Some Considerations on Vertical Differencing*)

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Abstract

A suitable way of distributing variables over the vertical grid points is studied from the standpoint of proper simulation of vertical wave propagations as well as eliminating any computational modes in a discrete model.

Based on the above considerations, a vertical differencing scheme is derived by use of Arakawa's conserving scheme. Some arbitrary factors in that scheme are eliminated by requiring a proper simulation of internal waves and accurate hydrostatic relations.

1. Introduction

In modeling a finite difference analog of (rotating) fluid motions, it is one of the most important points for the difference model to simulate dispersion characteristics of the (rotating) fluid properly. It is well-known that there exist two well-separable types of waves in the primitive equations under normal conditions. One is low-frequency quasi-geostrophic motion; and the other is high-frequency gravity-inertia waves. Because there is a considerable difference in the propagation speed into the horizontal direction between them in the extra-tropical latitudes, quasi-geostrophic balance is usually attained in those latitudes after the dispersion of gravity-inertia waves, and this process is called "geostrophic adjustment."

In designing a horizontal differencing scheme of UCLA-GCM (see Arakawa, 1972; Arakawa and Mintz, 1974*; or Arakawa and Lamb, 1976**), location of variables in the horizontal plane was chosen in such a way as to simulate "geostrophic adjustment" process properly, based on the study by Winninghoff (1968).

Waves, however, propagate into the vertical direction as well as into the horizontal direction under favorable conditions. It is known that most of the stratospheric wave motions are responsing rather passively to the forcings from the troposphere. For example, sudden warming, which is one of the characteristic features in the winter stratosphere, is explained satisfactorily by the propagation of quasi-stationary ultra-long waves into the stratosphere (Matsuno, 1971). Quasi-biennial oscillations observed in the equatorial lower stratosphere have also close connections with the propagation of equatorial waves (Wallace and Kousky, 1968; Maruyama, 1969; Kousky and Wallace, 1971). Holton and Lindzen (1972) have presented a theory to explain the quasi-biennial oscillations by acceleration and deceleration of the mean zonal flow due to equatorial waves propagated from below.

Therefore a difference model, especially when it is used for the simulation of the stratosphere, should have good characteristics in simulating dispersion processes in the vertical direction. Thus we study, in section 2, a suitable way of distributing variables over the vertical gird points in simulating dispersion processes into the vertical direction by use of a discrete model. Whether a scheme is free from any computational modes or not has also been studied in that section.

How we treat the upper boundary condition also matters in describing vertical wave propagations, and is one of the important problems left in designing numerical atmospheric models. However, that problem can be treated separately from the present study. Some trials to improve the treatment of the upper boundary condition have been made and will be discussed in a separate paper by Tokioka and Arakawa.

*) The main part of this study was performed while the author was a visitor to the Department of Atmospheric Sciences, UCLA, Los Angeles, Calif.

* This paper is abbreviated as "AM" in the following.

** This paper is abbreviated as "AL" in the following.

In section 3, we construct a vertical differencing scheme based on the best choice of location of variables in the vertical direction, studied in section 2. The scheme is derived to conserve momentum, kinetic energy, potential enthalpy and $F(\theta)$, *i.e.*, an arbitrary function of potential temperature θ, in the vertical advective process as well as mass itself. Hydrostatic relation system is derived to conserve total energy. In constructing a conserving scheme, the way introduced by Arakawa (1972) (or see AM or AL) has closely been followed.

There still remain several arbitrary factors in the vertical differencing scheme thus constructed. Some of them are eliminated by requiring proper simulation of waves in an isothermal atmosphere. This is discussed in section 4. Still remaining arbitrary factors are used to improve accuracy of the hydrostatic relations. Some trials for that have been discussed in section 5.

2. Vertical gridding and characteristics of vertical wave propagations

We study characteristics of the vertical wave propagations in a very simplified situation where there is no basic flow and the basic temperature is uniform in the horizontal direction.

By use of the Boussinesq approximation, we may write down

$$\rho-(\rho_{00}+\rho_0)=-\frac{\mu}{g}[\theta-(\theta_{00}+\theta_0)], \tag{1}$$

where ρ is density, θ: potential temperature, ρ_{00} and θ_{00}: representative value of ρ and θ, ρ_0 and θ_0: basic field of ρ and θ deviated from ρ_{00} and θ_{00} respectively, g: acceleration due to gravity, and μ is a constant. Then a linearized system of equations is written;

$$i\nu\hat{u}-2\Omega\sin\phi\hat{v}=-\frac{is}{a\cos\phi}\cdot\frac{\hat{p}}{\rho_{00}}. \tag{2}$$

$$i\nu\hat{v}+2\Omega\sin\phi\hat{u}=-\frac{1}{a}\frac{\partial}{\partial\phi}\left(\frac{\hat{p}}{\rho_{00}}\right), \tag{3}$$

$$0=-\frac{\partial\hat{p}}{\partial z}-\hat{\rho}g, \tag{4}$$

$$i\nu\hat{\theta}+\frac{d\theta_0}{dz}\hat{w}=0, \tag{5}$$

$$\frac{is}{a\cos\phi}\hat{u}+\frac{1}{a\cos\phi}(\cos\phi\hat{v})_\phi+\frac{\partial\hat{w}}{\partial z}=0, \tag{6}$$

$$\hat{\rho}=-\frac{\mu}{g}\hat{\theta}, \tag{7}$$

where an infinitesimal quantity q' deviated from the basic state is assumed the following form;

$$q'=R_e[\hat{q}(\phi,z)e^{i(\nu t+s\lambda)}], \tag{8}$$

where λ is longitude, ϕ: latitude, z: height, ν: frequency, s: zonal wavenumber, a: the radius of the earth, Ω: angular velocity of the earth, u, v and w are velocity components in λ, ϕ and z directions respectively, and p is pressure. Eliminating $\hat{u}$ and $\hat{v}$ among (2), (3) and (6), we obtain

$$L\left(i\nu\frac{p}{\rho_{00}}\right)=4a^2\Omega^2\frac{\partial\hat{w}}{\partial z}, \tag{9}$$

where

$$L=-\frac{d}{d\xi}\left(\frac{1-\xi^2}{f^2-\xi^2}\frac{d}{d\xi}\right)+\frac{1}{f^2-\xi^2}\times\left(\frac{s}{f}\cdot\frac{f^2+\xi^2}{f^2-\xi^2}+\frac{s^2}{1-\xi^2}\right), \tag{10}$$

$$\xi=\sin\phi,\quad f=\nu/2\Omega.$$

From (4), (5) and (7), we obtain

$$i\nu\frac{\partial\hat{p}}{\partial z}+\mu\frac{d\theta_0}{dz}\hat{w}=0. \tag{11}$$

Finally from (9) and (11),

$$\frac{\mu\dfrac{d\theta_0}{dz}}{\rho_{00}}\cdot L(\hat{w})+4a^2\Omega^2\frac{\partial^2\hat{w}}{\partial z^2}=0. \tag{12}$$

Because the operator L depends only on ϕ, we may assume a solution of the form,

$$\hat{w}=W(z)\cdot G(\phi). \tag{13}$$

After introducing a separation constant ϵ, defined by

$$L(G)=\epsilon G, \tag{14}$$

(12) can be written in the following form;

$$\frac{d^2W}{dz^2}+n^2W=0, \tag{15}$$

where

$$n^2=\frac{\epsilon}{4a^2\Omega^2}\cdot\frac{\mu\dfrac{d\theta_0}{dz}}{\rho_{00}}=\frac{\epsilon}{4a^2\Omega^2}\left(\frac{-g\dfrac{d\rho_0}{dz}}{\rho_{00}}\right). \tag{16}$$

The separation constant ϵ, or Lamb's parameter, is often written as $4\Omega^2a^2/gh$, and h is called as equivalent depth in the tidal wave theory. Eq. (14) is called as the horizontal structure equation. Eq. (15) describes wave propagations, when n^2 is positive, and is called as the vertical structure equation.

The solution of Eq. (15) is written,

$$W=Ae^{inz}+Be^{-inz}, \tag{17}$$

provided that n is constant with height.

Now let us derive a difference analog of (15) for a discrete model. There are eight choices of distribution of variables over the vertical grid points, provided that we define u and v at the same level. They are shown in Fig. 1.

