{R},\, D \leq \mathbb{R}^n$$

The function maps a value from an n-dimensional domain to the real numbers.

$$Range = \{f(\mathbf{x}) \,|\, \mathbf{x} \in dom(f)\}$$

We can plot multivariable functions that map from $\mathbb{R}^2$ to $\mathbb{R}$ in $\mathbb{R}^3$, with the domain being on the xy plane, and the z value is the value $f(x, y)$. Multivariable functions are surfaces in space. We can also use level curves to plot these graphs.

10 16.2

We start with some epsilon delta proofs:

$$\forall A > 0 \; \exists \delta \; s.t. \; 0 < |x - c| < \delta$$

To convert this to multivar, we don't treat the absolute value bars as absolute values, rather we just treat them as things that mean the function is close to c. In two dimensions:

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

or

$$\lim_{\mathbf{x}\to\mathbf{c}} \mathbf{x} = L$$

This maps from $\mathbb{R}^2$ to $\mathbb{R}$. The set of points that are less than δ away from the point (a,b) is given by a disk of radius δ. This will guarantee that we will land within ϵ of L. The limit definition becomes

$$\forall\epsilon > 0\ \exists\delta > 0\ \ s.t.\ \ \text{dist}(\mathbf{x},\mathbf{c}) < \delta \to \text{dist}(f(\mathbf{x}), c) < \epsilon$$

The second way to express this is:

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \to |f(\mathbf{x}) - C| < \epsilon$$

The third method is to find the vector $\mathbf{x} - \mathbf{c}$ and make sure that the magnitude of that vector is less than δ:

$$0 < |\mathbf{x} - \mathbf{c}| < \delta \to |f(\mathbf{x}) - L| < \epsilon$$

The fourth method is to solve for all vectors inside a ball of radius δ around $\mathbf{c}$, which maps to a ball around L of radius ϵ:

$$\mathbf{x} \in B_\delta(\mathbf{c}) \setminus \{\mathbf{c}\} \to f(\mathbf{x}) \in B_\epsilon(L)$$

Thing to note: In the case we want the output to multivariable (ch. 18), we can easily generalize these forms by adding a bunch of vector hats.
The definition of the multivariable limit is the same as it was before, so we expect limit rules to still hold:

$$\lim(f+g) = \lim f + \lim g$$

$$\lim(fg) = \lim f \lim g$$

$$\lim(kf) = k \lim f \text{ iff } k \in \mathbb{R}$$

Looking forward, the only thing with limits that doesn't work is L'Hopital's rule, which doesn't generalize to multiple dimensions.
Let's try to prove one of the identities:

$$\lim_{\mathbf{x}\to\mathbf{c}}(f+g) = \lim_{\mathbf{x}\to\mathbf{c}} f + \lim_{\mathbf{x}\to\mathbf{c}} g$$

Assume $\lim_{\mathbf{x}\to\mathbf{c}} f = K$ and $\lim_{\mathbf{x}\to\mathbf{c}} g = L$. This means:

$$\exists\delta_1 \text{ s.t. } 0 < |\mathbf{x} - \mathbf{c}| < \delta_1 \to |f(\mathbf{x}) - K| < \epsilon_1$$

$$\exists\delta_2 \text{ s.t. } 0 < |\mathbf{x} - \mathbf{c}| < \delta_2 \to |g(\mathbf{x}) - L| < \epsilon_2$$

We want to prove that

$$\exists\delta \text{ s.t. } 0 < |\mathbf{x} - \mathbf{c}| < \delta \to |(f+g) - (K+L)| < \epsilon$$

We want to split up the absolute value term so that we have terms that we can find in the two limits we already know. To do this, we use the triangle inequality:

$$|(f-K) + (g-L)| \le |f-K| + |g-L|$$

Let δ be the smaller of δ_1, δ_2. We know that

$$|(f-K) + (g-L)| \le |f-K| + |g-L| < \epsilon_1 + \epsilon_2$$

While this may not seem like what we want, we can choose the values of ϵ_1, ϵ_2 that make the sum $\epsilon_1 + \epsilon_2$ less than ϵ.
To compute limits, we first attempt to plug in the point (if the function is continuous at the point):

$$\lim_{(x,y)\to(1,2)} \frac{x-y}{x+y} = \frac{1-2}{1+2} = \frac{-1}{3}$$

Example 2:

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$$

Notice that we cannot plug it in since the function is not continuous at 0. What we do is take the limit from different directions, one of which is along the path $y = 0$:

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{x-y}{x+y} = \lim_{\mathbf{x}\to\mathbf{0}} \frac{x}{x}$$

Note that in prior math classes we noticed that selectively using limits (plugging in in only certain spots) was not allowed, i.e:

$$\lim_{x\to 0} \frac{x}{x} \neq \lim_{x\to 0} \frac{0}{x}$$

Computing our limit, we get:

$$\lim_{x\to 0} \frac{x}{x} = 1$$

We check this by approaching the point along the path $x = 0$:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(0,y)\to(0,0)} f(x)$$

$$= \lim_{y\to 0} \frac{0-y}{0+y} = -1$$

Since we got different answers, we know that the limit does not exist at the point $(0, 0)$.
Example 3:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Coming in along the $x = 0$ path, we get:

$$\lim_{y\to 0} \frac{0}{y^2} = 0$$

Using the second path, $y = 0$:

$$\lim_{x\to 0} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{0}{x^2} = 0$$

We did it! the limit is 0! Turns out, it isn't. There isn't just two paths, there are an infinite number of paths that approach the point. This one problem is the reason multivariable calculus is hard.
Let's try the path $y = x$:

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

This proves that the limit does not exist.
Example 4:

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{x^2y}{x^2+y^4}$$

On the curve where $x = 0$, we expect to get 0. On the curve where $y = 0$, we expect to get 0. On $y = x$, we expect to get 0 as well. The issue is that we have to check every single path of the form $y = mx$. Luckily, we can just check them all at once:

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{x^2 \cdot mx}{x^2 + m^4x^4} = \lim_{x\to 0} \frac{mx^3}{x^2 + m^4x^4} = \lim_{x\to 0} \frac{mx}{1 + m^4x^2} = 0$$

So we're good now! Right?
No.
We can't just check paths that are linear, we have to check every single other path in existence, including quadratic, polar, trig...
What do we do now?
Give up.
Generally, it is almost impossible to prove whether a multivariable limit exists, but we can prove that they don't exist. To do this, we just choose paths that will make things simple.
In single variable calculus, the limit $\lim_{x\to 0} \sqrt{x}$ can be considered either 0 or nonexistent (0 because the limit is taken on the domain and DNE because the left side has no limit). For multivar, we assume that the limit is restricted to the domain.
Some terms in the book that are important:

- Interior
- Boundary
- Closed
- Open

11 16.3

$$\begin{gathered}
\frac{d}{dx}[x^2] = 2x \\
\frac{d}{dx}[x^3] = 3x^2 \\
\frac{d}{dx}[x^4] = 4x^3 \\
\vdots \\
\frac{d}{dx}[x^n] = nx^{n-1}
\end{gathered}$$

We claim that this last one is actually multivariable calculus, because n is a variable but is treated as a constant, and that we can generate a multivariable function for it: $f(x, n) = x^n$. The derivative with respect to x of f is nx^{n-1}. The derivative with respect to n of f is $\ln(x)x^n$. We are treating the variable that we are differentiating with respect to as the only variable, and let everything else be a constant. This is known as a partial derivative:

$$\frac{\partial}{\partial x} f(x, n, a) = f_x(x, n, a) = \frac{\partial f}{\partial x} = D_x f$$

The formal definition:

$$\frac{\partial}{\partial x}g(x,y) := \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - g(x,y)}{\Delta x}$$

This can also be written as the derivative of the single variable function $h(x)$ if we hold one variable (y) constant, $h(x) = g(x, y_o)$.

$$\frac{\partial}{\partial x_i} f(\mathbf{x}) := \lim_{\Delta x_i \to 0} \frac{f(x_i, \ldots, x_{i-1}, x_i + \Delta x_i, \ldots, x_n) - f(x_i, \ldots, x_n)}{\Delta x_i}$$

We can take partial derivatives of different variables in order:

$$\frac{\partial}{\partial y}(\frac{\partial}{\partial x} f) = \frac{\partial^2}{\partial y \partial x} f$$

Or

$$\frac{\partial}{\partial y}(f_x) = (f_x)_y = f_{xy}$$

These may seem backwards, but you can think about it as starting from the one closest to the function. The difference between f_{xy} and f_{yx} is as follows:
Let

$$f(x,y) = 3xy^3 - \cos(x,y) + ye^x$$

$$f_{xy} = 9y^2 + \sin(xy) + xy\cos(xy) + e^x$$

$$f_{yx} = 9y^2 + xy\cos(xy) + \sin(xy) + e^x$$

11.1 Clairaut's Theorem (Equality of Mixed Partials)

Theorem 11.1 (Clairaut's Theorem)**.** *For $f(x,y)$ defined on domain D s.t. f_{xy} and f_{yx} are continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)\ \forall (a,b) \in D$.*

This generalizes to any number of variables.

Corollary 11.1.1. *If $f_{xy} \neq f_{yx}$ at point P, f is not defined at P (or near P, maybe) or f_{yx} or f_{xy} are not continuous at P.*

Remember that when you're taking multiple partials you can do them in whatever order you want. Look for terms that don't have any of the variables, which allows for us to just treat that term as a constant, leading to a 0 as the derivative.

12 16.4

Let $f(x,y)$ be defined as 0 at $(0,0)$ and $\frac{xy}{x^2+y^2}$ otherwise. To check whether this function is continuous at the origin, we use a limit:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

After trying the paths $x = 0$ and $y = 0$, we find that they are both 0. This doesn't really help us. Trying $y = x$, we get

$$\lim_{x \to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

We know that the limit doesn't exist because we checked some other paths, but even if we hadn't this tells us that we are not continuous, since it does not equal the value at $(0,0)$.
We can also define the partial with respect to x. We don't know what it will be at $(0,0)$, but we know what it will be everywhere else:

$$\frac{\partial}{\partial x}\frac{xy}{x^2+y^2} = y(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2})$$

To find the value of the partial at the point $(0,0)$, we know that $y = 0$, and for $x = 0$ or $x \neq 0$, the function will be 0. This means that when we take the limit:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x, 0) - f(x,0)}{\Delta x}|_{x=0}$$

Since the top is always 0, the value of the partial is 0:

$$f_x(0,0) = 0$$

However, this should be worrying, since both f_x and f_y exist, and so f must be differentiable. According to Analysis 1, this means that f is continuous right? But f is not continuous, so we need to either redefine the relationship between differentiability and continuity or redefine differentiable. The math gods have decided that we will redefine differentiability, and preserve the fact that differentiable functions are continuous. This motivates section 16.4.

Take a cube with dimensions x, y, and z. We know that the volume is given by

$$V = xyz$$

Extending each dimension slightly in one direction, adding on 3 slabs, 3 rods, and a cubelet. The change in volume is given by:

$$\Delta V = 3 \text{ slabs} + 3 \text{ rods} + 1 \text{ cubelet}$$

Let's put this example aside for a bit.
Take a function $f = f(x,y)$. The increment of w, $\Delta w := f(x+\Delta x, y+\Delta y) - f(x,y)$. For an independent variable $(x{,}y)$, $\Delta x =$ Change in x. We can generate a function that calculates the change:

$$\Delta w(x, y, \Delta x, \Delta y)$$

Or we could just do out the two computations and subtracting them. These aren't that fun, so we want to find another way to define an increment. Bad news, we're going to start with a really really bad method of incrementing things. Good news, we'll be able to generate a good approximation. Take a function $u = f(x)$. In single variable calculus, $\Delta u = f(x+\Delta x) - f(x)$. We also know another thing that uses the increment, the derivative:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

Notice that the LHS does not have any Δx in it, so we swing it to the other side (via limit rules):

$$0 = \lim_{\Delta x \to 0} (\frac{\Delta u}{\Delta x} - f'(x))$$

Let $\epsilon = \frac{\Delta u}{\Delta x} - f'(x)$. Notice, $\Delta x \to 0 \to \epsilon \to 0$. We can rewrite this equation:

$$\epsilon \Delta x = \Delta u - f'(x)\Delta x$$

$$\Delta u = f'(x)\Delta x + \epsilon \Delta x$$

Theorem 12.1. *For f_x and f_y cont. on an open box containing (x_o, y_o),*

$$\Delta w|_{(x_o,y_o)} = f_x(x_o, y_o)\Delta x + f_y(x_o, y_o)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

and $\epsilon_1 := \epsilon_1(\Delta x, \Delta y)$, $\epsilon_2 := \epsilon_2(\Delta x, \Delta y)$, and as $(x_o, y_o) \to (0, 0)$, $\epsilon_1, \epsilon_2 \to 0$.

Going back to the cube, we had three slabs:

$$xy\Delta z + xz\Delta y + yz\Delta x$$

three rods:

$$x\Delta y\Delta z + y\Delta x\Delta z + z\Delta x\Delta y$$

And a cubelet:

$$\Delta x\Delta y\Delta z$$

The function that we are incrementing is the volume:

$$V = xyz$$

The slabs, the most significant portion of the change, are represented by the partials in the increment formula, and the rods are represented by the epsilon terms.

Proof. Take a point (x_o, y_o) and increment it to $(x_o + \Delta x, y_o + \Delta t)$. We know the increment is given by:

$$\Delta w = f(x_o + \Delta x, y_o + \Delta y) - f(x_o, y_o)$$

However, we could also increment the coordinates in order, giving us a point $(x_o + \Delta x, y_o)$.

$$\Delta w = f(x_o + \Delta x, y_o + \Delta y) - f(x_o + \Delta x, y_o) + f(x_o + \Delta x, y_o) - f(x_o, y_o)$$

The first two terms are the path from the new point to the final point, and the last two terms are the path from the starting point to the intermediate point. Taking the first term, we can create a function $h(y) := f(x_o + \Delta x, y_o)$, which converts the first two terms into $h(y_o + \Delta y) - h(y_o)$. We can also do the same to the last two terms, defining a new function $g(x) := f(x, y_o)$, converting the last two terms into $g(x_o + \Delta x) - g(x_o)$. This converts the expression into something represented only by single variable functions.

$$\Delta w = h(y_o + \Delta y) - h(y_o) + g(x_o + \Delta x) - g(x_o)$$

By the MVT (h is cont. and diff. because f_y exists and $f_y = h'$, diff. implies cont.), $h(y_o + \Delta y) - h(y_o) = h'(v)\Delta y$, for some $v \in (y_o, y_o + \Delta y)$.

We now notice that taking the derivative of the h function is just the definition of the partial of f with respect to y. Similarly, we can rewrite the g function: $g(x_o + \Delta x) - g(x_o) = g'(u)\Delta x$, for some $u \in (x_o, x_o + \Delta x)$. This is also the definition of a partial, in this case the partial with respect to x. Rewriting our increment expression with these new expressions:

$$\Delta w = h'(u)\Delta y + g'(u)\Delta x$$

Rewriting in terms of f and its partials:

$$\Delta w = f_y(x_o + \Delta x, v)\Delta y + f_x(u, y_o)\Delta x$$

Recall that we wanted to show that $f_x(x_o, y_o)\Delta x + f_y(x_o, y_o)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$. Grouping the terms that are multiplied by Δx, we can equate the two expressions:

$$f_x(u, y_o) = f_x(x_o, y_o) + \epsilon_1$$

$$\epsilon_1 := f_x(u, y_o) - f_x(x_o, y_o)$$

The only thing we need to worry about is whether ϵ_1 goes to 0. In order for this term to go to 0, we need f_x to be continuous, which we have already assumed, so we're good. We can also group the terms that are multiplied by Δy, allowing us to define ϵ_2. Doing the same thing for ϵ_2, we find that it meets the properties of the theorem. This verifies the theorem, but does not really show the expression, only that it exists. □

Lets look at an example.

$$f = 2x^2 - xy^2 + 3y$$

$$\Delta f = 2(x + \Delta x)^2 - (x + \Delta x)(y + \Delta y)^2 + 3(y + \Delta y)$$

Expanding and cancelling:

$$\Delta f = -x(\Delta y)^2 - \Delta x(\Delta y)^2 + 3\Delta y$$

We can do it the other way too, taking the partials of f:

$$f_x = 4x - y^2$$

$$f_y = -2xy + 3$$

$$\Delta f = f_x\Delta x + f_y\Delta y + 2\Delta x^2 - x(\Delta y)^2 - 2y\Delta x\Delta y - \Delta x(\Delta y)^2$$

Option 1:

$$\epsilon_1 = 2\Delta x - 2y\Delta y - (\Delta y)^2$$

$$\epsilon_2 = -x\Delta y$$

Option 2:

$$\epsilon_1 = 2\Delta x$$

$$\epsilon_2 = -x\Delta y - 2y\Delta x - \Delta x\Delta y$$

Option 3:

$$\epsilon_1 = 2\Delta x - y\Delta y$$

$$\epsilon_2 = -x\Delta y - y\Delta x$$

A **differential** of an independent variable x is $dx := \Delta x$. For a dependent variable w, where $w = f(x, y)$, $dw = f_x(x, y)dx + f_y(x, y)dy$.
We can now also compute the difference between $\Delta w - dw$, and we find that:

$$\Delta w - dw = \epsilon_1\Delta x + \epsilon_2\Delta y$$

We claim that for small Δx and Δy, $\Delta w - dw \approx 0$.

