JOURNAL OF COMPUTATIONAL PHYSICS 17, 209–228 (1975)

On the Use of a Coordinate Transformation for the Solution of the Navier–Stokes Equations

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Received October 29, 1974

The equations of fluid motion have been formulated in a generalized noncartesian, nonorthogonal coordinate system. A particular coordinate transformation, which transforms a domain with an irregular lower boundary into a cube, has been constructed. The transformed system, unlike the original one, has flat boundaries and homogeneous boundary conditions. Where the topography is flat, the original and transformed systems are identical, and extra terms do not appear. A finite difference scheme for solving the transformed equations has been constructed and will be described in a subsequent issue of this journal.

I. INTRODUCTION

1.1. *General Statement of the Problem*

Considerable progress has been made in the last decade toward a better numerical solution of the Navier–Stokes equations without topography. The simulation of fluid flow above terrain (irregular lower boundary), however, has been hampered by the fact that with a Cartesian coordinate system and standard difference approximation to the Navier–Stokes equations, one has to deal with complicated boundaries. Thus it is very difficult to employ correct boundary conditions. In addition, boundary conditions which are essentially simple, such as no normal

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flow across rigid boundaries or continuity of the tangential components of the stress tensor, are difficult to apply since the normal to the surface is not the vertical z-axis. The purpose of this paper is to present a general method of solution for the Navier–Stokes equations above an irregular lower boundary, such as a mountain. The method is applicable to any topography with continuous second derivatives. Another distinguishing feature of the method is its ability to satisfy, identically, imposed boundary conditions.

In Section II the governing equations are presented and then reformulated in a generalized nonorthogonal, noncartesian coordinate system. We use the terminology and notations of tensor calculus to present the physical laws and boundary conditions clearly and compactly.

A particular coordinate transformation is then chosen, which transforms a domain with irregular lower boundary into a rectangular domain. Where the topography is flat, the original and transformed systems are identical, and extra terms do not appear.

In a subsequent publication we will present a difference scheme for solving the transformed equations. Because the transformed system has a flat lower boundary, it has been possible to employ many of the ideas and techniques of Cartesian difference schemes (e.g., [2, 8, 9, 18, 30, 36]).

The methods thus far developed have been used to calculate dry convective flow created by differential heating between top and bottom of mountainous terrain (mountain up-slope winds) [6].

1.2. *Review of Previous Numerical Modeling*

In principle, two avenues of approach are available in numerically simulating flow above a mountain. One approach is to retain the Cartesian framework and apply special techniques for the lower boundary. The second approach is to use a coordinate transformation which will transform the complicated domain into a rectangular one. The advantages of using the second approach have already been mentioned, and they will be discussed in detail in the following sections. The price paid for these advantages is the appearance of extra terms in the governing equations. This need not be a serious problem in a numerical approach. A more serious problem, however, can arise if the coordinate transformation causes new singularities to appear in the governing equations. Such singularities will not necessarily reflect genuine singularities in the solution itself and may be quite undesirable as far as numerical stability is concerned. An example of such singularities in spherical coordinates occurs at the poles, where the Jacobian of the transformation is zero. In the present study, singularities of this type were avoided by choosing a transformation with a Jacobian close to unity and by specifying a topography with continuous second derivatives.

The first approach (i.e., the Cartesian approach) has been utilized by Orville

[24–28], Hirt and Cook [11], and Viecelli [35]. Orville's approach is restricted to two-dimensional space and uses stream function and vorticity as dependent variables, rather than using the primitive equations (i.e., pressure and velocities as dependent variables). By confining himself to a simple geometry (a linear ridge sloped 45°), he avoided the problem of dealing with boundary points which are not grid points in his rectangular mesh. The extension of his work to arbitrary domain and/or to three dimensions does not seem straightforward. Hirt and Cook have used the primitive equations and have performed calculations for two and three dimensions. However, they approximated the topography as a series of blocks such that physical boundaries coincide with cell boundaries, thus avoiding a major difficulty associated with flow over rough terrain. Approximating topography by a series of blocks is a zero-order approximation to real topography (first term in Taylor expansion), and, since the shape of the body is important in determining fluid properties (e.g., stability, boundary layer separation), it is not clear to what extent their method could be utilized to calculate flow over real mountains. In Viecelli's calculations, the primitive equations are used, and mesh points do not coincide with boundary points. His method is applicable to cases where all the boundaries are irregular, and not only the lower one. In his calculations, it was necessary to define and "flag" all cells in the vicinity of the boundaries and to define the normal at each boundary. At every cell adjacent to the boundary, he used weighted interpolation formulas to calculate momentum and pressure on the mesh. Using the techniques developed for free surface calculations, at each time step he ingeniously specifies an internal pressure which forces the free surface to align with the desired boundary shape. The calculations reported thus far are for two dimensions, but in principle the same techniques could be used in three dimensions. The programming effort for such calculations, however, seems to be formidable.

These calculations also illustrate one typical and important source of difficulty. A common form of disturbance may arise when the boundary suddenly changes direction, in particular through an angle exceeding π. In this case the solution of the Poisson equation for the pressure p possesses a singularity in the tangential derivative, i.e., if t is the tangent to the boundary, $(\partial p/\partial t) \to \infty$ as one approaches the corner [4, 16]. Since the solution for an incompressible fluid typically involves a Poisson equation for pressure, and since velocities depend on pressure gradients, the singularities appear in the velocity fields. These singularites are genuine, and a standard finite difference scheme will either smooth out a singularity or perhaps blow up. It cannot resolve the flow in the immediate vicinity of the singularity accurately. At any event, large truncation errors are present near the singular point [4]. These errors may propagate destructively to other grid points [13]. In all the calculations mentioned above there is apparently some location where the angle exceeds π.

The second avenue of approach (that of a coordinate transformation) has been utilized successfully by Phillips [29] for the special case of fluid in hydrostatic balance. This coordinate transformation is used in almost all current atmospheric general circulation models (e.g. [20, 31, 32]) although the incorporation of very rough mountains (where the boundary suddenly changes direction) still remains a problem [7]. As already mentioned, the present study utilizes a coordinate transformation for the full time-dependent nonhydrostatic Navier–Stokes equations. Singularities and sharp corners are avoided by specifying a topography with continuous second derivatives.

II. Theoretical Development

2.1. *The Governing Equations in Cartesian Coordinate System and the Physical Model*

A particularly useful form of the Navier–Stokes equations is the so-called "anelastic" approximation [1, 22, 23]. The set may be written in Cartesian coordinates as follows.

Continuity equation,

$$(\rho_0 u^j)_{,j} = 0. \tag{2-1}$$

Momentum equations,

$$(\partial/\partial t)(\rho_0 u^i) + (\rho_0 u^i u^j)_{,j} = -(\delta^{ij} p')_{,j} + \delta^{i3}\rho' g + \tau^{ij}{}_{,j}\,. \tag{2-2}$$

Thermodynamic energy equation,

$$(\partial/\partial t)(\rho_0 \theta') + (\rho_0 \theta' u^j)_{,j} = H^j{}_{,j}\,. \tag{2-3}$$

Here the tensorial notation with the summation convention has been used (e.g. McConnel [19]). The operator $_{,j}$, is (in Cartesian coordinates) the derivative operator. Thus for example

$$(\rho_0 u^i u^j)_{,j} \equiv \partial(\rho_0 u^i u^j)/\partial x^j.$$

ρ_0, θ_0, p_0 are the "basic," prescribed density, potential temperature, and pressure which correspond to a hydrostatic adiabatic fluid. For an ideal gas their values are given by

$$\theta_0 = \text{constant}, \tag{2-4-a}$$

$$\rho_0 = \rho_0(z) = \rho_{00}(1 - z/Hi)^{1/\gamma\kappa}, \tag{2-4-b}$$

$$p_0 = P_{00}(\rho_0/\rho_{00})^{\gamma}, \tag{2-4-c}$$

where g is the acceleration of gravity, P_{00}, ρ_{00} are constants (usually ground pressure and density) related to θ_0 through the equation of state $P_{00} = \rho_{00} R \theta_0$. R is the gas constant for dry air ($=2.8704 \times 10^6$ cm^2 sec^{-2} deg^{-1}). C_p and C_v are the specific heats at constant pressure and volume, respectively; for an ideal gas their values are: $C_p = (7/2)R$, $C_v = (5/2)R$. Also, $\gamma = C_p/C_v$, $\kappa = R/C_p$, $Hi = C_p\theta_0/g$; and ρ', θ', p' are, respectively, small deviations of the density, potential temperature, and pressure from their "basic" state values. ρ' can be expressed in terms of p' and θ' by

$$\rho' = \rho_0(\theta'/\theta_0 - p'/(\gamma p_0)). \tag{2-5}$$

u^i is the velocity in the x^i direction, τ^{ij} is the i, jth stress component, H^j is the eddy heat flux in the x^j direction. The term $\delta^{i3}\rho' g$ in the momentum equations (2-2) is the deviation of the gravity force from its basic hydrostatic value. This term is properly called "bouyancy force" and is the driving force of convective elements.

