Representation Theory

Hersh Singh

October 9, 2014

Contents

Chapter 1

Representations of Finite Groups

1.1 Definitions

Let G be a finite group, V be a finite dimensional complex vector space.

A *representation* of G on V is a group homomorphism

$$\rho : G \to \mathrm{GL}(V), \tag{1.1}$$

where $\mathrm{GL}(V)$ is the group of automorphisms of V. Even though the "representation" is really the the vector space V *and* the homomorphism ρ, but it is common (especially by physicists) to refer to V itself as the representation.

> We say that this map gives V the structure of a G-module. This agrees with the definition of a R-module (R is a ring) that I have studied earlier. An R-module is simply a vector space defined over a ring, instead of a field. So the ring elements act as the *scalars*. Of course, the description must include a rule for the 'interaction' of the scalars with the vectors. To be a module, the interaction must be *linear*. In this case, the group homomorphism ρ gives us that rule of interaction between the vectors (elements of V) and the scalars (elements of G). For $g \in G, v \in V$,
> $$gv \equiv \rho(g)v \in V$$

A vector space homomorphism $\phi : V \to W$ is a morphism between the two representations V and W if the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{\phi} & W \end{array} \tag{1.2}$$

That is, $\phi g = g\phi$ for all $g \in G$. This makes the group elements *behave like scalars under module homomorphisms.* Such morphisms of representations are also called *G-linear map* or a *G intertwiner.*

> Why is this is a good definition? Seems to be inspired from module homomorphisms. This is natural in some sense - Figure that out.

$\operatorname{Ker}\phi$, $\operatorname{Im}\phi$, $\operatorname{Coker}\phi = V/\operatorname{Im}\phi$ are also G-modules. This is solely because of the commutativity of the above diagram.

- If $v \in \operatorname{Ker}\phi$, then $\phi(gv) = g\phi(v) = 0$. So gv also $\in \operatorname{Ker}\phi$. $\blacksquare$.
- If $v \in \operatorname{Im}\phi$, then let $\phi(v) = w$. So $\phi(gv) = g\phi(v) = gw \in W$. So gv also $\in \operatorname{Im}\phi$. $\blacksquare$.

One of the goals of our study is, given a representation, to develop tools for constructing other, preferably all, representations of the group. Some examples of representations that can be constructed from V and W

(a) Tensor product $V \otimes W$ via

$$g(v \otimes w) = g(v) \otimes g(w) \tag{1.3}$$

(b) Tensor power $V^{\otimes n}$ and the *exterior power* $\Lambda^n(V)$ and the *symmetric power* $\operatorname{Sym}^n(V)$ are its subrepresentations.

(c) the dual $V^* = \operatorname{Hom}(V, \mathbb{C})$. This is a little tricky though. The action of G on V^* must be such that it preserves the natural inner product, denoted by $\langle\ ,\ \rangle$, between them. So, if we have a representation (V^*, ρ^*) and (V, ρ), then we should have

$$\rho_g^*(u^*)(\rho_g v) = u^*(v) \tag{1.4}$$

where I have written $\rho_g = \rho(g)$ for a cleaner notation, and $u^* \in V^*$, $v \in V$.

> Remember the definition of the transpose map. If we have a map $f : V \to W$, then the transpose of ρ is a map ${}^t\rho : W^* \to V^*$ such that
>
> $${}^tf(\phi) = f \cdot \phi \qquad \forall \phi \in W^*. \tag{1.5}$$
>
> It's a good idea to make commutative diagrams out of statements like the above:
>
> $$\begin{tikzcd} & V \arrow[dl, "f"'] \arrow[dr, "{}^tf(\phi) = \phi \cdot f"] & \\ W \arrow[rr, "\phi"] & & \mathbb{C} \end{tikzcd}$$

If we now define

$$\rho^*(g) = {}^t\rho(g^{-1}) \tag{1.6}$$

we get

$$\begin{aligned} \rho_g^*(u*)(\rho_g v) &= {}^t\rho_{g^{-1}}(u^*)(\rho_g v) \\ &= u^* \cdot \rho_{g^{-1}}(\rho_g v) \\ &= u^*(\rho_{g^{-1}g} v) \\ &= u^*(v). \end{aligned}$$

The definition (1.6) preserves the inner product, and is a thus a sane definition.

