Estimate of Truncation Error in Transformed Coordinate, Primitive Equation Atmospheric Models

JOHN M. GARY[1]

National Center for Atmospheric Research,[2] Boulder, Colo. 80302

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ABSTRACT

A comparison of truncation error for second-order difference schemes based on the σ and z coordinate systems is given. We also test a transformed z system based on $s=(z-H)/(z_T-H)$, where $H\leqslant z\leqslant z_T$ and H is surface elevation. The error is estimated by substitution of a known pressure function into the pressure gradient term. The σ and z systems show comparable error in the case of a level surface. The presence of a mountain greatly increases the error in the σ and s system calculations. A modification of the σ and s systems is developed to reduce this error.

1. Introduction

In numerical weather prediction models the effects of mountains must be taken into account. This can be done by use of a coordinate transformation so that the domain of the model is still rectangular as in the "σ" coordinate system (Haltiner, 1971); or a boundary condition can be imposed in the "z" or "p" coordinates which leads to an irregular mesh (Oliger *et al.*, 1970). This produces coding problems, particularly on a parallel computer such as the Texas Instruments ASC or the Control Data STAR. Also, if a fourth-order accurate difference scheme is used, it will be difficult to maintain even third-order accuracy at the mountain boundary. This may not be too important since the boundary layer treatment itself is subject to considerable uncertainty. However, if a coordinate mapping scheme is used, then considerable error can be introduced into the pressure gradient term in the momentum equations. Since the pressure derivatives are taken on a sloping (in the vertical) surface, they contain a hydrostatic component. The hydrostatic component must cancel out between the two terms which make up the pressure gradient in the σ system, and this can introduce considerable error. Furthermore, this error extends throughout the entire atmosphere; it is not concentrated near the mountain as it is in the z system.

This error has been recognized for some time (Smagorinsky *et al.*, 1967). When the pressure gradient is computed in the σ coordinate system (i.e., along constant σ surfaces), a computational mode develops when the topography is not flat (Kurihara, 1968; Holloway and Manabe, 1971). This "checkerboard pattern" or computational mode was eliminated by computing the gradient of the geopotential along constant pressure surfaces following a suggestion by Smagorinsky (Smagorinsky *et al.*, 1967). Since the data are carried on σ surfaces in the model, interpolation must be used to transform the data to p surfaces. This may produce a smoothing or equivalent diffusion of the geopotential which would tend to eliminate the computational mode. While the interpolation will introduce an error, the gradient in the p system, on the other hand, is more accurate than in the σ system. This paper will not analyze the cause of the computational mode, since we do not consider any nonlinear effects. We only measure the truncation error for known analytic pressure or geopotential functions which are hopefully representative of model data. We do not analyze the influence of this error on the model.

By computing the pressure gradient for a known pressure function, we will estimate the truncation error for the σ system. We will also study another system based on the transformation $s=(z-H)/(z_T-H)$, where H is mountain height and z_T the height of the atmosphere. This also reduces the integration to a rectangular domain. This turns out to require more points in the vertical than the σ system, but it is free from a sharp error increase which the σ system frequently shows near $\sigma=0$ (Kurihara, 1968). We will also consider a method of subtracting out the hydrostatic variation in pressure in order to reduce the truncation error due to mountains in the s and σ systems.

2. The pressure gradient terms

We will study the primitive equations for atmospheric motion written in the σ and z coordinates (Haltiner,

[1] Present affiliation: Department of Computer Science, University of Colorado, Boulder.

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1971). We can measure the coordinates of a point in the atmosphere by (λ,θ,z) where λ is longitude, θ latitude, and z altitude. The vertical distance can also be measured by

$$\sigma=\sigma(\lambda,\theta,z,t)=p(\lambda,\theta,z,t)/p_*(\lambda,\theta,t), \qquad (1)$$

where p_* is the pressure at ground level. Then $1\geqslant\sigma\geqslant 0$, and the domain is rectangular, unlike the z system whose domain is defined by $H(\lambda,\theta)\leqslant z\leqslant z_T$, where H is surface height and z_T the level of the "top" or lid on the atmosphere. A regular domain can also be obtained from the z system by definition of the vertical coordinate as $s=[z-H(\lambda,\theta)]/[z_T-H(\lambda,\theta)]$; then, $0\leqslant s\leqslant 1$.

The σ system momentum equations are the following:

$$\frac{\partial(p_*u)}{\partial t}=-D_2(p_*u)-\frac{\partial(p_*u\omega)}{\partial\sigma}+Gp_*v-p_*p^{(\lambda)}+F_u, \qquad (2)$$

$$\frac{\partial(p_*v)}{\partial t}=-D_2(p_*v)-\frac{\partial(p_*v\omega)}{\partial\sigma}-Gp_*u-p_*p^{(\theta)}+F_v, \qquad (3)$$

where

$$D_2(\psi)=\frac{1}{\cos\theta}\left[\left.\frac{\partial(u\psi)}{\partial\lambda}\right|_\sigma+\left.\frac{\partial(v\cos\theta\psi)}{\partial\theta}\right|_\sigma\right], \qquad (4)$$

$$p^{(\lambda)}=\frac{1}{a\cos\theta}\left[\frac{RT}{p_*}\left.\frac{\partial p_*}{\partial\lambda}\right|_\sigma+\left.\frac{\partial\Phi}{\partial\lambda}\right|_\sigma\right], \qquad (5)$$

$$p^{(\theta)}=\frac{1}{a}\left[\frac{RT}{p_*}\left.\frac{\partial p_*}{\partial\theta}\right|_\sigma+\left.\frac{\partial\Phi}{\partial\theta}\right|_\sigma\right], \qquad (6)$$

$$G=2\Omega\sin\theta+\frac{u}{a}\tan\theta. \qquad (7)$$

Here Φ is the geopotential, and F_u and F_v are diffusion terms. We derived the s system equations following the presentation of Kasahara (1967). Here we assume the vertical coordinate is given by some function $s=s(\lambda,\theta,z)$ with $\partial s/\partial z>0$. We denote the inverse function by $z(\lambda,\theta,s)$, and $z_s\equiv\partial z/\partial s$. The momentum equations are:

$$\frac{\partial(\rho z_s u)}{\partial t}=-D_2(\rho z_s u)-\frac{\partial(\rho z_s\omega u)}{\partial s}+G\rho z_s v-z_s\hat{p}^{(\lambda)}+F_u, \qquad (8)$$

$$\frac{\partial(\rho z_s v)}{\partial t}=-D_2(\rho z_s v)-\frac{\partial(\rho z_s\omega v)}{\partial s}-G\rho z_s u-z_s\hat{p}^{(\theta)}+F_v, \qquad (9)$$

$$\hat{p}^{(\lambda)}=\frac{1}{a\cos\theta}\left[\left.\frac{\partial p}{\partial\lambda}\right|_s-\left.\frac{\partial p}{\partial\lambda}\frac{\partial z}{\partial\lambda}\right|_s\Big/z_s\right], \qquad (10)$$

$$\hat{p}^{(\theta)}=\frac{1}{a}\left[\left.\frac{\partial p}{\partial\theta}\right|_s-\left.\frac{\partial p}{\partial s}\frac{\partial z}{\partial\theta}\right|_s\Big/z_s\right]. \qquad (11)$$

In this s system the equations for velocity ω $(\omega=ds/dt)$ and pressure p are similar to the z system used by Oliger *et al.* (1970).

