PHYS624 Notes

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1 Motivation

Why do we need QFT? Consider a problem we already know, the Hydrogen atom. Consider an electron in the $2s$ state. If we wait long enough, it will decay to the $1s$ state. How long does it take for the electron to decay to decay from $2s \to 1s$? In QM, we spent time computing the energy splitting between the two states, which sets the frequency of the Lyman line. We never posed the question of how long it takes to transition between the two states. The reason for this is that in non-relativistic QM, the transition time is infinite, these are stationary states, they never decay. Since both mass and energy are conserved separately, the decay process

$$2s \to 1s + \text{photons}$$

can never occur, since particle number is conserved in non-relativistic QM. Usually in non-relativistic QM, we begin with a particle and we maintain that particle, the norm of the particle's wavefunction is conserved in time. In this case, we have photons that come into being, and we need some formalism that allows us to describe processes like this. Let us write down the Schrodinger equation for the Hydrogen atom:

$$\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi\left(\boldsymbol{r},t\right) - \frac{e^2}{r}\psi\left(\boldsymbol{r},t\right) = i\hbar\frac{\partial\psi}{\partial t}$$

We have a wavefunction that describes the electron, and the $\frac{e^2}{r}$ term comes from the electromagnetic field, and we are treating this field completely classically, we are using the classical Coulomb potential. The photons are quantum objects, and they are excitations in the EM field, which means we need to treat them quantum mechanically. Just like we quantized the motion of the electron into $\psi\left(\boldsymbol{r},t\right)$, we need to quantize the EM field in order to obtain a quantum mechanical formalism for the decay process.

QFT has many subtleties, but there is a central idea that we want to highlight. In QM, we discuss wave-particle duality: if we quantize the motion of particles, we observe wave behavior. What we will see in this course is that if we quantize the motion of waves, we get particles, the duality holds bidirectionally. Most of this course will be exploring the quantization of waves and how they generate particles.

What does this other direction of the duality mean? For every particle we think of in nature, we can start with a field description, and each particle will be an excitation of the field, i.e. an electron is an excitation of the electron field.

There are three ingredients that go into QFT:

1. Non-relativistic quantum mechanics
2. Special Relativity
3. Classical field theory

With these three things, we can produce relativistic quantum field theory. In fact, we can pick any of two of these, and we have a consistent subject. For example, if we put non-relativistic QM and special relativity together, we get relativistic QM. If we put classical field theory and special relativity together, we get relativistic classical field theory (such as E&M). Finally, if we put non-relativistic QM together with classical field theory, we will get non-relativistic QFT.

In this course, we will choose to discuss classical field theory and special relativity, to obtain relativistic classical field theory. We will then consider relativistic QM, then non-relativistic QFT, and then finally we will put them all together to look at relativistic QFT.

2 Classical Field Theory

This discussion is taken from the last chapter of Goldstein.

2.1 Discrete Systems

We can think of classical field theory as the mechanics of continuous media. The way we approach a continuous system is to take it as the limit of a discrete system with many degrees of freedom, we take the continuum limit to recover the continuous system.

Consider an infinitely long elastic rod that undergoes longitudinal vibrations, that is, compression waves. We will approximate this as an infinite chain of point masses spaced a distance a apart, connected by massless springs with spring constant k:

Suppose we have a vibration along this chain, we displace the masses from their equilibrium positions. We label the displacements by η, and we index each mass, that is, the displacement of the ith mass is η_i. At equilibrium, $\eta_i = 0$ for all i.

We can write down the kinetic energy in the chain:

$$T = \frac{1}{2} m \sum_i \dot{\eta}_i^2$$

And the potential energy:

$$V = \frac{1}{2} k \sum_i \left(\eta_{i+1} - \eta_i\right)^2$$

And then write down the Lagrangian:

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} \sum_i \left[m\dot{\eta}_i^2 - k\left(\eta_{i+1} - \eta_i\right)^2 \right] \end{aligned}$$

We can rewrite this to introduce the chain spacing:

$$L = \sum a L_i$$

Where

$$L_i = \frac{1}{2} \frac{m}{a} \dot{\eta}_i^2 - \frac{1}{2} ka \left(\frac{\eta_{i+1} - \eta_i}{a} \right)^2$$

We now want to relate the quantity ka to the Young's modulus of the material, Y. To do this, we first note that for an elastic rod, from Hooke's Law, the force is equal to the Young's modulus times ξ, the extension per unit length:

$$F = Y\xi$$

Let us apply this to our system. Consider a constant force being applied to one end of the rod. In this case, we have a uniform tension being applied to the springs, which is given by Hooke's Law for a spring:

$$F = k\left(\eta_{i+1} - \eta_i\right)$$

We can rewrite this:

$$F = ka\left(\frac{\eta_{i+1} - \eta_i}{a}\right)$$

Now recall that a is the chain spacing, and therefore ka is the Young's modulus, and the remaining term is exactly the displacement per unit length.

Now let us take the continuum limit of our discrete system. To do this, we move from the discrete index i to a continuous index x. When we make this replacement, we have that η_i becomes $\eta\left(x\right)$, and η_{i+1} becomes $\eta\left(x+a\right)$. Previously, we labelled each mass by its counted number. Instead, we now label each mass by its location at equilibrium. $\eta\left(x\right)$ is the displacement of the mass that, when the system is in equilibrium, would be sitting at location x. Note that x is not a dynamical variable, it is just a constant that labels the equilibrium locations, $\eta\left(x\right)$ is the dynamical variable. This is essentially downgrading x to the level of t, instead of a dynamical variable, it is a parameter that the actual dynamical variables depend on, which is foreshadowing the introduction of relativity, but everything here is completely classical.