(d) $\mathrm{Hom}(V, W)$ by the identification, $\mathrm{Hom}(V, W) = V^* \otimes W$ (see appendix A). If $(\phi : V \to W) \in \mathrm{Hom}(V, W)$ then let $\sum_i \phi_{ij} v_i^* \otimes w_j$. Writing out the action of this on $u \in V$,

$$\begin{aligned}
(g\phi)(gu) &= \left[g \sum \phi_{ij} v_i^* \otimes w_j\right](gu) \\
&= \left[\sum_{ij} \phi_{ij} g v_i^* \otimes g w_j\right](gu) \\
&= \sum \phi_{ij} \langle g v_i^*, gu \rangle g w_j \\
&= \sum \phi_{ij} \langle v_i^*, u \rangle g w_j
\end{aligned}$$

This gives us

$$(g\phi)(gu) = g \cdot \phi(u) \quad \forall u \in V. \tag{1.7}$$

> Note that you cannot multiply both sides by g^{-1} to imply that $\phi(gu) = \phi(u)$. This can't be done since the action of G on V^* is *not associative*. That means,
>
> $$(g\phi)(u) \neq g \cdot \phi(u),$$
>
> as can be seen by simply taking, say, $\phi = v^* \otimes w$. LHS becomes $\langle gv^*, u \rangle gw$ while RHS becomes $\langle v^*, w \rangle gw$.

$$\begin{tikzcd}
V \arrow[r, "\phi"'] \arrow[d, "g"'] & W \arrow[d, "g"] \\
V \arrow[r, "g\phi"'] & W
\end{tikzcd}$$

Ofcouse, if ϕ is a G-linear map (a map between reprentations V and W), then we also have $g \cdot \phi = \phi \cdot g$ for all $g \in G$. Consider the space $\mathrm{Hom}_G(V, W) \subset \mathrm{Hom}(V, W)$, which consists of maps from V to W invariant under the action G. If $\phi \in \mathrm{Hom}_G(V, W)$, then we have $(g\phi)(gu) = g\phi(u)$ and $(g\phi)(u) = \phi(u)$ (by invariance of the map under G). Thus we have $g\phi(u) = \phi(gu)$. The converse is also easily seen to be true.Therefore, $\mathrm{Hom}(V, W)$ is space of all G-linear maps $V \to W$. *[Solution to Exercise 1.2 in FH]*

- Todo: Regular Representation
- Exercise 1.3,1.4

1.2 Schur's Lemma

Lemma 1 (Schur's Lemma)**.** *Let V, W be irreps of a group G and $\phi : V \to W$ a G-linear map. Then,*

(i) either ϕ is 0 or an isomorphism;

(ii) if $V = W$, then $\phi = \lambda I$.

1.3 Examples

We observe that any $g \in G$ gives a map $\rho(g) : V \to V$. In general, this is not a G-linear map however. For $\rho(g)$ to be a G linear map, ...

Abelian Groups

If G is an abelian group, and V is an irrep, then $\rho(g)$ is a G-linear map.By Schur's Lemma, $\rho(g) = \lambda I$. That means that any proper subspace of V is actually invariant under the action of G, and is a thus a subrepresentation. But since V is irreducible, this can only mean that V has no non-trivial proper subspace, which implies that V is one-dimensional. Therefore, any representation of an abelian group is just an element of the *dual group*

$$\rho : G \to \mathbb{C}^*. \tag{1.8}$$

S_3

> Remember that S_3 is the group of permutations of three objects. Algebraically, it can be thought to be generated by the elements
>
> $$\{1,\ \tau,\ \tau^2, \sigma,\ \sigma\tau,\ \sigma\tau^2\} \tag{1.9}$$
>
> subject to the conditions
>
> $$\tau^3 = 1, \quad \sigma^2 = 1, \quad (\sigma\tau)^2 = 1. \tag{1.10}$$
>
> You can tell that σ is a 2-cycle and τ is a 3-cycle.

