Multidimensional Cubic Upwind-Biased Advection of Slanted Cell Grids

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1 Slanted cell grids

See figure 1

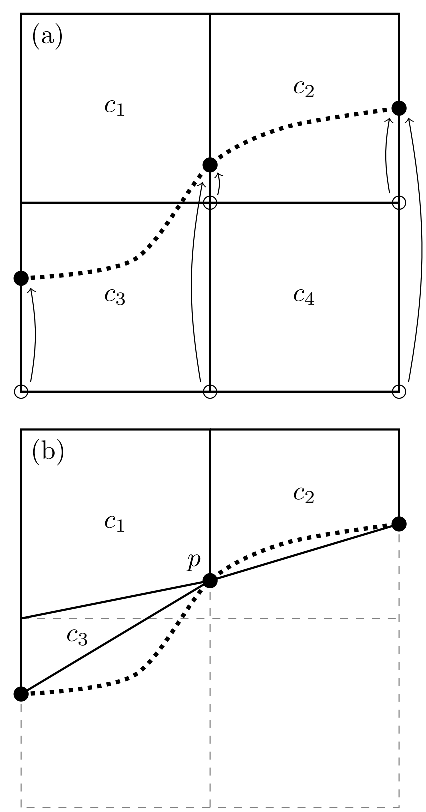

Figure 1: Illustration of a slanted cell grid (a) before, and (b) after construction. The terrain surface, denoted by a heavy dotted line, intersects a uniform rectangular grid comprising four cells, c_1, c_2, c_3 and c_4. The cell vertices, marked by open circles, are moved upwards to the points at which the terrain intersects closed vertical cell edges, marked by ~~open~~ circles. Cells that have no volume are removed. Where a cell has two vertices occupying the same point, the zero-length edge that joins those vertices is removed. In this illustration, cell c_4 is removed because it has no volume, and the zero-length edge at point p is removed to create a triangular cell, c_3.

I imagine that you will also include a picture of a stencil which includes a small slanted cell

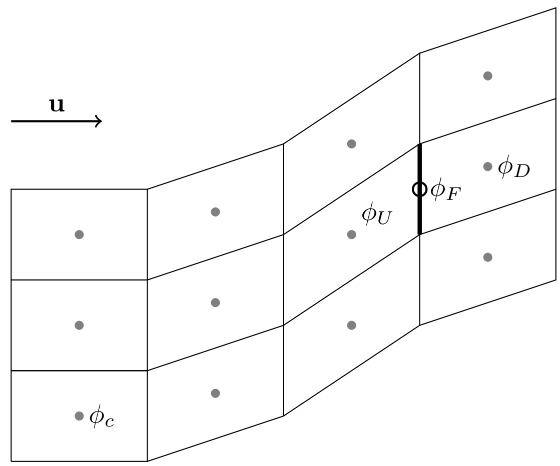

Figure 2: An upwind-biased stencil on a two-dimensional terrain-following grid. The stencil is used to fit a multidimensional polynomial to twelve cell centred values, ϕ_c, marked by grey circles, in order to approximate the value ϕ_F at the face centroid marked by an open circle. ϕ_U and ϕ_D are the values at the centroids of the upwind and downwind cells neighbouring the target face, drawn with a heavy line. These two cells are given a large weighting in the least squares fit so that their values lie almost exactly on the polynomial surface.

2 Multidimensional upwind-biased advection scheme

A tracer with density ϕ is advected in flux form

$$\partial\phi/\partial t + \nabla\cdot(\mathbf{u}\phi) = 0 \tag{1}$$

where $\mathbf{u}$ is the velocity field. We use the notation that, for a field ψ, ψ_f denotes the value of ψ at face f, ψ_c denotes the value at the centroid of cell c, and ψ_F is an interpolation onto a face from surrounding cell centre values. The divergence term in equation 1 is discretised using Gauss' divergence theorem:

$$\nabla\cdot(\mathbf{u}\phi) \approx \frac{1}{\mathcal{V}}\sum_{f\in c}\phi_F\mathbf{u}_f\cdot\mathbf{S}_f \tag{2}$$

where $\mathcal{V}$ is the cell volume, $f \in c$ denotes the faces of cell c, and $\mathbf{S}_f$ is the outward-pointing normal vector for face f with a magnitude equal to the face area. The value of ϕ_F is interpolated using a least squares fit of cell centre values from an upwind-biased stencil (figure 2). The upwind and downwind cell values, ϕ_U and ϕ_D, are ~~given a large weighting so that they~~ lie almost exactly on the polynomial surface. Weights* are iteratively adjusted to achieve a fit that is upwind-biased.

weighted in the least square fit so that the fit is nearly exact at cell centres U and D so that phi_U ar

* in the least squares fitting

This description is ok for now but would not make much sense to someone not already familiar with it. When you write