$w = f(x, y)$ is **differentiable** if Δw can be written as

$$f_x(x_o, y_o)\Delta x + f_y(x_o, y_o)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

This also comes with all the baggage that we found with the first theorem (epsilons are functions, go to 0, etc...).

Theorem 12.2. *f_x, f_y continuous implies f differentiable.*

Proof. Big obnoxious first theorem. □

This leads us to the theorem that we have been moving towards this whole time.

Theorem 12.3. *If f is differentiable, f is continuous.*

Proof.

$$\Delta w = (f_x(x_o, y_o) + \epsilon_1)\Delta x + (f_y(x_o, y_o) + \epsilon_2)\Delta y$$

Let $x = x_o + \Delta x$, $y = y_o + \Delta y$.

$$\Delta w = f(\mathbf{x_o} + \Delta\mathbf{x_o}) - f(\mathbf{x_o})$$

This statement is qual to the other definition of Δw. If we take a limit as $(\Delta x, \Delta y) \to \mathbf{0}$, the first expression goes to 0, and so the second must also be 0.

$$\lim_{(\Delta x, \Delta y)\to\mathbf{0}} f(x_o + \Delta x, y_o + \Delta y) - f(x_o, y_o) = 0$$

Since the second term doesn't have a Δx or Δy, we are left with:

$$\lim_{(\Delta x, \Delta y)\to\mathbf{0}} f(x_o + \Delta x, y_o + \Delta y) = f(x_o, y_o)$$

By plugging in our definitions of x and y, we are left with:

$$\lim_{(\Delta x, \Delta y)\to\mathbf{0}} f(x, y) = f(x_o, y_o)$$

This is the definition of continuous that we wanted. □

Corollary 12.3.1. *If f_x and f_y continuous, f continuous.*

13 16.5

Recall the following from Analysis 1:

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

However there is the other way of writing it. Let $g(x) = u$. The chain rule now becomes:

$$\frac{df}{du}\frac{dg}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

This bothers people because they think we are differentiating with respect to a function, but its not an issue. Note: **These are not fractions!** (They just sorta look like fractions... and act like fractions...)

13.1 Chain Rule

Theorem 13.1 (Chain Rule)**.** *Take a function $w = f(u, v)$, and let w be differentiable. Take two other functions, $u = g(x, y)$ and $v = h(x, y)$, which have continuous first partial derivatives. What is $\frac{\partial w}{\partial x}$? (Note that this does not mean to take x as a constant on this level of the function, as that would give us 0 and that's boring)*
This breaks down into two parts:

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = w_u u_x + w_v v_x$$

This can sort of be explained by saying that x changes u which changes w and x changes v which changes w as well.

Proof.

$$\Delta w = f(g(x + \Delta x, y), h(x + \Delta x, y)) - f(g(x, y), h(x, y))$$
$$\Delta u = g(x + \Delta x, y) - g(x, y)$$
$$\Delta v = h(x + \Delta x, y) - h(x, y)$$

These come from the definition of an increment on a function. We can also represent Δw via an increment using u and v:

$$\Delta w = f(u + \Delta u, v + \Delta v) - f(u, v)$$

We can also represent it via another definition (definition of differentiable):

$$\Delta w = w_u \Delta u + w_v + \Delta v + \epsilon_1 \Delta u + \epsilon_2 \Delta v$$

Taking this definition, we can divide by Δx:

$$\frac{\Delta w}{\Delta x} = w_u \frac{\Delta u}{\Delta x} + w_v \frac{\Delta v}{\Delta x} + \epsilon_1 \frac{\Delta u}{\Delta x} + \epsilon_2 \frac{\Delta v}{\Delta x}$$

Taking the limit:

$$\lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x} = w_u \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + w_v \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + (\lim_{\Delta x \to 0} \epsilon_1) \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + (\lim_{\Delta x \to 0} \epsilon_2) \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$$

However, there is a flaw in this, as ϵ_1 and ϵ_2 are functions of Δu and Δv, which means that the limits don't really make sense for the ϵ functions:

$$\lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x} = w_u \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + w_v \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + (\lim_{\Delta x \to 0} \epsilon_1(\Delta u, \Delta v)) \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + (\lim_{\Delta x \to 0} \epsilon_2(\Delta u, \Delta v)) \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$$

□

We have introduced the chain rule and proved it, except for a small issue. Now we wish to conclude the following:

$$\lim_{\Delta x \to 0} \epsilon_1(\Delta u, \Delta v) = 0$$
$$\epsilon_1(0, 0) = 0$$

The key to this is that we want to get rid of the ϵ_1 function anymore. It might be continuous, it might not be, and all we know is that it satisfies the following conditions:

$$\Delta w = w_u \Delta u + w_v \Delta v + \epsilon_1 \Delta u + \epsilon_2 \Delta v$$

$$\epsilon_1 := \epsilon_1(\Delta u, \Delta v)$$

$$\lim_{(\Delta u, \Delta v) \to (0,0)} \epsilon_1 = 0$$

What we will do is define a new $\bar{\epsilon}_1$ and force it to meet these previous conditions and force it to obey the conditions we want to show, and keep it continuous.

$$\bar{\epsilon}_1 := 0 \text{ for } (\Delta u, \Delta v) = (0,0)$$

Otherwise, we want to have a function that satisfies all these conditions. We already have a function that does this, ϵ_1.

$$\bar{\epsilon}_1 := \epsilon_1 \text{ o.w.}$$

Going through all of these conditions, $\bar{\epsilon}_1$ satisfies all of the conditions placed on ϵ_1, and is also continuous by design. to complete the chain rule proof, we go back through it and replace all ϵ_1s with a $\bar{\epsilon}_1$, and when we reach $\lim_{\Delta x \to 0} \epsilon_1(\Delta u, \Delta v) = 0$ and $\epsilon_1(0,0) = 0$, the issue is resolved via the new function, as these are true by definition of the function. Note that we are using the same technique for ϵ_2. Forcing the continuity of a function is a useful technique for proving stuff.

Lets express the chain rule in some different ways. Take a function $w = w(u_1, \ldots, u_n)$, and each $u_i = g_i(x_1, \ldots, x_m)$. We can write the chain rule as:

$$\frac{\partial w}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial w}{\partial u_i} \frac{\partial u_i}{\partial x_j}$$

Taking a function $w = f(u_i, \ldots, u_n)$, with each $u_i = g_1(t)$:

$$\frac{dw}{dt} = \sum_{i=1}^{n} \frac{\partial w}{\partial u_i} \frac{du_i}{dt}$$

Note that we use a total derivative on the u_i level because it is a single variable function.

Our older statement is just the two variable case of the first chain rule expression:

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial x}$$

Theorem 13.2 (Implicit Function Theorem in 2 Dimensions)**.** *Consider $F(x,y) = 0$. This defines a set of points in the xy plane. Note that this is a relationship between x and y, but it doesn't imply that you can set y as a function of x. However, if we assume that this can be solved for some $y = f(x)$, we are left with:*

$$w = F(x, f(x)) = 0$$

Let's rewrite it in a way that is nicer to look at for us, setting $u = x$ and $v = f(x)$:

$$w = F(u, v)$$

Note that setting $w = 0$ is the same as taking a single level curve of the function. Let's now consider the derivative of w with respect to x:

$$\frac{dw}{dx} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

We know that $\frac{du}{dx} = 1$, and that $\frac{dv}{dx} = f'(x)$. However, since $u = x$, $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}$, and we can replace $\frac{\partial w}{\partial y}$. This leaves us with something really weird, the total derivative of w with respect to x is equal to the partial plus some other stuff?

$$\frac{dw}{dx} \neq \frac{\partial w}{\partial x}$$

This issue comes up because we are using x as a variable on two different levels of the function, so the two expressions mean different things. Also notice that $\frac{\partial w}{\partial u}$ tells us to keep u constant and keep v constant, but u and v are a function of the same variable, x! This means that partials don't care about the connections on lower levels in regards to the variables. Partials are hypothetical, and so sometimes (like in this case) are not plausible. Total derivatives take all of the information in the system into account.

Taking a look at the original function, we know that the function is always 0, so $\frac{dw}{dx}$ has to be 0 as the y value balances out the change in the x value to keep the value of the function 0. Thus we can conclude:

$$0 = F_x + F_y y'$$

$$\boxed{y' = \frac{-F_x}{F_y}}$$

Lets look at an example.

$$x^2 + y^2 = 100$$

$$F(x, y) = x^2 + y^2 - 100$$

Implicitly differentiating:

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Using the new method, we can take the two partials (which ignore the relationship between the variables), and we find that we find the same value for y'. If we defined our function a little differently:

$$w = F(u, v, q)$$

with $u = x$, $v = y$, and $q = f(x, y)$. Looking at the chain rule again, this adds a new term, $\frac{dw}{dq}\frac{dq}{dx}$, and change a few of the total derivatives to partial derivatives, we find that

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$$

This isn't a totally rigorous extension to 3 dimensions, but the proof is pretty much the same, and this can extend to higher dimensions.

Let's do an example of the chain rule (16.5 #24):

$$w = f(x, y)$$

$$x = e^r \cos\theta$$
$$y = e^r \sin\theta$$

We work in the r, θ setting, get stuff with x and y, and then things cancel. We start by finding $\frac{\partial w}{\partial r}$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$
$$= w_x e^r \cos\theta + w_y e^r \sin\theta$$

We also need $\frac{\partial w_r}{\partial r}$.

$$\frac{\partial w_r}{\partial r} = \frac{\partial}{\partial r}(g(x,y)e^r \cos\theta) + \frac{\partial}{\partial r}(h(x,y)e^r \sin\theta)$$

We have replaced w_x with an arbitrary function of x and y. We can rewrite that piece as:

$$g_r = \frac{\partial g}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial r}$$

Note that this isn't the whole thing, just one part of the product rule bit. Doing the same thing for the $h(x,y)$ bit:

$$w_{rr} = e^r \cos\theta g + (g_x e^r \cos\theta + g_y e^r \sin\theta)e^r \cos\theta + e^r \sin\theta h + (h_x e^r \cos\theta + h_y e^r \sin\theta)e^r \sin\theta$$

Notice that $h_x = w_{yx}$, and we can make other substitutions.

14 16.6

We have taken partial derivatives in both the x and y directions separately (limits of Δx and Δy), but what if we take a point (x_o, y_o) and move out in both directions, to $(x_o + \Delta x, y_o + \Delta y)$. This induces a change in the value of the function, Δf, and a change in some combination of x and y. This is a multivariable limit, and that really sucks. What we do is take the vector between the two points and call it $\mathbf{AB}$. The limit now becomes:

$$\lim_{B\to A} \frac{f(B) - f(A)}{|\mathbf{AB}|}$$

We can creat a line l:

$$l = A + t\mathbf{AB}$$

This changes the limit to

$$\lim_{t\to 0} \frac{f(x_o + t(AB)_1, y_o + t(AB)_2) - f(x_o, y_o)}{t|\mathbf{AB}|}$$

We take $\mathbf{AB}$ to be a vector $\mathbf{u} = \langle u_1, u_2 \rangle$. If $\mathbf{u}$ is not a unit vector, we can just make it one. We now have

$$D_{\mathbf{u}} f := \lim_{t\to 0} \frac{f(\mathbf{x}_o + t\hat{u}) - f(\mathbf{x}_o)}{t}$$

This defines the derivative of f along the path that follows the vector $\mathbf{u}$. Note that this gets rid of the multivariable limit, reducing it to a single variable limit, something that we always try to do. However, doing this every time really sucks.

Theorem 14.1. *We need something that follows these:*

$$D_{\langle 1,0\rangle} = f_x$$
$$D_{\langle 0,1\rangle} = f_y$$

We know that certain things are going to show up in the expression: f_x, f_y, u_1, u_2. We want to combine these in such a way that they combine to follow the rules said above. We predict that the following is true:

$$\begin{aligned} D_{\mathbf{u}}f &= u_1 f_x + u_2 f_y \\ &= f_x u_1 + f_y u_2 \\ &= \langle f_x, f_y\rangle \cdot \hat{u} \\ &= \langle \frac{d}{dx}, \frac{d}{dy}\rangle f \cdot \hat{u} \\ &= \nabla f \cdot \mathbf{u} \\ D_{\mathbf{u}}f &= \nabla f \cdot \hat{u} \end{aligned}$$

where the del operator is

$$\nabla := \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \dots \rangle$$

*Another thing is the **gradient** of f:*

$$grad(f) := \nabla f$$

Proof. For $w = f(x, y)$, let $g(t) = f(x+tu_1, y+tu_2)$ at some point (x_o, y_o). We begin by computing $g'(0)$:

$$\lim_{t\to 0} \frac{g(t) - g(0)}{t}$$
$$= \lim_{t\to 0} \frac{f(x + tu_1, y + tu_2) - f(x, y)}{t}$$

This is by definition, the directional derivative $D_{\mathbf{u}}f(x, y)$. We now let $g(t) = f(r, v)$, where $r = x + tu_1$ and $v = y + tu_2$. To take the derivative, we use the chain rule:

$$g'(t) = \frac{d}{dt}f = \frac{\partial f}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} = f_r u_1 + f_v u_2$$

If we plug in $t = 0$ into r and v, we see that they go to x and y respectively. We can now conclude that

$$g'(0) = f_x u_1 + f_y u_2 = \nabla f \cdot \hat{u}$$

we have concluded that both sides are the directional derivative. □

We have proven as a theorem that the directional derivative

$$D_{\mathbf{u}} = \nabla f \cdot \hat{u} = |\nabla f||\hat{u}| \cos\theta$$

Since $\hat{u}$ is a unit vector, we find that the maximum value of $D_{\mathbf{u}}f$ is maximized when $\cos\theta = 1$, meaning that the gradient vector should be parallel to the direction vector. We don't get to choose the direction of gradient vector, but we can choose to move parallel to it via changing the direction vector, and the change is the magnitude of the gradient vector. The largest decrease occurs in the direction opposite of the gradient vector, by the same logic. We also know that being perpendicular to the gradient will give us a total change of 0, as the $\cos\theta = 0$, making the change 0. We will also see that this expression will be useful for finding extrema of a function.

15 16.7

Take a point on a surface. We can't really talk about the tangent line to the surface at that point, as there are an infinite number of tangent lines at that point. Instead, we talk about a tangent plane to the surface at that point. The plane tangent to surface S at point P is the plane composed of lines tangent to each curve C on S through P.

Our surface is some function $F(x, y, z) = 0$ (note that this is a level curve of $w = F(x, y, z)$). We have a curve $C = \mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$. Since the curve must be on the surface, we know that

$$\forall t,\ F(f(t), g(t), h(t)) = 0$$

If we take the total derivative in terms of t, by the chain rule, we get:

$$F(x)\frac{df}{dt} + F_y\frac{dg}{dt} + F_z\frac{dh}{dt} = 0$$

This looks like a dot product:

$$\langle F_x, F_y, F_z\rangle \cdot \langle f', g', h'\rangle$$

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

This is true at any given point, so we rewrite it:

$$\nabla F|_{P_o} \cdot \mathbf{r}'(t_o) = 0$$

if $\mathbf{r}(t_o) = P_o$. This equality is true of every single possible curve on the surface that is passing through the point. Recall that we defined a plane as the collection of vectors that go through a point and are orthogonal to the normal vector. We see that $\nabla F|_{P_o}$ is the normal vector of the plane, which we can use to define the plane. Written in vector form:

$$\nabla F|_{P_o} \cdot (\mathbf{r} - P_o) = 0$$

Converting to the normal way of writing the plane's equation:

$$\nabla F|_{P_o} \cdot (\langle x, y, z\rangle - \langle x_o, y_o, z_o\rangle)$$

$$F_x|_{P_o}(x - x_o) + F_y|_{P_o}(y - y_o) + F_z|_{P_o}(z - z_o) = 0$$

There is a special case, where the surface is of the form $z = f(x, y)$. We can rewrite this as $0 = f(x, y) - z$, and call this function $F(x, y, z) = 0$. Using the formula we found for the plane, we get that the plane's equation is

$$f_x|_{P_o}(x - x_o) + f_y|_{P_o}(y - y_o) - 1(x - x_o)$$

This simply makes one of the derivatives simpler.