Lets now discuss the case of the *simplest non-abelian group*, S_3. We already know three representations to begin with:

(i) the trivial representation,

$$\rho(g) = I \tag{1.11}$$

(ii) the alternating representation

$$\rho(g) = \mathrm{sgn}(g), \tag{1.12}$$

(iii) the natural permutation representation.
But this is not irreducible. It can be easily seen that the subspace spanned by the vector $(1, 1, 1)$ is invariant under G. So, the space V complimentary

to it is another (hopefully irreducible!) representation. If $v = (z_1, z_2, z_3) \in V$, then

$$(z_1, z_2, z_2) \cdot (1, 1, 1) = 0 \tag{1.13}$$
$$\implies z_1 + z_2 + z_3 = 0. \tag{1.14}$$

We thus have

$$V = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + z_2 + z_3 = 0\}. \tag{1.15}$$

Now if this further has an invariant subspace, it must be spanned by an element of the form $(z_1, z_2, -z_1 - z_2)$. Applying a few permutations will convince you immediately that this is not an invariant subspace. Therefore, the representation we have is irreducible, called the *standard representation* of S_3.

We now want to characterize any arbitrary representation W of S_3. To do so, we first look at the action of the abelian subgroup $U_3 = \mathbb{Z}/3 \in G$ (generated by x) on W. If $v \in W$ is an eigenvector of $\rho(x)$, then

$$\tau\sigma(v) = \sigma\tau^2(v) \tag{1.16}$$
$$= \omega^2(\sigma v) \tag{1.17}$$

This means that if v is eigenvector of τ with the eigenvalue ω, then σv is also an eigenvector with the eigenvalue ω^2. To find the decomposition of the W, we go through the following steps:

(i) Start with an eigenvector v of τ, which has the eigenvalue w^i.

(ii) If $w^i \neq 1$, then σv is an eigenvector independent of τ with the eigenvalue w^{2i}. In this case, $\{v, \sigma v\}$ form a two dimensional subspace of W invariant under S_3 (as σ just exhanges v and σv). In fact, this subrepresentation is isomorphic to the standard representation and is thus irreducible.

(iii) If $w^i = 1$, then σv may or may not be independent of v.

  (a) If σv is independent of v, then $v + \sigma v$ spans a subspace isomorphic to the trivial represntation and $v - \sigma v$ spans a subspace isomorphic to the alternating representation.

  (b) If σv and v are not linearly independent, then $\sigma v = \lambda v$ for some $\lambda \in \{1, -1\}$ (since $\sigma^2 = I$). If $\lambda = 1$, then $\mathbb{C}v$ is isomorphic to the trivial representation and if $\lambda = -1$, then $\mathbb{C}v$ is isomorphic to the alternating representation.

Note that this allows to find all irreps of a given representation W!

[Solution to Exercise 1.12(a) in FH] Lets use this approach to find out the irreps of th regular representation R of S_3. A general vector in the space looks like this

$$v = a_1 + a_2\tau + a_3\tau^2 + b_1\sigma + b_2\sigma\tau + b_3\sigma\tau^2 \tag{1.18}$$

The eigenvalues of τ are $\{1, \omega, \omega^2\}$.

1. For eigenvalue $= 1$, we have $v = 1 + \tau + \tau^2$ and $\sigma v = \sigma + \sigma\tau + \sigma\tau^2$. Thus $\mathbb{C}(v + \sigma v) \cong \mathrm{triv}$ and $\mathbb{C}(v - \sigma v) \cong \mathrm{sgn}$ are two irreps.
2. For eigenvalue $= \omega$, we get on solving $\tau\alpha = \omega\alpha$
$$\alpha = \omega^2\tau + \omega\tau + \tau^2 \tag{1.19}$$
With $\beta = \sigma\alpha$, we get the subspace spanned by $\{\alpha, \beta\}$ to be isomorphic to the standard representation.
3. For eigenvalue $= \omega^2$, we get
$$\alpha' = \omega^2 + \omega\tau + \tau^2 \tag{1.20}$$
$$\beta' = \sigma\alpha = \omega^2\sigma + \omega\sigma\tau + \sigma\tau^2 \tag{1.21}$$
The subspace spanned by $\{\alpha', \beta'\}$ is again isomorphic to the standard representation.