Now if we look at how our expressions change when we make this continuum limit:

$$\frac{\eta_{i+1} - \eta_i}{a} \rightarrow \frac{\eta\left(x+a\right) - \eta\left(x\right)}{a}$$

Now we note that in the continuum limit, this becomes $d\eta/dx$:

$$\frac{\eta\left(x+a\right) - \eta\left(x\right)}{a} = \frac{d\eta}{dx}$$

Now if we look at our summation in the continuum limit:

$$a\sum_i \rightarrow \int dx$$

And at our m/a, which now becomes the mass per unit length, μ:

$$\frac{m}{a} \rightarrow \mu$$

Putting all of these together, we find that the full continuum Lagrangian is given by

$$L = \frac{1}{2}\int \mathrm{d}x \left[\mu\dot{\eta}^2 - Y\left(\frac{\partial\eta}{\partial x}\right)^2\right]$$

Now using the Lagrangian, we can obtain the equation of motion. Let us first look at the discrete equations of motion for the ith mass:

$$m\ddot{\eta}_i - k\left(\eta_{i+1} - \eta_i\right) - k\left(\eta_i - \eta_{i-1}\right) = 0$$

Suppose we now take the continuum limit of this equation. In this case, we have that:

$$\eta_{i+1} - \eta_i \to a \left(\frac{\partial \eta}{\partial x}\right)\bigg|_x$$
$$\eta_i - \eta_{i-1} \to a \left(\frac{\partial \eta}{\partial x}\right)\bigg|_{x-a}$$

This gives us the continuum expression:

$$a \left[\mu \frac{\partial^2 \eta}{\partial t^2} - ka \frac{\partial^2 \eta}{\partial x^2}\right] = 0$$

Now recall that $Y = ka$, so we have the continuum equation of motion:

$$\mu \frac{\partial^2 \eta}{\partial t^2} - Y \frac{\partial^2 \eta}{\partial x^2} = 0$$

Which is the wave equation, and our wave velocity will be $v = \sqrt{Y/\mu}$. We obtained this by taking the continuum limit of the discrete equation of motion, but let us now recover this directly from the continuum Lagrangian that we derived earlier, rather than first discussing the discrete case.

2.2 Continuous Lagrangian Formalism

Usually, when we do particle mechanics, we write down an action, which is the time integral of a Lagrangian. In this case our Lagrangian is itself an integral over a variable. We denote the integrand as the Lagrangian density, $\mathcal{L}$:

$$L = \frac{1}{2} \int \mathrm{d}x \left[\mu \dot{\eta}^2 - Y \left(\frac{\partial \eta}{\partial x}\right)^2\right]$$
$$\mathcal{L} = \frac{1}{2} \left[\mu \dot{\eta}^2 - Y \left(\frac{\partial \eta}{\partial x}\right)^2\right]$$

Using this denotation, the action is the time integral and the spatial integral of $\mathcal{L}$.

We want to obtain the equation of motion directly from $\mathcal{L}$. In general, the Lagrangian density is a function of η and it its partials[1], along with the parameters that we have, x and t:

$$\mathcal{L} = \mathcal{L}\left(\eta, \frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial t}, x, t\right)$$

Starting from this, we define the action[2]:

$$S = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \mathrm{d}x \, \mathrm{d}t \, \mathcal{L}$$

We want to extremize the action with respect to variations of the dynamical variable, η. Note that we fix the endpoints in t and x, at both ends of the trajectory. We thus fix the variation of η at the endpoints to be zero.

[1]Note that we are not technically restricted to just the first order partials, but for higher order partials, we end up with differential equations that are harder to solve and produce spurious solutions.

[2]Chacko uses I to denote the action.

Suppose the variation is of the form:

$$\eta\left(x,t\right)=\eta_0\left(x,t\right)+\delta\eta\left(x,t\right)$$

In this form, our previous fixing of the variation is written as:

$$\begin{aligned}\delta\eta\left(x_1,t\right)=\delta\eta\left(x_2,t\right)=0\\ \delta\eta\left(x,t_1\right)=\delta\eta\left(x,t_2\right)=0\end{aligned}$$

We can now write out the variation in the action:

$$\delta S=\iint \mathrm{d}x\,\mathrm{d}t\,\left[\frac{\partial\mathcal{L}}{\partial\eta}\delta\eta+\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\eta}{\partial x}\right)}\delta\left(\frac{\partial\eta}{\partial x}\right)+\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\eta}{\partial t}\right)}\delta\left(\frac{\partial\eta}{\partial t}\right)\right]$$

Now thinking back to Lagrangian dynamics, we note that

$$\begin{aligned}\delta\left(\frac{\partial\eta}{\partial x}\right)&=\frac{\partial\left(\eta_0+\delta\eta\right)}{\partial x}-\frac{\partial\eta_0}{\partial x}\\ &=\frac{\partial}{\partial x}\delta\eta\end{aligned}$$

And similarly for $\frac{\partial\eta}{\partial t}$. This allows us to rewrite the change in our action:

$$\delta S=\iint \mathrm{d}x\,\mathrm{d}t\,\left[\frac{\partial\mathcal{L}}{\partial\eta}\delta\eta+\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\eta}{\partial x}\right)}\underbrace{\delta\left(\frac{\partial\eta}{\partial x}\right)}_{\frac{\partial}{\partial x}\delta\eta}+\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\eta}{\partial t}\right)}\underbrace{\delta\left(\frac{\partial\eta}{\partial t}\right)}_{\frac{\partial}{\partial t}\delta\eta}\right]$$