Because the basic temperature is constant in

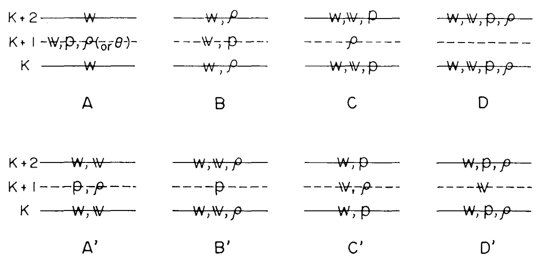

Fig. 1 Eight cases of distribution of variables over the vertical levels $K-1$, K, and $K+1$. $\mathbb{V}$, w, ρ and p indicate horizontal velocity vector, vertical velocity, density and pressure respectively.

the horizontal plane, we may assume separation of variables from the beginning and may write down relevant finite difference equations as follows;

<Scheme A>;

$$\left.\begin{aligned} & i\nu\hat{\rho}_{k+1}-\frac{1}{2}\cdot S(\hat{W}_{k+2}+\hat{W}_k)=0\,, \\ & (\hat{p}_{k+1}-\hat{p}_{k-1})/\Delta Z=-\frac{g}{2}(\hat{\rho}_{k+1}+\hat{\rho}_{k-1})\,, \\ & i\nu\hat{p}_{k+1}/\rho_{00}=\frac{gh}{\Delta Z}(\hat{W}_{k+2}-\hat{W}_k)\,, \end{aligned}\right\} \quad (18)$$

<Scheme B>;

$$\left.\begin{aligned} & i\nu\hat{\rho}_k-S\hat{W}_k=0\,, \\ & (\hat{p}_{k+1}-\hat{p}_{k-1})/\Delta Z=-g\hat{\rho}_k\,, \\ & i\nu\hat{p}_{k+1}/\rho_{00}=\frac{gh}{\Delta Z}(\hat{W}_{k+2}-\hat{W}_k) \end{aligned}\right\} \quad (19).$$

<Scheme C>;

$$\left.\begin{aligned} & i\nu\hat{\rho}_{k+1}-\frac{1}{2}S(\hat{W}_{k+2}+\hat{W}_k)=0\,, \\ & (\hat{p}_{k+2}-\hat{p}_k)/\Delta Z=-g\hat{\rho}_{k+1}\,, \\ & i\nu p_k/\rho_{00}=\frac{gh}{2\Delta Z}(\hat{W}_{k+2}-\hat{W}_{k-2})\,, \end{aligned}\right\} \quad (20)$$

<Scheme D>;

$$\left.\begin{aligned} & i\nu\rho_k-S\hat{W}_k=0\,, \\ & (\hat{p}_{k+2}-\hat{p}_k)/\Delta Z=-\frac{g}{2}(\hat{\rho}_{k+2}+\hat{\rho}_k)\,, \\ & i\nu\hat{p}_k/\rho_{00}=\frac{gh}{2\Delta Z}(\hat{W}_{k+2}-\hat{W}_{k-2}) \end{aligned}\right\} \quad (21)$$

<Scheme A′>;

$$\left.\begin{aligned} & i\nu\hat{\rho}_{k+1}-\frac{S}{2}(\hat{W}_{k+2}+\hat{W}_k)=0\,, \\ & (\hat{p}_{k+1}-\hat{p}_{k-1})/\Delta Z=-\frac{g}{2}(\hat{\rho}_{k+1}+\hat{\rho}_{k-1})\,, \\ & \frac{1}{2}i\nu(\hat{p}_{k+1}+\hat{p}_{k-1})/\rho_{00}=\frac{gh}{2\Delta Z}(\hat{W}_{k+2}-\hat{W}_{k-2})\,, \end{aligned}\right\} \quad (22)$$

<Scheme B′>;

$$\left.\begin{aligned} & i\nu\hat{\rho}_k-S\hat{W}_k=0\,, \\ & (\hat{p}_{k+1}-\hat{p}_{k-1})/\Delta Z=-g\hat{\rho}_k\,, \\ & \frac{1}{2}i\nu(\hat{p}_{k+1}+\hat{p}_{k-1})/\rho_{00}=\frac{gh}{2\Delta Z}(\hat{W}_{k+2}-\hat{W}_{k-2}) \end{aligned}\right\} \quad (23)$$

<Scheme C′>;

$$\left.\begin{aligned} & i\nu\hat{\rho}_{k+1}-\frac{1}{2}S(\hat{W}_{k+2}+\hat{W}_k)=0\,, \\ & (\hat{p}_{k+2}-\hat{p}_k)/\Delta Z=-g\hat{\rho}_{k+1}\,, \\ & \frac{1}{2}i\nu(\hat{p}_{k+2}+\hat{p}_k)/\rho_{00}=\frac{gh}{\Delta Z}(\hat{W}_{k+2}-\hat{W}_k) \end{aligned}\right\} \quad (24)$$

<Scheme D′>;

$$\left.\begin{aligned} & i\nu\hat{\rho}_k-S\hat{W}_k=0\,, \\ & (\hat{p}_{k+2}-\hat{p}_k)/\Delta Z=-\frac{g}{2}(\hat{\rho}_{k+2}+\hat{\rho}_k)\,, \\ & \frac{1}{2}i\nu(\hat{p}_{k+2}+\hat{p}_k)/\rho_{00}=\frac{gh}{\Delta Z}(\hat{W}_{k+2}-\hat{W}_k)\,, \end{aligned}\right\} \quad (25)$$

where

$$S=-\frac{d\rho_0}{dz}=\mu\frac{d\theta_0}{dz}\Big/g\,.$$

In the above schemes, we have used an arithmetic mean to interpolate physical variables at the undefined vertical levels, and have assumed that ΔZ and S are constant in height.

Eliminating $\hat{\rho}$ and $\hat{p}$ in each scheme, we obtain the following difference analogs of the vertical structure equation;

<Scheme A, C′, D′>

$$\frac{1}{\Delta Z^2}(\hat{W}_{k+2}-2\hat{W}_k+\hat{W}_{k+2}) + \frac{n^2}{4}(\hat{W}_{k+2}+2\hat{W}_k+\hat{W}_{k-2}) =0\,, \quad (26)$$

<Scheme B>

$$\frac{1}{\Delta Z^2}(\hat{W}_{k+2}-2\hat{W}_k+\hat{W}_{k-2})+n^2\hat{W}_k=0\,, \quad (27)$$

<Scheme C, D, B′>

$$\frac{1}{\Delta Z^2}(\hat{W}_{k+4}-\hat{W}_{k+2}-\hat{W}_k+\hat{W}_{k-2}) +n^2(\hat{W}_{k+2}+\hat{W}_k)=0\,, \quad (28)$$

<Scheme A′>

$$\frac{1}{\Delta Z^2}(\hat{W}_{k+4}-\hat{W}_{k+2}-\hat{W}_k+\hat{W}_{k-2}) +\frac{n^2}{4}(\hat{W}_{k+4}+3\hat{W}_{k+2}+3\hat{W}_k +\hat{W}_{k-2})=0\,, \quad (29)$$

where

$$n^2=\frac{1}{gh}\cdot\frac{gS}{\rho_{00}}\left(=\frac{\epsilon}{4a^2\Omega^2}\cdot\frac{\mu\dfrac{d\theta_0}{dz}}{\rho_{00}}\right)$$

Assuming the form of the solutions as

$$\hat{W}_k=e^{iN\cdot k\cdot \Delta Z/2}\,, \tag{30}$$

we obtain the following solutions to Eqs. (26)~(29);

In <Scheme A, C′, D′>,

$$N\Delta Z=\pm\tan^{-1}\left[\frac{4n\Delta Z}{4-n^2\Delta Z^2}\right], \tag{31}$$

<Scheme B>,

$$N\Delta Z=\pm\tan^{-1}\left[\frac{n\Delta Z(4-n^2\Delta Z^2)^{1/2}}{2-n^2\Delta Z^2}\right] \tag{32}$$

<Scheme C, D, B′>,

$$\left.\begin{aligned}&N\Delta Z=\pm\tan^{-1}\left[\frac{n\Delta Z(4-n^2\Delta Z^2)^{1/2}}{2-n^2\Delta Z^2}\right],\\ &\text{and}\\ &N\Delta Z=\pm\pi\,,\end{aligned}\right\} \tag{33}$$

<Scheme A′>,

$$\left.\begin{aligned}&N\Delta Z=\pm\tan^{-1}\left[\frac{4n\Delta Z}{4-n^2\Delta Z^2}\right],\\ &\text{and}\\ &N\Delta Z=\pm\pi\,.\end{aligned}\right\} \tag{34}$$

The solution $N\Delta Z=\pm\pi$ in Scheme C, D, A′ and B′ is a computational solution, because we do not have a corresponding solution in the continuous vertical structure equation (15). This computational solution can, however, be eliminated by use of the vertical boundary condition $W=0$.

We may have another type of computational solution in some of the above schemes, which has a tribial solution $\hat{W}_k=0$ in the vertical structure equation. For example, we may have a solution that satisfies

$$\left.\begin{aligned}\hat{\rho}_{k+1}+\hat{\rho}_{k-1}&=0\,,\\ \nu&=0\,,\end{aligned}\right\} \tag{35}$$

in Scheme A, D, A′ and D′. This solution is stationary, can have a non-zero value in ρ (and θ) that changes sign from level to level, and does not have a corresponding motion in the momentum equation.

In Scheme A, D, A′ and D′, pressure and density (or potential temperature) are defined at the same level. Therefore there is one additional freedom in density field, which allows the existence of the computational mode (35). This computational solution cannot be eliminated by use of the boundary condition $W=0$ because $\hat{W}_k=0$.