Our objective is to attempt an estimate of the truncation error in the pressure gradient terms, $p^{(\lambda)}$ and $\hat{p}^{(\lambda)}$. In the σ system Φ is obtained by the integration

$$\Phi=gH+R\int_\sigma^1\frac{Td\sigma}{\sigma}, \qquad (12)$$

where R is the gas constant and T the temperature. In the s system the pressure p is obtained by an integration in the vertical:

$$\frac{\partial p}{\partial t}=B+g\rho z_s\omega-g\int_s^1 D_2(\rho z_s)ds, \qquad (13)$$

where

$$B=\left.\frac{\partial p}{\partial t}\right|_{s=1}. \qquad (14)$$

This is adopted from the formulation of Oliger *et al.* Therefore, we will also try to estimate the error in the quadrature formulas used for these integrals, as well as the error in the finite-difference approximations for the derivatives. This will be done by substitution of known functions into the quadrature and difference formulas. Then the error can be computed. These functions can all be based on a pressure function $p(z,\lambda)$.

3. Estimation of the truncation errors

In the following we assume that a representative pressure distribution $p(z,\lambda)$ is known. We reduce the problem to one dimension by dropping θ. We used three such pressure distributions. We will also use a mountain height profile $H(\lambda)$. The s coordinate is then given by $s=(z-H)/(z_T-H)$, where z_T is the constant height of our model atmosphere. We used $z_T=18$ km.

The first pressure distribution is based on an isothermal atmosphere, namely,

$$p(z,\lambda)=p_1(z)F_1(z,\lambda), \qquad (15)$$

$$p_1(z)=p_0e^{-(gz/RT_0)}, \qquad (16)$$

$$F_1(z,\lambda)=1+(\delta p/p_0)F_2(z)\sin m\lambda, \qquad (17)$$

$$F_2(z)=\begin{cases}c_1+c_2(1-(z-z_1)^2/z_1^2), & 0\leqslant z\leqslant z_1\\ c_1+c_2, & z_1\leqslant z\end{cases} \qquad (18)$$

This gives a sinusoidal perturbation on an isothermal atmosphere. The amplitude is determined by the parameters δp, c_1, c_2. For most runs these values were $p_0=1013$ mb, $T_0=288$C, $\delta p=13.3$ mb, $c_1=0.75$, $c_2=1.5$. We used $z_1=z_T=18$ km, and $m=6$. The other constants are defined in the Appendix. The constants c_1 and c_2 are chosen to control the variation in s of the sinusoidal

TABLE 1. Vertical profiles (cgs units), no mountain ($h_0=0$). We use 5.6 (+5) to denote 5.6×10^5.

Height (z)	1.5 (+5)	4.5 (+5)	7.5 (+5)	10.5 (+5)	13.5 (+5)	16.5 (+5)
$p(z)$	8.3 (+5)	5.6 (+5)	3.7 (+5)	2.3 (+5)	1.5 (+5)	9.2 (+4)
$\partial p/\partial\lambda$	3.3 (+4)	3.2 (+4)	2.6 (+4)	1.9 (+4)	1.3 (+4)	8.4 (+3)
Height (σ)	0.92	0.75	0.58	0.39	0.25	0.08
$z(\sigma)$	0.71 (+5)	2.3 (+5)	4.2 (+5)	6.6 (+5)	10.0 (+5)	19.1 (+5)
$g(\partial p/\partial\lambda)/p_z$	2.8 (+7)	3.4 (+7)	4.0 (+7)	4.5 (+7)	5.1 (+7)	5.7 (+7)

perturbation. The values of the derivative $(\partial p/\partial\lambda)(z,\lambda)$ at $\lambda=-\pi/18$ when $H(\lambda)\equiv0$ are given in Table 1. With $\delta p=13.3$, the horizontal variation of pressure has a peak-to-peak amplitude of 20 mb at ground level.

A second pressure distribution is based on a constant lapse rate atmosphere. In this case $p_1(z)$ is defined by

$$p_1(z)=p_0\left(\frac{T_0-\gamma z}{T_0}\right)^{g/(\gamma R)}. \tag{19}$$

We take $T_0=288$C, $\gamma=6.5$C km^{-1}. The function $F_1(z,\lambda)$ remains the same.

The third profile is based on a periodic temperature gradient:

$$T_2(z)=\begin{cases}T_0+\beta_1 z(z-2z_2), & 0\leqslant z\leqslant z_2\\ T_1, & z_2\leqslant z\end{cases} \tag{20}$$

where $\beta_1=(T_0-T_1)/z_2^2$. Then the pressure is obtained by the quadrature

$$p_1(z)=p_0e^{I(z)}, \tag{21}$$

$$I(z)=-\int_0^z\frac{gdz}{RT_2(z)}. \tag{22}$$

We evaluated this integral on an equally spaced mesh over the interval $[0,z_2]$ using the trapezoid rule. For $z>z_2$ we compute the integral from

$$I(z)=I(z_2)-\int_{z_2}^z\frac{gdz}{RT_1}=I(z_2)-\frac{g(z-z_2)}{RT_1}. \tag{23}$$

To obtain values for z between mesh points we use linear interpolation. This is a crude procedure, but we used a mesh of 2000 points. Dropping this to 1000 points did not change the truncation error in the second digit. We used the parameters $T_0=288$C, $T_1=218$C, $z_2=15$ km, $p_0=1013$ mb. The same function $F_1(z,\lambda)$ was used.

In all cases we defined the surface height as follows:

$$H(\lambda)=\begin{cases}0, & \lambda\leqslant-\lambda_0\\ h_0(\lambda-\lambda_0)^2(\lambda+\lambda_0)^2/\lambda_0^4, & -\lambda_0\leqslant\lambda\leqslant\lambda_0\\ 0, & \lambda_0<\lambda\end{cases} \tag{24}$$

using $\lambda_0=\pi/9$, $h_0=4.5$ km. This produces a maximum change in H of about 1.5 km in 5° of longitude.

For the s system we measure the truncation error in the term

$$\left.\frac{\partial p}{\partial\lambda}\right|_z=\left.\frac{\partial p}{\partial\lambda}\right|_s-\left(\frac{\partial p}{\partial s}\left.\frac{\partial z}{\partial\lambda}\right|_s\bigg/\frac{\partial z}{\partial s}\right). \tag{25}$$

We assume a mesh in the (s,λ) plane (s_k,λ_i) where $s_k=k/K$, $0\leqslant k\leqslant K$. Since in most cases we used only three points in the λ direction,

$$\lambda_i=\bar\lambda+i\Delta\lambda,\quad -1\leqslant i\leqslant1, \tag{26}$$

where

$$\bar\lambda=-\frac{\lambda_0}{2},\quad \Delta\lambda=5°, 2.5°, \cdots. \tag{27}$$

The values of p_λ are computed on the half-integer mesh $s_{k+\frac12}=(k+\frac12)\Delta s$, where $\Delta s=1/K$. We used the finite-difference expression:

$$\left.\frac{\partial p}{\partial\lambda}\right|_z\approx\frac{p_{k+\frac12,i+1}-p_{k+\frac12,i-1}}{2\Delta\lambda}-\frac{p_{k+1,i}-p_{k,i}}{\Delta s}z', \tag{28}$$

where

$$z'=\frac{z_{k+\frac12,i+1}-z_{k+\frac12,i-1}}{2\Delta\lambda}\bigg/z_s. \tag{29}$$

Here $z(s,\lambda)$ is obtained from $z=H+s(z_T-H)$ and $z_s=z_T-H(\lambda)$. We can compute the values of $p_{k,i}$ exactly from the functions defined above. Then the only error is due to the finite-difference expressions in the above formula for $\partial p/\partial\lambda$. We also compute $p_{k,i}$ by use of various quadrature methods for the equation

$$p(s,\lambda)=p(1,\lambda)-\int_s^1\frac{\partial p}{\partial z}z_s ds, \tag{30}$$

using the exact values of $\partial p/\partial z$ at the half-integer points $s_{k+\frac12}=(s_{k+1}+s_k)/2$ in the quadrature formulas.