We have enumerated six linearly independent eigenvectors and therefore exhausted all of them. We thus get

$$R \cong \mathrm{triv} \oplus \mathrm{sgn} \oplus (\mathrm{std})^2 \qquad \square \tag{1.22}$$

[Solution to Exercise 1.14 in FH] For an irrep V of a finite group G, there is a unique Hermitian inner product preserved by G. Say there are two Hermitian products, H and H', preserved by G. Any Hermitian inner product sets up an isomorphism between V and V^*. Let that map be given by

$$\phi_1(v) = H_1(v, \cdot) \in V^* \tag{1.23}$$
$$\phi_2(v) = H_2(v, \cdot) \in V^* \tag{1.24}$$

> Write details about the isomorphism between V and V^*. Crucial is the positive-definiteness of the inner product. Not sure where the Hermiticity of the inner product is important.

$$V^* \xleftarrow{\phi_1} V \xrightarrow{\phi_2} V^* \tag{1.25}$$

Now, the map $\phi = \phi_2 \cdot \phi_1^{-1}$ is an isomorphism of vector spaces. It is also a G-linear map since its the composition of G-linear maps. Therefore, ϕ is an isomorphism between two irreps. By Schur's lemma, $\phi = \lambda \cdot I$, where λ is any scalar. This gives us

$$H_2(v, \cdot) = \phi(H_1(v, \cdot)) \tag{1.26}$$
$$= \lambda H_1(v, \cdot) \qquad \square \tag{1.27}$$

[Solution to Exercise 1.13(a) in FH]
[Solution to Exercise 1.13(b) in FH] Is $\mathrm{Sym}^n(\mathrm{Sym}^m V) \cong \mathrm{Sym}^m(\mathrm{Sym}^n V)$? No, since the dimensions don't match on both sides.

$$\dim \mathrm{Sym}^n(\mathrm{Sym}^m V) = {}^{{}^{\dim V - 1 + m}C_m - 1 + n} C_n \tag{1.28}$$

The above equation is clearly not symmetric in m and n. So the isomorphism can not hold in general.

Chapter 2

Character Theory

Definition 1. If V is a representatin of G, its *character* χ_V is the complex-valued functin on the group defined by

$$\chi_V(g) = \mathrm{Tr}(g|_V), \tag{2.1}$$

the trace of g on V.

We shall drop the subscript V when it's obvious.
Some properties

1. χ_V is a class function, which means that it is constant on conjugacy classes of G.

$$\chi(hgh^{-1}) = \chi(g) \tag{2.2}$$

2.

2.1 First Projection Formula

Let us define V^G as the elements of V fixed under the action of G.

$$V^G = \{v \in V : gv = v \ \forall g \in G\} \tag{2.3}$$

The sub space V^G is actually the direct sum of the trivial subrepresentations.

Now, the endomorpism $\varphi \in \mathrm{End}(V)$,

$$\varphi = \frac{1}{|G|} \sum_{g \in G} g \tag{2.4}$$

is G linear, since $h\varphi h^{-1} = \frac{1}{|G|} \sum_{g \in G} hgh^{-1} = \varphi$. Infact, φ is projection of V to V^G.

So, now we have a way of finding the direct sum of the trivial subreprentations of V. The trace of this map is would be the dimension of V_G [Why?], which is the number of copies of trivial representations in V^G.

> For an arbitrary linear map (projection?) $T : V \to W$, do we have $\operatorname{Tr} T = \dim W$?

$$\begin{aligned} \dim V^G &= \operatorname{Tr}(\varphi) && (2.5)\\ &= \frac{1}{|G|}\sum_{g\in G}\operatorname{Tr}(g) && (2.6)\\ &= \frac{1}{|G|}\sum_{g\in G}\chi_V(g) && (2.7) \end{aligned}$$

In particular, if V is irreducible, then it has no trivial subrepresentations and thus

$$\sum_{g\in G}\chi_V(g) = 0 \qquad \text{if } V \text{ is an nontrivial irrep.} \tag{2.8}$$

We know that $\operatorname{Hom}(V,W)^G$, the set of all homorphisms $V \to W$ fixed under G, is just the space of all G-linear maps $V \to W$. [reference here.]

> Given a G-linear map $\phi : V \to W$, where V is a an irrep, ϕ defines an isomorphism between V and $\operatorname{Im}(\phi)$ (since $\ker\phi$ is a subrepresentation of V, and so must be $\{0\}$) Now, other such ϕ would give us copies of V in W. The dimensionality of the space $\operatorname{Hom}_G(V,W)$ thus tells us the multiplicity of V in W.

