An Improved Numerical Approximation of the Horizontal Gradients in a Terrain-Following Coordinate System

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ABSTRACT

The use of a fine vertical grid resolution near the surface may lead to a numerically inconsistent approximation of the horizontal gradient terms in a terrain-following coordinate system. This occurs when the distance between two vertical grid points is smaller than the elevation difference between two horizontally adjacent (in the terrain-following coordinate system) points. In this paper an improved numerical procedure is proposed which eliminates this inconsistency and significantly increases the accuracy of the numerical approximation. Results are compared with those obtained with the conventional forward and centered schemes.

1. Introduction

The use of the terrain-following coordinate system (TFCS), in meteorological numerical models, has been shown to be very effective when topographic features are considered. Generally, this coordinate system is defined by using the transformation

$$x^* = x, \quad y^* = y, \quad z^* = s\frac{z - Z_G}{s - Z_G}$$

where x, y and z are the horizontal and vertical coordinates in a Cartesian coordinate system; x^*, y^*, and z^* are the horizontal and vertical coordinates in the TFCS; Z_G is the terrain height and s is a reference height (usually the top of the model). This form of a TFCS has been used in recent years in large-scale and mesoscale numerical models (e.g., Kasahara, 1974; Mahrer and Pielke, 1975; Gal-Chen and Somerville, 1975; McNider and Pielke, 1981; Yamada, 1983; among others).

It will be shown in Section 2, that when using a numerical scheme for the horizontal gradients in the TFCS, care must be taken that the distance between two vertical grid points will not be less than the elevation difference between two horizontally (in the TFCS) adjacent points. This situation is mostly pertinent in mesoscale models that use a relatively fine vertical grid resolution in the atmospheric boundary layer.

It is also worth noting that models which use the *sigma* (σ) coordinate system, where the vertical coordinate σ is defined as a function of pressure are subject to a similar minimum vertical grid size requirement. The σ coordinate system was first introduced by Phillips (1957) and has been adopted in mesoscale models by Anthes and Warner (1978), Alpert *et al.* (1982), among others.

In some of the aforementioned models, the investigators have used relatively fine grid resolution near the surface, assuming implicitly that the variables vary linearly with height within the elevation difference between two horizontally adjacent points. For example: in Mahrer and Pielke (1975) the smallest grid interval was 50 m while the maximum elevation difference between two horizontal grid points was 400 m; in Yamada (1983) the corresponding values were 2 m and 100 m, while in Gal-Chen and Somerville (1975) they were approximately 100 m and 120 m respectively.

Several investigators (e.g., Smagorinsky *et al.*, 1967; Gary, 1973; Janjic, 1977) have recognized, in the σ coordinate system, an increased truncation error in the pressure gradient terms in the presence of very steep topography. Janjic (1977) specifically pointed out the inconsistency in the pressure gradient calculation in cases of very steep slopes of surfaces and thin layers. This inconsistency, however, could be tolerated in Janjic's experiments because his smallest vertical spacing was 100 mb ($\sim$1000 m).

Although the errors associated with a TFCS are most significant in the pressure gradient terms, they may be also important in other horizontal gradient terms, mainly within the planetary boundary layer.

In the present study, it is demonstrated that by using an improved numerical expression for the horizontal gradient terms in the TFCS, errors in the horizontal gradients of the meteorological fields are reduced substantially when the vertical grid interval is very small.

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Results are compared with those obtained by the forward and centered conventional schemes.

2. Evaluation of the numerical difference scheme for the horizontal gradients in TFCS

The transform equations of the partial derivatives with respect to x and z from the Cartesian coordinate system to the TFCS are:

$$\left.\frac{\partial}{\partial x}\right|_z = \left.\frac{\partial}{\partial x}\right|_{z*} + \left.\frac{\partial z^*}{\partial x}\right|_z \frac{\partial}{\partial z^*}$$

$$= \left.\frac{\partial}{\partial x}\right|_{z*} + \frac{z^* - s}{s - Z_G}\frac{\partial Z_G}{\partial x}\frac{\partial}{\partial z^*}, \qquad (1)$$

$$\frac{\partial}{\partial z} = \frac{\partial z^*}{\partial z}\frac{\partial}{\partial z^*}. \qquad (2)$$