For the σ system we estimate the truncation error in the equation

$$\frac{1}{\rho}\left.\frac{\partial p}{\partial\lambda}\right|_z=-\frac{g}{\partial p/\partial z}\left.\frac{\partial p}{\partial\lambda}\right|_z=-\frac{g}{p_z}[p_\lambda|_\sigma+p_\sigma\sigma_\lambda|_z], \tag{31}$$

where we use the notation $\partial p/\partial\lambda=p_\lambda$. Using the relation $\sigma_\lambda=-z_\lambda/z_\sigma$, we obtain

$$-\frac{gp_\lambda}{p_z}+\frac{gp_\sigma z_\lambda}{p_z z_\sigma}=-\frac{gp_\lambda}{p_z}+gz_\lambda. \tag{32}$$

Making use of the relations $\sigma=p/p_*$, where $p_*=p[H(\lambda),\lambda]$, we have

$$-\frac{g}{p_z}p_\lambda\bigg|_z=-\frac{g\sigma}{p_z}p_{*\lambda}+gz_\lambda. \tag{33}$$

If we define a function $\hat{T}$ by $\hat{T}=-g\sigma/Rp_z$, then this equation can be written

$$-\frac{g}{p_z}p_\lambda\bigg|_z=R\hat{T}p_{*\lambda}+gz_\lambda, \tag{34}$$

which is the usual form for the gradient term in the σ system. The left side of the equation can be computed exactly from the functions $p(z,\lambda)$ described above. The function $z(\sigma,\lambda)$ can be computed by solving

$$p[z(\sigma,\lambda),\lambda]=\sigma p_*(\lambda)=\sigma p[H(\lambda),\lambda], \tag{35}$$

using a Newton-Raphson iteration.

We use the same mesh as above, namely $\sigma_k=k/K$, $\lambda_i=\bar{\lambda}+i\Delta\lambda$. We assume the gradient must be computed at half-integer mesh points, $\sigma_{k+\frac{1}{2}}$. The finite-difference approximation is

$$R\hat{T}_{k+\frac{1}{2},i}\frac{(p_{*i+1}-p_{*i-1})}{2\Delta\lambda}+g\frac{z_{k+\frac{1}{2},i+1}-z_{k+\frac{1}{2},i-1}}{2\Delta\lambda}. \tag{36}$$

Values of $\hat{T}_{k+\frac{1}{2},i}$ are always computed "exactly" (i.e., with no truncation error, only the error from the Newton-Raphson iteration). Values of $z_{k+\frac{1}{2},i}$ are either computed exactly or from a quadrature formula described below. The quadrature is based on the formula

$$z(\sigma,\lambda)=H(\lambda)-\int_\sigma^1\frac{\partial z}{\partial\sigma}d\sigma, \tag{37}$$

where we use the following relation to evaluate the derivative:

$$\frac{\partial z}{\partial\sigma}=\frac{1}{\sigma_z}=\frac{p_*}{p_z}=\frac{p}{\sigma p_z}. \tag{38}$$

Therefore, we can write the integral in two ways:

$$z(\sigma,\lambda)=H(\lambda)-\int_\sigma^1\frac{p}{p_z}\frac{d\sigma}{\sigma}=H(\lambda)-\int_\sigma^1\frac{p}{p_z}d(\ln\sigma). \tag{39}$$

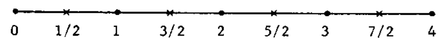
The quadrature formulas require the exact values of p/p_z at half-integer mesh points $\sigma_{k+\frac{1}{2}}$. This corresponds to a finite-difference scheme for the σ system with the velocity, and therefore the pressure gradient computed at half-integer mesh points.

4. The quadrature formulas

The first formula is the "midpoint" rule combined with an interpolation. We compute

$$F(s)=F(1)-\int_s^1 f(s)ds \tag{40}$$

for s_k using values $f(s_{k+\frac{1}{2}})\equiv f_{k+\frac{1}{2}}$; the mesh is shown below. The midpoint formula is



$$F_k=F_{k+1}-\Delta s f_{k+\frac{1}{2}}, \tag{41}$$

and the interpolation formula

$$F_{k+\frac{1}{2}}=\tfrac{1}{2}(F_{k+1}+F_k). \tag{42}$$

In later sections we refer to this as the midpoint rule with average interpolation. The midpoint rule for the σ system is similar. We also used an interpolation formula based on the assumption of an isothermal atmosphere. Then

$$p=p_*e^{-\alpha_1 s}, \tag{43}$$

$$z=H+\alpha_2\ln\sigma, \tag{44}$$

for some constants α_1 and α_2. In this case the following formulas are exact:

$$p_{k+\frac{1}{2}}=(p_{k+1}p_k)^{\frac{1}{2}} \tag{45}$$

$$z_{k+\frac{1}{2}}=\frac{\beta_1 z_{k+1}+\beta_0 z_k}{\beta_1+\beta_0}, \tag{46}$$

$$\beta_1=\ln(\sigma_{k+\frac{1}{2}}/\sigma_k),\quad \beta_0=\ln(\sigma_{k+1}/\sigma_{k+\frac{1}{2}}). \tag{47}$$

In later sections we refer to this as the midpoint rule with square root or logarithmic interpolation. We cannot use the logarithmic interpolation at the top of the mesh since $\sigma_{k+1}=0$, so we always average at the top.

The second quadrature formula is a variant of Simpson's rule and is more accurate than the first. To compute the integral in the interval $[s_k,s_{k+\frac{1}{2}}]$ and also $[s_{k+\frac{1}{2}},s_{k+1}]$, we use values at the three points $s_{k-\frac{1}{2}}$, $s_{k+\frac{1}{2}}$, $s_{k+\frac{3}{2}}$. We find the quadratic $A+Bs+Cs^2$ which has the values $f_{k-\frac{1}{2}}$, $f_{k+\frac{1}{2}}$, $f_{k+\frac{3}{2}}$ at the three points and then we integrate this quadratic. This yields the approximation

$$\int_{s_k}^{z_{k+\frac{1}{2}}} f(s)ds\approx\alpha_1 f_{k-\frac{1}{2}}+\alpha_2 f_{k+\frac{1}{2}}+\alpha_3 f_{k+\frac{3}{2}}. \tag{48}$$

A similar formula is obtained for $\int f(\sigma)d(\ln\sigma)$. At the ends we use the three nearest points, so that the formula is no longer centered. We assume the integrand $f(s)$ is known at the ground level $s=0$ (or $\sigma=1$). This would be the case with a primitive equation model. With this formula we can compute values at both integer and half-integer points; no interpolation is required. Note

TABLE 2. Surface pressure variations with λ, $h_0=0$, $\delta p=13.3$ mb, $\Delta\lambda=2.5°$.

