Lattice Field Theory Notes

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1 Overview of Continuum Theory

1.1 Path Integrals in the Continuum

Consider a general quantum mechanical system in 1D with Hamiltonian:

$$H = \frac{p^2}{2m} + V(x)$$

The quantity of interest is the partition function:

$$\begin{aligned} Z &= \mathrm{Tr}\left[e^{-iHt}\right] \\ &= \int dx \ \langle x|e^{-iHt}|x\rangle \end{aligned}$$

which will give us the generating functional for correlation functions. First, let us compute the free matrix element $\langle x|e^{-iH_0t}|y\rangle$:

$$\begin{aligned} \langle x|e^{-iH_0t}|y\rangle &= \langle x|e^{-ip^2t/2m}|y\rangle \\ &= \int dp \ \langle x|e^{-ip^2t/2m}|p\rangle \langle p|y\rangle \\ &= \int dp \ \langle x|p\rangle \langle p|y\rangle \, e^{-ip^2t/2m} \\ &= \int \frac{dp}{2\pi} e^{-ip^2t/2m + i(x-y)} \\ &= \sqrt{\frac{m}{2\pi i t}} e^{(x-y)^2 m/2t} \end{aligned}$$

This gives us the free matrix elements, but to introduce the potential we can Trotter decompose the full time evolution operator:

$$\begin{aligned} e^{-iH\delta t} &= e^{-iV(x)\delta t/2} e^{-iH_0\delta t} e^{-iV(x)\delta t/2} + \mathcal{O}\left(\delta t^2\right) \\ &= \int dx\, dy\, e^{-iV(x)\delta t/2} \left|x\right\rangle \left\langle x\right| e^{-iH_0\delta t} \left|y\right\rangle \left\langle y\right| e^{-iV(x)\delta t/2} + \mathcal{O}\left(\delta t^2\right) \end{aligned}$$

We can now reconstruct our time evolution operator:

$$e^{-iHt} = \underbrace{e^{-iHt/N} e^{-iHt/N} \dots e^{-iHt/N}}_{N \text{ times}}$$

Where each term uses the above decomposition, with $\delta t = t/N$.

$$e^{-iHt} = \int d\boldsymbol{x} d\boldsymbol{y}\, e^{-iV(x)\delta t/2} \left|x_1\right\rangle \left\langle x_1\right| e^{-iH_0 t/2} \left|y_1\right\rangle \left\langle y_1\right| e^{-iV(x)\delta t/2} \left|x_2\right\rangle \dots \left\langle x_N\right| e^{iH_0\delta t} \left|y_N\right\rangle \left\langle y_N\right| e^{-iV(x)\delta t/2} + \mathcal{O}\left(\delta t^2\right)$$

We can continue the derivation to show that:

$$Z = \left(\frac{m}{2\pi i \delta t}\right)^{N/2} \int dx_1 \dots dx_{N-1} e^{iS(x, \dots x_{N-1})}$$

Where

$$S\left(x_1,\ldots x_{N-1}\right)=\frac{1}{2m}\frac{1}{\delta t}\sum_{j=1}^{N}\left(x_{j+1}-x_j\right)^2-\delta t\sum_{j=1}^{N}\frac{V\left(x_j\right)+V\left(x_{j+1}\right)}{2}$$

We can see that we can take the limit of this:

$$\lim_{\delta t\to 0}\lim_{N\to\infty}S\left(x_1,\ldots,x_{N-1}\right)=\int_0^t dt'\,\left(\frac{1}{2}m\dot{x}\left(t'\right)^2-V\left(x\left(t'\right)\right)\right)$$

Which is the continuum action!

Since we have a finite number of integration measures, we can compute this path integral numerically. However, the integrand is an oscillating phase, which leads to the Monte Carlo sign problem. To avoid this, we perform an analytic continuation to imaginary time, $t=-i\tau$:

$$Z=\int dx_1\ldots dx_{N-1}e^{-S_E\left(x_1,\ldots x_{N-1}\right)}$$

Where

$$S_E\left(x_1,\ldots x_{N-1}\right)=\int_0^t d\tau'\,\left(\frac{1}{2}m\dot{x}\left(\tau'\right)^2+V\left(x\left(\tau'\right)\right)\right)$$

This is very similar to the Boltzmann distribution from statistical mechanics. Thus we can interpret the Euclidean path integral with periodic boundary conditions as the canonical partition function of the corresponding thermal system.