We used the tangent line in 2d to approximate the value of the curve close to the point that we took the tangent to. This means that we want to approximate the value of the surface by finding the value of the tangent plane. Remember in 2d we go over Δx from the point of tangency and go up a Δy. Using the tangent line, we approximate that $\Delta y \approx dy$. In the 3d case, we move some $\Delta \mathbf{r}$, and measure Δz to be close to dz. This process is known as linearization (same as what it was called in 2d).

Just as we can make a tangent plane, we can make a normal line as well. The direction vector of the line is just the gradient vector.

Lets consider level curves of a curve $z = f(x, y)$. The level curves are two dimensional. At any given point on any of the level curves, the gradient points orthogonally to the level curve outwards. While it may not seem like the gradient is pointing in the direction of the closest path, it is pointing to the next infinitesimally larger level curve. Thus following the gradient may not always be the shortest path from any two level curves, but following the gradient is the locally optimized method for shortest paths.

16 16.8

This section is about optimization. There are three major types of optimization problems that are worth doing, finding local extrema, finding absolute extrema on bounded regions, and finding absolute extrema on certain special bounded regions.

Surfaces continue to have the 4 properties of local/absolute max and mins. The definition of the absolute max is:

$$\forall x \in D,\ \exists x_o | f(x_o) = M,\ M \geq f(x)$$

For the local maximum:

$$\exists I \text{ open, containing } x_o \text{ s.t. } \forall x \in I,\ f(x) \leq f(x_o)$$

These definitions remain the same in multiple dimensions, except that the interval I becomes a ball:

$$I \to B_\epsilon(P_o),\ f(\mathbf{x} \leq f(\mathbf{x_o}))$$

We can define a critical point in 2d as a point where the derivative is 0 or undefined. These were possible locations for extrema (along with endpoints). We note that in 3d, if a point is not a minimum in the x direction, it cannot be the x value of the minimum of the surface, and this is true for the y direction as well. This means that we want $f_x(a, b) = 0$, AND $f_y(a, b) = 0$. We realize that we actually have to check every single direction, of which there are an infinite number. However, we can't really check an infinite number of directions. This provides a balancing act, since adding more directions narrows down the possible extrema, but makes it much hard to find points. In general, the list of just the x and y directions is just fine. We can also check for points where $f_x(a, b) = DNE$ OR $f_y(a, b) = DNE$.

1. Bad News: There's no First Derivative Test
2. Good News: You don't have to use the FDT

However, there is still a Second Derivative Test:
For a critical point (a, b) of $f(x, y)$, consider $g(x, y) = f_{xx}f_{yy} - (f_{xy})^2$.

- If $g > 0$ and $f_{xx} < 0$, (a, b) is a relative maximum.
- If $g > 0$ and $f_{xx} > 0$, (a, b) is a relative minimum.
- If $g < 0$, (a, b) is a saddle point (Could be either).
- If $g = 0$, inconclusive.

Turns out that we can also think of this as the determinant of the matrix

$$g = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

16.1 Example 1

Let's do an example. The problems work the same way as in single variable.
Take the function $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.
First we consider what would happen if we take x and y to positive and negative infinity. We notice that this probably won't have an absolute maximum. We then compute the partials:

$$f_x = 2x - 2$$

$$f_y = 2y - 6$$

$$f_{xx} = 2$$

$$f_{yy} = 2$$

$$f_{yx} = 0$$

f_x and f_y are never undefined, so there are no critical points of that type. We then set f_x and f_y to be 0:

$$2x - 2 = 0 \to x = 1$$

$$2y - 6 = 0 \to y = 3$$

We see that there is only one critical point, $(1, 3)$. We now want to compute $g(x, y)$:

$$g(x, y) = 4 - 0^2 = 4$$

$$g(1, 3) = 4$$

This means that we are either in case 1 or case 2. Looking at f_{xx}, we see that it is positive, meaning that $(1, 3)$ is the location of a local minimum.

16.2 Example 2

Let's do another example:

$$f(x, y) = x^2 - 2xy + 2y$$

restricted to the rectangle given by the corners $(0, 2)$ and $(3, 0)$.

The only thing that changed is that we now have to check the edges of the rectangle. We can split the boundary into 4 parts, B_1 through B_4. This reduces the problem down into a set of 4 Analysis 1 problems. We have only 1 possible extrema inside, $(1, 1)$. For each boundary, we find the critical points, and check the endpoints. We know that the boundary along the bottom, $B1$ can be categorized as $f(x, 0)$, and we can find that the critical point occurs at $x = 0$, meaning that we add the point $(0, 0)$ to the list. We can be proactive and just list all the vertices in the first place, although as we can see with boundary 1, sometimes the critical points just are the endpoints of the boundaries. The right side boundary, $B2$ is classified as $f(3, x)$, and we see that there are no critical points along the boundary. Doing the same process for the remaining two boundaries, we are left with the final list of possible critical points:

- $(1, 1)$
- $(0, 0)$
- $(2, 2)$
- $(0, 2)$
- $(3, 2)$
- $(3, 0)$

Now to determine what the extrema are, we simply compute the function at all of the points:

- $f(1, 1) = 1$
- $f(0, 0) = 0$
- $f(2, 2) = 0$
- $f(0, 2) = 4$
- $f(3, 2) = 1$
- $f(3, 0) = 9$

We now see that the maximum occurs at $(3, 0)$, and the minimum occurs at $(0, 0)$ as well as $(2, 2)$. Note that when plugging in points on the boundaries, we can just plug the point into the equation for the boundary, rather than the original function, as it might be slightly easier. Now let's move to the next example.

16.3 Example 3

Take the function $f(x, y) = 3 + xy - x - 2y$. Find the extrema if the function is bounded by the triangle with vertices $(1, 0)$, $(5, 0)$, and $(1, 4)$. We take the partials:

$$f_x = y - 1$$

$$f_y = x - 2$$

The only critical point inside the bound is $(2, 1)$. We can easily do the horizontal line boundary and the vertical line boundary, and find that they have no critical points, leaving us with just the last boundary, where both of the variables are changing. We can find the equation of the line via point-slope form (you can use whatever method you want to find the equation of the line):

$$y = -x + 5, \quad 1 \leq x \leq 5$$

This leaves us with the equation for the boundary:

$$f|_{B1} = f(x, -x + 5)$$

We find that this function has a critical point of $(3, 2)$. This leaves us with our completed list of possible extrema points:

- $(2, 1)$
- $(3, 2)$

- $(1, 0)$
- $(5, 0)$
- $(1, 4)$

We now evaluate the critical points just like last time.

16.4 Example 4

Say we have a open-top box. We want to maximize the volume of the box, given that the surface area is 60. We want to optimize $V = xyz$, given that $60 = xy + 2xz + 2yz$ (no last term because its an open top box). Solving for z:

$$z = \frac{60 - xy}{2(x + y)}$$

Plugging this into the Volume equation:

$$V = \frac{xy(60 - xy)}{2(x + y)}$$

To find the extrema, we need to find V_x and V_y. Notice that this function is symmetric, meaning that it is the same when we swap the two variables.

$$V_x = \frac{2(x + y)\frac{\partial}{\partial x}(xy(60 - xy)) - 2xy(60 - xy)}{4(x + y)^2}$$

$$= \frac{(x + y)(60y - xy^2 - xy^2) - 60xy + x^2y^2}{2(x + y)^2}$$

$$= \frac{60y^2 - x^2y^2 - 2xy^3}{2(x + y)^2}$$

We can now find V_y by just swapping x and y:

$$V_y = \frac{60x^2 - x^2y^2 - 2x^3y}{2(x + y)^2}$$

For the denominator, we'll get a type 2 point if $2(x+y)^2 = 0$ Since lengths are all positive, the only point that fulfils this is $(0, 0)$. We also know that that point is the absolute minimum. Looking at the numerators:

$$60y^2 - x^2y^2 - 2xy^3 = 0$$

$$60x^2 - x^2y^2 - 2x^3y = 0$$

We know ways to do this, but they are annoying. Solving that system of equations will give us critical points, one of which will be the absolute maximum. It ends up that (as we should have expected) the volume is maximized when we approach a cube (not entirely because of the missing top), and we find that the lengths are $(x, x, \frac{60-x^2}{4x})$, and we can find that $x = \sqrt{20}$, leaving us with $(\sqrt{20}, \sqrt{20}, \frac{10}{\sqrt{20}})$.

16.5 Proof of the Multivariable Second Derivative Test

Proof. In order for us to prove the second derivative test, we need the second derivative in every direction to be a certain sign, either always negative, always positive, or a mix of both (saddle point). Luckily, we know about directional derivatives:

$$D_{\mathbf{u}}f = \nabla f \cdot \langle h, k\rangle$$

$$D_{\mathbf{u}}f = f_x h + f_y k$$

We now take the directional derivative in the same direction again:

$$D_{\mathbf{u}}(f_x h + f_y k) = \langle \frac{\partial}{\partial x}(f_x h + f_y k), \frac{\partial}{\partial y}(f_x h + f_y k)\rangle \cdot \langle h, k\rangle$$

This leaves us with a bunch of mixed partials:

$$= f_{xx}h^2 + f_{yx}kh + f_{xy}hk + f_{yy}k^2$$

$$= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2$$

This looks a lot like a quadratic. Factoring:

$$D^2_{\mathbf{u}}f = (f_{xx})h^2 + (2f_{xy}k)h + (f_{yy}k^2)$$

We can now complete the square:

$$= f_{xx}(h^2 + \frac{2f_{xy}k}{f_{xx}}h) + f_{yy}k^2 + (\frac{f_{xy}k}{f_{xx}})^2 - \frac{(f_{xy}k)^2}{f_{xx}}$$

We can rewrite this whole thing:

$$= f_{xx}(h + \frac{f_{xy}k}{f_{xx}})^2 + (\frac{f_{xx}f_{yy}}{f_{xx}} - \frac{f_{xy}^2}{f_{xx}})k^2$$

Combining the right side actually gives use something we want, g:

$$= f_{xx}(h + \frac{f_{xy}k}{f_{xx}})^2 + (\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}})k^2$$

We now break this down into cases. For the first case, we treat $g < 0$, and we can see that there exists some (h, k) such that the whole derivative is positive, and there are some (h, k) that force the derivative is negative. We can also think of this as a parabola with a vertex below 0 and opens upward. We can see that the parabola has values that are less than 0 as well as more than 0.
In the case that $g > 0$, we see that the parabola has a vertex above 0 and opens downward, also having values greater than and less than 0.

If we look at the case where $g > 0$ and $f_{xx} >$, we see that the parabola has a negative vertex position and opens downwards. In the case where $f_{xx} > 0$, we see that the parabola has a positive vertex position and opens upwards, meaning that it is always positive. □

17 16.9

Theorem 17.1. *Lagrange's Theorem Assume $f(\mathbf{x})$, $g(\mathbf{x})$ have continuous first partials. Then, if the $f(\mathbf{x})$ function is subject to the constraint $g(\mathbf{x}) = 0$ and has an extremum at point P, then*

$$\exists\lambda \in \mathbb{R} \; s.t. \; \nabla f|_P = \lambda \nabla g|_P$$

17.1 Example 1

Let's do an example. Lets take a box with an open top, and only 60 total materials to construct it.

$$f(x, y, z) = xyz$$

$$g(x, y, z) = xy + 2xz + 2yz - 60 = 0$$

Let's take the gradient of the two:

$$\nabla f = \langle yz, xz, xy\rangle$$

$$\nabla g = \langle y + 2z, x + 2z, 2x + 2y\rangle$$

We don't have an extremum though! What do we do?
We use the contrapositive of the theorem, which allows us to say that the list of points at which the gradients are parallel is the list of all possible extrema. Thus, we can set the gradients equal to each other:

$$\langle yz, xz, xy\rangle = \lambda\langle y + 2z, x + 2z, 2x + 2y\rangle$$

$$\begin{cases} \quad yz = y + 2z \\ xz \quad = x + 2z \\ xy \quad = 2x + 2y \end{cases}$$

Solving this system of equations (make all the left sides the same by multiplying by x, y, and z) gives us $(x, x, \frac{x}{2})$. Plugging this back into the constraint, we find that $x = \sqrt{20}$, as we expected from our solving this problem earlier.

17.2 Example 2

We would like to find points on $x^2 + y^2 + z^2 = 4$ closest to $(3, 1, -1)$. To optimize this, we minimize the distance from an arbitrary point (x, y, z) to $(3, 1, -1)$. Our restriction is the equation $g = x^2 + y^2 + z^2 - 4 = 0$. We know that the distance is $f = \sqrt{(x-3)^2 + (y-1)^2 + (z+1)^2}$. We don't want to take derivatives of that, so we decide to minimize the square of the distances.

$$\nabla((x-3)^2 - (y-1)^2 + (z+1)^2) = \lambda\nabla(x^2 + y^2 + z^2 - 4)$$

$$\langle 2(x-3), 2(y-1), 2(z+1)\rangle = \lambda\langle 2x, 2y, 2z\rangle$$

This gives us:

$$\begin{cases} 2x - 6 = 2x\lambda \\ 2y - 2 = 2y\lambda \\ 2z + 2 = 2z\lambda \end{cases}$$

$$x = \frac{3}{1-\lambda} \quad y = \frac{1}{1-\lambda} \quad z = \frac{-1}{1-\lambda}$$

Using the constraint, we can plug in all of these values and we find that

$$11 = 4(1-\lambda)^2$$

$$1-\lambda = \sqrt{\frac{11}{4}}$$

$$\lambda = 1 \pm \sqrt{\frac{11}{4}}$$

After this you can just plug in the value for λ to find the values for (x, y, z).

17.3 Proof of Lagrange's Theorem (Geometric)

Proof. Consider the level curves of $f(x, y)$. Now consider the set of points that satisfy the claim $g(x, y) = 0$. Notice that $g(x, y) = 0$ is a level curve of $w = g(x, y)$. We know that the gradient of a function is always orthogonal to the level curve of the function. Now consider an intersection point between $g(x, y) = 0$ and the level curves of f. We want the gradient vectors to be parallel for both curves. After some inspection, we find that we want $g(x, y) = 0$ to hit a level curve of f but not go through/past it. These are the possible extrema for f. The gradient of f at these points is orthogonal to the direction of direction of f, and the gradient of g is orthogonal to the direction of g. If the directions are parallel, then the gradients must also be parallel. □

17.4 Proof of Lagrange's Theorem (Algebraic)

Proof. Suppose $f(\mathbf{x})$ has an extremum at $\mathbf{x_o}$ on the surface S denoted by $g(\mathbf{x}) = k$. For any curve on S passing through $\mathbf{x_o}$,

$$C = \mathbf{r}(t) = \langle x_1(t), x_2(t), \dots, x_n(t)\rangle$$

with $\mathbf{r}(t_o) = x_o$. Our claim is that if x_o is an extremum on S, it must be an extremum on $\mathbf{r}$. We know that t_o provides us with a critical point, so the derivative must be 0:

$$\frac{d}{dt} f(\mathbf{r}(t))|_{t_o} = 0$$

$$\frac{d}{dt} f(\mathbf{r}(t)) = f_{x_1}\frac{dx_1}{dt} + f_{x_2}\frac{dx_2}{dt} + \cdots + f_{x_n}\frac{dx_n}{dt}$$

We can see that this is a dot product, with the first piece being the gradient:

$$= \nabla f \cdot \frac{d}{dt}\mathbf{r}(t)$$

$$= \nabla f \cdot \mathbf{r}'(t) = 0$$

So at t_o, we find that the gradient of f is orthogonal to $\mathbf{r}'(t)$. This means that for any curve passing through the point, we have a vector that is orthogonal to the surface. This gives us a normal vector for S's tangent plane! This is important because when we normally find the normal vector to a function, we use the gradient of the function itself, but in this case we are using the gradient of f, but only at extrema. This tells us that the gradient of g is parallel to the gradient of f. □

18 17.1

When we define definite single variable calculus, we have certain different parts, the symbol that says integrate, the bounds, the function we are integrating, and the variable of integration:

$$\int_a^b f(x)\,dx$$

Remember that this is equivalent to a Riemann Sum:

$$= \lim \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

This essentially defines an integral as the sum of the signed area of a bunch of boxes. What we do is take the width of the boxes to 0 to approach the true area under the curve. What is different between making more boxes as opposed to making them smaller is that when using noneven partitions (same size boxes), the sum of the areas does not approach the true value. For this reason, we have the limit as the size of the largest box goes to 0:

$$= \lim_{||\Delta||\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

Let us now define multivariable integration. We still need a symbol that says integrate, as well as some region in space over which we are integrating. We also need a function to integrate, and then a variable of integration, the area of a small rectangle:

$$\int_R f(x,y)\,dA$$

However, we do change the symbol in math, to a double integral:

$$\iint_R f(x,y)\,dA$$

Remember that this is not two single variable integrals, this is a singular double integral. Let's now define it:

$$\iint_R f(x,y)\,dA = \lim \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta R_i$$

Since we have multiple variables, the limit is multivariable, which is something that we really don't want. We now need to find a property of the box that we are adding that only requires one variable. We decide to use the length of the diagonal of the box.

$$\iint_R f(x,y)\,dA = \lim_{||\Delta||\to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta R_i$$

There is another definition:

$$\lim_{n\to\infty} \lim_{m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)\Delta x_i \Delta y_j$$

Note that the way that we are writing it here implies that we are using regular partitions.
When we partition an arbitrary region R into a grid, we don't always get perfect rectangles. The section of the grid containing rectangles that are fully inside the region is called the **inner partition**. Thus, when we make the boxes small, we not only increase accuracy, we also approach the

true region. We posit (without proof), that for any ϵ, we can make the inner partition large enough that the remaining area inside the region but outside the inner partition is smaller than *epsilon*. This is a pain, so we won't prove it.
We can characterize spaces into 2 types: type 1 and type 2 spaces. If we have a space where x is bound by constants and y is bound by functions, we hae a Type 1 region. If y is bound by constants and x is bound by functions, we have a Type 2 region. Certain regions (like boxes and circles) can be classified as both Type 1 and Type 2 regions. There are also shapes that are neither, such as some piecewise functions. However, we can then just break up the region into smaller regions that can be classified as Type 1 or Type 2. From now on, unless explicitly stated, "a Region R" means a region of $\mathbb{R}^2$ which can be expressed as a finite (disjoint) union of Type 1 and Type 2 regions.

