Séries geométricas

Progressão geométrica

Sequência $(a_k)_k$ da forma $a_k = a_1 q^{k-1}$

$q \rightarrow$ razão de $(a_k)_k$

$$\frac{a_{k+1}}{a_k} = q \quad se \quad a_k \neq 0$$

Exemplo 1

$(a_k) = (2,6,18,54,162,\ldots)$

$a_1 = 2$

$$q = \frac{a_2}{a_1} = \frac{6}{2} = 3$$

$$a_k = 2.3^{k-1}$$

Exemplo 2

$$(a_k) = \left(\frac{1}{2}; \frac{1}{2^2}; \frac{1}{2^3}; \frac{1}{2^4} \dots.\right)$$

$$a_1 = \frac{1}{2}$$

$$q = \frac{1}{2}$$

Exemplo 3

$$(a_k) = (5; 0; 0; 0; \dots)$$

a_1 =5

$$q = 0$$

Séries geométricas

Séries cujos termos são uma progressão geométrica

$$\sum_{k=1}^{\infty} a_1 q^{k-1}$$

$q \quad \rightarrow \quad$ razão de $\displaystyle\sum_{k=1}^{\infty} a_1 q^{k-1}$

Exemplo 4

$$\sum_{k=1}^{\infty} 2.3^{k-1} = 2 + 6 + 18 + 54 + 162 + \cdots$$

Exemplo 5

$$\sum_{k=1}^{\infty} 7.\left(\frac{1}{3}\right)^{k-1} = 7 + \frac{7}{3} + \frac{7}{3^2} + \frac{7}{3^3} + \frac{7}{3^4} \ldots$$

$$s_n = \sum_{k=1}^{n} a_1 q^{k-1} = a_1 + a_1 q + a_1 q^2 + \cdots + a_1 q^{n-2} + a_1 q^{n-1}$$

$$\Leftrightarrow$$

$$s_n q = a_1 q + a_1 q^2 + a_1 q^3 + \cdots + a_1 q^{n-1} + a_1 q^n$$

$$\Leftrightarrow$$

$$a_1 + q s_n = \underbrace{a_1 + a_1 q + a_1 q^2 + \cdots + a_1 q^{n-1}}_{s_n} + a_1 q^n$$

$$\Leftrightarrow$$

$$a_1 + q s_n = s_n + a_1 q^n$$

$$\Leftrightarrow$$

$$q s_n - s_n = a_1 q^n - a_1$$

$$\Leftrightarrow$$

$$(q-1) s_n = a_1 (q^n - 1)$$

$Se\ q \neq 1:$

$$s_n = a_1 \left(\frac{q^n - 1}{q - 1} \right) = a_1 \left(\frac{1 - q^n}{1 - q} \right) = \frac{a_1}{1 - q} (1 - q^n)$$

Exemplo 6

$$\sum_{k=1}^{\infty} 7.\left(\frac{1}{3}\right)^{k-1} = 7 + \frac{7}{3} + \frac{7}{3^2} + \frac{7}{3^3} + \frac{7}{3^4} \ldots$$

$$s_n = \sum_{k=1}^{n} 7.\left(\frac{1}{3}\right)^{k-1} = 7 + \frac{7}{3} + \frac{7}{3^2} + \cdots + \frac{7}{3^{n-1}} = 7\left(\frac{1 - \frac{1}{3^n}}{1 - \frac{1}{3}}\right) = 7\,\frac{1 - \frac{1}{3^n}}{\frac{2}{3}} = \frac{21}{2}\left(1 - \frac{1}{3^n}\right)$$

$$s_4 = \sum_{k=1}^{4} 7.\left(\frac{1}{3}\right)^{k-1} = 7 + \frac{7}{3} + \frac{7}{3^2} + \frac{7}{3^3} = \frac{21}{2}\left(1 - \frac{1}{3^4}\right) = \frac{21}{2}\frac{80}{81} = \frac{280}{27}$$

$$\sum_{k=1}^{\infty} 7.\left(\frac{1}{3}\right)^{k-1} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{21}{2}\left(1 - \frac{1}{3^n}\right) = \frac{21}{2}\left(1 - \frac{1}{\lim_{n\to\infty} 3^n}\right) = \frac{21}{2}$$

$$\sum_{k=1}^{\infty} a_1 q^{k-1} = ? \quad q \neq 1, a_1 \text{ constantes reais,} \qquad n \text{ variável natural}$$

$$\sum_{k=1}^{\infty} a_1 q^{k-1} = \lim s_n = \lim \left(\frac{a_1}{1-q}\right)(1-q^n) = \left(\frac{a_1}{1-q}\right)\lim(1-q^n) = \left(\frac{a_1}{1-q}\right)(1-\lim q^n)$$

Ou seja:

$$\sum_{k=1}^{\infty} a_1 q^{k-1} < \infty \quad se\ e\ somente\ se\ (q^n)_n \text{ é } convergente$$

Mas

$$(q^n)_n \text{ é } convergente \text{ se e somente se } |q| < 1$$

Conclusão:

$$\sum_{k=1}^{\infty} a_1 q^{k-1} < \infty \quad se\ e\ somente\ se\ |q| < 1$$

E mais, caso $|q| < 1$:

$$\sum_{k=1}^{\infty} a_1 q^{k-1} = \left(\frac{a_1}{1-q}\right)(1 - \lim q^n) = \frac{a_1}{1-q}$$

Exemplo 7

$$\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k-1} = ? \qquad a_1 = 1 \quad e \quad |q| = \frac{2}{3} < 1$$

$$|q| = \frac{2}{3} < 1 \;\Rightarrow\; \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k-1} < \infty; \qquad \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k-1} = \frac{1}{1-\frac{2}{3}} = \frac{1}{\frac{1}{3}} = 3$$

Exemplo 8

$$\sum_{k=1}^{\infty}\left(\frac{3}{2}\right)^{k-1} = ?$$

$$|q| = \frac{3}{2} > 1 \;\Rightarrow\; \sum_{k=1}^{\infty}\left(\frac{3}{2}\right)^{k-1} \nless \infty; \quad \text{como} \;\; a_k > 0 \;\; \text{então} \;\; \sum_{k=1}^{\infty}\left(\frac{3}{2}\right)^{k-1} = \infty.$$

Exercícios

1. Calcule, quando possível, a soma das seguintes séries:

$a)\ \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{k-1}$

$b)\ \sum_{k=1}^{\infty} \left(\frac{2}{5}\right)^{k}$

$c)\ \sum_{k=1}^{\infty} \left(\frac{7}{10}\right)^{k}$

$d)\ \sum_{k=1}^{\infty} 2\left(\frac{3}{5}\right)^{k}$

$e)\ \sum_{k=1}^{\infty} \left(\frac{2}{5}\right)^{k}$

$f)\ \sum_{k=1}^{\infty} 5\left(\frac{1}{8}\right)^{k} + 2\left(\frac{2}{7}\right)^{k}$

$g)\ \sum_{k=1}^{\infty} 4\left(\frac{4}{5}\right)^{k-3}$

$h)\ \sum_{k=1}^{\infty} \left(-\frac{5}{8}\right)^{k-1}$

2. Uma série geométrica tem como primeiro termo 4 e soma igual à 6. Qual a expressão desta série?