1.2 Correlation Functions

Correlation functions are of great importance as they give us infromation about the mass spectrum of a particle, hadronic contributions to $\mu\ g-2$, weak decays, etc. Correlation functions follow an operator expectation value. Let us first consider a Euclidean two point correlator:

$$\begin{aligned}\left\langle x\left(t\right)x\left(0\right)\right\rangle &=\frac{1}{Z}\int dx_1\ldots dx_N\,x\left(t\right)x\left(0\right)e^{-S\left(x_1,\ldots x_N\right)}\\ &=\frac{1}{Z}\mathrm{Tr}\left[xe^{-Ht}xe^{-H(T-t)}\right]\\ &=\frac{1}{Z}\sum_{m,n}\left\langle n|e^{-HT}x|m\right\rangle\left\langle m|e^{-H(T-t)}|n\right\rangle\\ &=\frac{1}{Z}\sum_{m,n}\left\langle n|x|m\right\rangle\left\langle m|x|n\right\rangle e^{-t\Delta E_n}e^{-(T-t)\Delta E_m}\\ &=\frac{\sum_{n,m}\left\langle n|x|m\right\rangle\left\langle m|x|n\right\rangle e^{-t\Delta E_n}e^{-(T-t)\Delta E_m}}{1+e^{-T\Delta E_1}+e^{-T\Delta E_2}+\cdots}\end{aligned}$$

In the limit of large T, we recover the two point correlator:

$$\left\langle x\left(t\right)x\left(0\right)\right\rangle=\sum_{n,m}\left\langle n|x|m\right\rangle\left\langle m|x|n\right\rangle e^{-t\Delta E_n}$$

From this, we can extract the mass/energy spectrum from a two-point function.

2 Unit 2: Scalar Field Theory to Gauge Theory

We previously had a lightning review of QCD, and a brief intro to scalar field theory with a lattice regulator.

We have a lattice Λ:

$$\Lambda = \{n \in (n_1, n_2, n_3, n_4) \mid n_1, n_2, n_3 \in \{0, 1, \ldots, N\}, n_4 \in \{0, 1, \ldots, N_T - 1\}\}$$

This describes an isotropic hypercubic lattice, with a potentially different temporal extent. We can then define the Euclidean action for a free scalar field:

$$S_E = \int d^4x \left(\frac{1}{2}\phi\left(-\partial^2 + m^2\right)\phi\right)$$

Which we discretized:

$$S_E^{\text{lattice}} = a^4 \sum_{n\in\Lambda}\left[\sum_{\nu=1}^{4}\frac{1}{2}\left(\frac{\phi\left(n+\hat{\nu}\right) - \phi\left(n\right)}{a}\right)^2 + \frac{m_0^2}{2}\phi\left(n\right)^2\right]$$

Where a is the lattice spacing. If we expand this out and take the $a \to 0$ limit, we reproduce the continuum action. We can look at the scalar propagator, the Fourier Transform of the Euclidean two point function, $\langle 0|\phi\left(x\right)\phi\left(0\right)|0\rangle$:

$$G\left(q\right) = \frac{1}{q^2 + m_0^2} \to G_{\text{lat}}\left(q\right) = \left[m_0^2 + \sum_{\nu=1}^{4}\frac{4}{a^2}\sin^2\left(\frac{q_\nu a}{2}\right)\right]^{-1}$$

Again, if we take $a \to 0$, we recover the continuum propagator. The dispersion relation of the continuum propagator (how the 4th component of the momentum depends on the other 3). The lattice propagator for small q looks the same as the continuum propagator, but then diverges from the continuum dispersion relation for large momenta.

Now let us continue discussing scalar field theory, focusing on some expansions that we can make. We will begin by discussing $O\left(n\right)$ models and interactions.