We can now define a Riemann Sum. For inner partition $\{R_i\}$ of region R, a Riemann Sum is

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A(R_i)$$

Note that we will only ever work with inner partitions, and never with the full partition R.
Lets do some definitions:

$$\lim_{||\Delta||\to 0} \sum_i f(x_i^*, y_i^*)\Delta A(R_i) = L$$

means that

$$\forall \epsilon > 0 \,\exists \delta \;\; s.t. \;\; ||\Delta|| < \delta \to |(\sum_i f(x_i^*, y_i^*)\Delta A(R_i) - L)| < \epsilon$$

Lets now define the double integral:

$$\iint_R f(x,y)\, dA = \lim_{||\Delta||\to 0} \sum_i f(x_i^*, y_i^*)\Delta A(R_i)$$

if the RHS exists.
$f(x, y)$ is integrable on R means that $\iint_R f(x,y)\, dA$ exists.
Something else that is important is that continuity still implies integrability. We're not going to prove this but it's true.
We haven't really talked about what the double integral is really computing. The integral computes the signed value of the volume under the surface, because we have some areas and a height and we're multiplying them together, giving volume.
Lets list some properties:

$$\iint_R cf(x,y)\, dA = c\iint_R f(x,y)\, dA$$

$$\iint_R f(x,y) + g(x,y)\, dA = \iint_R f\, dA + \iint_R g\, dA$$

These two properties provide the linearity of the double integral operator.
Remember that for singlevar integrals, we could separate the intervals of an integral, and for double integrals we can split up the region into two regions:

$$\iint_R f(x,y)\, dA = \iint_{R_1} f\, dA + \iint_{R_2} f\, dA \;\; ifR = R_1 \sqcup r_2$$

19 17.2

19.1 Generalized Fubini's Theorem

Theorem 19.1. *For R, a type 1 region on which f is continuous, and for $g_1(x)$,$g_2(x)$ continuous on $[a,b]$:*

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

This second term is called an iterated integral. There is also a second expression that holds for when R is a Type 2 region:

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,dx\,dy$$

Essentially, Fubini's Theorem allows us to take double integrals and convert them to iterated single variable integrals. This is nice because we can now compute double integrals. This takes the concept of volume and converts it to something that is "easily" computable.

19.2 Fubini's Theorem

Theorem 19.2. *For a region R defined to be a set of points $R = \{(x,y)|\, a \leq x \leq b \text{ and } c \leq y \leq d\}$ and with f continuous on R,*

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Notice that this essentially defines a rectangular region, which is both type 1 and type 2. Fubini's Theorem only works for the special case of the rectangular region, so we'll be using Generalized Fubini's for other shapes. Let's do some example integrals now:
Take $f(x,y) = y$, and take the region $0 \leq x \leq 3$, $0 \leq y \leq 2$. We can set up two different integrals to compute the double integral:

$$\iint_R f(x,y)\,dA = \int_0^3 \int_0^2 y\,dy\,dx = \int_0^3 \frac{y^2}{2}|_{y=0}^{y=2}\,dx = \int_0^3 2\,dx = 6$$

Computing it the other way:

$$= \int_0^2 \int_0^3 y\,dx\,dy = \int_0^2 xy|_{x=0}^{x=3}\,dy = \int_0^2 3y\,dy = 6$$

Example 2:
Take the function $z = x^2 + y^2$. Let the region of interest be a triangle defined by the points $(0,0)$,$(0,2)$, and $(2,0)$. We can define the bounds as $0 \leq x \leq 2$ and $0 \leq y \leq 2-x$. We can then setup the integral:

$$\int_0^2 \int_0^{2-x} (x^2+y^2)\,dy\,dx = \int_0^2 x^2 y + \frac{y^3}{3}|_{y=0}^{y=2-x}\,dx$$

$$= \int_0^2 2x^2 - x^3 + \frac{(2-x)^3}{3}\,dx = [\frac{2}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{12}(2-x)^4]_0^2$$

We could set it up the other way, but the bounds change, because now $0 \leq x \leq 2-y$ and $0 \leq y \leq 2$.
Example 3:
$f(x,y) = 3x + xy$, in the region governed by $y = x$ and with $0 \leq x \leq 2$. This means that the bound

on why can be written as $0 \leq y \leq x$. If we want to use the Type 2 boundaries, we would have $0 \leq y \leq 2$ and $y \leq x \leq 2$. We can then setup the two different iterated integrals:

$$\int_0^2 \int_0^x 3x + xy \, dy \, dx$$

$$\int_0^2 \int_y^2 3x + xy \, dx \, dy$$

Example 4:
$f(x, y) = e^{y^2}$, on the region $0 \leq x \leq 2$, $x \leq y \leq 2$. Setting up the integral:

$$\int_0^2 \int_x^2 e^{y^2} \, dy \, dx$$

We can't do this integral symbolically! But what if we did the type 2 region? We would get

$$\int_0^2 \int_0^y e^{y^2} \, dx \, dy = \int_0^2 xe^{y^2} |_{x=0}^{x=y} \, dy = \int_0^2 ye^{y^2} \, dy$$

This is an integral that we can solve. Thus it is possible that only one of the two ways will work.
Example 5:
Take a region that is a trapezoid (in this case $y = x$ for $0 \leq x \leq 1$ and 1 until $x = 3$). We now have to split up the region in order to make an iterated integral:

$$\int_0^1 \int_0^x f \, dy \, dx + \int_0^1 \int_2^3 f \, dx \, dy$$

20 17.4

Take a beam on a fulcrum. Place two different masses at different distances from the fulcrum. let $\bar{x}$ be the location of the fulcrum. Doing torques, we know that if we want the system to be in equilibrium, we need

$$(x_1 - \bar{x})m_1 + (x_2 - \bar{x})m_2 = 0$$

$$\bar{x} = \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2}$$

From here we can generalize to any number of masses. This gives us the center of mass:

$$\bar{x} = \frac{\sum x_i m_i}{\sum m_i}$$

We can turn this into something involving linear density:

$$\bar{x} = \frac{\sum x_i \lambda_i \Delta x_i}{\sum \lambda_i \Delta x_i}$$

We can turn this into an integral!

$$\bar{x} = \frac{\int_a^b x\lambda(x) \, dx}{\int_a^b \lambda(x) \, dx}$$

Take the seesaw once more. We know that the pipe has some linear density $\lambda(x)$, meaning that we can separate the seesaw into a conglomerate of smaller masses:

$$m_i = \lambda(x_i^*) \Delta x_i$$

$$\bar{x} = \frac{\sum(\lambda(x_i^*)\Delta x_i)x_i^*}{\sum \lambda(x_i^*)\Delta x_i}$$

We can convert this into an integral:

$$\bar{x} = \frac{\int x\lambda(x)\,dx}{\int \lambda(x)\,dx}$$

If we take a 2d lamina (sheet of material) with some area density $\sigma(x)$, that is bounded by functions above and below and is bound by the interval $a \leq x \leq b$. We can partition the lamina into a bunch of strips, and then calculate the center of mass by taking an integral:

$$\bar{x} = \frac{\int_a^b x\sigma(x)(g_2(x) - g_1(x))\,dx}{\int_a^b \sigma(x)(g_2(x) - g_1(x))\,dx}$$

To get the y coordinate, we would have to find an expression for the density in terms of y, because we would be using Δys. What we can do is to approximate the center of mass of the strip as the midpoint, allowing us to approximate the value of $\bar{y}$ via:

$$\frac{\int_a^b (\frac{g_1+g_2}{2})\sigma(x)(g_2 - g_1)\,dx}{\int_a^b \sigma(g_2 - g_1)\,dx}$$

This second equation is known as $\frac{M_x}{M}$, where M_X is the moment with respect to x. Likewise, the first equation is $\frac{M_y}{M}$, where M_y is the moment with respect to y. This method is pretty terrible, so lets move to the multivar method. We now know how to use little squares instead of strips, allowing us to vary both variables instead of just one of the two.

$$M_x : y\sigma(x,y)\Delta A$$

$$M_y : x\sigma(x,y)\Delta A$$

When summing over all of the little squares, we can compute the value of $\bar{x}$ and $\bar{y}$:

$$\bar{x} = \frac{\iint_R x\sigma(x,y)\,dA}{\iint \sigma(x,y)\,dA}$$

$$\bar{y} = \frac{\iint_R y\sigma(x,y)\,dA}{\iint \sigma(x,y)\,dA}$$

Some more notation:

- λ is the line density
- σ is the surface density
- ρ is the volume density

The moment of inertia I_x can be defined as:

$$I_x = \iint_R y^2\sigma(x,y)\,dA$$

$$I_y = \iint_R x^2\sigma(x,y)\,dA$$

$$I_o = \iint_R (x^2 + y^2)\sigma(x,y)\,dA = I_x + I_y$$

We can also compute things without needing to use the mass density. We can for example use a probability density. There are two rules for a probability density $f(x,y)$:

$$f(x,y) \geq 0$$

$$\iint_R f(x,y)\,dA = 1$$

If we want the probability that the event happens given an interval D, we can integrate using those bounds:

$$P(D) = \iint_D f(x,y)\,dA\,,\quad (x,y) \in D$$

Note that the center of mass of the distribution turns out to be the expected value.

21 17.5

Find the volume over the circle of radius 1 around the origin up to the paraboloid $z = 1 - x^2 - y^2$: We can split this up into quarters in order to make it easier:

$$V = 4\iint_R (1 - x^2 - y^2)\,dA = 4\int_0^1 \int_0^{\sqrt{1-x^2}} 1 - x^2 - y^2\,dy\,dx$$

$$= 4\int_0^1 y - x^2 y - \frac{y^3}{3}\Big|_0^{\sqrt{1-x^2}}\,dx$$

This turns into a mess, so lets not do this. We can make this worse by taking a region that has a circle cut out of it. One way to do this is to just compute the integrals for the larger region and then subtract the volume of the smaller one. However, this can sometimes be an issue due to discontinuities. What we do instead is switch to polar to solve problems that have circles. Redoing the first integral but in polar:

$$r = 0 \to 1$$

$$\theta = 0 \to 2\pi$$

We can then set up the integral:

$$V = \int_0^{2\pi} \int_0^1 1 - r^2\ dr\,d\theta$$

However, notice that $drd\theta$ is not equivalent to dA. This is an issue, as this means that our integral isn't really volume.

Consider some bounds $\theta \in [\alpha, \beta]$ and $r \in [g_1(\theta), g_2(\theta)]$. Partitioning whatever is in between g_1 and g_2, the partitions are wedges that are small rectangles. Each wedge has the same Δr, as well as the same $\Delta\theta$, but they change in size based on how far they are from the origin. Therefore, without a formal proof, we claim that $\Delta A = r\Delta r\Delta\theta$. We can then modify our integral:

$$V = \int_0^{2\pi} \int_0^1 r - r^3\ dr\,d\theta$$

Notice that if there are no θs, we can do the θ integral first, leaving us with

$$V = 2\pi\int_0^1 r - r^3 dr$$

From here we can just evaluate.
Remember that we had a formula for the area of a polar wedge under a function in analysis 1:

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$$

Putting the one-half back in, we find that

$$\int_{\alpha}^{\beta}\int_{0}^{f} 1\,r\,dr\,d\theta$$

This sort of justifies our choice of using $r\,dr\,d\theta$.
Take a cone with a center tube cut out of it, going through the center, subdivided into 3 sections. We know that the side of the cone is the regular cone:

$$z^2 = x^2 + y^2$$

Let's say that it goes to a height of $z = 5$, and the cylinder is of radius 1. If we tried to set this up in Cartesian, we'd have a negative space of radius 1, and then a circle of radius 5 containing the area that we really want to know about. This donut-like part is subdivided into 3 different parts. Setting this part up would be very painful. However, polar saves the day once again. Take just one of the regions, which goes from $\theta : 0 \to \frac{2\pi}{3}$, and a radius of $r : 1 \to 5$. This isn't very difficult to set up:

$$V = \int_{0}^{\frac{2\pi}{3}}\int_{1}^{5}(5-r)\ r\,dr\,d\theta$$

We know how to convert to polar double integrals:

$$\iint_{(x,y)} f(x,y)\,dx\,dy = \iint_{(r,\theta)} f(r\cos\theta, r\sin\theta)\,rdrd\theta$$

Let's find a use for this. Take the normal distribution from statistics:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx$$

We learned that the outside constant term was meant to be a normalization term that sets the value of the entire integral to 1. So lets solve the actual integral and see if it turns into the thing that we want. However we can't seem to actually solve this integral, so we do the extremely obvious thing and multiply the function by itself:

$$C^2 = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx\ \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx$$

We can also replace the xs in the second function 2 ys:

$$C^2 = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx\ \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy$$

We since the second term is now just a constant with respect to the x, so we can move it into the integral:

$$C^2 = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy\,dx$$

We now see the function with the x in it is a constant to the second term, so we can pull that in:

$$C^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}}\, dy\, dx$$

We can now smash them together and we see that we have something that looks like it would be nice in polar:

$$C^2 = \iint_{\mathbb{R}_2} e^{-\frac{x^2+y^2}{2}}\, dA$$

$$\int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2}}\, rdrd\theta$$

Setting $u = r^2$:

$$C^2 = 2\pi \int_0^{\infty} \frac{1}{2} e^{-\frac{u}{2}}\, du$$

$$C^2 = 2\pi$$

$$C = \sqrt{2\pi}$$

21.1 Proof of $rdrd\theta$

Let's now prove that we want to use $rdrd\theta$. Take the polar wedge bounded by $r = r_1$ and $r = r_2$, between $\theta = \alpha$ and $\theta = \beta$. We can say that the wedge that we care about is the total wedge minus the inner wedge:

$$\Delta A = \pi r_2^2 \frac{\Delta\theta}{2\pi} - \pi r_1^2 \frac{\Delta\theta}{2\pi}$$

$$= \frac{1}{2}(r_2^2 - r_1^2)\Delta\theta$$

$$= \frac{1}{2}(r_2 - r_1)(r_2 + r_1)\Delta\theta$$

$$\Delta A = \frac{r_1 + r_2}{2}\Delta r \Delta\theta$$

This first term is just the average r value, or $\bar{r}$:

$$\Delta A = \bar{r}\Delta r\Delta\theta$$

If we replace ΔA with what we have:

$$V \approx \sum_i f(r_i^*, \theta_i^*)\bar{r}\Delta r\Delta\theta$$

The fact that $\bar{r} \neq r^*$ doesn't really matter because as we take the limit to get the integral the two values turn into the same value.

$$V = \iint_{(r,\theta)} f(r,\theta)\, rdrd\theta$$

22 17.6

We can do single integrals and double integrals, so why don't we try doing a triple integral? We find that the jump from 1d to 2d is the hardest, and it's not that hard to generalize multiple integrals to n dimensions. Take a triple integral over a region Q:

$$\iiint_Q f(x,y,z)\,dV := \lim_{||\Delta||\to 0}\sum_i f(x_i^*,y_i^*,z_i^*)\Delta V_i$$

We can also think of this integral the other way, using three limits:

$$\lim\lim\lim\sum\sum\sum f(x_i^*,y_i^*,z_i^*)\Delta x_i\Delta y_i\Delta z_i$$

For double integration, we have 4 ways to do integrals ($dx\,dy$, $dy\,dx$, $r\,dr\,d\theta$, $r\,d\theta\,dr$). In 3 dimensions, there will end up to be 18 different ways to integrate, 6 in Cartesian, 6 in spherical, and 6 in cylindrical. Take a region in space, with x bounded by a and b and y bounded by $g_1(x)$ and $g_2(x)$. Let there be two surfaces above the region, $z = h_1(x,y)$ and $z = h_2(x,y)$. First we partition the region into rectangles and choose a rectangle (just like a double integral). We then go up from h_1 to h_2, and compute the value of the function for each little rectangular prism. This gives us the integral:

$$\int_a^b\int_{g_1(x)}^{g_2(x)}\int_{h_1(x,y)}^{h_2(x,y)} f\,dzdydx$$

We can rewrite this via a double integral and another integral:

$$\iint_R\int_{h_1(x,y)}^{h_2(x,y)} f\,dz\,dA$$

What if our region is in the xz, we don't have an elevator anymore, we have a Wonkavator. We partition the region into little squares in the xz plane, which we then move along the Wonkavator track.

$$\int_a^b\int_{g_1(x)}^{g_2(x)}\int_{h_1 x,z}^{h_2(x,z)} f\,dydzdx$$

Note that for both of these integrals, we are left with values, not expressions with variables.

