We have the potential $V(x) = \frac{1}{2}m\omega^2 x^2 + \lambda x^4$, and we want to compute the ground state energy of the system. Since we know the ground state solution to the harmonic oscillator, we let $H^0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2$, and let $H^1 = \lambda x^4$. We then have that $\psi_0^0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$, and $E_0^0 = \frac{1}{2}\hbar\omega$. We can now compute the first order corrections to the energy, E_0^1.

$$\begin{aligned}
E_0^1 &= \langle \psi_0^0 | H^1 | \psi_0^0 \rangle \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \lambda \int e^{-m\omega x^2/\hbar} x^4 \, dx \\
&= \frac{3}{4}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \lambda \left(\frac{\pi\hbar^5}{m^5\omega^5}\right)^{1/2} \\
&= \frac{3}{4}\lambda
\end{aligned}$$

Where we have let $m = \omega = \hbar = 1$, and we have used the known solution to the gaussian integral

$$\int x^4 e^{-ax^2} \, dx = \frac{3}{4}\frac{\sqrt{\pi}}{a^{5/2}}$$

Now applying the first order corrections, we have that the ground state energy is given by

$$E_0 = \frac{1}{2} + \frac{3}{4}\lambda$$

We now want to compute the second order corrections to the energy. From perturbation theory, we have that the second order corrections to the nth energy level are given by

$$E_n^2 = \sum_{m \neq n} \frac{|\langle \psi_m^0 | H^1 | \psi_n^0 \rangle|^2}{E_n^0 - E_m^0}$$

We can represent $H^1 = x^4$ using the harmonic oscillator raising and lowering operators:

$$\hat{x} = \sqrt{\frac{1}{2}}(\hat{a} + \hat{a}^\dagger)$$

$$\hat{x}^4 = \frac{1}{4}(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger)^2$$

If we do this out, we have 16 terms. We note that we are interested in the ground state, where $n = 0$. We also note that the lowering operator a cannot lower below the ground state, and thus any of the 16 terms that try to lower past the ground state can be discarded. This leaves us with 6 terms, two of which can be further removed because the end state is the ground state. Since the summation for the energy correction disallows $m = n = 0$, this means that we can disregard those two terms, leaving us with 4 terms:

$$\hat{x}^4 = \frac{1}{4}(aa^\dagger a^\dagger a^\dagger + a^\dagger a a^\dagger a^\dagger + a^\dagger a^\dagger a a^\dagger + a^\dagger a^\dagger a^\dagger a^\dagger)$$

Now computing the numerator of the correction (where I have replaced ψ_0^0 with $|0\rangle$ for simplicity's sake):

$$\langle m | H^1 | 0 \rangle = \langle m | \hat{x}^4 | 0 \rangle = \frac{1}{4}(3\sqrt{2}\delta_{m,2} + 2\sqrt{2}\delta_{m,2} + \sqrt{2}\delta_{m,2} + 2\sqrt{6}\delta_{m,4})$$

Now computing the corrections to the energy:

$$E_0^{(2)} = \frac{\left(\frac{1}{4}\cdot 6\sqrt{2}\right)^2}{-2} + \frac{\left(\frac{1}{4}\cdot 2\sqrt{6}\right)^2}{-4} = -\frac{21}{8}$$

Thus, using the corrections to the energy, we have that

$$E_0 \approx \frac{1}{2} + \frac{3}{4}\lambda - \frac{21}{8}\lambda^2$$