λ (deg)	−20	−17.5	−15.0	−12.5	−10.0	−7.5	−5.0	−2.5	0.0
$p^*(\lambda)$	100.40	1003.0	1003.0	1003.0	1004.0	1006.0	1008.0	1010.0	1013.0

that the s system requires values at both half-integer and integer points; the σ system at only half-integer points.

The last quadrature method uses the trapezoid rule. To evaluate the integral in the s system at the half-integer points we use

$$F_{k-\frac{1}{2}}=F_{k+\frac{1}{2}}-\frac{\Delta s}{K}(f_{k+\frac{1}{2}}+f_{k-\frac{1}{2}}), \tag{49}$$

and start from a known value of F_K:

$$F_{K-\frac{1}{2}}=F_K-\frac{\Delta s}{4}(f_K+f_{K-\frac{1}{2}}). \tag{50}$$

To compute F_k at integer points we use the midpoint quadrature rule rather than interpolation. With the σ system, the computation is similar, except the values of z are not required at integer points, but only at the half-interger points.

5. A comparison of the s and σ systems

We next show the result of a measurement of the truncation error in the pressure gradient terms [Eqs. (25) and (34)], using the finite-difference form of these equations [Eqs. (28) and (36)]. We use the parabolic temperature profile and derive the pressure or height from it. Thus, $p(z)=p_1(z)F_1(z,\lambda)$, where $F_1(z,\lambda)$ is defined in Eq. (17) and $p_1(z)$ in Eq. (21). We use the values $\delta p=13.3$ mb, $c_1=0.75$, $c_2=1.25$. The top of the atmosphere is set at 18 km ($z_T=18$). In the mountain cases we use Eq. (24) for the height field with $h_0=4.5$ km, while for the no-mountain cases we have $h_0=0$. The parameter K measures the number of vertical levels ($s_k=k/K$, $\sigma_k=k/K$) with $0\leqslant k\leqslant K$. We measure the error at a single value of λ, namely $\bar{\lambda}=\lambda_0/2$ where λ_0 is used in Eq. (24). The error in the pressure gradient is normalized according to its maximum over all λ. For the s system

$$\operatorname*{Max}_{\lambda}\left.\frac{\partial p}{\partial\lambda}\right|_z=p_1(z)\operatorname*{Max}_{\lambda}\frac{\partial F_1}{\partial\lambda}=p_1(z)m\frac{\delta p}{p_0}F_2(z). \tag{51}$$

Since this term is much smaller at the top of the atmosphere than at the bottom, our use of such a relative error weights the error at the top of the atmosphere more heavily. In the σ system the error is normalized by use of the divisor

$$\operatorname*{Max}_{\lambda}\left.\frac{g}{p_z}\frac{\partial p}{\partial\lambda}\right|_z.$$

Since this term contains the factor $1/\rho$ (or $-g/p_z$), it does not decrease with height so rapidly. The error given in the tables is the maximum over all vertical levels, i.e.,

$$\max_{\frac{1}{2}\leqslant k\leqslant K-\frac{1}{2}}|e_k|.$$

In Table 1 we show the variation with height of p, and the gradient terms

$$\frac{\partial p}{\partial\lambda}(z_1,\lambda_0),\quad \frac{g}{p_z}\frac{\partial p}{\partial\lambda}(z,\lambda_0).$$

In Table 2 we show the variation of p with λ at the surface for the no-mountain case. The parameters are as stated above. Table 2 shows the magnitude of the surface pressure wave (for $\delta p=13.3$). The amplitude of the wave is 20 mb peak to peak at $z=0$.

In Tables 3 and 4 the values of the relative error in the pressure gradient are shown for the s and σ systems. When $p(s_k,\lambda_k)$ is calculated exactly, the error drops by a factor of 4 as $\Delta\lambda$ is halved. Since $\partial z/\partial\lambda=0$ when $h_0=0$, we expect no vertical truncation error in this case. However, if we compute p by a quadrature as discussed in Section 4, we will have vertical truncation error. The results for the four methods defined in

TABLE 3. Error in s system [Eq. (28)], no mountain ($h_0=0$). Parabolic $T_2(z)$, $K=6$.

	Horizontal resolution $\Delta\lambda$			
Integration method	5°	2.5°	1.25°	0.63°
Exact	2.3 (−2)	5.7 (−3)	1.4 (−3)	3.6 (−4)
Simpson's rule	2.5 (−2)	7.7 (−3)	3.5 (−3)	2.4 (−3)
Midpoint, square-root interpolation	2.6 (−2)	9.1 (−3)	4.8 (−3)	3.8 (−3)
Trapezoid rule	3.5 (−2)	1.8 (−2)	1.4 (−2)	1.3 (−2)
Midpoint, average interpolation	3.7 (−2)	2.1 (−2)	1.7 (−2)	1.6 (−2)

TABLE 4. Error in σ system [Eq. (36)], $h_0=0$. Parabolic $T_2(z)$, $K=6$.*

Integration method	Horizontal resolution $\Delta\lambda$ 5°	2.5°	1.25°	0.63°
Exact	2.4 (−2)	6.1 (−3)	1.5 (−3)	3.8 (−4)
Simpson's rule	3.1 (−2) [2.2(−2)]	1.3 (−2) [5.7(−3)]	8.8 (−3) [1.4(−3)]	7.6 (−3) [3.5(−4)]
Midpoint, log interpolation	3.4 (−2) [2.6(−2)]	1.6 (−2) [9.1(−3)]	1.2 (−2) [4.9(−3)]	1.1 (−2) [3.8(−3)]
Trapezoid rule	2.2 (−2)	5.6 (−3)	1.9 (−3)	3.1 (−3)
Midpoint, average interpolation	3.4 (−2) [2.3(−2)]	1.6 (−2) [6.1(−3)]	1.2 (−2) [1.8(−3)]	1.1 (−2) [1.0(−3)]

* See text for discussion of terms in brackets in this and other tables.

Section 4 are listed in Tables 3 and 4. Simpson's rule seems to be slightly superior to the midpoint rule with square-root interpolation and considerably superior to the other methods. In later examples with $h_0\neq0$, Simpson's rule will be considerably superior to the midpoint rule by a factor approaching 4, so we will largely restrict our examples to Simpson's rule. However, the use of Simpson's rule in a numerical model rather than our smooth test case might produce poor results because it is higher order, using three points rather than two. It should be tested in a model.