23 17.7

The mass of a three dimensional object can be defined as follows:

$$M = \iiint_Q \rho(x,y,z)\,dV$$

where ρ is the density. We can also define the moments, but we're not quite sure what distances we care about:

$$M_? = \iiint_Q ?\,\rho(x,y,z)\,dV$$

Turns out, we now care about the distance from the different planes, giving us the following moments:

$$M_{yz} = \iiint_Q x\rho(x,y,z)\,dV$$

$$M_{xz} = \iiint_Q y\rho(x,y,z)\,dV$$

$$M_{xy} = \iiint_Q z\rho(x,y,z)\,dV$$

We can once again talk about the moments of inertia:

$$I_x = \iiint y^2 + z^2 \rho\,dV$$

$$I_y = \iiint x^2 + z^2 \rho\,dV$$

$$I_z = \iiint x^2 + y^2 \rho\,dV$$

We can also define the coordinates of the center of mass $(\bar{x}, \bar{y}, \bar{z})$:

$$\bar{x} = \frac{M_{yz}}{M}$$

$$\bar{y} = \frac{M_{xz}}{M}$$

$$\bar{z} = \frac{M_{xy}}{M}$$

Let's do an example. Take the following boundaries: $z = e^{x+y}$, $y = 3x$, $x = 2$, $y = 0$, $z = 0$. We can see that in the xy plane we have a triangle as our region, but the exponential is pretty hard to picture. If we think about the different regions, we definitely want to work with the triangle. This gives us the bounds $x : 0 \to 2$, $y : 0 \to 3x$. Setting up the integral:

$$V = \int_0^2 \int_0^{3x} \int_0^{e^{x+y}} 1\,dydxdz$$

Let's do another example:
Let the density function be defined as follows:

$$\rho = k|y|$$

And let the region be bound by the coordinate planes and the plane of $2x + 5y + z = 10$. Just looking at the region, we see that the y value is always positive, so we can just drop the absolute values. Thinking about the regions, if we look down onto the xy plane, we get a triangle, which is pretty nice. In fact, we get triangles for all of the possible pairs for the outside. If we look at the leftover variables, all of them move from 0 to the plane, meaning that every one of the integrals is nice and easy to set up.

24 17.8

This section is about doing triple integrals in cylindrical and spherical. In 2D, we had $dA = dydx = rdrd\theta$, so we want to find a similar relationship for dV in 3D. We can find the cylindrical conversion quite easily:

$$dV = dzdydx = dzdA = dz\,rdrd\theta = rdzdrd\theta$$

The harder one is finding a conversion factor for spherical:

$$dV = ?\,dpd\theta d\phi$$

Bad news, we don't have a proof of this until after chapter 18, but the good news is that we don't have to do the proof. We begin with 2 rays in the $z = 0$ plane, which in spherical is the $\phi = \frac{\pi}{2}$ plane. We can then incline the rays up and down (we can't just move them straight up because that doesn't exist in spherical). As we incline them up higher and higher, we see that the angle swept out between the rays must decrease to 0 (attained when $\phi = 0$). This presents a complication. This means that we can't just do $\rho\theta$ to represent the distance swept out between them, as the length varies with respect to ϕ. If we take two values of ρ at two separate inclination angles, we can see that we have a spherical rectangle (a wedge) that forms. We have 3 length based dimensions, $\Delta\rho$, $\rho\Delta\phi$, and $\rho\sin\phi\Delta\theta$. This last one occurs because we need the length of the dimension to be maximized when we are at $\phi = \frac{\pi}{2}$, and 0 at π and 0. This suggests that we should use sin (we're not going to prove this just yet). Thus, we have found that the conversion factor (after converting all the variables to differentials) is $\rho^2 \sin\theta$.

25 17.9

Think of a surface in space. If we put a bunch of post-it notes on it, we see that we approximate the surface area. However, becuase our post-it notes are flat, we have to use tiny little post-it notes to accurately solve for the surface area. What we want is to find the area of the little planes. We know that the planes are given by the cross product of 2 vectors that are on the plane, and we can remember that the magnitude of the cross product is the area of the parallelogram fromed by the two vectors.

We could start with a function $z = f(x, y)$, but not all surfaces in 3d meet this requirement (a sphere for example). We are essentially generalizing the idea of arclength ($\int_a^b \sqrt{1 + (g')^2}\, dx$) to more dimensions, but remember that arclength doesn't work in every case. We generalized arclength by using parametric:

$$x = f(t)$$

$$y = g(t)$$

We could then represent the arclength:

$$\int_a^b \sqrt{(f')^2 + (g')^2}\, dt$$

We can also think of this as:

$$\int_a^b |\mathbf{r}'(t)|\, dt$$

We want to do something similar for 3d, so we can generalize surface area. Our current paramaterization is as follows:

$$x = x$$

$$y = y$$

$$z = f(x, y)$$

However, we can let x and y be functions of some arbitrary parameters u and v:

$$x = u$$

$$y = v$$

$$z = f(u, v)$$

However, they don't have to be exactly u and v:

$$x = f(u,v)$$

$$y = g(u,v)$$

$$z = h(u,v)$$

We can now define our surface:

$$S := \mathbf{r}(u,v) := \langle x(u,v), y(u,v), z(u,v)\rangle$$

Note that $\mathbf{r}$ is always defined in terms of x,y, and z, no matter what coordinate system u and v are in. Let's now take the point $\mathbf{r}(u,v)$. We have one point, but now we need 2 more. We can increment both variables in order to find these points:

$$P_1 = \mathbf{r}(u,v)$$

$$P_2 = \mathbf{r}(u+\Delta u, v)$$

$$P_3 = \mathbf{r}(u, v+\Delta v)$$

To get the actual vectors, we have to subtract the initial points from the terminal points:

$$\mathbf{v_1} = \mathbf{r}(u, v+\Delta v) - \mathbf{r}(u,v)$$

$$\mathbf{v_2} = \mathbf{r}(u+\Delta u, v) - \mathbf{r}(u,v)$$

We could just cross these, but that's a lot of work. Instead, we notice that for each vector, we can just use the Mean Value Theorem:

$$\mathbf{v_2} = \mathbf{r_u}(u^*, v)\Delta u$$

We can do the same thing for $\mathbf{v_1}$. Now we can take the cross product:

$$\Delta S \approx |\mathbf{r_u}\Delta u \times \mathbf{r_v}\Delta v|$$

$$\approx |\mathbf{r_u} \times \mathbf{r_v}|\Delta u\Delta v$$

This is one little post-it note, so we need to sum them up:

$$\sum \Delta S = \lim_{(\Delta u,\Delta v)\to(0,0)} \sum |\mathbf{r_u} \times \mathbf{r_v}|\Delta u\Delta v$$

Which then turns into an integral:

$$S = \iint_R |\mathbf{r_u} \times \mathbf{r_v}|\, dA$$

Generally, we abuse notation and write it as

$$\boxed{S = \iint_R |\mathbf{r_u} \times \mathbf{r_v}|\, dudv}$$

If we look at the book, we see that we have done better than them. Let's do it their way. We take the special case where $z = f(x,y)$. We can use the parameterization of $u = z$, $v = y$. This gives us our $\mathbf{r}$:

$$\mathbf{r}(x,y) = \langle x, y, f(x,y)\rangle$$

$$\mathbf{r_x} = \langle 1, 0 f_x \rangle$$

$$\mathbf{r_y} = \langle 0, 1, f_y \rangle$$

Taking the necessary cross product:

$$\mathbf{r_x} \times \mathbf{r_y} = \langle -f_x, -f_y, 1 \rangle$$

Taking the magnitude, we find that the surface area is:

$$S = \iint \sqrt{1 + f_x^2 + f_y^2}\, dA$$

Note that this is very similar to the 2d arclength formula. In fact, we have essentially just generalized arclength to 3d. Our version is better than theirs, as our parameterization works in cases where their special case does not.
Let's do an example.
Take a sphere of radius k. Let's do this in spherical, where the sphere can be given by $\rho = k$. We need to find 2 parameters to use for our $\mathbf{r}$. Since ρ is constant, we can just use ϕ and θ:

$$x = k \sin\phi \cos\theta$$

$$y = k \sin\phi \sin\theta$$

$$z = k \cos\phi$$

$$\mathbf{r} = \langle k \sin\phi \cos\theta, k \sin\phi \sin\theta, k \cos\theta \rangle$$

$$\mathbf{r}_\phi = \langle k \cos\phi \cos\theta, k \cos\phi \sin\theta, -k \sin\phi \rangle$$

$$\mathbf{r}_\theta = \langle -k \sin\phi \sin\theta, k \sin\phi \cos\theta, 0 \rangle$$

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = |\langle k^2 \sin^2\phi \cos\theta, k^2 \sin^2\phi \sin\theta, k^2 \sin\phi \cos\phi \cos^2\theta - (-k^2 \sin\phi \cos\phi \sin^2\theta) \rangle|$$

Cancelling stuff out and rewriting:

$$= k^2 \sqrt{\sin^4\phi + \sin^2\phi \cos^2\phi} = k^2 \sqrt{\sin^2\phi} = k^2 \sin\phi$$

We don't need the absolute value because ϕ goes from $0 \to \pi$, so $\sin\phi$ will always be positive or 0 in that region.

$$\iint k^2 \sin\phi\, dA$$

We can say that $dA = d\theta d\phi$ because we started in spherical, not in Cartesian, so there is no need to convert.

$$\int_0^{2\pi} \int_0^{\pi} k^2 \sin\phi\, d\phi d\theta = 2\pi k^2 \int_0^{\pi} \sin\phi\, d\phi = 4\pi k^2$$

26 18.8/9

We can think about u-subs in multivar as the opposite of singlevar u-subs:

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du = \int_c^d f(x(u))\,\frac{dx}{du}du$$

Taking this into multivar:

$$\iint_{R_{xy}} f(x,y)\,dxdy = \iint_{R_{uv}} f(x(u,v),y(u,v))\,dudv$$

However, there is a conversion factor, which corresponds to the derivative term in the singlevar one. This can also be thought of as the conversion factor between $dxdy$ and $dudv$.
If we have a transformation $T(u,v) = (x,y)$ where $x = g(u,v)$ and $y = h(u,v)$. This T function taks us from a $u\,v$ region into an $x\,y$ region. We hope that the transformation keeps things nice (1-to-1 or not 1-to-1 in an obvious way). We can also represent T as a vector, $\mathbf{r}(u,v)$. If the $u\,v$ region is a nice rectangle, we can take two adjacent edges as vectors and use the transformation to map them to the $x\,y$ space, and cross them to approximate the area of the new region. We can then write the area of the region in $x\,y$:

$$A(R) = |\mathbf{r}(u_0,v_1) - \mathbf{r}(u_0,v_0) \times \mathbf{r}(u_1,v_0) - \mathbf{r}(u_0,v_0)|$$

This is just a surface area computation, so we can write:

$$A(R) \approx |\mathbf{r}_u \times \mathbf{r}_v|\Delta u \Delta v$$

We know that a double integral is defined as:

$$\iint_R f(x,y)\,dA = \lim \sum_i \sum_j f(x_i^*, y_j^*)\Delta R_{ij}$$

$$= \lim \sum_i \sum_j f(x(u,v),y(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\Delta u \Delta v$$

$$= \iint_{u\,v} f(x(u,v),y(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,dudv$$

Let's compute the actual cross product now:

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v)\rangle$$

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 0, 0, x_u y_v - x_v y_u\rangle$$

Taking the magnitude:

$$= |\frac{\partial x}{\partial y}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}|$$

If we take the transpose, we get what is known as the Jacobian of the transformation T:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Let T be a 1-to-1, C^1 transformation with non-zero Jacobian mapping S in the $u\,v$ plane to R in the $x\,y$ plane. For a cont. f on R and if R and S and Type 1 or Type 2 regions:

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v))|J_T|\,dudv$$

Let's do an example:
Let us define $x = u\cos v$ and $y = u\sin v$. Let's now find the magnitude of the Jacobian:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = u$$

Notice that we have just proven that the conversion between $x\,y$ and $dr\,d\theta$ is r. Let's do examples of change of coordinates.

26.1 Example 1

Take the parameterization:

$$x = u^2 - v^2$$

$$y = 2uv$$

and let the bounds on u and v be the square bounded by $(0, 0)$ and $(1, 1)$.
We first look at the first boundary:

$$v = 0$$

$$u : 0 \to 1$$

We want to convert this region to a boundary in x and y. We plug into the parameterization:

$$x = u^2 - 0 = u^2$$

$$y = 2u * 0 = 0$$

This is a line segment on the x axis. The second boundary is governed by:

$$u = 1$$

$$v : 0 \to 1$$

Once again plugging this in:

$$x = 1 - v^2$$

$$y = 2v$$

We then eliminate the parameter, solving for v and plugging it back in:

$$x = 1 - \frac{y^2}{4}$$

Repeating this process for the other 2 boundaries, we get

$$x = \frac{y^2}{4} - 1$$

$$x = -v^2$$

We have now converted a bounded region in uv into an xy region.