By Schur's lemma, if V and W are both irreducible then

$$\dim \operatorname{Hom}_G(V,W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \ncong W \end{cases} \tag{2.9}$$

Since we have $\operatorname{Hom}_G(V,W) \cong V^* \otimes W$,

$$\chi_{\operatorname{Hom}_G(V,W)}(g) = \overline{\chi_V(g)} \cdot \chi_W(g) \tag{2.10}$$

Using (2.7) here, we get the nice result

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g)} \cdot \chi_W(g) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \ncong W \end{cases} \tag{2.11}$$

This inspires us to define a dot product on the space of all class functions (functions that are defined on the conjugacy classes of a group) of G,

$$\mathbb{C}_{\text{class}}(G) = \{\text{class functions on } G\}, \tag{2.12}$$

as

$$(\alpha,\beta) = \frac{1}{|G|}\sum_{g\in G}\overline{\alpha(g)}\beta(g). \tag{2.13}$$

In respect this dot product, the characters of irreps are orthornormal.

Chapter 3

Induced Reprentations

3.1 Overview

1. Define induced representations
2. Mackay formula for the character of an induced representation.
3. Frobenius Reciprocity

3.2 Definitions

Don't really understand the motivation for this section. *Reference: Etingof.*

Given a subgroup $H \subset G$, and (ρ_V, V), a representation of G, we can easily construct a representation for H by simply restricting ρ_V to H, which we denote by $\rho_V|_H$.

However, if want to construct a representation of G from a representation of H, then we need to think a little more about it.

Think of the elements of V as functions of G, that is, every element $v \in V$ defines a map $v : G \to V$ given by

$$v(g) = \rho(g)v. \tag{3.1}$$

Now, we already know how $v(\cdot)$ acts on H. We want to extend that action to G. The least we could do is to ensure that the action on G is consistent with the known action on H. So, we enforce that

$$v(hg) = h \cdot v(g) = \rho_V(h)v(g) \quad \forall h \in H,\ g \in G. \tag{3.2}$$

> Why is $\mathrm{Ind}_H^G V$ naturally isomorphic to $\mathrm{Hom}_H(k[G], V)$?

Chapter 4

Representations of S_n

4.1 Overview

1. Represenations of S_n
2. Frobenius Formula for the character/dimension of a representation of S_n
3. Schur Functors

4.2 Representations

1. Define the standard Young tableau, given by a partition λ.
2. Define the subgroups A_λ, B_λ of S_n that act only on rows and columns of the tableau, respectively.
3. Define the **Young symmtrizer**,

$$c_\lambda = \sum_{\alpha \in A,\, \beta \in B} \alpha\beta \operatorname{sgn}(\beta). \tag{4.1}$$

4. For some $x \in G$, if $\alpha x \operatorname{sgn}(\beta)\beta = x$ for all $\alpha \in A$, $\beta \in B$, then x is a scalar multiple of c_λ.
5. The set $\{(\mathbb{C}S_n)c_\lambda : \lambda \text{ is a partition of } S_n\}$, with S_n acting on the left by multiplication, is the set of all representations of S_n.

Chapter 5

A Detour to Differential Geometry

1. Manifolds
2. Morphism of manifolds - Smooth mappings etc
3. Types of submanifolds - Immersion, Embedding etc
4. Tangent Spaces
5. Differential forms

Chapter 6

Lie Groups

6.1 Overview

1. Define Lie groups
2. Two principles
3. Vector fields, differential forms
4. The Lie bracket: two ways to define it.
5. The transition: Lie groups $\rightarrow$ Lie algebras
6. Adjoint representation
7. Examples of Lie algebras - from classical groups
8. Exponential map
9. Apply the exponential map to Lie groups

Chapter 7

Campbell-Baker-Hausdorff

7.1

We describe the algebraic proof of the CBH formula.

1. Define universal enveloping algebra
2. Define the tensor algebra
3. Show that universal enveloping algebra and the tensor algebra are canonically isomorphic
4. Poincare-Birkhoff-Witt theorem

Universal enveloping algebra

A *universal enveloping algebra* is, in some sense, the most general associative algebra which contains a given lie algebra.

