the grand potential (6.34). Let us also revert the definition $x = p/m = \sqrt{\mu^2 - m^2}/m$, in order to make the dependency on μ and $\langle\sigma\rangle$ explicit. Thus, the grand potential is

$$\begin{aligned}
\omega(\langle\sigma\rangle, \mu_u, \mu_d, \mu_e) = &-\frac{1}{2}m^2\langle\sigma\rangle^2 + \frac{\lambda_P}{4!}\langle\sigma\rangle^4 - h\langle\sigma\rangle + N_c N_f \frac{m_q^4}{16\pi^2}\left[\frac{3}{2} + \log\left(\frac{\Lambda^2}{m_q^2}\right)\right] \\
&- \frac{N_c}{24\pi^2}\sum_{f=\{u,d\}}\left[\left(2\mu_f^2 - 5m_q^2\right)\mu_f\sqrt{\mu_f^2 - m_q^2} + 3m_q^4 \operatorname{asinh}\left(\sqrt{\frac{\mu_f^2}{m_q^2} - 1}\right)\right] \\
&- \frac{1}{24\pi^2}\left[\left(2\mu_e^2 - 5m_e^2\right)\mu_e\sqrt{\mu_e^2 - m_e^2} + 3m_e^4 \operatorname{asinh}\left(\sqrt{\frac{\mu_e^2}{m_e^2} - 1}\right)\right].
\end{aligned} \tag{6.34}$$

Remember that the effective quark mass $m_q = g\langle\sigma\rangle$ also contains dependence on the field! In addition, we implicitly take the real part of every square root $\sqrt{\mu^2 - m^2}$, or equivalently consider them as step functions $\Theta(\mu - m)\sqrt{\mu^2 - m^2}$. The rigorous understanding of this can be traced back to the zero-temperature calculation of the pressure (4.10c), related to the grand potential density by $\omega = -P$, in which the integrand contained a step function $\Theta(\mu - E(p)) = \Theta(\mu - \sqrt{m^2 + p^2})$ and we took for granted that $\mu > m$. In the opposite case $\mu < m$, the step function in the integrand would be deactivated for all p, and the integral would be 0.

We require that the mean field $\langle\sigma\rangle$ always takes on a value that minimizes grand potential according to

$$\frac{\partial\omega}{\partial\langle\sigma\rangle} = 0. \tag{6.35}$$

To get a feeling for the general shape of the potential, we visualize the special case with $\mu_e = 0$ and $\mu_u = \mu_d = 0$ in figure 6.2. In this case, the potential is effectively a two-dimensional function $\omega(\langle\sigma\rangle, \mu)$, and minimizing the potential yields a curve $[\langle\sigma\rangle(\mu), \mu, \omega(\langle\sigma\rangle, \mu))]$ through three-dimensional $\langle\sigma\rangle - \mu - \omega$-space. From the figure, we see that:

Silver Blaze.

- For $\mu < 300\,\text{MeV} = m_q(f_\pi)$, all square roots are "deactivated", so we are in vacuum with minima at $\langle\sigma\rangle = f_\pi = 93\,\text{MeV}$, as we required in equation (6.33).
- For $300\,\text{MeV} < \mu \lesssim 400\,\text{MeV}$, the square roots are "activated" and the minimum quickly and continuously transitions closer to 0, corresponding to a second-order phase transition.
- For $\mu \gtrsim 400\,\text{MeV}$, the minimum asymptotically approaches $\langle\sigma\rangle \to 0$.

Our objective, however, is to determine the equation of state $\epsilon(P)$ in the general case where we also take the conditions of charge neutrality **(TODO: ref)** and β-equilibrium **(TODO: ref)** into account. Then $\omega(\langle\sigma\rangle, \mu_u, \mu_d, \mu_e)$ is a four-dimensional function, and the three interdependent chemical potentials are reduced to one independent one by charge neutrality and chemical equilibrium. Using that up and down quarks have respective charges $+2/3$ and $-1/3$ and remembering their color degeneracy, we must now solve the *system* of equations