26.2 Example 2

$$\iint_R xy\,dA$$

Where R is defined as $xy = 1$,$xy = 3$, $y = x$, and $y = 3x$. If we take the transformation $R(u, v) = (x, y)$ where $x = \frac{u}{v}$ and $y = v$. Lets first find the integrand. That's pretty easy, we just convert via our parameterization to get u. Now let's do the region. We have some lines and two hyperbolas. If we graph this region, we see that there are two different regions, although one is just the double of the other, so we can just look at the other. Taking a look at the different functions:

$$xy = 1 \to u = 1$$

$$xy = 3 \to u = 3$$

$$y = x \to v^2 = u$$

$$y = 3x \to \frac{v^2}{3} = u$$

We see that when we plot these, we still have 2 regions, so once again we can just look at one of the regions and double it. This give us our bounds:

$$u : 1 \to 3$$

$$v : \sqrt{u} \to \sqrt{3u}$$

Now all we need is to get the correct Jacobian. We want a term that is similar to $\frac{dx\,dy}{du\,dv}$, so we want the Jacobian to be

$$J_T = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \frac{1}{v}$$

Now putting all the parts together:

$$2\int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v}\,dvdu = 4\ln 3$$

26.3 Example 3

$$\iint_R \frac{x - y}{x + y}\,dA$$

bounded by the region defined by the points $(0, 2)$, $(1, 1)$, $(2, 2)$, and $(1, 3)$. We can look at the lines that govern this rectangle:

$$y = -x + 2$$

$$y = x$$

$$y = -x + 2$$

$$y = -x + 4$$

We can now mess with these to get some nicer things:

$$x - y = 0$$

$$x - y = -2$$

$$x + y = 2$$

$$x + y = 4$$

Looking at this, we can see a pattern that allows for a nice transformation. If we let $u = x - y$ and $v = x + y$, we can easily find the bounds, $u : -2 \to 0$ and $v : 2 \to 4$. We see that the integrand also simplifies down to $\frac{u}{v}$. To find the Jacobian, we want to find

$$J_T = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

To get the partials, we just do some algebra and solve for x and y in terms of just u and v, allowing us to find the partials and compute the Jacobian (it's $\frac{1}{2}$).

$$\int_{-2}^{0} \int_{2}^{4} \frac{u}{v} \frac{1}{2} \, dv du$$

26.4 Example 4

$$\iint_R (4x - 4y + 1)^{-2} \, dx dy$$

Bounded by $x = \sqrt{-y}$, $x = y$, and $x = 1$. We are using the transformation $x = u + v$ and $y = v - u^2$. Converting the integrand:

$$(4(u + v) - 4(v - u^2) + 1)^{-2} = (2u + 1)^{-4}$$

We now need to find the Jacobian:

$$J_T = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = 2u + 1$$

We now know the Jacobian and the integrand, so we just need to find the bounds.
We start with $x = \sqrt{-y}$, which becomes $u + v = \sqrt{u^2 - v}$. Simplifying, we are left with $v = 0$ or $v = -2u - 1$. This one curve has turned into 2 separate boundaries

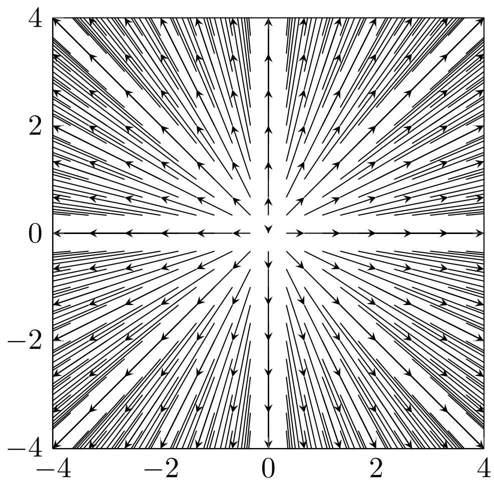
27 18.1

We will be talking about vector functions, which are equal to the sum of multivariable functions, and output a vector:

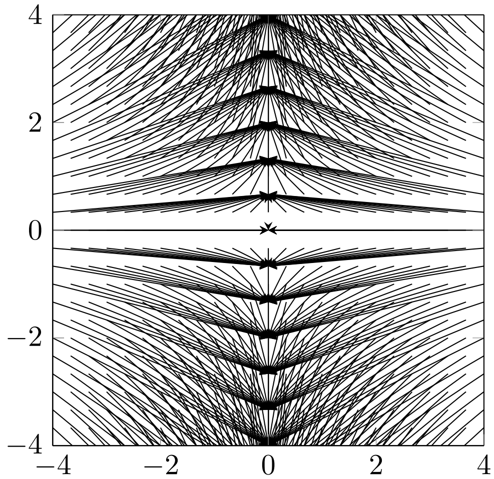
$$\mathbf{F}(x, y, z) = M(x, y, z)\hat{i} + N(x, y, z)\hat{j} + P(x, y, z)\hat{k}$$

These functions take in points in space, and associate with each point a vector. We have already probably worked with equations like this, such as wind and ocean currents. Vector funtions map from $\mathbb{R}^n \to \mathbb{R}^m$. Let's look at a very basic vector function:

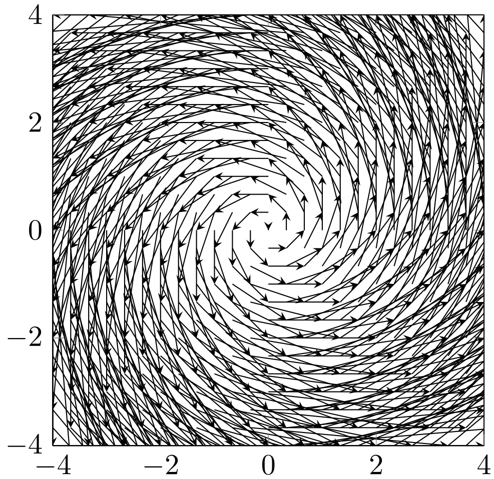
$$\mathbf{F} = \langle x, y \rangle$$


4
2
0
−2
−4
−4
−2
0
2
4

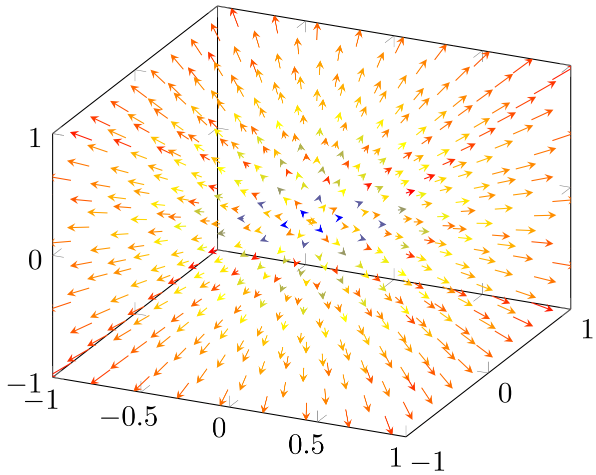
Lets look at the function $\mathbf{G}(x, y) = -x\hat{\imath} + y\hat{\jmath}$:


4
2
0
−2
−4
−4
−2
0
2
4

and the function $\mathbf{H}(x,y) - y\hat{\imath} + x\hat{\jmath}$:


4
2
0
−2
−4
−4
−2
0
2
4

If we look at what would happen if we put an animal in our vector field? That provides some properties about the vector fields. If we look at the first one, we would find that the animal would explode outwards, while in the second one the animal is squished (negatively exploded). The third one shows a whirlpool sort of effect, which we call whirl. We can also expand these vector fields to 3d, such as the following vector field, which represents $\mathbf{F} = \langle x, y, z \rangle$:


1
0
−1
−1
−0.5
0
0.5
1
−1
0
1

Let's do some algebra now. Since these functions can be represented by vectors, we can use the ∇ operator to operate on them:

$$\nabla \cdot \mathbf{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle x, y, z \rangle$$

We can also use it to cross vector functions:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

If we go back to the original 3 2d vector fields and take the dot product with ∇, we find that the dot product for the first vector function and 0 for the others. This gives insight to the fact that the first vector function was exclusively exploding, while the others were either not exploding or not exclusively exploding. When we take the cross product with ∇, we find that only the whirly vector field has a nonzero vector as the output. We see that the output is $2\hat{k}$. From this we can deduce that the 2 indicates how strong the whirl is, as well as the axis around which the whirl is rotating, in this case, the $\hat{k}$ direction. We have actual names for these. The $\nabla\cdot$ operation is known as **divergence**, denoted as $div\mathbf{F}$, and the $\nabla \times \mathbf{F}$ is known as the **curl**, and is denoted $curl\mathbf{F}$. Remember that we also have the gradient, $\nabla f = gradf$. The three of these operations together are the important operations of multivar.

28 18.2

Remember that a vector $\mathbf{r}(t)$ is smooth iff $\mathbf{r}'(t) \neq \mathbf{0}$ for any value of t. Given a $\mathbf{r}(t)$ for $t \in [a, b]$, the positive orientation is the direction going from $\mathbf{r}(a)$ to $\mathbf{r}(b)$. On a path integral, we can subdivide the path into many different small vectors. The measure of the change between $\mathbf{r}(t_i)$ and $\mathbf{r}(t_{i+1})$ is a small bit of arclength. We can then write the path integral on a curve:

$$\int_{\mathcal{C}} f(x, y)\, dS := \lim_{|\Delta S_i| \to 0} \sum_i f(x_i^*, y_i^*) \Delta S_i$$

However, this parameterization in terms of arclength is actually a pain, so we want to find a way to actually evaluate this. We have the curve $\mathcal{C} = \mathbf{r}(t)$, which can be parameterized in t:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle$$

We now need to take into account the fact that x and y are functions:

$$\int_{\mathcal{C}} f(x, y)\, dS = \int_a^b f(x(t), y(t)) \sqrt{(x')^2 + (y')^2}\, dt$$

Where we have input the formula for arclength of a function defined in x and y. We have also been able to convert the path integral to a regular integral. We can also look at how x changes along the path:

$$\int_{\mathcal{C}} f(x, y)\, dx$$

Note that this method does not hold y as a constant, as all the (x, y) points must be on the curve, which does not guarantee constant y values.

28.1 Scalar Path Integration

$$\int_{\mathcal{C}} dS = \int_a^b f(\mathbf{r}(t))\sqrt{(x')^2 + (y'^2)}\, dt$$

28.2 Vector Path Integration

Remember from physics that we can define Work as $\mathbf{F} \cdot \mathbf{d}$. If we take this concept and apply it back to path integration, the derivative of $\mathbf{r}$ gives us the direction that the path is moving in, and the force that we are looking at is given by some vector function. This means that we can get the value for some small change in work:

$$\Delta W = \mathbf{F}(x_i, y_i) \cdot \Delta \mathbf{r}(t_i)\Delta t$$

If we now take limits and turn it into an integral:

$$W = \int_{\mathcal{C}} \mathbf{F}(x, y) \cdot \mathbf{r}'\, dt$$

We have 4 different forms of writing this, and this one is known as the vector form of the path integral.

28.3 Conceptual Path Integration

We know that

$$\mathbf{r}' = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle = \frac{1}{dt}\langle dx, dy\rangle = \frac{1}{dt} d\langle x, y\rangle = \frac{1}{dt} d\mathbf{r}$$

This now gives the conceptual form of the path integral:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

28.4 Phantom Path Integrals

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{r}\, dt = \int_{\mathcal{C}} \langle M, N\rangle \cdot \frac{1}{dt}\langle dx, dy\rangle\, dt = \int_{\mathcal{C}} M\, dx + N\, dy$$

28.5 Tangent Vector Path Integration

We know that the tangent vector can be defined as $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$, and we can rewrite this:

$$\mathbf{r}' = \mathbf{T} \cdot \text{speed}$$

$$\mathbf{r}' = \mathbf{T} \cdot \frac{dS}{dt}$$

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, dS$$

There are some properties that are intuitive about path integrals, such as the fact that switching directions in a path integral negates the value of the path integral, and that path integrals of curves can be split into sums of the path integrals of component curves.

28.6 Example 1

Take the path integral:

$$\int_{\mathcal{C}} z^2\,dx - z\,dy + 2y\,dz$$

With the path being defined by 3 linear segments defined by the points

$$(0,0,0)$$

$$(0,1,1)$$

$$(1,2,3)$$

$$(1,2,4)$$

This essentially computes the work done by the vector field as we move across this path. Looking at the first linear segment, we can do this many different ways. We could define it as a line:

$$(0,0,0) + t\langle 0,1,1\rangle$$

Telling us that

$$x = 0 \to dx = 0$$

$$y = t \to dy = dt$$

$$z = t \to dz = dt$$

If we now just plug these in, we get the value for the first line segments:

$$\int_0^1 t^2 \cdot 0 - t\,dt + 2t\,dt = \frac{1}{2}$$

Looking at the second line segment, we can again define line:

$$(0,1,1) + t\langle 1,1,2\rangle$$

as $t : 0 \to 1$. This tells us:

$$x = t$$

$$y = 1 + t$$

$$z = 1 + 2t$$

Note that we can just use x instead of t, letting us say that:

$$x : 0 \to 1$$

$$y = 1 + x \to dy = dx$$

$$z = 1 + 2x \to dz = 2dx$$

We can now write the integral for $\mathcal{C}_2$:

$$\int_0^1 (1+2x)^2\,dx - (1+2x)\,dx + 2(1+x)(2dx)$$

We have now reduced the integral into a single variable integral, which we can compute quite easily. For curve $\mathcal{C}_3$, we notice that only 1 variable is changing, z. Thus we can simply say:

$$z = 3 \to 4$$

$$dx = 0$$

$$dy = 0$$

Writing the integral, plugging in 2 for y since the integral takes place on the curve and $y = 2$ for the duration of the entire curve:

$$\int_3^4 4\,dx = 4$$

28.7 Example 2

The path integral

$$\int_{\mathcal{C}} \frac{y}{x}\,dS$$

With $\mathcal{C}$ defined as follows:

$$x = t^4$$

$$y = t^3$$

$$\frac{1}{2} \leq t \leq 1$$

Note that this is a scalar path integral. Plugging in the curve, replacing dS with the surface area:

$$\int_{\frac{1}{2}}^1 \frac{t^3}{t^4}\sqrt{(4t^3)^2 + (3t^2)^2}\,dt$$

28.8 Example 3

We have a force $\mathbf{F}$ acting along a curve $\mathcal{C} = \mathbf{r}(t)$, from A to B. Note that $\mathbf{F}|_{\mathbf{r}} = m\mathbf{a}$, from Newton's Second Law. We know that

$$W = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{t_A}^{t_B} m\mathbf{r}'' \cdot \mathbf{r}'\,dt = m\int_{t_A}^{t_B} (\mathbf{r}'' \cdot \mathbf{r})\,dt$$

The inside looks like the inside of a product rule applied to $\mathbf{r}' \cdot \mathbf{r}'$. Thus we can just rewrite it:

$$m\int_{t_A}^{t_B} \frac{1}{2}\frac{d}{dt}(\mathbf{r}' \cdot \mathbf{r}')\,dt = m\int_{t_A}^{t_B} \frac{d}{dt}|\mathbf{r}'|^2\,dt$$

Using the FTC, we find that

$$W = (\frac{1}{2}m|\mathbf{r}'|^2)_{start}^{end} = \Delta KE$$

We have just proved the Work-Energy Theorem!
This leads into another definition.
F is **conservative** means

$$\exists f \quad s.t. \quad \mathbf{F} = \nabla f$$

In such a case, f is called the **potential** of **F**.

29 18.3

$$\mathbf{F} = \langle M, N \rangle$$

$$\exists f?\ s.t.\ \nabla f = \mathbf{F}$$

This means that we need a function f that satisfies the following:

$$\langle f_x, f_y \rangle = \langle M, N \rangle$$

This leads to a TFAE theorem (The Following Are Equivalent).

29.1 Fundamental Theorem of Path Integration

For vector function $\mathbf{F}$ with potential f, on path $\mathcal{C} = \mathbf{r}(t)$ for $t \in [a, b]$, and $\mathbf{F}$ is continuous on domain D containing $\mathcal{C}$,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = f|_{\mathbf{r}(a)}^{\mathbf{r}(b)}$$

Proof.

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}_{\mathbf{r}} \cdot \mathbf{r}'\, dt$$

$$= \int_a^b \nabla f|_{\mathbf{r}} \cdot \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle\, dt$$

$$\int_a^b (\frac{d}{dt} f)|_{\mathbf{r}}\, dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

□

29.2 TFAE

1. $\mathbf{F}$ is conservative
2. $\exists f$ s.t. $\nabla f = \mathbf{F}$
3.
4.
5. $M_y = N_x$

Proof. We can obtain statement 5 from 2:

$$f_x = M \quad f_y = N$$

$$f_{xy} = M_y \quad f_{yx} = N_x$$

By Clairaut's:

$$M_y = N_x$$

□

We can work the other way and prove that 5 implies 2, but we will save that for when we learned a couple more theorems. However, there are some caveats for when 5 implies 2. We need to be in 2D, and R has to be simply connected. We don't yet know what R even is, or what simply connected means. If we take 5, we can see that we have $M_y - N_x = 0$, which is the $\hat{k}$ portion of the curl cross product, which makes sense, since in 2D the axis of rotation is always the z axis.

29.3 Example 1

$$\mathbf{F} = \langle y^2 \cos z, 2xy \cos z, -xy^2 \sin z \rangle$$

We can first just compute M_y and N_x, and we see that they are both equal to $2y\cos z$, telling us that trying to prove the rest might not be a waste of time.
If $f_y = 2xy\cos z$, $f = xy^2 \cos z + D$. If $f_z = -xy^2 \sin z$, we need $f = xy^2 \cos z + E$. This tells us that we need something of the form $f = xy^2 \cos z$. This tells us that this is indeed a conservative vector function, so we can use FToPI.