An important point concerning the anelastic system is that the variables u^i, θ', p' are not completely independent. The distribution of p' must always be such that the velocities computed from the momentum equations (2-2) continue to satisfy the continuity equation (2-1). This is pointed out by Batchelor [1] and is also encountered in the theory of incompressible flow. The net result is that taking the divergence of the momentum equations shows that, unlike the fully compressible case, p' can no longer be determined from an equation of state $p = p(\rho, \theta)$, but must satisfy the elliptic equation

$$(\rho_0 u^i u^j)_{,ji} = -(\delta^{ij} p')_{,ji} + (\delta^{i3}\rho' g)_{,i} + \tau^{ij}{}_{,ji}\,. \tag{2-6}$$

If the vertical extent of the domain d is musch smaller than Hi (see formula (2-4-b)), the basic density can be treated as constant. In this case the density deviation ρ' is, to a first-order approximation, a function of the potential temperature deviation only, and one gets the well-known and more frequently used Boussinesq system [33].

So far we have not yet specified explicitly the fluxes of momentum and heat. Following Lilly [17], we assume that these terms are proportional to mean gradients, by eddy viscosity and eddy diffusion coefficients, constant and isotropic within a space–time grid, and we express these fluxes as follows.

$$\tau^{ij} = \rho_0 K_M(e^{ij} - (2\delta^{ij}/\delta^{ii})(u^k{}_{,k})), \tag{2-7}$$

$$H^i = \rho_0 K_H \delta^{ij}(\partial\theta'/\partial x^j). \tag{2-8}$$

δ^{ij} is the Kronecker delta, K_M and K_H are the (variable) eddy viscosity and heat diffusion coefficients, and are determined from the explicit flow parameters as follows.

$$K_M = (k\Delta)^2 \mid \mathrm{Def} \mid (1 - (K_H/K_M)(Ri)')^{1/2}, \tag{2-9}$$

$$(Ri)' = \begin{cases} Ri & \text{when} \quad \mid \theta' \mid \leqslant 10^{-3}\,\theta_0, \\ Ri - \left(\left|\dfrac{\partial p'}{\partial x}\right|\left|\dfrac{\partial \theta'}{\partial x}\right| + \left|\dfrac{\partial p'}{\partial y}\right|\left|\dfrac{\partial \theta'}{\partial y}\right|\right)\Big/(\mid \rho_0\theta' \mid (\mathrm{Def})^2), & \text{otherwise,} \end{cases} \tag{2-10}$$

$$Ri = (g/\theta_0)(\partial\theta'/\partial z)/(\mathrm{Def})^2, \tag{2-11}$$

$$(\mathrm{Def})^2 = 0.5\tau^{ij}e_{ij}/(\rho_0 K_M) = \tfrac{1}{2}e^{ij}e_{ij} - (2/\delta^{ij})(u^k{}_{,k})^2, \tag{2-12}$$

$$e^{ij} = (\partial u^i/\partial x^j) + (\partial u^j/\partial x^i), \tag{2-13}$$

$$K_H/K_M = \text{constant} = 1/P_r\,. \tag{2-14}$$

P_r is the Prandtl number, k is a presumed universal constant, Δ is the grid resolution. Ri is the Richardson number, $(Ri)'$ is the modified Richardson number. The representations of e^{ij} and e_{ij} are the same for Cartesian coordinates, and their general form will be discussed in the next section. The quantity $\rho_0 K_M(\mathrm{Def})^2$ is the dissipation function. This formulation is strictly applicable only to three-dimensional turbulence [3]. In two-dimensional turbulence the energy is cascading up-scale [15].

Our representation is slightly different from that of Lilly, since we consider a "modified" Richardson number $(Ri)'$, while he introduces the original Richardson number Ri. Our reasons for modifying the Richardson number are purely numerical. We have observed that the mountain is able to induce discontinuities in the potential temperature field, and therefore in all other fields. In order to prevent postshock oscillation, the term

$$-\left(\left|\frac{\partial p'}{\partial x}\right|\left|\frac{\partial \theta'}{\partial x}\right| + \left|\frac{\partial p'}{\partial y}\right|\left|\frac{\partial \theta'}{\partial y}\right|\right)\Big/(\mid \rho_0\theta' \mid (\mathrm{Def})^2)$$

was added to the Richardson number. We have found by numerical experimentation that the contribution of this extra term was significant only in places where strong temperature gradients are present.

The value of the presumed universal constant k, and the ratio K_H/K_M is uncertain. Numerical experiments (e.g. [3, 6]) seem to suggest $0.21 \leqslant k \leqslant 0.5$, $1 \leqslant K_H/K_M \leqslant 3$. We have used $k = 0.42$, $1 \leqslant K_H/K_M \leqslant 3$. As a rule of thumb, viscosity and diffusion must be large enough so that the truncation grid interval, below which scales cannot be resolved, must be within the dissipation range of the flow.

If the scale of the motion is fine enough, molecular viscosity and diffusion become important. In this scale K_M, K_H and ρ_0 can be treated as constants and the forms of the frictional and diffusive terms become

$$\partial\tau^{ij}/\partial x^j = \rho_0 K_M \nabla^2 u_i\,,$$

for the friction terms, and

$$\partial H^j/\partial x^j = \rho_0 K_H \nabla^2\theta,$$

for the diffusive terms. These forms considerably simplify the numerical treatment and they are the more commonly used, (e.g. [2, 9, 11]). Nevertheless we did not use these forms, since we were concerned with larger-scale flow, in which nonlinear eddy viscosity prescriptions are more appropriate.

2.2. *The Governing Equations in Generalized Coordinate System*

In principle, the Navier–Stokes equations can be written and formulated in any coordinate system. The guiding principle is that the laws of physics are independent of any particular choice of coordinates. Thus if a certain physical law is expressed in a particular coordinate system x^r as

$$A^r_{st} = B^r_{st}\,,$$

then the same physical law expressed in any other coordinate system $\bar{x}^r$ should be

$$\bar{A}^r_{st} = \bar{B}^r_{st}\,.$$

In order for this to be true, a certain relation must exist between A and $\bar{A}$, namely,

$$\bar{A}^r_{st} = (\partial\bar{x}^r/\partial x^u)\cdot(\partial x^v/\partial\bar{x}^s)\cdot(\partial x^w/\partial\bar{x}^t)\cdot A^u_{vw}\,. \tag{2-15}$$

The same relation must exist between B and $\bar{B}$. In this case A^r_{st} is said to be a mixed tensor of the third order with one contravariant suffix and two covariant suffixes. Tensors of any order are defined in the same fashion (e.g. [10, 19]).

Our governing equations (2-1) to (2-3) are tensorial relations only in Cartesian frames of reference. Let us now find a tensorial form of the Navier–Stokes equations which will be invariant in any frame of reference. All Cartesian frames of reference will be then a special case.

