Differential Geometry

Hersh Singh

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Chapter 1

Basics

Definition 1. *A topological space M is called Haursdoff if any two points in it can be seperated by open sets. Explicitly, for any two points $p, q \in M$, there must be open neighbourhoods of U, U' of p, q respectively such that $U \cap U' = \varnothing$.*

1.1 Manifold

M is an n dimensional topological manifold if it

(i) is *Haursdoff*,
(ii) has a countable basis for topology, that is, it is *second countable*,
(iii) is locally homeomorphic to $\mathbb{R}^n$.

A basis for a topological space M is just an a family of open sets $\{U_\alpha | \alpha \in I\}$ such that $M = \bigcup_{\alpha \in I} U_\alpha$.

> The dimension of an n-dimensional manifold is unique.

Coordinate Charts

One of main motivations of a manifold is to find a space which looks locally euclidean. This section makes that explicit.

A pair (U, φ) where $U \in M$ is open and $\varphi : U \to \mathbb{R}^n$ is a homeomorphism is called a *coordinate chart*. By definition of a manifold, any point $p \in M$ is contained in at least one such open set $U \in M$.

Given a pair (U, φ), we call U the *coordinate domain* or *coordinate neighbourhood* of each of its points. The set U is called a *coordinate ball* or a *coordinate cube* if the image $\varphi(U)$ is an open ball or an open cube in $\mathbb{R}^n$, respectively.

Topological Properties of Manifolds

Smooth Structures

We can intuitively guess what a smooth map on the euclidean space means. A map $F : V \to U$, where $U \in \mathbb{R}^n$, $V \in \mathbb{R}^m$ are open subsets, is said to be *smooth* if all its partial derivatives are continuous in all orders. If in addition F is bijective and has a smooth inverse map, then it's called a *diffeomorphism*.

Now, here's how we extend concept of smoothness to manifolds.

We have M, a topological n-manifold.

$$\begin{tikzcd}
 & U \cup V \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\
\varphi(U \cup V) \in \mathbb{R}^n \arrow[rr, "\psi \cdot \varphi^{-1}"] & & \psi(U \cup V) \in \mathbb{R}^n
\end{tikzcd}$$

We already know how to talk about the smoothness of the map $\psi \cdot \varphi^{-1}$, since its just a map on the euclidean space. Lets call this the *transition map* from φ to ψ. Two charts (U, φ) and (V, ψ) are said to be smoothly compatible if either $U \cup V = \varnothing$ or the if transition map $\psi \cdot \varphi^{-1}$ is a diffeomorphism.

We define an *atlas for M* as a collection of charts whose domains cover M. An atlas $\mathcal{A}$ is called a *smooth atlas* if any two charts in $\mathcal{A}$ are smoothly compatible with each other.

more here

1.2 Tangent Space

Appendix A

References

1. Introduction to Differential Topology by Brocker and Janich