29.4 Example 2

$$\mathbf{F} = \langle 2xz + y^2, 2xy, x^2 + 3z^2 \rangle$$

Once again we first check M_y and M_x, and indeed we find that they are both equal to $2y$. We need a function that satisfies:

$$f_x = 2xz + y^2 \rightarrow f = x^2 z + xy^2 + C$$
$$f_x = 2xy \rightarrow f = xy^2 + D$$
$$f_z = x^2 + 3z^2 \rightarrow f = x^2 z + z^3 + E$$

It seems that we have messed up! We see repeated terms, but there we can't merge these together! The flaw inherent in our solution is that the constants are constants with respect to the variables of integration, meaning that we can actually have the other variables inside the constants!

$$f_x = 2xz + y^2 \rightarrow f = x^2 z + xy^2 + C(y, z)$$
$$f_x = 2xy \rightarrow f = xy^2 + D(x, z)$$
$$f_z = x^2 + 3z^2 \rightarrow f = x^2 z + z^3 + E(x, y)$$

This allows us to find a final potential:

$$f = x^2 z + xy^2 + z^3$$

There is however another method that we can use instead of the method that we just used. We first calulate by looking at f_x that $f = x^2 z + xy^2 + C(y, z)$. We can now simply take the derivative with respect to y and compare it to the f_y we want to look at.

$$f_y = 0 + 2xy + \frac{\partial}{\partial y} C(y, z) = 2xy$$
$$C_y = 0$$

This tells us that C is simply a function of z:

$$f_z = x^2 + 0 + \frac{\partial}{\partial z} C = x^2 + 3z^2$$

We can then determine that $C = z^3$.

29.5 Example 3

$$\mathbf{F} = \langle y, x, xyz\rangle$$
$$M_y = 1 = N_x$$

Now we can be confident that there may be a potential, so lets find it:

$$f_x = y \to f = xy + C(y,z)$$
$$f_y = x \to f = xy + D(x,z)$$
$$f_z = xyz \to f = \frac{xyz^2}{2} + E(x,y)$$

We see that there is no way to mesh these together, which tells us that there is no potential f. This tells us that $\mathbf{F}$ is not conservative.

29.6 Example 4

Given conservative $\mathbf{F}$ with physics potential $-p$, then $-\nabla p = \mathbf{F}$. If we do a path integral:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = p_{start} - pend$$

We also know that the path integral is the same as the change in kinetic energy. This tells us that $p_{start} + KE_{start} = KE_{end} + p_{end}$. Turns out that the definitions of the path integral actually prove conservation of energy!

29.7 TFAE (again)

Moving back to the TFAE, lets try to do 2 implies 3:
By 2, we have a potential f such that $\mathbf{F} = \nabla f$. We know that through the FToPI:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = f|_{\mathbf{r}(a)}^{\mathbf{r}(b)}$$

Note that the right side of the expression does not care about what path we take, so it is known as being **independent of path**. Often, we these path integrals as

$$\int_a^b \mathbf{F} \cdot d\mathbf{r}$$

For example, if we have an $\mathbf{r}(t)$ s.t $\mathbf{r}_o = (1,5,4)$ and $\mathbf{r}_f = (2,2,-9)$:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{(1,5,4)}^{(2,2,-9)} \mathbf{F} \cdot d\mathbf{r}$$

Path independence is really nice, since it means that wwe can just ignore terrible paths if the force is path independent. Leaving us with the new and updated TFAE:

1. $\mathbf{F}$ is conservative
2. $\exists f$ s.t. $\nabla f = \mathbf{F}$

3. For all curves in $\mathbb{R}$, the path integral is independent of path
4.
5. $M_y = N_x$

C We would now like to truly link 3 to something else, in this case, we choose 2:

Proof. If we have a point (x_o, y_o), a point (x, y), and a point (x_1, y), and we let $f(x, y) = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, we can rewrite the path integral:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

Note that the path $\mathcal{C}_2$ is a horizontal line, so it's a nice path. Since we've assumed path independence, we can rewrite the path integrals:

$$f(x, y) = \int_{(x_o,y_o)}^{(x_1,y)} \mathbf{F} \cdot d\mathbf{r} + \int_{(x_1,y)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r}$$

Now taking the partial with respect to x in order to get f_x:

$$f_x = \frac{\partial}{\partial x} \int_{(x_1,y)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r}$$

Note that the first integral dropped out because there are no xs in it. Rewriting the second integral:

$$f_x = \frac{\partial}{\partial x} \int_{(x_1,y)}^{(x,y)} M\, dx + N\, dy$$

If we choose our path to be $\mathcal{C}_2$, the second term goes to 0, as y is constant. This leaves us with:

$$f_x = \frac{\partial}{\partial x} \int_{(x_1,y)}^{(x,y)} M\, dx$$

This integral is just a singlevar integral, so taking the derivative of it just invokes the FTC:

$$f_x = M(x, y)$$

This is exactly what we're trying to prove! We now need to prove that f_y is equivalent to the N part of the path integral. We can see that the proof of the second part is almost exactly the same thing, we simply move from (x_o, y_o) to (x, y_1) and from there to (x, y). $\square$

We now want to find 4. Let's assume 3 (and technically 2):

Proof. For an arbitrary smooth, closed curve (end where you started) $\mathcal{C}$ that begins at point P, we split up the curve into two arbitrary parts, $\mathcal{C}_1$ and $\mathcal{C}_2$, which touch at P_1. ($\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$). This tells us that the path integral can be split up:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

Since we are path independent:

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{-\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

Therefore:

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$$

The $\oint$ symbol means that the curve is closed. □

In order to truly add 4 to the TFAE, we have to prove that 4 implies something else. In this case, we chose to prove that 4 implies 3.

Proof. Take some $\mathcal{C}_1$ from A to B. Take some curve $\mathcal{C}_2$ from A to B. We know that $\mathcal{C}_1 \cup -\mathcal{C}_2$ is a closed loop. 4 tells us that $\oint_{\mathcal{C}_1 \cup -\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = 0$. Thus, we know that

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = -\int_{-\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

□

This ends the TFAE:

1. $\mathbf{F}$ is conservative
2. $\exists f$ s.t. $\nabla f = \mathbf{F}$
3. For all curves in $\mathbb{R}$, the path integral is independent of path
4. Closed curves have path integrals equal to 0
5. $M_y = N_x$

30 18.4

All of the calculus we've been doing has led to these three theorems. This is the first of the three.

30.1 Green's Theorem

Theorem 30.1. *For positively oriented, (finitely) piecewise-smooth, simple, closed, planar curve $\mathcal{C}$ with $R = \mathcal{C} \cup int(\mathcal{C})$ and if M and N have continuous first partials throughout an open domain $D \supseteq R$ (D contains R), then*

$$\oint_{\mathcal{C}} M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA$$

To tell whether a curve is positively oriented or negatively oriented, walk along the region and see which direction your left arm is pointing. If it is pointing in towards the region, the region is positively oriented. Notice that if $N_x - M_y = 1$, we see that we get the area of the region. We have stated Green's theorem, and we have realized that we can find the area:

Corollary 30.1.1. *If* $N_x - M_y = 1$,

$$\oint_{\partial R} M\,dx + N\,dy = Area$$

We can do a couple things, we can set $N_x = 1$*, and get* $M_y = 0$*, meaning that* $N = x$ *and* $M = 0$*. Or we can do* $N_x = 0$*,*$M_y = -1$*, leaving us with* $N = 0$ *and* $M = -y$*. We can also split them evenly and getting some sort of symmetric thing,* $N_x = \frac{1}{2}$*,* $M_x = \frac{1}{2}$*. This all leads into the statement of the corollary:*

$$\oint_{\partial R} x\,dy = \oint_{\partial R} -y\,dx = \frac{1}{2}\oint_{\partial R} x\,dy - y\,dx = A(R)$$

Proof. Use Green's theorem. □

Let's do an example:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Remember that the area of this shape is πab. We will prove this using the Green's theorem corollary. We can parameterize this in terms of θ:

$$\frac{x}{a} = \cos\theta$$

$$\frac{y}{b} = \sin\theta$$

$$x = a\cos\theta \to dx = -a\sin\theta\,d\theta$$

$$y = b\sin\theta \to dy = b\cos\theta\,d\theta$$

We know x and dy, so let's do the first integral:

$$\int_0^{2\pi} a\cos\theta \cdot b\cos\theta\,d\theta = ab\int_0^{2\pi}\cos^2\theta\,d\theta$$

This is ugly and requires a power reducing formula, so lets do the other one.

$$\int_0^{2\pi} -b\sin\theta(-a\sin\theta\,d\theta) = ab\int_0^{2\pi}\sin^2\theta\,d\theta$$

This would also require a power reducing formula, so lets do the right side instead:

$$\frac{1}{2}\int_0^{2\pi} ab\cos^2\theta\,d\theta + ab\sin^2\theta\,d\theta = \frac{ab}{2}\int_0^{2\pi}\cos^2\theta + \sin^2\theta\,d\theta$$

$$= \pi ab$$

As part of Green's theorem, we want to compute

$$\iint_R N_x - M_y\,dA$$

If the integrand goes to 0, we see that the integral is 0, regardless of the path R (as long as it's nice). This tells us that if $N_x - M_y = 0$ and R is nice:

$$\oint_{\partial R} \mathbf{F}\cdot d\mathbf{r} = 0$$

We just proved that TFAE 5 implies statement 4.
We have 2 lemmas that we are sort of going to prove, and then going to amend Green's theorem.

Lemma 30.2. *Green's theorem holds for more complicated regions (both Type 1 and Type 2).*

Proof. Assume Green's theorem holds. Let R be both a Type 1 and a Type 2 region. We can break up the region into regions that are Type 1 and Type 2. Say we have 5 regions, R_1 through R_5. We know that we can split the double integral up:

$$\iint_R N_x - M_y \, dA = \sum_{i=0}^{5} \iint_{R_i} N_x - M_y \, dA$$

Green's theorem now says for R_1:

$$\iint_{R_1} N_x - M_y \, dA = \oint_{\partial R_1} M \, dx - N \, dy$$

We can break this into multiple path integrals:

$$= \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{r}$$

Looking at R_2:

$$\iint_{R_2} N_x - M_y \, dA = \oint_{\partial R_2} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_4} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_5} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_6} \mathbf{F} \cdot d\mathbf{r}$$

However, in our picture (I don't have it here sorry guys), we can see that many of the curves cancel with each other, as the positively oriented paths move in different directions, leading to cancellation. We are left with only the paths that make up the boundary of R. $\square$

Lemma 30.3. *Green's Theorem applies for non-simple regions.*

Proof. If we take a circle, we have justified that we can use Green's theorem on this. If we cut a hole out of it and get a torus cross-section, we see that we have 2 boundary curves. We see that the two curves are oriented in different directions, the outside boundary going counter-clockwise and the inside boundary going clockwise.

$$\partial R = \mathcal{C}_1 \cup \mathcal{C}_2$$

Note that this is not a simple region because we can draw a loop inside the region that contains a part that is not part of the region. Separate the region into multiple regions... $\square$

Let's now prove the Pokeball theorem:

Proof. We have initial curves $\mathcal{C}_1$ and $\mathcal{C}_2$, which are oriented differently. We can split the torus cross section into two parts, each part having 4 boundaries, and then use Green's theorem:

$$\iint_{R_{top}} N_x - M_y \, dA = \oint_{\partial R_{top}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_{1A}} + \int_{\mathcal{C}_3} + \int_{\mathcal{C}_{2A}} + \int_{\mathcal{C}_4}$$

We can do the same thing for the bottom part of the torus. Note that $\mathcal{C}_5$ and $\mathcal{C}_5$ are the same curve but moving in the same direction, and the same thing is true for $\mathcal{C}_3$ and $\mathcal{C}_6$, which means that when we add the path integrals for both regions we see that we are left with

$$\int_{\mathcal{C}_{1A}} + \int_{\mathcal{C}_{1B}} + \int_{\mathcal{C}_{2A}} + \int_{\mathcal{C}_{2B}}$$
$$= \int_{\partial R}$$

□

We still need to do 3 things before we have proved Green's theorem.

Lemma 30.4. *Call the region D. The following identities are true:*

$$\int_{\mathcal{C}} M\,dx = -\iint_D M_y\,dA$$
$$\int_{\mathcal{C}} N\,dy = \iint_D N_x\,dA$$

Proof. Let $D = \{(x,y)|z \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$.

$$\iint_D M_y\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial}{\partial y} M\,dy\,dx$$

By the single variable FTC:

$$\int_a^b M|_{y=g_1(x)}^{y=g_2(x)}\,dx$$
$$= \int_a^b M(x, g_2(x)) - M(x, g_1(x))\,dx$$

We can't do this integral! we don't know M, so we can't really go any farther. Instead, we look at the other side of the equation, by splitting the region D into different boundaries:

$$\int_{\mathcal{C}} M\,dx = \int_{\mathcal{C}_1} M\,dx + \int_{\mathcal{C}_2} M\,dx + \int_{\mathcal{C}_3} M\,dx + \int_{\mathcal{C}_4} M\,dx$$

By looking at the different paths, we see that the first and third integrals disappear, and we are left with

$$\oint_{\mathcal{C}} M\,dx = \int_b^a M(x,y)|_{y=g_2(x)}\,dx + \int_a^b M(x,y)|_{y=g_1(x)}\,dx$$

Negating the first integral and swapping the bounds:

$$= -\int_a^b M(x, g_2(x))\,dx + \int_a^b M(x, g_1(x))\,dx$$
$$= i\int_a^b M(x, g_2(x)) - M(x, g_1(x))\,dx = -\iint_D M_y dA$$

The other half of the proof is the exact same thing, except that we get the double integral itself, without the negative. □

There are two types of Green's theorem problems. The first type you are asked to verify Green's theorem, by computing the path integral and computing the double integral. On the other type, you are asked to compute either the double integral or the path integral, and you have to decide whether you want to convert to the other form or compute it as is.

30.2 Verification Example

Take the vector function $\mathbf{F} = \langle x^2y^2, 4xy^3\rangle$. Verify that Green's theorem holds:

$$\iint_R 4y^3 - 2x^2y\, dA$$

$$\oint_{\partial R} x^2y^2\, dx + 4xy^3\, dy$$

Computing the double integral:

$$\int_0^3 \int_0^{y/3} 4y^3 - 2x^2y\, dxdy$$

$$\int_0^3 \frac{4y^4}{3} - \frac{2y^4}{3^4}\, dy = \int_0^3 (\frac{4}{3} - \frac{2}{3^4})y^4\, dy$$

$$= \frac{318}{5}$$

Setting the top boundary to be $\mathcal{C}_1$, we see that it is horizontal, so $y = 3 \to dy = 0$ and $x : 1 \to 0$. Setting up the integral:

$$\int_1^0 x^2 \cdot 3^2\, dx + 4x(3)^3 \cdot 0 = -3$$

Looking at the vertical boundary ($\mathcal{C}_2$), we see that $x = 0$ at all times so the integral will evaluate to 0. Taking the final boundary, we know that $y = 3x \to dy = 3dx$, with $x : 0 \to 1$. Setting up the integral:

$$\int_0^1 x^2(3x)^2\, dx + 4x(3x)^3 \cdot 3\, dy = \frac{333}{5}$$

Summing the different integrals together, we get $\frac{318}{5}$, exactly what the double integral got us. We see that nice boundaries (especially vertical and horizontal boundaries) are a big plus when using the path integrals.