Now integrating by parts, and noting that the boundary terms vanish because of the fixed boundary conditions:

$$\delta S=\iint \mathrm{d}x\,\mathrm{d}t\,\left[\frac{\partial\mathcal{L}}{\partial\eta}-\frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\eta}{\partial x}\right)}\right)-\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\eta}{\partial t}\right)}\right)\right]\delta\eta$$

If we set $\delta S=0$, then we see that the only way for this to be true is if everything in the square brackets is zero, which is the same as the usual Lagrangian argument. This leaves us with the Euler-Lagrange equation for the continuous case:

$$\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\eta}{\partial t}\right)}\right)+\frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\eta}{\partial x}\right)}\right)-\frac{\partial\mathcal{L}}{\partial\eta}=0$$

This is the equation that must be satisfied on the classical trajectory, the one that extremizes the action. Note that in the discrete case, we had a set of coupled ODEs, but in the continuous case we have a single PDE.

Let us apply this to our elastic rod, and see if this recovers the previously obtained equation of motion. We have our Lagrangian density:

$$\mathcal{L}=\frac{1}{2}\mu\left(\frac{\partial\eta}{\partial t}\right)^2-\frac{1}{2}Y\left(\frac{\partial\eta}{\partial x}\right)^2$$

Now computing our partials:

$$\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \eta}{\partial x}\right)} = -Y\frac{\partial \eta}{\partial x}$$
$$\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \eta}{\partial t}\right)} = \mu\frac{\partial \eta}{\partial t}$$

Similarly, we can look at $\frac{\partial \mathcal{L}}{\partial \eta}$:

$$\frac{\partial \mathcal{L}}{\partial \eta} = 0$$

Which, when inserted into our equation, gives us:

$$\mu\frac{\partial^2 \eta}{\partial t^2} - Y\frac{\partial^2 \eta}{\partial x^2} = 0$$

Which is exactly what we obtained from taking the discrete system to the continuum limit.

In this case, we have only used a single dynamical field, η. How do we generalize this to multiple fields?

Suppose we are now working with more spatial dimensions. In this case, we move from t, x to x^μ, where μ is an index, $\mu = 0, 1, 2, 3$, where $x^0 = t$, $x^1 = x$, $x^2 = y$, and $x^3 = z$. Suppose we have a general number of fields, so η becomes η^ρ, where ρ indexes over some arbitrary number of indices, it may be a Lorentz index, or it could be any number of scalar fields. We keep this arbitrary so that we can derive all cases at once.

We can write down the general Lagrangian density:

$$\mathcal{L} = \mathcal{L}\left(\eta^\rho, \partial_\nu \eta^\rho, x^\nu\right)$$

We want to extremize the action, which is now an integral over all spacetime:

$$S = \int \mathrm{d}^4x\, \mathcal{L}$$

Note that in this formalism, space and time are on equal footing, so it will be easy to generalize to relativity.

Now looking at variations in η^ρ:

$$\eta^\rho = {\eta_0}^\rho + \delta\eta^\rho$$

We again fix the endpoints, $\delta\eta^\rho = 0$ at the endpoints in spacetime.

We can look at variations in the action, where we use Einstein notation, summation over repeated indices is implied:

$$\delta S = \int \mathrm{d}^4x\, \left[\frac{\partial \mathcal{L}}{\partial \eta^\rho}\delta\eta^\rho + \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \eta^\rho\right)}\delta\left(\partial_\nu \eta^\rho\right)\right]$$

Again noting that $\delta\left(\partial_n u\eta^\rho\right) = \partial_\nu\left(\delta\eta^\rho\right)$, and integrating by parts, we have that

$$\delta S = \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial\eta^\rho}\delta\eta^\rho - \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\eta^\rho\right)}\right)\right]$$

Setting this equal to zero, and using the same argument as the single field case, we have the general Euler-Lagrange equation in the continuous formalism:

$$\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\eta^\rho\right)}\right) - \frac{\partial\mathcal{L}}{\partial\eta^\rho} = 0$$

With this, we can take a very general Lagrangian density, and then obtain the equation of motion.

Recall the classical dynamics of a single point particle. In this case, we have the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

We see that this is very similar to our continuous result, we just make space take the same footing as time, and we recover the same form.

2.3 Energy-Momentum Tensor

Recall from point particle mechanics, that we have energy conservation if the Lagrangian does not explicitly depend on time. In our continuous formalism, we will show that if the Lagrangian density does not depend on x^0, we have energy conservation, and if the density does not depend on x^i then p^i is conserved.

Let us first recall the classical proof of this, which we will then generalize to the field formalism.

If we have no explicit dependence of L on t, then we have that

$$L = L\left(q_i, \dot{q}_i\right)$$

If this is the case, then

$$\begin{aligned}\frac{dL}{dt} &= \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\frac{d}{dt}\left(\dot{q}_i\right) + \frac{\partial L}{\partial q_i}\frac{dq_i}{dt}\\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\dot{q}_i\right) - \dot{q}_i\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) + \frac{\partial L}{\partial q_i}\frac{dq_i}{dt}\end{aligned}$$

Where we have rewritten the first term. Now applying the equation of motion, we know that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

Inserting this, we see that the second and third terms cancel:

$$\frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\dot{q}_i\right)$$

Which can be rewritten:

$$\frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right) = 0$$

Now nothing that this is just the time derivative of the Hamiltonian:

$$\frac{d}{dt}H = 0$$

We see that the total energy (the Hamiltonian) is a constant of the motion.