A commonly used finite differencing scheme for (1) in TFCS is a forward-in-space approximation, namely:

$$\left.\frac{\partial F}{\partial x}\right|_z = \frac{F(i+1,j) - F(i,j)}{\Delta x} + \left.\frac{\partial z^*}{\partial x}\right|_{i,j}\frac{F(i+1,j) - F(i+1,j-1)}{\Delta z^*}, \qquad (3)$$

where F represents the meteorological field, and

$$\left.\frac{\partial z^*}{\partial x}\right|_{i,j} = \frac{z^*(j) - s}{s - Z_G(i+1)}\frac{Z_G(i+1) - Z_G(i)}{\Delta x}.$$

As illustrated in Fig. 1, in which the vertical resolution is relatively coarse, the first-order approximation for F_0 can be written as

$$F_0 = F(i+1,j) - \left.\frac{\partial F}{\partial z}\right|_{i+1,j}\Delta z. \qquad (4)$$

Since the forward difference scheme for the horizontal gradient in the Cartesian coordinate system is given by

$$\left.\frac{\partial F}{\partial x}\right|_z = \frac{F_0 - F(i,j)}{\Delta x}, \qquad (5)$$

Eq. (4) can be written as

$$\left.\frac{\partial F}{\partial x}\right|_z = \frac{F(i+1,j) - F(i,j)}{\Delta x} + \frac{\Delta z}{\Delta x}\frac{F(i+1,j) - F(i+1,j-1)}{z(j) - z(j-1)}. \qquad (6)$$

After substituting z^* for z in Eq. (6) we obtain Eq. (3).

However, let us consider now a case in which a finer vertical resolution, as illustrated in Fig. 2, is adopted. Here Δx is the same as in Fig. 1 but, the vertical grid distances were reduced five times. In this situation the best first-order approximation for F_0 will be

$$F_0 = F(i+1,j-3) - \left.\frac{\partial F}{\partial z}\right|_{i+1,j-3}\Delta z \qquad (7)$$

and not the relation given in Eq. (4). Under the current refined vertical grid resolution, Eq. (4) provides a first-order approximation to F_0 only when the field is linear

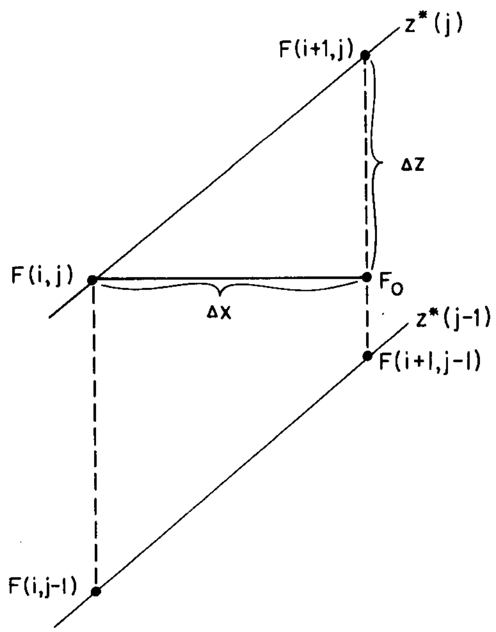

FIG. 1. A schematic illustration of the geometry involved in the calculation of the horizontal derivative in the terrain-following coordinate system.

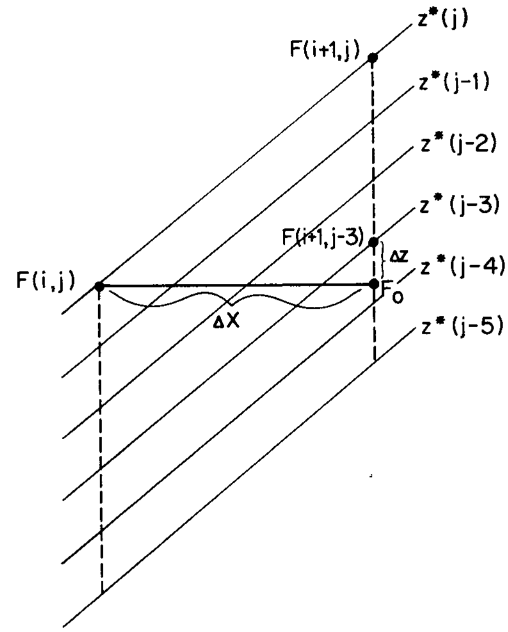

FIG. 2. As in Fig. 1, but with a five times finer vertical resolution.

between the points j and $j-4$. Eq. (4), however, is indirectly commonly used in numerical models over complex terrain, since, in general the numerical scheme given by Eq. (3) is utilized.

