Integrais Multiplas

Integrais duplas

Exemplo 1

Encontre o conjunto de todas as funções $f\colon \mathbb{R}^2 \to \mathbb{R}$ tais que:

$$\frac{\partial^2 f}{\partial y \partial x} = x$$

Solução

$$\frac{\partial^2 f}{\partial y \partial x} = x \Leftrightarrow \frac{\partial \left(\frac{\partial f}{\partial x}\right)}{\partial y} = x \Rightarrow \int \frac{\partial \left(\frac{\partial f}{\partial x}\right)}{\partial y} dy \Rightarrow \frac{\partial f}{\partial x} = xy + c(x)$$

$$\Rightarrow \int \frac{\partial f}{\partial x} dx = xy + c(x) \Rightarrow f(x,y) = \frac{x^2}{2} + C(x) + d(y)$$

Onde $C(x)$ é uma primitiva de $c(x)$

Exemplo 2

Calcule:

$$\int_0^2 \int_x^{x^2} x^2y + 2y \, dydx$$

Solução

$$\int_0^2 \int_x^{x^2} x^2y + 2y \, dydx = \int_0^2 \frac{x^2y^2}{2} + y^2 \Big|_x^{x^2} dx = \int_0^2 \frac{x^2(x^2)^2}{2} + (x^2)^2 - \left(\frac{x^2x^2}{2} + x^2\right) dx$$

$$= \int_0^2 \frac{x^6}{2} + x^4 - \frac{x^4}{2} - x^2 dx = \int_0^2 \frac{x^6}{2} + \frac{x^4}{2} - x^2 dx$$

$$= \frac{x^7}{14} + \frac{x^5}{10} - \frac{x^3}{3}\Big|_0^2 = \frac{2^7}{14} + \frac{2^5}{10} - \frac{2^3}{3} = \frac{1016}{105}$$

Exemplo 3

Calcule:

$$\int_{1}^{4}\int_{y-1}^{2y+1} 4xy\, dxdy$$

Solução

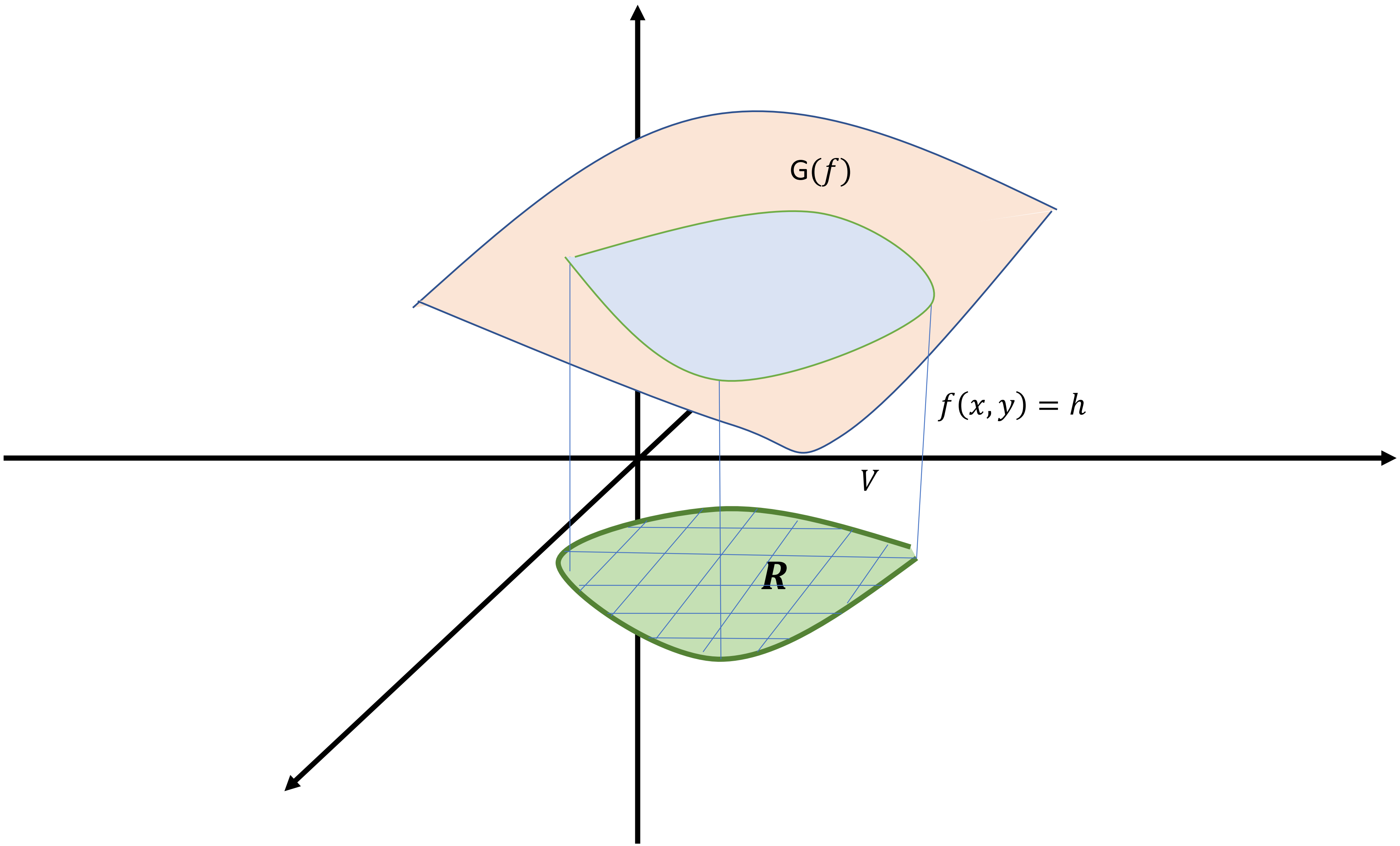
$$\int_{1}^{4}\int_{y-1}^{2y+1} 4xy\, dxdy = \int_{1}^{4} 2x^2y \Big|_{y-1}^{2y+1} dx = \int_{1}^{4} 2(2y+1)^2 - (y-1)^2 dx$$