Now let us generalize this to the field formalism. We have a Lagrangian density, which is a function of our fields η^ρ, their partials, $\partial_\nu \eta^\rho$, but explicitly not a function of x^μ:

$$\mathcal{L} = \mathcal{L}\left(\eta^\rho, \partial_\nu \eta^\rho\right)$$

We can look at $\frac{\partial L}{\partial x^\nu}$:

$$\frac{\partial \mathcal{L}}{\partial x^\nu} = \frac{\partial \mathcal{L}}{\partial \eta^\rho}\partial_\nu \eta^\rho + \frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha \eta^\rho\right)}\partial_\nu\left(\partial_\alpha \eta^\rho\right)$$

Now rewriting the second term, just as we did in the classical derivation:

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha \eta^\rho\right)}\partial_\nu \partial_\alpha \eta^\rho = \partial_\alpha\left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha \eta^\rho\right)}\partial_\nu \eta^\rho\right] - \partial_\alpha\left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha \eta^\rho\right)}\right]\partial_\nu \eta^\rho$$

By the equation of motion, we see that the second term here cancels with the first term in the equation above.

Thus we are left with

$$\frac{\partial \mathcal{L}}{\partial x^\nu} = \partial_\alpha\left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha \eta^\rho\right)}\partial_\nu \eta^\rho\right]$$

Which we can rewrite as:

$$\partial_\alpha\left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha \eta^\rho\right)}\partial_\nu \eta^\rho - \mathcal{L}\delta^\alpha{}_\nu\right] = 0$$

We define the quantity in brackets as $T^\alpha{}_\nu$, which is known as the energy-momentum tensor (or the stress-energy tensor):

$$T^\alpha{}_\nu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha \eta^\rho\right)}\partial_\nu \eta^\rho - \mathcal{L}\delta^\alpha{}_\nu$$

Our equation tells us that this tensor is a constant of the motion:

$$\partial_\alpha T^\alpha{}_\nu = 0$$

We can compare this to the continuity equation from electromagnetism:

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}$$

Which can be rewritten in tensor notation:

$$\partial_\mu j^\mu = 0$$

Where $j^0 = \rho$, and $j^i = \boldsymbol{j}$. This equation looks very similar, except for the fact that our recently derived equation has two indices, rather than 1. The continuity equation tells us that the net current inflow is equal to the charge enclosed in the region. We can show this by integrating the equation over some region:

$$\int d^3x\,\frac{\partial\rho}{\partial t} = \int d^3x\,(-\boldsymbol{\nabla}\cdot\boldsymbol{J})$$
$$\frac{\partial}{\partial t}\int d^3x\,\rho = -\int d^3x\,\boldsymbol{\nabla}\cdot\boldsymbol{J}$$
$$\frac{\partial}{\partial t}Q_{\text{enclosed}} = -\int d\boldsymbol{A}\cdot\boldsymbol{J}$$

We see that this equates the change in enclosed charge to the flux of the current flowing out of the region. In the case where the volume is all of space, current vanishes at the boundary, and we have global conservation of charge:

$$\frac{\partial}{\partial t}Q_{\text{enclosed}} = 0$$

We will show that the energy-momentum tensor relation that we derived behaves exactly the same, but with 4 independent relations, due to the extra index. In the case where $\nu = 0$, we get energy conservation, and when $\nu = 1, 2, 3$, we get momentum conservation in the respective direction. If we look at some volume, we get similar continuity relations for energy and momentum. Let us show this. We can write out the relation and then integrate over a volume:

$$\partial_t T^0{}_\nu + \partial_j T^j{}_\nu = 0$$
$$\partial_t\int d^3x\,T^0{}_\nu + \int d^3x\,\partial_j T^j{}_\nu = 0$$

If we integrate over all of space, then the second term becomes a surface integral, and vanishes:

$$\int d^3x\,\partial_j T^j{}_\nu \to \int ds\,n_j T^j{}_\nu = 0$$

Then we have just the first term:

$$\partial_t \underbrace{\int d^3x\,T^0{}_\nu}_{R_\nu} = 0$$

We denote the 4 quantities R_ν the "conserved charges". We denote $T^0{}_\nu$ as the "charge densities", and $T^j{}_\nu$ are the "charge current densities". These are all named via direct analogy to electromagnetism. We have shown that these quantities are conserved, but we have not brought in any ideas of energy or momentum.

Let us consider again the elastic rod. In this case, we can compute $T^0{}_0$:

$$T^0{}_0 = \frac{1}{2}\mu\left(\frac{\partial\eta}{\partial t}\right)^2 + \frac{1}{2}Y\left(\frac{\partial\eta}{\partial x}\right)^2$$

We see that these are the kinetic energy density and potential energy density, respectively. Thus $T^0{}_0$ is the total energy density of the system. Thus we see that the conservation of R_0 over all of space indicates that energy is conserved over all of space.

