Alguns critérios de convergência de séries com termos não negativos

Parte II

Critério da comparação

Teorema 1

Sejam $\sum_{k=1}^{\infty} a_k$ e $\sum_{k=1}^{\infty} b_k$ séries reais. Assim:

1. $\sum_{k=1}^{\infty} b_k < \infty \quad e \quad a_k \leq b_k \quad \Rightarrow \quad \sum_{k=1}^{\infty} a_k < \infty;$
2. $\sum_{k=1}^{\infty} b_k = \infty \quad e \quad a_k \geq b_k \quad \Rightarrow \quad \sum_{k=1}^{\infty} a_k = \infty.$

Exemplo 1

A série $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ é convergente.

$* \quad \dfrac{1}{k^2+1} \leq \dfrac{1}{k^2}, \forall k \in \mathbb{N};$

$* \quad \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$

$$0 \leq \sum_{k=1}^{\infty} \frac{1}{k^2+1} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

Exemplo 2

A série $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k-\frac{1}{2}}$ é divergente.

$* \quad \dfrac{1}{k-\frac{1}{2}} > \dfrac{1}{k}, \forall k \in \mathbb{N};$

$* \quad \displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = \infty$

$$\sum_{k=1}^{\infty} \frac{1}{k-\frac{1}{2}} \geq \sum_{k=1}^{\infty} \frac{1}{k}$$

Exemplo 3

Verifique se a série $\sum_{k=1}^{\infty} \frac{1}{k!}$ é convergente.

Solução

$$k! = k.(k-1).(k-2)\dots 3.2.1 \qquad k \neq 0$$
$$0! = 1$$

Podemos mostrar que $2^k \leq k!$ para todo $k \geq 4$. Com isso:

$$\frac{1}{k!} \leq \frac{1}{2^k}, \forall k \geq 4$$
$$\Rightarrow$$
$$\sum_{k=4}^{\infty} \frac{1}{k!} < \infty \quad \left(\text{Pois} \quad \sum_{k=4}^{\infty} \frac{1}{2^k} < \infty\right) \quad \text{Logo:} \sum_{k=1}^{\infty} \frac{1}{k!} < \infty \left(\text{Pois} \quad \sum_{k=1}^{\infty} \frac{1}{k!} \; converge\ apartir\ de\ um\ dado\ k\right.$$

Atenção!!!

$$* \sum_{k=1}^{\infty} \frac{1}{k+1} < \infty \,?$$

$$* \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} < \infty \,?$$

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

$$\frac{1}{k+1} < \frac{1}{k} \quad \text{e} \quad \frac{1}{(k+1)^2} < \frac{1}{k}$$

Porém:

$$\sum_{k=1}^{\infty} \frac{1}{k+1} = \infty \quad \text{e} \quad \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} < \infty \qquad \text{Pois} \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

Critério da razão (ou de D'Alambert)

Teorema 2

Seja $\sum_{k=1}^{\infty} a_k$ uma série real e $L = \lim_{k \to \infty} \frac{a_{k+1}}{a_k}$. Assim:

$*$ $$L < 1 \Rightarrow \sum_{k=1}^{\infty} a_k < \infty;$$

$*$ $$L > 1 \Rightarrow \sum_{k=1}^{\infty} a_k = \infty.$$

$$L = 1 \Rightarrow ?$$

Exemplo 4

$$\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k} = ?$$

$$L = \lim \frac{\left(\frac{2}{3}\right)^{k+1}}{\left(\frac{2}{3}\right)^{k}} = \lim \frac{\left(\frac{2}{3}\right)^{k}\left(\frac{2}{3}\right)}{\left(\frac{2}{3}\right)^{k}} = \left(\frac{2}{3}\right) < 1$$

Logo:

$$\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k} < \infty$$

Exemplo 5

$$\sum_{k=1}^{\infty} \frac{3^k}{k2^k} = ?$$

$$L = \lim \frac{\frac{3^{k+1}}{(k+1)2^{k+1}}}{\frac{3^k}{k2^k}} = \lim \frac{\frac{3^k 3}{(k+1)2^k.2}}{\frac{3^k}{k2^k}} = \lim \frac{\cancel{3^k}.3.k.\cancel{2^k}}{(k+1).\cancel{2^k}.2.\cancel{3^k}} = \lim \frac{k}{(k+1)}\frac{3}{2} = \frac{3}{2} > 1$$

Logo:

$$\sum_{k=1}^{\infty} \frac{3^k}{k2^k} = \infty$$

Exemplo 6

$$\sum_{k=1}^{\infty} \frac{2^k}{k!} = ?$$

$$L = \lim \frac{\frac{2^{k+1}}{(k+1)!}}{\frac{2^k}{k!}} = \lim \frac{2^{k+1}.k!}{2^k(k+1)!} = \lim \frac{\cancel{2^k}.2.\cancel{k!}}{\cancel{2^k}(k+1)\cancel{k!}} = \lim \frac{2}{(k+1)} = 0 < 1$$