In Table 4 for the σ system we have sometimes listed two results, one in brackets. In these cases, there was a large error at the highest mesh level near $\sigma=0$ ($k=1$); the first number is then the maximum over all levels, the number in brackets being the maximum over all levels except $k=1$. The trapezoid rule did not show this error growth at $k=1$, but in a mountain case ($h_0=4.5$ km) it demonstrated greater error than the Simpson's rule case. This behavior is probably due to the singularity in the σ system at $\sigma=0$. The geopotential is obtained from an integrand containing a $d\sigma/\sigma$ or $d(\ln\sigma)$ factor. We will have more to say about this singularity later. Unless otherwise stated, we use the logarithmic integral for z, i.e.,

$$z=H+\int\frac{p}{p_z}d(\ln\sigma).$$

In Table 5 we vary the vertical resolution for a Simpson's rule case. For $K=12$ the truncation error drops by a factor of 4 each time $\Delta\lambda$ is halved. Thus, the vertical truncation appears to be negligible down to $\Delta\lambda=0.63°$. For the σ system the effect of vertical truncation is negligible at $K=6$ provided we ignore the top level. As we will see below, this conclusion also holds for the mountain case. For the s system with $h_0=0$ (or the z system with mountains), we appear to need six vertical levels at 2.5° horizontal resolution and 12 levels at 1.25° or 0.63°.

TABLE 5. Error in s and σ systems, $h=0$. Parabolic $T_2(z)$, Simpson's rule integration.

Vertical resolution	Horizontal resolution $\Delta\lambda$ 5°	2.5°	1.25°	0.63°
		s system		
$K=6$	2.5 (−2)	7.7 (−3)	3.5 (−3)	2.4 (−3)
$K=12$	2.3 (−2)	5.7 (−3)	1.5 (−3)	3.9 (−4)
$K=24$	2.3 (−2)	5.7 (−3)	1.4 (−3)	3.6 (−4)
		σ system		
$K=6$	3.1 (−2) [2.3(−2)]	1.3 (−2) [5.7(−3)]	8.8 (−3) [1.5(−3)]	7.6 (−3) [3.5(−4)]
$K=12$	2.3 (−2)	5.9 (−3)	7.7 (−3) [1.5(−3)]	8.8 (−3) [3.6(−4)]
$K=24$	3.1 (−2) [2.3(−2)]	1.3 (−2) [6.4(−3)]	9.0 (−3) [1.9(−3)]	7.8 (−3) [9.0(−4)]

TABLE 6. Error in s system, mountain ($h_0=4.5$ km). Parabolic $T_2(z)$, $K=6$.

Integration method	Horizontal resolution $\Delta\lambda$ 5°	2.5°	1.25°	0.63°
Exact	3.0 (−1)	5.9 (−2)	2.9 (−2)	3.8 (−2)
Simpson's rule	3.1 (−1)	6.4 (−2)	2.2 (−1)	3.1 (−1)
Midpoint, square-root interpolation	8.6 (−2)	1.7 (−1)	2.4 (−1)	2.6 (−1)
Trapezoid rule	6.1 (−1)	3.9 (−1)	3.2 (−1)	3.0 (−1)
Midpoint, average interpolation	5.5 (−1)	3.3 (−1)	2.6 (−1)	2.4 (−1)

TABLE 7. Error in σ system, $h_0=4.5$ km. Parabolic $T_2(z)$, $K=6$.

Integration method	Horizontal resolution $\Delta\lambda$ 5°	2.5°	1.25°	0.63°
Exact	3.8 (−1)	1.1 (−1)	2.9 (−2)	7.4 (−3)
Simpson's rule	3.8 (−1)	1.7 (−1) [1.1(−1)]	1.1 (−1) [2.9(−2)]	1.0 (−1) [7.4(−3)]
Midpoint, log interpolation	3.8 (−1)	2.0 (−1) [1.0(−1)]	1.4 (−1) [2.5(−2)]	1.3 (−1) [4.1(−2)]
Trapezoid rule	5.1 (−1)	3.4 (−1) [1.1(−1)]	2.8 (−1) [3.0(−2)]	2.7 (−1) [1.4(−2)]
Midpoint, average interpolation	3.9 (−1)	2.0 (−1) [1.2(−1)]	1.4 (−1) [3.4(−2)]	1.3 (−1) [1.6(−2)]

TABLE 8. Error in s system, $h_0=4.5$ km. Parabolic $T_2(z)$, $K=12$.

Integration method	Horizontal resolution $\Delta\lambda$ 5°	2.5°	1.25°	0.63°
Exact	3.6 (−1)	9.9 (−2)	1.8 (−2)	7.4 (−3)
Simpson's rule	3.6 (−1)	1.0 (−1)	1.9 (−2)	6.6 (−3)
Midpoint, square-root interpolation	3.0 (−1)	3.9 (−2)	4.2 (−2)	6.4 (−2)
Trapezoid rule	4.3 (−1)	1.7 (−1)	8.9 (−2)	6.8 (−2)
Midpoint, average interpolation	4.4 (−1)	1.8 (−1)	1.0 (−1)	8.3 (−2)

If we introduce a mountain into the calculation, the error is greatly increased. This is shown in Tables 6–9. The mountain elevation is given by Eq. (24). The bracketed figures in the σ table give the error when the top level (for $K=12$ the top three levels) is neglected. From Tables 6 and 8 we see that Simpson's rule is superior to the others for the s system. From Table 10 we see that 12 vertical levels are needed in order that the horizontal and vertical truncation error should balance for $\Delta\lambda=0.63°$. If we cannot ignore the error at the top levels, then the σ system is rather poor. If we do ignore this error, then the σ system seems to require fewer points in the vertical than the s system, although the results are somewhat inconsistent. At $K=6$ Simpson's rule seems to be best, while at $K=12$ the midpoint rule with logarithmic interpolation seems to be equally good.

In Tables 11 and 12 we show the error which results when an isothermal basic atmosphere is used. That is, $p_1(z)$ in Eq. (16) is obtained from an isothermal atmosphere. The error for the σ system is about the same as it is for the parabolic $T_2(z)$; however, the s system error has increased by a factor of 2. Note from Table 13 that the s system again requires more points in the vertical

TABLE 9. Error in σ system, $h_0=4.5$ km. Parabolic $T_2(z)$, $K=12$.

Integration method	Horizontal resolution $\Delta\lambda$ 5°	2.5°	1.25°	0.63°
Exact	4.0 (−1)	1.2 (−1)	3.0 (−2)	7.6 (−3)
Simpson's rule	4.0 (−1)	1.2 (−1)	8.9 (−2) [3.0(−2)]	1.0 (−1) [7.6(−3)]
Midpoint, log interpolation	3.9 (−1)	1.1 (−1)	4.5 (−2) [2.7(−2)]	3.0 (−2) [4.2(−3)]
Trapezoid rule	4.0 (−1)	1.2 (−1)	7.5 (−2) [3.0(−2)]	6.0 (−2) [7.6(−3)]
Midpoint, average interpolation	4.0 (−1)	1.2 (−1)	4.0 (−2) [4.1(−2)]	2.5 (−2) [8.7(−3)]

TABLE 10. Error in s and σ systems, $h_0=4.5$ km. Parabolic $T_2(z)$, Simpson's rule integration.