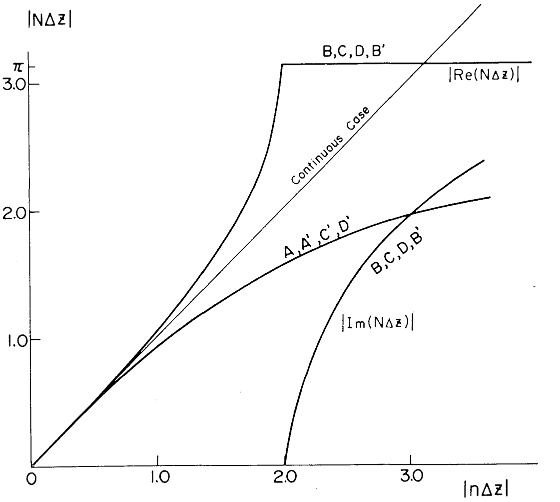

Fig. 2 The relation between $N\Delta Z$ and $n\Delta Z$ for eight schemes shown in Fig. 1. Only the physical solution in each scheme is shown in this figure. The relation in the continuous model ($N\Delta Z=n\Delta Z$) is also shown as a reference.

The relations between $N\Delta Z$ and $n\Delta Z$ are shown in Fig. 2. Computational solution is not shown in it. Physical solution in any scheme simulates true one (thin line) quite well so long as $|n\Delta Z|$ is small. However, when $|n\Delta Z|$ exceeds 2, $N\Delta Z$ becomes complex in Scheme B, C, D and B′. As the real part of $N\Delta Z$ is equal to π, damped and amplified oscillations occur in that range.

In Scheme A, A′, C′ and D′, on the other hand,

Table 1

Scheme	NΔz Physical sol.	NΔz Computational sol.	Computational mode in temperature
A	$\pm N_1\Delta z$		✓
B	$\pm N_2\Delta z$		
C	$\pm N_2\Delta z$	π	
D	$\pm N_2\Delta z$	π	✓
A′	$\pm N_1\Delta z$	π	✓
B′	$\pm N_2\Delta z$	π	
C′	$\pm N_1\Delta z$		
D′	$\pm N_1\Delta z$		✓

where

$N_1\Delta z=\tan^{-1}[4n\Delta z/(4-n^2\Delta z^2)]$,

$N_2\Delta z=\tan^{-1}[n\Delta z(4-n^2\Delta z^2)/(2-n^2\Delta z^2)]$

$N\Delta Z$ remains real for the real value of $n\Delta Z$, although there is an unavoidable error due to descretization in $N\Delta Z$ for a large value of $|n\Delta Z|$. In other words, internal waves ($n^2>0$) are treated as internal in Scheme A, A′, C′, and D′, though there is an unavoidable error in the vertical wavelength for a large value of $|n\Delta Z|$.

Fig. 2 indicates also that $|dn/dN|$ is less than 1 in Scheme B, C, D and B′. On the other hand, $|dn/dN|$ is greater than 1 in Scheme A, A′, C′ and D′. This means that the vertical group velocity in Scheme B, C, D and B′ is slower than that in the continuous model so far as only the vertical differencing is concerned; and vice versa in Scheme A, A′, C′ and D′.

Main results obtained in the above study have been summarized in Table 1. From that table, we may point out that Scheme C′ is superior to other schemes, because Scheme C′ describes internal modes as internal and there is no computational mode in the density (temperature) field. It is mentioned that Scheme C′ has been used as the basic structure for the Shuman-Hovermale models at NMC (Shuman and Hovermale, 1968) and for the Kasahara-Washington model at NCAR (Kasahara and Washington, 1967).

3. Formulation of the vertical differencing based on Scheme C’

Although the vertical differencing scheme designed by Arakawa (1972, or see AM) has good features in satisfying several integral constraints, it is essentially based on Scheme A. This defect, however, may be overcome as in the following formulation, by simply replacing $\wedge$ from the even level geopotential to the odd level geopotential in Arakawa's formulation. Most of the equations in the following formulations, therefore, are just reproduction of those in Arakawa's formulation except the definition of potential temperature (see (49)).

3.1 Vertical coordinate

We use σ-coordinate defined by

$$\sigma_k=\frac{p_k-p_{k_I}}{\pi_k}, \quad (k: \text{ even}) \tag{36}$$

where

$$\pi_k=\begin{cases} p_{k_I}-p_0 & (k<k_I) \\ p_S-p_{k_I} & (k>k_I). \end{cases}$$

p_S and p_o are pressures at the lower and upper boundaries. The vertical index is shown in Fig. 3. Above the level $k=k_I$, π_k is just a constant.

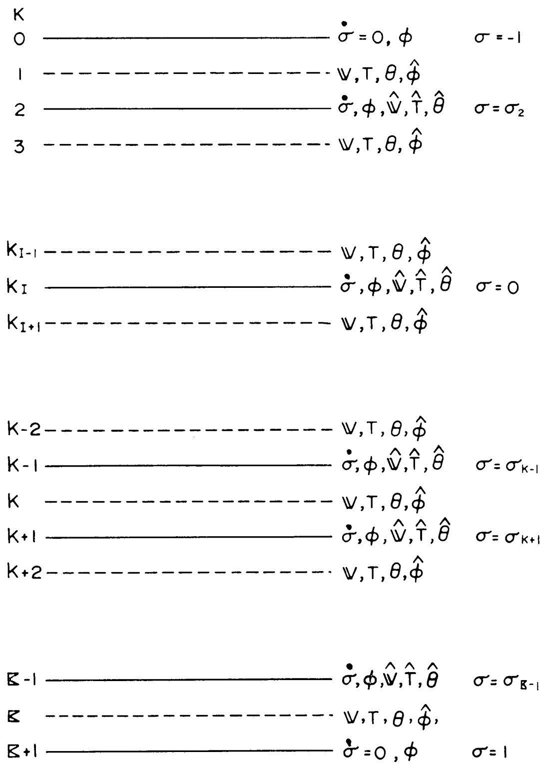

Fig. 3 Vertical index and distribution of variables based on Scheme C'. $\wedge$ is a reminder that the variable is an interpolated one.

Therefore a constant σ surface coincides with a constant pressure surface in that region. From the definition (36), $\sigma_0=-1$, $\sigma_{kI}=0$ and $\sigma_{K+1}=1$.

We define $\Delta\sigma_k$ by

$$\Delta\sigma_k=\sigma_{k+1}-\sigma_{k-1}. \tag{37}$$

3.2 The flux form of variable A

Let A_k be a variable A defined at odd level k, and define a notation;

$$\frac{D}{Dt}(\pi_k * A_k)\equiv\frac{\partial}{\partial t}(\pi_k A_k)+\nabla\cdot(\pi_k V_k A_k) + \frac{1}{\Delta\sigma_k}[(\pi\dot{\sigma})_{k+1}\hat{A}_{k+1}-(\pi\dot{\sigma})_{k-1}\hat{A}_{k-1}]. \tag{38}$$

This is the flux form of variable A, and conserves mass weighted integral of A under the vertical boundary condition $\sigma_0=\sigma_{K+1}=0$. An interpolated value of A at the even level $k+1$, $\hat{A}_{k+1}$, is defined as

$$\hat{A}_{k+1}=\frac{(G'_{k+2}A_{k+2}-G_{k+2})-(G_k'A_k-G_k)}{G'_{k+2}-G_k'} \tag{39}$$

where $G_k\equiv G(A_k)$ is an arbitrary function of variable A_k and $G' \equiv dG(A_k)/dA_k$. Arakawa (1972) has shown that we can further conserve mass weighted integral of $G(A)$ by use of (39).

3.3 The equation of continuity

By use of the definition (38), the equation of continuity is written as

$$\frac{D}{Dt}(\pi_k * 1) = 0\,. \tag{40}$$

The advective form of variable A can now be defined from (38) and (40) as

$$\frac{D}{Dt}(\pi_k * A_k) - A_k \frac{D}{Dt}(\pi_k *1) \equiv \pi_k \left(\frac{\partial}{\partial t} + V_k \cdot \nabla\right) A_k + \frac{1}{\Delta\sigma_k}[(\pi\dot{\sigma})_{k+1}(\hat{A}_{k+1} - A_k) + (\pi\dot{\sigma})_{k-1}(A_k - \hat{A}_{k-1})]\,. \tag{41}$$

3.4 The acceleration term

We write the acceleration term in the momentum equation as,

$$\frac{D}{Dt}(\pi_k * V_k)\,. \tag{42}$$

In order to conserve total kinetic energy in the process of vertical advections, we define

$$V_{k+1} = \frac{1}{2}(\hat{V}_k + V_{k+2})\,, \tag{43}$$

which is derived from (39) by setting $G(A) = A^2$.