We could also consider ${T^j}_0$, which are energy current densities[3]. Instead, let us come to ${T^0}_x$ in the case of the elastic rod:

$$
{T^0}_x = \mu\dot{\eta}\frac{\partial\eta}{\partial x}
$$

Let us consider the momentum density of the elastic rod. There is a contribution $\mu\dot{\eta}$ (mass times velocity), which is there even for rigid body motion. We want the contribution to the momentum from wave motion inside the rod. We first notice that when wave motion happens, there is a net change in the mass of the element of the rod between x and $x + dx$. If $\eta(x) > \eta(x + dx)$, then the mass between the two points has decreased. This change is given by:

$$
\mu\left[\eta(x) - \eta(x + dx)\right] \to -\mu\frac{\partial\eta}{\partial x}\,dx
$$

Where we have taken the continuum limit. The net change in momentum between x and $x + dx$ associated with this motion is:

$$
\left(-\mu\frac{\partial\eta}{\partial x}\,dx\right)\dot{\eta} = \left(-\mu\dot{\eta}\frac{\partial\eta}{\partial x}\right)\,dx
$$

Which matches exactly what we got for ${T^0}_x$ (with a minus sign). This is known as the wave (or field) momentum density, and the integral of this over all space is conserved. Thus we have connected our conserved R_ν quantities to the conservation of energy and wave momentum in the elastic rod case.

2.4 Hamiltonian Formulation

Up until now, we have been working with the Lagrangian formalism, but at various points we will consider the Hamiltonian formulation. Let us once again consider the discrete chain of masses and springs. Conjugate to each η_i, we have a canonical momentum, p_i:

$$
\begin{aligned}
p_i &= \frac{\partial L}{\partial\dot{\eta}_i} \\
&= a\frac{\partial L_i}{\partial\dot{\eta}_i}
\end{aligned}
$$

When we take the continuum limit of this, we see that p_i vanishes, since $L_i \sim \frac{1}{2}\mu\dot{\eta}_i^2$, and $a \to 0$. The derivative is finite, but a goes to zero, so $p_i = 0$. However, we can define the momentum density π, which is nonzero in the continuum limit:

$$
\begin{aligned}
\pi &= \lim_{a\to 0}\frac{p_i}{a} \\
&= \frac{\partial\mathcal{L}}{\partial\dot{\eta}}
\end{aligned}
$$

Note that in general, π is a function of x. Now that we have the conjugate momentum, we can write down the Hamiltonian:

$$
H = \sum_i p_i\dot{\eta}_i - L
$$

[3]Goldstein shows these in the case of the elastic rod, and discusses the physical intuition for them, as well as the momentum current densities.

$$= a \sum_i \left(\frac{\partial L_i}{\partial \dot{\eta}_i} \dot{\eta}_i - L_i \right)$$

Writing this in the continuum limit:

$$H = \int dx \, \left(\pi \left(x \right) \dot{\eta} \left(x \right) - \mathcal{L} \right)$$

Which defines the Hamiltonian density, $\mathcal{H}$:

$$\mathcal{H} = \pi \left(x \right) \dot{\eta} \left(x \right) - \mathcal{L}$$

Essentially, when we define a valid continuum momentum, the Hamiltonian density is exactly what we would expect. We can generalize this to more than 1 field, where we now have multiple momenta:

$$\pi_\rho = \frac{\partial \mathcal{L}}{\partial \left(\dot{\eta}^\rho \right)}$$

For each field, we have a conjugate momentum. We can then write the multiple-field Hamiltonian density:

$$\mathcal{H} = \sum_\rho \pi_\rho \dot{\eta}^\rho - \mathcal{L}$$

Note that this is exactly what we found for ${T^0}_0$:

$$\mathcal{H} = {T^0}_0$$

This should not be very surprising, since this is exactly the same result we find in classical mechanics, as long as our Lagrangian is not explicitly time-dependent, the Hamiltonian is a constant of the motion.

2.5 Noether's Theorem

Let us now discuss Noether's theorem, which is a connection between the symmetry properties of the Lagrangian, and conserved quantities, known as currents. We have seen this theorem from classical mechanics, let us now discuss the form that this theorem takes in classical field theory.

Symmetry in Physics

For some historical background, at the turn of the 20th century, we had several revolutionary ideas. The first of these was special relativity, which not only introduced a new set of physical laws, but also changed the way that physicists do physics. When Maxwell's equations were written down, these laws came out of a lot of experimentation, a long series of experiments. The way that Einstein derived $E = mc^2$ was to realize that Maxwell's equations have a symmetry property, they are invariant under Lorentz transformations. He then imposed the requirement of Lorentz symmetry onto the other laws of physics and explored what happened. He was the first physicist to put symmetry first. After Einstein did this, Dirac required that the laws of quantum mechanics be invariant under Lorentz transformations, and derived the existence of the positron. Pauli predicted the neutrino via the symmetry of beta decays, Gell-Mann generalized isospin symmetry to discover the Ω^-. This method was used extensively since Einstein in the 20th century, the application of symmetries to the laws of physics.

Theorem 2.1. *Noether's Theorem. For every continuous symmetry transformation of the Lagrangian, there exists a conserved current.*

Note that this is only in one direction, symmetries imply conserved quantities, but conserved quantities do not imply symmetries. Another thing to note is that the transformations that produce conserved currents are necessarily continuous, there must be some infinitesimal generator of the symmetry transformation.