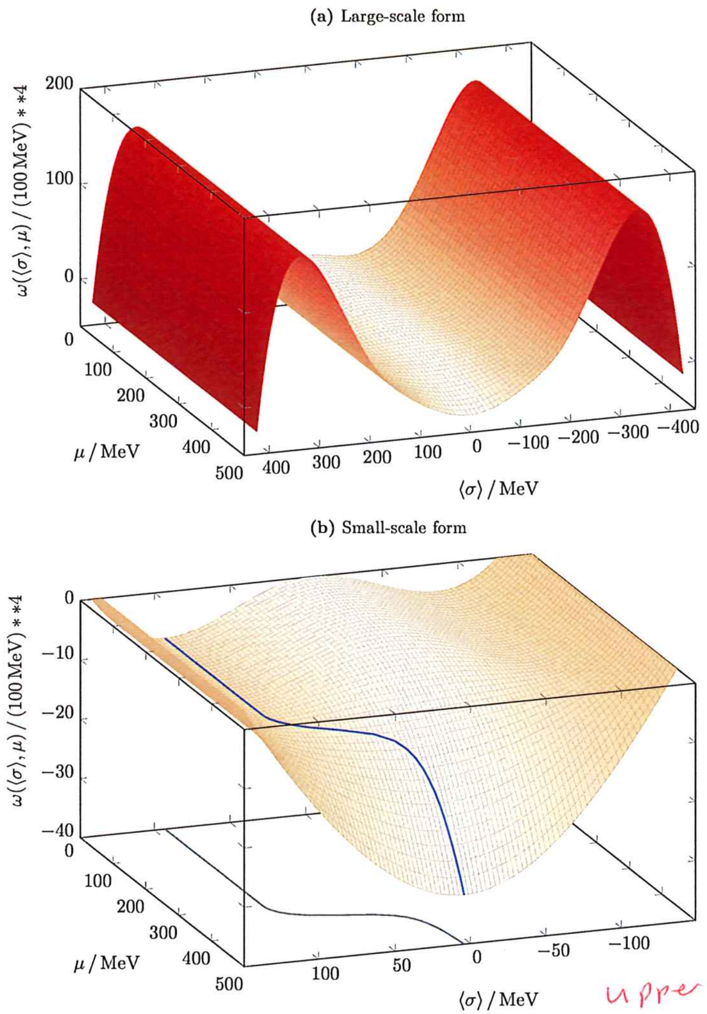
$$0 = \frac{\partial\omega}{\partial\langle\sigma\rangle}, \tag{6.36a}$$

$$0 = +N_c\frac{2}{3}\left(\mu_u^2 - m_q^2\right)^{3/2} - N_c\frac{1}{3}\left(\mu_d^2 - m_q^2\right)^{3/2} - 1\left(\mu_e^2 - m_e^2\right)^{3/2}, \tag{6.36b}$$

$$\mu_d = \mu_u + \mu_e. \tag{6.36c}$$

These three equations constrains the four variables down to one independent one – say μ_u – that parametrizes the other ones $\mu_d(\mu_u)$, $\mu_e(\mu_u)$ and $\langle\sigma\rangle(\mu_u)$, and thus also the potential $\omega'(\mu_u) = \omega(\langle\sigma\rangle(\mu_u), \mu_u, \mu_d(\mu_u), \mu_e(\mu_u))$. We now shift the grand potential

$$\omega'(\mu_u) \to \omega'(\mu_u) - \omega'(\mu_u < 300\,\text{MeV}) \tag{6.37}$$



upper panel

Figure 6.2: The grand potential (6.34) with $\mu_e = 0$ and $\mu_u = \mu_d = \mu$. Subfigure (a) shows its asymptotic form as $\langle\sigma\rangle \to \pm\infty$, while subfigure (b) highlights the interesting region around $0\,\text{MeV} \lesssim \langle\sigma\rangle \lesssim 100\,\text{MeV}$. The blue line and its gray projection corresponds to the fields $\langle\sigma\rangle(\mu)$ for which the potential has a minimum $\partial\omega/\partial\langle\sigma\rangle = 0$.

lower...

Non zero ?

so that it is measured relative to vacuum. Glancing back at equation (3.1) and (3.2), we now compute the pressure

$$P(\mu_u) = -\omega'(\mu_u) \tag{6.38}$$

relative to vacuum, and the corresponding energy density

$$\epsilon(\mu_u) = -P(\mu_u) + \mu_u n_u(\mu_u) + \mu_d(\mu_u) n_d(\mu_u) + \mu_e(\mu_u) n_e(\mu_u) \tag{6.39}$$

$\sum \mu_i n_i$?

with the zero-temperature densities (4.10a) that we calculated in equation (4.10a), (**TODO:** factor N_c for quarks?)

$$n_u(\mu_u) = \frac{1}{3\pi^2}\left[\mu_u^2 - m_q^2(\langle\sigma\rangle(\mu_u))\right]^{3/2}, \tag{6.40a}$$

$$n_d(\mu_u) = \frac{1}{3\pi^2}\left[\mu_d^2(\mu_u) - m_q^2(\langle\sigma\rangle(\mu_u))\right]^{3/2}, \tag{6.40b}$$

$$n_e(\mu_u) = \frac{1}{3\pi^2}\left[\mu_e^2(\mu_u) - m_e^2\right]^{3/2}. \tag{6.40c}$$

Then we finally eliminate μ_u to produce the equation of state $\epsilon(P)$. The numerical implementation of this whole procedure is described in (**TODO:** add and ref to appendix).