3.5 The pressure gradient force and the kinetic energy generation

We introduce the pressure gradient force at the odd level k as

$$\nabla(\pi_k \hat{\phi}_k) - \frac{1}{\Delta\sigma_k}(\phi_{k+1}\sigma_{k+1} - \phi_{k-1}\sigma_{k-1})\nabla\pi_k\,. \tag{44}$$

If we replace $\wedge$ from the odd level geopotential to the even level one, (44) is the same as Arakawa (1972) introduced in his formulation. Because geopotentials are defined at even levels, we have to interpolate geopotential $\hat{\phi}_k$ at the odd level k. We leave the interpolation form unspecified at this stage. Arakawa (1972) has shown that the form (44), when integrated vertically, accelerates a circulation only when there is a non-horizontal boundary surface; which is one of the important integral constraints in σ-coordinate to avoid spurious acceleration through the pressure gradient force. Keeping in mind the identity $\nabla(\pi\phi) - \frac{\partial(\phi\sigma)}{\partial\sigma}\nabla\pi \equiv \pi\nabla\phi + \pi\sigma\alpha\nabla\pi$, we define $(\sigma\alpha)_k$ by

$$\pi_k(\sigma\alpha)_k = \hat{\phi}_k - \frac{1}{\Delta\sigma_k}(\phi_{k+1}\sigma_{k+1} - \phi_{k-1}\sigma_{k-1})\,, \tag{45}$$

where α is specific volume.

The kinetic energy generation is calculated, by use of (40) and (45), as follows;

$$-V_k \cdot \left[\nabla(\pi_k\hat{\phi}_k) - \frac{1}{\Delta\sigma_k}(\phi_{k+1}\sigma_{k+1} - \phi_{k-1}\sigma_{k-1})\nabla\pi_k\right] = -\nabla\cdot(\pi_k V_k \hat{\phi}_k) - \frac{1}{\Delta\sigma_k} \times \left[\left\{(\pi\dot{\sigma})_{k+1} + \sigma_{k+1}\frac{\partial\pi_k}{\partial t}\right\}\phi_{k+1} - \left\{(\pi\dot{\sigma})_{k-1} + \sigma_{k-1}\frac{\partial\pi_k}{\partial t}\right\}\phi_{k-1}\right] - \pi_k(\omega\alpha)_k\,, \tag{46}$$

where

$$(\omega\alpha)_k = (\sigma\alpha)_k\left(\frac{\partial}{\partial t} + V\cdot\nabla\right)\pi_k - \frac{1}{\pi_k\Delta\sigma_k} \times \{(\pi\dot{\sigma})_{k+1}(\phi_{k+1} - \hat{\phi}_k) + (\pi\dot{\sigma})_{k-1}(\hat{\phi}_k - \phi_{k-1})\}\,. \tag{47}$$

(47) is just a definition of $(\omega\alpha)_k$ derived from the kinetic energy generation term.

3.6 The first law of thermodynamics

We write the first law of thermodynamics as,

$$\frac{D}{Dt}(\pi_k * \theta_k) = 0\,, \tag{48}$$

to conserve mass weighted integral of potential enthalpy in an adiabatic process. θ_k in (48) is potential temperature at the odd level k. If we define temperature at that level by

$$\left.\begin{aligned} T_k &= \theta_k \cdot P_k\,, \\ P_k &= P(p_{k+1}\,,\ p_{k-1})\,, \end{aligned}\right\} \tag{49}$$

where P_k is an analog to $(p/p_{00})^\kappa$ for the layer k and p_{00} is a reference pressure, the following enthalpy equation is derived from (48);

$$\frac{D}{Dt}(\pi_k * c_p T_k) = \pi_k c_p T_k \frac{\partial \ln P_k}{\partial \pi_k} \cdot \left(\frac{\partial}{\partial t} + V\cdot\nabla\right)\pi_k + \frac{1}{\Delta\sigma_k}[(\pi\dot{\sigma})_{k+1}c_p(\hat{T}_{k+1} - P_k\hat{\theta}_{k+1}) + (\pi\dot{\sigma})_{k-1}c_p(P_k\hat{\theta}_{k-1} - \hat{T}_{k-1})]\,, \tag{50}$$

where c_p is the specific heat of the atmosphere at constant pressure. In (50), $\hat{T}_{k+1}$, an interpolated temperature at the even level $k+1$, has been introduced. The interpolation form of $\hat{T}$ is left unspecified. As for $\hat{\theta}_{k+1}$, it is defined as;

$$\hat{\theta}_{k+1} = \frac{(F'_{k+2}\theta_{k+2} - F_{k+2}) - (F'_k\theta_k - F_k)}{F'_{k+2} - F'_k}\,, \tag{51}$$

in order to conserve $F(\theta)$ besides θ (see (39)).

3.7 Total energy conservation and the hydrostatic equation

In order to conserve total energy in an adiabatic, frictionless process, the *r. h. s.* of Eq. (50) should be identical to $\pi_k(\omega\alpha)_k$, defined by

(47). Thus we require

$$c_pT_k\frac{\partial \ln P_k}{\partial \pi_k}=(\sigma\alpha)_k\,, \tag{52}$$

$$\left.\begin{aligned} &c_p(\hat{T}_{k+1}-P_k\hat{\theta}_{k+1})=\hat{\phi}_k-\phi_{k+1}\,,\\ &\text{and}\\ &c_p(P_k\hat{\theta}_{k-1}-\hat{T}_{k-1})=\phi_{k-1}-\hat{\phi}_k\,. \end{aligned}\right\} \tag{53}$$

(52) is required only for $k>k_I$, because π_k is constant above the level $k=k_I$. Combining (45) and (52), we obtain an hydrostatic equation

$$\left.\begin{aligned} &\left[(\hat{\phi}_k-\phi_{k+1})-c_pT_k\frac{\partial \ln P_k}{\partial p_{k+1}}\Delta p_k\right]\frac{\sigma_{k+1}}{\Delta\sigma_k}\\ &+\left[(\phi_{k-1}-\hat{\phi}_k)-c_pT_k\frac{\partial \ln P_k}{\partial p_{k-1}}\right.\\ &\left.\times\Delta p_k\right]\frac{\sigma_{k-1}}{\Delta\sigma_k}=0\,,\\ &\text{where}\\ &\Delta p_k=\Delta\sigma_k\cdot\pi_k=p_{k+1}-p_{k-1}\,. \end{aligned}\right\} \tag{54}$$

(53) is transformed into the following form,

$$\left.\begin{aligned} &\hat{s}_{k+1}-s_k=c_pP_k(\hat{\theta}_{k+1}-\theta_k)\\ &\text{and}\\ &s_k-\hat{s}_{k-1}=c_pP_k(\theta_k-\hat{\theta}_{k-1})\,,\\ &\text{where}\\ &s_k=c_pT_k+\hat{\phi}_k \text{ and } \hat{s}_{k+1}=c_p\hat{T}_{k+1}+\phi_{k+1}. \end{aligned}\right\} \tag{55}$$

The following form is also derived from either (53) or (55),

$$\hat{\phi}_k-\hat{\phi}_{k-2}=c_p(P_{k+2}-P_k)\hat{\theta}_{k+1}\,. \tag{56}$$

3.8 *Freedoms left in the vertical differencing scheme described above*

The scheme, described so far, conserves momentum, kinetic energy, potential enthalpy and $F(\theta)$ through the vertical advective process as well as mass itself. The total energy is conserved by use of the hydrostatic equations derived in the subsection 3.7, and no spurious acceleration of a vertical column occurs through the pressure gradient force. In the above formulations, we have not specified the followings yet;

(I) Functional form of $F(\theta)$ introduced in (51),

(II) Functional form of $P\,(p_{k+1}, p_{k-1})$ introduced in (49),

(III) Position of even levels, σ_{k+1}.

Let us suppose, here, that temperatures at all odd levels are known. Then we can calculate θ_k and $\hat{\theta}_{k+1}$ by use of (49) and (51). Multiplying $\Delta\sigma_k$ to (54), taking summation over odd values of k below the level k_I and using the relation (56), we obtain

$$\left.\begin{aligned} &\hat{\phi}_K-\phi_s=\sum_{k_{I+1}}^{K}{}' c_pa_k\theta_k-\sum_{k_{I+1}}^{K-2}{}' c_pb_{k+1}\hat{\theta}_{k+1}\,,\\ &\text{where}\\ &a_k=\left(\frac{\partial P_k}{\partial p_{k+1}}\sigma_{k+1}+\frac{\partial P_k}{\partial p_{k-1}}\sigma_{k-1}\right)\Delta p_k\,,\\ &b_{k+1}=(P_{k+2}-P_k)\sigma_{k+1}\,. \end{aligned}\right\} \tag{57}$$

ϕ_S is the surface geopotential height, and $\sum'$ indicates summation over odd values of k. The equation (57) gives thickness between the lowest odd level and the surface in terms of all θ_k and $\hat{\theta}_{k+1}$ below the level $k=k_I$. Once we know ϕ_k by use of (57), odd level geopotentials are determined by (56). Summing up (54) from the level $k=k_{I+1}$ down, even level geopotentials are determined. By use of (53), even level temperatures $\hat{T}$ are determined. However, even level geopotentials $\phi_{k+1}(k\leqslant k_I-1)$, or $\hat{T}_{k+1}(k\leqslant k_I-1)$, are left undetermined, though $\hat{s}_{k+1}$ are determined by (55). If we use (54) above the level $k=k_I$, ϕ_{k+1} and $\hat{T}_{k+1}$ $(k\leq k_I-1)$ are determined except those at the level $k=k_I$. However the use of (54) in that region is not required. Thus we add the following two as yet unspecified factors;

IV) A hydrostatic relation that determine either ϕ_{kI} or $\hat{T}_{kI}$,

V) A hydrostatic relation that determines either ϕ_{k+1} or $\hat{T}_{k+1}$ for $k<k_I-3$.