Let us denote a Cartesian frame of reference by x^i, and a generalized frame of reference by $\bar{x}^i$, $\bar{x}^i = f^i(x^j)$. In an orthogonal Cartesian system, the length of a differential line element is expressed by,

$$ds^2 = dx^i\,dx^i,$$

but

$$dx^i = (\partial x^i/\partial\bar{x}^j)\,d\bar{x}^j,$$

hence,

$$ds^2 = G_{mn}\,d\bar{x}^m\,d\bar{x}^n,$$

where we have put

$$G_{mn} = (\partial x^i/\partial\bar{x}^m)\cdot(\partial x^i/\partial\bar{x}^n). \tag{2-16}$$

G_{mn} is called the metric tensor. The quantity

$$G^{mn} = (\partial \bar{x}^m/\partial x^i) \cdot (\partial \bar{x}^n/\partial x^i) \tag{2-17}$$

is evidently the inverse of G_{mn}, and is called the conjugate tensor. G_{mn} and G^{mn} are seen to be symmetric. The metric tensor and its conjugate are important quantities, since they relate contravariant quantities to covariant quantities in the same coordinate system, namely (e.g. [10, 19])

$$\bar{A}^i = G^{im}\bar{A}_m, \qquad \bar{A}_i = G_{im}\bar{A}^m. \tag{2-18}$$

$$\bar{A}^{ij} = G^{im}G^{jm}\bar{A}_{mn}, \qquad \bar{A}_{ij} = G_{im}G_{jn}\bar{A}^{mn}. \tag{2-19}$$

This process applies to tensors of any order and type.

If $\bar{A}^{i_1,\dots,i_n}_{j_1,\dots,j_m}$ is a tensor, $\partial \bar{A}^{i_1,\dots,i_n}_{j_1,\dots,j_m}/\partial \bar{x}^u$ is not necessarily a tensor; however, the quantity defined by

$$\bar{A}^{i_1,\dots,i_n}_{j_1,\dots,j_m,s} \equiv \frac{\partial \bar{A}^{i_1,\dots,i_n}_{j_1,\dots,j_m}}{\partial \bar{x}^s} + \begin{Bmatrix} i_1 \\ l \quad s \end{Bmatrix} \bar{A}^{li_2,\dots,i_n}_{j_1,\dots,j_m} + \cdots + \begin{Bmatrix} i_n \\ l \quad s \end{Bmatrix} \bar{A}^{i_1,\dots,i_{m-1}l}_{j_1,\dots,j_m}$$
$$- \begin{Bmatrix} l \\ j_1 \quad s \end{Bmatrix} \bar{A}^{i_1,\dots,i_n}_{lj_2,\dots,j_m} - \cdots - \begin{Bmatrix} l \\ j_m \quad s \end{Bmatrix} \bar{A}^{i_1,\dots,i_n}_{j_1,\dots,j_{m-1}l} \tag{2-20}$$

can be shown to be a tensor of order $n + m + 1$, with n contravariant components and $m + 1$ covariant components [19]. The quantity defined by (2-20) is called the covariant derivative of $\bar{A}^{i_1,\dots,i}_{j_1,\dots,j_m}$ and it replaces the usual derivative. $\{ {}_{m}{}^{i}{}_{n} \}$ are the so-called Christophel's symbol of the second kind defined by

$$\begin{Bmatrix} i \\ m \quad n \end{Bmatrix} = G^{ip}[mn, p], \tag{2-21}$$

$$[mn, p] = \frac{\partial^2 x^i}{\partial \bar{x}^m\, \partial \bar{x}^n} \cdot \frac{\partial \bar{x}^i}{\partial \bar{x}^p} = \frac{1}{2}\left[\frac{\partial G_{mp}}{\partial \bar{x}^n} + \frac{\partial G_{np}}{\partial \bar{x}^m} - \frac{\partial G_{mn}}{\partial \bar{x}^p}\right]. \tag{2-22}$$

Another important quantity in a coordinate transformation is the Jacobian of the transformation, which we denote by $G^{1/2}$. It is defined as

$$G^{1/2} \equiv \mathrm{Det}(\partial x^i/\partial \bar{x}^m), \tag{2-23}$$

and from the definition of G^{ij} and G_{ij} in (2-17) and (2-16), it can be readily shown that

$$G^{1/2} = +(\mathrm{Det}(G_{ij}))^{1/2} = +(\mathrm{Det}(G^{ij}))^{-1/2}. \tag{2-24}$$

$G^{1/2}$ is related to the Christophel's symbol via the relation

$$(1/G^{1/2})(\partial G^{1/2}/\partial \bar{x}^p) = \begin{Bmatrix} m \\ m \quad p \end{Bmatrix} \tag{2-25}$$

[19, p. 155].

In Cartesian coordinates $\delta^{ij} = G^{ij} = G_{ij}$, and therefore in those coordinates $\partial G^{ij}/\partial x^t = \partial G_{ij}/\partial x^t = 0$. Consequently, we must have in any coordinate system the tensor equation

$$G^{ij}{}_{,t} = G_{ij,t} = 0. \tag{2-26}$$

Using the definition (2-20) of covariant derivative we get

$$(\partial G^{ij}/\partial \bar{x}^t) + G^{mj}\begin{Bmatrix} i \\ m \quad t \end{Bmatrix} + G^{im}\begin{Bmatrix} j \\ m \quad t \end{Bmatrix} = 0. \tag{2-27}$$

By using (2-25) and putting $j = t$ we get

$$(1/G^{1/2})(\partial/\partial \bar{x}^j)(G^{1/2}G^{ij}) + \begin{Bmatrix} i \\ m \quad n \end{Bmatrix} G^{mn} = 0. \tag{2-28}$$

From (2-20) and (2-25) we can get a special form for the divergence of a tensor A^{ij}, namely,

$$\bar{A}^{ij}{}_{,j} = (1/G^{1/2})(\partial/\partial \bar{x}^j)(G^{1/2}\bar{A}^{ij}) + \begin{Bmatrix} i \\ m \quad n \end{Bmatrix} \bar{A}^{mn}. \tag{2-29}$$

Relations (2-18) to (2-29) will now be used to derive a contravariant representation of the Navier–Stokes equations. For the sake of clarity we denote any quantity in a generalized coordinate system by an overbar ($^-$), to distinguish it from Cartesian quantities. Formally the equations are nearly unaltered, and we need only replace the Cartesian components of the appropriate tensors and vectors by their contravariant components, and the usual differentiation by covariant differentiation (2-20). Let us do this replacement in detail and term-by-term. The continuity equation becomes (using (2-20) and (2-25))

$$(1/G^{1/2})(\partial/\partial \bar{x}^i)(G^{1/2}\rho_0\bar{u}^i) = 0.$$

The momentum flux density tensor is still written as $\rho_0\bar{u}^i\bar{u}^j$, and its divergence is, according to (2-29),

$$(1/G^{1/2})(\partial/\partial \bar{x}^j)(G^{1/2}\rho_0\bar{u}^i\bar{u}^j) + \begin{Bmatrix} i \\ m \quad n \end{Bmatrix} \rho_0\bar{u}^m\bar{u}^n.$$

The covariant components of the pressure gradient are $\partial p/\partial \bar{x}^i$; therefore its contravariant components are, according to (2-18), $G^{ij}(\partial p/\partial \bar{x}^j)$, but from relation (2-26) they can be written as $(G^{ij}p)_{,j}$ and according to (2-29) and (2-28),

$$(G^{ij}p)_{,j} = (1/G^{1/2})(\partial/\partial \bar{x}^j)(G^{1/2}G^{ij}p) - (p/G^{1/2})(\partial/\partial \bar{x}^j)(G^{1/2}G^{ij}).$$

The strain tensor has been defined by (2-13) and it can be written in Cartesian frame of reference as $e_{ij} = (\partial u_i/\partial x^j) + (\partial u_j/\partial x^i)$. Consequently the covariant components of this tensor are in general

$$\bar{e}_{ij} = \bar{u}_{i,j} + \bar{u}_{j,i}\,.$$

In order to find its contravariant components, we use the prescription (2-19) to get

$$\bar{e}^{ij} = G^{jn}(G^{im}\bar{u}_{m,n}) + G^{im}(G^{jn}\bar{u}_{n,m}).$$

By using (2-18), (2-20), and (2-26), this is equal to

$$G^{jn}\left(\frac{\partial \bar{u}^i}{\partial \bar{x}^n} + \begin{Bmatrix} i \\ n \quad s \end{Bmatrix} \bar{u}^s\right) + G^{im}\left(\frac{\partial \bar{u}^j}{\partial \bar{x}^m} + \begin{Bmatrix} j \\ m \quad s \end{Bmatrix} \bar{u}^s\right),$$

and finally by using (2-27), we can write

$$\bar{e}^{ij} = G^{jn}(\partial \bar{u}^i/\partial \bar{x}^n) + G^{in}(\partial \bar{u}^j/\partial \bar{x}^n) - (\partial G^{ij}/\partial \bar{x}^n)\,\bar{u}^n.$$