2.1 Stencil construction

the upwind

The upwind-biased stencil is constructed by finding the opposing faces for a given face, f, belonging to a cell, c. Defining G to be the set of other faces in cell c, we calculate the 'opposedness', Opp, between faces f and $g \in G$, defined as

$$\mathrm{Opp}(f, g) \equiv -\frac{\mathbf{S}_f \cdot \mathbf{S}_g}{|\mathbf{S}_f|^2} \tag{3}$$

where $\mathbf{S}_f$ and $\mathbf{S}_g$ are the surface normal vectors pointing outward from cell c for faces f and g respectively. Using the fact that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\theta)$ we can rewrite equation 3:

$$\mathrm{Opp}(f, g) = -\frac{|\mathbf{S}_g|}{|\mathbf{S}_f|} \cos(\theta) \tag{4}$$

where θ is the angle between faces f and g. In this form, it can be seen that Opp is a measure of the area of g and how closely it parallels face f.

Now, let OF be the set of faces opposing face f, which is defined as

$$\mathrm{OF}(f) \equiv \{g : \max(\mathrm{Opp}(f, g))\} \cup \{g : \mathrm{Opp}(f, g) \geq 0.5\} \tag{5}$$

TODO: *can the opposed face be a boundary face? I think this is desirable in order to find the upwind cell centre...* The stencil includes cells adjacent to the faces in OF, and their vertex neighbours.

?

Figure 3 illustrates a stencil construction for face f and cell c. The two opposing faces are denoted by thick dashed lines and the centres of the three neighbouring cells are marked by grey circles. The stencil is extended outwards from the central cells by including the cells neighbouring the vertices marked by black squares. The resultant stencil contains 13 cells.

2.2 Singular value decomposition

Once the stencil has been found, a multidimensional polynomial is fitted to the cell centre values. In two dimensions, the polynomial is

$$\phi = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^3 + a_8 x^2 y + a_9 xy^2 \tag{6}$$

note which terms are or

where $\mathbf{a} = [a_1, \ldots, a_n]^\intercal$ is the vector of unknown coefficients with $n = 9$ in two dimensions. A local coordinate system is established in which x is in the direction of $\mathbf{S}_f$ and y is perpendicular to x. TODO: *in 3D, how do we choose y and z? does it even matter?* The origin of the local coordinate system is fixed to be the target face centroid. Note that the term involving y^3 is omitted.

it doesn't matter

A matrix equation is constructed to calculate a least squares fit:

$$\mathbf{B}\left(\mathbf{w_p'}\mathbf{a}\right) = \mathbf{w_c}\boldsymbol{\phi} \tag{7}$$

where $\mathbf{B}$ is a rectangular matrix with one row for each cell in the stencil and one column for each term in the polynomial, $\mathbf{w_p'} = [w_{p1}^{-1}, \ldots, w_{pn}^{-1}]^\intercal$ is the polynomial

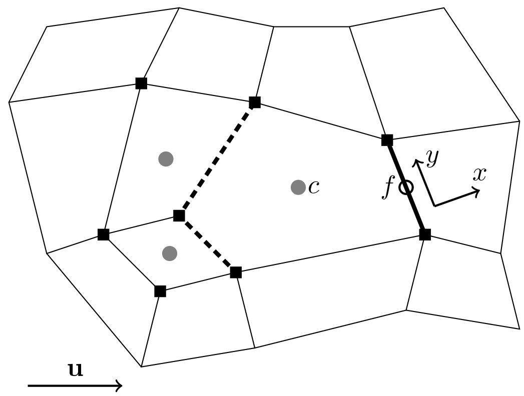

Figure 3: A thirteen-cell, upwind-biased stencil for face f belonging to a pentagonal cell, c. The dashed lines denote the two faces of cell c that oppose f, and grey circles mark the centroids of the cells neighbouring these two opposing faces. The stencil is expanded outwards by including cells that neighbour the vertices of the three central cells, where black squares mark these vertices. The local coordinate system (x, y) has its origin at the centroid of face f, marked by an open circle, with x normal to f and y perpendicular.

weight reciprocal vector with n being the number of polynomial terms, $\mathbf{w_c} = [w_{c1}, \dots, w_{cm}]^\intercal$ is the cell weighting vector with m being the number of cells in the stencil, and $\boldsymbol{\phi} = [\phi_1, \dots, \phi_m]^\intercal$ is the vector of cell centre values. The matrix $\mathbf{B}$ takes the form

$$\mathbf{B} = \left(\mathbf{W_C}\tilde{\mathbf{B}}\right)^\intercal \mathbf{W_P} \tag{8}$$

where $\mathbf{W_P} = \operatorname{diag}(\mathbf{w_p})$ is an $n \times n$ matrix of polynomial weights and $\mathbf{w_p} = [w_{p1}, \dots, w_{pn}]^\intercal$, and $\mathbf{W_C} = \operatorname{diag}(\mathbf{w_c})$ is an $m \times m$ matrix of cell weights. Equation 8 is constructed so that the rows of $\tilde{\mathbf{B}}$ are multiplied by the corresponding cell weights in $\mathbf{w_c}$, and the columns of $\tilde{\mathbf{B}}$ are multiplied by the corresponding polynomial weights in $\mathbf{w_p}$.