4. Reconsiderations of the characteristics of the vertical wave propagation within the framework of the vertical differencing scheme described in section 3

Based on the results obtained in section 2, Scheme C′ has been adopted in deriving the vertical differencing scheme described in the previous section. In the discussion of section 2, however, the Boussinesq approximation has been used and an arithmetic mean has been adopted as an interpolation scheme *a priori*. Therefore we consider, again, the characteristics of the vertical wave propagation in more detail, within the framework of the vertical differencing scheme described in section 3. It will be shown, in the following, that some of the freedoms mentioned in 3.8 may be utilized to guarantee proper descriptions of internal waves.

We choose the rest, isothermal atmosphere as the basic state, and consider perturbed motions of the form

$$q_k=R_e[q'_k\cdot e^{i(\nu t+s\lambda)}]\,,$$

in the domain above the level $k=k_I$. The same notations as were introduced in section 2 have been used. Because a constant σ surface coincides with a constant p surface above the level $k=k_I$, we may write down perturbation equations, in p-coordinate, as follows;

$$i\nu T'_k=\frac{T_0}{\Delta p_k}(R_k\omega'_{k+1}+S_k\omega'_{k-1}), \tag{58}$$

$$\hat{\phi}_{k+2}-\hat{\phi}'_k=-c_p(C_{k+2}T'_{k+2}+D_kT'_k), \tag{59}$$

$$i\nu\hat{\phi}'_k=\frac{gh}{\Delta p_k}(\omega'_{k+1}-\omega'_{k-1}), \tag{60}$$

where

$$\omega'_{k+1}=\dot{\sigma}'_{k+1}\cdot\pi_{k+1}.$$

Eqs. (58) and (59) are linearized forms of the thermodynamic equation (48) and the hydrostatic equation (56) respectively. The temperature T_0 in (58) is the basic temperature. The symbols R_k, S_k, C_{k+2} and D_k are defined by;

$$\left.\begin{aligned}
&R_k=1-\bar{\bar{\theta}}_{k+1}/\hat{\theta}_k,\\
&S_k=\bar{\bar{\theta}}_{k-1}/\bar{\theta}_k-1,\\
&C_{k+2}=(1-P_k/P_{k+2})\cdot\partial\bar{\bar{\theta}}_{k+1}/\partial\bar{\theta}_{k+2},\\
&\text{and}\\
&D_k=(P_{k+2}/P_k-1)\cdot\partial\bar{\bar{\theta}}_{k+1}/\partial\bar{\theta}_k,
\end{aligned}\right\} \tag{61}$$

where $\bar{\theta}_k$ and $\bar{\bar{\theta}}_{k+1}$ are basic potential temperatures at the level k and $k+1$. The symbols can be written as follows;

$$\left.\begin{aligned}
C_{k+2}=&\frac{(1-P_k/P_{k+2})F''_{k+2}}{(F'_{k+2}-F'_k)^2}[(F_{k+2}-F_k)\\
&-F'_k(\bar{\theta}_{k+2}-\bar{\theta}_k)],\\
D_k=&\frac{(P_{k+2}/P_k-1)F''_k}{(F'_{k+2}-F'_k)^2}[F'_{k+2}(\bar{\theta}_{k+2}\\
&-\bar{\theta}_k)-(F_{k+2}-F_k)],\\
R_{k+2}=&\frac{(F_{k+4}-F_{k+2})-F'_{k+4}(\bar{\theta}_{k+4}-\bar{\theta}_{k+2})}{\bar{\theta}_{k+2}(F'_{k+4}-F'_{k+2})}\\
S_k=&\frac{F'_{k-2}(\bar{\theta}_k-\bar{\theta}_{k-2})-(F_k-F_{k-2})}{\bar{\theta}_k(F'_k-F'_{k-2})}
\end{aligned}\right\} \tag{61'}$$

where the use has been made of (51), with the replacement of θ_k by $\bar{\theta}_k$. Because $\bar{\theta}_k$ is a function of pressure at even levels under the present choice of the basic state, C_{k+2}, D_k, R_{k+2} and S_k are, also, functions of pressure at even levels. Eq. (60) can be derived from (40) and (44) by use of the fact that q'_k can be separable into vertically dependent part and horizontally dependent part, because of the isothermal assumption about the basic state. The parameter h in Eq. (60) is the equivalent depth defined in a similar way as was done in section 2.

From (58), (59) and (60), we obtain the following analog of the vertical structure equation;

$$\left.\begin{aligned}
&(\omega'_{k+3}-\omega'_{k+1})-d_{k+1}(\omega'_{k+1}-\omega_{k-1})\\
&+\frac{C_{k+2}}{H}(R_{k+2}\omega'_{k+3}+S_{k+2}\omega'_{k+1})\\
&+d_{k+1}\frac{D_k}{H}(R_k\omega_{k+1}+S_k\omega'_{k-1})=0,
\end{aligned}\right\} \tag{62}$$

where

$$H=gh/c_pT_0,\quad d_{k+1}-\Delta p_{k+2}/\Delta p_k.$$

In the continuous case, the vertical structure equation has the form;

$$\frac{d^2\omega}{dp^2}+\frac{\kappa^2}{Hp^2}\omega=0,\quad \kappa=R/c_p,$$

where R is the gas constant of the dry air. The solution of the above equation is written as;

$$\omega=A_1p^{\mu_1}+A_2p^{\mu_2},$$

where

$$\mu_1=\frac{1}{2}+il,$$

$$\mu_2=\frac{1}{2}-il,$$

$$l=\left(\frac{\kappa^2}{H}-\frac{1}{4}\right)^{1/2}.$$

Therefore we assume a solution of the form

$$\omega'_{k+1}=p^{\mu}{}_{k+1}, \tag{63}$$

to Eq. (62), and require the followings;

i) two solutions for μ,

ii) $Re(\mu)=\frac{1}{2}$,

iii) $Im(\mu)$ is constant and not a function of pressure.

By substituting (63) into (62), we may rewrite (62) as follows;

$$a_{k+1}X^{2+q}+b_{k+1}X+c_{k+1}=0, \tag{64}$$

where

$$\begin{aligned}
&X=(p_{k+1}/p_{k-1})^{\mu},\\
&q=ln(p_{k+3}p_{k-1}/p^2_{k+1})/ln(p_{k+1}/p_{k-1}),\\
&a_{k+1}=1+C_{k+2}R_{k+2}/H,\\
&b_{k+1}=-1-d_{k+1}+(C_{k+2}S_{k+2}+d_{k+1}D_kR_k)/H,\\
&c_{k+1}=d_{k+1}(1+D_kS_k/H).
\end{aligned}$$

The coefficients a_{k+1}, b_{k+1} and c_{k+1} in (64) depend only on even level pressures but not on μ. In order to satisfy the condition i), therefore, Eq. (64) should be a second order equation with respect to X, *i.e.*, $q=0$. We may rewrite the condition $q=0$ as follows;

$$\frac{p_{k+3}}{p_{k+1}}=\frac{p_{k+1}}{p_{k-1}}\left(=d\right). \tag{65}$$

When even levels are located in such a way as to satisfy (65), *i.e.*, in equal interval in lnp, d_{k+1}

becomes a constant and is equal to d, and the solutions of (64) are written as follows;

$$X=d^{\mu}=\frac{1}{2(1+C_{k+2}R_{k+2}/H)}[(d+1)-(C_{k+2}S_{k+2}+dD_kR_k)/H \pm \sqrt{\{(d+1)-(C_{k+2}S_{k+2}+dD_kR_k)/H\}^2-4d(1+C_{k+2}R_{k+2}/H)(1+D_kS_k/H)}]\,, \tag{66}$$

By use of (66), the conditions ii) and iii) are now written respectively as;

$$C_{k+2}R_{k+2}=D_kS_k\,, \tag{67}$$

and

$$\frac{(d+1)-\frac{1}{H}(C_{k+2}S_{k+2}+dD_kR_k)}{1+\frac{C_{k+2}R_{k+2}}{H}} \text{ is constant in height.} \tag{68}$$

We have not specified the functional form of P_k, introduced in (49), yet. However, it may not be unreasonable to assume, *a priori*, that P_{k+2}/P_k is a function of d under the condition (65), *i.e.*, $Q(d)$. Then the condition (67) is transformed, by use of (61)$'$, into

$$\frac{F''_{k+2}}{F''_k}\times\frac{F'_k-F'_{k-2}}{F'_{k+4}-F'_{k+2}} \times\frac{F'_{k+4}\bar{\theta}_{k+4}(1-Q)-(F_{k+4}-F_{k+2})}{F'_{k+2}\bar{\theta}_{k+2}(1-Q)-(F_{k+2}-F_k)} \times\frac{F'_k\bar{\theta}_{k+2}(1-Q)-(F_{k+2}-F_k)}{F'_{k-2}\bar{\theta}_k(1-Q)-(F_k-F_{k-2})}=1\,. \tag{69}$$

This is a differential equation for F_k. Although it is hard to solve it, it is easy to test a particular form of F_k. If we substitute a form $F_k(\bar{\theta})=(\bar{\theta}_k)^m$; $m\neq 0$, 1, (69) is not satisfied. On the other hand,

$$F_k(\bar{\theta})=ln\bar{\theta}_k \tag{70}$$

satisfies it.