To express the buoyancy term $\delta^{i3}\rho' g$ we note that a contravariant vector is related to its "old" Cartesian components by a relation such as (2-15). Consequently, the contravariant components of the buoyancy force are

$$(\partial \bar{x}^i/\partial x^3)\,\rho' g.$$

Summing up, the general representation of the anelastic system (2-1), (2-2), (2-3), (2-7), (2-8), and (2-13) is converted to

$$(1/G^{1/2})(\partial/\partial \bar{x}^j)(G^{1/2}\rho_0 \bar{u}^j) = 0, \qquad (2\text{-}30)$$

$$\frac{\partial}{\partial t}(\rho_0 \bar{u}^i) + \frac{1}{G^{1/2}}\frac{\partial}{\partial \bar{x}^j}(G^{1/2}\rho_0 \bar{u}^i \bar{u}^j) + \begin{Bmatrix} i \\ m \quad n \end{Bmatrix} \rho_0 \bar{u}^m \bar{u}^n = -\frac{1}{G^{1/2}}\frac{\partial}{\partial \bar{x}^j}(G^{ij}G^{1/2}p')$$
$$+\frac{p'}{G^{1/2}}\frac{\partial}{\partial \bar{x}^j}(G^{1/2}G^{ij}) + \frac{1}{G^{1/2}}\frac{\partial}{\partial \bar{x}^j}(G^{1/2}\bar{\tau}^{ij}) + \begin{Bmatrix} i \\ m \quad n \end{Bmatrix} \bar{\tau}^{mn} + \rho' \frac{\partial \bar{x}^i}{\partial x^3} g, \qquad (2\text{-}31)$$

$$\frac{\partial}{\partial t}(\rho_0 \theta') + \frac{1}{G^{1/2}}\frac{\partial}{\partial \bar{x}^j}(G^{1/2}\rho_0 \theta' \bar{u}^j) = \frac{1}{G^{1/2}}\frac{\partial}{\partial \bar{x}^j}\left(\rho_0 K_H G^{1/2} G^{ij}\frac{\partial \theta'}{\partial \bar{x}^j}\right), \qquad (2\text{-}32)$$

$$\bar{\tau}^{ij} = \rho_0 K_M\left(\bar{e}^{ij} - \frac{2G^{ij}}{\delta^{ii}}\frac{1}{G^{1/2}}\frac{\partial}{\partial \bar{x}^j}(G^{1/2}\bar{u}^j)\right). \qquad (2\text{-}33)$$

Note that $\bar{\tau}^{ii} = \tau^{ii} = 0$ (see also Eq. (2-7)).

$$\bar{e}^{ij} = G^{jt}(\partial \bar{u}^i/\partial \bar{x}^t) + G^{it}(\partial u^j/\partial \bar{x}^t) - (\partial G^{ij}/\partial \bar{x}^t)\,\bar{u}^t. \qquad (2\text{-}34)$$

G^{ij}, $G^{1/2}$ and $\{{}_m{}^i{}_n\}$ have been defined by (2-17), (2-24), and (2-21), respectively. K_M and K_H have been defined by (2-9) and (2-14), respectively. ρ' and ρ_0 have been defined by (2-5) and (2-4-b), respectively. The diagnostic pressure equation (2-6) becomes

$$(\rho_0 \bar{u}^i \bar{u}^j)_{,ji} = -(G^{ij}p')_{,ji} + (\rho'(\partial \bar{x}^i/\partial x^3)\,g)_{,i} + \bar{\tau}^{ij}{}_{,ji}\,. \qquad (2\text{-}35)$$

2.3. *A Particular Coordinate Transformation for Calculating Flow Fields Above an Irregular Lower Boundary*

We are now in a position to apply a particular coordinate transformation suitable for our needs. Suppose that we want to solve the anelastic set (2-1) to (2-3) in the domain

$$0 \leqslant x \leqslant D, \qquad 0 \leqslant y \leqslant D, \qquad 0 \leqslant z_s(x, y) \leqslant z \leqslant H, \qquad (2\text{-}36)$$

where x, y, z are Cartesian coordinates, D is the lateral extent of the domain, H is its vertical extent, and $z_s(x, y)$ is the topography.

We now look for a coordinate transformation which will have the following properties.

(a) The domain defined by (2-36) will be transformed into a rectangular domain.

(b) The transformation should be reversible, that is a one-to-one relationship should exist between the "old" coordinates and the transformed coordinates.

(c) In cases where the topography is flat, the transformation should become the identity transformation, that is, the original Cartesian coordinates.

(d) The transformation should become the identity transformation at the upper boundary $z = H$.

(e) The transformation should be continuous up to second derivatives.

It is well known that if (b) is satisfied then the Jacobian of the transformation $G^{1/2}$ (see formula (2–23)) is different from zero. A spherical coordinate in cases of a spherical mountain will be then excluded, because the Jacobian is zero at the poles, and the transformation is then irreversible. The rationale behind requirement (e) is that the Christophel's symbols defined by (2-21) and (2-22) require evaluation of second derivatives, and linear stability properties are strongly dependent on the fact that extra nonderivative terms are $O(\Delta t)$. Requirement (e) will force us to confine ourselves to topographies which possess continuous second derivatives. In particular, we will require that the second derivative (when put into nondimensional form) must be of an order not greater than the first derivative, that is,

$$f''(x)/f'(x) \approx \epsilon, \qquad 0 \leqslant |\epsilon| \leqslant 1.$$

frame of reference as $e_{ij} = (\partial u_i/\partial x^j) + (\partial u_j/\partial x^i)$. Consequently the covariant components of this tensor are in general

$$\bar{e}_{ij} = \bar{u}_{i,j} + \bar{u}_{j,i}\,.$$

In order to find its contravariant components, we use the prescription (2-19) to get

$$\bar{e}^{ij} = G^{jn}(G^{im}\bar{u}_{m,n}) + G^{im}(G^{jn}\bar{u}_{n,m}).$$

By using (2-18), (2-20), and (2-26), this is equal to

$$G^{jn}\left(\frac{\partial \bar{u}^i}{\partial \bar{x}^n} + \begin{Bmatrix} i \\ n\ s \end{Bmatrix} \bar{u}^s\right) + G^{im}\left(\frac{\partial \bar{u}^j}{\partial \bar{x}^m} + \begin{Bmatrix} j \\ m\ s \end{Bmatrix} \bar{u}^s\right),$$

and finally by using (2-27), we can write

$$\bar{e}^{ij} = G^{jn}(\partial \bar{u}^i/\partial \bar{x}^n) + G^{in}(\partial \bar{u}^j/\partial \bar{x}^n) - (\partial G^{ij}/\partial \bar{x}^n)\,\bar{u}^n.$$