Let us now prove Noether's Theorem. Let us first consider a transformation[4]:

$$x^\mu \to x^{\mu'} = x^\mu + \delta x^\mu \left(x^\nu\right)$$

Where the second term is a function x^μ of x^ν. This can be any general transformation. This transformations affects our field:

$$\eta^\rho \left(x^\mu\right) \to \eta^{\rho'} \left(x^{\mu'}\right) = \eta^\rho \left(x^\mu\right) + \delta \eta^\rho \left(x^\mu\right)$$

This changes our Lagrangian density, but we can claim that the functional form is the same if the transformation is a symmetry of the Lagrangian:

$$\mathcal{L}\left[\eta^\rho \left(x^\mu\right), \partial_\nu \eta^\rho \left(x^\mu\right), x^\mu\right] = \mathcal{L}\left[\eta^{\rho'} \left(x^{\mu'}\right), \partial'_\nu \eta^{\rho'} \left(x^{\mu'}\right), x^{\mu'}\right] \tag{1}$$

This is known as *form invariance*, the form of the Lagrangian does not change. If the transformation is not a symmetry, it would be $\mathcal{L}'$ on the right side, rather than the original $\mathcal{L}$. The other claim that we could make is that the action remains the same:

$$S' = \int_{\Omega'} dx^{\mu'}\, \mathcal{L}'\left[\eta^\rho \left(x^{\mu'}\right), \partial'_\nu \eta^{\rho'} \left(x^{\mu'}\right), x^{\mu'}\right] = \int_\Omega dx^\mu\, \mathcal{L}\left[\eta^\rho \left(x^\mu\right), \partial_\nu \eta^\rho \left(x^\mu\right), x^\mu\right] \tag{2}$$

This is known as *scale invariance*. Note that the Jacobian determinant in the left integral is not present, this is what makes the transformation a symmetry. If we had explicitly written out the Jacobian determinant then this statement would be true for all transformations. According to Goldstein, a transformation must require both of these invariances for the transformation to be a symmetry. However, according to Chacko, there is a weaker claim that still holds. Consider the case where form invariance is not met, but scale invariance is. The argument for this case existing is that a scaling transformation might change d^4x, but the functional form of the Lagrangian might be changed in the exact opposite way, in order to keep the action integral invariant. Thus we would not have the exact same functional form of the Lagrangian, breaking form invariance, but the action would be invariant, giving us scale invariance. However, let us just go with what Goldstein says, because the most common case is that both conditions are met.

Let us do an example of a transformation and Lagrangian. Consider the Lagrangian:

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi + m^2 \left(\phi^2 + \phi^{\dagger 2}\right)$$

This is a scalar field, the dagger represents complex conjugation. Now consider a field

[4] Note that this is a *passive* transformation, but can be done in the language of active transformations as well.

transformtaion:

$$\phi \to \phi' = e^{i\alpha}\phi$$

In this case, we can look at $\phi'^{\dagger}$:

$$\phi'^{\dagger} = e^{-i\alpha}\phi^{\dagger}$$

With this, we can rewrite our Lagrangian density in terms of ϕ' and $\phi'^{\dagger}$:

$$\begin{aligned}
\mathcal{L}' &= \partial_\mu \left[e^{i\alpha}\phi'^{\dagger}\right]\partial^\mu\left[e^{-i\alpha}\phi'\right] - m^2\left[e^{i\alpha}\phi'^{\dagger}\right]\left[e^{-i\alpha}\phi'\right] + m^2\left(e^{-2i\alpha}\phi'^2 + e^{2i\alpha}\phi'^{\dagger 2}\right) \\
&= \partial_\mu\phi'^{\dagger}\partial^\mu\phi' - m^2\phi'^{\dagger}\phi' + m^2\left(e^{-2i\alpha}\phi'^2 + e^{2i\alpha}\phi'^{\dagger 2}\right)
\end{aligned}$$

We see that this transformation is not a symmetry transformation, since we have an extra $e^{\pm 2i\alpha}$ in the third term. If we removed the last term from the Lagrangian, then this would be a symmetry transformation of the modified Lagrangian, since the first two terms remain form invariant.

Let us now continue proving Noether's theorem. Suppose we satisfy both Defn. 1 and Defn. 2, we have both form and scale invariance. In this case, the statement of scale invariance can be rewritten as:

$$\int_{\Omega'} dx^\mu\, \mathcal{L}\left[\eta^{\rho'}\left(x^\mu\right), \partial_\nu\eta^{\rho'}\left(x^\mu\right), x^\mu\right] - \int_{\Omega} dx^\mu\, \mathcal{L}\left[\eta^{\rho}\left(x^\mu\right), \partial_\nu\eta^{\rho}\left(x^\mu\right), x^\mu\right] = 0$$

Where we have changed the Lagrangian in the left integral to $\mathcal{L}$, via form invariance. Note that we have also changed the dummy variable of integration of the left integral from x' to x. These integrals differ in their region of integration, as well as the fields that are in the Lagrangian. In other words, we have an of $\Delta\mathcal{L}$ over the region Ω, and we have an integral of $\mathcal{L}$ over the region $\Omega' - \Omega$. Thus we can rewrite the equation as:

$$\underbrace{\int_{\Omega} dx^\mu\, \left(\mathcal{L}\left[\eta^{\rho'}, \partial_\nu\eta^{\rho'}, x^\mu\right] - \mathcal{L}\left[\eta^{\rho}, \partial_\nu\eta^{\rho}, x^\mu\right]\right)}_{\Delta\mathcal{L}\text{ in }\Omega} + \underbrace{\int_{\Omega'-\Omega} dx^\mu\, \mathcal{L}\left[\eta^{\rho'}, \partial_\nu\eta^{\rho'}, x^\mu\right]}_{\mathcal{L}\text{ in disjoint region}} = 0 \tag{3}$$