Let $U(L)$ be an associative algebra with unit and L be a lie algebra. $U(L)$ is the universal enveloping algebra if we have a map $\epsilon : L \to U(L)$ such that

1. ϵ is Lie algebra homomorphism. This means that it is linear and it preserves the Lie bracket:
$$\epsilon[x, y] = \epsilon(x)\epsilon(y) - \epsilon(y)\epsilon(x) \tag{7.1}$$
2. it satisfies a *universal property*. If A is any associative algebra with unit and $\alpha : L \to A$ is any Lie algebra then there exists a unique $\phi : U(L) \to A$ such that the following diagram commutes.

$$\begin{tikzcd}
 & L \arrow[dl, "\alpha"'] \arrow[dr, "\epsilon"] & \\
A & & U(L) \arrow[ll, dashed, "\exists!\ \phi"]
\end{tikzcd}$$

> Given (ϵ_1, U_1) and (ϵ_2, U_2) we can see get maps $\phi_1 : U_1 \to U_2$ and $\phi_2 : U_2 \to U_1$ such that $\phi_1 \cdot \phi_2$ is identity on $\epsilon_1(L)$ and $\phi_2 \cdot \phi_1$ is identity on $\epsilon_2(L)$. But $\epsilon(L)$ generates $U(L)$, and ϕ is a associative

> algebra homomorphism, so
>
> $$\begin{aligned} \phi\left(\sum \epsilon(x_1)\cdots\epsilon(x_n)\right) &= \sum \phi(\epsilon(x_1))\cdots\phi(\epsilon(x_n)) && (7.2) \\ &= \sum \epsilon(x_1)\cdots\epsilon(x_n). && (7.3) \end{aligned}$$
>
> Thus the compositions are identity on the whole of U_1 and U_2.

Universal enveloping algebras are unique upto an isomorphism due to this universal property.

Tensor Algebra

$i : V \to T(V)$ such that for any map $\alpha : V \to A$, where A is an associative algebra, there exists an a map linear map $\psi : T(V) \to A$ such that the following diagram commutes

$$\begin{tikzcd} & V \arrow[dl, "\alpha"'] \arrow[dr, "i"] & \\ A & & T(V) \arrow[ll, dashed, "\exists!\ \psi"] \end{tikzcd}$$

Construction

Let L be a Lie algebra and let I be the two sided ideal generated by the elements $[x, y] - x \otimes y + y \otimes x$, then

$$U(L) = T(L)/I \tag{7.4}$$

is a universal enveloping algebra for L.

We need to show that this satisfies the universal property. Set $V = L$ in the diagram for tensor algebra $T(V)$ to get

$$\begin{tikzcd} & L \arrow[dl, "\alpha"'] \arrow[dr, "i"] & \\ A & & T(L) \arrow[ll, dashed, "\exists!\ \psi"] \end{tikzcd}$$

If, additionally, α is a Lie algebra homomorphism, then

$$\alpha[x, y] = \alpha(x)\alpha(y) - \alpha(y)\alpha(x) \quad \forall x, y \in L. \tag{7.5}$$

But since $\alpha = \psi \cdot i$

$$\psi([x, y] - x \otimes y - y \otimes x) = 0 \tag{7.6}$$

Therefore $I \subset \ker \psi$, and we get that there is a unique map $\overline{\phi} : T/I \to A$ such that the following diagram commutes

$$\begin{tikzcd}
 & L \arrow[dl, "\alpha"'] \arrow[dr, "i"] \arrow[d, phantom, "\circlearrowright"] & \\
A & \arrow[d, phantom, "\circlearrowright"] & T(L) \arrow[ll, "\psi"] \arrow[dl, "\phi"] \\
 & T/I \arrow[ul, dashed, "\overline{\phi}"] &
\end{tikzcd}$$

Thus, we have the a lie algebra T/I and a map $\epsilon = \phi \cdot i : L \to T/I$ such that the universal property is satisfied. $\square$

Extension of Lie algebra homorphism to its UEA

Given map $h : L \to M$ between two lie algebras L and M, the universal property implies the existence of a associative algebra homomorphism $U(L) \to U(M)$ as shown in the diagram:

$$\begin{tikzcd}
L \arrow[r, "h"] \arrow[d, "\epsilon_L"'] \arrow[dr, "\epsilon_M \cdot h"] & M \arrow[d, "\epsilon_M"] \\
U(L) \arrow[r, dashed, "\exists!"'] & U(M)
\end{tikzcd}$$