We do not know whether (70) is a unique solution of (69) or not. However, the solution gives us several additional advantages (see AM or AL). First of all, entropy is conserved by use of (70), because the interpolation form (51) gives us a conservation of F. Secondly, the hydrostatic relation (56) gives exact thickness for a wide range of stratification including isothermal and isentropic cases. Furthermore, the statistical distribution of mass in potential temperature space is maintained well through the conservation of entropy.

As for the remaining condition (68), we can show, by use of (61)$'$, that the condition is automatically satisfied when F_k is given by (70).

In summary, if we choose even levels to satisfy (65), *i.e.*, in equal interval in lnp, and use (70) in the interpolation formula (51), within the framework of the vertical differencing scheme described in section 3, there exist no internal reflections due to descretization, and thus the amplitude of internal waves is described exactly in an isothermal situation.

5. Accuracy of the hydrostatic equation system

The freedoms (I) and (III), in the vertical differencing scheme described in section 3, have been used, in the previous section, for the proper description of internal waves in an isothermal atmosphere. Although the functional form (70) has been derived for the use in the region above the level $k=k_I$, it may be used in the region below that level as well, because several advantages are guaranteed by the use of it, as mentioned already. Now the remaining freedoms are (II), (IV) and (V). It will be shown, in this section, that those remaining freedoms may be utilized to improve accuracy of the hydrostatic relation.

5.1 *Improvement of the hydrostatic relation (57)*

Eq. (57) is the hydrostatic equation which determines thickness between the surface and the lowest odd level. Although the thickness should be determined by the local temperature, the right hand side of (57) depends on all potential temperatures θ_k and $\hat{\theta}_{k+1}$ below the level $k=k_I$. This is because we have used, in deriving (57), two sets of hydrostatic relations (54) and (56) which have been derived to conserve total energy, and because the accuracy of those two hydrostatic relations is not formally the same.

Suppose if we could eliminate or reduce dependency of Eq. (57) on higher level temperatures, we may regard it as a logically better analog of the hydrostatic relation.

The accuracy of (54) depends on how P_k is defined, and that of (56) on how $\hat{\theta}_{k+1}$ is defined. Because we have already defined $F(\theta)$ by Eq. (70), $\hat{\theta}_{k+1}$ has the following form;

$$\hat{\theta}_{k+1}=\frac{ln(\theta_k/\theta_{k+2})}{1/\theta_{k+2}-1/\theta_k}\,. \tag{71}$$

However, we use a linear interpolation form,

$$\hat{\theta}_{k+1}=\frac{1}{2}(\theta_k+\theta_{k+2})\,, \tag{72}$$

for the time being to facilitate an analysis.

Let us consider a polytropic atmosphere, in which

$$\vartheta_k = T_k p^{*}{}_k{}^{-a}, \tag{73}$$

is a constant ($=\vartheta$), where

$$p^{*}{}_k = p_{00} P(p_{k+1}, p_{k-1})^{1/k}. \tag{74}$$

The parameter a in (73) is just a constant, and not the radius of the earth. By use of (72) and (73), Eq. (57) is reduced to

$$\frac{1}{C_q}(\hat{\phi}_k - \phi_s) = \vartheta \sum_{k_I+1}^{K-1}{}' \left(A_k \frac{p^{*}{}_k{}^a}{P_k} - B_{k+2} \frac{p^{*}{}_{k+2}{}^a}{P_{k+2}} \right) \cdot \sigma_{k+1} + \vartheta \frac{\partial P_K}{\partial p_{K+1}} \Delta p_K \frac{p^{*}{}_K{}^a}{P_K}, \tag{75}$$

where

$$A_k = \frac{\partial P_k}{\partial p_{k+1}} \Delta p_k - \frac{1}{2}(P_{k+2} - P_k),$$

$$B_k = \frac{1}{2}(P_k - P_{k-2}) - \frac{\partial P_k}{\partial p_{k-1}} \Delta p_k.$$

Let us require

$$A_k \frac{p^{*}{}_k{}^a}{P_k} - B_{k+2} \frac{p^{*}{}_{k+2}{}^a}{P_{k+2}} = 0 \tag{76}$$

so that the surface thickness does not depend on the upper level temperature at all. Eq. (76) is rewritten as

$$\frac{\partial P_k}{\partial p_{k+1}} \cdot \frac{p^{*}{}_k{}^a}{P_k} \Delta p_k + \frac{\partial P_{k+2}}{\partial p_{k+1}} \frac{p^{*}{}_{k+2}{}^a}{P_{k+2}} \Delta p_{k+2} = \frac{1}{2}(p^{*}{}_{k+2}{}^a - p^{*}{}_k{}^a) + \frac{1}{2}(p^{*}{}_k{}^{a-\kappa} p^{*}{}_{k+1}{}^{\kappa} - p^{*}{}_k{}^{\kappa} p^{*}{}_{k+2}{}^{a-\kappa}). \tag{77}$$

The first term of the *l. h. s.* of (77) depends on p_{k+1} and p_{k-1}, but not on p_{k+3}; the second term of the *l. h. s.* of (77) depends on p_{k+3} and p_{k+1}, but not on p_{k-1}. The second term of the *r. h. s.* of (77) depends on p_{k+3}, p_{k+1} and p_{k-1} except when $a=\kappa$ or $a=2\kappa$. It is difficult to solve (77) for P_k for an arbitrary value of a, therefore we consider a special case where $a=\kappa$. (We discard $a=2\kappa$ because it gives unstable stratification). In that case, (77) reduces to;

$$\frac{\partial P_k}{\partial p_{k+1}} \Delta p_k + \frac{\partial P_{k+2}}{\partial p_{k+1}} \Delta p_{k+2} = P_{k+2} - P_k. \tag{78}$$

Under the assumption that P_k depends only on p_{k+1} and p_{k-1}, we may rewrite (78) as follows;

$$\left. \begin{aligned} \frac{\partial P_{k+2}}{\partial p_{k+1}} \Delta p_{k+2} &= P_{k+2} - E(p_{k+1}), \\ \frac{\partial P_k}{\partial p_{k+1}} \Delta p_k &= E(p_{k+1}) - P_k, \end{aligned} \right\} \tag{79}$$

where $E(p_{k+1})$ can be an arbitrary function of p_{k+1}. However, it will be reasonable to assign a functional form of E as

$$E(p_{k+1}) = (p_{k+1}/p_{00})^{\kappa}. \tag{80}$$

Then we obtain

$$P_k = \frac{p_{00}{}^{-\kappa}}{1+\kappa} \cdot \frac{p_{k+1}{}^{\kappa+1} - p_{k-1}{}^{\kappa+1}}{p_{k+1} - p_{k-1}}. \tag{81}$$

This P_k is nothing but π_k derived by Phillips (1974). By substitution of (81) into (77) without assuming $a=\kappa$, we obtain

$$\left[\left(\frac{p_{k+1}}{p_{00}} \right)^{\kappa} - \frac{1}{2}(P_k + P_{k+2}) \right] \left(\frac{p^{*}{}_{k+2}{}^a}{P_{k+2}} - \frac{p^{*}{}_k{}^a}{P_k} \right) = 0. \tag{82}$$

If we determine pressure levels to satisfy

$$\left(\frac{p_{k+1}}{p_{00}} \right)^{\kappa} = \frac{1}{2}(P_k + P_{k+2}),$$

the coefficients of σ_{k+1} in (75) vanishes regardless of the value of a, and (75) reduces to

$$\frac{1}{C_p}(\hat{\phi}_K - \phi_S) = (E(p_S) - P_K) \cdot \theta_K, \tag{84}$$

which gives us an exact hydrostatic relation, when θ is constant between p_S and p_K. Even when pressure levels are not chosen to satisfy (83), (81) guarantees an exact θ in an isentropic case. It is pointed out that an interpolation form (71), instead of (72), also gives us the same relation as (78) when $a=\kappa$. Thus the use of (71) and (81) guarantees exact θ in an isentropic case.

