Representation Theory

Hersh Singh

August 27, 2014

Chapter 1

Representations of Finite Groups

1.1 Definitions

Let G be a finite group, V be a finite dimensional complex vector space.

A *representation* of G on V is a group homomorphism

$$\rho : G \to \mathrm{GL}(V), \tag{1.1}$$

where $\mathrm{GL}(V)$ is the group of automorphisms of V. Even though the "representation" is really the homomorphism, but it is common (especially by physicists) to refer to V itself as the representation.

> We say that this map gives V the structure of a G-module. This agrees with the definition of a R-module (R is a ring) that I have studied earlier. An R-module is simply a vector space defined over a ring, instead of a field. So the ring elements act as the *scalars*. Of course, the description must include a rule for the 'interaction' of the scalars with the vectors. To be a module, the interaction must be *linear*. In this case, the group homomorphism ρ gives us that rule of interaction between the vectors (elements of V) and the scalars (elements of G). For $g \in G, v \in V$,
>
> $$gv \equiv \rho(g)v \in V$$

A vector space homomorphism $\phi : V \to W$ is a morphism between the two representations V and W if the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{\phi} & W \end{array} \tag{1.2}$$

That is, $\phi g = g\phi$ for all $g \in G$. This makes the group elements *behave like scalars under module homomorphisms*. Such morphisms of representations are also called *G-linear map* or a *G intertwiner*.

> Why is this is a good definition? Seems to be inspired from module homomorphisms. This is natural in some sense - Figure that out.

$\operatorname{Ker}\phi$, $\operatorname{Im}\phi$, $\operatorname{Coker}\phi = V/\operatorname{Im}\phi$ are also G-modules. This is solely because of the commutativity of the above diagram.

- If $v \in \operatorname{Ker}\phi$, then $\phi(gv) = g\phi(v) = 0$. So gv also $\in \operatorname{Ker}\phi$. $\blacksquare$.
- If $v \in \operatorname{Im}\phi$, then let $\phi(v) = w$. So $\phi(gv) = g\phi(v) = gw \in W$. So gv also $\in \operatorname{Im}\phi$. $\blacksquare$.

One of the goals of our study is, given a representation, to develop tools for constructing other, preferably all, representations of the group. Some examples of representations that can be constructed from V and W

- Tensor product $V \otimes W$ via

$$g(v \otimes w) = g(v) \otimes g(w) \tag{1.3}$$

- Tensor power $V^{\otimes n}$ and the *exterior power* $\Lambda^n(V)$ and the *symmetric power* $\operatorname{Sym}^n(V)$ are its subrepresentations.
- the dual $V^* = \operatorname{Hom}(V, \mathbb{C})$. This is a little tricky though. The action of G on V^* must be such that it preserves the natural inner product, denoted by $\langle\ ,\ \rangle$, between them. This forces us to define the action of g such that

$$\rho^*(v^*) =$$

Chapter 2

General Information

2.1 Resources

Here are some resources that I found useful while preparing these notes

- Representation Theory - A First Course by *Fulton, Harris*