$\tilde{\mathbf{B}}$ is an $m \times n$ matrix of the geometric terms in the polynomial which, in two dimensions, is given by

this in not clear

$$\tilde{\mathbf{B}} = \begin{pmatrix} 1 & \mathbf{X}_{1,1} & \mathbf{X}_{1,1} & \mathbf{X}_{1,1}^2 & \mathbf{X}_{1,1}\mathbf{X}_{1,2} & \mathbf{X}_{1,2}^2 & \mathbf{X}_{1,1}^3 & \mathbf{X}_{1,1}^2\mathbf{X}_{1,2} & \mathbf{X}_{1,1}\mathbf{X}_{1,2}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \mathbf{X}_{m,1} & \mathbf{X}_{m,1} & \mathbf{X}_{m,1}^2 & \mathbf{X}_{m,1}\mathbf{X}_{m,2} & \mathbf{X}_{m,2}^2 & \mathbf{X}_{m,1}^3 & \mathbf{X}_{m,1}^2\mathbf{X}_{m,2} & \mathbf{X}_{m,1}\mathbf{X}_{m,2}^2 \end{pmatrix} \tag{9}$$

where $\mathbf{X}$ is a matrix with m rows and one column for each geometric dimension. Each row of the $\mathbf{X}$ matrix contains the local coordinates of a cell centre so

To clarify, you could describe the method without weighting first, then describe how the weighting works that, in two dimensions, the matrix is

$$\mathbf{X} = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_m & y_m \end{pmatrix} \tag{10}$$

A singular value decomposition is used to calculate the pseudo-inverse of $\mathbf{B}$, $\mathbf{B}^+$, such that

$$\mathbf{B}^+ (\mathbf{w_c} \boldsymbol{\phi}) = \mathbf{w_p'} \mathbf{a} \tag{11}$$

and noting that $\mathbf{w_p'}$ is the reciprocal of $\mathbf{w_p}$, we find the unknown polynomial coefficients, $\mathbf{a}$:

what is the reciprocol of

$$\mathbf{a} = \mathbf{B}^+ (\mathbf{w_c} \mathbf{w_p} \boldsymbol{\phi}) \tag{12}$$

Since the target face centroid is positioned at the origin of the local coordinate system then $\phi_F = a_1$, hence

$$\phi_F = \mathbf{c} \boldsymbol{\phi} \tag{13}$$

where the cell value coefficients vector, $\mathbf{c} = \mathbf{B}_1^+ (\mathbf{w_c} \mathbf{w_p})$ and $\mathbf{B}_1^+$ is the first row of $\mathbf{B}^+$. $\mathbf{c}$ can be calculated once during model initialisation because it depends only upon the cell weights, polynomial weights and grid geometry, which is assumed to be static.

2.3 Iterative weight adjustment for upwind-biased fit

The cell weight and polynomial weight vectors, $\mathbf{w_c}$ and $\mathbf{w_p}$, are used in the least squares fit to control the weightings of the cells and polynomial terms respectively. These weights are adjusted iteratively to ensure that the fit is upwind-biased. In the first iteration, a weighting of 1000 is given to the two cells sharing the target face, and to the constant term and linear term in x in the polynomial. That is, $w_{cU} = w_{cD} = w_{p1} = w_{p2} = 1000$ where U and D are the upwind and downwind cell indexes respectively. All other entries in $\mathbf{w_c}$ and $\mathbf{w_p}$ are 1.

The cell value coefficients, $\mathbf{c}$, must satisfy four criteria for the fit to be considered upwind-biased:

I guess that ^ means "and". E

$$(\max(\mathbf{c}) < 1) \wedge (\mathbf{c}_U \geq \mathbf{c}_D) \wedge \left(\mathbf{c}_U > \sum_{1 \leq i < m,\, i \neq U} \max(0, \mathbf{c}_i) \right) \wedge (|\mathbf{c}_U - 1| < l) \tag{14}$$

where U and D are the indices of the upwind and downwind cells respectively, and the linear limit factor $l = 3$. TODO: *what's the rationale for this linear limit factor?*

If an upwind-biased fit is not found after the first iteration then three adjustments are made. First, the upwind cell weighting, w_{cU}, is increased ten-fold. Second, the weightings are increased ten-fold for the constant term and linear term in x, w_{p1} and w_{p2} respectively. Third, the large cell weighting is removed from the downwind cell so that $w_{cD} = 1$.

The adjustment procedure is executed up to seven times until the fit is upwind-biased. If the fit is still not considered upwind-biased after these iterations, a pure upwind approximation is made so that $\phi_F = \phi_U$.