Vertical resolution	Horizontal resolution $\Delta\lambda$ 5°	2.5°	1.25°	0.63°
		s system		
$K=6$	3.1 (−1)	6.4 (−2)	2.2 (−2)	3.1 (−2)
$K=12$	3.6 (−1)	1.0 (−1)	1.9 (−2)	6.6 (−3)
$K=24$	3.9 (−1)	1.1 (−1)	2.7 (−2)	5.0 (−3)
		σ system		
$K=6$	3.8 (−1)	1.7 (−1) [1.1(−1)]	1.1 (−1) [2.9(−2)]	1.0 (−1) [7.4(−3)]
$K=12$	4.0 (−1)	1.2 (−1)	8.9 (−2) [3.0(−2)]	1.0 (−1) [7.6(−3)]
$K=24$	4.0 (−1)	1.2 (−1)	3.1 (−2)	1.1 (−2) [7.8(−3)]

TABLE 11. Error in s system, $h_0=4.5$ km, $T(z)=T_0$, $K=6$.

Integration method	Horizontal resolution $\Delta°$ 5°	2.5°	1.25°	0.63°
Exact	3.1 (−1)	3.1 (−2)	5.3 (−2)	7.5 (−2)
Simpson's rule	3.1 (−1)	3.4 (−2)	4.9 (−2)	7.1 (−2)
Midpoint, square-root interpolation	2.3 (−1)	4.4 (−2)	1.3 (−1)	1.5 (−1)
Trapezoid rule	6.9 (−1)	4.3 (−1)	3.5 (−1)	3.3 (−1)
Midpoint, average interpolation	7.4 (−1)	4.8 (−1)	4.0 (−1)	3.8 (−1)

TABLE 12. Error in σ system, $h_0=4.5$ km. Isothermal [$T(z)=T_0$], $K=6$.

Integration method	Horizontal resolution $\Delta\lambda$ 5°	2.5°	1.25°	0.63°
Exact	4.8 (−1)	1.4 (−1)	3.7 (−2)	9.3 (−3)
Simpson's rule	4.8 (−1)	1.4 (−1)	3.7 (−2)	2.6 (−2) [9.3(−3)]
Midpoint, log interpolation	4.8 (−1)	1.4 (−1)	3.8 (−2)	1.0 (−2)
Trapezoid rule	4.8 (−1)	1.4 (−1)	3.7 (−2)	2.4 (−2) [9.3(−3)]
Midpoint, average interpolation	4.8 (−1)	1.4 (−1)	3.5 (−2)	7.6 (−3)

direction than does the σ system. With an isothermal atmosphere the σ system does not show as sharp an increase in the error at the top of the atmosphere.

In Table 15 we show the results for the σ system with a constant lapse rate atmosphere (see Section 3) of 6.5C km^{-1}. Here there is a very sharp rate of increase in the error at the top of the atmosphere for the midpoint quadrature rule. This might be due to the fact that z does not approach infinity as σ approaches zero, since

$$\lim_{\sigma\to 0} z(\sigma)=T_0/\gamma=44 \text{ km}. \tag{52}$$

However, where $K=6$, $\sigma_{5\frac{1}{2}}=18.2$ km which is far removed from this limit, and yet the error in Table 15 is very large for the midpoint rule. Unlike the isothermal case in Table 12, the midpoint rule in Table 15 shows a very large error at the top compared with Simpson's rule. In Table 14 the results for the s system with a constant lapse rate atmosphere are shown. They differ rather little from the parabolic $T_2(z)$ case shown in Table 6.

TABLE 13. Error in s system, $h_0=4.5$ km, $T(z)=T_0$, Simpson's rule.

Vertical resolution	Horizonal resolutions $\Delta\lambda$ 5°	2.5°	1.25°	0.63°
$K=6$	3.1 (−1)	3.4 (−2)	4.9 (−2)	7.1 (−2)
$K=12$	4.2 (−1)	1.1 (−1)	1.2 (−2)	1.4 (−2)
$K=24$	4.8 (−1)	1.4 (−1)	3.1 (−2)	3.3 (−3)

Table 16 shows the effect of increasing the amplitude of the pressure perturbation. We use $\delta p=33.3$ which results in a peak-to-peak amplitude of 50 mb. Comparison with Table 10 shows that the error has been halved from the 20 mb peak-to-peak perturbation. The larger perturbation is not affected as much by the cancellation caused by the hydrostatic pressure gradient.

6. Reduction of the hydrostatic variation

The large truncation error is probably caused by cancellation of the hydrostatic component in the pressure gradient. When we differentiate along an s or σ surface, the derivative contains a large component due to the vertical hydrostatic variation since the surface is not level in the mountain case. This large component must cancel out between the two terms which represent the pressure gradient since the gradient is at constant z or p. For the s system, the gradient is the difference

TABLE 14. Error in s system, $h_0=4.5$ km. Constant lapse rate [$T(z), =T_0-6.5z$], $K=6$.

Integration method	Horizontal resolution $\Delta\lambda$ 5°	2.5°	1.25°	0.63°
Exact	3.0 (−1)	4.7 (−2)	2.9 (−2)	4.9 (−2)
Simpson's rule	3.0 (−1)	5.1 (−2)	2.6 (−2)	4.5 (−2)
Midpoint, square-root interpolation	1.4 (−1)	1.2 (−1)	1.9 (−1)	2.1 (−1)
Trapezoid rule	5.8 (−1)	3.5 (−1)	2.8 (−1)	2.6 (−1)
Midpoint, average interpolation	5.6 (−1)	4.2 (−1)	3.5 (−1)	3.2 (−1)

TABLE 15. Error in σ system, $h_0=4.5$ km. Constant lapse rate $[T(z)=T_0-6.5z]$, $K=6$.

Integration method	Horizontal resolution $\Delta\lambda$ 5°	2.5°	1.25°	0.63°
Exact	4.1 (−1)	1.2 (−1)	3.1 (−2)	7.9 (−3)
Simpson's rule	4.1 (−1)	1.2 (−1)	3.2 (−2)	2.0 (−2) [7.9(−3)]
Midpoint, log interpolation	4.1 (−1) [9.2(−1)]	1.2 (−1) [7.9(−1)]	7.6 (−1) [3.1(−2)]	7.5 (−1) [7.1(−3)]
Trapezoid rule	4.1 (−1)	1.2 (−1)	3.1 (−2)	1.7 (−2) [7.9(−3)]
Midpoint, average interpolation	4.3 (−1) [9.2(−1)]	1.4 (−1) [7.9(−1)]	7.6 (−1) [1.7(−1)]	7.5 (−1) [1.5(−1)]

of two terms, each about 40 times larger than their difference. Thus, we might expect the error to increase by this factor, although it actually increases by a factor of ~12 (compare Tables 3 and 6). To reduce this error we will suggest a scheme which removes the hydrostatic pressure from each of the two terms in Eq. (25) [and Eq. (34) for the σ system].