Having described we *do* and *should do*, let us also remark on what we *did* and *shouldn't do*. A tempting and similar approach to minimizing ω with respect to $\langle\sigma\rangle$ is to rather construct the potential $\omega'(\langle\sigma\rangle, \mu_u) = \omega(\langle\sigma\rangle, \mu_u, \mu_d(\mu_u, \langle\sigma\rangle), \mu_e(\mu_u, \langle\sigma\rangle))$, then minimize ω' with respect to $\langle\sigma\rangle$. In this approach, we could create two functions $\mu_{d/e}(\langle\sigma\rangle, \mu_u)$ that calculate the two remaining chemical potentials by solving the charge neutrality condition (6.36b) with a simple scalar root finder, both of which would be invoked upon evaluating the new potential $\omega'(\langle\sigma\rangle, \mu_u)$. This sounds really nice, because we could then use a simple minimization algorithm on ω', sparing us from calculating $\partial\omega/\partial\langle\sigma\rangle$ and ensuring that we always find *minima*, instead of maxima that we can obtain when solving $\partial\omega/\partial\langle\sigma\rangle = 0$. In effect, we would completely circumvent the need of ever solving a *system* of equations! Even better, the two-dimensional potential ω' would be possible to visualize as in figure 6.2, allowing us to verify our solution visually. Unfortunately, $\partial\omega'/\partial\langle\sigma\rangle \neq \partial\omega/\partial\langle\sigma\rangle$, because the two differs by terms arising from the chain rule, so this approach is different and – like many nice-sounding things – *wrong*! The cause of this source of confusion is that *the charge neutrality condition* (6.36b) *depends on the same variable that the potential should be minimized with respect to.* Hopefully, this remark will save someone from enduring this painful realization again.

Doing things the right way, we end up with the results shown in figure 6.3, and in particular the equation of state in figure 6.3c.

(**TODO:** fix parameter choices. so far I have only gotten $m_\sigma = 900\,\text{MeV}$ to work.)

(**TODO:** we have $d\epsilon/dP < 1$, contradicting stability, for larger P. is this where the bag constant comes in?)

(**TODO:** determine all 4 couplings from 4 requirements, only use physical values $h \neq 0$ (?))

(**TODO:** redefine sign in $m^2 \to -m^2$?)

(**TODO:** why renormalize vacuum? seems like it only adds minor corrections to grand potential?)

(**TODO:** do I determine λP from

$$\langle\sigma\rangle_{\min} = f_\pi = \frac{\sqrt{6}m}{\sqrt{\lambda}} = \frac{\sqrt{6}m}{\sqrt{\lambda_P\left(1+\frac{\delta\lambda}{\lambda_P}\right)}} \simeq \frac{\sqrt{6}m}{\lambda_P} \tag{6.41}$$

???)

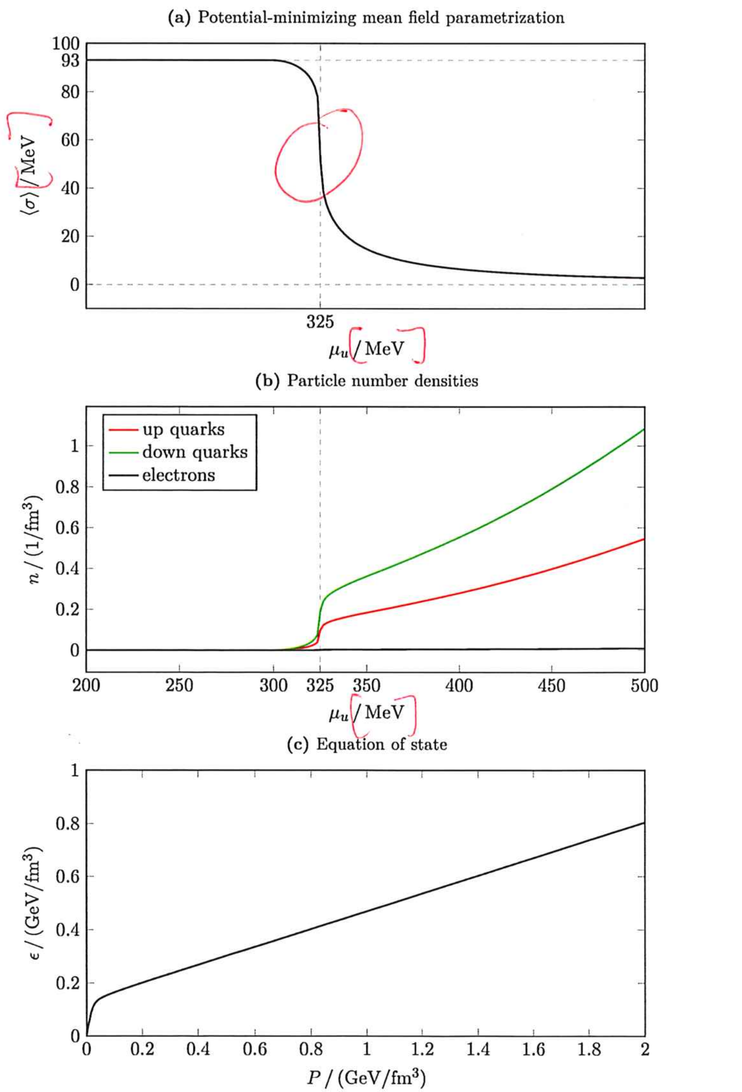


Figure 6.3: (a) Potential-minimizing mean field parametrization, (b) particle number densities and (c) equation of state corresponding to the solution of the system of equations (6.36).

Draw free gas result to same!