Let us take a set of real scalar fields $\phi = \left(\phi_0, \phi_1, \ldots, \phi_{n-1}\right)^T$. The Lagrangian will have kinetic terms for each component, as well as a potential term:

$$\mathcal{L} = \sum_i \partial_\mu\phi_i\partial_\mu\phi_i + V\left(\phi^T\phi\right)$$

This is invariant under a global transformation (not dependent on spacetime):

$$\phi\left(x\right) \to \phi'\left(x\right) = \Omega\left(\theta\right)\phi\left(x\right)$$

with $\Omega\left(\theta\right) \in O\left(n\right)$, which means that Ω is an $n \times n$ matrix such that $\Omega^T\Omega = \mathbb{I}$. θ is some set of parameters that parameterizes the rotation.

Let us now define a quartic potential:

$$V = \frac{m_0^2}{2}\phi^T\phi + \frac{\tilde{\lambda}_0}{4}\left(\phi^T\phi\right)^2$$

This has two parameters, m_0 and $\tilde{\lambda}_0$, and in order for it to be well defined, we will need $\tilde{\lambda}_0 \geq 0$. The sign of m_0 is not fixed. If $m_0^2 \geq 0$, then this potential is convex up, and the ground state will have field expectation value zero, and this preserved $O(n)$ symmetry. If on the other hand, $m_0^2 \leq 0$, then the potential will be minimized when ϕ_g (the ground state field) satisfies $\phi_g^T \phi_g = V^2 = -m_0^2/\lambda_0 > 0$. This is spontaneous symmetry breaking, the vacuum state spontaneously breaks $O(n)$ symmetry. This is connected to the existence of massless scalar Goldstone bosons.

After discretizing, we have:

$$S_E^{\text{lattice}} = a^4 \sum_{n\in\Lambda} \left[\frac{1}{2} \sum_{\nu=1}^{4} \left(\frac{\phi^i_{n+\hat{\nu}} - \phi^i_n}{a} \right)^2 + \frac{m_0^2}{2} \phi_n^{i2} + \frac{\tilde{\lambda}_0}{4} \left(\phi_n^{i_2} \right)^2 \right]$$

If we rewrite the finite difference term:

$$\begin{aligned} \frac{1}{2} \sum_{\nu=1}^{4} \left(\frac{\phi^i_{n+\hat{\nu}} - \phi^i_n}{a} \right)^2 &= \frac{1}{2a^2} \sum_n \left(\phi^{i^2}_{n+\hat{\nu}} + \phi^{i^2}_n - 2\phi^i_{n+\hat{\nu}} \phi^2_n \right)^2 \\ &= 4\phi_n^{i^2} - \sum_\nu \phi^i_{n+\hat{\nu}} \phi^i_n \end{aligned}$$

If we now rescale our parameters:

$$\begin{aligned} a\phi_n^i &= \sqrt{2\kappa}\phi_n^i \\ am_0^2 &= \frac{1-2\lambda_0}{\kappa} - 8 \\ \tilde{\lambda}_0 &= \frac{\lambda_0}{\kappa^2} \end{aligned}$$

We can rewrite our lattice action:

$$\begin{aligned} S_E^{\text{lattice}} &= \sum_{n\in\Lambda} \left(-2\kappa \sum_\nu \phi^i_{n+\hat{\nu}} \phi^i_n + \phi_n^{i^2} + \lambda_0 \left(\phi_n^{i^2} - 1 \right)^2 - \lambda_0 \right) \\ &= \sum_n s(\phi_n, \lambda) - 2\kappa \sum_{\langle n,m \rangle} \phi^i_n \phi^i_m \end{aligned}$$

Where

$$s(\phi_n, \lambda) = \phi_n^{i^2} + \lambda_0 \left(\phi_n^{i^2} - 1 \right)^2 - \lambda_0$$

and where $\langle n, m \rangle$ indicates nearest neighbors. We have mapped from parameters $\left(m_0, \tilde{\lambda}_0\right)$ to (κ, λ_0), these are the "bare" parameters. Note that $\kappa \to 0$ as $m_0 \to \infty$. When looking at the phase structure, there is also some $\kappa_{\text{crit}}(\lambda_0)$ at which we have a second order phase transition, between the symmetric and symmetry broken phases.

Now let us look back at the original lattice action we wrote down. We can do a weak coupling expansion, where $\lambda_0 \to 0$. This is valid in the small λ_0 region. If we look at the second action we wrote, we can consider the small κ expansion, which is the same as the large m_0 expansion, and is known as a hopping expansion. Note that we are not specifying what small and large mean, relative to other quantities.