The first integral seems workable, but what do we do with the second integral? We will not treat this integral honestly, instead we will deal with it in 1 dimension and then claim that it generalizes to the 4D integral. Consider the analogous 1D case of our equation:

$$\begin{aligned}
\int_{a+\delta a}^{b+\delta b} dx\, \left[f\left(x\right) + \delta f\left(x\right)\right] - \int_a^b dx\, f\left(x\right) &= 0 \\
\int_a^b dx\, \delta f\left(x\right) + \int_b^{b+\delta b} dx\, \left[f\left(x\right) + \delta f\left(x\right)\right] - \int_a^{a+\delta a} dx\, \left[f\left(x\right) + \delta f\left(x\right)\right] &= 0
\end{aligned}$$

Now dropping the $\delta f\left(x\right)$ in the integrals that are over infinitesimal regions, and claiming that $f\left(x\right)$ is approximately constant over such small regions:

$$\begin{aligned}
\int_a^b dx\, \delta f\left(x\right) + \int_b^{b+\delta b} dx\, \left[f\left(x\right) + \delta f\left(x\right)\right] - \int_a^{a+\delta a} dx\, \left[f\left(x\right) + \delta f\left(x\right)\right] &= 0 \\
\int_a^b dx\, \delta f\left(x\right) + f\left(b\right)\delta b - f\left(a\right)\delta a &= 0
\end{aligned}$$

Now rewriting this to "undo" an integration:

$$\int_a^b dx\, \left(\delta f(x) + \frac{d}{dx}\left[f(x)\,\delta x\right]\right) = 0$$

Where δx is any smooth function of x that satisfies the boundary conditions of the integral. Now what is the higher dimensional analog of our newly derived 1D equation? We claim that it is:

$$\int_{\Omega'-\Omega} dx^\mu\, \mathcal{L}\left[\eta^{\rho'}, \partial_\nu \eta^{\rho'}, x^\mu\right] = \int_S dS_\mu\, \mathcal{L}\left[\eta^\rho, \partial_\nu \eta^\rho, x^\mu\right]\delta x^\mu$$

Where the term on the right is the result of the integral over the infinitesimal disjoint region. Similarly to the 1D case, we can write this as a total derivative:

$$\int_S dS^\mu\, \mathcal{L}\left[\eta^\rho, \partial_\nu \eta^\rho, x\right]\delta x_\mu = \int_\Omega dx^\nu\, \frac{\partial}{\partial x^\nu}\left[\mathcal{L}\left[\eta^\rho, \partial_\nu \eta^\rho, x^\mu\right]\delta x^\mu\right]$$

This is essentially Gauss's Law in 3+1 dimensions. We are saying that the Lagrangian does not have enough time to change over the small disjoint region, and we can integrate it over the surface of the disjoint region. Now let us return to Equation 3, where we have now dealt with the second integral. Let us now consider the first term. We can rewrite this as:

$$\int_\Omega dx^\mu\, \left(\mathcal{L}\left[\eta^{\rho'}, \partial_\nu \eta^{\rho'}, x^\mu\right] - \mathcal{L}\left[\eta^\rho, \partial_\nu \eta^\rho, x^\mu\right]\right) = \int_\Omega dx^\mu\, \left[\frac{\partial \mathcal{L}}{\partial \eta^\rho}\bar{\delta}\eta^\rho + \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \eta^\rho\right)}\bar{\delta}\left(\partial_\nu \eta^\rho\right)\right]$$

Where $\bar{\delta}\eta$ is the change in η at the point with coordinates x^μ (as opposed to $\delta\eta$, which is the change in η at the same physical point). To demonstrate the difference between our two deltas, consider a field $\phi(x)$, which is a scalar under rotations. We take some physical point x, and we rotate our coordinate system, and the new coordinate of the same physical point is now x'. The field at the physical point must always be the same, regardless of the coordinate system:

$$\phi(x) = \phi'\left(x'\right)$$

Knowing this, we can find ϕ':

$$\begin{aligned}\phi(x) &= \phi'\left(x'\right)\\ \phi(x) &= \phi'\left(R^{-1}x\right)\end{aligned}$$

Recall that

$$\eta^{\rho'}\left(x^{\mu'}\right) = \eta^\rho\left(x^\mu\right) + \delta\eta^\rho\left(x^\mu\right)$$

For a scalar under rotations, the value at the same physical point does not change, so $\delta\eta^\rho\left(x^\mu\right) = 0$. However, there is a point in our new coordinate system that has the same label as our physical point in the original coordinates. This is a *different* physical point, but they have the same coordinates. This is how we define $\bar{\delta}\eta^\rho$:

$$\bar{\delta}\eta^\rho = \eta^{\rho'}(x) = \eta^\rho(x)$$

Note that for our scalar field, this is *not* zero, unlike $\delta\eta^\rho$. We can write out what $\bar{\delta}\eta^\rho$ is:

$$\bar{\delta}\eta^\rho = \eta^{\rho'}(x) - \eta^\rho(x)$$

$$\begin{aligned}
&= \eta^{\rho'}\left(x' - \delta x\right) - \eta^{\rho}\left(x\right) \\
&= \eta^{\rho'}\left(x'\right) - \delta x^{\alpha}\frac{\partial \eta^{\rho'}}{\partial x'^{\alpha}} - \eta^{\rho}\left(x\right) \\
&= -\delta x^{\alpha}\frac{\partial \eta^{\rho}}{\partial x^{\alpha}}
\end{aligned}$$

Note that we drop the primes in the derivative because we already have a δx, the difference between the primed and unprimed coordinates is subleading relative to δx. We can now compute $\bar{\delta}\left(\partial_\nu \eta^\rho\right)$ via analogy to $\bar{\delta}\eta^\rho$:

$$\begin{aligned}
\bar{\delta}\left(\partial_\nu \eta^\rho\right) &= \partial_\nu \eta^{\rho'}\left(x\right) - \partial_\nu \eta^{\rho}\left(x\right) \\
&= \partial_\nu\left[\bar{\delta}\eta^\rho\left(x\right)\right]
\end{aligned}$$