To express the buoyancy term $\delta^{i3}\rho' g$ we note that a contravariant vector is related to its "old" Cartesian components by a relation such as (2-15). Consequently, the contravariant components of the buoyancy force are

$$(\partial \bar{x}^i/\partial x^3)\,\rho' g.$$

Summing up, the general representation of the anelastic system (2-1), (2-2), (2-3), (2-7), (2-8), and (2-13) is converted to

$$(1/G^{1/2})(\partial/\partial \bar{x}^j)(G^{1/2}\rho_0 \bar{u}^j) = 0, \tag{2-30}$$

$$\frac{\partial}{\partial t}(\rho_0 \bar{u}^i) + \frac{1}{G^{1/2}}\frac{\partial}{\partial \bar{x}^j}(G^{1/2}\rho_0 \bar{u}^i \bar{u}^j) + \begin{Bmatrix} i \\ m\ n \end{Bmatrix} \rho_0 \bar{u}^m \bar{u}^n = -\frac{1}{G^{1/2}}\frac{\partial}{\partial \bar{x}^j}(G^{ij}G^{1/2}p')$$
$$+ \frac{p'}{G^{1/2}}\frac{\partial}{\partial \bar{x}^j}(G^{1/2}G^{ij}) + \frac{1}{G^{1/2}}\frac{\partial}{\partial \bar{x}^j}(G^{1/2}\bar{\tau}^{ij}) + \begin{Bmatrix} i \\ m\ n \end{Bmatrix} \bar{\tau}^{mn} + \rho' \frac{\partial \bar{x}^i}{\partial x^3} g, \tag{2-31}$$

$$\frac{\partial}{\partial t}(\rho_0 \theta') + \frac{1}{G^{1/2}}\frac{\partial}{\partial \bar{x}^j}(G^{1/2}\rho_0 \theta' \bar{u}^j) = \frac{1}{G^{1/2}}\frac{\partial}{\partial \bar{x}^j}\left(\rho_0 K_H G^{1/2} G^{ij}\frac{\partial \theta'}{\partial \bar{x}^j}\right), \tag{2-32}$$

$$\bar{\tau}^{ij} = \rho_0 K_M \left(\bar{e}^{ij} - \frac{2G^{ij}}{\delta^{ii}}\frac{1}{G^{1/2}}\frac{\partial}{\partial \bar{x}^j}(G^{1/2}\bar{u}^j)\right). \tag{2-33}$$

Note that $\bar{\tau}^{ii} = \tau^{ii} = 0$ (see also Eq. (2-7)).

$$\bar{e}^{ij} = G^{jt}(\partial \bar{u}^i/\partial \bar{x}^t) + G^{it}(\partial u^j/\partial \bar{x}^t) - (\partial G^{ij}/\partial \bar{x}^t)\,\bar{u}^t. \tag{2-34}$$

G^{ij}, $G^{1/2}$ and $\{{}_{m}{}^{i}{}_{n}\}$ have been defined by (2-17), (2-24), and (2-21), respectively. K_M and K_H have been defined by (2-9) and (2-14), respectively. ρ' and ρ_0 have been defined by (2-5) and (2-4-b), respectively. The diagnostic pressure equation (2-6) becomes

$$(\rho_0 \bar{u}^i \bar{u}^j)_{,ji} = -(G^{ij}p')_{,ji} + (\rho'(\partial \bar{x}^i/\partial x^3)\,g)_{,i} + \bar{\tau}^{ij}{}_{,ji}\,. \tag{2-35}$$

2.3. *A Particular Coordinate Transformation for Calculating Flow Fields Above an Irregular Lower Boundary*

We are now in a position to apply a particular coordinate transformation suitable for our needs. Suppose that we want to solve the anelastic set (2-1) to (2-3) in the domain

$$0 \leqslant x \leqslant D, \qquad 0 \leqslant y \leqslant D, \qquad 0 \leqslant z_s(x, y) \leqslant z \leqslant H, \tag{2-36}$$

where x, y, z are Cartesian coordinates, D is the lateral extent of the domain, H is its vertical extent, and $z_s(x, y)$ is the topography.

We now look for a coordinate transformation which will have the following properties.

(a) The domain defined by (2-36) will be transformed into a rectangular domain.

(b) The transformation should be reversible, that is a one-to-one relationship should exist between the "old" coordinates and the transformed coordinates.

(c) In cases where the topography is flat, the transformation should become the identity transformation, that is, the original Cartesian coordinates.

(d) The transformation should become the identity transformation at the upper boundary $z = H$.

(e) The transformation should be continuous up to second derivatives.

It is well known that if (b) is satisfied then the Jacobian of the transformation $G^{1/2}$ (see formula (2–23)) is different from zero. A spherical coordinate in cases of a spherical mountain will be then excluded, because the Jacobian is zero at the poles, and the transformation is then irreversible. The rationale behind requirement (e) is that the Christophel's symbols defined by (2-21) and (2-22) require evaluation of second derivatives, and linear stability properties are strongly dependent on the fact that extra nonderivative terms are $O(\Delta t)$. Requirement (e) will force us to confine ourselves to topographies which possess continuous second derivatives. In particular, we will require that the second derivative (when put into nondimensional form) must be of an order not greater than the first derivative, that is,

$$f''(x)/f'(x) \approx \epsilon, \qquad 0 \leqslant |\epsilon| \leqslant 1.$$

The following transformation will satisfy requirements (a)–(e):

$$\bar{x} = x, \qquad \bar{y} = y, \qquad \bar{z} = H(z - z_s)/(H - z_s). \tag{2-37}$$

The inverse transformation is

$$x = \bar{x}, \qquad y = \bar{y}, \qquad z = [\bar{z}(H - z_s)/H] + z_s\,. \tag{2-38}$$

It is now a straightforward matter to calculate all the quantities defined in the previous section. The results are as follows. The conjugate tensor G^{mn} defined in (2-17) is

$$G^{mn} = \begin{pmatrix} 1, & 0, & \dfrac{\partial z_s}{\partial x}\dfrac{\bar{z} - H}{H - z_s} \\ 0, & 1, & \dfrac{\partial z_s}{\partial y}\dfrac{\bar{z} - H}{H - z_s} \\ \dfrac{\partial z_s}{\partial x}\dfrac{z - H}{H - z_s}, & \dfrac{\partial z_s}{\partial y}\dfrac{\bar{z} - H}{H - z_s}, & \left\{\left(\dfrac{H}{H - z_s}\right)^2 + \left(\dfrac{\bar{z} - H}{H - z_s}\right)^2\left[\left(\dfrac{\partial z_s}{\partial x}\right)^2 + \left(\dfrac{\partial z_s}{\partial y}\right)^2\right]\right\} \end{pmatrix}. \tag{2-39}$$

The Jacobian of the transformation defined in (2-23) is

$$G^{1/2} = (H - z_s)/H. \tag{2-40}$$

The Christophel's symbols of the second kind are

$$\left\{ {r \atop m \;\; n} \right\} = \frac{\partial \bar{x}^r}{\partial x^j} \cdot \frac{\partial \bar{x}^p}{\partial x^j} \cdot \frac{\partial^2 x^i}{\partial \bar{x}^m\, \partial \bar{x}^n} \cdot \frac{\partial x^i}{\partial \bar{x}^p} = \frac{\partial \bar{x}^r}{\partial x^j}\, \delta_j{}^i\, \frac{\partial^2 x^i}{\partial \bar{x}^m\, \partial \bar{x}^n}\,.$$

By using (2-38) we get

$$\left\{ {1 \atop m \;\; n} \right\} = 0, \qquad \left\{ {2 \atop m \;\; n} \right\} = 0, \tag{2-41-a}$$

$$\left\{ {3 \atop 1 \;\; 1} \right\} = \frac{\partial^2 z_s}{\partial x^2}\frac{\bar{z} - H}{H - z_s}, \qquad \left\{ {3 \atop 1 \;\; 2} \right\} = -\frac{\partial^2 z_s}{\partial x\, \partial y}\frac{\bar{z} - H}{H - z_s}, \tag{2-41-b}$$

$$\left\{ {3 \atop 2 \;\; 2} \right\} = -\frac{\partial^2 z_s}{\partial y^2}\frac{\bar{z} - H}{H - z_s}, \qquad \left\{ {3 \atop 1 \;\; 3} \right\} = -\frac{\partial z_s}{\partial x}\frac{1}{H - z_s}, \tag{2-41-c}$$

$$\left\{ {3 \atop 2 \;\; 3} \right\} = -\frac{\partial z_s}{\partial y}\frac{1}{H - z_s}, \qquad \left\{ {3 \atop 3 \;\; 3} \right\} = 0. \tag{2-41-d}$$

The matrix $a_j{}^i = \partial \bar{x}^i/\partial x^j$ relates the contravariant components of an arbitrary

vector in the new coordinate system to its old Cartesian components. In particular, the relation between the contravariant components $\bar{u}$, $\bar{v}$, $\bar{w}$ of the velocity vector in the "new" coordinate system defined by (2-37), and the "old" Cartesian components u, v, w is

$$\begin{pmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{pmatrix} = \begin{pmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ \dfrac{\partial z_s}{\partial x}\dfrac{\bar{z} - H}{H - z_s}, & \dfrac{\partial z_s}{\partial y}\dfrac{\bar{z} - H}{H - z_s}, & \dfrac{\bar{z} - H}{H - z_s} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{2-42}$$