UEA of a direct sum

$$U(L_1 \oplus L_2) \cong U(L_1) \otimes U(L_2) \tag{7.7}$$

Bialgebra structure

Definition 2 (Bialgebra)**.** A vector space C with a map (*comultiplication*) $\Delta : C \to C \otimes C$ and a map (*co-unit*) $\varepsilon : D \to k$ satisfying

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} \quad \text{and} \tag{7.8}$$

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id} \tag{7.9}$$

is called a *co-algebra*. If C is an algebra and both Δ and ε are algebra homomorphisms, we say that C is a *bi-algebra*.

Let L be any lie algebra. The map $f : L \to U(L) \otimes U(L)$ defined by

$$x \mapsto f(x) = x \otimes 1 + 1 \otimes x \tag{7.10}$$

is a Lie algebra homomorphism. This can be seen by

$$f(x)f(y) - f(y)f(x) = [x, y] \otimes 1 + 1 \otimes [x, y] \tag{7.11}$$

$$= f[x, y]. \tag{7.12}$$

Thus this map induces a map $\Delta : U(L) \to U(L) \otimes U(L)$ as seen in the following commutative diagram.

$$\begin{tikzcd} & L \arrow[dl, "\epsilon"'] \arrow[dr, "f"] & \\ U(L) \arrow[rr, dashed, "\exists!\Delta"'] & & U(L)\otimes U(L) \end{tikzcd}$$

Now, define the map $\varepsilon : U(L) \to k$ as

$$\epsilon(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{for all } x \in L \end{cases} \tag{7.13}$$

and extend this as an algebra homorphism.

Proposition 1. *$(U(L), \Delta, \varepsilon)$ is a bialgebra.*

Proof. complete this □

The Poincaré-Birkhoff-Witt Theorem.

Complete this section.

Theorem 1 (Poincaré-Birkhoff-Witt)**.**

$$S(L) \cong \operatorname{gr} U(L) \tag{7.14}$$

Chapter 8

Lie Algebras

In this chapter, we undertake a systematic study of Lie algebras and attempt to classify them.

Theorem 2 (Ado's theorem)**.** *Every finite dimensional lie algebra is a subalgebra of* $\mathfrak{gl}(V)$*.*

Let's make a few definitions.

$$D\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \tag{8.1}$$

$$D_k\mathfrak{g} = [\mathfrak{g}, D_{k-1}\mathfrak{g}] \qquad \text{lower central series} \tag{8.2}$$

$$D^k\mathfrak{g} = [D^{k-1}\mathfrak{g}, D^{k-1}\mathfrak{g}] \qquad \text{derived series} \tag{8.3}$$

Definition 3. A lie algebra $\mathfrak{g}$ is

(i) **nilpotent** if $D_k\mathfrak{g} = 0$ for some k,
(ii) **solvable** if $D^k\mathfrak{g} = 0$ for some k, and
(iii) **semi-simple** if it has no non nonzero solvable ideals.

Definition 4 (Radical)**.** Sum of all solvable ideals in $\mathfrak{g}$ is again a solvable ideal, called the the **radical** of $\mathfrak{g}$ and denoted by $\operatorname{Rad}(\mathfrak{g})$

Definition 5 (Reductive)**.** A lie algebra $\mathfrak{g}$ is **reductive** if

Lemma 2. $(\mathfrak{a} + \mathfrak{b})/\mathfrak{b} \cong \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{b})$

Proof. The kernel of the composite map

$$\mathfrak{a} + \mathfrak{b} \twoheadrightarrow \mathfrak{a} \twoheadrightarrow \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{b}) \tag{8.4}$$

is the ideal $\mathfrak{b} \in \mathfrak{a} + \mathfrak{b}$. □

Notice that the lie algebra $\mathfrak{g}/\operatorname{Rad}(\mathfrak{g})$ is semisimple. Any lie algebra fits into the exact sequence

$$0 \to \operatorname{Rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\operatorname{Rad}(\mathfrak{g}) \to 0 \tag{8.5}$$

where the first algebra is solvable and the last algebra is semisimple. Our approach to classify representations of Lie algebras is then to study the representations of solvable and semisimple Lie algebras.