Logo:

$$\sum_{k=1}^{\infty} \frac{2^k}{k!} < \infty$$

Critério da raíz (ou de Cauchy)

Teorema 3

Seja $\sum_{k=1}^{\infty} a_k$ uma série real e $L = \lim_{k \to \infty} \sqrt[k]{a_k}$. Assim:

* $\quad L < 1 \Rightarrow \sum_{k=1}^{\infty} a_k < \infty;$

* $\quad L > 1 \Rightarrow \sum_{k=1}^{\infty} a_k = \infty.$

$L = 1 \Rightarrow ?$

Exemplo 7

$$\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k}=?$$

$$L=\lim\sqrt[k]{\left(\frac{2}{3}\right)^{k}}=\frac{2}{3}<1$$

Logo:

$$\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k}<\infty$$

Exemplo 8

$$\sum_{k=1}^{\infty}\left(\frac{K+1}{2k}\right)^{k} = ?$$

$$L = \lim \sqrt[k]{\left(\frac{K+1}{2k}\right)^{k}} = \lim\left(\frac{K+1}{2k}\right) = \frac{1}{2} < 1$$

Logo:

$$\sum_{k=1}^{\infty}\left(\frac{K+1}{2k}\right)^{k} < \infty$$

Proposição

$$Sejam\ a, b \in \mathbb{R}, onde\ a \neq 0\ ou\ b \neq 0.\ Assim\colon \lim \sqrt[k]{ak+b} = 1$$

Idéia da prova:

$$y = \lim \sqrt[k]{ak+b} \Leftrightarrow \ln y = \ln\left(\lim \sqrt[k]{ak+b}\right) = \lim \ln \sqrt[k]{ak+b} = \lim \frac{\ln(ak+b)}{k} = 0$$

Logo:

$$\lim \sqrt[k]{ak+b} = y = 1$$

Exemplo 9

$$\sum_{k=1}^{\infty} \frac{k}{2^k} = ?$$

$$L = \lim \sqrt[k]{\frac{k}{2^k}} = \lim \frac{\sqrt[k]{k}}{2} = \frac{1}{2} < 1$$

Logo:

$$\sum_{k=1}^{\infty} \frac{k}{2^k} < \infty$$

Exercícios

1. Verifique se as seguintes séries convergem ou não.

$a) \sum_{k=1}^{\infty} \frac{k!}{k^k}$

$b) \sum_{k=1}^{\infty} \frac{3^{k+4}.k^2}{2^{k^2}}$

$c) \sum_{k=1}^{\infty} \frac{3^k(k^3+k)}{4^k+1}$

$d) \sum_{k=1}^{\infty} \frac{4^{2k}}{k!+k^2}$

$e) \sum_{k=1}^{\infty} \frac{k^k}{2^k k!}$

$f) \sum_{k=1}^{\infty} \frac{k^k}{3^k k!}$

2. Mostre que:

i. Para todo $c > 0$:

$$\lim \frac{c^k}{k!} = 0$$

ii. $\lim \frac{k!}{k^k} = 0$

iii. $\lim \frac{k^3 4^k}{5^k} = 0$

iv. Para todo $n \in \mathbb{N}$:

$$\lim_{k \to \infty} \frac{k^n 4^k}{5^k} = 0$$

v. Para todo $n \in \mathbb{N}$ e $a \in]0,1[$:

$$\lim_{k \to \infty} k^n a^k = 0$$

3. Verifique se as seguintes séries convergem ou não.

$a) \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

$b) \sum_{k=1}^{\infty} \frac{1}{3^k + 1}$

$c) \sum_{k=1}^{\infty} \frac{k + 1}{k^3}$

$d) \sum_{k=1}^{\infty} \frac{2^k}{3^k + 1}$

$e) \sum_{k=1}^{\infty} \frac{4^k}{3^k - 2}$

$f)$ $\displaystyle\sum_{k=1}^{\infty} \frac{4^k}{3^k + 2^k}$

$g)$ $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^3}$

4 Cite um exemplo de três séries $\sum_{k=1}^{\infty} a_k$; $\sum_{k=1}^{\infty} b_k$ e $\sum_{k=1}^{\infty} c_k$ com termos não negativos, que atendam (todas)as características abaixo:

$i.$ $\displaystyle\sum_{k=1}^{\infty} a_k < \infty$

$ii.$ $b_k \geq a_k$

$iii.$ $c_k \geq a_k$

$iv.$ $\displaystyle\sum_{k=1}^{\infty} b_k = \infty$

$v.$ $\displaystyle\sum_{k=1}^{\infty} c_k < \infty$