With these two quantities computed, we now look at the integral we had, and we can replace the $\frac{\partial \mathcal{L}}{\partial \eta^\rho}$ using the equations of motion:

$$\int_\Omega dx^\mu\left[\frac{\partial \mathcal{L}}{\partial \eta^\rho}\bar{\delta}\eta^\rho + \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \eta^\rho\right)}\bar{\delta}\left(\partial_\nu \eta^\rho\right)\right] = \int_\Omega dx^\mu\ \left[\frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \eta^\rho\right)}\bar{\delta}\eta^\rho\right)\right]$$

Now recombining this term with the disjoint integral term:

$$\int_\Omega dx^\mu \frac{\partial}{\partial x^\nu}\left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \eta^\rho\right)}\bar{\delta}\eta^\rho + \mathcal{L}\delta x^\mu\right] = 0$$

Now we recall that we never specified what Ω was, it can be any 4-volume. Thus the integrand must vanish for all volumes, and therefore the quantity in the total derivative must be constant. This is denoted j:

$$j^\nu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \eta^\rho\right)}\bar{\delta}\eta^\rho + \mathcal{L}\delta x^\mu \tag{4}$$

This is the Noether current, and is conserved, $\partial_\mu j^\mu = 0$.

Noether Current Examples

The first example we wil look at is translations. Consider a Lagrangian density that is invariant under translations. In this case, the value of the field at the physical point does not change, regadless of whether it is a scalar field or a vector field. Thus we have that

$$\eta^{\rho'}\left(x'\right) = \eta^\rho\left(x\right)$$

Which gives us that $\delta\eta^\rho\left(x\right) = 0$. We can explicitly define our translation:

$$x'^{\nu} = x^\nu + a^\nu$$

and so $\delta x^\nu = a^\nu$. We now want to find $\bar{\delta}\eta^\rho$:

$$\begin{aligned}
\bar{\delta}\eta^\rho &= \eta^{\rho'}\left(x\right) - \eta^\rho\left(x\right) \\
&= -a^\nu \partial_\nu \eta^\rho
\end{aligned}$$

From this, we can write out the Noether current:

$$j^\nu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \eta^\rho\right)}\left[-a^\alpha \partial_\alpha \eta^\rho\right] + \mathcal{L}a^\nu$$

$$= -a^{\alpha} \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \left(\partial_{\nu} \eta^{\rho}\right)} \partial_{\alpha} \eta^{\rho} - \mathcal{L} \delta^{\nu}{}_{\alpha}\right]}_{T^{\nu}{}_{\alpha}}$$

We see that we recover the energy-momentum tensor as our conserved quantity, spacetime translational invariance leads to the conservation of the energy-momentum tensor.
Let us do another example. Consider the Lagrangian density:

$$\mathcal{L} = \partial_{\mu} \phi^{\dagger} \partial^{\mu} \phi - m^2 \phi^{\dagger} \phi$$

Now consider the transformation:

$$\begin{aligned} \phi &\to e^{i\alpha} \phi = \phi' \\ \phi^{\dagger} &\to e^{-i\alpha} \phi^{\dagger} = \phi'^{\dagger} \end{aligned}$$

We can first show that this is a symmetry, and then we can find the Noether current. To show that it is a symmetry, we must show form invariance:

$$\begin{aligned} \mathcal{L} &= \partial_{\mu} \left(e^{i\alpha} \phi'^{\dagger}\right) \partial^{\mu} \left(e^{-\alpha} \phi'\right) - m^2 \left(e^{i\alpha \phi'^{\dagger}} e^{-i\alpha} \phi'\right) \\ &= \partial_{\mu} \phi'^{\dagger} \partial^{\mu} \phi' - m^2 \phi'^{\dagger} \phi' \end{aligned}$$

We see that if we insert the transformed coordinates, we obtain the same functional form, giving us form invariance. Scale invariance in this case is trivially obtained, since the transformation does not depend on the coordinates. Thus this transformation is indeed a symmetry.
Now let us compute the Noether current. We can compute $\overline{\delta} \phi$, which, since we haven't changed the coordinates, is the same as $\delta \phi$:

$$\begin{aligned} \overline{\delta} \phi &= \phi'(x) - \phi(x) \\ &= e^{i\alpha} \phi(x) - \phi(x) \\ &= i\alpha \phi(x) \end{aligned}$$

Where we have Taylor expanded the exponential, since we consider an infinitesimal transformation. We can do the same for $\phi^{\dagger}$, and we find that

$$\overline{\delta} \phi^{\dagger} = -i\alpha \phi^{\dagger}$$

We can now insert these into the definition of the Noether current:

$$\begin{aligned} j^{\nu} &= \frac{\partial \mathcal{L}}{\partial \left(\partial_{\nu} \phi\right)} \overline{\delta} \phi + \frac{\partial \mathcal{L}}{\partial \left(\partial_{\nu} \phi^{\dagger}\right)} \overline{\delta} \phi^{\dagger} \\ &= \left(\partial^{\nu} \phi^{\dagger}\right) (i\alpha \phi) + \partial_{\nu} \phi \left(-i\alpha \phi^{\dagger}\right) \\ &= -i\alpha \left[\phi^{\dagger} \partial^{\nu} \phi - \phi \partial^{\nu} \phi^{\dagger}\right] \end{aligned}$$