The direction of $\bar{w}$ can be seen to be the direction of the normal to the surface $\bar{z} =$ const. The inverse transformation is

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ -\dfrac{\partial z_s}{\partial x}\dfrac{\bar{z} - H}{H}, & -\dfrac{\partial z_s}{\partial y}\dfrac{\bar{z} - H}{H}, & \dfrac{H - z_s}{H} \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{pmatrix}. \tag{2-43}$$

Let us now denote the contravariant components of the strain tensor, in the coordinate system defined by (2-37), as $\bar{e}^{ij}$. Inspection of (2-12) reveals that in order to calculate the so-called deformation function $(\mathrm{Def})^2$, we have to know not only the contravariant representation $\bar{e}^{ij}$, but also the covariant representation $\bar{e}_{ij}$. We note, however, that $(\mathrm{Def})^2$ is an invariant, so we can calculate e^{ij}, the Cartesian components of the strain tensor, and since in Cartesian coordinates $e^{ij} = e_{ij}$, the quantity $e^{ij}e_{ij}$ will give us the deformation function. Since

$$e^{ij} = (\partial x^i/\partial \bar{x}^k) \cdot (\partial x^j/\partial \bar{x}^l)\, \bar{e}^{kl},$$

we find by using (2-37) and (2-38) that

$$e^{11} = \bar{e}^{11}, \qquad e^{12} = \bar{e}^{12}, \qquad e^{22} = \bar{e}^{22}, \tag{2-44-a}$$

$$e^{13} = -\frac{\partial z_s}{\partial x}\frac{\bar{z} - H}{H}\bar{e}^{11} - \frac{\partial z_s}{\partial y}\frac{\bar{z} - H}{H}\bar{e}^{12} + \bar{e}^{13}\frac{H - z_s}{H}, \tag{2-44-b}$$

$$e^{23} = -\frac{\partial z_s}{\partial x}\frac{\bar{z} - H}{H}\bar{e}^{12} - \frac{\partial z_s}{\partial y}\frac{\bar{z} - H}{H}\bar{e}^{22} + \bar{e}^{23}\frac{H - z_s}{H}. \tag{2-44-c}$$

To calculate e^{33} note that $e^{33} = 2(\partial w/\partial z) = 2(\partial w/\partial \bar{z})(\partial \bar{z}/\partial z)$ and by using (2-43) and (2-37) we get

$$e^{33} = 2\frac{H}{H - z_s}\frac{\partial}{\partial \bar{z}}\left(-\frac{\partial z_s}{\partial x}\frac{\bar{z} - H}{H}\bar{u} - \frac{\partial z_s}{\partial y}\frac{\bar{z} - H}{H}\bar{v} + \frac{H - z_s}{H}\bar{w}\right). \tag{2-44-d}$$

2.4. *Boundary Conditions for the Anelastic Set, Differential and Integral Constraints of the Motion*

Thus far we were concerned with transforming the equations in a consistent way. The next logical step is to transform the boundary conditions. We consider two types of boundary conditions: (a) no-slip, and (b) free-slip.

The no-slip conditions are simply

$$u^i = 0 \text{ on the boundary.}$$

Consequently, we must have at any coordinate system

$$\bar{u}^i = 0 \text{ at the boundary.}$$

The implementation of these conditions for a machine calculation is straightforward, as is evident from (2-30) to (2-34). It must be emphasized that the no-slip conditions are consistent with the equation of motion only if the frictional forces $\tau^{ij}{}_{,j}$ are different from zero. In this case the equations are of parabolic type, and no-slip conditions are one of the boundary conditions for which these equations are well posed. In the limiting case, however, in which the frictional forces are identically zero (i.e., the equations become the Euler equations), the no-slip boundary condition is incompatible with the governing equations. A complete mathematical discussion of this question can be found, e.g., in [12].

The physical significance of our remark about the no-slip boundary conditions is that no matter how small the viscosity is, there are always regions in the flow, close to the physical boundaries, in which the viscosity is important. The correct matching of the boundary layer, and the inviscid Eulerian flow outside the boundary layer, is a major area of research in fluid dynamics.

A natural lower boundary for an atmospheric flow is the solid earth. Consequently, the no-slip conditions should be imposed on the lower boundary. Unfortunately, the explicit viscosity of the air, even an eddy viscosity, is so small that the widths of the resulting boundary layers are too small for the resolution of present finite difference networks. While parameterization of atmospheric boundary layers is an active field in meteorology, many investigators, such as Lilly [17], Ogura [21], Orville [24], Fox [5], and Steiner [34], to mention but a few, have "filtered" the boundary layer out of their computation by specifying a "free-slip/rigid" boundary condition. While it is easy to criticize this approach, it is very difficult to propose a viable alternative. The hope is that the behavior of the fluid "far" from the lower boundary is not very sensitive to the conditions imposed on the lower boundary. In addition, the free-slip/rigid boundary conditions are compatible with the Euler equations.

The free-slip/rigid boundary conditions are expressed mathematically as

$$\mathbf{u} \cdot \mathbf{n} = 0, \tag{2-45-a}$$

$$\boldsymbol{\xi} \times \mathbf{n} = 0, \tag{2-45-b}$$

$\mathbf{n}$ is a unit vector normal to the boundary and $\boldsymbol{\xi}$ is the vorticity pseudovector. If the lower boundary is flat, the free-slip/rigid conditions become $\partial u/\partial z = \partial v/\partial z = 0$, $w = 0$.

To express the free-slip/rigid boundary conditions in a generalized coordinate system, let us first write the condition (2-45-a-b) in a tensorial form

$$\bar{u}^i \bar{n}_i = 0, \tag{2-46-a}$$

$$\bar{\epsilon}_{ijk} \bar{\xi}^j \bar{n}^k = 0. \tag{2-46-b}$$

In Cartesian form $\bar{\epsilon}_{ijk} = \bar{\epsilon}^{ijk} = e^{ijk} = e_{ijk}$, and these are the skew-symmetric systems with values $+1$ for even permutation, -1 for odd permutation, and 0 when two or more indices are equal. The relations (2-46-a-b) are true for any coordinate system provided that one modifies the definition of $\bar{\epsilon}^{ijk}$ and $\bar{\epsilon}_{ijk}$ [19, p. 135] to be

$$\bar{\epsilon}^{ijk} = (1/G^{1/2})\, e^{ijk}, \qquad \bar{\epsilon}_{ijk} = G^{1/2} e_{ijk}. \tag{2-47}$$

For the particular transformation that we choose, the covariant components of the normal to the surface $\bar{z} = \text{const.}$ are

$$\bar{n}_i = \left(\frac{\partial \bar{z}}{\partial x}, \frac{\partial \bar{z}}{\partial y}, \frac{\partial \bar{z}}{\partial z}\right) \Big/ |\nabla \bar{z}| = (0, 0, 1)/(G^{33})^{1/2}. \tag{2-48}$$

So that the condition on the normal velocity simply becomes $\bar{w} = 0$. This relation could of course be derived using the Cartesian representation.

To express the condition of (2-46-b) in a more expanded form, note that

$$\bar{\xi}^j = \bar{\epsilon}^{jlm} \bar{u}_{l,m}.$$

Thus (2-46-b) becomes, using (2-47),

$$e_{ijk} e^{jlm} \bar{u}_{l,m} \bar{n}^k = 0.$$

Since $e_{ijk} = -e_{ikj}$ and since $e_{ijk} e^{jlm} = \delta_i{}^m \delta_k{}^l - \delta_i{}^l \delta_k{}^m$, we get

$$(\bar{u}_{i,k} - \bar{u}_{k,i})\, \bar{n}^k = 0,$$

but

$$\bar{n}^k = G^{kl} \bar{n}_l.$$

Multiplying by G^{mi} and using (2-26) and (2-18), we finally get

$$(G^{kl} \bar{u}^m{}_{,k} - G^{im} \bar{u}^l{}_{,i})\, n_l = 0.$$

Using Eq. (2-48), the definition of covariant derivative (2-20), the fact that $\{{}_{m}{}^{i}{}_{n}\} = 0$, $i \neq 3$ and relation (2-27) we get the free-slip/rigid condition for our particular transformation as

$$\bar{w} = 0 \quad \text{at} \quad \bar{z} = 0, \tag{2-49-a}$$

$$G^{k3}(\partial \bar{u}^m/\partial \bar{x}^k) = G^{im}(\partial \bar{u}^3/\partial \bar{x}^i) - (\partial G^{m3}/\partial \bar{x}^l)\,\bar{u}^l, \quad \text{at} \quad \bar{z} = 0, \quad m \neq 3. \tag{2-49-b}$$

Boundary conditions are also needed for the potential temperature. We shall again consider two types:

(a) θ' given on the boundary,
(b) $\partial\theta'/\partial n$ given on the boundary.