By substituting F_0 according to Eq. (7) into Eq. (5) and replacing z with z^* we obtain

$$\left.\frac{\partial F}{\partial x}\right|_z = \frac{F(i+1, j-3) - F(i,j)}{\Delta x} + \left[\left.\frac{\partial z^*}{\partial x}\right|_{i,j} - \frac{z^*(j-3) - z^*(j)}{\Delta x}\right] \times \frac{F(i+1, j-3) - F(i+1, j-4)}{z^*(j-3) - z^*(j-4)}, \quad (8)$$

and in the general case

$$\left.\frac{\partial F}{\partial x}\right|_z = \frac{F(i+1, M) - F(i,j)}{\Delta x} + \left[\left.\frac{\partial z^*}{\partial x}\right|_{i,j} - \frac{z^*(M) - z^*(j)}{\Delta x}\right] \times \frac{F(i+1, M) - F(i+1, M-1)}{z^*(M) - z^*(M-1)}. \quad (9)$$

Notice that the elevation at grid point (i, j) satisfies the condition

$$z(i+1, M-1) \leqslant z(i,j) \leqslant z(i+1, M)$$

and that M may be larger, smaller or equal to j, depending on the slope of the topography. Eq. (9) will be referred to later as the modified scheme.

Similarly, the expression for the horizontal gradient can be easily modified when using a centered or a backward differencing scheme. For the centered scheme, two points M and N (to the left and to the right of the point in question) have to be determined.

In the aforementioned numerical approximations, special care must be taken when $M = 1$ (Fig. 3), since then, F_0 does not exist (it is an imaginary point inside the topography). Under these conditions the distance of a point to the boundary must be calculated so that

$$\left.\frac{\partial F}{\partial x}\right|_z = \frac{F(i,j) - F_e}{\text{DIS}}.$$

Here F_e is the boundary value of F, and DIS is the horizontal distance (in the Cartesian coordinate system) of point (i, j) to the boundary.

It should be pointed out, that when using a numerical scheme (like Eq. (3), for example) for the horizontal derivative at point $(i, 3)$, as illustrated in Fig. 3, a value for F_0 is extrapolated and the actual boundary condition is not imposed.

Conclusion of the evaluation in this section suggests that in order that the numerical approximation given in Eq. (3) will be appropriate, the smallest vertical grid spacing, (Δz), should satisfy the condition:

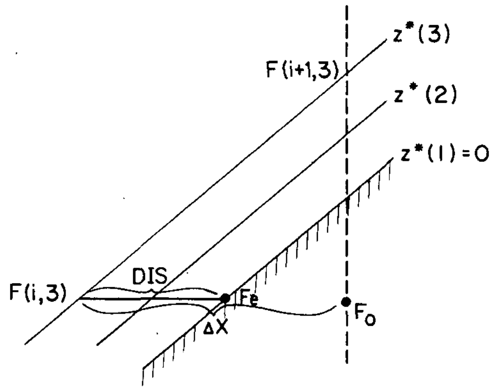

FIG. 3. A schematic illustration of the geometry involved in the calculation of the horizontal derivative in the terrain-following coordinate system near the terrain boundary with a fine vertical grid resolution.

$$\Delta z \geqslant \frac{s - z^*}{s} \Delta Z_G \quad \text{or} \quad \Delta z^* \geqslant \frac{s - z^*}{s - Z_G} \Delta Z_G,$$

where ΔZ_G is the height change of the terrain per one horizontal grid interval. Thus, at the surface, where $z^* = 0$, the smallest vertical grid interval must fulfill the requirement $\Delta z \geqslant \Delta Z_G$.

With the *sigma* coordinate system where σ is defined as

$$\sigma = \frac{P - P_t}{P_s - P_t},$$

where P is the pressure, P_s and P_t are the surface and top pressures respectively. Here, the condition of the minimum vertical grid interval will be

$$\Delta\sigma \geqslant \frac{\sigma}{P_s - P_t} \Delta P_s,$$

where ΔP_s is the change in the surface pressure per one horizontal grid interval.

3. Numerical test evaluations

In order to compare the accuracy of the "old" (Eq. 3) and the "modified" (Eq. 9) schemes, the horizontal gradients of hydrostatic pressure and temperature fields were calculated in the presence of topography. The calculations were performed in a two-dimensional (x, z) domain, consisting of 30×20 grid points. A bell-shaped mountain of the form

$$h(X) = \frac{H}{1 + (X - X_0)^2},$$

with $H = 1$ km and centered at $X = X_0$, was considered. The vertical z^* grid points were set at the heights of 0, 5, 15, 50, 100, 200, 300, 500, 700, 1000, 1500, 2000, 2500, . . . , 4500, and 5000 m. An initial horizontally uniform (in the Cartesian coordinate system) temperature field with a lapse rate of 0.01°C m^{-1} in the first

TABLE 1. Horizontal pressure differences (mb) along one grid interval at selected heights near the mountain crest as obtained with: (a) forward differencing scheme (Eq. 3), (b) centered differencing scheme, (c) modified forward scheme (Eq. 4), (d) modified centered scheme.