$$= \int_{1}^{4} 8y^2 + 8y + 2 - (y^2 - 2y + 1)dx = \int_{1}^{4} 7y^2 + 10y + 1dx$$

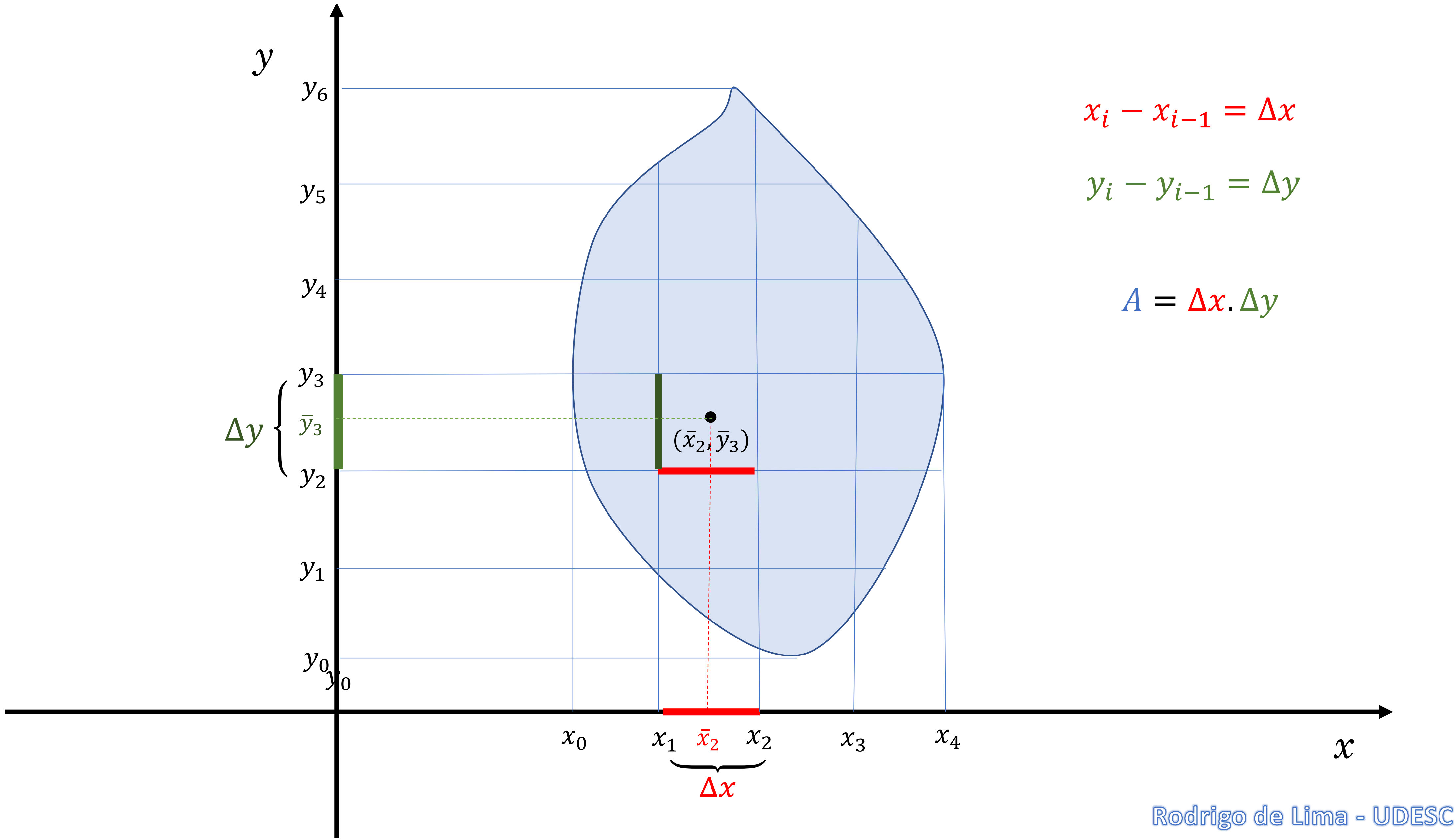
$$= \frac{7y^3}{3} + 5y^2 + y \Big|_{1}^{4} = 7\frac{4^3}{3} + 5.4^2 + 4 - \left(7\frac{1^3}{3} + 5.1^2 + 1\right) = 225$$