In the generalized coordinate system, the covariant components of the temperature gradient are $\partial\theta'/\partial\bar{x}^i$; the contravariant components are by (2-18) $G^{ij}(\partial\theta'/\partial\bar{x}^j)$. For our particular transformation, condition (b) becomes, using (2-48),

$$G^{3j}(\partial\theta'/\partial\bar{x}^j) = (G^{33})^{1/2} f3(\bar{x}, \bar{y}), \quad \text{at} \quad \bar{z} = 0, \tag{2-50}$$

where $f3(\bar{x}, \bar{y})$ is the normal heat flux. For flat topography our transformation is the identity transformation, and one gets the frequently used condition

$$(\partial\theta/\partial\bar{z}) \text{ given on } \bar{z} = 0. \tag{2-51}$$

Our next step is to discuss the constraints which must be imposed on the motion. The most important constraint is the differential constraint

$$(\rho_0 u^i)_{,i} = 0.$$

It is this constraint which makes the numerical solution of an incompressible fluid more difficult than the numerical solution of a compressible fluid. We have already shown in Section 2.1 that in order to satisfy the constraint mentioned above, the pressure must satisfy the elliptic equation (2-6) or its equivalent form (2-35). Usually the pressure boundary condition, as deduced from the momentum equation, is of the Neumann type. Therefore the resulting finite difference analog of the pressure equation will be singular (e.g. [4, 14]).

Our next consideration is energy integrals of the motion. Since those integrals are invariants of the motion, they can be derived in a Cartesian coordinate system and will remain true for any coordinate system.

Starting from the conservative form of the momentum Eq. (2-2) and subtracting from this the continuity Eq. (2-1), then multiplying by u^i, summing and using (2-4-b) and (2-4-c) we get (e.g. [6, 22])

$$\iiint_v \frac{\partial}{\partial t}(\rho_0 E_K) - \iint_s (pu^i - \tau^{ij}u_j + E_K u^i)\, n_i = \iiint_v - \rho_0 K_M (\mathrm{Def})^2 + \iiint_v w\theta' \frac{g\rho_0}{\theta_0},$$

Where the direction of the normal is taken inward. We have defined the kinetic energy E_K per unit mass as, $E_K = \frac{1}{2}u^i u_i = \frac{1}{2}\bar{u}^i \bar{u}_i$. $(\mathrm{Def})^2$ has been defined by (2-12). In particular, if the normal components of the surface integral vanish, we get

$$\iiint_v \frac{\partial}{\partial t}(\rho_0 E_K) = - \iiint_v \rho_0 K_M (\mathrm{Def})^2 + \iiint_v w\theta' \frac{g\rho_0}{\theta_0}. \tag{2-52}$$

The term $w\theta' g\rho_0/\theta_0$ can be interpreted physically as the work per unit time done by the part of the buoyancy force associated with the potential temperature deviation. It also can be interpreted as the conversion of potential to kinetic energy. The term $-\rho_0 K_M(\mathrm{Def})^2$ is the kinetic energy dissipation due to subgrid motion, and it is always negative according to our parameterization of eddy viscosity. The kinetic energy equation (2-52) can be brought into another useful form by noting that, for any arbitrary physical quantity, and in the case of no normal velocity across boundaries,

$$\iiint_v \rho_0 \frac{df}{dt} = \iiint_v \rho_0 \left(\frac{\partial f}{dt} + u^i \frac{\partial f}{\partial x^i}\right) = \iiint \rho_0 \frac{\partial f}{\partial t} + \frac{\partial}{\partial x^i}\rho_0 u^i f = \iiint_v \rho_0 \frac{\partial f}{\partial t}.$$

Taking $f = z$, (2-52) can then be written as

$$\iiint_v \frac{\partial}{\partial t}\rho_0(E_K + E_P) = \iiint_v - \rho K_M(\mathrm{Def})^2, \tag{2-53}$$

where we have denoted E_P as $E_P \equiv -(g/\theta_0)\,\theta' z$. E_P can be interpreted physically as the available potential energy of the convective system. The sum $E_K + E_P$ is the total energy of the perturbation, where the unperturbed state is the state of hydrostatic adiabatic atmosphere.

We have given the derivation of the well-known result (2-53), because it is important to note that this derivation was possible only because $\rho_0(z)$ and $p_0(z)$ are given by (2-4-b) and (2-4-c), i.e., the basic atmosphere must be adiabatic and hydrostatic. It is also interesting to observe that although we have assumed that the density deviations are functions of both the pressure and temperature deviation, we have ended up with an energy integral in which the changes in the kinetic energy were associated only with the potential temperature deviation. If instead of (2-5) we were to assume that

$$\rho' = \rho_0(\theta'/\theta_0), \tag{2-54}$$

we could not have obtained the closed energy integral (2-53).

In the actual computation, however, we found that continued experiment (200 time steps) using (2-54) yields the same results (up to the second significant digit)

Using Eq. (2-48), the definition of covariant derivative (2-20), the fact that $\{{}_{m}{}^{i}{}_{n}\} = 0$, $i \neq 3$ and relation (2-27) we get the free-slip/rigid condition for our particular transformation as

$$\bar{w} = 0 \quad \text{at} \quad \bar{z} = 0, \tag{2-49-a}$$

$$G^{k3}(\partial \bar{u}^m/\partial \bar{x}^k) = G^{im}(\partial \bar{u}^3/\partial \bar{x}^i) - (\partial G^{m3}/\partial \bar{x}^l)\, \bar{u}^l, \quad \text{at} \quad \bar{z} = 0, \quad m \neq 3. \tag{2-49-b}$$

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Our next step is to discuss the constraints which must be imposed on the motion. The most important constraint is the differential constraint

$$(\rho_0 u^i)_{,i} = 0.$$

It is this constraint which makes the numerical solution of an incompressible fluid more difficult than the numerical solution of a compressible fluid. We have already shown in Section 2.1 that in order to satisfy the constraint mentioned above, the pressure must satisfy the elliptic equation (2-6) or its equivalent form (2-35). Usually the pressure boundary condition, as deduced from the momentum equation, is of the Neumann type. Therefore the resulting finite difference analog of the pressure equation will be singular (e.g. [4, 14]).

Our next consideration is energy integrals of the motion. Since those integrals are invariants of the motion, they can be derived in a Cartesian coordinate system and will remain true for any coordinate system.