Let us begin with the weak coupling expansion. In QFT, we have Feynman rules as loops, and we can compute quantities order by order in the coupling λ. We have the same thing here, but a bit more cumbersome. We have vertices that come from the Lagrangian, edges that come from the free propagator, and the integrals of the momenta are from $-\frac{\pi}{a}$ to $\frac{\pi}{a}$.

Perturbation theory allows us to relate the bare and renormalized coupling constants:

$$\begin{aligned}
m_R^2 &= m_0^2 + \lambda_0 (n+2) \left(\frac{r_0}{a^2} + r_1 m_0^2 + \frac{m_0^2}{16\pi^2} \ln\left(a^2 m_0^2\right) \right) + \mathcal{O}\left(\lambda_0^2\right) \\
\lambda_R &= \lambda_0 + \lambda_0^2 (\cdots) + \mathcal{O}\left(\lambda_0^3\right)
\end{aligned}$$

Now let us discuss the hopping expansion. For notational brevity, we are dropping our $O(n)$ indices here. We have that

$$\begin{aligned}
\langle 0|\phi_{n_1}\phi_{n_2}\dots\phi_{n_k}|0\rangle &= \frac{1}{Z}\int \mathcal{D}\phi e^{-S_E^{\text{latt}}[\phi]}\phi_{n_1}\dots\phi_{n_k} \\
&= \frac{1}{Z}\int \mathcal{D}\phi \prod_n e^{-s(\phi_n,\lambda_0)} \text{exp}\left[-2\kappa \sum_{\langle n,m\rangle}\phi_n\phi_m\right]\phi_{n_1}\dots\phi_{n_k} \\
&= \frac{1}{Z}\sum_j \frac{(-2\kappa)^j}{j!}\int \mathcal{D}\phi \prod_n e^{-s(\phi_n,\lambda_0)} \left(\sum_{\langle n,m\rangle}\phi_n\phi_m\right)^j \phi_{n_1}\dots\phi_{n_k}
\end{aligned}$$

We can write out the partition function:

$$Z = \sum_j \int \mathcal{D}\phi \prod_n e^{-s(\phi_n,\lambda_0)} \left(\sum_{\langle n,m\rangle}\phi_n\phi_m\right)^j \frac{(-2\kappa)^j}{j!}$$

We can break this into a sum:

$$Z = Z_0 + Z_1 + \dots$$

Where $Z_j \propto \kappa^j$:

$$\begin{aligned}
Z_0 &= \int \mathcal{D}\phi \prod_n e^{-s(\phi_n,\lambda_0)} \\
&= \left(\int_{-\infty}^{\infty} d\phi e^{-s(\phi,\lambda_0)}\right)^\Omega
\end{aligned}$$

Where Ω is the number of sites. We can look at Z_1:

$$\begin{aligned}
Z_1 &= \int \mathcal{D}\phi \prod_n e^{-s(\phi_n,\lambda_0)} \sum_{\langle l,m\rangle}\phi_l\phi_m \\
&= (Z_0)^{\Omega-2}\sum_{\langle l,m\rangle}\left(\int_{-\infty}^{\infty} d\phi_l\, e^{-s(\phi_l)}\phi_l\right)\left(\int_{-\infty}^{\infty} d\phi_m e^{-s(\phi_m,\lambda_0)}\phi_m\right) \\
&= 0
\end{aligned}$$

This is zero because the second integral must be zero. We can look at a more interesting expression:

$$Z_2 = \int \mathcal{D}\phi \prod_n e^{-s(\phi_n,\lambda_0)} \frac{4\kappa^2}{2}\left(\sum_{\langle l,m\rangle}\phi_l\phi_m\right)\left(\sum_{\langle p,q\rangle}\phi_p\phi_q\right)$$

We have 3 possible cases for this integral. Either l and m are neighbors and are not adjacent to p and q, or we can have that $p = l$ and $q = m$, or we can have $p = m$ but $l \neq q$. The only one of these cases that is nonzero is the case where $p = l$ and $q = m$. In this case, we find that

$$Z_2 = 2\kappa^2 \left(Z_0\right)^{\Omega-2} \gamma_2^2 \left(\lambda\right)$$

Where $\gamma_2 \left(\lambda\right) = \int d\phi \, e^{-s(\phi,\lambda_0)} \phi^n$.