Rodrigo de Lima - UDESC

Cálculo de uma integral dupla



R → Região no $\mathbb{R}^2$



y
y_6
y_5
y_4
y_3
Δy
$\bar{y}_3$
y_2
y_1
y_0
y_0
$(\bar{x}_2, \bar{y}_3)$
x_0
x_1
$\bar{x}_2$
x_2
x_3
x_4
Δx
x
$x_i - x_{i-1} = \Delta x$
$y_i - y_{i-1} = \Delta y$
$A = \Delta x . \Delta y$
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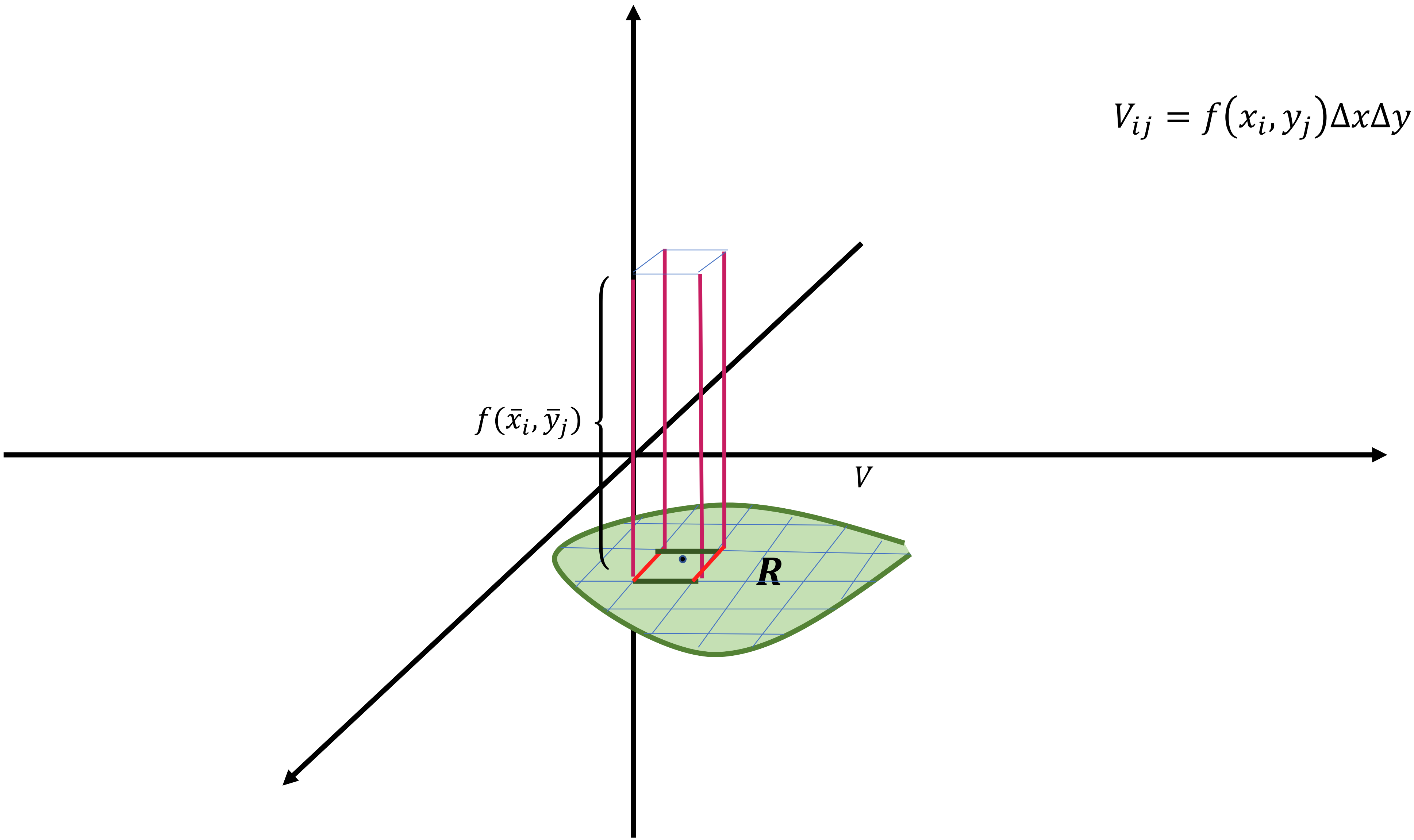