We first consider the s system. We define a pressure perturbation by

$$p'(s,\lambda)=p(s,\lambda)-\bar{p}(s,\lambda), \tag{53}$$

where $p(s,\lambda)$ is the complete or calculated pressure and $\bar{p}$ is a typical average pressure. {In an actual model we would use $p'(\lambda,\theta,s,t)=p(\lambda,\theta,s,t)-\bar{p}(\lambda,\theta,s)$ and probably $\bar{p}=\bar{p}[\theta,z(\lambda,\theta,s)]$.} In this experiment we use an isothermal hydrostatic term, that is

$$\bar{p}(z)=\bar{p}_0\exp[-gz/(R\bar{T}_0)]. \tag{54}$$

We can easily transform to the s coordinates by using $z(\lambda,s)=H(\lambda)+s[z_T-H(\lambda)]$; thus,

$$\bar{p}(s,\lambda)=\bar{p}_0\exp[-gz(\lambda,s)/(R\bar{T}_0)]. \tag{55}$$

If we substitute $\bar{p}$ into the gradient term [Eq. (25)], we obtain exact cancellation, i.e.,

$$\left.\frac{\partial\bar{p}}{\partial\lambda}\right|_s-\left(\left.\frac{\partial\bar{p}}{\partial s}\frac{\partial z}{\partial\lambda}\right|_s\bigg/\frac{\partial z}{\partial s}\right)=\left.\frac{\partial\bar{p}}{\partial\lambda}\right|_z=0. \tag{56}$$

Therefore, we can write

$$\frac{\partial p}{\partial\lambda}-\left(\frac{\partial p}{\partial s}\frac{\partial z}{\partial\lambda}\bigg/\frac{\partial z}{\partial s}\right)=\frac{\partial p'}{\partial\lambda}-\left(\frac{\partial p'}{\partial s}\frac{\partial z}{\partial\lambda}\bigg/\frac{\partial z}{\partial s}\right). \tag{57}$$

TABLE 16. Error in s system, $h_0=4.5$ km. Parabolic $T_2(z)$, Simpson's rule, $\delta p=33.3$.

Vertical resolution	Horizontal resolution $\Delta\lambda$ 5°	2.5°	1.25°	0.63°
$K=6$	1.5 (−1)	3.1 (−2)	9.8 (−3)	1.4 (−2)
$K=12$	1.8 (−1)	4.8 (−2)	9.0 (−3)	2.4 (−3)
$K=24$	1.9 (−1)	5.4 (−2)	1.3 (−2)	2.4 (−3)

We hope the right side will have less truncation error since the two terms should not be so large.

The procedure for the σ system is similar, although more complicated. We assume we have a pressure profile $\bar{p}(z,\lambda)$ as before. Then we define $\bar{p}_*(\lambda)=\bar{p}[H(\lambda),\lambda]$. The function $\bar{z}(\sigma,\lambda)$ is obtained by solution of the system

$$\bar{p}[\bar{z}(\sigma,\lambda),\lambda]=\sigma\bar{p}_*(\lambda), \tag{58}$$

which is solved by a Newton-Raphson iteration. Then we define a perturbation from the calculated $z(\sigma,\lambda)$ by $z'(\sigma,\lambda)=z(\sigma,\lambda)-\bar{z}(\sigma,\lambda)$. We define the coefficient $\bar{c}$ by

$$\bar{c}=\frac{R\bar{T}(\sigma,\lambda)}{\bar{p}_*(\lambda)},\quad\text{where }\bar{T}=-\frac{g}{R}\frac{\bar{p}[\bar{z}(\sigma,\lambda),\lambda]}{\bar{p}_z[\bar{z}(\sigma,\lambda),\lambda]}. \tag{59}$$

Now since we have cancellation in Eq. (34), i.e.,

$$R\bar{T}\frac{\partial\bar{p}_*}{\partial\lambda}+g\frac{\partial\bar{z}}{\partial\lambda}=-\frac{g}{\bar{p}_z}\bar{p}_\lambda\bigg|_z=0, \tag{60}$$

TABLE 17. Error with "average" pressure removal, $h_0=4.5$ km. Parabolic $T_2(z)$, Simpson's rule.

Vertical resolution	Horizontal resolution $\Delta\lambda$ 5°	2.5°	1.25°	0.63°
		s system		
$K=6$	7.0 (−2)	3.7 (−2)	4.0 (−2)	4.1 (−2)
$K=12$	5.9 (−2)	1.8 (−2)	9.6 (−3)	1.1 (−2)
$K=24$	5.7 (−2)	1.6 (−2)	4.3 (−3)	2.3 (−3)
		σ system		
$K=6$	9.9 (−2)	8.0 (−2) [3.0(−2)]	9.1 (−2) [1.2(−2)]	9.4 (−2) [8.7(−3)]
$K=12$	1.4 (−1) [1.0(−1)]	1.2 (−1) [3.1(−2)]	1.1 (−1) [8.0(−3)]	1.1 (−1) [4.3(−3)]
$K=24$	1.0 (−1)	3.1 (−2)	1.3 (−2) [8.2(−3)]	1.2 (−2) [2.0(−3)]

TABLE 18. Error with "average" pressure removal, h_0+4.5 km, $T(z)=T_0-\gamma z$, Simpson's rule.

Vertical resolution	Horizontal resolution Δλ 5°	2.5°	1.25°	0.63°
		s system		
$K=6$	5.4 (−2)	2.5 (−2)	2.4 (−2)	2.5 (−2)
$K=12$	4.9 (−2)	1.4 (−2)	6.3 (−3)	6.7 (−3)
$K=24$	4.8 (−2)	1.3 (−2)	3.5 (−3)	1.5 (−3)
		σ system		
$K=6$	1.4 (−1) [8.7(−2)]	3.5 (−2) [2.7(−2)]	7.5 (−3)	1.2 (−2) [2.2(−3)]
$K=12$	2.1 (−1) [9.5(−2)]	7.3 (−2) [2.9(−2)]	2.8 (−2) [7.7(−3)]	1.6 (−2) [2.0(−3)]
$K=24$	2.6 (−1) [1.0(−1)]	6.9 (−2) [3.1(−2)]	1.9 (−2) [8.1(−3)]	8.6 (−3) [2.0(−3)]

TABLE 19. Error with "average" pressure removal, h_0=4.5 km. Parabolic $T_2(z)$, Simpson's rule, $\bar{T}_0=318$, $\bar{P}_0=983$.

Vertical resolution	Horizontal resolution Δλ 5°	2.5°	1.25°	0.63°
		s system		
$K=6$	8.8 (−2)	3.8 (−2)	2.6 (−2)	2.3 (−2)
$K=12$	7.9 (−2)	2.4 (−2)	9.8 (−3)	7.5 (−3)
$K=24$	7.9 (−2)	2.2 (−2)	6.0 (−3)	2.5 (−3)
		σ system		
$K=6$	1.1 (−1) [4.1(−2)]	1.3 (−1) [1.3(−2)]	1.3 (−1) [1.1(−2)]	1.3 (−1) [1.0(−2)]
$K=12$	4.3 (−2)	1.5 (−2) [1.3(−3)]	1.3 (−2) [5.1(−3)]	1.8 (−2) [4.2(−3)]
$K=24$	4.4 (−2)	1.4 (−2)	7.6 (−3) [3.5(−3)]	9.0 (−3) [1.4(−3)]

we may write Eq. (34) as

$$RT\frac{\partial p_*}{\partial\lambda}+g\frac{\partial z}{\partial\lambda}=RT\frac{\partial p'_*}{\partial\lambda}+R(T-\bar{T})\frac{\partial\bar{p}_*}{\partial\lambda}+g\frac{\partial z'}{\partial\lambda}, \quad (61)$$

where

$$T(\sigma,\lambda)=-\frac{g}{R}\frac{p[z(\sigma,\lambda),\lambda]}{p_z[z(\sigma,\lambda),\lambda]}, \quad (62)$$

$$z'=z-\bar{z}, \quad (63)$$

$$p'_*=p-\bar{p}. \quad (64)$$

The results using Eqs. (57) and (61) are given in Tables 17–19. Table 17 should be compared with Table 10, and Table 18 with Tables 14 and 15. Removal of an "average" pressure clearly results in a considerable improvement at low resolution, i.e., at 5° and 2.5°. This improvement might not prevail in a real model since the vertical variation would probably be much less regular. In the above cases, we used the same ground level temperature and pressure to determine $p(z,\lambda)$ and $\bar{p}(z)$, *viz.* $p_0=\bar{p}_0=1013$ and $T_0=\bar{T}_0=288$K. We also ran a case in which these parameters did not agree. The results are shown in Table 19. The error seems to be about the same.