	Height (m)									
Scheme	5	15	50	100	200	300	500	700	1000	2000
a	1.065	0.962	0.967	0.882	0.938	0.722	0.831	0.601	0.507	0.313
b	0.298	0.266	0.278	0.219	0.283	0.129	0.252	0.103	0.176	0.139
c	0.019	0.024	0.039	0.059	0.082	0.086	0.044	0.026	0.023	0.039
d	0.017	0.019	0.025	0.030	0.045	0.041	0.022	0.015	0.006	0.003

1000 m and of 0.0065°C m^{-1} above 1000 m was prescribed. With this thermal stratification the horizontal gradients of the pressure and temperature fields should be identically zero.

The numerical evaluations for pressure and temperature differences along one horizontal grid interval with the forward and centered schemes are summarized in Tables 1 and 2 at grid points $(14, j)$ where the largest errors occurred. It can be seen that with the "old" forward scheme horizontal pressure differences were as large as 1.065 mb per one horizontal grid interval, while with the "modified" forward scheme they were at least 10 times smaller. For temperatures, the differences with the modified forward scheme were practically zero, while with the "old" forward scheme they were larger than 1°C at the lower layers. In the upper layers, where the prescribed temperature distribution is continuously linear, both schemes evaluations were correct.

With the centered scheme, the errors were somewhat reduced (Tables 1 and 2, b) but were still substantially larger than the errors with the "modified" centered scheme (Tables 1 and 2, d).

In order to test the effect of increasing the vertical grid interval on the accuracy of the schemes the horizontal pressure gradients were evaluated with a coarser vertical grid spacing. The lowest z^* points were set at the heights of 0, 5, 100, 300, and 700 m. From Table 3 it can be seen that with the centered scheme the horizontal differences were somewhat reduced at equivalent heights. With the modified scheme, however, errors have increased, but were still less than with the centered scheme. These results show that the modified scheme is consistent with convergence of numerical approximations to the true value as the grid interval is reduced. The "old" scheme, however, suffers from the problem that the errors do not necessarily decrease when the grid interval is reduced (Mesinger, 1982).

4. Conclusions

It has been shown that the proposed numerical schemes for the approximation of the horizontal gradients in a terrain-following coordinate system significantly improves the accuracy when the vertical grid is highly refined. This method works well with very steep slopes and when the vertical grid interval is smaller than the elevation difference between two horizontally adjacent terrain points. Further improvement can be made, when the vertical resolution near the surface is very small, by using the actual horizontal distance between a grid point and the terrain boundary, instead of the original horizontal grid interval. In this paper the sources of the numerical deficiencies associated with the application of conventional schemes in TFCS are pointed out. The proposed procedure can be easily applied when using the forward, backward or centered differencing schemes. It is important to notice that reducing the horizontal grid interval can be a way to eliminate the errors. However, when steep slopes are considered this will not be practical because of the significant increase in computational expenses. Finally, it is worth noting that models that use the *sigma* coordinate system are also subject to a similar error.

TABLE 2. Horizontal temperature differences (°C) along one grid interval at selected heights near the mountain crest as obtained with: (a) forward differencing scheme (Eq. 3), (b) centered differencing scheme, (c) modified forward scheme (Eq. 4), (d) modified centered scheme.

	Height (m)									
Scheme	5	15	50	100	200	300	500	700	1000	2000
a	1.734	1.702	1.590	1.429	1.108	0.787	0.146	0.000	0.000	0.000
b	0.867	0.851	0.794	0.715	0.554	0.394	0.073	0.018	0.000	0.000
c	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.008	0.007	0.000
d	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.000

TABLE 3. Horizontal pressure differences (mb) along one grid interval at selected heights near the mountain crest as obtained for the coarse vertical grid spacing with: (a) centered differencing scheme, (b) modified centered scheme.

	Height (m)					
Scheme	5	100	300	700	1000	2000
a	0.209	0.174	0.113	0.096	0.150	0.139
b	0.024	0.029	0.050	0.024	0.007	0.003

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