R → Região no $\mathbb{R}^2$

$$V \approx \sum_{j=1}^{n} \sum_{i=1}^{m} f(\bar{x}_i, \bar{y}_j) \Delta x \Delta y$$

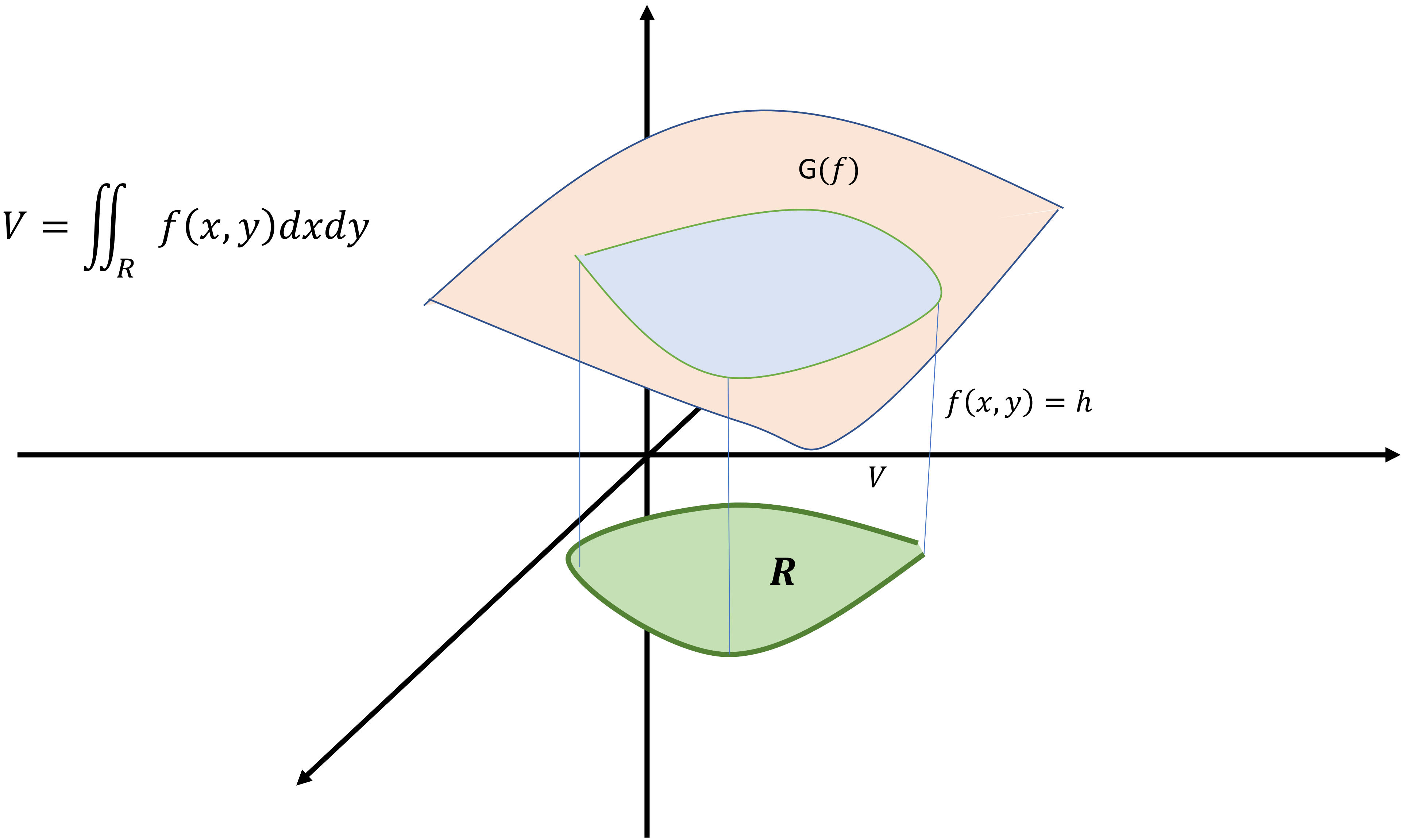
Onde $(\bar{x}_i, \bar{y}_j) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$.