Starting from the conservative form of the momentum Eq. (2-2) and subtracting from this the continuity Eq. (2-1), then multiplying by u^i, summing and using (2-4-b) and (2-4-c) we get (e.g. [6, 22])

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Where the direction of the normal is taken inward. We have defined the kinetic energy E_K per unit mass as, $E_K = \frac{1}{2}u^i u_i = \frac{1}{2}\bar{u}^i \bar{u}_i$. $(\mathrm{Def})^2$ has been defined by (2-12). In particular, if the normal components of the surface integral vanish, we get

$$\iiint_v \frac{\partial}{\partial t}(\rho_0 E_K) = - \iiint_v \rho_0 K_M(\mathrm{Def})^2 + \iiint_v w\theta' \frac{g\rho_0}{\theta_0}. \tag{2-52}$$

The term $w\theta' g\rho_0/\theta_0$ can be interpreted physically as the work per unit time done by the part of the buoyancy force associated with the potential temperature deviation. It also can be interpreted as the conversion of potential to kinetic energy. The term $-\rho_0 K_M(\mathrm{Def})^2$ is the kinetic energy dissipation due to subgrid motion, and it is always negative according to our parameterization of eddy viscosity. The kinetic energy equation (2-52) can be brought into another useful form by noting that, for any arbitrary physical quantity, and in the case of no normal velocity across boundaries,

$$\iiint_v \rho_0 \frac{df}{dt} = \iiint_v \rho_0 \left(\frac{\partial f}{dt} + u^i \frac{\partial f}{\partial x^i}\right) = \iiint_v \rho_0 \frac{\partial f}{\partial t} + \frac{\partial}{\partial x^i} \rho_0 u^i f = \iiint_v \rho_0 \frac{\partial f}{\partial t}.$$

Taking $f = z$, (2-52) can then be written as

$$\iiint_v \frac{\partial}{\partial t} \rho_0(E_K + E_P) = \iiint_v - \rho K_M(\mathrm{Def})^2, \tag{2-53}$$

where we have denoted E_P as $E_P \equiv -(g/\theta_0)\,\theta' z$. E_P can be interpreted physically as the available potential energy of the convective system. The sum $E_K + E_P$ is the total energy of the perturbation, where the unperturbed state is the state of hydrostatic adiabatic atmosphere.

We have given the derivation of the well-known result (2-53), because it is important to note that this derivation was possible only because $\rho_0(z)$ and $p_0(z)$ are given by (2-4-b) and (2-4-c), i.e., the basic atmosphere must be adiabatic and hydrostatic. It is also interesting to observe that although we have assumed that the density deviations are functions of both the pressure and temperature deviation, we have ended up with an energy integral in which the changes in the kinetic energy were associated only with the potential temperature deviation. If instead of (2-5) we were to assume that

$$\rho' = \rho_0(\theta'/\theta_0), \tag{2-54}$$

we could not have obtained the closed energy integral (2-53).

In the actual computation, however, we found that continued experiment (200 time steps) using (2-54) yields the same results (up to the second significant digit) as one in which (2-5) is used. This was probably so because the term $\rho_0 p'/\gamma p_0$ (see relation (2-5)) was small compared to the other terms, such as advection, thermal buoyancy, or vertical pressure gradient.

In the actual computation the finite difference analog of the pressure equation is solved by relaxation, and we have found that more relaxation steps are needed if relation (2-5) is used, rather than (2-54). For computational expediency we therefor use relation (2-54), being aware of course, of the inherent inconsistency implied by doing so.

Let us now examine the other invariants of the motion. From the thermodynamic energy equation (2-3) and the continuity equation (2-1), one can deduce that in the absence of heat sources, and with vanishing normal velocities at the boundaries,

$$\iiint_v \frac{\partial}{\partial t}(\rho_0\theta') = -\iint_s K_H\rho_0 \frac{\partial\theta'}{\partial x^i} n_i\,, \tag{2-55}$$

$$\iiint_v \frac{\partial}{\partial t}\rho_0\theta'^2 = -2\iiint_v K_H\rho_0\left(\frac{\partial\theta'}{\partial x^i}\right)^2 - \iint_s K_H\rho_0\frac{\partial\theta'}{\partial x^i} n_i\,. \tag{2-56}$$

In a generalized coordinate system, these relations are

$$\iiint_v \frac{\partial}{\partial t}\rho_0\theta' = -\iint_s \rho_0 K_H G^{ij}\frac{\partial\theta'}{\partial \bar{x}^j} n_i\,, \tag{2-57}$$

$$\iiint_v \frac{\partial}{\partial t}\rho_0\theta'^2 = -\iiint_v 2K_H G^{ij}\frac{\partial\theta'}{\partial\bar{x}^j}\frac{\partial\theta'}{\partial\bar{x}^i} - \iint_s \rho_0 K_H G^{ij}\frac{\partial\theta'^2}{\partial x^i} n_j\,. \tag{2-58}$$

In the case where there is no heat flux normal to the boundaries, the density-weighted perturbation potential temperature is conserved, even though diffusion is present. If diffusion is absent ($K_H = 0$), the density-weighted square of potential temperature is conserved, as well as the density-weighted potential temperature. In fact, it can be shown that in the latter case, all moments are conserved.

Associating f with any of the invariants mentioned previously, the conservation law in the generalized coordinate system has the form

$$\frac{\partial}{\partial t}\iiint f\, G^{1/2}\, d\bar{x}\, d\bar{y}\, d\bar{z} \leqslant 0.$$

Due to the complexity and nonlinearity of the full Navier–Stokes equations, numerical methods of solution generally involve some degree of approximation. This will probably be the case in the near future. The question which must be raised, then, is to what degree the differential and integral constraints discussed in this chapter must be satisfied.

We have already said that the most important constraint is the continuity equation

$$(\rho_0 u^i)_{,i} = 0.$$

This is so for several reasons. First, this constraint must be satisfied locally as well as globally. Second, only when this constraint is satisfied is the "advective" form of the governing equation equivalent to the flux form. That is to say, if f^i is an arbitrary vector quantity, then

$$\rho_0((\partial f^i/\partial t) + u^j f^i_{\,,j}) = (\partial/\partial t)(\rho_0 f^i) + (\rho_0 f^i u^j)_{,j} \qquad \text{iff} \quad (\rho_0 u^i)_{,i} = 0.$$

In the case of thermal convection, the importance of constraint (2-57) cannot be underestimated, since thermal convection is characterized by a physically unstable situation in which small deviations in the potential temperature are the driving force. It is therefore important that no spurious heat fluxes be present.

The quantities $\iiint_v \rho_0\theta'^2$ and $\iiint_v E_K + E_P$ which appear on the left-hand side of (2-58) and (2-53) will be conserved only in the limiting case where dissipation and diffusion are absent. In that sense, constraints on the first moments, such as (2-57), are more general than constraints on the second moments, such as (2-58) and (2-53). Consequently, approximate solutions which contain implicitly some sort of dissipation are not wrong outright, provided that the net effect of these artificial devices will be much smaller than the effect of what one considers to be "real viscosity," or "real diffusion."

2.5. *Summary of the Governing Equations*

The detailed scalar version of the governing equations, for our particular coordinate transformation can be found in [6].

Acknowledgments

This work, based largely on Tzvi Gal-Chen's Ph.D. thesis, was supported by NASA Grant NGR 33-008-191 to Columbia University.

This manuscript was professionally and skillfully typed by Ms. J. Prangley and Ms. U. Rosner.

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Journal of Computational Physics **17**, 229–233 (1975)

Note

The Evaluation of Integrals with Oscillatory Integrands*

We present here an extension to Filon's method for the evaluation of oscillatory integrals with infinite limits. The method amounts to producing an asymptotic expansion for the original integral. The qualitative relevance of our results is indicated.

There are a few methods available for the evaluation of integrals of the form

$$I(r) = \int_0^\infty f(k) \, {}^{\cos}_{\sin} kr \, dk, \tag{1}$$

which frequently occur in investigations in mathematical physics.

The methods adopted by Hurwitz and Zweifel [1], Hurwitz, Pfeifer, and Zweifel [2], Saenger [3], and Balbine and Franklin [4], converting the infinite integral into a summation by subdividing the range and performing the integration between successive zeros of ${}^{\sin}_{\cos} kr$ prove to be not completely satisfactory in all cases.

This is particularly true for cases where the approximating series does not converge rapidly.

The same applies to Longmann's [5] approach in which he used a variation of Euler's transformation to accelerate convergence.

A further development of this approach was proposed by Alaylioglou, Evans, and Heyslop [6] who investigated the use of the more general nonlinear transformation of Shank [7] and reported quite satisfactory results. Perhaps, a disadvantage of this approach (and indeed of all subdivision/acceleration algorithms) is that the integration limits (a_n, a_{n+1}) depend on the parameter r, thus the Gaussian–Legendre Integration points have to be retransformed for each of its values. This is not apparent when one seeks to evaluate the integral (1) for a few values of r, but it might become a laborious task when r assumes a sufficiently large number of values.

The original method, proposed by Filon [8], of approximating $f(k)$ with a polymonial of a low degree does not provide for the evaluation of integrals of infinite range and cannot be applied directly.

* This paper is based on part of a MS. thesis by G. Pantis, University of South Africa, unpublished.