In an actual model we could define $\bar{p}(s,\lambda)$ or $\bar{p}(\sigma,\lambda)$ to be an isothermal atmosphere whose temperature $\bar{T}(\theta)$ and ground pressure were dependent on latitude (and longitude if desired). Then $\bar{p}$ (or $\bar{z}$) would no longer cancel out the pressure term but could be differentiated as accurately as the $\bar{T}(\theta)$ function could be differentiated.

7. Error in the σ system

We made a somewhat unsuccessful attempt to understand the precise reason for the error increase at the top of the atmosphere in the σ system. It is probably due to the logarithmic singularity in the height field $z(\sigma,\lambda)$ at $\sigma=0$. The $z(\sigma,\lambda)$ field must be differenced to approximate $\partial z/\partial\lambda$ and any error in z due to the quadrature method will be amplified by this difference. One would expect the relative error in z to increase near $\sigma=0$ due

TABLE 20. Relative error in z, h_0=4.5 km, $K=6$, Simpson's rule.

$T(z)$	k = 1	2	3	4	5	6
Parabolic $T_2(z)$	1.8 (−3)	1.1 (−4)	8.5 (−6)	4.9 (−6)	3.6 (−6)	1.2 (−6)
T_0	1.8 (−4)	2.2 (−5)	3.7 (−8)	1.2 (−8)	6.1 (−9)	1.4 (−9)
$T_0-\gamma z$	5.4 (−5)	4.4 (−6)	5.6 (−7)	1.7 (−7)	7.2 (−8)	1.4 (−8)
Parabolic $T_2(z)$ using $d\sigma/\sigma$	3.7 (−2)	1.7 (−2)	1.9 (−3)	5.9 (−4)	2.6 (−4)	5.6 (−5)

TABLE 21. Error in σ system, $h_0=4.5$ km. Parabolic $T_2(z)$, using $d\sigma/\sigma$ in z quadrature.

Integration method	5°	2.5°	1.25°	0.63°
		Horizontal resolution $\Delta\lambda$		
		$K=6$		
Simpson's rule	3.9 (−1)	2.1 (−1) [1.1(−1)]	1.5 (−1) [2.9(−2)]	1.4 (−1) [1.0(−2)]
Midpoint, log interpolation	3.7 (−1)	1.7 (−1) [9.7(−2)]	1.1 (−1) [4.9(−2)]	9.8 (−2) [6.5(−2)]
Trapezoid rule	3.8 (−1)	1.7 (−1) [1.1(−1)]	1.1 (−1) [2.9(−2)]	9.6 (−2) [3.1(−2)]
Midpoint, average interpolation	3.9 (−1)	1.7 (−1) [1.2(−1)]	1.1 (−1) [3.3(−2)]	9.8 (−2) [2.9(−2)]
		$K=12$		
Simpson's rule	4.0 (−1)	1.2 (−1)	3.9 (−2) [3.0(−2)]	2.4 (−2) [7.6(−3)]
Midpoint, log interpolation	3.9 (−1)	1.1 (−1)	3.8 (−2) [2.6(−2)]	2.3 (−2) [1.8(−2)]
Trapezoid rule	4.0 (−1)	1.2 (−1)	3.0 (−2)	1.8 (−2) [1.2(−2)]
Midpoint, average interpolation	4.0 (−1)	1.2 (−1)	3.4 (−2) [3.1(−2)]	1.9 (−2) [1.1(−2)]

to the $d(\ln\sigma)$ or $d\sigma/\sigma$ term in the integrand used to compute z. This is the case, but this error does not seem to be correlated with the error in the pressure gradient. Table 20 shows the relative error in the value of $z(\sigma_k,\bar{\lambda})$ computed by Simpson's rule. The relative error is defined by

$$[z(\sigma_k,\bar{\lambda})-z_e(\sigma_k,\bar{\lambda})]/[z_e(\sigma_k,\bar{\lambda})],$$

where z_e is the exact value. The error in z for the parabolic $T_2(z)$ profile is greater than that for a constant lapse rate, $T(z)=T_0-\gamma z$; and the error in the pressure gradient at the top level ($k=1$) is greater for the parabolic $T(z)$. However, we also ran a case in which z was computed from the integral

$$z=H+\int_\sigma^1 \frac{p}{p_z}d\sigma/\sigma, \tag{65}$$

rather than

$$z=H+\int_\sigma^1 \frac{p}{p_z}d(\ln\sigma). \tag{66}$$

The error in z for this case is given in the bottom row of Table 20. It is considerably greater than for the integrand using $d(\ln\sigma)$. The error in the pressure gradient using $d\sigma/\sigma$ is given in Table 21. This should be compared with Tables 7 and 9, where $d(\ln\sigma)$ was used. The error in the pressure gradient for $K=6$ is about the same for both forms of the integral, but the error in z is 20 times as large for the $d\sigma/\sigma$ form. Thus, the quadrature error for z does not provide a complete explanation for the error growth in the pressure gradient near $\sigma=0$.

If we compare Table 21 at $K=12$ with Table 9, we see that use of $d(\ln\sigma)$ does not always produce more accurate results than $d\sigma/\sigma$. For Simpson's rule at $\Delta\lambda=1.25°$ and $\Delta\lambda=0.63$, the $d\sigma/\sigma$ form seems to be superior.

APPENDIX

List of Symbols

γ	lapse rate
δp	Eq. (15)
θ	latitude
λ	longitude
σ	vertical coordinate [$=p/p_*$]
c_1	Eq. (15)
c_2	Eq. (15)
$F_1(z,\lambda)$	pressure perturbation [Eq. (15)]
$F_2(z)$	Eq. (15)
g	gravitational constant
h_0	maximum surface elevation [Eq. (24)]
H	surface elevation [Eq. (24)]
p	pressure
p_*	surface pressure
p_0	pressure constant [Eq. (15)]
$p_1(z)$	basic pressure profile [Eq. (15)]
R	gas constant
s	vertical coordinate [$=(z-H)/(z_T-H)$]
T	temperature
$T_2(z)$	parabolic temperature profile
T_0	temperature constant [Eq. (20)]
$\bar{T}_0$	temperature constant
z	vertical coordinate
z_1	Eq. (15)

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