It is noted here that it is impossible to determine σ_{k+1} in such a way as to satisfy (83) regardless of the surface pressure, because of the definition of σ (see Eq. (36)). It is also noted that a representative stratification in the troposphere is approximated better by $a<\kappa$ rather than $a=\kappa$.

As repeatedly mentioned, the second term of the *r. h. s.* of Eq. (57), with the use of (71), yields exact thickness for the wide range of stratification. Therefore, as the next approach, we require that the first term of the *r. h. s.* of Eq. (57) should give exact thickness when the atmosphere is polytropic.

Keeping in mind that the hydrostatic relation in the polytropic atmosphere is;

$$-\frac{1}{C_p} \delta\phi = \frac{\kappa}{a} \vartheta \delta p^a, \tag{85}$$

we require

$$\frac{\partial P_k}{\partial p_{k+1}} \theta_k \Delta p_k + \frac{\partial P_{k+2}}{\partial p_{k+1}} \theta_{k+2} \Delta p_{k+2} = \frac{\kappa}{a} \vartheta (p^{*}{}_{k+2}{}^a - p^{*}{}_k{}^a). \tag{86}$$

We can solve Eq. (86) for P_k, in a similar way as was done in solving Eq. (78), as;

$$\left.\begin{aligned} p^*{}_k{}^a &= \frac{1}{1+a}\cdot\frac{p_{k+1}{}^{a+1}-p_{k-1}{}^{a+1}}{p_{k+1}-p_{k-1}}, \\ P_k &= \left(\frac{p^*{}_k}{p_{00}}\right)^{\kappa}. \end{aligned}\right\} \tag{87}$$

It is mentioned that this form includes (81) as a special case.

5.2 A hydrostatic relation that determines either ϕ_{k_I} or $\hat{T}_{k_I}$

As mentioned in subsection 3.8, ϕ_{k_I} remains undetermined even when temperature at all odd levels are known. This is caused by the fact $\sigma_{k_I}=0$. When $k=k_I+1$, (54) reduces to

$$\hat{\phi}_{k_I+1}-\phi_{k_I+2}=C_pT_{k_I+1}\frac{\partial lnP_{k_I+1}}{\partial p_{k_I+2}}\cdot \Delta p_{k_I+1}\,. \tag{88}$$

This relation gives exact thickness, between the levels k_I+1 and k_I+2, in the polytropic atmosphere provided that the use has been made of (87). Similarly, $C_pT_{k_I+1}\frac{\partial lnP_{k_I+1}}{\partial p_{kI}}\Delta p_{k_I+1}$ gives an exact thickness between the levels k_I and k_I+1 under the same condition. Therefore, we may introduce

$$\phi_{k_I}-\hat{\phi}_{k_I+1}=CpT_{k_I+1}\frac{\partial lnP_{k_I+1}}{\partial p_{k_I}}\Delta p_{k_I+1}\,, \tag{89}$$

because (89), with the use of (88), yields exact thickness between the levels k_I and k_I+2 when ϑ is constant between them.

5.3 Test of the accuracy of the hydrostatic relation system below the level $k=k_I$

The hydrostatic relations below the level $k=k_I$ now form a complete set with the inclusion of (89). They are (53), (54), (89), (49), (71) and (87). The numerical constant a introduced in (87) still remains undetermined. If we replace a by κ, (87) reduces to (81). Therefore the above closed system of hydrostatic relations give exact thickness between any two levels and exact temperature at both even and odd levels in an isentropic case.

In order to describe hydrostatic relation exactly in an isothermal case ($a=0$), we have to replace (87) by

$$ln(p_k{}^*/p_{k-1})=\frac{p_{k+1}}{p_{k+1}-p_{k-1}}\cdot ln(p_{k+1}/p_{k-1})-1\,, \tag{90}$$

by taking $\lim\limits_{a\to 0}$ (87).

In order to have some idea about a proper value of a, we will check the accuracy of the hydrostatic relations (53), (54), (89), (49), (71) and (87) for three stratifications. They are;

(I): isothermal case;

$$\begin{aligned} \phi/c_p &= 260\,\kappa z \text{ (degree)}, \\ \theta &= 260\, e^{\kappa z} \ (^\circ K), \end{aligned} \tag{91}$$

(II): normal case;

$$\begin{aligned} \phi/c_p &= 1.11\,[0.95+z\{72.43+z(z-6.9)\}] \\ &\quad \text{(degree)}, \\ \theta &= \frac{1110}{R}e^{\kappa z}\{72.43+z(3z-13.8)\} \quad (^\circ K), \end{aligned} \tag{92}$$

(III): isentropic case;

$$\begin{aligned} \phi/c_p &= 300(1-e^{-\kappa z}) \text{ (degree)}, \\ \theta &= 300 \ (^\circ K), \end{aligned} \tag{93}$$

where $z=-ln(p/p_{00})$.

The normal stratification (II) is the same as was used by Phillips (1974) in his test. The parameters used in the following test are; $\sigma_{kI}=0$, $\sigma_{kI+2}=0.111111$, $\sigma_{kI+4}=0.333333$, $\sigma_{kI+6}=0.555556$, $\sigma_{kI+8}=0.777778$, $\sigma_{kI+10}=1.0$ and $p_{00}=p_s=1{,}000$ mb. Exact geopotentials at the even levels are calculated by use of (91), (92) or (93). Those

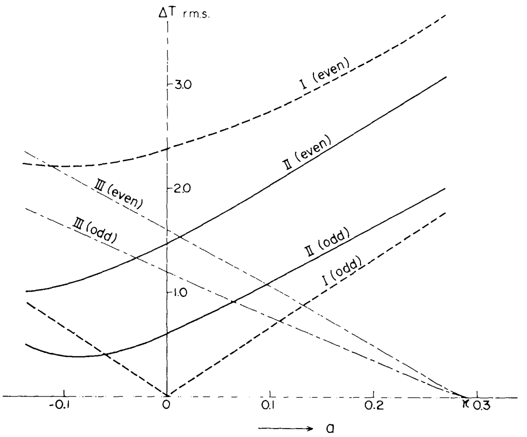

Fig. 4 The root mean square errors (in unit °K) in temperature, at odd and even levels, calculated by use of the hydrostatic equation system (53), (54), (89), (49), (71) and (87) as a function of "a" (see Eq. (87)). Even level geopotentials calculated by (91), (92) and (93) are given to the system as input. The dashed lines in the figure are the *r. m. s.* error in isothermal case (I), the solid lines in normal case (II) and the dash dotted lines in isentropic case (III). Parameters used in the calculations are; $\sigma_{k1}=0$, $\sigma_{k1+2}=0.111111$, $\sigma_{k1+4}=0.333333$. $\sigma_{k1+6}=0.555556$, $\sigma_{k1+8}=0.777778$, $\sigma_{k1+10}=1.0$ and $p_{00}=p_s=1{,}000$ mb.

values are given to the hydrostatic equation system as input, and all other thermodynamical quantities are determined by use of the hydrostatic equation system. The accuracy of the system is measured by the *r. m. s.* error in temperature at odd and even levels, and by the *r. m. s.* error in geopotential at odd levels. The parameter $p_k{}^*$ defined by (87) is used as a reference odd level pressure for the evaluation of errors at the odd level.

The *r. m. s.* errors in temperature are shown in Fig. 4 as a function of a. When $a=\kappa$, temperatures at both even and odd levels are determined exactly in isentropic case (III). In that case, errors increase almost linearly with the decrease of a.

In isothermal case (I), odd level temperature is exact when $a=0$. Errors in even level temperature decrease monotonically with the decrease of a and attains its minimum at $a\approx -0.1$.

In normal case (II), errors decrease monotonically as a decreases down to about -0.08, where the error in the odd level temperature is minimum. Although errors in temperature at both even and odd levels depend on a in every case, we may say that their dependency on a is relatively weak and that the errors themselves are not so large for the range of a shown in Fig. 4.

Fig. 5 shows the *r. m. s.* error in geopotential at odd levels. In isothermal case (I), the error is zero when $a=0$. In isentropic case (III), it is zero when $a=\kappa$. In normal case (II), *r. m. s.* error is minimum around $a=0.1$, and the value

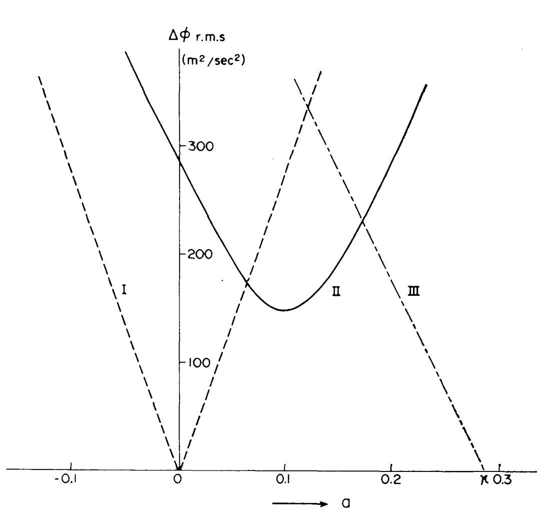

Fig. 5 The same as those in Fig. 4 except the *r.m.s.* error (in unit m^2/sec^2) in geopotential at odd levels.