Theorem 3 (Lie's theorem)**.**

Appendix A

Tensor Product

Let $V,\ W$ be vector spaces. A tensor product is a vector space $V \otimes W$ equipped with a bilinear map

$$V \times W \to V \otimes W \tag{A.1}$$

$$(v, w) \mapsto v \otimes w \tag{A.2}$$

such that it is *universal*. That is, given any other vector space U and a bilinear map $\beta : V \times W \to U$, there is unique map $\beta' : V \otimes W \to U$ such that $\beta'(v \otimes w) = \beta(v, w)$. The following diagram commutes:

$$\begin{tikzcd}
& V \times W \arrow[dl, "\beta"'] \arrow[dr, "\otimes"] & \\
U & & V \otimes W \arrow[ll, dotted, "\exists!\ \beta'"]
\end{tikzcd}$$

The universality requirement means that the tensor product thus defined is unique upto an isomorphism. Let there be another tensor product $V \otimes' W$ (a vector space and a correspoding map that takes $(v, w) \mapsto v \otimes' w$) that is also universal. Then it immediately implies that there is a bijection between $V \otimes W$ and $V \otimes' W$ since

$$\begin{tikzcd}
& V \times W \arrow[dl, "\otimes'"'] \arrow[dr, "\otimes"] & \\
V \otimes' W \arrow[rr, dotted, shift left, "\exists!"] & & V \otimes W \arrow[ll, dotted, shift left, "\exists!"]
\end{tikzcd}$$

Theorem 4. $\mathrm{Hom}(V, W) \cong V^* \otimes W$ *as vector spaces.*

Proof. Quick Check: Set $W = \mathbb{C}$ so that we get $\mathrm{Hom}(V, \mathbb{C}) \cong V^* \otimes 1 = V^*$, which is the definition of the dual space V^*.

Given $(\beta : V \to W) \in \mathrm{Hom}(V, W)$, define the map $\phi : \mathrm{Hom}(V, W) \to V^* \otimes W$ such that

$$\phi(\beta) = \sum_{i,j} \beta_{ij} v_i^* \otimes w_j \tag{A.3}$$

where $\{v_i\}$ and $\{w_j\}$ are orthonormal basis for V and W respectively and $\beta_{ij} = \beta(v_i)^T w_j$. This is a homorphism of vector spaces. The inverse map is obvious. $\square$

A.1 Exterior Products and Symmetric Products

Definitions

Some Combinatorics

Lets find out the dimensions of the vector space $\Lambda^k V$, where V is an n-dimensional vector space. If $\{e_i\}$ is a basis for V, we know that a basis for $\Lambda^k V$ is

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} : i_1 < i_2 < \cdots < i_k\}. \tag{A.4}$$

How many vectors are there in this basis? This can be asnwered by looking at how many ways are there to simply choose k objects from a set of n different objects. Since there is just one of arranging them (in increasing order), that would be the number of vectors in the basis. So,

$$\dim \Lambda^k(V) = {}^{\dim V}C_k \tag{A.5}$$

What's the dimension of $\mathrm{Sym}^k(V)$? The basis is

$$\{e_{i_1} \cdot e_{i_2} \cdot \cdots \cdot e_{i_k} : i_1 \leq i_2 \leq \cdots \leq i_k\}. \tag{A.6}$$

Another way to write that is

$$\{e_1^{a_1} \cdot e_2^{a_2} \cdots e_k^{a_k}\}, \quad \text{such that } \sum_i a_i = n,\, a_i \geq 0. \tag{A.7}$$

We just need to know how many ways are there to distribute n identical things among k different people with no restriction on the number of things anyone can get. To solve this, introduce $k-1$ identical barriers (denoted by $|$) between n things (denoted by $\circ$)and look at the number of permutations.

$$\circ | \circ | \cdots | \circ \tag{A.8}$$

Every permutation here gives possible choice of $a_1, \ldots, a_k$. Therefore,

$$\dim(\mathrm{Sym}^k(V)) = {}^{\dim V + k - 1}C_k \tag{A.9}$$

Appendix B

General Information

B.1 Resources

Here are some resources that I found useful while preparing these notes

- Representation Theory - A First Course by *Fulton, Harris*