31 18.5

This section is about surface integration. We take a chunk of a surface and partition it up, and we associate a value with it, and sum those values up. Essentially, these are the double integration form of path integrals:

$$\sum f \cdot \Delta S_i \to \sum f \cdot \Delta x_i$$

$$\sum f \cdot \Delta S_i \to \sum f \cdot \Delta A_i$$

When doing path integration, we knew that

$$ds = \sqrt{(x')^2 + (y')^2} = |\mathbf{r}'|\, dt$$

We now need to define a little bit of surface area in order to convert the sums into integrals. We use what we learned about surface area:

$$dS = |\mathbf{r}_u \times \mathbf{r}_v|\, dudv$$

Now comparing them in full integral form:

$$\int_{\mathcal{C}} f(x, y)\, ds \to \int_I f(x)\, dx$$

$$\iint_{(u,v)} f(x(u, v), y(u, v), z(u, v))|\mathbf{r}_u \times \mathbf{r}_v|\, dudv \to \iint_R f(x, y)\, dxdy$$

In essence, we're finding a sort of weighted surface area. Let's do an example of this now:

31.1 Example of Surface Integrals

Find the surface integral of the function $f(x, y, z) = 2z + 1$ over the cylinder of radius 2 and height 5 centered at the origin.
Note that calling this a surface is a slight misnomer, as it is truly 3 different surfaces put together, the top, the bottom, and the lateral surface. Taking the bottom surface, we see that r and θ are a good way of dealing with this surface:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = 0$$

We could now compute $\mathbf{r}_r \times \mathbf{r}_\theta$, but we should notice that we would be doing the surface integral of $2(0) + 1\,dS$, which leads to just the surface area of the surface, which we know is just 4π.
Similarly, looking at the top, we know that the function restricted to the surface can be given via:

$$f(x, y, z)|_S = f(x, y, 5) = 11$$

$$\iint_R 11\,dS = 11 \cdot 4\pi$$

Now looking at surface 3, we definitely want to use cylindrical, but we need two parameters, but that's easy since r is constant, so we can use θ and z:

$$\mathbf{r}(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z\rangle$$

$$\int_0^5 \int_0^{2\pi} f(x, y, z)|_{\mathbf{r}} \cdot |\mathbf{r}_\theta \times \mathbf{r}_z|\,d\theta dz$$

Computing the cross product:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2\cos\theta, 2\sin\theta, 0\rangle$$

Taking the magnitude of this:

$$\sqrt{4\cos^2\theta + 4\sin^2\theta} = 2$$

We can now plug this into te integral:

$$\int_0^{2\pi} \int_0^5 (2z + 1)|_{\mathbf{r}(\theta, z)} \cdot 2\,dzd\theta$$

$$= 2 \cdot 2\pi \int_0^{\pi} 2z + 1\,dz = 4\pi(z^2 + z)|_0^5$$

Note that us plugging in $\mathbf{r}(\theta, z)$ essentially means that we are replacing (x, y, z) with (r, θ, z), although in this case our function is already in terms of z, and the parameterization maps $z \to z$, so we don't really have to change anything.
Let's talk about vector surface integrals! These show up in physics as flux. We can think of flux as the amount of a vector field that is passing through a surface. We are interested in the vector form of surface integration:

$$\mathbf{F} \cdot \hat{N}\,dS$$

where $\mathbf{F}$ describes the strength and direction of the vector field and $\hat{N}$ represents the orientation of the flux collector.

$$\Phi = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

If we take the force of the vector field to be an electric field, we see that we have obtained Gauss's Law:

$$\frac{q_{enc}}{\epsilon_0} = \iint_R \mathbf{E} \cdot d\mathbf{S}$$

If we take a tube with a liquid flowing through it, we can place a stuff detector (think of a membrane or grate) in the tube. This will detect the amount of liquid that will be flowing through. We can have the grate at an angle as well, and we can disregard the components of the velocities that are parallel to the grate, which is another way to think about the fact that the orientation relative to the detector is important. We can't actually compute these integrals like this, so we need to find a way to reparameterize.
We need to evaluate $\iint_S \mathbf{F} \cdot \hat{n}\, dS$. We can easily convert dS:

$$dS = |\mathbf{r}_u \times \mathbf{r}_v|\, dudv$$

And then write the force as a vector:

$$F = \langle M, N, P \rangle$$

We can get the normal vector by normalizing the vector $\mathbf{r}_u \times \mathbf{r}_v$:

$$\hat{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

$$\iint_R \mathbf{F}|_{\mathbf{r}(u,v)} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dudv$$

31.2 Example

Take a cylinder at the origin of radius 2 and height 5.

$$\langle x, y+1, 2z+1 \rangle$$

Taking the bottom surface, we can see that the normal vector will be $-\hat{k}$:

$$\langle 0, 0, -1 \rangle$$

This gives us a double integral:

$$\iint_S \mathbf{F}|_{\mathbf{r}} \cdot \langle 0, 0, -1 \rangle\, dS$$

Since the vector is 0 for two of the components:

$$\iint_S -2z - 1\, dS$$

On this surface, the value of $z = 0$:

$$\iint_S -1\, dS = -4\pi$$

Surface two is the top potion of the cylinder:

$$\iint_S \mathbf{F} \cdot \langle 0, 0, 1 \rangle\, dS$$

$$\iint_S 2z + 1 = 44\pi$$

Note that we have just done the same thing we did for the first surface, just changed the value of z from 0 to 5. For the last surface, we need to do it out. We know that

$$\mathbf{r} = \langle 2\cos\theta, 2\sin\theta, z\rangle$$

Doing $\mathbf{r}_\theta \times \mathbf{r}_z$:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2\cos\theta, 2\sin\theta, 0\rangle$$

Plugging this in:

$$\iint_S \mathbf{F} \cdot \langle 2\cos\theta, 2\sin\theta, 0\rangle \, dS = \int_0^5 \int_0^{2\pi} \langle 2\cos\theta, 2\sin\theta + 1, 2z + 1\rangle \cdot \langle 2\cos\theta, 2\sin\theta, 0\rangle \, d\theta dz$$

Doing this out, we get 40π.

32 18.6

Theorem 32.1. *For a closed S enclosing volume Q, with positively oriented normal vector $\hat{n}$ and $\mathbf{F}$ has continuous partials throughout Q, then*

$$\Phi = \iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_Q div\mathbf{F} \, dV$$

Given an epsilon ball $B_\epsilon(P_o)$:

$$\forall P \in B_\epsilon(P_o), \; div\mathbf{F}|_P \approx div\mathbf{F}|_{P_o}$$

This tells us that the divergence can only change so much in a small region. From here we can claim that

$$\Phi \approx \iiint_{B_\epsilon(P_o)} (dvi\mathbf{F}|_{P_o}) \, dV$$

$$\Phi \approx div\mathbf{F}|_{P_o} \cdot \iiint_{B_\epsilon(P_o)} 1 \, dV$$

$$div\mathbf{F}|_{P_o} \approx \frac{\Phi}{V}$$

This tells us that divergence is the tendency of things to move away from where you are, which we have already sort of intuited when we drew vector fields.
We can do the same thing that we did for Green's Theorem, where we can split the solid into multiple solids and the interior faces have opposite orientations, so the faces cancel out. We can also do the Pokeball theorem, where we take a 3 dimensional pokeball and notice that we have 6 faces when we cut the pokeball in half, and we see that 2 of them cancel out.
Suppose we hae a weird solid that we want to find the flux through. What we can do is create a sphere that encloses the solid, and call the area around the solid but inside the sphere Q. We know that if $div\mathbf{F} = 0$ throughout Q, then

$$\iiint_Q div\mathbf{F} \, dV = 0$$

Which by the divergence theorem tells us that

$$\iint_{\partial Q} \mathbf{F} \cdot d\mathbf{S} = 0$$

where ∂Q is the combination of the sphere oriented out, and the bad thing oriented into the bad thing (out from Q). We can then set up the surface integral:

$$\iint \mathbf{F} \cdot d\mathbf{S} + \iint \mathbf{F} \cdot d\mathbf{S}$$

Where the first integral is for the sphere and the second integral is for the bad thing. However, the second integral is currently going into the bad thing, so we're going to want to switch the orientation of the second integral so that it is oriented positively:

$$\iint \mathbf{F} \cdot d\mathbf{S} - \iint \mathbf{F} \cdot d\mathbf{S} = 0$$

$$\iint \mathbf{F} \cdot d\mathbf{S} = \iint \mathbf{F} \cdot d\mathbf{S}$$

32.1 Divergence Theorem Example

Take the vector function $\langle xz, yz, 3z^2 \rangle$, with the solid $z = 1$ and $z = x^2 + y^2$. We can do this both as a surface integral or as a double integral. Let's do it as a surface integral first:

$$\iint_S \mathbf{F} \cdot \hat{n}\, dS$$

$$\iint \mathbf{F}|_S \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dudv$$

We know that $\hat{n} = \langle 0, 0, 1 \rangle$ for surface 1, the top disk of the surface. We know that on this disk, a little bit of $dS = dxdy$, so we can just use it without a conversion factor. We can now set up the integral:

$$\iint \langle xz, yz, 3z^2 \rangle|_S \cdot \langle 0, 0, 1 \rangle\, dxdy$$

Note that we can convert this to polar, but we can make that decision later. We know that on the surface, $z = 1$, so we can plug that in:

$$\iint 3\, dxdy = 3 \iint 1\, dxdy = 3\pi$$

We see that converting to polar doesn't really matter in this case. Let's move on to surface 2, the paraboloid. We know that $z = x^2 + y^2$. We see that we could convert to r and θ, but we don't like the function we have in polar, so let's once again hold off on converting. We can get $\mathbf{r}$:

$$\mathbf{r} = \langle x, y, x^2 + y^2 \rangle$$

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = \langle -2x, -2y, 1 \rangle$$

Notice that this vector is pointing the wrong way, so we have to switch the orientation of the vector, by negating it.

$$\iint \langle xz, yz, 3z^2 \rangle|_{z=x^2+y^2} \cdot \langle 2x, 2y, -1 \rangle\, dxdy$$

$$\iint 2x^2(x^2+y^2)+2y^2(x^2+y^2)-3(x^2+y^2)^2\,dxdy$$
$$=\iint -(x^2+y^2)^2\,dxdy$$

We look at this and we definitely want to do this in polar:

$$\int_0^{2\pi}\int_0^1 -(r^2)^2\,rdrd\theta$$
$$=-\frac{\pi}{3}$$

We see that the net flux through the solid is $3\pi-\frac{\pi}{3}$, which is a net positive, so we see that things are being generated inside the solid and are then exiting the solid.
We can then do it via a triple integral:

$$\iiint_Q z+z+6z\,dV$$
$$\iint\int_{x^2+y^2}^1 8z\,dz\,dA$$

Converting the outer integrals to polar:

$$\int_0^{2\pi}\int_0^1 (4-4r^2)\,rdrd\theta$$
$$=2\pi(2r^2-\frac{4r^6}{6}|_0^1)=\frac{8\pi}{3}$$

32.2 Stokes' Theorem

Theorem 32.2. *For S oriented and piecewise smooth, bounded by $\mathcal{C}$ which is simple, closed, piecewise smooth, and positively oriented (with respect to S), and $\mathbf{F}$ with continuous first partials on open region D containing S, then*

$$\oint_{\mathcal{C}}\mathbf{F}\cdot d\mathbf{r}=\iint_S\nabla\times\mathbf{F}\cdot d\mathbf{S}$$

32.3 Fixing the TFAE

Remember that we had a fake statement 5, which stated that $N_x=M_y$, which told us that $N_x-M_y=0$, which showed up in Green's theorem. We know that statement 4 tells us that every possible loop gives a path integral of 0, which means that the right hand side must have an integrand of 0:

$$\nabla\times\mathbf{F}=\mathbf{0}$$

Which is the real statement 5.

32.4 $S\subseteq xy$

If our surface is a subset of the xy-plane, we know that

$$\iint_S\nabla\times\mathbf{F}\cdot\mathbf{S}=\iint_{xy}\nabla\times curl\mathbf{F}\cdot\hat{k}\,dxdy=\iint_R N_x-M_y\,dxdy$$

We see that we have obtained Green's Theorem. Stokes' can be thought of as the generalized form of Green's Theorem.

32.5 Uses

Take an open box, which is not a closed region of space. Since it is not closed, we can't use the divergence theorem on it (although there are workarounds like considering it to be closed and subtracting the last face out). If we wanted to compute

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

We assert that there is some "out" (technically we haven't defined an inside or outside so we aren't quite sure). We can actually find out the orientation of the removed top by just looking at the other faces, and we see that it forces the correct orientation upon us. We can now compute the integral:

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial \mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

If we look at just this right side, we notice that it is actually a boundary for just the top piece, which actually condenses the 4 part path integral into a single double integral:

$$\oint_{\partial \mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{S_1}$$

Notice that the positive orientation is actually facing downwards.
Let $\mathbf{F}$ denote the velocities of some fluid. If we have a surface S, at any given point we have som value for $\mathbf{F}$, as well as some tangent vector $\mathbf{T}$, which is parallel to the $d\mathbf{r}$. We claim that $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is the tendency of the fluid to circulate around the edge of the surface. By Stokes' we can say that this is equal to $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$, and if we can approximate that the curl of $\mathbf{F}$ is pretty much the same at all points on the surface, we can pull out the dot product from the integral and get an expression for the curl:

$$\nabla \times \mathbf{F}_{P_o} \cdot \hat{n} \approx \frac{Tendency\ to\ circulate}{Surface\ Area}$$

If we go back to the first time we talked about curl, we considered it to be the "strength of spinniness", which we see is just a sort of calculation for the spin, but we see that the dot product only works out perfectly when it is orthogonal to the axis of rotation, but when $\hat{n}$ isn't perfect we don't really have a good intuition of what it is.

32.6 Example 1

$$\mathbf{F} = \langle y^2, x, z^2 \rangle$$

And S is the portion of $x^2 + y^2 = z$ that is under $z = 1$, oriented up.
We will want to paramaterize this:

$$\langle x, y, x^2 + y^2 \rangle$$

We can then compute $\mathbf{r}_x \times \mathbf{r}_y = \langle -2x, -2y, 1 \rangle$. We can then set up the integral:

$$\iint \nabla \times \mathbf{F}|_{z=x^2+y^2} \cdot \langle -2x, -2y, 1 \rangle \, d(x, y)$$

We can just compute the curl:

$$\nabla \times \mathbf{F} = \langle 0, 0, 1 - 2y \rangle$$

$$\iint \langle 0, 0, 1 - 2y \rangle \cdot \langle -2x, -2y, 1 \rangle \, d(x, y)$$

$$\iint 1 - 2y \, d(x, y) = \pi$$

At this point we just compute the integral.
Let's do this one with path integrals. We see that the boundary is the circle $z = 1$. We want to parameterize in terms of θ:

$$\mathbf{r}(\theta) = \langle \cos\theta, \sin\theta, 1 \rangle$$

We can easily get the $\mathbf{r}'$:

$$\langle -\sin\theta, \cos\theta, 0 \rangle$$

$$\int_0^{2\pi} \langle y^2, x, z^2 \rangle|_{\mathbf{r}} \cdot \langle -\sin\theta, \cos\theta, 0 \rangle \, d\theta$$

Notice that we have to be careful when we get the bounds in order to keep orientation correct. We can do the dot product out:

$$\int_0^{2\pi} -\sin^3\theta + \cos^2\theta \, d\theta$$

This is OIGOIG, and for the cosine term we'd need to power reduce. We see that doing it the way with fewer terms is not always the easiest way to do the problem, although in this case it is sometimes difficult to tell which one will be worse.
We can actually do the surface integral of any surface with the same boundary by Stokes' theorem, so let's do this with the disk with the same boundary. We know that in order to satisfy the boundaries orientation we need up to be the direction of the normal:

$$\hat{n} = \hat{k}$$

$$d\mathbf{S} = dxdy$$

$$\iint_{xy} \langle 0, 0, 1 - 2y \rangle \cdot \langle 0, 0, 1 \rangle \, dxdy$$

We see that this just gets us what we had before, but with a lot less work.

32.7 Example 2

Verify Stokes' theorem for $\mathbf{F} = \langle x, y, xyz \rangle$, and S is the portion of $2x + y + z = 2$ in octant 1, oriented upwards.
We want to parameterize $\mathbf{r}$ in terms of x and y:

$$\mathbf{r}(x, y) = \langle x, y, 2 - 2x - y \rangle$$

$$|\mathbf{r}_x \times \mathbf{r}_y| = \langle 2, 1, 1 \rangle$$

Note that this gives us the correct direction and the correct magnitude, but you couldn't just pull this out of the equation because you wouldn't necessarily get the correct magnitude (think $4x + 2y + 2z = 4$ would not get you $\langle 4, 2, 2 \rangle$). We could now compute the curl of $\mathbf{F}$:

$$\nabla \times \mathbf{F} = \langle xz, -yz, 0 \rangle$$

$$\iint \langle xz, -yz, 0 \rangle_{z=2-2x-y} \cdot \langle 2, 1, 1 \rangle \, dxdy$$

$$= \iint 4x - 4x^2 - 2xy - 2y + 2xy + y^2 \, dxdy$$

$$= \int_0^1 \int_0^{y=-2x+2} 4x - 4x^2 - 2y + y^2 \, dydx$$

We could now just do this out, and we see that we get -2.
Now doing the path integral, we have 3 separate pieces to worry about, but we see that each is on one of the coordinate axes. Doing the one along the xz plane, $x : 0 \to 1$ and $z = -2x + 2$.

$$\int_0^1 \langle x, y, xyz \rangle_{y=0, z=-2x+2} \cdot \langle 1, 0, -2 \rangle = \int_0^1 x \, dx = \frac{1}{2}$$

Doing the boundary that is in the xy plane, $z = 0$, $x : 1 \to 0$, $y = -2x + 2$.

$$\int_1^0 \langle x, y, xyz \rangle_{z=0, y=-2x+2} \cdot \langle 1, -2, 0 \rangle$$

$$= \int_1^0 5x - 4 \, dx$$

$$= \frac{3}{2}$$

Doing the last boundary, which is in the yz plane, we know that $y : 2 \to 0$, $x = 0$, $z = -y + 2$:

$$\int_2^0 \langle x, y, xyz \rangle_{x=0, z=-y+2} \cdot \langle 0, 1, -1 \rangle \, dy$$

$$= \int_2^0 y \, dy$$

$$= -2$$