$$V = \lim_{\substack{m \to \infty \\ n \to \infty}} \sum_{j=1}^{n} \sum_{i=1}^{m} f(\bar{x}_i, \bar{y}_j) \Delta x \Delta y = \lim_{n,m \to \infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f(\bar{x}_i, \bar{y}_j) \Delta x \Delta y$$

Definimos então:

$$\iint_R f(x, y) dxdy = \lim_{n,m \to \infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f(\bar{x}_i, \bar{y}_j) \Delta x \Delta y$$

$$V = \iint_R f(x,y)\,dx\,dy$$



R → Região no $\mathbb{R}^2$

$$V \approx \sum_{j=1}^{n} \sum_{i=1}^{m} f\left(\bar{x}_i, \bar{y}_j\right) \Delta x \Delta y$$

Fixando j e fazendo m tender ao infinito :

$$\lim_{m \to \infty} \sum_{i=1}^{m} f\left(\bar{x}_i, \bar{y}_j\right) \Delta x \Delta y = \Delta y \lim_{m \to \infty} \sum_{i=1}^{m} f\left(\bar{x}_i, \bar{y}_j\right) \Delta x = \Delta y \int_{a_j}^{b_j} f\left(x, \bar{y}_j\right) dx$$

Tomando agora o resultado acima e fazendo n tender ao infinito :

$$\lim_{n \to \infty} \sum_{j=1}^{n} \left(\int_{a_j}^{b_j} f\left(x, \bar{y}_j\right) dx \right) \Delta y = \int_{c}^{d} \left(\int_{a(y)}^{b\,(y)} f(x, y) dx \right) dy = \int_{c}^{d} \int_{a(y)}^{b(y)} f(x, y) dx\, dy$$

Onde $a(y)$ e $b(y)$ são funções que dependem apenas de y

$$V \approx \sum_{j=1}^{n} \sum_{i=1}^{m} f\left(\bar{x}_i, \bar{y}_j\right) \Delta x \Delta y$$

De modo análogo, fixando primeiro i e fazendo n tender ao infinito e depois m tender ao infinito :

$$\lim_{n \to \infty} \sum_{i=1}^{m} \left(\int_{c_i}^{d_i} f(\bar{x}_i, y) dx \right) \Delta y = \int_{a}^{b} \left(\int_{d(x)}^{c(x)} f(x, y) dy \right) dx = \int_{a}^{b} \int_{d(x)}^{d(x)} f(x, y) dy\, dx$$

Onde $c(x)$ e $d(x)$ são funções que dependem apenas de x.

Exercício

1. Encontre o conjunto de todas as funções $f: \mathbb{R}^2 \to \mathbb{R}$ tais que:

$$\frac{\partial^2 f}{\partial x \partial y} = x$$

Teorema (de Fubini)

Seja f *integrável no retângulo* $R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2; x \in [a,b] \text{ e } y \in [c,d]\}$. *Desse modo:*

$$\iint_R f(x,y)dxdy = \iint_R f(x,y)dydx =$$