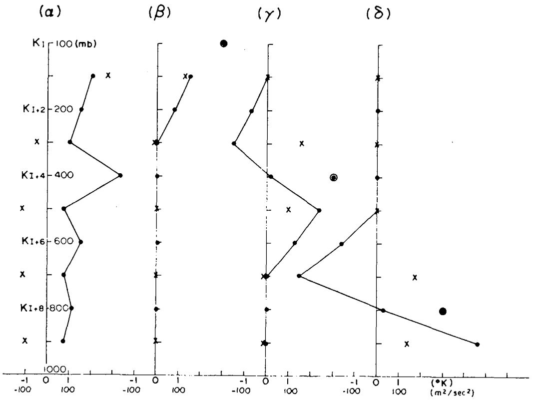

Fig. 6 (α) Vertical distribution of errors in temperature (thin line) and geopotential (crosses) for the normal case (II) when $a=0.11$. (β), (γ) and (δ) show errors deviated from that of (α) when an artificial error (double circles) is included to the input geopotential value at the level k_I, k_I+4 and k_I+8 respectively. $a=0.11$ is used in every case. The same even level pressures as those in Fig. 4 are used.

is about $150\,m^2/sec^2$ (or about 15 g.p.m.). The large value of the error does not directly cause spurious acceleration (or, deceleration) in the momentum equation, because the horizontal gradient of the errors matters.

Fig. 6 gives us some informations about the vertical distribution of errors. Fig. 6 (α) shows the vertical distribution of errors in temperatures and geopotentials in normal case (II) when $a=0.11$. Temperature errors are shown by dots connected by thin lines; and geopotential errors at odd levels, by crosses. Fig. 6 (β), (γ) and (δ) show differences in temperature and geopotential between the case where an error $\Delta\phi^*$ is added to the exact geopotential value, in normal case (II) when $a=0.11$, at an even level $k=k^*$ and the case without errors, for $k^*=k_I$, k_I+4 and and k_I+8 respectively. The characteristic value of $\Delta\phi^*=300\,m^2/sec^2$ is used in every case. We notice that the differences in odd level geopotentials are localized at the adjacent odd levels to k^*, and are about a half of $\Delta\phi^*$.

Temperature errors induced by the error in the input geopotential at the level $k=k^*$ are localized at adjacent odd and even levels. We also notice that the error in geopotential induces more serious errors in temperature, when it is included at a lower level. Fig. 6 (δ) shows the results when $\Delta\phi^*=300\,m^2/sec^2$ is included at the level k_I+8

(800 mb). The maximum error in temperature in that case is as much as 4.6°.

Although it is not easy to determine a preferable value of a from the test discussed so far, $a\sim 0.1$ may not be a bad choice, because the *r. m. s.* error in geopotential is minimum in normal case (II) and the error in temperature is not so large.

5.4 A hydrostatic relation that determines either ϕ_{k+1} or $\hat{T}_{k+1}$ for $k<k_I-3$

In the region above the level $k=k_I$, *i.e.*, $k<k_I$, we are not forced to use the hydrostatic relation (54). We may replace it with another expression which gives exact thickness in both isothermal and isentropic cases. Then, thickness between any two levels, above $k=k_I$, is exact in both isothermal and isentropic cases, because the hydrostatic relation (53), with the use of (71), also gives exact thickness in both cases.

If the following hydrostatic relation is adopted, for example, the thickness between the levels $k+1$ and $k-1$ is exact provided that $\theta_a\equiv T\cdot p^{-a}$ is constant between them;

$$\phi_{k+1}-\phi_{k-1}=-c_p\frac{\kappa}{a}\vartheta_k(p_{k+1}{}^a-p_{k-1}{}^a)\,, \quad (94)$$

where

$$\vartheta_k=T_k\cdot(p_k{}^*)^{-a}\,.$$

The symbol $p^*{}_k$ follows the definition of (74). We can further require that Eq. (94) should give exact thickness when $\vartheta_b\equiv T\cdot p^{-b}\,(a\neq b)$ is constant between the levels $k+1$ and $k-1$. In order to satisfy the above requirement, the *r. h. s.* of Eq. (94) should be equal to $-c_p(\kappa/b)\vartheta_b\,\delta(p^b)$, *i.e.*,

$$\frac{1}{a}\vartheta_k(p_{k+1}{}^a-p_{k-1}{}^a)$$

$$=\frac{1}{b}\vartheta_k(p_k{}^*)^{a-b}[p_{k+1}{}^b-p_{k-1}{}^b]\,,$$

or

$$(p_k{}^*)^{a-b}=\frac{b}{a}\cdot\frac{p_{k+1}{}^a-p_{k-1}{}^a}{p_{k+1}{}^b-p_{k-1}{}^b}\,. \quad (95)$$

The use of (94) under the condition (95) therefore guarantees exact thickness in two polytropic situations.

In the discussion of section 4, we assumed, *a priori*, that P_{k+2}/P_k is a function of *d*, *i.e.*, $Q(d)$. It is almost clear that the definition (95), with the aid of (74), causes no contradiction to that assumption.

If we replace a by κ and taking $\lim_{b\to 0}$ (95), we may obtain

$$P_k=\frac{1}{\kappa}\frac{p_{k+1}{}^\kappa-p_{k-1}{}^\kappa}{\ln p_{k+1}{}^\kappa-\ln p_{k-1}{}^\kappa}\cdot\frac{1}{p_{00}{}^\kappa}\,, \quad (96)$$

and Eq. (94) reduces to

$$\phi_{k+1}-\phi_{k-1}=-C_p\theta_k\cdot\left[\left(\frac{p_{k+1}}{p_{00}}\right)^\kappa-\left(\frac{p_{k-1}}{p_{00}}\right)^\kappa\right] \quad (97)$$

Because the thickness between the levels $k+1$ and $k-1$ is determined exactly, by use of (96) and (97), in both isothermal and isentropic cases, the thickness between any two levels is determined exactly, by use of the hydrostatic relation system above $k=k_I$ [(49), (53), (71), (96), and (97)], in both isothermal and isentropic cases.

It will be reasonable to define odd level pressure $p_k(k<k_I)$ by

$$p_k=p_k{}^*=p_{00}P_k{}^{1/\kappa}\,. \quad (98)$$

By doing so, temperatures at odd levels are also determined exactly in both isothermal and isentropic cases.

6. Summary

The followings are summarized from the present study;

i) Scheme C′ (see Fig. 1) is the best way in distributing variables over the vertical grids, because internal waves in the continuous model are treated as internal and because there are no computational modes at all in it.

ii) Vertical differencing scheme has been derived, based on Scheme C′, by use of Arakawa's conserving scheme.

Concerning the vertical differencing scheme thus derived, we may point out the followings;

iii) If we choose even levels in equal interval in *ln p* above the level $k=k_I$ and if we adopt (70) to define an interpolated potential temperature, $\hat{\theta}_{k+1}$, by (51), the amplitude of internal waves is expressed exactly, without any internal reflections due to descretization in an isothermal case.

iv) If a functional form P_k, given by (87), is introduced below the level $k=k_I$, thickness between two adjacent even levels is expressed exactly in terms of the odd level temperature when $\vartheta=T\cdot p^{-a}$ is constant in pressure. And if an additional hydrostatic relation (89) is introduced in the hydrostatic system, the thickness between the level k_I and k_I+1 is expressed exactly when ϑ is constant between them.

v) The accuracy of the hydrostatic systems below $k=k_I$ has been tested for the three cases of stratification (I), (II) and (III) (see (91), (92), and (93)). The *r. m. s.*

errors in temperature at even and odd levels and that in geopotential at odd levels are calculated as a function of a, with even level geopotentials as input. The value $a \sim 0.1$ may not be a bad choice, because the geopotential error is minimum in normal case for that value, and because the error in temperature is not so large for that value of a.

vi) It is not necessary for the conservation of total energy to use the hydrostatic equation (54) above the level $k=k_I$. An alternative form, (97) and (96), is introduced, instead of (54) and (87), in that region. If we use them, and define odd level pressure by (98), the thickness between the levels $k+1$ and $k-1$ and odd level temperature T_k are related exactly in both isothermal and isentropic cases.

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垂直差分法に関する考察

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垂直方向のグリッド上に物理変数を配列させるやり方について，波動の垂直伝播をうまく記述する観点から検討を加えた。同時に，差分化によるにせの解についても調べた。

次に，変数の最適な配列法に基づいて，プリミティブ方程式系の垂直差分式を導いた。その際，荒川(1972)によって考察された，種々の物理量を保存する方法が用いられている。こうして導かれた垂直差分式には，まだいくつかの任意な要素が残されている。それらの要素は，内部波のより良い表現のため，又静力学の関係式の精度を上げるために使う事ができる。